MTH 101

CALCULUS & ANALYTICAL GEOMETRY

LESSON 01

What is calculus?

The study of continuous rates of change of quantities.

Natural numbers:-

$\{1, 2, 3, 4, \ldots\}$

Whole numbers:-

$\{0, 1, 2, 3, 4, \ldots\}$

Integers:-

$\{0, \pm 1, \pm 2, \pm 3, \ldots\}$

Rational no.:-

which can be written in form of p/q $\left[\frac{2}{3}, \frac{4}{5}\right]$

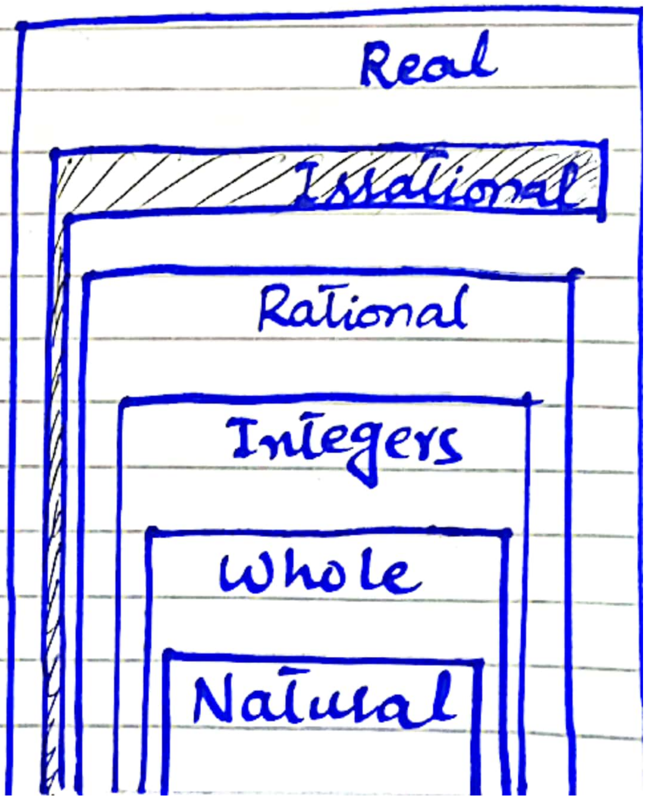
Irrational no.

which cannot be written as $\frac{p}{q}$ $(\sqrt{2}, \pi)$

Prime no.

$\{2, 3, 5, 7, 11, 13, 17, \ldots\}$

non-recurring & non-terminating numbers are called irrational numbers.



Sets:-

=) A collection of things

=) Always use curley brackets for representing sets.

=) The objects of a set are called elements.

=) If set A lies in set B then A is called the subset of B. denoted as $A \subseteq B$. For example if $A = \{1, 2, 3\}$ & $B = \{1, 2, 3, 4, 5\}$ then $A \subseteq B$.

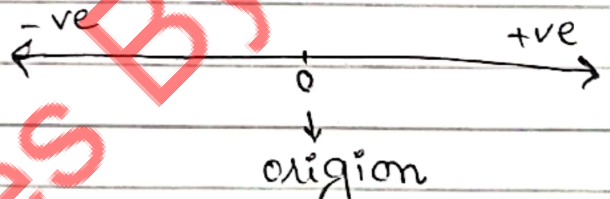
=) $\in$ symbol is used for showing an element of a set.

For example:

$$A = \{a, b, c\}$$

Then $a \in A$ but $d \notin A$.

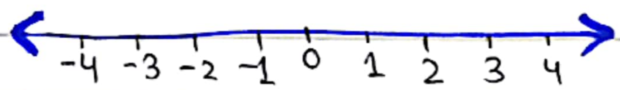
Co-ordinate line:-



The real number ~~core~~ corresponding to a point on the line is called co-ordinate of the point.

$\sqrt{2} = 1.41$

$\pi = 3.14$



Inequality:-

An expression involving $<, >, \leq, \geq, \neq$

For example $3 < 8$, $-3 > -8$

Remark:-

$a > 0$ is a positive number.
$a \geq 0$ is non-negative.

"A non-negative no. is either zero or positive."

THEOREM 1.2.1

a) If $a < b$ & $b < c$ Then $a < c$.

b) If $a < b$ & $a + c < b + c$ Then $a - c < b - c$

c) If $a < b$ & $ac < bc$ when c is +ve
& $ac > bc$ if c is −ve.

d) If $a < b$ & $c < d$ Then $a + c < b + d$

e) If a & b are both +ve or both −ve
& $a < b$ Then $\frac{1}{a} > \frac{1}{b}$

Union & Intersection of a Set

In union we combine all elements.

$B \cup A = \{1, 2, 3, 6\} \cup \{1, 2, 3, 4, 5\}$

$= \{1, 2, 3, 4, 5, 6\}$

$B \cap A = \{1, 2, 3, 6\} \cap \{1, 2, 3, 4, 5\}$

$= \{1, 2, 3\}$.

Empty set is represented by $\{\ \}$ or ϕ read as phi.

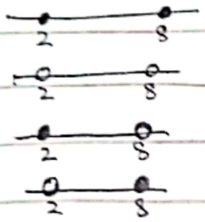
Intervals:-

geometrically an interval is a line segment on the co-ordinate line.

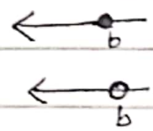
The interval of this line segment b/w 2 & 8 will be written as

$[2, 8]$ or $\{x : 2 \le x \le 8\}$

2 & 8 are end points.



$[2, 8]$	closed interval	$\{x : 2 \le x \le 8\}$
$(2, 8)$	open interval	$\{x : 2 < x < 8\}$
$[2, 8)$	half open	$\{x : 2 \le x < 8\}$
$(2, 8]$	half open	$\{x : 2 < x \le 8\}$



$(-\infty, b]$	closed	$\{x : x \le b\}$
$(-\infty, b)$	open	$\{x : x < b\}$

Solving inequalities:-

Example:-

$$3+7x \le 2x-9$$

$7x-2x \le -9-3$

$5x \le -12$

$x \le \frac{-12}{5}$

OR

$3+9 \le 2x-7x$

$12 \le -5x$

$-\frac{12}{5} \ge x$

Example:- $7 \le 2-5x \le 9$

$7 \le 2-5x$

$5x \le 2-7$

$5x \le -5$

$x \le -1$

$2-5x \le 9$

$2-9 \le 5x$

$-7 \le 5x$

$-7/5 \le x$

$$-1 \geq x \leq -\frac{7}{5}$$

alternate:-

$$7 \leq 2-5x \leq 9$$

$$7-2 \leq \cancel{2}-5x \leq 9-2$$

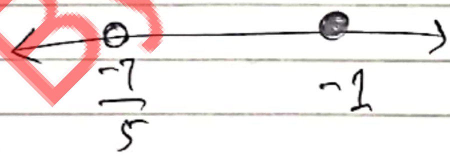
$$5 \leq -5x \leq 7$$

$$\frac{5}{5} \leq -x \leq \frac{7}{5}$$

$$1 \leq -x \leq \frac{7}{5}$$

$$-1 \geq x \geq -\frac{7}{5}$$

if $-1 \geq x > -\frac{7}{5}$



LECTURE 02

Absolute value:-

denoted by $|a|$

$$|a| = \begin{cases} a & \text{if } a \geq 0 \text{ (non-negative)} \\ -a & \text{if } a < 0 \text{ (negative)} \end{cases}$$

For example

$|5| = 5$, $|-5| = 5$ } It is basically the no. of points moved from origion on either side.

Example:-

$$|x-3| = 4$$

$x-3 = \pm 4$

$x-3 = 4$	$x-3 = -4$
$x = 4+3$	$x = -4+3$
$x = 7$	$x = -1$

Example:-

$$|3x-2| = |5x+4|$$

$(3x-2) = \pm(5x+4)$

$3x-2 = 5x+4$	$3x-2 = -(5x+4)$
$3x-5x = 4+2$	$3x-2 = -5x-4$
$-2x = 6$	$3x+5x = -4+2$
$x = -3$	$8x = -2$
	$x = -2/8$
	$x = -\frac{1}{4}$

Relation b/w square root & absolute value:-

$$\sqrt{a^2} = |a|$$

because

$\sqrt{a^2} = \pm a$, $|a| = \pm a$

(e.g) $\sqrt{(7)^2} = \sqrt{(-7)^2} = \sqrt{49}$

Properties:-

$$|-a| = |a|$$

$$|ab| = |a||b|$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

Distance Formula:-

The distance b/w a & b can be found by

$$d = |b-a|$$

$$d = \begin{cases} b-a & \text{if} \quad a<b \\ a-b & \text{if} \quad a>b \\ 0 & \text{if} \quad a=b \end{cases}$$

$|x-a|$ = distance b/w x & a

$|x+a|$ = distance b/w x & $-a$

$|x|$ = distance b/w x & origion

Solve:- $|x-3| < 4$

$x-3 < \pm 4$

$x-3 < 4$	$x-3 < -4$
$x < 4+3$	$x < -4+3$
$x < 7$	$x < -1$

Solve:-

$|x+4| \geq 2$

$x+4 \geq \pm 2$

$x+4 \geq 2$	$x+4 \geq -2$
$x \geq 2-4$	$x \geq -2-4$
$x \geq -2$	$x \geq -6$

$$-2 \leq x \geq -6$$

interval notation:-

$$(-\infty, -6] \cup [-2, +\infty)$$

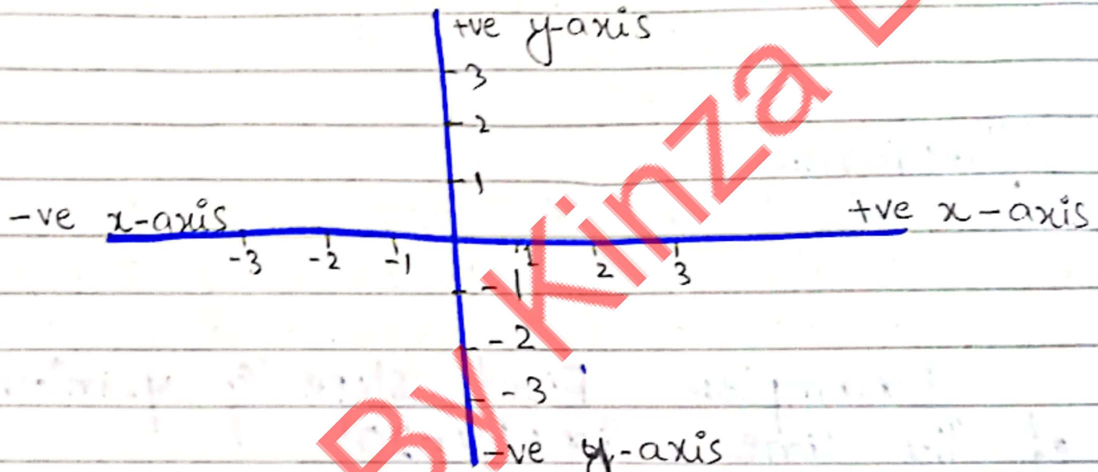
Triangle inequality:-

$$|a+b| \leq |a| + |b|$$

LECTURE 03

Plane:-

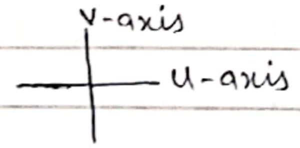
A Plane is intersection of two co-ordinate lines at 90 degrees. It is called a co-ordinate plane.



called xy-plane.

always write the name of horizontal axis first & then the name of y-axis (vertical a

uv-plane



Example:-

Does (3,2) satisfy $6x-4y=10$

$6x-4y=10$

put $x=3$ & $y=2$

$6(3)-4(2)=10$

$18-8=10$

$10=10$

So (3,2) is solution of $6x-4y=10$.

Intercepts:-

In $(a,0)$ & $(0,b)$ [redacted] a & b are called x-intercept & y-intercept respectively.

Example:

Find all intercepts of

a) $3x+2y=6$

For x-intercept put $y=0$

$3x+2(0)=6$

$3x=6$

$\boxed{x=2}$

For y-intercept put $x=0$

$3(0)+2y=6$

$2y=6$

$\boxed{y=3}$

b) $x=y^2-2y$

x-intercept

put $y=0$

$x=0^2-2(0)$

$\boxed{x=0}$

y-intercepts:

put $x=0$

$0=y^2-2y$

$0=y(y-2)$

$\boxed{y=0}$ or $y-2=0$

$\boxed{y=2}$

(c) $y=\dfrac{1}{x}$

x-intercept

$0=\dfrac{1}{x}$

$0\cdot x=1$

It is undefined.

y-intercept

$y=\dfrac{1}{0}$

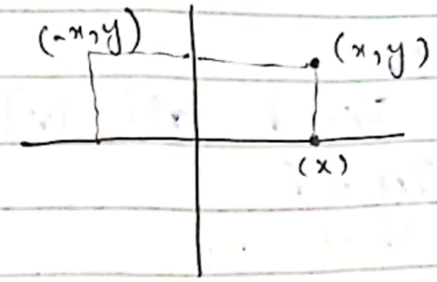
$y=\infty$

It is also undefined.

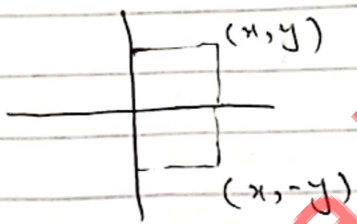
No x & y-intercepts.

Symmetry:-

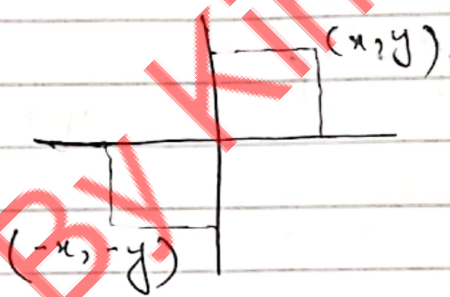
(x,y) & $(-x,y)$ are symmetric about y-axis.



(x,y) & $(x,-y)$ are symmetric about x-axis.



(x,y) & $(-x,-y)$ are symmetric about origion.



Example:-

$$y = \frac{1}{8}x^4 - x^2$$

put $x = -x$

$$y = \frac{1}{8}(-x)^4 - (-x)^2$$

$$y = \frac{1}{8}x^4 - x^2$$

As the answer is not changed so this graph will be symmetric about y-axis.

Example:- $x = y^2$

Here if we put $y = -y$

The equation remains unchanged.

So $x = (-y)^2$

$x = y^2$

So its graph is symmetric about x-axis.

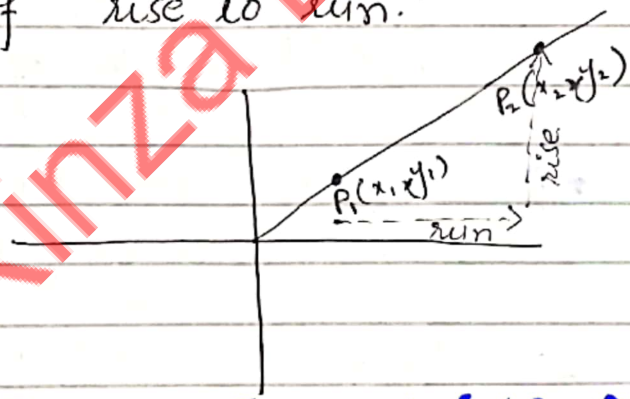
LECTURE 04

Slope:- ("rate of change of y with respect to x".)

"The ratio of rise to run."

represented by "m"

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$



Example:-

Find slope of The (i) points (6, 2) & (9, 8)

$(x_1, y_1) = (6, 2)$ & $(x_2, y_2) = (9, 8)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow m = \frac{8-2}{9-6} = \frac{6}{3} = 2$$

(ii) points (-2, 7) & (5, 7)

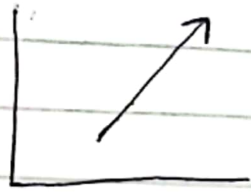
$(x_1, y_1) = (-2, 7)$ & $(x_2, y_2) = (5, 7)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow m = \frac{7-7}{5-(-2)} = \frac{0}{5+2} = \frac{0}{7} = 0$$

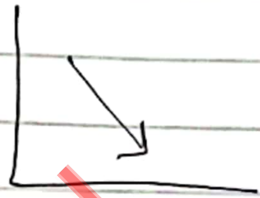
It's formula can also be written as

rise = m. run

⇒ A **+ve slope** means that the line is inclined upward to the right. (e.g $m=2$)



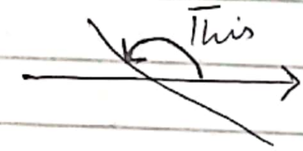
⇒ A **-ve slope** means that the line is inclined downward to the right. (e.g $m=-3$)



⇒ A **zero slope** means that the line is horizontal. (e.g $m=0$)

Angle of inclination:-

The angle of the line with +ve x-axis is called angle of inclination.



⇒ For a line parallel to x-axis this angle is always zero. ($\phi=0$)

⇒ It's value is alway between $0°$ & $180°$.
(i-e) $0° \leq \phi \leq 180°$ or $0 \leq \phi \leq \pi$

Theorem:-

For a non-vertical line, the slope m & angle of inclination ϕ are related by
$$m = \tan\phi$$

⇒ If line is parallel to y-axis (perpendicular to x-axis) then $\phi = 90°$ or $\phi = \frac{\pi}{2}$

So $\tan\phi = \tan 90° = $ undefined.

Example:-

Find angle of inclination for a line of slope $m=1$ & also for a line of slope $m=-1$.

For $m = 1$

$m = \tan\phi$

$1 = \tan\phi$

$\tan^{-1}(1) = \phi$

$45^\circ = \phi$

$\frac{\pi}{4} = \phi$

For $m = -1$

$m = \tan\phi$

$-1 = \tan\phi$

$\tan^{-1}(-1) = \phi$

$\frac{3\pi}{4} = \phi$

Theorem:-

Slope for parallel lines:-

The lines are parallel if & only if their slopes are same.

$$m_1 = m_2$$

Slope for perpendicular lines:-

The lines are perpendicular if & only if their slopes are negative Reciprocals of one another or the product of their slopes is always -1.

$$m_1 m_2 = -1$$

or

$$m_1 = \frac{-1}{m_2}$$

Example:-

Use slopes to show $A(1,3)$, $B(3,7)$ & $C(7,5)$ are vertices of right Triangle.

Slope of A & B $= m_1 = \frac{7-3}{3-1} = \frac{4}{2} = 2$

Slope of B & C $= m_2 = \frac{5-7}{7-3} = \frac{-2}{4} = \frac{-1}{2}$

Here

$$m_1 m_2 = (2)\left(\frac{-1}{2}\right) = -1$$

So the $\overline{AB}$ is perpendicular to BC.
So ABC is a right Triangle.

Vertical line:-

In which $(a, 0)$ where $y = 0$
for example in $(-5, 0)$ $x = -5$ & $y = 0$

Horizontal line:-

In which $(0, b)$ where $x = 0$
for example in $(0, 7)$ $x = 0$ & $y = 7$

Theorem:-

point-slope form

$$y - y_1 = m(x - x_1)$$

Example:-

Write an equation of line Through The point $(2, 3)$ with slope $\frac{-3}{2}$.

$(x_1, y_1) = (2, 3)$ & $m = \frac{-3}{2}$

using point-slope form

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{-3}{2}(x - 2)$$

$$2y - 6 = -3(x - 2)$$

$$2y - 6 = -3x + 6$$

$$3x + 2y - 6 - 6 = 0$$

$$3x + 2y - 12 = 0$$

or

$$2y = -3x + 12$$

$$y = \frac{-3x + 12}{2}$$

$$y = \frac{-3}{2}x + 6$$

Example:- Write an eq. for the line through the point $(-2,-1)$ & $(3,4)$.

$$m = \frac{4-(-1)}{3-(-2)} = \frac{4+1}{3+2} = \frac{5}{5} = 1$$

we can use either of the two points in point-slope form.

for $(3,4)$

$$y - y_1 = m(x - x_1)$$
$$y - 4 = 1(x - 3)$$
$$y - 4 = x - 3$$
$$y - x - 4 + 3 = 0$$
$$y - x - 1 = 0$$
or
$$y = x + 1$$

Theorem:-

Slope intercept form.

$$y = mx + b \quad (b \text{ is } y\text{-intercept})$$

Example:-

$$y = 2x - 5$$

here comparing with $y = mx + b$

$$m = 2 \quad \& \quad b = -5$$

Example: Find ~~slope~~ equation where slope is -9 & point is $(0,-4)$

here $m = -9$ & $b = -4$

$$y = mx + b$$
$$y = -9x - 4$$

Example:-

Find slope intercept form of the eq. of line that passes through $(3,4)$ & $(-2,-1)$

$$\text{Slope} = m = \frac{-1-4}{-2-3} = \frac{-5}{-5} = 1$$

Slope intercept form

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = 1(x - (-2))$$

$$y + 1 = x + 2$$

General equation of a Line:-

$$Ax + By + C = 0$$ where A & B are non-zero.

It is also called first degree equ.

For example:- $4x + 6y - 5 = 0$

Theorem:-

Every first degree equation in x, y has a straight line graph or every straight line graph has a first degree equ.

Example:- Find slope & y-intercept of the line $8x + 5y = 20$

$$5y = -8x + 20$$

$$y = \frac{-8x}{5} + \frac{20}{5}$$

$$y = -\frac{8}{5}x + 4$$

here $m = \frac{-8}{5}$ & $b = 4$.

Slope is important bcz it gives us way to say how steep is something.

Lecture 05

Distance Formula:-

distance between two points $A(x_1, y_1)$ & $B(x_2, y_2)$ will be

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Example:

Find distance b/w $(-2, 3)$ & $(1, 7)$.

$$d = \sqrt{[1-(-2)]^2 + (7-3)^2}$$

$$= \sqrt{(1+2)^2 + (4)^2}$$

$$= \sqrt{3^2 + 4^2}$$

$$= \sqrt{9+16}$$

$$= \sqrt{25}$$

$$\boxed{d = 5}$$

Example:- Show that the points $A(4,6)$, $B(1,-3)$ $C(7,5)$ are vertices of a right triangle.

$$\overline{AB} = \sqrt{(1-4)^2 + (-3-6)^2} = \sqrt{(-3)^2 + (-9)^2} = \sqrt{9+81} = \sqrt{90}$$

$$\overline{AC} = \sqrt{(7-4)^2 + (5-6)^2} = \sqrt{3^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$$

$$\overline{BC} = \sqrt{(7-1)^2 + (5-(-3))^2} = \sqrt{6^2 + (5+3)^2} = \sqrt{6^2 + 8^2} = \sqrt{36+64} = \sqrt{100}$$

using Pythagorous Theorm

$$(\overline{BC})^2 = (\overline{AB})^2 + (\overline{AC})^2$$

$$(\sqrt{100})^2 = (\sqrt{90})^2 + (\sqrt{10})^2$$

$$100 = 90 + 10$$

$$100 = 100$$

So proved that A, B, C are vertices of right Triangle.

Mid-Point Formula:-

mid point of two points $A(x_1, y_1)$, $B(x_2, y_2)$ is

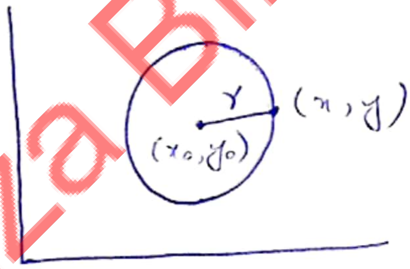
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Example:- Find mid-point of $(3,-4)$ & $(7,2)$.

$$\left(\frac{3+7}{2}, \frac{-4+2}{2}\right)$$

$$= \left(\frac{10}{2}, \frac{-2}{2}\right) \Rightarrow (5,-1)$$

Circles:-



by distance formula

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = r$$

or $(x-x_0)^2+(y-y_0)^2 = r^2$ (called standard form of equation of circle).

Example:-

Find equation of circle of radius 4 centered at $(-5,3)$.

here $(x_0, y_0) = (-5,3)$ & $r = 4$

using standard eqn of circle.

$$[x-(-5)]^2+(y-3)^2 = (4)^2$$

$$(x+5)^2+(y-3)^2 = 16$$

$$x^2+25+10x+y^2+9-6y-16=0$$

$$x^2+y^2+10x-6y+18=0$$

Here is the equation of circle.

Example:- Find equ of circle with center $(1,-2)$ That passes Through $(4,2)$.

using distance formula

$$r = \sqrt{(1-4)^2 + (-2-2)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25}$$

$$\boxed{r = 5}$$

Here $(x_0, y_0) = (1,-2)$ & $r = 5$

Put in equ of circle

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

$$(x-1)^2 + (y-(-2))^2 = r^2$$

$$(x-1)^2 + (y+2)^2 = r^2$$

$$x^2 - 2x + 1 + y^2 + 4y + 4 = (5)^2$$

$$x^2 + y^2 - 2x + 4y + 5 - 25 = 0$$

$$\boxed{x^2 + y^2 - 2x + 4y - 20 = 0}$$

Equation of circle	center	radius
$(x-2)^2 + (y-5)^2 = 9$	$(2,5)$	3
$(x+7)^2 + (y+1)^2 = 16$	$(-7,-1)$	4
$x^2 + y^2 = 25$	$(0,0)$	5
$(x-4)^2 + y^2 = 5$	$(4,0)$	$\sqrt{5}$

$x^2 + y^2 = 1$ This circle is centered at origion & has radius 1. It is called <u>unit circle</u>.

Another form of equ of circle:-

$$x^2 + y^2 + dx + ey + f = 0$$

or $Ax^2 + By^2 + Dx + Ey + F = 0$

Example:- Find center & radius of

(i) $x^2+y^2-8x+2y+8=0$

$$(x^2-8x)+(y^2+2y)+8=0$$

$$x^2-2(x)(4)+(4)^2-(4)^2+y^2+2(y)(1)+1^2-1^2+8=$$

$$(x-4)^2-16+(y+1)^2-1+8=0$$

$$(x-4)^2+(y+1)^2-9=0$$

$$(x-4)^2+(y+1)^2=9$$

here $\boxed{(x_0,y_0)=(4,-1)}$ & $\boxed{r=3}$

(ii) $2x^2+2y^2+24x-81=0$

divide b/s by 2

$$\frac{2x^2}{2}+\frac{2y^2}{2}+\frac{24x}{2}-\frac{81}{2}=\frac{0}{2}$$

$$x^2+y^2+12x-\frac{81}{2}=0$$

$$x^2+12x+y^2-\frac{81}{2}=0$$

$$x^2+2(x)(6)+(6)^2-(6)^2+y^2-\frac{81}{2}=0$$

$$(x+6)^2-36+y^2-\frac{81}{2}=0$$

$$(x+6)^2+y^2=\frac{81}{2}+36$$

$$(x+6)^2+y^2=\frac{153}{2}$$

Here $\boxed{(x_0,y_0)=(-6,0)}$, $\boxed{r=\sqrt{\frac{153}{2}}}$

Degenerate cases of circle:-

In $(x-x_0)^2+(y-y_0)^2=k$

(k>0) The graph is a circle & centered at (x_0,y_0) & Radius k.

(k=0) The only solution of the equation is $x=x_0$, $y=y_0$. So graph is a single point (x_0,y_0)

(k<0) The equation has ~~only~~ no real graph & no real Solutions.

Example:- Describe graph of

(a) $(x-1)^2+(y+4)^2=-9$

As $k<0$, so This equation has no real solution & no graph.

(b) $(x-1)^2+(y+4)^2=0$

The only values of x & y that will make the left side 0 are (1,-4). So graph is a single point at (1,-4).

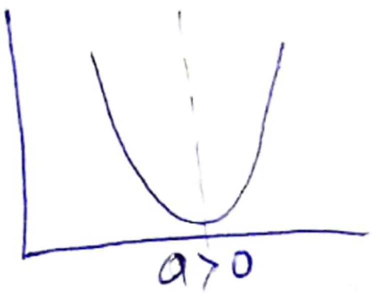
Quadratic equation:-

The equation of the form $y=ax^2+bx+c$ is called a quadratic equation.

Graph of quadratic equ.

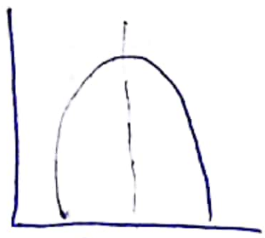
The graph of quadratic equ is a parabola.

If a is +ve then graph is



$a>0$

If a is -ve then graph is



$a<0$

parabola is symmetric about a line parallel to y-axis.

x-coordinate of vertex of parabola can be found as $\boxed{x = \frac{-b}{2a}}$ & vertex has highest point if $a<0$ & a lowest point if $a>0$.

Example: Find vertex of $y = -x^2+4x-5$.

$$x = \frac{-b}{2a} \qquad a=-1,\ b=4,\ c=-5$$

$$x = \frac{-4}{2(-1)} \Rightarrow \frac{-4}{-2} \Rightarrow \boxed{x=2}$$

Example: (on page 55). see statement.

$s = 24.5t - 4.9t^2$

(a) vertex(t) $= \frac{-b}{2a} = \frac{-24.5}{2(-4.9)} = 2.5$ sec

(b) height of the graph is max or min, depending on depending on whether the graph opens up or down.

$$S = 24.5t - 4.9t^2$$
$$S = 24.5(2.5) - 4.9(2.5)^2$$
$$S = 30.626\,m$$

So the ball rises 30.626m above the ground.

In $x = ay^2+by+c$ the vertex is $y = -b$ & graphs will be symmetric about a line parallel to x-axis.

$a>0$

$a<0$

Lecture 06

Function: This term was used first by French Mathematician Leibiniz. He used it to denote dependence of one quantity on other.

Example:- In $A = \pi r^2$, Area of circle depends upon radius r. So Area is a function of r.

Defination: If a quantity y depends upon other quantity x in such a way that each value of x determines exactly one value of y. We say y is a function of x.

$y = 4x + 1$ is a function.

$y = \pm\sqrt{x}$ is not a function bcz y gives two values if one value of x is put.

Notations:- Euler introduced the notation $y = f(x)$ & y is a function of x. read as "y equals f of x" here x is independent variable & y is a dependent variable.

Example:- $y = f(x) = x^2$ find $f(3)$ & $f(-2)$

$f(3) = (3)^2 \Rightarrow 9$ & $f(-2) = (-2)^2 = 4$

Remark:-

Any letter can be used instead of f.

$y = f(x)$, $y = g(x)$, $y = h(x)$ all are same

In $s = f(t)$ t is independent & s is dependent

Example:- If $\phi(x) = \frac{1}{x^3-1}$

find $\phi(5^{\frac{1}{6}})$ & $\phi(1)$

$$\phi(5^{\frac{1}{6}}) = \frac{1}{(5^{\frac{1}{6}})^3-1} = \frac{1}{5^{\frac{1}{2}}-1} = \frac{1}{\sqrt{5}-1}$$

$$\phi(1) = \frac{1}{(1)^3-1} = \frac{1}{1-1} = \frac{1}{0} = \infty \text{ (undefined)}.$$

Example:- $f(x) = 2x^2-1$, find $f(d)$ & $f(t-1)$

$$f(d) = 2d^2-1$$

$$f(t-1) = 2(t-1)^2-1 = 2(t^2+1-2t)-1$$
$$= 2t^2+2-4t-1 = 2t^2-4t+1$$

Remark:- If two functions look alike in all aspects other than variables, then they are same. (e.g) $g(x) = x^2-4x$ & $g(c) = c^2-4c$

Domain of a function:-

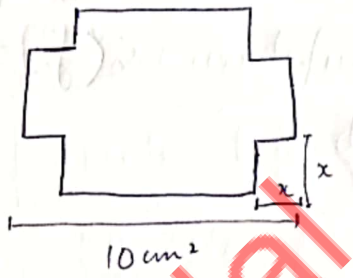
The independent ~~value~~ variable is not ~~on~~ allowed to take any value.

It may be restricted to take some values. This is called domain. It is the set consisting of all elements/allowable values for independent variable.

In other words "The value to be put" is called domain.

Example:- Suppose that a square with a side of length x cm is cut from four corners of a piece of cardboard that is $10cm^2$ & let y be area of the cardboard that remain. Find its domain.

Solution:-



Area of cardboard = ~~side~~ side x side

$= 10 \times 10$

$= 100cm^2$

area of pieces that are cut $= 4(x \times x)$

$= 4x^2$

Area of carboard that remain $= 100 - 4x^2$

Here we cannot put values greater than 5 bcz it will give -ve answer that is not possible in the case of area. So its domain'll be

$0 \le x \le 5$ or $[0, 5]$.

Types of Domain:- There are two types

1. Natural Domain 2. Restricted Domain

Natural domain:- It comes out as a result of the formula of function. for example in

$h(x) = \dfrac{1}{(x-1)(x-3)}$ The denominator shouldn't be zero so we can't put 1 & 3 as value of x. So 1 & 3 is not part of domain. The domain is

$(-\infty, 1) \cup (1, 3) \cup (3, +\infty)$

Restricted domain:-

It is the domain which can be altered by doing some manipulations.

for example:- $h(x) = \frac{x^2-4}{x-2}$ (here we cannot put $x=2$ bcz it makes the denominator zero).

But we can do some algebric manipulation.

$$h(x) = \frac{x^2-4}{x-2} = \frac{(x)^2-(2)^2}{x-2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$$

$$h(x) = x+2$$

Now here we can put any value of x, even 2 also. So we altered the domain.

Range of a function:-

The set of y values that come out as a result of x values are called Range.

In other words Range is set of all possible values for $f(x)$ as x varies over domain.

Techniques for finding Range:-

In $y=x^2$, the value of y always comes out positive. So its Range is all positive real numbers.

⇒ In $2+\sqrt{x-1}$ The domain will be $[1,+\infty)$

So The range will be $[2,+\infty)$

⇒ In $y=\frac{x+1}{x-1}$ The domain is all real numbers except 1. bcz 1 makes The denominator zero.

Solve it for x.

$$y=\frac{x+1}{x-1}$$

$$y(x-1)=x+1$$

$$xy-y=x+1$$

$$xy-x=1+y$$

$$x(y-1)=y+1$$

$$x=\frac{y+1}{y-1}$$

So here we can see That y can also be not equal to one. So its range will be

$(-\infty,1)\cup(1,+\infty)$.

Functions devided piecewise:-

Sometimes The function need to be defined by formulas That have been "pieced together."

As
$$f(x)=\begin{cases}1.75 & 0<x\leq 1\\ 1.75+0.50(x-1) & 1<x \text{ or } x>1\end{cases}$$

Reversing roles of x & y:

Usually x is independent & y is dependent. But the role can be reversed for our convenience.

For example $x = 4y^5 - 2y^3 + 7y - 5$

For example

In $3x + 2y = 6$, either x or y can be seperated on our own choice.

as

$3x = 6 - 2y$	$2y = 6 - 3x$
$x = \frac{6}{3} - \frac{2y}{3}$	$y = \frac{6}{2} - \frac{3x}{2}$
$x = -\frac{2}{3}y + 2$	$y = -\frac{3}{2}x + 3$

lecture 07

Operations on functions

Example:- $f(x) = x^2$ & $g(x) = x$, find $f(x) + g(x)$

Simply add the functions as

$$f(x) + g(x) = x^2 + x$$

Definations:

$$(f+g)x = f(x) + g(x)$$

$$(f-g)x = f(x) - g(x)$$

$$(f \cdot g)x = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$$

$\Rightarrow$ For $f+g$, $f-g$ & $f \cdot g$ domains are intersection of domains of f & g.

$\Rightarrow$ For $\frac{f}{g}$ The domain is intersection of f & g except for the points where $g(x)=0$.

Example:- $f(x) = 1+\sqrt{x-2}$ & $g(x) = x-1$

find domain of $(f+g)x$:-

$$(f+g)x = f(x)+g(x) = 1+\sqrt{x-2}+x-1 = x+\sqrt{x-2}$$

Domain of $f(x) = [2, +\infty)$

Domain of $g(x) = (-\infty, +\infty)$

Domain of $(f+g)x = [-2, +\infty) \cap (-\infty, +\infty)$
$= [-2, +\infty)$

Example:- $f(x) = 3\sqrt{x}$ & $g(x) = \sqrt{x}$

find $f \cdot g(x)$

$$f \cdot g(x) = f(x) \cdot g(x)$$
$$= (3\sqrt{x})(\sqrt{x}) = 3(\sqrt{x})^2 = 3x$$

The natural domain of $f \cdot g(x)$ is $(-\infty, +\infty)$ but resitricted domain will be intersection of domains of f & g.

restricted domain intersection lene se ati hy

Dom of $f(x) = [0, +\infty)$

Dom of $g(x) = [0, +\infty)$

Dom of $f \cdot g(x) = [0, +\infty) \cap [0, +\infty)$
$= [0, +\infty)$

Notation:-

$f^2(x) = f(x) \cdot f(x)$

$f^3(x) = f(x) \cdot f(x) \cdot f(x)$

$f^n(x) = f(x) \cdot f(x) \ldots\ldots f(x).$

(e.g) $(\sin x)^2 = \sin x \cdot \sin x = \sin^2 x$

Composition of functions:- Imp

Composition has no analog with arithmatic operations.

When two functions are composed, One is assigned as a value to the independent variable of other. $f(x) = x^3$, $g(x) = x+4$

$$f \circ g(x) = f(g(x))$$

First $f(x) = x^3$

put $x = g(x)$, $f(g(x)) = (g(x))^3$

$\quad // \quad = (x+4)^3$

Domain of $f \circ g(x)$ consists of all x in the domain of g for which $g(x)$ is in domain of f.

Example:

$f(x) = x^2+3$, $g(x) = \sqrt{x}$

$f \circ g(x) = f(g(x))$

First $f(x) = x^2+3$

$f(g(x)) = [g(x)]^2+3$

$= (\sqrt{x})^2 + 3$

$= x + 3$

Dom of $g = [0, +\infty)$

Dom of $f = (-\infty, +\infty)$

Dom of $f \circ g(x) = [0, +\infty) \cap (-\infty, +\infty)$

$= [0, +\infty)$.

Remark:-

$f \circ g(x) \neq g \circ f(x)$

Decomposing The function:-

Sometimes we decompose a function (i-e) break the function into simpler ones.

(e-g) $h(x) = (x+1)^2$

is a composition of $f(x) = x+1$ & $g(x) = x^2$. So $h(x) = g(f(x)) = g \circ f(x)$

Example:-

$(x^2+1)^{10} = [(x^2+1)^2]^5 = f(g(x))$, $g(x) = (x^2+1)^2$ & $f(x) = x^5$

or $(x^2+1)^{10} = [(x^2+1)^3]^{\frac{10}{3}} = f(g(x))$, $g(x) = (x^2+1)^3$ & $f(x) = x^{\frac{10}{3}}$

Example:- $T(x) = \sqrt{\left(\frac{x}{3}\right)^3} = f(g(h(x)))$

$f(x) = \sqrt{x}$, $g(x) = x^3$, $h(x) = \frac{x}{3}$

Some more examples

Function	$g(x)$ inside	$f(x)$ outside	composition
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10} = f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x = f(g(x))$
$\frac{1}{x+1}$	$x+1$	$\frac{1}{x}$	$\frac{1}{x+1} = f(g(x))$
$\tan x^5$	x^5	$\tan x$	$\tan x^5 = f(g(x))$

Constant function:-

These assign the same number to every x in the domain. (e.g) $f(x) = 2$, $f(-2) = 2$, $f(t) = 2$

Monomial in x:-

Anything like cx^n (c is constant & n is any non-negative integer) e.g. $2x^5$, $\sqrt{3}\,x^{55}$

$4x^{-5}$, $5x^{\frac{2}{3}}$ are not monomials

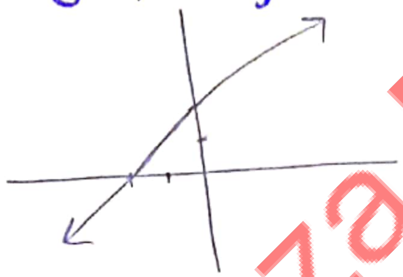
Polynomial in x:- which have more than one terms like $2x^2 + 3x + 1$ or $2 + \frac{5}{3}x^2$.

Lecture 08 Graphs of functions

Defination:- A graph of an equation is just the points on xy-plane that satisfy the equation. Similarly graph of a function f in xy-plane is graph of $y = f(x)$

Example:- Sketch graph of (i) $f(x) = x+2$

$y = x+2$

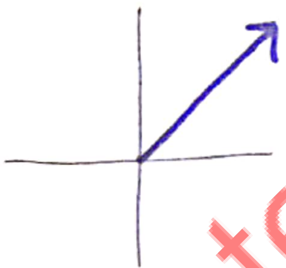


(ii) $g(x) = |x|$

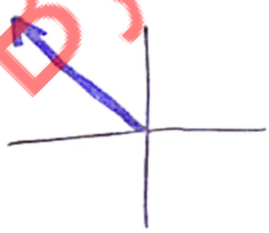
As $|x| = \pm x$

So $g(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

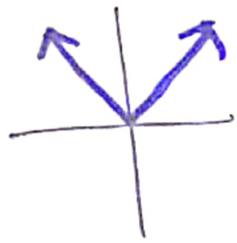
Imp



$g(x) = x$



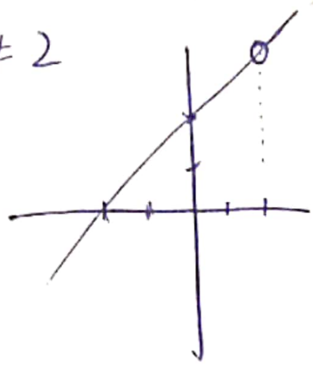
$g(x) = -x$



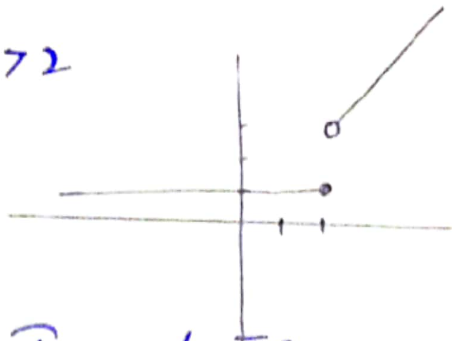
$g(x) = |x|$

(iii) $f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x+2)(x-2)}{(x-2)} = (x+2)$

So $y = x+2, \; x \neq 2$



(iv) $y = \begin{cases} 1 & \text{if } x \leq 2 \\ x+2 & \text{if } x > 2 \end{cases}$



Exp

Graphing functions by Translations:-

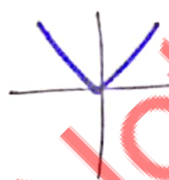
$y = f(x)+c$ graph of $f(x)$ Translates UP by c units.

$y = f(x)-c$ graph of $f(x)$ Translates DOWN by c units.

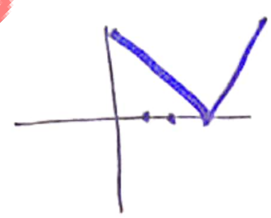
$y = f(x+c)$ graph of $f(x)$ Translates LEFT by c units.

$y = f(x-c)$ graph of $f(x)$ Translates RIGHT by c units.

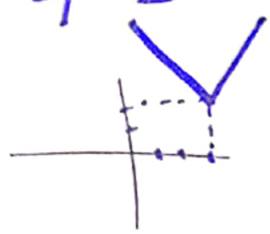
Example:- Graph of $f(x) = |x-3|+2$



$f(x)=|x|$



$f(x)=|x-3|$



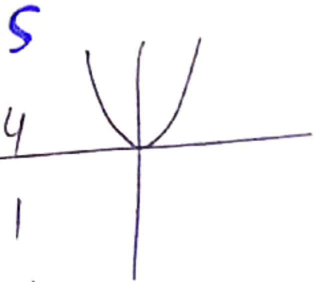
$f(x)=|x-3|+2$

$y = x^2-4x+5$

$y = x^2-4x+5+4-4$

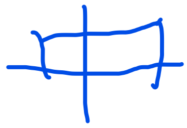
$= (x^2-4x+4)+1$

$= (x-2)^2+1$



$f(x) = x^2-4x+5$

Reflections



imp

- $(-x, y)$ is reflection of (x, y) about y-axis
- $(x, -y)$ is reflection of (x, y) about x-axis.

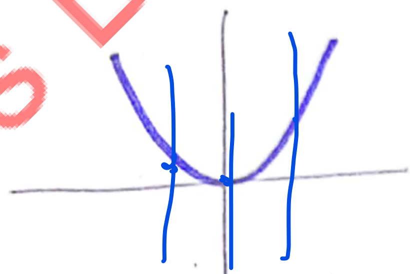
⇒ Graphs of $y = f(x)$ & $y = f(-x)$ are reflections of one another about y-axis.

Graphs of $y = f(x)$ & $y = -f(x)$ are reflections ⇒ of one another about x-axis.

Vertical line Test:-

A graph in the plane is graph of a function if & only if NO VERTICAL line intersects the graph more than once.

For example



Every vertical line cuts/intersects the graph only at one point. So it is graph of a function.

Lecture 09. Limits

In calculus we find areas of plane regions & find Tangent lines to curves.

⇒ Calculus that comes out of Tangent problem is called differential Calculus.

⇒ Calculus That comes out of Area problems is called Integral Calculus.

Limits:- Limits are basically a way to study the behaviour of y-values of a function in response to x-values as they approach some number or go to infinity.

Imp Formulas:-

Notation	How to read.
$\lim_{x \to x_0^+} f(x) = L_1$	Limit of f(x) as x approaches x_0 from the right is equal L_1.
$\lim_{x \to x_0^-} f(x) = L_2$	Limit of f(x) as x approaches x_0 from the left is equal to L_2.
$\lim_{x \to x_0} f(x) = L$	Limit of f(x) as x approaches x_0 is equal to L.

Example:- $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$

Its graph has no limiting value as its graph oscillates b/w 1 & -1.

Existence of limit:-

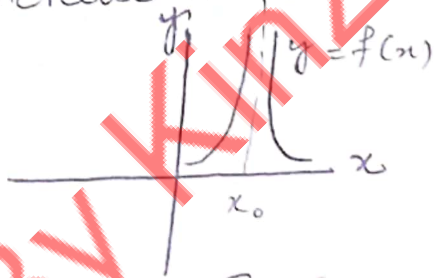
Function does not always have a limit as x value approaches some number. Here we say that limit does not exist./DNE.

limit does not exist due to

$\Rightarrow$ Oscillations

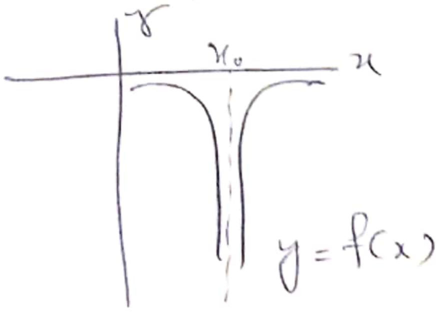
$\Rightarrow$ Unbounded increase or decrease

For example:-



There is increase in y without bound. So limit does not exist due to unboundedness.

In the graph



There is decrease in y without bound

So limit ~~decrease~~ does not exist.

Lecture 10

Limits & Computational Techniques

limit:

$\lim_{x\to a} k = k$ (e.g $\lim_{x\to 2} 3 = 3$)

left hand limit:

$\lim_{x\to a^-} k = k$

Right hand limit:

$\lim_{x\to a^+} k = k$

limits of constant functions:-

$\lim_{x\to a} k = k$ (e.g $\lim_{x\to 2} 3 = 3$)

$\lim_{x\to \infty} k = k$ (e.g $\lim_{x\to \infty} 3 = 3$)

$\lim_{x\to -\infty} k = k$ (e.g $\lim_{x\to -\infty} 3 = 3$)

limits of (variable) functions:-

$\lim_{x\to a} x = a$

$\lim_{x\to +\infty} x = +\infty$

$\lim_{x\to -\infty} x = -\infty$

Theorem:-

If $L_1 = \lim f(x)$ & $L_2 = \lim g(x)$ both exists

$\Rightarrow \lim[f(x)+g(x)] = \lim f(x) + \lim g(x) = L_1 + L_2$

$\Rightarrow \lim[f(x)-g(x)] = \lim f(x) - \lim g(x) = L_1 - L_2$

$\Rightarrow \lim[f(x) \cdot g(x)] = \lim f(x) \cdot \lim g(x) = L_1 L_2$

$\Rightarrow \lim\left[\frac{f(x)}{g(x)}\right] = \frac{\lim f(x)}{\lim g(x)} = \frac{L_1}{L_2}, (L_2 \neq 0)$

- limit of the sum is the sum of the limits
- limit of the difference is difference of limits.
- Similarly for product & division.
- We can also write
  $\lim[f_1(x) + f_2(x) + f_3(x) + \cdots + f_n(x)]$
  $= \lim f_1(x) + \lim f_2(x) + \lim f_3(x) + \cdots + \lim f_n(x)$
- If $f_1 = f_2 = \cdots = f_n$ Then $\lim[f(x)]^n = [\lim f(x)]^n$

Results:-

$\Rightarrow \lim\limits_{x \to a}(x^n) = \left[\lim\limits_{x \to a} x\right]^n = a^n$

$\Rightarrow \lim[k\, f(x)] = k \cdot \lim[f(x)]$

A constant can be moved through a limit sign.

Example:- $\lim\limits_{x \to 5} (x^2 - 4x + 3)$

$$= \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3$$

$$= \lim_{x \to 5} x^2 - 4 \lim_{x \to 5} x + \lim_{x \to 5} 3$$

$$= (5)^2 - 4(5) + 3$$

$$= 25 - 20 + 3$$

$$= 5 + 3$$

$$= 8$$

Theorem:- $\lim\limits_{x \to a} P(x) = \lim\limits_{x \to a} (c_0 + c_1 x + \cdots + c_n x^n)$

$$= c_0 + c_1 a + \cdots\cdots + c_n a^n$$

$$= P(a)$$

Limits involving $\frac{1}{x}$:-

$$\lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0} = \underline{+\infty} \qquad \lim_{x \to +\infty} \frac{1}{x} = \frac{1}{+\infty} = \underline{0}$$

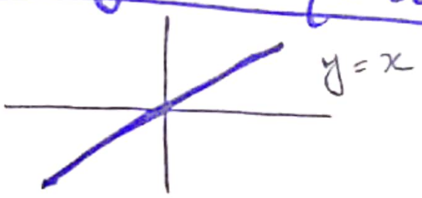
$$\lim_{x \to 0^-} \frac{1}{x} = \frac{1}{0} = \underline{-\infty} \qquad \lim_{x \to -\infty} \frac{1}{x} = \frac{1}{-\infty} = \underline{0}$$

Remark:- For every real no. a

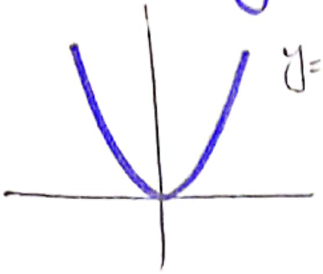
$g(x) = \dfrac{1}{x-a}$ is a translation of $f(x) = \dfrac{1}{x}$

Graph of 1st degree equation:-

MCQS k liye

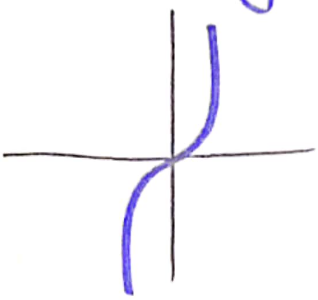


Graph of 2nd degree equation:-



$y = x^2$

Graph of 3rd degree equation:-



$y = x^3$

Example:-

$$\lim_{x \to +\infty} 2x^5 = 2(+\infty)^5 = +\infty$$

$$\lim_{x \to +\infty} -7x^6 = -7(+\infty)^6 = -\infty$$

$$\lim_{x \to +\infty} \frac{1}{x^n} = \left(\lim_{x \to +\infty} \frac{1}{x}\right)^n = \left(\frac{1}{+\infty}\right)^n = (0)^n = 0$$

$$\lim_{x \to -\infty} \frac{1}{x^n} = \left(\lim_{x \to -\infty} \frac{1}{x}\right)^n = \left(\frac{1}{-\infty}\right)^n = (0)^n = 0$$

Rational Function:-

A Rational function is a function defined by the ratio of two polynomials.

Example:-

$$\lim_{x \to 2} \frac{5x^3+4}{x-3} = \frac{\lim_{x \to 2}(5x^3+4)}{\lim_{x \to 2}(x-3)} \Rightarrow \frac{5(2)^3+4}{2-3} \Rightarrow \frac{44}{-1} \Rightarrow -44$$

Example:- $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$

$$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)}$$

$$= \lim_{x\to 2} (x+2) \Rightarrow (2+2) \Rightarrow 4$$

Limit of functions of Type $\frac{1}{x}$.

⇒ The limit may be $+\infty$.

⇒ The limit may be $-\infty$.

⇒ The limit may be $+\infty$ from one side & $-\infty$ from other side.

$$\lim_{x\to 4^+} \frac{2-x}{(x-4)(x+2)} \Rightarrow \frac{2-4}{(4-4)(4+2)}$$

$$= \frac{-2}{(0)(6)} \Rightarrow \frac{-2}{0} \Rightarrow \underline{-\infty}$$

Example:- $\lim_{x\to -\infty} \dfrac{4x^2-x}{2x^3-5} = \lim_{x\to-\infty} \left(\dfrac{\frac{4x^2}{x^3} - \frac{x}{x^3}}{\frac{2x^3}{x^3} - \frac{5}{x^3}} \right)$

$$= \frac{\lim_{x\to-\infty} \frac{4}{x} - \frac{1}{x^2}}{\lim_{x\to-\infty} 2 - \frac{5}{x^3}} \Rightarrow \frac{4\lim_{x\to\infty}\frac{1}{x} - \lim_{x\to-\infty}\frac{1}{x^2}}{\lim_{x\to-\infty} 2 - 5\lim_{x\to-\infty}\frac{1}{x^3}}$$

$$= \frac{4\left(\frac{1}{\infty}\right) - \frac{1}{\infty^2}}{2 - 5\left(\frac{1}{\infty^3}\right)} \Rightarrow \frac{4(0)-0}{2-5(0)} \Rightarrow \frac{0}{2} = \underline{0}$$

Quick Rule:- (only for limits of $+\infty$ & $-\infty$)

$$\lim_{x\to+\infty} \frac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_nx^n} = \lim_{x\to+\infty} \frac{c_nx^n}{d_nx^n}$$

$$\lim_{x\to-\infty} \frac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_nx^n} = \lim_{x\to-\infty} \frac{c_nx^n}{d_nx^n}$$

Not True if limit $x\to a$.

Example:- $\lim_{x\to+\infty} \frac{4x^2-x}{2x^3-5} = \lim_{x\to+\infty} \frac{4x^2}{2x^3} \Rightarrow \frac{2}{x} \Rightarrow \frac{2}{\infty} \Rightarrow \underline{0}$

Lecture 11

Limits: A Rigorous Approach.

$$\lim_{x\to a} f(x) = L$$

The concept of approaches is intuitive. For any number $\varepsilon > 0$ if we can find an open interval (x_0, x_1) on x-axis containing a point "a" such that $L-\varepsilon < f(x) < L+\varepsilon$ for each x in (x_0, x_1) except possibly $x=a$ Then

$$\lim_{x \to a} f(x) = L$$

So $f(x)$ is in interval $(L-\varepsilon, L+\varepsilon)$

Remark:- $\dfrac{1}{10^{10^{100}}}$ The bottom is called a GOOGOLPLEX

$L-\varepsilon < f(x) < L+\varepsilon$ can also be written as $|f(x)-L| < \varepsilon$ & $(a-\delta, a) \cup (a, a+\delta)$ as

$0 < |x-a| < \delta$

Example:- Find ε in $\lim_{x \to 2} (3x-5) = 1$

$$|(3x-5)-1| < \varepsilon \quad \text{if } 0 < |x-2| < \delta$$

here $f(x) = 3x-5$, $L = +1$, $a = 2$

we have to find ε.

$$|3x-5-1| < \varepsilon$$

$$|3x-6| < \varepsilon$$

$$3|x-2| < \varepsilon$$

$$|x-2| < \frac{\varepsilon}{2} \quad \text{if } 0 < (x-2) < \frac{\varepsilon}{3}$$

Lecture 12 Continuity.

Continuous Function:-

Function $f(x)$ is continuous if (i) $f(c)$ is defined. (ii) $\lim_{x \to c} f(x)$ exists. (iii) $\lim_{x \to c} f(x) = f(c)$

If any of these conditions fails to hold for a function then $f(x)$ is called discontinuous.

c is called point of discontinuity.

If function is continuous on all points in (a,b) then it is called continuous over (a,b)

If function is continuous on $(-\infty, +\infty)$ then its called continuous function.

Example:- $f(x) = y = \dfrac{x^2-4}{x-2}$ it is discontinuous at $x=2$ because $f(2)$ is undefined.

Example:- $g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

$g(x)$ is discontinuous because

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)}$$

$$= \lim_{x \to 2} (x+2) = 2+2 = 4$$

but in $g(x)$, $g(2) = 3$. That's not True.

Show That $f(x) = x^2 - 2x + 1$ is continuous.

$$\lim_{x \to c} x^2 - 2x + 1 = c^2 - 2c + 1$$

function is continuous.

Theorem:-

Polynomials are continuous functions.

Example:- $f(x) = |x|$ is continuous or not?

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to c} f(x) = f(c)$$

let $c \geq 0$ Then $f(c) = c$

$$\lim_{x \to c} f(x) = \lim_{x \to c} |x| = \lim_{x \to c} x = c$$

let $c \leq 0$ Then $f(c) = -c$

$$\lim_{x \to c} f(x) = \lim_{x \to c} |x| = \lim_{x \to c} (-x) = -(-c) = c$$

So function is continuous.

Properties:- If f & g are continuous, Then

a) $f+g$ is continuous at c.

b) $f-g$ is continuous at c.

c) $f \cdot g$ is continuous at c.

d) f/g is continuous at c if $g(c) \neq 0$ & dis continuous if $g(c) = 0$

Continuity of Rational functions:-

Example:- $h(x) = \dfrac{x^2-9}{x^2-5x+6}$

we'll find factors of denominator.

$$x^2-5x+6 = x^2-(3+2)x+6$$
$$= x^2-3x-2x+6$$
$$= x(x-3)-2(x-3)$$
$$= (x-2)(x-3)$$

So $h(x) = \dfrac{x^2-9}{(x-2)(x-3)}$

So $h(x)$ is discontinuous at $x=2$ & $x=3$.

Theorem:-

If $\lim g(x) = L$ & if the function f is continuous at L, Then $\lim f(g(x)) = f(L)$

(i-e) $\lim f(g(x)) = f(\lim g(x))$.

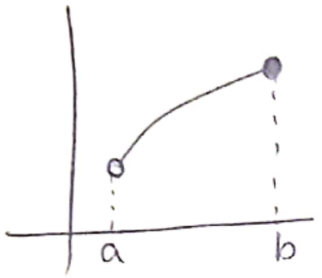
Example:- $f(x) = \lim\limits_{x\to 3} |5-x^2|$

Here $f(x) = |x|$, $g(x) = 5-x^2$

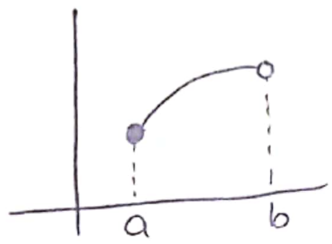
So, as

$$\lim_{x\to 3} f(g(x)) = \cancel{\lim_{x\to 3} f}$$
$$= \lim_{x\to 3} |5-x^2| \Rightarrow \left|\lim_{x\to 3}(5-x^2)\right|$$
$$= |5-3^2| \Rightarrow |5-9| = |-4| = 4$$

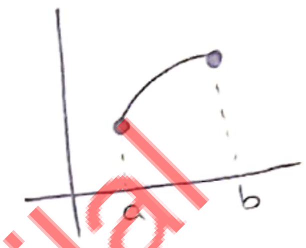
Theorem:- If function g is continuous at point c & function f is continuous at point $g(c)$ Then composition $f(g(x)) = f\circ g(x)$ is continuous.



discontinuous at "a"



discontinuous at "b".



continuous at a & b.

Theorem:-

1. A function is called continuous from the left at point c if
   a) $f(c)$ is defined.
   b) $\lim_{x\to c^-} f(x)$ exists.
   c) $\lim_{x\to c^-} f(x) = f(c)$

2. A function is called continuous from the right at point c if
   a) $f(c)$ is defined.
   b) $\lim_{x\to c^+} f(x)$ exists.
   c) $\lim_{x\to c^+} f(x) = f(c)$

Theorem:- $f(x)$ is continuous on $[a, b]$ (closed interval) if

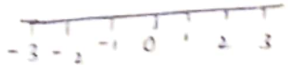
1. f is continuous on (a, b)
2. f is continuous from left on (a,b)
3. f is continuous from right on (a,b).

Example:- Show that $f(x) = \sqrt{9-x^2}$ is continuous on $[3, -3]$.

1. $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \sqrt{9-x^2} = \sqrt{\lim\limits_{x \to c}(9-x^2)} = \sqrt{9-c^2} = f(c)$
2. $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} \sqrt{9-x^2} = \sqrt{\lim\limits_{x \to 3^-}(9-x^2)} = \sqrt{9-3^2} = 0 = f(3)$
3. $\lim\limits_{x \to -3^+} f(x) = \lim\limits_{x \to -3^+} \sqrt{9-x^2} = \sqrt{\lim\limits_{x \to -3^+}(9-x^2)} = \sqrt{9-(-3)^2} = 0 = f(-3)$

(We approach 3 from left & -3 from right)

So f is continuous on $[-3, 3]$.



Intermediate value Theorem:-

If f is continuous on closed interval $[a,b]$ & c is any number b/w $f(a)$ & $f(b)$ inclusive, then there is at least one number x in interval $[a,b]$ such that $f(x) = c$.

If f is continuous on $[a, b]$ & if $f(a)$ & $f(b)$ have opposite signs then there is at least one solution of $f(x) = 0$ in interval (a, b).

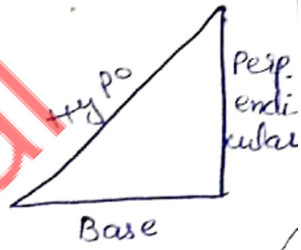
Lecture 13

Limits & continuity of Trigonometric functions

Continuity of sine & cosine:-

As $\sin\theta = \dfrac{\text{oppsite side / Perpendicular / altitude}}{\text{Hypotaneous}}$

$\cos\theta = \dfrac{\text{Adjacent side / Base}}{\text{Hypotaneous}}$



Theorem:-

The functions $\sin x$ & $\cos x$ are continuous.

$$\lim_{x\to 0} \sin x = 0 \quad \& \quad \lim_{x\to 0} \cos x = 1$$

- let $h = x - c$, So $x = h + c$ Then $x \to c$ is equivalent to $h \to 0$, So

function is continuous at c if

1. $f(c)$ is defined.
2. $\lim_{h\to 0} f(h+c)$ exists.
3. $\lim_{h\to 0} f(h+c) = f(c)$

Proof:- $\lim_{h\to 0} \sin(c+h) = \sin c$

$$\lim_{h\to 0} \sin(c+h) = \lim_{h\to 0}\left[\sin(c)\cos(h) + \cos(c)\sin(h)\right]$$

$$= \lim_{h\to 0} \sin(c)\cos(h) + \lim_{h\to 0} \cos(c)\sin(h)$$

$$= \sin c \lim_{h\to 0} \cos(c) + \cos(c) \lim_{h\to 0} \sin(c)$$

$$= \sin(c)\,(1) + \cos(c)\,(0)$$

$$= \sin(C) + 0$$

$$\lim_{h\to 0} \sin(c+h) = \sin(c)$$

Similarly we can do proof for $\lim_{h\to 0} \cos(c+h) = \cos(C)$

Continuity for other Trigonometric functions:-

If $f(x)$ & $g(x)$ are continuous then

$h(x) = \dfrac{f(x)}{g(x)}$ is also continuous. (except $g(x) = 0$)

Similarly $\sin x$ & $\cos x$ are continuous, so

$\mathrm{Tan}(x) = \dfrac{\sin x}{\cos x}$ is also continuous. (except $\cos x = 0$)

Like wise

$\cot(x) = \dfrac{\cos(x)}{\sin(x)}$, $\sec(x) = \dfrac{1}{\cos(x)}$, $\mathrm{cosec}(x) = \dfrac{1}{\sin(x)}$

Squeeze Theorem:-

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

Squeezing Theorem:-

Let f, g & h be functions satisfying $g(x) \le f(x) \le h(x)$ for all x in some open interval containing point a, with possible exception that inequality need not to hold at a.

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

Then f also has this limit as x approaches to a $\lim_{x\to a} f(x) = L$

Example:-

$$\lim_{x\to 0} x^2 \sin^2\left(\frac{1}{x}\right)$$

Remember $0 \le \sin(x) \le 1$

So $0 \le \sin^2(x) \le 1$

Also $0 \le \sin^2\left(\frac{1}{x}\right) \le 1$

multiply by x^2 on all sides

$$(0)(x^2) \le x^2 \sin^2\left(\frac{1}{x}\right) \le (1)(x^2)$$

$$0 \le x^2 \sin^2\left(\frac{1}{x}\right) \le x^2$$

But $\lim_{x\to 0} 0 = \lim_{x\to 0} x^2 = 0$

Both L.H.S & R.H.S are zero So by Squeezing Theorem

$$\lim_{x\to 0} x^2 \sin^2\left(\frac{1}{x}\right) = 0$$

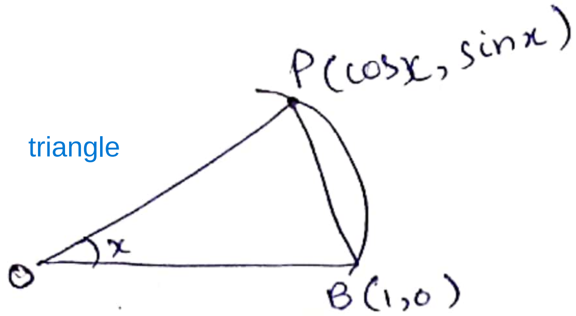
Theorem:-

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

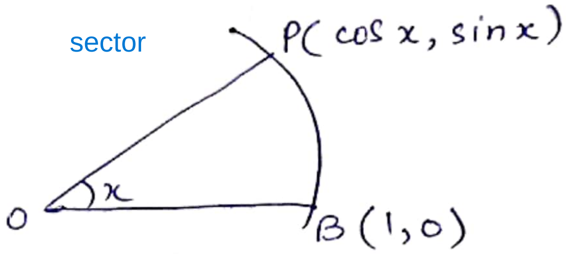
Proof:-

let $0 < x < \frac{\pi}{2}$ & take the following shapes.

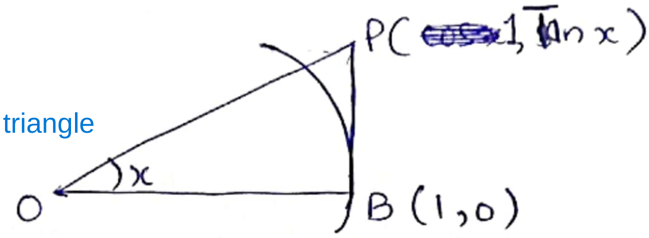
triangle



sector



right triangle



$0 <$ area of $\Delta OBP <$ area of sector $OBP <$ area of ΔOBP

$$0 < \frac{1}{2}(\text{base})(\text{height}) < \frac{1}{2} r^2 \theta < \frac{1}{2}(\text{base})(\text{height})$$

$$0 < \frac{1}{2}(1)(\sin x) < \frac{1}{2}(1)^2(x) < \frac{1}{2}(1)(\text{Tan } x)$$

$$0 < \frac{1}{2} \sin x < \frac{1}{2} x < \frac{1}{2} \text{Tan } x$$

multiplying by $\frac{2}{\sin x}$

$$0 < \frac{1}{2} \sin x \left(\frac{2}{\sin x}\right) < \frac{1}{2} x \left(\frac{2}{\sin x}\right) < \frac{1}{2} \text{Tan} x \left(\frac{2}{\sin x}\right)$$

$$0 < 1 < \frac{x}{\sin x} < \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x}$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Take Reciprocal

$$1 < \frac{\sin x}{x} < \cos x$$

As $0 < x < \frac{\pi}{2}$

$$1 < \frac{\sin x}{x} < \cos x \quad \text{becomes}$$

As $\lim_{x\to 0} \cos x = 1$ & $\lim_{x\to 0} 1 = 1$

Take limits

$$\lim_{x\to 0} 1 < \lim_{x\to 0} \frac{\sin x}{x} < \lim_{x\to 0} \cos x$$

$$1 < \lim_{x\to 0} \frac{\sin x}{x} < 1$$

By Squeezing Theorem

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Lecture 14

Tangent lines & rates of Change

limit of a secant line (Slope)

$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Slope of secant line will approach that of Tangent line at P.

$$m_{Tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Also

$$\text{Average velocity} = \frac{\text{distance}}{\text{Time}} = \frac{d_1 - d_0}{t_1 - t_0} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

Defination:-

If $y=f(x)$, Then average rate of change of y w.r.t x over The interval $[x_0, x_1]$ is slope m_{sec} of secant line joining The points $(x_0, f(x_0))$ & $(x_1, f(x_1))$ on graph of f.

$$m_{sec} = \frac{f(x_1)-f(x_0)}{x_1 - x_0}$$

Defination:- If $y=f(x)$ Then instantaneous rate of change of y with respect to x at point x_0 is The slope

$$m_{Tan} = \lim_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1 - x_0}$$

Example:- $y = f(x) = x^2+1$

a) Find average rate of y w.r.t x over $[3,5]$

b) Find instantaneous rate of y w.r.t x at $x=x_0$ point $x_0=-4$

c) Find instantaneous rate of change of y w.r.t x at a general point.

a) $y=f(x)=x^2+1$, $x_0=3$, $x_1=5$

$$m_{sec} = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(5)-f(3)}{5-3} = \frac{(5^2+1)-(3^2+1)}{2}$$

$$= \frac{26-10}{2} = \frac{16}{2} = 8$$

(b)

$$y = f(x) = x^2+1 \quad , \quad x_0 = -4$$

$$m_{Tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^2 + 1 - (x_0^2+1)}{x_1 - x_0}$$

$$\lim_{x_1 \to x_0} = \frac{x_1^2 + 1 - (-4)^2 - 1}{x_1 - (-4)} = \lim_{x \to -4} \frac{x_1^2 + 1 - 16 - 1}{x_1 + 4}$$

$$= \lim_{x_1 \to -4} \frac{x_1^2 - 16}{x_1 + 4} = \lim_{x \to -4} \frac{(x_1+4)(x_1-4)}{(x_1+4)}$$

$$= \lim_{x_1 \to -4} (x_1 - 4) = -4-4 = -8$$

(c)

$$m_{Tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \lim_{x_1 \to x_0} \frac{(x_1^2+1) - (x_0^2 - 1)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^2 + 1 - x_0^2 - 1}{x_1 - x_0}$$

$$= \lim_{x_1 \to x_0} \frac{x_1^2 - x_0^2}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{(x_1 + x_0)(x_1 - x_0)}{x_1 - x_0}$$

$$= \lim_{x_1 \to x_0} (x_1 + x_0) = (x_0 + x_0) = 2x_0$$

Lecture 15

The Derivative

We can rewrite $m_{Tan} = \lim_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

as

$h = x_1 - x_0$, $x_1 = x_0 + h$ & $h \to 0$ as $x_1 \to x_0$

$$m_{Tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If $P(x_0, y_0)$ is a point on graph of a function f then Tangent line to graph of f at P is defined to be line through P. with slope.

$$m_{Tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Example:- Find slope & equation of Tangent line to graph of $f(x) = x^2$ at point $(3, 9)$

$f(x) = x^2$, $x_0 = 3$, $y_0 = 9$

$$m_{Tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{9 + h^2 + 6h - 9}{h}$$

$f(x) = x^2$

$f(x_0 + h) = (x_0 + h)^2$

$= (3 + h)^2$

&

$f(x_0) = x_0^2$

$= (3)^2$

$= 9$

$$= \lim_{h\to 0} \frac{h^2+6h}{h}$$

$$= \lim_{h\to 0} \frac{h(h+6)}{h}$$

$$= \lim_{h\to 0} (h+6) = 0+6 = 6$$

Equation of Tangent line

$$y - y_0 = m(x - x_0)$$

$$y - 9 = 6(x-3)$$

$$y - 9 = 6x - 18$$

$$y = 6x + 9 - 18$$

$$y = 6x - 9$$

Defination:- The function f is denoted by formula $f' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ is called with respect to x of function f. The domain of f consists of all x for which limit exists.

⇒ We can interpret derivative in 2-ways.

- Geometric interpretation of derivative
- Rate of change is an interpretation of derivative also.

Example:- $f(x) = x^2+1$

Find $f'(x)$.

$$f' = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f(x+h) = (x+h)^2+1$$
$$= x^2+h^2+2hx+1$$

Put values

$$f'(x) = \lim_{h\to 0} \frac{x^2+h^2+2hx+1-(x^2+1)}{h}$$

$$= \lim_{h\to 0} \frac{x^2+h^2+2hx+1-x^2-1}{h}$$

$$= \lim_{h\to 0} \frac{h^2+2hx}{h}$$

$$= \lim_{h\to 0} \frac{h(h+2x)}{h}$$

$$= \lim_{h\to 0} (h+2x)$$

$$= 0+2x$$

$$f'(x) = 2x$$

$$f'(2) = 2(2) = 4$$

$$f'(-2) = 2(-2) = -4$$

$$f'(0) = 2(0) = 0$$

Example:- $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = \sqrt{x+h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$ rationalizing

$$= \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Derivative Notation:-

The process of finding the derivative is called Differentiation.

It is written as $\frac{d}{dx}[f(x)]$ read as " The derivative of f w.r.t x.

$\frac{d}{dx}[f(x)]$ can also be written as $f'(x)$.

$$Y = f(x)$$

$$\frac{d}{dx}(y) = \frac{d}{dx}[f(x)]$$

$$\frac{dy}{dx} = f'(x)$$

Example:- $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

or $$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

Existence of derivative:-

⇒ Derivative exists at points where limit exists.

⇒ f is differentiable on an open interval (a, b) if it is differentiable at each point in (a, b).

⇒ f is not differentiable, we say derivative does not exists.

Non-differentiability occurs when graph of f(x) has

a) corners

b) Vertical Tangents

c) Points of discontinuity.

Theorem:- If f is differentiable at a point x_0 Then f is also continuous at x_0.

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Proof:-

$$\lim_{h \to 0} f(x_0+h) = f(x_0)$$

$$\lim_{h \to 0} \left(f(x_0+h) - f(x_0)\right) = 0$$

L.H.S

$$\lim_{h \to 0} \left(f(x_0+h) - f(x_0)\right)$$

= multiply & divide by "h"

$$= \lim_{h \to 0} \left(\left[\frac{f(x_0+h) - f(x_0)}{h}\right] \cdot h\right)$$

$$= \lim_{h \to 0} \left[\frac{f(x_0+h) - f(x_0)}{h}\right] \cdot \lim_{h \to 0} h.$$

$$= f'(x_0) \cdot 0$$

$$= 0 = R.H.S.$$

Example:- $f(x) = |x|$, $f'(x) = ?$

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

MTH 101

Lecture 16

ayse functions jin me x ki value put karne se answer change na hota hn constant fun

Techniques of differentiation:-

Derivatives of constant functions:-

$f(x) = 2$
$= 2$

If f is a constant function $f(x) = c$ for all x, then $f'[x] = \frac{d}{dx}[f(x)] = \frac{d}{dx}[c] = 0$

Proof:- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

as $f(x) = c$

so $f(x+h) = c$

Then $f'(x) = \lim_{h \to 0} \left(\frac{c - c}{h}\right) = 0$

Example:- $f(x) = 5$

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(5)$$

$$f'(x) = 0$$

Theorem:- If n is +ve integer, then

$$\frac{d}{dx}(x^n) = n(x^{n-1})$$

Proof:- Let $f(x) = x^n$

$$\frac{d}{dx}(x^n) = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{\left[x\left(1+\frac{h}{x}\right)\right]^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{x^n\left(1+\frac{h}{x}\right)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{x^n\left[\left(1+\frac{h}{x}\right)^n - 1\right]}{h}$$

$$= \lim_{h\to 0} \frac{x^n\left[1 + (n)\left(\frac{h}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2 + \cdots - 1\right]}{h}$$

$$= \lim_{h\to 0} \frac{x^n\left[n\left(\frac{h}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2 + \cdots\right]}{h}$$

$$= \lim_{h\to 0} \frac{x^n}{h}\cdot\frac{h}{x}\left[n + \frac{n(n-1)}{2!}\left(\frac{h}{x}\right) + \cdots\right]$$

$$= x^{n-1}\left[n + 0\right] \Rightarrow x^{n-1}(n) \Rightarrow n(x^{n-1})$$

Example:-

(i) $\frac{d}{dx}(x^5)$

$= 5(x^{5-1}) \Rightarrow 5x^4$

(ii) $\frac{d}{dx}(x) = 1(x^{1-1}) \Rightarrow 1(x^0) \Rightarrow 1(1) \Rightarrow 1$

Theorems:- $\frac{d}{dx}[c\,f(x)] \Rightarrow c\frac{d}{dx}[f(x)]$

Proof:-

$$\frac{d}{dx}[c\,f(x)] = \lim_{h\to 0}\frac{c f(x+h) - c f(x)}{h}$$

$$= \lim_{h\to 0} c\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= c\lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= c\,f'(x) \text{ or } c\frac{d}{dx}(f(x))$$

Example:- $\frac{d}{dx}(3x^8)$

$= 3\frac{d}{dx}(x^8) \Rightarrow 3[8(x^{8-1})]$

$\Rightarrow 3[8x^7] = 24x^7$ Ans.

Derivative of Sum & differences:-

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Proof:- $\frac{d}{dx}[f(x)+g(x)]$

$$= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)-f(x)-g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}$$

$$= \lim_{h\to 0} \left\{ \frac{[f(x+h)-f(x)]}{h} + \frac{[g(x+h)-g(x)]}{h} \right\}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

Example:- $\dfrac{d}{dx}(x^4 + x^3)$

$$= \frac{d}{dx}(x^4) + \frac{d}{dx}(x^3)$$

$$= 4(x^{4-1}) + (3x^{3-1})$$

$$= 4x^3 + 3x^2$$

In general $\dfrac{d}{dx}[f_1(x) + f_2(x) + \cdots + f_n(x)]$

$$= \frac{d}{dx}(f_1(x)) + \frac{d}{dx}(f_2(x)) + \cdots + \frac{d}{dx}(f_n(x))$$

Derivative of a product:-

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

~~**Proof:-**~~ $\cancel{\dfrac{d}{dx}[f(x) \cdot g(x)]}$

Proof:- $\dfrac{d}{dx}[f(x) \cdot g(x)] = \lim\limits_{h\to 0}\left[\dfrac{f(x+h)\cdot g(x+h) - f(x)g(x)}{h}\right]$

add & subtract $f(x+h)g(x)$

$$= \lim_{h\to 0}\left[\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}\right]$$

$$= \lim_{h\to 0}\left\{\frac{f(x+h)[g(x+h) - g(x)]}{h} + \frac{g(x)[f(x+h) - f(x)]}{h}\right\}$$

$$= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0}\left[\frac{g(x+h) - g(x)}{h}\right] + \lim_{h\to 0} g(x) \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

or

$$(f\cdot g)' = f\cdot g' + g\cdot f'$$

Example:- $\frac{d}{dx}[(4x^2)(3x)]$

$$= 4x^2\frac{d}{dx}(3x) + 3x\frac{d}{dx}(4x^2)$$

$$= (4x^2)(3)(1) + (3x)(4\{2x^{2-1}\})$$

$$= (4x^2)(3) + (3x)(8x)$$

$$= 12x^2 + 24x^2$$

$$= 36x^2 \quad \underline{\text{Ans.}}$$

Derivative of Quotient:-

If f & g are differentiable functions at x; $g(x) \neq 0$.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

or

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$$

Example:- $\frac{d}{dx}\left(\frac{3x}{5x^2}\right)$

$$= \frac{5x^2 \frac{d}{dx}(3x) - 3x\frac{d}{dx}(5x^2)}{(5x^2)^2}$$

$$= \frac{5x^2(3) - 3x\left(5(2x)\right)}{25x^4} \Rightarrow \frac{15x^2 - 3x(10x)}{25x^4}$$

$$= \frac{15x^2 - 30x^2}{25x^4} \Rightarrow \frac{-15x^2}{25x^4} \Rightarrow \frac{-3}{5x^2} \quad \underline{\text{Ans.}}$$

Derivative of Reciprocal.

If we have a function $g(x)$ & $g(x) \neq 0$

Then

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

or $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$

Lecture 17

Derivatives of Trigonometric functions.

Derivative of $f(x) = \sin x$

$$\frac{d}{dx}[\sin(x)] = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin x \cosh - \sin x}{h} + \frac{\cos x \sinh}{h} \right]$$

$$= \lim_{h\to 0} \left(\frac{\sin x \cosh - \sin x}{h} \right) + \lim_{h\to 0} \left(\cos x \frac{\sinh}{h} \right)$$

$$= \frac{\sin x \cos 0^\circ - \sin x}{h} + \cos x \lim_{h\to 0} \frac{\sinh}{h}$$

$$= \frac{\sin x - \sin x}{h} + \cos x (1)$$

$$= 0 + \cos x \Rightarrow \cos x \quad \underline{\text{Ans.}}$$

So $\dfrac{d}{dx}[\sin x] = \cos x$

Derivative of $f(x) = \cos x$

$$\frac{d}{dx}[\cos x] = -\sin x$$

Derivative of $f(x) = \tan x$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

$$= \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{(\cos x)^2}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \Rightarrow \frac{1}{\cos^2 x} \Rightarrow \sec^2 x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Derivative of $f(x) = \sec x$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) \Rightarrow \frac{-\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{-(-\sin x)}{\cos^2 x} \Rightarrow \frac{\sin x}{\cos x \cdot \cos x} \Rightarrow \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$$

Derivative of $f(x) = \operatorname{cosec} x$

$$\frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{-\frac{d}{dx}(\sin x)}{\sin^2 x} \Rightarrow -\frac{\cos x}{\sin x \sin x} \Rightarrow -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$\boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$$

Derivative of $f(x) = \cot x$

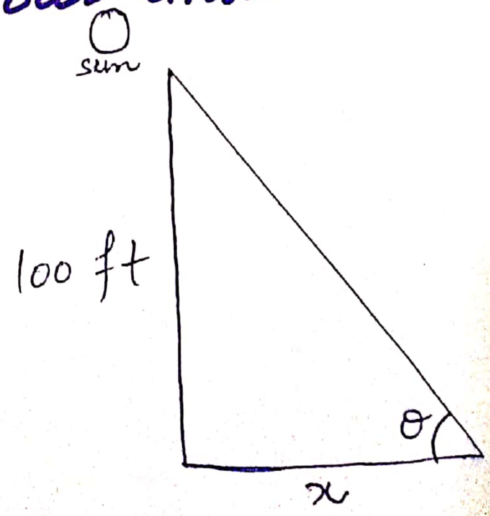
$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right)$$

$$= \frac{-\frac{d}{dx}(\tan x)}{\tan^2 x} \Rightarrow \frac{-\sec^2 x}{\tan^2 x}$$

$$= -\frac{1}{\cos^2 x} \times \frac{\cos^2 x}{\sin^2 x} \Rightarrow -\operatorname{cosec}^2 x$$

$$\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

Example:- Suppose that the rising Sun passes directly over a building that is 100 feet high & let θ be angle of elevation of the sun. Find the rate at which the length x of the building's shadow is changing w.r.t to θ. when $\theta = 45°$. Express your answer in feet/deg.



$$\frac{dx}{d\theta} = ?$$

Solution:-

$$\tan\theta = \frac{\text{Perp}}{\text{Base}} = \frac{100}{x}$$

$$x \tan\theta = 100$$

$$x = \frac{100}{\tan\theta}$$

$$x = 100 \cot\theta$$

now
$$\frac{dx}{d\theta} = \frac{d}{d\theta}(100 \cot\theta)$$

$$= 100 \frac{d}{d\theta}(\cot\theta)$$

$$= 100(-\operatorname{cosec}^2\theta) \Rightarrow -100 \operatorname{cosec}^2\theta$$

put $\theta = 45° = \frac{\pi}{4}$

$$\frac{dx}{d\theta} = -100 \operatorname{cosec}^2\left(\frac{\pi}{4}\right) \Rightarrow -100 \frac{1}{\sin^2\left(\frac{\pi}{4}\right)}$$

$$= -100\left(\frac{1}{(0.7071)^2}\right) \Rightarrow -100\left(\frac{1}{0.5}\right) \Rightarrow -100(2)$$

$= -200$ feet / radian

$= -200 \times \frac{\pi}{180} \Rightarrow -\frac{10}{9}\pi$ feet/degree.

Lecture 18 The Chain Rule

Derivative of composition of functions.

$f \circ g(x) \Rightarrow f(g(x))$

Let $g(x) = u$, Then $y = f(g(x)) = f(u)$

As $y = f(u)$ & $u = g(x)$

$\frac{dy}{du} = f'(u)$ & $\frac{du}{dx} = g'(x)$

So

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example:- Find $\frac{dy}{dx}$ if $y = 4\cos x^3$

$y = 4\cos(x^3)$

Let $u = x^3$ Then $y = 4\cos u$

$\frac{du}{dx} = 3x^2$, $\frac{dy}{du} = 4(-\sin u)$

$= -4\sin u$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \Rightarrow (-4\sin u)(3x^2) \Rightarrow -12x^2\sin u$$

$$\frac{dy}{dx} = -12x^2\sin(x^3)$$

Generalized formula:-

$$\frac{dy}{dx} = \frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$

Example:- $f(x) = (x^2 - x + 1)^{23}$

Let $u = x^2 - x + 1$ Then $f(u) = u^{23}$

$$\frac{d}{dx}[f(u)] = \frac{d}{dx}[u^{23}] \Rightarrow 23\,u^{22}.\frac{du}{dx}$$

$$= 23\,u^{22}.\frac{d}{dx}(x^2 - x + 1)$$

$$= 23\,u^{22}(2x - 1 + 0) \Rightarrow 23\,u^{22}(2x - 1)$$

$$= 23(x^2 - x + 1)(2x - 1)$$

Table:-

important formulae

$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$	$\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}\frac{du}{dx}$
$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$	$\frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u\tan u\frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u\cot u\frac{du}{dx}$

Examples:-

(i) $\frac{d}{dx}[\sin 2x]$

$=$ let $u = 2x$

$\frac{d}{dx}\{\sin u]$

$= \cos u \cdot \frac{du}{dx}$

$= \cos(2x) \cdot \frac{d(2x)}{dx}$

$= \cos 2x \ (2)$

$= 2 \cos 2x$

(ii) $\frac{d}{dx}[\tan(x^2+1)]$

let $u = x^2+1$

$\frac{d}{dx}[\tan u]$

$= \sec^2 u \cdot \frac{du}{dx}$

$= \sec^2(x^2+1) \frac{d(x^2+1)}{dx}$

$= \sec^2(x^2+1)(2x+0)$

$= 2x \sec^2(x^2+1)$

make your Substitution So That result comes out to be a function That you already know how to differentiate.

Example:- $\frac{d}{dx}\left(\sqrt{x^3+\operatorname{cosec}(x)}\right)$

let $u = x^3 + \operatorname{cosec} x$

$\frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx} \Rightarrow \frac{1}{2\sqrt{x^3+\operatorname{cosec} x}} \cdot \frac{d}{dx}(x^3+\operatorname{cosec} x)$

$= \frac{1}{2\sqrt{x^3+\csc x}} \cdot (3x^2+(-\operatorname{cosec} x \cot x))$

$= \frac{3x^2 - \operatorname{cosec} x \cot x}{2\sqrt{x^3+\operatorname{cosec} x}}$ Ans.

Alternate to chain rule:-

like generalized formula

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Example:- $\frac{d}{dx}[\cos(3x+1)]$

here $f(x) = \cos x$ & $g(x) = 3x+1$

$$\frac{d}{dx}[\cos(3x+1)] = -\sin(3x+1) \cdot \frac{d}{dx}(3x+1)$$

$$= -\sin(3x+1) \cdot [3(1)+0] \Rightarrow -3\sin(3x+1)$$

Lecture 19.

$3 = xy$
$\frac{3}{x} = y$
$y = 2x^2$
$xy = 2x^2y^3 y$

Implicit differentiation:-

If we find $\frac{dy}{dx}$ without solving for y first. This is called implicit differentiation.

Example:-

Find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$

$$\frac{d}{dx}(\underbrace{5y^2} + \underbrace{\sin y}) = \frac{d}{dx}(x^2)$$

$$5\frac{d}{dx}y^2 + \frac{d}{dx}\sin y = 2x$$

$$5(2y)\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}\left[10y + \cos y\right] = 2x$$

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

Example:- Find slope of Tangent line at point $(4,0)$ on graph of $7y^4 + x^3y + x = 4$.

$$\frac{d}{dx}(7y^4 + x^3y + x) = \frac{d}{dx}(4)$$

$$7\left(4y^3\frac{dy}{dx}\right) + \left[x^3\frac{dy}{dx} + y\frac{d}{dx}x^3\right] + 1 = 0$$

$$28y^3 \frac{dy}{dx} + x^3 \frac{dy}{dx} + 3x^2 y \frac{\cancel{dy}}{\cancel{dx}} = -1$$

$$\frac{dy}{dx}\left[28y^3 + x^3\right] = -1 - 3x^2 y$$

$$\frac{dy}{dx} = -\frac{(1+3x^2y)}{28y^3 + x^3}$$

now put $(4,0)$

$$m_{Tan} = \left.\frac{dy}{dx}\right|_{\substack{x=4\\y=0}} = -\frac{\left[1+3(4)^2(0)\right]}{28(0)^3 + (4)^3}$$

$$= \frac{-(1+0)}{0+64} \Rightarrow \frac{-1}{64} \quad \underline{\text{Ans.}}$$

Example:- Find $\dfrac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$

$$\frac{d}{dx}(4x^2 - 2y^2) = \frac{d}{dx}(9)$$

$$4(2x) - 2(2y)\frac{dy}{dx} = 0$$

$$8x - 4y\frac{dy}{dx} = 0$$

$$8x = 4y\frac{dy}{dx}$$

$$\frac{8x}{4y} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{8x}{4y}$$

$$\frac{dy}{dx} = \frac{2x}{y}$$

now

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[2\frac{x}{y}\right]$$

$$\frac{d^2y}{dx^2} = 2\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= 2\left[\frac{y\frac{dx}{dx} - x\frac{dy}{dx}}{y^2}\right]$$

$$= \frac{2}{y^2}\left[y - x\frac{dy}{dx}\right]$$

$$= \frac{2}{y^2}\left[y - x\left(\frac{2x}{y}\right)\right]$$

$$= \frac{2}{y^2}\left[\frac{y}{1} - \frac{2x^2}{y}\right]$$

$$= \frac{2}{y^2}\left[\frac{y^2 - 2x^2}{y}\right]$$

$$= \frac{2y^2 - 4x^2}{y^3} \Rightarrow -\frac{(4x^2 - 2y^2)}{y^3}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-9}{y^3}}$$

Lecture 20

Derivative of The logarithmic, exponential & inverse functions:-

Logarithmic functions:-

In $y = f(x) = \log_b(x)$

b is called the base.

"Log base b of x is y" $(y = \log_b x)$

Properties of log:-

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$\Rightarrow \log_b 1 = 0$

$\Rightarrow \log_b \frac{a}{c} = \log_b a - \log_b c$

$\Rightarrow \log_b b = 1$

$\Rightarrow \log_b a^r = r \log_b a$

$\Rightarrow \log_b ac = \log_b a + \log_b c$

$\Rightarrow \log_b \frac{1}{c} = -\log_b c$

Natural log functions:-

If $b = e$ (i-e) log with base "e" is called natural log.

$$\log_e x = \ln(x)$$

Derivative of ln:-

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

if $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$

Example:- Find $\frac{d}{dx}\left[\ln(x^2+1)\right]$

$$= \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1)$$

$$= \frac{1}{x^2+1} \cdot (2x+0) \Rightarrow \frac{2x}{x^2+1} \quad \underline{\text{Ans.}}$$

Example:- $\frac{d}{dx}\left[\ln\left(\frac{x^2\sin(x)}{\sqrt{1+x}}\right)\right]$

$$= \frac{d}{dx}\left[\ln(x^2\sin x) - \ln\sqrt{1+x}\right]$$

$$= \frac{d}{dx}\left[\ln(x^2) + \ln(\sin x) - \ln(1+x)^{1/2}\right]$$

$$= \frac{d}{dx}\left[2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)\right]$$

$$= 2\left(\frac{1}{x}\right) + \frac{1}{\sin x}(\cos x) - \frac{1}{2} \cdot \frac{1}{1+x}(0+1)\Big]$$

$$= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)} \quad \underline{\text{Ans.}}$$

Logarithmic Differentiation:-

We can use log properties to find derivative of messy functions.

Example:- $$y = \frac{x^2 \cdot \sqrt[3]{7x-14}}{(1+x^2)^4}$$

Apply $\ln$ on b/s

$$\ln y = \ln\left[\frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}\right]$$

$$\ln y = \ln\left(x^2 \sqrt[3]{7x-14}\right) - \ln (1+x^2)^4$$

$$= \ln x^2 + \ln \sqrt[3]{7x-14} - 4\ln(1+x^2)$$

$$\ln y = 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

Take derivative on b/s

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2\left(\frac{1}{x}\right) + \frac{1}{3} \cdot \frac{1}{7x-14} \cdot (7) - 4 \cdot \frac{1}{1+x^2} \cdot (0+2x)$$

$$\frac{dy}{dx} = \left[\frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2}\right] y$$

$$\frac{dy}{dx} = \left(\frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2}\right) \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}$$

Power rule:-

$$y = x^r$$

$$\frac{dy}{dx} = r\,x^{r-1}$$

Derivative of exponential function:-

$$\frac{d}{dx}(b^x) = b^x . \ln b$$

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If $b = e$

$$\frac{d}{dx}(e^x) = e^x . \ln e = e^x(1) \Rightarrow e^x$$

$$\frac{d}{dx}(e^u) = e^u . \frac{du}{dx}$$

Defination:-

If the function f & g satisfy the two condition

$f(g(x)) = x$

$g(f(x)) = x$

Then we say that f is an inverse of g & g is an inverse of f. or we can say that f & g are inverse of each other.

Ex $f(x) = 2x$ & $g(x) = \frac{1}{2}x$

$$f(x) = 2x$$
$$f(g(x)) = 2(g(x)) \Rightarrow 2\left(\frac{1}{2}x\right) \Rightarrow x$$
$$g(x) = \frac{1}{2}x$$
$$g(f(x)) = \frac{1}{2}(f(x)) = \frac{1}{2}(2x) = x$$

Derivative of inverse functions:-

Suppose $f(x)$ has an inverse $f^{-1}(x)$ over an interval on which f has a non-zero derivative as x varies over interval. Then $f^{-1}(x)$ is differentiable on I & the formula for its derivative is

$$\left[f^{-1}\right]'(x) = \frac{1}{f'(f^{-1}(x))}$$

or more simply, if $y = f^{-1}(x)$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Applications of differentiation:-

Lecture 21

Related Rates Related rates are real life problems. For example, we want to know how fast a satellite is changing altitude w.r.t time or gravity.

Example:- Assume that oil spilled from a ruptured tanker spread in a circular pattern whose radius increases at a constant rate of 2 ft/sec. How fast is the area of spill increasing when the radius of the spill is 60 ft?

let $t=$ time

$r=$ radius

$A=$ area

$\frac{dA}{dt}=?$

$\frac{dr}{dt}=2\ ft/sec$

spill is circular in shape

So $A=\pi r^2$

now $\frac{dA}{dt}=2\pi r\frac{dr}{dt}$

$\left.\frac{dA}{dt}\right|_{r=60}=2\pi(60)(2)$

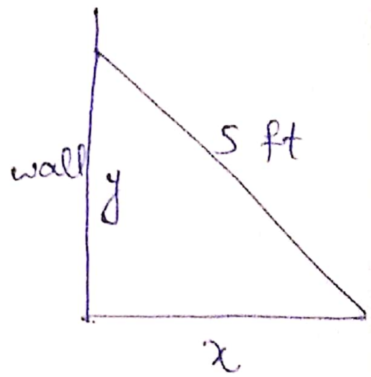
$=240\pi\ ft^2/sec$

Example:- A five foot ladder is leaning against a wall. It slips in such a way that its base is moving away from the wall at a rate of 2 ft/sec at instant when the base is 4 ft from the wall. How fast is the top of the ladder moving down the wall at that instant?

let $t =$ time (no. of seconds)

$x =$ distance from base (feet)

$x =$ distance from top (feet) of ladder to floor.



$$\frac{dx}{dt} = 2 \text{ (horizontal movement)}$$

$$\frac{dy}{dt} = ? \text{ (vertical movement)}$$

using Pythagorous Theorem.

$$(H)^2 = (P)^2 + (B)^2$$

$$(5)^2 = (y)^2 + (x^2)$$

$$25 = x^2 + y^2$$

$$x^2 + y^2 = 25$$

differentiate w.r.t t

put $x = 4$ in

$$25 = (4)^2 + y^2$$

$$25 = 16 + y^2$$

$$25 - 16 = y^2$$

$$9 = y^2$$

$$\sqrt{9} = \sqrt{y^2}$$

$$\boxed{3 = y}$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(25)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2y\frac{dy}{dt} = -2x\left(\frac{dx}{dt}\right)$$

$$\frac{dy}{dt} = \frac{-2x}{2y}\left(\frac{dx}{dt}\right)$$

$$= -\frac{2x}{2y}(2)$$

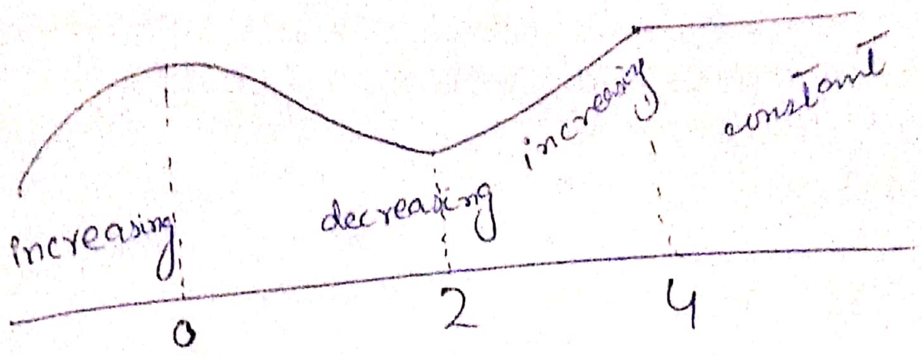
$$\left.\frac{dy}{dt}\right|_{x=4} = \frac{-2x}{y}$$

$$= -\frac{2(4)}{3} \Rightarrow -\frac{8}{3}\ ft/sec$$

Increasing & decreasing functions.

increasing function on an interval means that as we move from left to right in x-direction the y-values increase in magnitude.

decreasing function on an interval means that as we move from left to right in x-direction, the y-values decrease in magnitude.

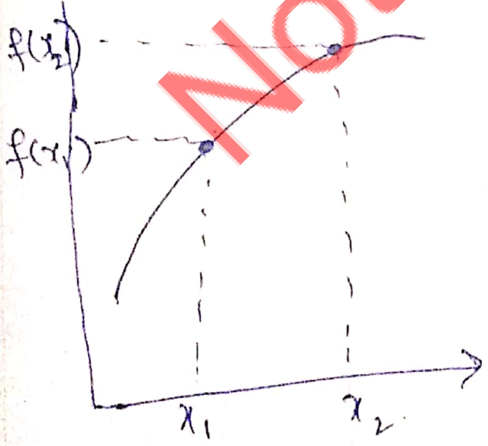


Defination:- let f be defined on an interval & let x_1 & x_2 denote points in that interval.

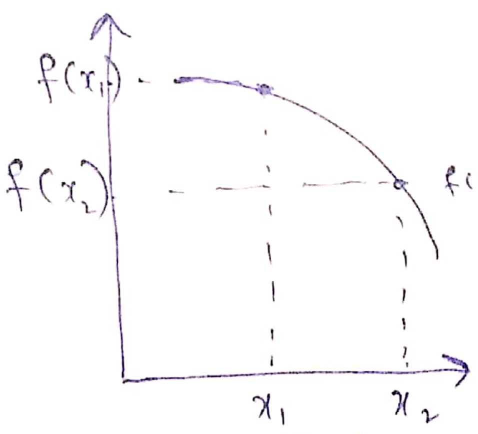
(a) f is increasing on that interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(b) f is decreasing on that interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

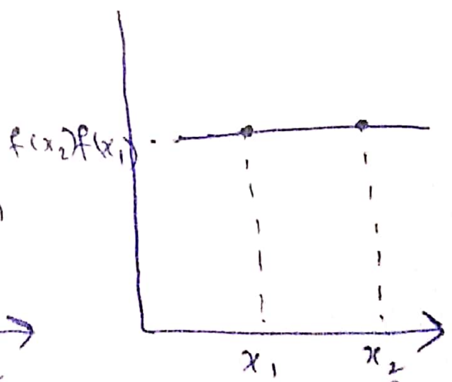
(c) f is constant on the interval if $f(x_1) = f(x_2)$ for all x_1 & x_2.



Tangent line has +ve slope



Tangent line has -ve slope.



Tangent line has zero slope

Theorem:- let f is a continuous function on $[a, b]$ & differentiable function on (a, b)

If $f'(x) > 0$, Then function is increasing.

If $f'(x) < 0$, Then function is decreasing.

If $f'(x) = 0$, Then function is constant.

Example:- Find intervals on which function is decreasing or increasing. $f(x) = x^2 - 4x + 3$

$$f(x) = x^2 - 4x + 3$$
$$f'(x) = 2x - 4$$

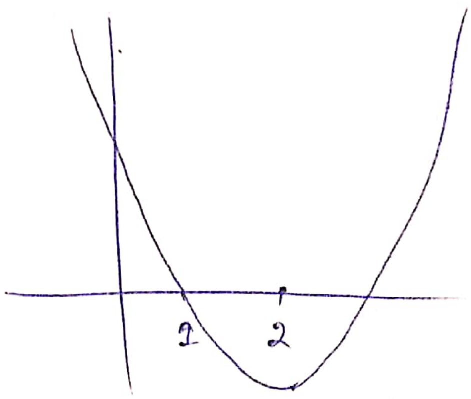
now

$$f'(x) = 0$$
$$2x - 4 = 0$$
$$2x = 4$$
$$x = 4/2$$
$$\boxed{x = 2}$$

So

$f'(x) < 0$ on $(-\infty, 2)$

$f'(x) > 0$ on $(2, +\infty)$



Definations Let f be differentiable on an interval

(a) f is called concave up on interval if f' is increasing on the interval.

(b) f is called concave down on interval if f' is decreasing on the interval.

slope of Tangent lines is decreasing.

Slope of Tangent line is increasing.

Example:- Find interval on which f(x) is concave up or concave down.

$$f(x) = x^2 - 4x + 3$$

$$f'(x) = 2x - 4$$

$$f''(x) = 2 > 0$$

So graph of this function is concave up on interval $(-\infty, +\infty)$

Theorem:-

(a) if $f''(x) > 0$ Then f is concave up.

(b) if $f''(x) < 0$ Then f is concave down.

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Lecture 22 Relative Extrema

Relative minimum:-

A function is said to have a relative minimum at x_0, if $f(x_0) \leq f(x)$ for all x in some open interval containing x_0.

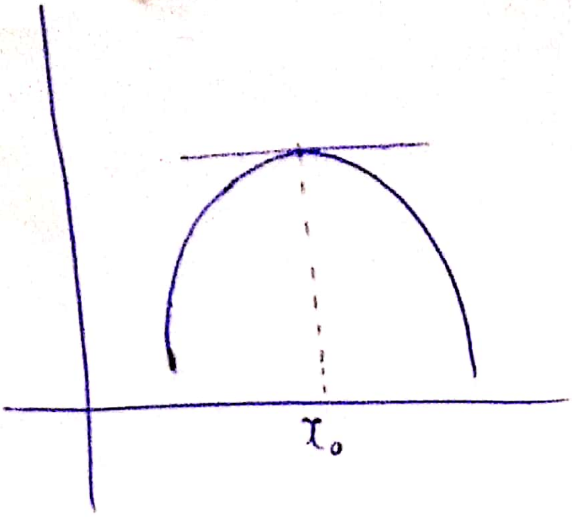
Relative Maximum:-

A function is said to ~~be~~ have a relative maximum at x_0 if $f(x_0) \geq f(x)$ for all x in some open interval containing x_0.

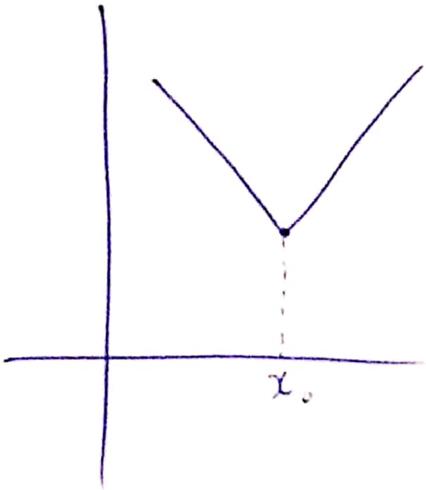
Relative extremum:-

A function is said to have a relative extremum at x_0 if it has either a relative minimum or relative ~~extremum~~ maximum at x_0.

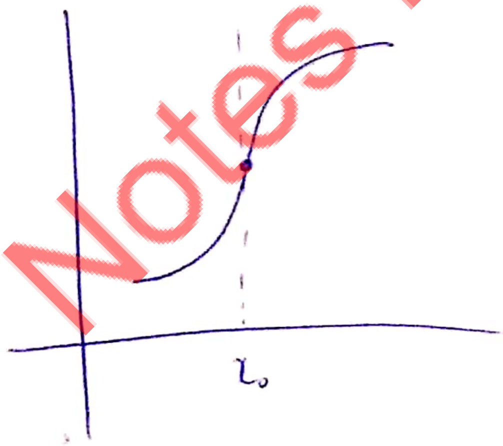
Critical point:- A critical point for function f is any value of x in the domain of f at which $f'(x) = 0$ or at which f is not differentiable; The critical points where $f'(x) = 0$ are called stationary points.



x_0 is critical & stationary point because Tangent line has zero slope.



x_0 is critical & it has minimum value at that point but Tangent line is not defined.



x_0 is critical point but not stationary because derivative does not exist. (Slope of vertical Tangent line is undefined).

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First derivative Test:-

$\Rightarrow$ If $f'(x) > 0$ on an open interval Then function has relative maximum.

$\Rightarrow$ If $f'(x) < 0$ on an open interval Then function has relative minimum.

$\Rightarrow$ If $f'(x)$ has no sign (+ve or -ve) Then function does not have relative extremum.

Example:- Locate relative extrema of

$$f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$$

$$f'(x) = 3\left(\frac{5}{3}x^{\frac{5}{3}-1}\right) - 15\left(\frac{2}{3}x^{\frac{2}{3}-1}\right)$$

$$= 5x^{2/3} - 10x^{-1/3}$$

$$= 5x^{2/3} - \frac{10}{x^{1/3}}$$

$$= 5\left(x^{2/3} - \frac{2}{x^{1/3}}\right)$$

$$= \frac{5}{x^{1/3}}\left(x^{\frac{2}{3}} \cdot x^{\frac{1}{3}} - \frac{2}{x^{1/3}} \cdot x^{1/3}\right)$$

$$= \frac{5}{x^{1/3}}\left(x^{\frac{2}{3}+\frac{1}{3}} - 2\right)$$

$$f'(x) = \frac{5}{x^{1/3}}(x-2)$$

Now

$$f'(x) = 0$$

$$\frac{5}{x^{1/3}}(x-2) = 0$$

$$\frac{5}{x^{1/3}} = 0$$

no value

$$x - 2 = 0$$

$$\boxed{x = 2}$$

So derivative does not exist

Second derivative Test:-

Suppose f is twice differentiable at a stationary point x_0

(a) if $f''(x_0) > 0$. Then f has relative minimum.

(b) if $f''(x_0) < 0$ Then f has relative maximum.

Example:- Locate the relative extrema of $f(x) = x^4 - 2x^2$

$f'(x) = 4x^3 - 2(2x)$

$= 4x^3 - 4x$

put $f'(x) = 0$

$4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$4x = 0, \quad x^2 - 1 = 0$

$\boxed{x = 0}, \quad (x+1)(x-1) = 0$

$x + 1 = 0, \quad x - 1 = 0$

$\boxed{x = -1}, \quad \boxed{x = 1}$

$f''(x) = 12x^2 - 4$

put $x = 0$

$f''(0) = 12(0)^2 - 4 \Rightarrow 0 - 4 \Rightarrow -4 < 0$

put $x = 1$

$f''(1) = 12(1)^2 - 4 \Rightarrow 12 - 4 \Rightarrow 8 > 0$

put $x = -1$

$f''(-1) = 12(-1)^2 - 4 \Rightarrow 12 - 4 \Rightarrow 8 > 0$

So f has relative maxima at $x = 0$ & relative minima at $x = 1$ & $x = -1$.

Graphs of Polynomial.

1. Calculate $P'(x)$ & $P''(x)$. 2. using P', calculate stationary points & the intervals of increase & decrease. 3. Use P'' to determine, if graph is concave up or concave down. 4. Plot all of the above & x & y intercepts.

Sketch the graph of

$$P(x) = y = x^3 - 3x + 2$$

$$P'(x) = \frac{dy}{dx} = 3x^2 - 3$$

$$P''(x) = \frac{d^2y}{dx} = 6x$$

Put

$P'(x) = 0$

$3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$x^2 - 1 = 0$

$(x + 1)(x - 1) = 0$

$x + 1 = 0$ | $x - 1 = 0$

$\boxed{x = -1}$ | $\boxed{x = 1}$

now

Put $P''(x) = 0$

$6x = 0$

$\boxed{x = 0}$

Put it in y

$y = x^3 - 3x + 2$

$y = (0)^3 - 3(0) + 2$

$\boxed{y = 2}$

$(0, 2)$ inflection point

Put $x = 1$ in

$f''(x) = 6x$

$f''(1) = 6(1)$

$f''(1) = 6 > 0$

concave up

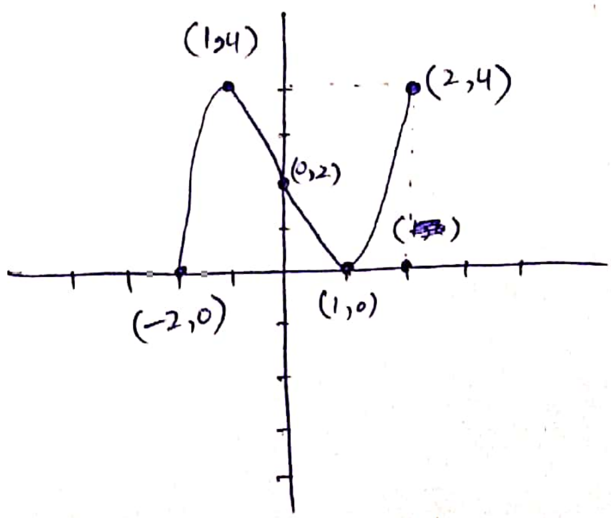
Put $x = -1$

$f''(-1) = 6(-1)$

$f''(-1) = -6 < 0$

concave down

x	$y = x^3 - 3x + 2$
-2	0
-1	4
0	2
1	0
2	4



Graph of Rational Functions:-

Rational functions are those which can be written as ratio of two polynomials.

$$R(x) = \frac{P(x)}{Q(x)}$$

if $Q(x) = 0$ then $R(x)$ has discontinuity at those values of x.

For example $f(x) = \frac{x}{x-2}$

function is discontinuous at $x = 2$.

Vertical asymptot:- A line $x = x_0$ is called a ~~verticle~~ vertical asymptots for the graph of a function f if $f(x) \to +\infty$ or $f(x) \to -\infty$ as x approaches x_0 from right or from the left.

Horizontal asymptot:- A line $y = y_0$ is called horizontal asymptot for the graph of f if $\lim_{x \to +\infty} f(x) = y_0$ or $\lim_{x \to -\infty} f(x) = y_0$

Example:- Find horizontal & vertical asymptotes of $f(x) = \dfrac{x^2+2x}{x^2-1}$

vertical asymptot occurs at points where $x = 1$ & $x = -1$.

Since

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^2+2x}{x^2-1}$$

$$= \lim_{x \to +\infty} \frac{x^2}{x^2} \Rightarrow 1$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2+2x}{x^2-1}$$

$$= \lim_{x \to -\infty} \frac{x^2}{x^2} \Rightarrow 1$$

So ~~vertical~~ horizontal asymptot is $y = 1$.