THE STRAIGHT LINE (3)

Art. 12. The equation of a straight line

(a) Parallel to the x axis is y = constant

(b) Parallel to the y axis is x = constant

Art. 13. Slope intercept form : The equation of a straight line which cuts of a intercept on the axis of y and is inclined at a given angle θ to the positive direction of x axis is

$y = x\tan\theta + c$

or, $y = mx + c$ Where $m = \tan\theta$ ... ... (1)

Cor. If the straight line passes through the orgin (0, 0) then the equation of the line is $y = mx$ ... ... (2)

Art. 14. Points-Slope Form : The equation of the straight line of slope m passing through the point P (x, y) is

$y - y_1 = m(x - x_1)$ ... ... (1)

Cor. $y - y_1 = m(x - x_1)$ or, $y - y_1 = \tan\theta\ (x - x_1)$

$\frac{y-y_1}{\sin\theta} = \frac{x-x_1}{\cos\theta} = r$ (say) or $\frac{x-x_1}{l} = \frac{y-y_1}{m} = r$ ... ... (2)

or, $x = x_1 + rl$ ... ... (3)

$y = y_1 + rm$

Equations in (3) are called Parametric equation of a straight line. It is generally written as $\frac{y-y_1}{l} = \frac{x-x_1}{m} = r$ ... ... (4)

Note : r is the distance between any point (x, y) and the given point (x_1, y_1) on the line

Art. 15. Two point Form : The equation of a straight line passing through the two points P (x_1, y_1) and Q (x_2, y_2) is $\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$

Art. 16. Intercept Form :

The equation of a straight line which cuts off intercepts a and b from the axis is $\frac{x}{a} + \frac{y}{b} = 1$

Since this straight line passes through $(a, 0)$ and $(0, b)$: put $x_1 = a$, $x_2 = 0$, $y = 0$, $y_2 = b$ in eq (1) of [Art. 15]

Art. 17. Normal form of the equation of a straight line is $x\cos\theta + y\sin\theta = p$

where, p is the length of the perpendicular form the origin to the line and θ is the angle made by perpendicular with the positive direction of x-axis

Art. 18. Equations of the line passing through the intersection of two lines.

The equation of the line passing through the intersection of two lines $P = ax + by + c = 0$ and $Q = a_1x + b_1y + c = 0$

$$ax + by + c + \lambda(a_1x + b_1y + c_1) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where λ is an arbitrary constant.

$$\text{or } P + \lambda Q = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Art. 19. Every first degree equation in x and y represents a straight line and conversely every straight line can be represented by a first degree equation in x and y.

The general equation of the first degree is of the form $Ax + By + C = 0$ where A, B, C are constants.

Art. 20. Perpendicular distance :

The length of the perpendicular distance d from a point $P(x_1, y_1)$ on the line, $ax + by + c = 0$ is $d = \pm \dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$ ... ... (1)

Positive sign is choosen when c is positive and the negative sign when c is negative. This formula gives positive when P lies on same side of the line of the origin, a negative result when P lies on the opposite sides of the origin.

Art. 21. Bisector of angles between two lines :

The equation of the bisectors of the angle between two straight lines $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$ are

$$\frac{ax + y + c}{\sqrt{(a^2 + b^2)}} = \frac{a_1x + b_1y + c_1}{\pm\sqrt{(a_1^2 + b_1^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Art. 22. Condition for the Concurrency of three lines.

Let the equations to the three lines be

$$a_1x + b_1y + c_1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$a_3x + b_3 + c_3 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Let (x_1, y_1) be the common point of intersection. Since all the straight lines are passing through this point.

$$a_1x_1 + b_1y_1 + c_1 = 0,\ a_2x_1 + b_2y_1 + c_2 = 0,\ a_3x_1 + b_3y_1 + c_3 = 0$$

Eliminating x_1 and y_1 from the above equation, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

which is the required condition of the concurency of three straight lines

Art. 23. Condition for Collinearity of three points.

$P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$

If each of them lies on the line $ax + by + c = 0$, then

$$ax_1 + by_1 + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$ax_2 + by_2 + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$ax_3 + by_3 + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Eliminating a, b and c, From (1) (2) and (3), we have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

which is the required condition for collinearity of the points.

Art. 24. Area of a triangle formed by three given lines.

$$a_1x + b_1y + c_1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$a_3x + b_3y + c_3 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad B_1 = \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}, \quad C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

and similar expression for A_2, B_2, C_2, A_3, B_3, C_3. It is evident that A_1 is the co-factor a_1 in Δ

Solving (2) and (3) the point of intersection (x_1, y_1) is given by $x_1 = A_1/C_1$, $y_1 = B_1/C_1$

Similarly, the point of intersection (x_2, y_2) of (3) and (1) is given by $x_2 = A_2/C_2$, $y_2 = B_2/C_2$ and the point of intersection (x_3, y_3) of (1) and (2) is given by $x_3 = A_3/C_3$ and $y_3 = B_3/C_3$

The area of the triangle with three points as vertices is given by

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2C_1C_2C_3}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \frac{\Delta^2}{2C_1C_2C_3}$$

(See Author's Algebra)

Art. 25. Angle between two lines.

The angle ϕ between the two lines $y = m_1x + c_1$ ... ... (i) and $y = m_2x + c_2$ is ... (ii)

$\tan\phi = \pm\dfrac{m_1 - m_2}{1 + m_1m_2}$ ... ... (1)

Where m_1 and m_2 are the slopes of the lines.

If the two straight lines are $a_1x + b_1y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$, then

$\tan\phi = \pm\dfrac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$

Art. 26. (a) Condition of parallelism of these (i) and (ii)

is $m_1 = m_2$ ... ... (i)

or, $a_2b_1 - a_1b_2 = 0$ or, $a_1/a_2 = b_1/b_2$

(b) Condition of Perpendicularity of these lines (i) and (ii) is $m_1m_2 = -1$

or, $a_1a_2 + b_1b_2 = 0$ ... . (3)

(1) The equation to the line parallel to $ax + by + c = 0$ is $ax + by + k = 0$

where k is a constant to be determined by a suitable condition. ... ... (1)

(2) The equation to the line Perpendicular to $ax + by + c = 0$ is $bx - ay + k = 0$

where k is an arbitrary constant to be determined by a suitable condition

Rule : To obtain the equation of a line perpendicular to a line whose equation is given, interchange the co-efficients of x and y, change the sign of y and suitably change the constant term.

Art. 27. Equation of a straight line in polar co-ordinates.

Let LM be the given straight line, OA the initial line.

ON the perpendicular from the pole O on the line LM. Let ON = p. and the angle which P makes with the initial line viz the $\angle AON = \alpha$. are given.

Let P be any Pt. (r, θ) on LM. Then OP = r, and $\angle AOP = \theta$ so that $\angle NOP = \theta - \alpha$

But from the ΔONP, right angled at N, we have

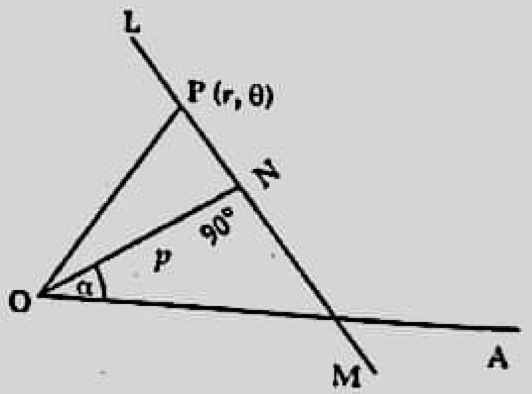

Fig. 8

OP cos NOP = ON, Therefore

$r\cos(\theta - \alpha) = p$

which is the required equation.

Cor. 1. The equation of the line LM in perpendicular form is

$x\cos\alpha + y\sin = p$

Putting $x = r\cos\theta$, $y = r\sin\theta$ (to transform into polar co-ordinates), we get $r\cos(\theta - \alpha) = p$

Cor. 2. The length of the perpendicular from (r_1, θ_1) on this straight line is evidently $= r_1 \cos(\theta_1 - \alpha) - p_1$

Cor. 3. The general equation of the straight line in Cartesian co-ordinate is $Ax + By + C = 0$

Transforming into polar co-ordinates, the equation becomes

$A r \cos\theta + B r \sin\theta + C = 0$

or, $A\cos\theta + B\sin\theta + C/r = 0$ ... ... (1)

which is the most general form of the equation of a straight line in polar co-ordinates.

Cor. 4. The equation to the line perpendicular to (i) is

$A\cos(\theta + \pi/2) + B\sin(\theta + \pi/2) + C'/r = 0$

or, $B\cos\theta - A\sin\theta + C'/r = 0$

Art. 28. Equation of a straight line through two given points whose co-ordinates are (r_1, θ_1) and (r_2, θ_2)

Let the given points be $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$, P be any point on the line P_1P_2 and has the co-ordinates (r, θ).

Then, since, $\Delta P_1OP_2 = \Delta P_1OP + \Delta POP_2$

$\therefore \frac{1}{2} r_1 r_2 \sin(\theta_1 - \theta_2) = \frac{1}{2} r r_1 \sin(\theta_1 - \theta) + \frac{1}{2} r r_2 \sin(\theta - \theta_2)$

or, $r_1 r_2 \sin(\theta_1 - \theta_2) = r r_1 \sin(\theta_1 - \theta) + \theta\, r r_2 \sin(\theta - \theta_2)$

or, $\frac{\sin(\theta_1 - \theta_2)}{r} = \frac{\sin(\theta - \theta_2)}{r_1} + \frac{\sin(\theta_1 - \theta)}{r_2}$

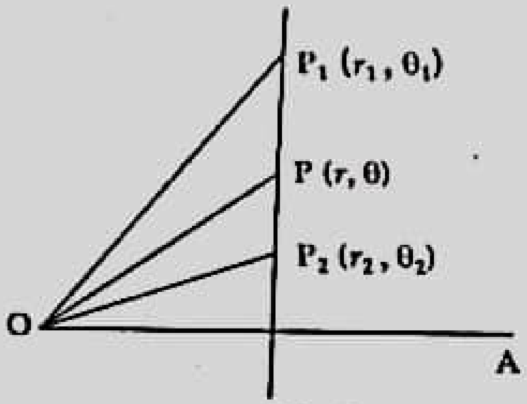

Fig. 9

Art. 29. Equation of the line passing through the intersection of given lines.

Let the equation to the given line be $r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \alpha') = p'$. The equation to any straight line through the intersection of the given lines is.

$\{r\cos(\theta - \alpha) - p\} + \lambda\{r\cos(\theta - \alpha') - p'\} = 0$

If this line passes through a given point (r_1, θ_1) then

$\{r_1\cos(\theta_1 - \alpha) - p\} + \lambda\{r_1\cos(\theta_1 - \alpha') - p'\} = 0$

Hence λ is obtained.

Ex. 1. Find the equations to lines passing through (–5, 6)and (a) parallel (b) perpendicula to $7x - 8y = 9$.

(a) Let the equation parallel to $7x - 8y - 9 = 0$ be $7x - 8y = k$ since it passes throug $(-5, 6)$: $7(-5) - 8(6) = k$ or, $-35 - 48 = k$ or, $k = -83$

The required equation is $7x - 8y + 83 = 0$

(b) Let the equation perpendicular to $7x - 8y - 9 = 0$ be $8x + 7y = k$. since it passes throug $(-5, 6)$ of $8(-5) + 7(6) = k$ or, $-40 + 42 = k$ or, $k = 2$

The required equations is $8x + 7y - 2 = 0$