MTH 301/201.

Calculus:-

↳ tool used to analyze changes in physical quantities.

↳ mathematics of motion & change.

↳ used where variable forces are at work, producing acceleration.

Differential Calculus:-

deals with the problems of

⇒ Rate of change

⇒ Slope of curve

(e.g) acceleration is rate of change of velocity.

Integral Calculus:-

deals with problem of determining a function from information about its rates of change.

(e.g) finding length of curve, area, volume, mass etc.

Reference axis system:-

Remember real no. (real number line) consists of both rational & irrational numbers.

(Real no. are union of rational & irrationals).

Abscissa: In an ordered pair (x, y), the first term 'x' is called abscissa.

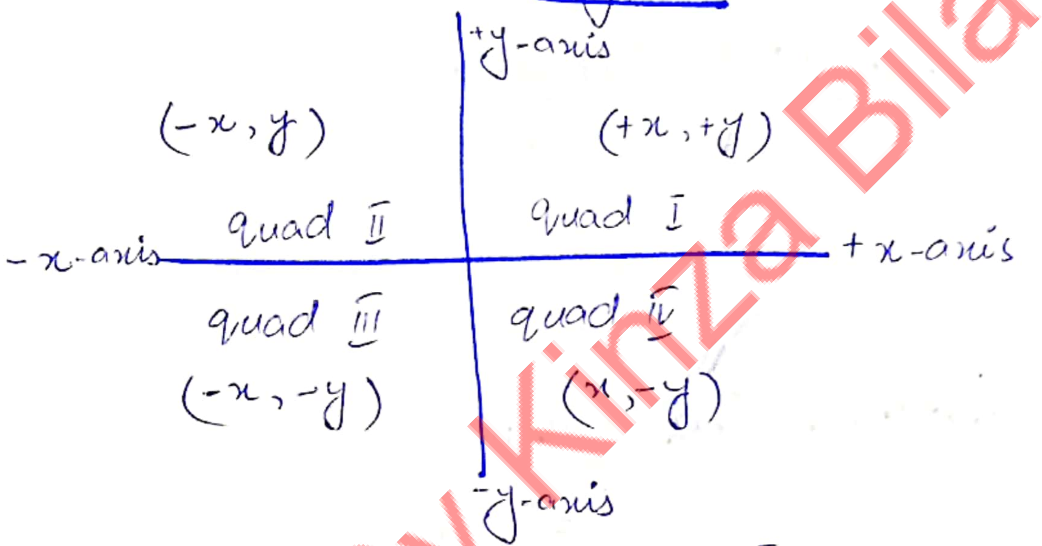
Ordinate:- In an ordered pair (x, y), the second number 'y' is called ordinate.

↳ When in two dimension, we take two mutually perpendicular lines, they are called axis.

↳ The horizontal axis is called x-axis.

↳ The vertical line/axis is called y-axis.

↳ The point where x-axis & y-axis intersect each other is called origion.



↳ In two dimensional system the origion is (0,0) while in 3 dimensional system (3d/space) the origion is (0,0,0). Here the third coordinate/axis is called z-axis. In space the signs in ~~quadrants~~ octants are:

1st Octant	$(+x, +y, +z)$
2nd Octant	$(-x, y, z)$
3rd Octant	$(-x, -y, z)$
4th octant	$(x, -y, z)$
5th octant	$(x, y, -z)$
6th octant	$(-x, y, -z)$
7th octant	$(-x, -y, -z)$
8th octant	$(x, -y, -z)$

Function:-

- dependence of one quantity on other.
- quantity x is called independent variable & the quantity y is called dependent variable.
- we write $y = f(x)$
- we read y is a function of x.
- each value assigned to x denotes a unique value of y.

Examples:- In $A = \pi r^2$, area of a circle depends upon the radius. So A is function of r.

In $v = \frac{d}{t}$, velocity depends upon time.

In volume formula, the volume depends upon length ($V = x^3$). So V is a function of x.

Function of several variables:-

A function can have more than one independent variable.

Example:- In Area of a rectangle $A = l \times w$

Area depends upon length & width, So A is a function of l & w.

Lecture 02

Values of function:-

If $f(x) = 2x^2 - 1$

Then $f(1) = 2(1)^2 - 1 = 2 - 1 = 1$

$f(-2) = 2(-2)^2 - 1 = 2(4) - 1 = 8 - 1 = 7$

$f(4) = 2(4)^2 - 1 = 2(16) - 1 = 32 - 1 = 31$

These are values of $f(x)$ at some points.

Function of two variables:-

If $f(x,y) = x^2y + 1$

Then $f(2,1) = (2)^2(1) + 1 = 4(1) + 1 = 4 + 1 = \boxed{5}$

Example:- ~~f(x)~~

$$f(x,y) = x + \sqrt[3]{xy}$$

(a) $f(2,4) = 2 + \sqrt[3]{(2)(4)} = 2 + \sqrt[3]{8} = 2 + 2 = \boxed{4}$

(b) $f(t,t^2) = t + \sqrt[3]{(t)(t^2)} = t + \sqrt[3]{t^3} = t + t = \boxed{2t}$

(c) $f(2y^2, 4y) = 2y^2 + \sqrt[3]{(2y^2)(4y)} = 2y^2 + \sqrt[3]{8y^3} = \boxed{2y^2 + 2y}$

Function of Three variables:-

$f(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}$

$f(0, \frac{1}{2}, \frac{1}{2}) = \sqrt{1 - 0^2 - (\frac{1}{2})^2 - (\frac{1}{2})^2}$

$= \sqrt{1 - \frac{1}{4} - \frac{1}{4}}$

$= \sqrt{\frac{4-1-1}{4}} = \sqrt{\frac{2}{4}} = \boxed{\sqrt{\frac{1}{2}}}$

Example:-

$$f(x,y,z) = xy^2z^3 + 3$$

(a) $f(2,1,2) = (2)(1)^2(2)^3 + 3$

$= (2)(1)(8) + 3 \Rightarrow 16 + 3 \Rightarrow \boxed{19}$

(b) $f(0,0,0) = (0)(0)^2(0)^3 + 3$

$= 0 + 3 \Rightarrow \boxed{3}$

(c) $f(a,a,a) = (a)(a)^2(a)^3 + 3$

$= \boxed{a^6 + 3}$

(d) $f(t,t^2,-t) = (t)(t^2)^2(-t)^3 + 3$

$= (t)(t^4)(-t^3) + 3$

$= \boxed{-t^8 + 3}$

Example:- If $f(x,y,z) = x^2y^2z^4$

where $x(t) = t^3$, $y(t) = t^2$ & $z(t) = t$

(a) $f(x(t), y(t), z(t)) = [x(t)]^2[y(t)]^2[z(t)]^4$

$= (t^3)^2(t^2)^2(t)^4$

$= t^6 . t^4 . t^4$

$= \boxed{t^{14}}$

(b) $f[x(0), y(0), z(0)] = [x(0)]^2[y(0)]^2[z(0)]^4$

$= (0^2)(0^2)(0^4)$

$= \boxed{0}$

Example:-

$$f(x,y,z) = xyz + x$$

$f(xy, \frac{y}{x}, xz) = (xy)(\frac{y}{x})(xz) + (xy)$

$= \boxed{xy^2z + xy}$

Example:-

$$g(x,y,z) = z\sin(xy)$$

$$u(x,y,z) = x^2z^3$$

$$v(x,y,z) = Pxyz$$

$$w(x,y,z) = \frac{xy}{z}$$

$$g[u(x,y,z), v(x,y,z), w(x,y,z)] = ?$$

As

$$g(x,y,z) = z\sin(xy)$$

$$g[u(x,y,z), v(x,y,z), w(x,y,z)] = \frac{xy}{z}\sin[(x^2z^3)(Pxyz)]$$

$$= \boxed{\frac{xy}{z}\sin Px^3yz^4}$$

Example:-

$$g(x,y) = y\sin x^2y \quad , \quad u(x,y) = x^2y^3$$

$$v(x,y) = \bar{\pi}xy \quad , \quad g[u(x,y), v(x,y)] = ?$$

As

$$g(x,y) = y\sin x^2y$$

$$g[u(x,y), v(x,y)] = \bar{\pi}xy\sin(x^2y^3)^2(\bar{\pi}xy)$$

$$= \bar{\pi}xy\sin(x^4y^6\bar{\pi}xy)$$

$$= \boxed{\bar{\pi}xy\sin(\bar{\pi}x^5y^7)}$$

Function of one variable:-

function of one real variable x assigns value to $f(x)$ for each point x.

Function of two variables:-

function of two real variables x & y assing value to $f(x,y)$ for each (x,y) point.

Function of Three variables:-

function of Three real variables x, y & z assign value to $f(x,y,z)$ for each point (x,y,z).

Function of n variables:-

function of n real variables $x_1, x_2, \ldots, x_n$ assign value to $f(x_1, x_2, \ldots, x_n)$ for each point $(x_1, x_2, \ldots, x_n)$.

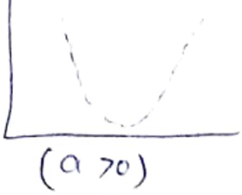
Remarks:-

general equation of Parabola is $ax^2+bx+c=0$ (quadratic equ.)

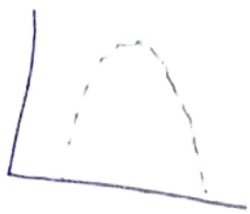
parabola opens upward if $a>0$

parabola opens downward if $a<0$

$\boxed{\text{vertex} = \frac{-b}{2a}}$



$(a>0)$



Lecture 03

Elements of Three dimensional geometry

Distance Formula:- (for 3-dimension).

Let two points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$

Then

$$\overline{PQ} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Example:-

Find $\overline{AB}$, $\overline{BC}$ & $\overline{AC}$ if $A(3,2,4)$, $B(6,10,-1)$ & $C(9,4,1)$.

$$\overline{AB} = \sqrt{(6-3)^2+(10-2)^2+(-1-4)^2} = \sqrt{(3)^2+(8)^2+(-5)^2}$$

$$= \sqrt{9+64+25} = \boxed{\sqrt{98}}$$

$$\overline{BC} = \sqrt{(9-6)^2+(4-10)^2+(1+1)^2} = \sqrt{3^2+(-6)^2+2^2}$$

$$= \sqrt{9+36+4} = \sqrt{49} = \boxed{7}$$

$$\overline{AC} = \sqrt{(9-3)^2+(4-2)^2+(1-4)^2} = \sqrt{6^2+2^2+(-3)^2}$$

$$= \sqrt{36+4+9} = \sqrt{49} = \boxed{7}$$

Mid point:- (for 3-dimensions)

Mid point of $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ will be

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Example:- Find mid-point of $A(3,2,4)$ & $B(6,10,-1)$

$$M\left(\frac{3+6}{2}, \frac{2+10}{2}, \frac{4+(-1)}{2}\right) \Rightarrow \left(\frac{9}{2}, \frac{12}{2}, \frac{4-1}{2}\right)$$

$$= \boxed{\left(\frac{9}{2}, 6, \frac{3}{2}\right)}$$

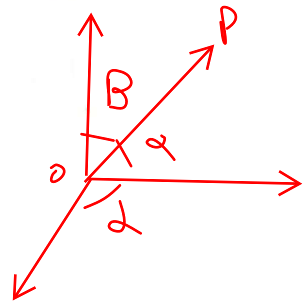
Direction angles:-

Direction angles α, β, γ are defined as

α = angle b/w line & +ve x-axis

β = angle b/w line & +ve y-axis

γ = angle b/w line & +ve z-axis



α, β, γ lies b/w 0 & π (180°).

Direction ratios:-

⟹ cosine of angles are called direction cosines.

⟹ Any multiple of direction cosines are called direction ratio of direction number.

Direction cosines:-

$$\cos\alpha = \frac{x}{OP} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\cos\beta = \frac{y}{OP} = \frac{y}{x^2+y^2+z^2}$$

$$\cos\gamma = \frac{z}{OP} = \frac{z}{x^2+y^2+z^2}$$

Always $\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$

Direction ratios for two points:-

For $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$
direction ratios will be $x_2 - x_1, y_2 - y_1, z_2 - z_1$

Direction ~~...~~ cosines for (two points:-)

For $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$
direction cosines will be

$$\cos\alpha = \frac{x_2 - x_1}{\overline{PQ}}, \cos\beta = \frac{y_2 - y_1}{PQ}, \cos\gamma = \frac{z_2 - z_1}{PQ}$$

Here $\overline{PQ} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Example:- Find direction ratios & direction cosines of $P(1, 3, 2)$ & $Q(7, -2, 3)$.

direction ratios:-

$$x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1$$

$$7 - 1, \quad -2 - 3, \quad 3 - 2$$

$$6, \quad -5, \quad 1$$

direction cosines:- $\overline{PQ} = \sqrt{(7-1)^2 + (-2-3)^2 + (3-2)^2}$

$$= \sqrt{6^2 + (-5)^2 + (1)^2}$$

$$= \sqrt{36 + 25 + 1}$$

$$= \sqrt{62}$$

Now

$$\cos\alpha = \frac{x_2 - x_1}{PQ} = \boxed{\frac{6}{\sqrt{62}}}$$

$$\cos\beta = \frac{y_2 - y_1}{PQ} = \boxed{\frac{-5}{\sqrt{62}}}$$

$$\cos\gamma = \frac{z_2 - z_1}{PQ} = \boxed{\frac{1}{\sqrt{62}}}$$

Intersection of two surfaces:-

a) Intersection of two surfaces is a curve in three dimensional space.

b) Curve in three-dimensional space is represented by two equations.

Intersection of two planes:-

If two planes are not parallel, then they intersect & their intersection is a straight line.

(or we can say that two non-parallel planes represent a straight line.

Region	Description	Equation
xy-plane	consists of all points of the form $(x, y, 0)$	$z=0$
xz-plane	all points of form $(x, 0, z)$	$y=0$
yz-plane	all points of form $(0, y, z)$	$x=0$
x-axis	all points of form $(x, 0, 0)$	$y=0$, $z=0$
y-axis	all points of form $(0, y, 0)$	$x=0$, $z=0$
z-axis	all points of form $(0, 0, z)$	$x=0$, $y=0$

General equ. of plane:-

$$ax+by+cz+d=0$$

General Equ. of Sphere:-

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2$$

Right circular cone:-

$$z=\sqrt{x^2+y^2}$$

$$\& \quad \phi=\frac{\pi}{4}$$

Horizontal Circular Cyllinder:-

$$x^2+z^2=1$$

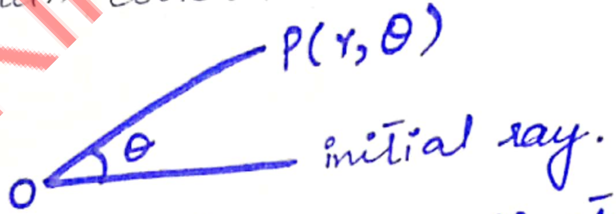
Lecture 04

Polar Coordinates

xy-plane is called cartesian system/Plane.

x & y are called cartesian coordinates.

O is known as Pole.



Conversion from polar to Cartesian coordinates.

$x=r\cos\theta$ — (α) , $y=r\sin\theta$ — (β)

Take square on b/s

$x^2=r^2\cos^2\theta$ — ①

Take square on b/s

$y^2=r^2\sin^2\theta$ — ②

adding eq ① & ②

$$x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta$$

$$x^2+y^2=r^2(\cos^2\theta+\sin^2\theta)$$

$$x^2+y^2=r^2(1)$$

$$x^2+y^2=r^2$$

dividing α by β

$$\frac{y}{x}=\frac{r\sin\theta}{r\cos\theta}$$

$$\frac{y}{x}=\tan\theta$$

$$\tan^{-1}\left(\frac{y}{x}\right)=\theta$$

Polar coordinates:- $P(r, \theta)$

Rectangular coordinates:- $P(x, y, z)$

Cylindrical coordinates:- $P(r, \theta, z)$ $r \geq 0$, $0 \leq \theta \leq 2\pi$

Spherical coordinates:- $P(\rho, \theta, \phi)$ $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$

Conversion b/w rectangular & cylindrical coordinates

$$(r, \theta, z) \longrightarrow (x, y, z)$$

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad z = z$$

$$r^2 = (x^2 + y^2)$$

$$r = \sqrt{x^2+y^2} \quad , \quad \tan\theta = \frac{y}{x} \quad , \quad z = z$$

conversion formula b/w cylindrical & spherical coordinates

$$(\rho, \theta, \phi) \longrightarrow (r, \theta, z)$$

$$\rho = \sqrt{r^2+z^2} \quad , \quad \theta = \theta \quad , \quad \tan\phi = \frac{r}{z}$$

conversion b/w rectangular & spherical coordinates:-

spherical se direct rectangular me nai ja

spherical -> cylandrical -> rectangular

$$(\rho, \theta, \phi) \longrightarrow (x, y, z)$$

$$r = \rho\sin\phi \quad , \quad \theta = \theta \quad , \quad z = \rho\cos\phi$$

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad z = z$$

Now $(x, y, z) \longrightarrow (\rho, \theta, \phi)$

$$x^2+y^2+z^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2$$

$$= \rho^2\{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi\}$$

$$= \rho^2\{\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi\}$$

$$= \rho^2\{\sin^2\phi + \cos^2\phi\}$$

$$= \rho^2(1)$$

$$x^2+y^2+z^2 = \rho^2$$

& or

$$\rho^2 = x^2+y^2+z^2$$

Take square on b/s

$$\sqrt{\rho^2} = \sqrt{x^2+y^2+z^2}$$

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

Constant surfaces:- (in rectangular coordinates).

- In $x = x_0$ (plane parallel to yz-plane)
- In $y = y_0$ (plane parallel to xz-plane).
- In $z = z_0$ (plane parallel to xy-plane).

Constant surfaces:- (in cylindrical coordinates).

- $r = r_0$ is a right cylinder.
- $\theta = \theta_0$ is a half plane attached to z-axis.
- $z = z_0$ is a horizontal plane.

Constant surfaces:- (in spherical coordinates).

$\rho = \rho_0$ consists of all points whose distance ρ from origin is ρ_0.

$\theta = \theta_0$ is a half plane along z-axis.

$\phi = \phi_0$ consists of all points from which a line segment to the origin makes an angle of ϕ_0.

→ Spherical coordinates are related to longitude & latitude coordinates used in navigation.

Domain of a function:-

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=) The values to be put in a function are called domain of that function.

=) The set of all values which the assigns for every element of the domain is called Range.

=) When domain consist of real numbers, the function is called real valued function.

Natural domain:-

consists of all points at which the formula has no division by zero. & produces only real numbers.

Example:-

Find domain of $\sqrt{y-x^2}$

The answer becomes imaginary if $y < x^2$

So the domain will be $y \geq x^2$ at which the answer is zero or a real number.

Lecture 05

Limit of multivariable Function:

Domain & Range:-

Function	Domain	Range
$\sqrt{x^2+y^2+z^2}$	Entire Space	$[0,\infty)$
$\dfrac{1}{x^2+y^2+z^2} \neq 0,0,0$	$(x,y,z)\neq(0,0,0)$	$(0,\infty)$
$xy \ln z$	Half space $(z>0)$	$(-\infty,\infty)$

Examples of Domain:-

(i) $f(x,y)=xy\sqrt{y-1}$ its domain consists of region in xy-plane where $y\geq 1$

(ii) $f(x,y)=\sqrt{x^2+y^2-4}$ its domain consists of region in xy-plane where $x^2+y^2\geq 4$

(iii) $f(x,y)=\dfrac{\sqrt{4-x^2}}{y^2+3}$ its domain consists of region in xy-plane where $x^2\leq 4$ & $-2\leq x\leq 2$

(iv) $f(x,y)=\dfrac{x^3+2x^2y-xy-2y^2}{x+2y}$ its domain consists of points where $f(x,y)\neq f(0,0)$.

(v) $f(x,y,z)=\sqrt{25-x^2-y^2-z^2}$ domain consists of 3d space centered at $(0,0,0)$ & radius 5.

Example 1:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

let $(x,y) \to (0,0)$ along line $y=x$

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$$= \frac{x \cdot x}{x^2+x^2} \Rightarrow \frac{x^2}{x^2+x^2}$$

$$= \frac{(1)^2}{(1)^2+(1)^2} \Rightarrow \frac{1}{2}$$

$$f(x,y) = 0.5$$

let $(x,y) \to (0,0)$ along line $y=0$

$$f(x,y) = \frac{xy}{x^2+y^2} = \frac{x(0)}{x^2+(0)^2}$$

$$= \frac{0}{x^2} = 0$$

$$f(x,y) = 0$$

As $f(x,y)$ gives two different values at two different paths so

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Rule:-

If $\lim_{(x,y)\to(a,b)} f(x,y)$ gives two or more different values along two different paths, Then this limit does not exist.

The path may be straight line or curve.

Example:-

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

let

$x = r\cos\theta$, $y = r\sin\theta$

$$\text{Then} = \frac{xy}{\sqrt{x^2+y^2}} = \frac{r\cos\theta \cdot r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$= \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}} = \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2}}$$

$$= \frac{r^{\cancel{2}}\cos\theta\sin\theta}{\cancel{r}}$$

$$= r\cos\theta\sin\theta$$

& $r\to 0$ as $(x,y)\to(0,0)$

So $\boxed{\lim_{r\to 0} r\cos\theta\sin\theta}$

Rules:-

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$

&

$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$

(a) $\lim_{(x,y)\to(x_0,y_0)} c\,f(x,y) = cL_1$

(b) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)+g(x,y)] = L_1+L_2$

(c) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)-g(x,y)] = L_1-L_2$

$f(x,y) = L$

(d) $\lim_{(x,y)\to(x_0,y_0)} \{f(x,y)\,g(x,y)\} = L_1 L_2$

(e) $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2} \quad (L_2 \neq 0)$

(i) $\lim_{(x,y)\to(x_0,y_0)} c = c \quad$ (c is constant).

(ii) $\lim_{(x,y)\to(x_0,y_0)} x_0 = x_0$

(iii) $\lim_{(x,y)\to(x_0,y_0)} y_0 = y_0$

Lecture 06

Geometry of continuous functions

A function is continuous if we draw its graph by a pen then the pen is not raised so that there is no gap in the graph.

Continuity of function of two variables:-

Function of two variables is continuous if

$f(x_0, y_0)$ is defined.

$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

$$f(x,y) = x^2+y^2 \ln(x^2+y^2)$$

Here when $x=0$ & $y=0$ (i-e) at $(0,0)$ this function will be undefined. Because $\ln(0)$ is undefined.

So the graph of $x^2+y^2 \ln(x^2+y^2)$ will have a hole at origion $(0,0)$.

Similarly $\frac{1}{\sqrt{x^2+y^2}}$ becomes undefined at $(0,0)$

So it will also have hole at origion $(0,0)$

$$z = \begin{cases} 0 & \text{if } x \ge 0,\ y \ge 0 \\ 1 & \text{otherwise} \end{cases}$$

Its graph has a jump (vertical) at the origin.

Example:- Check whether limit exists or not?

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+y^2}$$

Along x-axis:- (put $y=0$) & $x=x$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+0} = \frac{x^2}{x^2} = 1$$

Along y-axis:- (put $x=y$) & $y=0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{y^2}{y^2+0} = \frac{y^2}{y^2} = 1$$

Along $y=x$:-

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

So the limit has different values for different paths. The limit does not exist. Also $f(x,y)$ is undefined at $(0,0)$

Example:- Check continuity of

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

Here $f(0,0) = 1$

& Taking $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$ $\left(\text{bcz } \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1\right)$

So

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

The function is continuous.

Continuity of function of Three variables:-

function of Three variables will be continuous if

$f(x_0, y_0, z_0)$ is defined.

$\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$ exists.

$\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z) = f(x_0, y_0, z_0)$.

Check continuity of $f(x,y,z) = \frac{y+1}{x^2+y^2-1}$

Here the function is undefined at $x^2+y^2-1=0$

So it is continuous at all points of domain except $x^2+y^2-1=0$.

Rules for continuous function:-

(a) If g & h are continuous functions, Their product is also continuous. $f(x,y) = g(x)h(y)$ will be continuous.

(b) If $g(x)$ is a continuous function in one variable & $h(x,y)$ is continuous in two variables Then $f(x,y) = g(h(x,y))$ is also continuous in x & y.

$\Rightarrow$ Composition of continuous functions is continuous.

$\Rightarrow$ A sum, difference or product of continuous functions is continuous.

$\Rightarrow$ A Quotient of continuous function is continuous if denominator is not zero.

Continuous everywhere

A function that is continuous at every point of 2-d & 3-d space is called continuous everywhere.

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for example

$f(x,y) = \ln(2x-y+1)$ is continuous where $2x > y-1$

$f(x,y) = e^{1-xy}$ is continous everywhere.

$f(x,y) = \tan^{-1}(y-x)$ is continuous in whole region.

$f(x,y) = \sqrt{y-x}$ is continuous where $x \leq y$.

Partial derivative:-

If in $f(x,y)$ we held y as constant then derivative will be $f_x(x_0,y_0)$ called Partial derivative.

Similarly if we held x as constant then derivative will be $f_y(x_0,y_0)$ called Partial derivative.

Example:- $f(x,y) = 2x^3y^2 + 2y + 4x$

Taking y as constant, Take derivative

$$f_x(x,y) = 2(3x^2)y^2 + 0 + 4(1)$$
$$= 6x^2y^2 + 4$$

Treating x as constant, Take derivative

$$f_y(x,y) = 2x^3(2y) + 2(1) + 0$$
$$= 4x^3y + 2$$

Also Find $f_x(1,2)$ & $f_y(1,2)$.

$$f_x(1,2) = 6(1)^2(2)^2+4 = 6(1)(4)+4 = 28$$

$$f_y(1,2) = 4(1)^3(2)+2 = 4(1)(2)+2 = 10$$

Example:- $z = 4x^2 - 2y + 7x^4y^5$

$$\frac{\partial z}{\partial x} = 4(2x) - 0 + 7(4x^3)y^5$$
$$= 8x + 28x^3y^5$$

&

$$\frac{\partial z}{\partial y} = 0 - 2(1) + 7x^4(5y^4)$$
$$= -2 + 35x^4y^4$$

Example:-
$f(x,y) = z = x^2\sin^2 y$

$$f_x = \frac{\partial z}{\partial x} = (2x)\sin^2 y$$

&

$$f_y = \frac{\partial z}{\partial y} = x^2(2\sin y)(\cos y)$$
$$= x^2 2\sin y\cos y$$
$$= x^2 \sin 2y$$

Example:- $z = \ln\left(\frac{x^2+y^2}{x+y}\right)$

using ln property

$$Z = \ln(x^2+y^2) - \ln(x+y)$$

Now
$$\frac{\partial Z}{\partial x} = \frac{1}{x^2+y^2}\cdot(2x+0) - \frac{1}{x+y}(1+0)$$

$$= \frac{2x}{x^2+y^2} - \frac{1}{x+y}$$

$$= \frac{2x(x+y) - 1(x^2+y^2)}{(x^2+y^2)(x+y)}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x^2+y^2)(x+y)}$$

$$\frac{\partial Z}{\partial x} = \frac{x^2+2xy-y^2}{(x^2+y^2)(x+y)}$$

Similarly

$$\frac{\partial Z}{\partial y} = \frac{1}{x^2+y^2}(0+2y) - \frac{1}{x+y}(0+1)$$

$$= \frac{2y}{x^2+y^2} - \frac{1}{x+y}$$

$$= \frac{2y(x+y) - 1(x^2+y^2)}{(x^2+y^2)(x+y)}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x^2+y^2)(x+y)}$$

$$\frac{\partial Z}{\partial y} = \frac{y^2+2xy-x^2}{(x^2+y^2)(x+y)}$$

Example:- $z = x^4 \sin(xy^3)$

$$\frac{\partial z}{\partial x} = x^4 \frac{\partial}{\partial x}(\sin xy^3) + \sin xy^3 \frac{\partial}{\partial x}(x^4)$$

$$= x^4(\cos xy^3)\frac{\partial}{\partial x} xy^3 + \sin xy^3 (4x^3)$$

$$= x^4 \cos xy^3 \cdot y^3(1) + 4x^3 \sin xy^3$$

$$= x^4 y^3 \cos xy^3 + 4x^3 \sin xy^3$$

$$\frac{\partial z}{\partial y} = x^4 \frac{\partial}{\partial y}(\sin xy^3)$$

$$= x^4 \cos xy^3 \frac{\partial}{\partial y} xy^3$$

$$= x^4 \cos xy^3 (x \cdot 3y^2)$$

$$= 3x^5 y^2 \cos xy^3.$$

Example:- $z = \cos(x^5 y^4)$

$$\frac{\partial z}{\partial x} = -\sin(x^5y^4) \cdot \frac{\partial}{\partial x}(x^5y^4)$$

$$= -\sin(x^5y^4)(5x^4y^4)$$

$$\frac{\partial z}{\partial y} = -\sin(x^5y^4)(x^5 \cdot 4y^3)$$

$$= -4x^5y^3 \sin x^5y^4.$$

Example:- $w = x^2 + 3y^2 + 4z^2 - xyz$

$$\frac{\partial w}{\partial x} = 2x + 0 + 0 - (1)yz$$

$$= 2x - yz$$

$$\frac{\partial w}{\partial y} = 0 + 3(2y) + 0 - x(1)z$$

$$= 6y - xz$$

$$\frac{\partial w}{\partial z} = 0 + 0 + 4(2z) - (1)xy$$

$$= 8z - xy$$

lecture 07

geometric meaning of Partial derivative:-

If there is a change in x, called Δx Then $\frac{\Delta z}{\Delta x}$ is

$$\frac{\Delta z}{\Delta x} = f(x+\Delta x, y) - f(x,y)$$

& if there is a change in y, called Δy Then $\frac{\Delta z}{\Delta y}$ is

$$\frac{\Delta z}{\Delta y} = \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

If we apply limit aproaching to zero (i-e $\lim_{\Delta x \to 0}$ & $\lim_{\Delta y \to 0}$) respectively. Then the limit is called Partial derivative.

- $\frac{\partial z}{\partial x}$ = slope of Tangent to curve.
- $\frac{\partial z}{\partial y}$ = gradient of Tangent to curve

Partial derivatives of higher order:-

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(f_x) = f_{xx} = f_x^2$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(f_x) = f_{xy}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = f_{yx}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(f_y) = f_{yy} = f_y^2$$

So There are four second order derivatives of a function $f(x,y)$. f_{xy} & f_{yx} are called mixed second order derivatives & They are not equal generally

Example:- $z = \sin^{-1}\left(\frac{x}{y}\right)$ Prove $f_{xy} = f_{yx}$ $\left(\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}\right)$

$$f_x = \frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \frac{1}{y}(1)$$

$$= \frac{\cancel{y}}{\sqrt{y^2-x^2}} \cdot \frac{1}{\cancel{y}} = \boxed{\frac{1}{\sqrt{y^2-x^2}}}$$

$$f_y = \frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot x\frac{\partial}{\partial y}(y^{-1})$$

$$= \frac{1}{\frac{\sqrt{y^2-x^2}}{y}} \cdot \frac{-x}{y^2}$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{-x}{y^2}$$

$$= \boxed{\frac{-x}{y\sqrt{y^2-x^2}}}$$

$$f_{xy} = \frac{\partial z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y^2-x^2}}\right) = \frac{\partial}{\partial y}(y^2-x^2)^{-\frac{1}{2}}$$

$$= \frac{-1}{2}(y^2-x^2)^{-3/2} \cdot \frac{\partial}{\partial y}(y^2-x^2)$$

$$= \frac{-1}{2(y^2-x^2)^{3/2}} \cdot (2y-0)$$

$$= \frac{-1}{2(y^2-x^2)^{3/2}} \cdot 2y \Rightarrow \boxed{\frac{-y}{(y^2-x^2)^{3/2}}}$$

$$f_{yx} = \frac{\partial z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-x}{y\sqrt{y^2-x^2}}\right) \Rightarrow \frac{-1}{y}\left[\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{y^2-x^2}}\right)\right]$$

$$= \frac{-1}{y}\left[\frac{\sqrt{y^2-x^2}\cdot(1) - x\left[\frac{+1}{2\sqrt{y^2-x^2}}\right]\cdot(0-2x)}{\left(\sqrt{y^2-x^2}\right)^2}\right]$$

$$= \frac{-1}{y(y^2-x^2)}\left[\sqrt{y^2-x^2} + \frac{x^2}{\sqrt{y^2-x^2}}\right]$$

$$= \frac{-1}{y(y^2-x^2)}\left[\frac{y^2-x^2+x^2}{\sqrt{y^2-x^2}}\right]$$

$$= \frac{-y^2}{y(y^2-x^2)(\sqrt{y^2-x^2})} \Rightarrow \boxed{\frac{-y}{(y^2-x^2)^{3/2}}}$$

Example:- $f(x,y) = x\cos y + ye^x$ Find all double partial derivatives

$$\frac{\partial f}{\partial x} = (1)\cos y + ye^x(1) \Rightarrow \boxed{\cos y + ye^x}$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\cos y + ye^x\right) \Rightarrow \boxed{-\sin y(1) + e^x(1)}$$

$$= -\sin y + e^x$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\cos y + ye^x\right) \Rightarrow 0 + ye^x(1) \Rightarrow \boxed{ye^x}$$

Now

$$\frac{\partial f}{\partial y} = x(-\sin y)(1) + e^x(1) \Rightarrow \boxed{-x\sin y + e^x}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-x\sin y + e^x\right) \Rightarrow -(1)\sin y + e^x(1)$$

$$= \boxed{-\sin y + e^x}$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(-x\sin y + e^x\right) \Rightarrow -x(\cos y) + 0$$

$$= \boxed{-x\cos y}$$

Laplace's Equation:- For a function $f(x,y,z)$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Example:- $f(x,y) = e^x \sin y + e^y \cos x$

$$\frac{\partial f}{\partial x} = e^x(1)\sin y + e^y(-\sin x)(1) \Rightarrow e^x \sin y - e^y \sin x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(e^x \sin y - e^y \sin x\right) \Rightarrow e^x \sin y - e^y(\cos x)$$

Now

$$\frac{\partial f}{\partial y} = e^x(\cos y) + e^y(1)\cos x \Rightarrow e^x \cos y + e^y \cos x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(e^x \cos y + e^y \cos x\right) \Rightarrow e^x(-\sin y) + e^y(1)\cos x$$

$$= -e^x \sin y + e^y \cos x$$

using Laplace's Equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\cancel{e^x \sin y} - e^y \cos x + (\cancel{-e^x \sin y} + e^y \cos x) = 0$$

$$e^y \cancel{\cos x} - e^y \cancel{\cos x} = 0$$

$$0 = 0$$

So proved

Euler's Theorm:-

Mixed derivative Theorm:-

If $f(x,y)$ & f_x, f_y, f_{xy}, f_{yx} are defined & continuous at (a,b) Then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Advantage

$$w = xy + \frac{e^y}{y^2+1}$$

$$\frac{\partial w}{\partial x} = y(1)+0 = y$$

$$\frac{\partial^2 w}{\partial y \partial x} = 1$$

Here we had to find $\frac{\partial^2 w}{\partial x \partial y}$ but it was lengthy & difficult so by using euler's Theorm we found $\frac{\partial^2 w}{\partial y \partial x}$, which gives answer more quickly.

Lecture 08 Euler Theorm

Order of differentiation can be changed without changing the final result. For example

$$f_{xyy} = f_{yxy} = f_{yyx}$$

or

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^3 f}{\partial y \partial x \partial y} = \frac{\partial^3 f}{\partial x \partial y^2}$$

$$f(x,y) = y^2 x^4 e^x + 2$$

Find $\frac{\partial^5 f}{\partial y^3 \partial x^2}$

According to Euler's Theorm we can also find $\frac{\partial^5 f}{\partial x^2 \partial y^3}$. Becaus y has smaller degree & single Term.

$$\frac{\partial f}{\partial y} = 2y x^4 e^x , \quad \frac{\partial^2 f}{\partial y^2} = 2x^4 e^x$$

$$\frac{\partial^3 f}{\partial y^3} = 0 , \quad \frac{\partial^4 f}{\partial x \partial y^3} = 0 , \quad \frac{\partial^5 f}{\partial x^2 \partial y^3} = 0$$

Example:-

$$f(x,y) = \frac{x+y}{x-y}$$

Find f_x & f_y.

$$f_x = \frac{\partial f}{\partial x} = \frac{(x-y)\frac{\partial}{\partial x}(x+y) - (x+y)\frac{\partial}{\partial x}(x-y)}{(x-y)^2}$$

$$f_x = \frac{(x-y)(1+0) - (x+y)(1-0)}{(x-y)^2}$$

$$= \frac{x-y-x-y}{(x-y)^2} \Rightarrow \boxed{\frac{-2y}{(x-y)^2}}$$

Now

$$f_y = \frac{\partial f}{\partial y} = \frac{(x-y)\frac{\partial}{\partial y}(x+y) - (x+y)\frac{\partial}{\partial y}(x-y)}{(x-y)^2}$$

$$= \frac{(x-y)(0+1) - (x+y)(0-1)}{(x-y)^2}$$

$$= \frac{x-y+x+y}{(x-y)^2} \Rightarrow \boxed{\frac{2x}{(x-y)^2}}$$

Example:-

$$f(x,y) = x^3e^{-y} + y^3\sec\sqrt{x}$$

$$f_x = 3x^2e^{-y} + y^3\sec\sqrt{x}\tan\sqrt{x}\cdot\frac{\partial}{\partial x}(\sqrt{x})$$

$$= \boxed{3x^2e^{-y} + y^3\sec\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}}}$$

$$f_y = x^3e^{-y}(-1) + 3y^2\sec\sqrt{x}$$

$$= \boxed{-x^3e^{-y} + 3y^2\sec\sqrt{x}}$$

Example: $f(x,y) = x^2 y e^{xy}$

Find $f_x(1,1)$ & $f_y(1,1)$.

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 y e^{xy})$$

$$= y\left[x^2 \frac{\partial}{\partial x}(e^{xy}) + e^{xy} \frac{\partial}{\partial x}(x^2)\right]$$

$$= y\left[x^2 e^{xy} \cdot \frac{\partial}{\partial x}(xy) + e^{xy} \cdot 2x\right]$$

$$= y\left[x^2 e^{xy} y + e^{xy} 2x\right]$$

$$f_x(x,y) = xy e^{xy}[xy + 2]$$

$$f_x(1,1) = (1)(1)e^{(1)(1)}[(1)(1)+2] \Rightarrow e(1+2) \Rightarrow \boxed{3e}$$

Now $f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 y e^{xy})$

$$= x^2\left[y \frac{\partial}{\partial y}(e^{xy}) + e^{xy} \frac{\partial}{\partial y}(y)\right]$$

$$= x^2\left[y e^{xy} \frac{\partial}{\partial y}(xy) + e^{xy}(1)\right]$$

$$= x^2\left[y e^{xy} \cdot x + e^{xy}\right]$$

$$= x^2 e^{xy}[xy + 1]$$

$$f_y(1,1) = 1^2 e^{(1)(1)}[(1)(1)+1] \Rightarrow e(1+1) \Rightarrow \boxed{2e}$$

Example $f(x,y) = x^2\cos(xy)$

Find $f_x(\frac{1}{2}, \pi)$ & $f_y(\frac{1}{2}, \pi)$.

$$f_x(x,y) = \frac{\partial}{\partial x}\left(x^2\cos xy\right)$$

$$= x^2\frac{\partial}{\partial x}\cos(xy) + \cos xy\frac{\partial}{\partial x}(x^2)$$

$$= x^2(-\sin xy)\cdot(1)y + \cos(xy)\cdot 2x$$

$$f_x(x,y) = -x^2y\sin(xy) + 2x\cos xy$$

$$f_x\left(\tfrac{1}{2}, \pi\right) = -\left(\tfrac{1}{2}\right)^2(\pi)\sin\left(\tfrac{1}{2}\pi\right) + 2\left(\tfrac{1}{2}\right)\cos\left(\tfrac{1}{2}\pi\right)$$

$$= -\frac{1}{4}\pi\sin\frac{\pi}{2} + \cos\frac{\pi}{2}$$

$$= -\frac{\pi}{4}(1) + (0)$$

$$\boxed{f_x\left(\tfrac{1}{2}, \pi\right) = -\frac{\pi}{4}}$$

Now

$$f_y(x,y) = \frac{\partial}{\partial y}\left(x^2\cos xy\right)$$

$$= x^2(-\sin xy)\frac{\partial}{\partial y}(xy)$$

$$= -x^2\sin xy\cdot x(1)$$

$$f_y(x,y) = -x^3\sin xy$$

$$f_y\left(\tfrac{1}{2}, \pi\right) = -\left(\tfrac{1}{2}\right)^3\sin\left(\tfrac{1}{2}\pi\right)$$

$$= -\frac{1}{8}(1)$$

$$\boxed{f_y\left(\tfrac{1}{2}, \pi\right) = -\frac{1}{8}}$$

Example:- $w = (4x - 3y + 2z)^5$

Find $\dfrac{\partial^4 w}{\partial z^2 \partial y \partial x}$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(4x - 3y + 2z)^5$$

$$= 5(4x - 3y + 2z)^4 \cdot \frac{\partial}{\partial x}(4x - 3y + 2z)$$

$$= 5(4x - 3y + 2z)^4 (4 - 0 + 0)$$

$$= 20(4x - 3y + 2z)^4$$

$$\frac{\partial w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(20(4x - 3y + 2z)^4\right)$$

$$= 20 \times 4 (4x - 3y + 2z)^3 \cdot (0 - 3 + 0)$$

$$= -240 (4x - 3y + 2z)^3$$

$$\frac{\partial w}{\partial z \partial y \partial x} = \frac{\partial}{\partial z}\left(-240(4x - 3y + 2z)^3\right)$$

$$= -240 \times 3 (4x - 3y + 2z)^2 \cdot (0 - 0 + 2)$$

$$= -1440 (4x - 3y + 2z)^2$$

$$\frac{\partial w}{\partial z^2 \partial y \partial x} = \frac{\partial}{\partial z}\left(-1440(4x - 3y + 2z)^2\right)$$

$$= -1440 \times 2 (4x - 3y + 2z)$$

$$= \boxed{-5760(4x - 3y + 2z)}$$

Chain rule:- If $y = f(x)$ & $x = g(t)$

Then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Example $w = x+4$, $x = \sin t$

find $\frac{dw}{dt}$.

$w = x+4$

$\frac{dw}{dx} = 1+0$

$\frac{dw}{dx} = 1$

$x = \sin t$

$\frac{dx}{dt} = \cos t$

Alternate:-

By Substitution

$w = \sin t + 4$

$\frac{dw}{dt} = \cos t + 0$

$\frac{dw}{dt} = \cos t$

By Chain rule

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

$$= 1 \cdot \cos t$$

$$\boxed{\frac{dw}{dt} = \cos t}$$

$\Rightarrow$ If y is function of u, u is function of v, v is function of w, w is function of z, z is function of x, y is a function of x

So by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dw} \frac{dw}{dz} \frac{dz}{dx}$$

$w = f(x, y)$, $x = g(t)$, $y = f(t)$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Example:- $w = xy$, $x = \cos t$, $y = \sin t$

Solve by Substitution as well as by chain rule

Substitution Method

$w = xy \Rightarrow \cos t \sin t$

$$\frac{dw}{dt} = \frac{d}{dt}(\cos t \sin t) \Rightarrow \cos t \frac{d}{dx}\sin t + \sin t \frac{d}{dx}\cos t$$

$$= \cos t(\cos t) + (\sin t)(-\sin t) \Rightarrow \cos^2 t - \sin^2 t$$

$$\boxed{\frac{dw}{dt} = \cos 2t}$$ (double angle Formula for cos).

Chain rule:-

$w = xy$

$\frac{\partial w}{\partial y} = x(1)$

$$\boxed{\frac{\partial w}{\partial y} = x}$$

$$\boxed{\frac{\partial w}{\partial x} = y}$$

$x = \cos t$

$$\boxed{\frac{dx}{dt} = -\sin t}$$

$y = \sin t$

$$\boxed{\frac{dy}{dt} = \cos t}$$

Now

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t)$$

$$= -\sin^2 t + \cos^2 t \Rightarrow \cos^2 t - \sin^2 t$$

$$\boxed{\frac{dw}{dt} = \cos 2t}$$

Example:- $z = 3x^2y^3,\ x = t^4,\ y = t^2,\ \frac{dz}{dt} = ?$

$$\frac{\partial z}{\partial x} = 3(2x)y^3 \Rightarrow 6xy^3 \Rightarrow 6(t^4)(t^2)^3 \Rightarrow 6t^{10}$$

$$\frac{\partial z}{\partial y} = 3x^2(3y^2) \Rightarrow 9x^2y^2 \Rightarrow 9(t^4)^2(t^2)^2 \Rightarrow 9t^{12}$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^4) \Rightarrow 4t^3$$

$$\frac{dy}{dt} = \frac{d}{dt}(t^2) \Rightarrow 2t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= (6t^{10})(4t^3) + (9t^{12})(2t)$$

$$= 24t^{13} + 18t^{13}$$

$$\boxed{\frac{dz}{dt} = 42t^{13}}$$

Example:- $z = \sqrt{1+x-2xy^4},\ x = \ln t,\ y = t,\ \frac{dz}{dt} = ?$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{1+x-2xy^4}}\cdot(0+1-2(1)(y^4)) \Rightarrow \boxed{\frac{1-2y^4}{2\sqrt{1+x-2xy^4}}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{1+x-2xy^4}}\cdot(0+0-2x(4y^3)) \Rightarrow \boxed{\frac{-8xy^3}{2\sqrt{1+x-2xy^4}}}$$

$$\frac{dx}{dt} = \frac{d}{dt}(\ln t) \Rightarrow \boxed{\frac{1}{t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}(t) \Rightarrow \boxed{1}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1-2y^4}{2\sqrt{1+x-2xy^4}}\cdot\frac{1}{t} + \frac{-8xy^3}{2\sqrt{1+x-2xy^4}}\cdot 1$$

$$= \frac{1}{\sqrt{1+x-2xy^4}}\left[\frac{1-2y^4}{2t} - 4xy^3\right]$$

$$= \frac{1}{\sqrt{1+\ln t-2(\ln t)(t^4)}}\left[\frac{1-2t^4}{2t} - 4(\ln t)(t^3)\right]$$

$$= \frac{1}{\sqrt{1+\ln t-2t^4\ln t}}\left[\frac{1-2t^4}{2t} - 4t^3\ln t\right]$$

$$= \boxed{\frac{1}{\sqrt{1+\ln t-2t^4\ln t}}\left[\frac{1}{2t} - t^3 - 4t^3\ln t\right]}$$

Example:- $z = \ln(2x^2+y)$, $x=\sqrt{t}$, $y = t^{2/3}$,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\ln(2x^2+y)\right) \Rightarrow \frac{1}{2x^2+y}\cdot 4x+0 \Rightarrow \frac{4x}{2x^2+y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\ln(2x^2+y)\right) \Rightarrow \frac{1}{2x^2+y}\cdot 0+1 \Rightarrow \frac{1}{2x^2+y}$$

$$\frac{dx}{dt} = \frac{d}{dt}(\sqrt{t}) \Rightarrow \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}(t^{2/3}) \Rightarrow \frac{2}{3}t^{-1/3}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{4x}{2x^2+y}\cdot\frac{1}{2\sqrt{t}} + \frac{1}{2x^2+y}\cdot\frac{2}{3t^{1/3}}$$

$$= \frac{\cancel{4}^{2}\sqrt{t}}{\cancel{2}(\sqrt{t})^2+(t^{2/3})}\cdot\frac{1}{\cancel{2}_{1}\sqrt{t}} + \frac{1}{2(\sqrt{t})^2+t^{2/3}}\cdot\frac{2}{3t^{1/3}}$$

$$\frac{dz}{dt} = \boxed{\frac{2}{2t+t^{2/3}} + \frac{2}{3t^{1/3}}\left(\frac{1}{2t+t^{2/3}}\right)}$$

Lecture 09 Examples

If $w = f(x, y, z)$ where $x = g(t)$, $y = f(t)$, $z = h(t)$

Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

Example:- $w = x^2 + y + z + 4$

$x = e^t$, $y = \cos t$, $z = t + 4$

$\frac{\partial w}{\partial x} = 2x + 0 + 0 + 0 \Rightarrow 2x$, $\frac{\partial w}{\partial y} = 1$, $\frac{\partial w}{\partial z} = 1$

$\frac{dx}{dt} = e^t(1) \Rightarrow e^t$, $\frac{dy}{dt} = -\sin t$, $\frac{dz}{dt} = 1 + 0 \Rightarrow 1$

Now

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

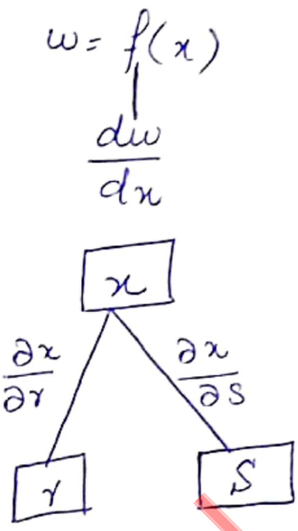
$$= (2x)(e^t) + (1)(-\sin t) + (1)(1)$$

$$= \boxed{2e^{2t} - \sin t + 1}$$

If $w = f(x)$ & $x = g(r,s)$

So $\dfrac{\partial w}{\partial r} = \dfrac{dw}{dx} \cdot \dfrac{\partial x}{\partial r}$

$\dfrac{\partial w}{\partial s} = \dfrac{dw}{dx} \cdot \dfrac{\partial x}{\partial s}$



Example:-

$w = \sin x + x^2$, $x = 3r + 4s$

$\dfrac{\partial w}{\partial r} = ?$ $\quad \dfrac{\partial w}{\partial s} = ?$

$\dfrac{dw}{dx} = \dfrac{d}{dx}(\sin x + x^2) \Rightarrow \cos x + 2x$

$\dfrac{\partial x}{\partial r} = \dfrac{\partial}{\partial r}(3r+4s) \Rightarrow 3(1) + 0 \Rightarrow 3$

$\dfrac{\partial x}{\partial s} = \dfrac{\partial}{\partial s}(3r+4s) \Rightarrow 0 + 4(1) \Rightarrow 4$

$\dfrac{\partial w}{\partial r} = \dfrac{dw}{dx} \cdot \dfrac{\partial x}{\partial r} \Rightarrow (\cos x + 2x)(3) \Rightarrow 3\cos x + 6x$

$= 3\cos(3r+4s) + 6(3r+4s)$

$= \boxed{3\cos(3r+4s) + 18r + 24s}$

Now

$\dfrac{\partial w}{\partial s} = \dfrac{dw}{dx} \cdot \dfrac{\partial x}{\partial s} \Rightarrow (\cos x + 2x)\,4 \Rightarrow 4\cos x + 8x$

$= 4\cos(3r+4s) + 8(3r+4s)$

$= \boxed{4\cos(3r+4s) + 24r + 32s}$

$\Rightarrow$ If $w = f(x,y)$ where $x = g(r,s)$ & $y = h(r,s)$

Then $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$

Similarly. $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$

$\Rightarrow$ If $w = f(x,y,z)$ where $x = g(r,s)$, $y = h(r,s)$, $z = k(r,s)$

Then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

Similarly

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Example:- $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$

$\frac{\partial w}{\partial r} = ?$, $\frac{\partial w}{\partial s} = ?$

- $\frac{\partial w}{\partial x} = \underline{1}$, $\frac{\partial w}{\partial y} = 0 + 2(1) + 0 \Rightarrow \underline{2}$, $\frac{\partial w}{\partial z} = 0 + 0 + 2z \Rightarrow \underline{2z}$
- $x = \frac{r}{s}$, $\frac{\partial x}{\partial r} = \frac{1}{s}(1) \Rightarrow \underline{\frac{1}{s}}$, $\frac{\partial x}{\partial s} = r(-1)s^{-2} \Rightarrow -rs^{-2} \Rightarrow \underline{\frac{-r}{s^2}}$
- $y = r^2 + \ln s$, $\frac{\partial y}{\partial r} = \underline{2r}$, $\frac{\partial y}{\partial s} = 0 + \frac{1}{s} \Rightarrow \underline{\frac{1}{s}}$
- $z = 2r$, $\frac{\partial z}{\partial r} = \underline{2}$, $\frac{\partial z}{\partial s} = \underline{0}$

Now

$$\frac{\partial w}{\partial r} = (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + 4z \Rightarrow \frac{1}{s} + 4r + 4(2r) \Rightarrow \boxed{\frac{1}{s} + 12r}$$

Now $\frac{\partial w}{\partial s} = (1)\left(\frac{-r}{s^2}\right) + 2\left(\frac{1}{s}\right) + (2z)(0)$

$= \frac{-r}{s^2} + \frac{2}{s} + 0 \Rightarrow \boxed{\frac{2}{s} - \frac{r^2}{s}}$

Chain rule for many variables:-

if $w = f(x, y, \ldots, i)$ & w is function of variables from p to t. Then

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial i}\cdot\frac{\partial w}{\partial p}$$

Similarly we can replace p by q or any other variable from p to t.

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Example:- $w = \ln(e^r + e^s + e^t + e^u)$, $w_{rstu} = ?$

Take $\ln$ on b/s

$e^w = e^r + e^s + e^t + e^u$ ($\because \ln$ is inverse of e)

$e^w w_r = e^w \frac{\partial w}{\partial r} = e^r$, $e^w \frac{\partial w}{\partial s} = e^s$, $e^w \frac{\partial w}{\partial t} = e^t$, $e^w \frac{\partial w}{\partial u} = e^u$

$\frac{\partial w}{\partial r} = \frac{e^r}{e^w}$, $\frac{\partial w}{\partial s} = \frac{e^s}{e^w}$, $\frac{\partial w}{\partial t} = \frac{e^t}{e^w}$, $\frac{\partial w}{\partial u} = \frac{e^u}{e^w}$

$\frac{\partial w}{\partial r} = e^{r-w}$, $\frac{\partial w}{\partial s} = e^{s-w}$, $\frac{\partial w}{\partial t} = e^{t-w}$, $\frac{\partial w}{\partial u} = e^{u-u}$

$w_r = e^{r-w}$, $w_s = e^{s-w}$, $w_t = e^{t-w}$, $w_u = e^{u-u}$

Now

$$w_r = e^{r-w}$$

$$\frac{\partial}{\partial s} w_r = e^{r-w} \cdot \frac{\partial}{\partial s}(r-w)$$

$$w_{rs} = e^{r-w}(-w_s) \Rightarrow e^{r-w}(-e^{s-w}) \Rightarrow -e^{r-w+s-w}$$

$$w_{rs} = -e^{r+s-2w}$$

Now

$$\frac{\partial}{\partial t} w_{rs} = \frac{\partial}{\partial t}\left(-e^{r+s-2w}\right)$$

$$w_{rst} = -e^{r+s-2w}(-2w_t) \Rightarrow -e^{r+s-2w}\left[-2(e^{t-w})\right]$$

$$= 2e^{r+s-2w+t-w} \Rightarrow 2e^{r+s+t-3w}$$

Similarly.

$$\frac{\partial}{\partial u} w_{rst} = \frac{\partial}{\partial u}\left(2e^{r+s+t-3w}\right)$$

$$w_{rstu} = 2e^{r+s+t-3w}(-3w_u)$$

$$= 2e^{r+s+t-3w}\left[-3(e^{u-w})\right]$$

$$= -6e^{r+s+t-3w+u-w}$$

$$\boxed{w_{rstu} = -6e^{r+s+t+u-4w}}$$

Lecture 10 Intro to vectors

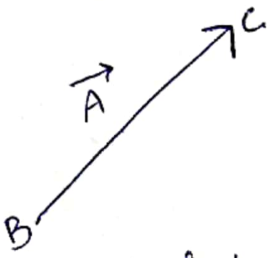
Physical Quantities that can be determined by their **magnitude only** are called **Scalars**. While the Quantities which are determined by their **magnitude & direction** are called **vectors**.

For example Time, length, mass etc are scalar, & Force, velocity, displacement etc are vectors.

Representation Vectors are usually represented by single bold face letters or letter with an arrow on it. like **A** or $\vec{A}$.

A vector in plane is a directed line segment.

Magnitude:- Magnitude of vector $\vec{A}$ is represented by $|\vec{A}| = |\vec{BC}|$ is length of line segment BC. $|A| = \sqrt{x^2 + y^2 + z^2}$



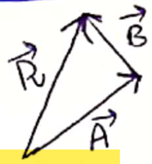
Unit vector

Any vector whose magnitude is 1 is called unit vector. It is denoted by $\hat{V}$.

$$\boxed{\hat{V} = \frac{\vec{V}}{|V|}}$$

Addition of vectors:-

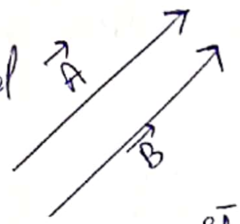
Vectors are added using head to tail rule. We place the tail of second vector on the head of first vector & the tail of third vector on the head of second vector & so on. At the end join the tail of first vector with the head of last vector. This is called Resultant vector. For example $\vec{A}$ & $\vec{B}$ are added like this



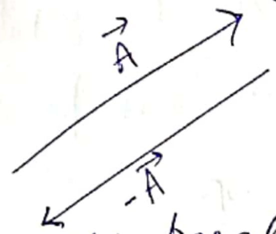
$\vec{R} = \vec{A} + \vec{B}$

Now $\vec{R}$ is the Resultant.

Equal vectors:- Vectors having same magnitude & same direction are called Equal vectors. $(\vec{A} = \vec{B})$



Opposite vectors:- Vector having same magnitude but opposite direction are called Opposite vectors. $(\vec{A} = -\vec{A})$



Parallel vectors:- Two vectors are parallel if one vector is scalar multiple of the other.

$\underline{b} = \lambda \underline{a}$ where λ is a non-zero scalar no.

Rectangular Components:-

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$

Addition & Subtraction:-

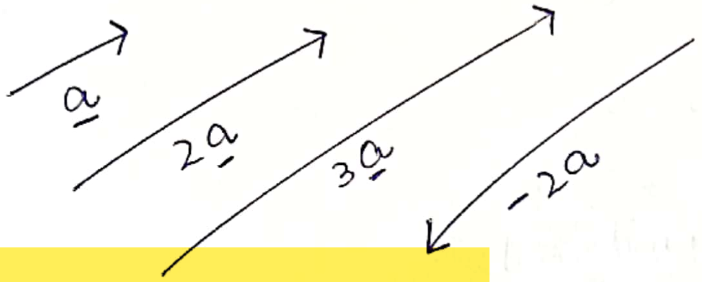
if $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ & $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$

Then

$$\underline{a}+\underline{b} = (a_1 i + a_2 \underline{j} + a_3 \underline{k}) + (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k})$$
$$= (a_1 + b_1) i + (a_2 + b_2) j + (a_3 + b_3) \underline{k}$$

$$\underline{a}-\underline{b} = (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) - (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k})$$
$$= (a_1 - b_1) \underline{i} + (a_2 - b_2) \underline{j} + (a_3 - b_3) \underline{k}$$

Multiplication of vector by Scalar:-



Scalar Product:- also called dot Product.

$$\underline{a} \cdot \underline{b} = |a||b| \cos\theta \quad (\theta \text{ is angle b/w } a \text{ \& } b)$$

$\underline{a} \cdot \underline{b}$ is a times length of b times cosine of angle b/w a & b.

Characteristics:-

$\Rightarrow$ $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ (commutative property).

$\Rightarrow$ if $\underline{a} \perp \underline{b}$ Then $\underline{a} \cdot \underline{b} = 0$

Also $i \cdot j = j \cdot i = 0$, $j \cdot k = k \cdot j = 0$, $k \cdot i = i \cdot k = 0$

$\Rightarrow$ if $a \parallel b$ Then $a \cdot b = |a||b| \cos\theta$. ($\theta = 0$ bcz vectors are parallel)

replace $\underline{b}$ by $\underline{a}$ $\quad \underline{a} \cdot \underline{a} = |a||a| \cos 0°$

$$\underline{a} \cdot \underline{a} = |a|^2$$

Also $i \cdot i = j \cdot j = k \cdot k = 1$

Example:-

$\theta = \frac{\pi}{4}$, $a = 3k$ & $b = \sqrt{2}\,i + \sqrt{2}\,k$, $\underline{a}\cdot\underline{b} = ?$

$|a| = \sqrt{x^2+y^2+z^2} \Rightarrow \sqrt{0^2+0^2+(3)^2} \Rightarrow \sqrt{3^2} \Rightarrow \underline{3}$

$|b| = \sqrt{x^2+y^2+z^2} \Rightarrow \sqrt{(\sqrt{2})^2+0^2+(\sqrt{2})^2} \Rightarrow \sqrt{2+2} \Rightarrow \sqrt{4} \Rightarrow \underline{2}$

$\underline{a}\cdot\underline{b} = |a||b|\cos\theta$

$= (3)(2)\cos\frac{\pi}{4} \Rightarrow (3)(2)\left(\frac{1}{\sqrt{2}}\right) \Rightarrow 3(\sqrt{2})^2\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow 3(\sqrt{2})(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) \Rightarrow \boxed{3\sqrt{2}}$

$\underline{a}\cdot\underline{b}$ in component form:-

$\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ & $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$

$\underline{a}\cdot\underline{b} = (a_1\underline{i} + a_2\underline{j} + a_3 k)\cdot(b_1\underline{i} + b_2\underline{j} + b_3\underline{k})$

$= a_1b_1 + a_2b_2 + a_3b_3$

Angle b/w two vectors:-

As $\underline{a}\cdot\underline{b} = |a||b|\cos\theta$

$\frac{\underline{a}\cdot\underline{b}}{|a||b|} = \cos\theta$

$\boxed{\cos^{-1}\left(\frac{\underline{a}\cdot\underline{b}}{|a||b|}\right) = \theta}$ $(0 \le \theta \le \pi)$

Example:-

$\underline{a} = \underline{i} - 2\underline{j} - 2\underline{k}$ & $\underline{b} = 6\underline{i} + 3\underline{j} + 2\underline{k}$, $\theta = ?$

$|a| = \sqrt{(1)^2+(-2)^2+(-2)^2} \Rightarrow \sqrt{1+4+4} \Rightarrow \sqrt{9} \Rightarrow \underline{3}$

$|b| = \sqrt{6^2+3^2+2^2} \Rightarrow \sqrt{36+9+4} \Rightarrow \sqrt{49} \Rightarrow \underline{7}$

$$\underline{a}\cdot\underline{b} = (\underline{i}-2\underline{j}-2\underline{k})\cdot(6\underline{i}+3\underline{j}+2\underline{k})$$
$$= (1)(6)+(-2)(3)+(-2)(2)$$
$$= 6-6-4 \Rightarrow \underline{-4}$$

Now

$$\theta = \cos^{-1}\left(\frac{\underline{a}\cdot\underline{b}}{|a||b|}\right)$$
$$= \cos^{-1}\left[\frac{-4}{(3)(7)}\right] \Rightarrow \cos^{-1}\left(\frac{-4}{21}\right)$$
$$\boxed{\theta = 1.76 \text{ rad}}$$

Perpendicular vectors:-

Two vectors are perpendicular if & only if Their dot product is zero. (i-e) $\underline{a}\cdot\underline{b} = 0$

Vector Projection:-

As $\underline{a}\cdot\underline{b} = |a||b|\cos\theta$

$$\boxed{\frac{\underline{a}\cdot\underline{b}}{|a|} = |b|\cos\theta}$$

$|b|\cos\theta$ is called scalar component of $\underline{b}$ in direction of $\underline{a}$.

Example:- $\underline{a} = \underline{i}-2\underline{j}-2\underline{k}$ & $\underline{b} = 6\underline{i}+3\underline{j}+2\underline{k}$

$$\underline{a}\cdot\underline{b} = (1)(6)+(-2)(3)+(-2)(2) \Rightarrow 6-6-4 \Rightarrow \underline{-4}$$
$$|a| = \sqrt{1^2+(-2)^2+(-2)^2} \Rightarrow \sqrt{1+4+4} \Rightarrow \sqrt{9} \Rightarrow \underline{3}$$
$$|b|\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|a|} \Rightarrow \boxed{\frac{-4}{3}}$$

Vector Product:- also called cross product.

$\underline{a} \times \underline{b} = |a||b| \sin\theta \ \hat{n}$

$\hat{n}$ is a vector determined by right hand rule.

Characteristics:-

$\Rightarrow$ If $\underline{a} \parallel \underline{b}$ Then $\underline{a} \times \underline{b} = 0$

Also • $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$

$\Rightarrow$ If $a \perp b$ Then $a \times b = |a||b| \sin 90^\circ \hat{n}$

$\underline{a} \times \underline{b} = |a||b| \hat{n}$

Also
- $i \times j = k$, $j \times i = -k$
- $j \times k = i$, $j \times k = -i$
- $k \times i = j$, $i \times k = -j$

$\Rightarrow$ $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$ (not commutative).

Area of parallelogram:-

if $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ & $\underline{b} = b_1 i + b_2 j + b_3 k$

Then area of parallelogram will be

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example:- Find area of parallelogram if $\underline{a} = 2\underline{i} + \underline{j} + \underline{k}$ & $\underline{b} = -4\underline{i} + 3\underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$\underline{a} \times \underline{b} = i\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - j\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \Rightarrow i(1-3) - j(2+4) + k(6+4)$

$\Rightarrow -2i - 6j + 10k$

Lecture 11

Box Product

$(a \times b) \cdot c$ is called Triple Scalar or Box Product

$$a \cdot (b \times c) = (a \times b) \cdot c = b \cdot (c \times a)$$

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example: $a = i + 2j - k$, $b = -2i + 3k$, $c = 7j - 4k$

$a \cdot (b \times c) = ?$ (volume)

$$a \cdot (b \times c) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}$$

$$= 1(0-21) - 2(8-0) - 1(-14-0)$$

$$= -21 - 16 + 14 \Rightarrow \boxed{-23}$$

As the volume is always +ve so $a \cdot (b \times c) = 23$

Gradient of a Scalar Function:-

As $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

∇ is called del operator & ϕ is called gradient. gradient ϕ is a vector operator. ∇ is also vector quantity.

$$\text{grad } \phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$\boxed{\text{grad } \phi = \nabla \phi}$$

ϕ is scalar but $\nabla\phi$ (Grad ϕ) is vector quantity.

Directional Derivatives

$$D_u f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$$

Keep in mind that there are infinitely many directional derivatives of $f(x,y)$. at (x_0,y_0) for each possible direction of u.

Example:- Find directional derivatives of $f(x,y) = 3x^2y$ at point $(1,2)$ in direction of $a = 3i+4j$

$$f(x,y) = 3x^2y$$

$f_x(x,y) = 3(2x)y \Rightarrow 6xy$ $\qquad$ $f_y(x,y) = 3x^2(1) \Rightarrow 3x^2$

$f_x(1,2) = 6(1)(2) \Rightarrow 12$ $\qquad$ $f_y(1,2) = 3(1)^2 \Rightarrow 3$

As $a = 3i+4j$

$$|a| = \sqrt{3^2+4^2} \Rightarrow \sqrt{9+16} \Rightarrow \sqrt{25} \Rightarrow 5$$

$$\hat{a} = \frac{a}{|a|} \Rightarrow \frac{3i+4j}{5} \Rightarrow \frac{3i}{5} + \frac{4j}{5}$$

Now

$$D_u f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$$

$$D_u f(1,2) = f_x(1,2)a_1 + f_y(1,2)a_2$$

$$= (12)\left(\frac{3}{5}\right) + (3)\left(\frac{4}{5}\right)$$

$$= \frac{36}{5} + \frac{12}{5} \Rightarrow \boxed{\frac{48}{5}}$$

Formula for directional derivative can also be written as gradient notation.

$$D_u f(x,y) = \nabla f(x,y) \cdot \hat{u}$$

$$\because \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial x} + k\frac{\partial}{\partial z}$$

Example:-

$f(x,y) = 2x^2+y^2$, $P_0(-1,1)$, $u = 3\underline{i} - 4\underline{j}$

$D_u f(-1,1) = ?$

$f(x,y) = 2x^2+y^2$

$f_x(x,y) = 4x+0 \Rightarrow 4x$

$f_x(-1,1) = 4(-1) \Rightarrow \underline{-4}$

$f_y(x,y) = 0+2y \Rightarrow 2y$

$f_y(-1,1) = 2(1) \Rightarrow \underline{2}$

$|u| = \sqrt{3^2+(-4)^2}$

$= \sqrt{9+16} = \sqrt{25}$

$|u| = \underline{5}$

$\hat{u} = \frac{u}{|u|} \Rightarrow \frac{3\underline{i} - 4\underline{j}}{5} \Rightarrow \underline{\frac{3i}{5} - \frac{4j}{5}}$

$D_u(f(-1,1)) = f_x(-1,1)u_1 + f_y(-1,1)u_2$

$= (-4)\left(\frac{3}{5}\right) + (2)\left(\frac{-4}{5}\right)$

$= -\frac{12}{5} - \frac{8}{5}$

$= -\frac{20}{5}$

$$\boxed{D_u f(-1,1) = -4}$$

OR

$f(x,y) = 2x^2+y^2$

$\frac{\partial f}{\partial x} = 4x$

$\frac{\partial f}{\partial y} = 2y$

$\nabla f(x,y) = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial x}$

$\nabla f(-1,1) = 4x\underline{i} + 2y\underline{j}$

$= \underline{-4i+2j}$

$\underline{\hat{u} = \frac{3i}{5} - \frac{4}{5}j}$

$D_u f(x,y) = \nabla f(x,y) \cdot \hat{u}$

$= \frac{3}{5}$

$= (4x\underline{i} + 2y\underline{j}) \cdot \left(\frac{3}{5}i - \frac{4}{5}\underline{j}\right)$

$= (-4)\left(\frac{3}{5}\right) + (2)\left(\frac{-4}{5}\right)$

$= -\frac{12}{5} - \frac{8}{5}$

$= -\frac{20}{5}$

$$\boxed{D_u f(-1,1) = -4}$$

If $\underline{u}$ is a unit vector making angle θ with +ve x-axis, then $u_1 = \cos\theta$ & $u_2 = \sin\theta$

Formula for directional derivative can be written as

$$D_u f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$$

Find directional derivative of e^{xy} at $(-2,0)$ in direction of unit vector u that makes an angle of $\pi/3$ with +ve x-axis.

$\theta = \frac{\pi}{3}$

$f(x,y) = e^{xy}$

$f_x(x,y) = e^{xy} \cdot y$

$f_x(-2,0) = e^{(-2)(0)} \cdot (0)$

$\boxed{f_x(-2,0) = 0}$

$f(x,y) = e^{xy}$

$f_y(x,y) = e^{xy} \cdot x$

$f_y(-2,0) = e^{(-2)(0)}(-2)$

$\boxed{f_y(-2,0) = (1)(2) = 2}$

$$D_u f(-2,0) = f_x(-2,0)\cos\frac{\pi}{3} + f_y(-2,0)\sin\frac{\pi}{3}$$

$$= (0)\left(\frac{1}{2}\right) + (-2)\left(\frac{\sqrt{3}}{2}\right)$$

$$= 0 - \sqrt{3}$$

$$\boxed{D_u f(-2,0) = -\sqrt{3}}$$

Example:- $f(x,y) = 2xy - 3y^2$, $P_0(5,5)$

$u = 4i + 3j$

As $D_u f(x,y) = \nabla f(x,y) \cdot \hat{u}$

& $\nabla f(x,y) = f_x(x,y)\underline{i} + f_y(x,y)\underline{j}$

$\hat{u} = \frac{u}{|u|} = \frac{4i+3j}{\sqrt{4^2+3^2}} \Rightarrow \frac{4i+3j}{\sqrt{25}} \Rightarrow \frac{4i}{5} + \frac{3j}{5}$

$f_x(x,y) = 2(1)y + 0$
$= 2y$
$f_x(5,5) = 2(5)$
$f_x(5,5) = 10$

$f_y(x,y) = 2x(1) - 6y$
$= 2x - 6y$
$f_y(5,5) = 2(5) - 6(5)$
$= 10 - 30$
$f_y(5,5) = -20$

Now

$D_u f(5,5) = \nabla f(5,5) \cdot \hat{u}$
$= (f_x(5,5)i + f_y(5,5)j) \cdot \hat{u}$
$= [10i + (-20j)] \cdot (\frac{4}{5}i + \frac{3}{5}j)$
$= (10)(\frac{4}{5}) + (-20)(\frac{3}{5})$
$= 8 - 12$

$\boxed{D_u f(5,5) = -4}$

Example:- Find directional derivative of $f(x,y) = xe^y - \cos(xy)$ at the point $(2,0)$ in the direction of $a = 3i - 4j$

$f_x(x,y) = (1)e^y - \sin xy(1 \cdot y) \Rightarrow e^y - y\sin xy$

$f_y(x,y) = xe^y - \sin xy \cdot (x \cdot 1) \Rightarrow xe^y - x\sin xy$

$f_x(2,0) = e^0 - (0)\sin(2)(0) \Rightarrow 1 - 0 \Rightarrow 1$

$f_y(2,0) = 2e^0 - 2\sin(2)(0) \Rightarrow 2(1) - 2\sin 0 \Rightarrow 2$

$$\nabla f(2,0) = f_x(2,0)\underline{i} + f_y(2,0)\underline{j}$$
$$= (1)(\underline{i}) + 2(\underline{j})$$
$$\nabla f(2,0) = \underline{i} + 2\underline{j}$$

$$\hat{a} = \frac{a}{|a|} = \frac{3i-4j}{\sqrt{3^2+(-4)^2}}$$
$$\hat{a} = \frac{3i}{5} - \frac{4j}{5}$$

Now

$$D_u f(2,0) = \nabla f(x,y) \cdot \hat{u}$$
$$= \nabla f(2,0) \cdot \hat{a}$$
$$= (\underline{i} + 2\underline{j}) \cdot \left(\frac{3i}{5} - \frac{4j}{5}\right)$$
$$= (1)\left(\frac{3}{5}\right) + (2)\left(-\frac{4}{5}\right)$$
$$= \frac{3}{5} - \frac{8}{5}$$
$$= \frac{-5}{5} \Rightarrow -1$$

Properties of directional derivatives:-

$$D_u f = \nabla f \cdot \hat{u} = \nabla f \cdot \cos\theta$$

⇒ f increases most rapidly in the direction of gradient vector ∇f at P. (when $\cos\theta = 1$)

$$D_u f = |\nabla f| \cos 0^\circ = \boxed{|\nabla f|}$$

⇒ Similarly f decreases most rapidly in direction of $-\nabla f$. (when $\cos\theta = -1$)

$$D_u f = |\nabla f| \cos(\pi) = \boxed{-|\nabla f|}$$

⇒ Any direction $\hat{u}$ orthogonal of the gradient is a direction of zero change in f (when $\cos\theta = 0$)

$$D_u f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = \boxed{0}$$

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} \text{ at } (1,1)$$

Find direction of rapid change $(\hat{u})$.

$f_x(x,y) = \frac{2x}{2} + 0 = x$ $\quad$ $f_y(x,y) = 0 + \frac{2y}{2} = y$

$f_x(1,1) = \underline{1}$ $\quad$ $f_y(1,1) = \underline{1}$

$$\nabla f(1,1) = f_x(1,1)i + f_y(1,1)j$$

$$= 1i + 1j$$

$$= \underline{i + j}$$

$$\hat{u} = \frac{u}{|u|} = \frac{i+j}{\sqrt{1^2+1^2}} = \frac{i+j}{\sqrt{2}}$$

$$\boxed{\hat{u} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j}$$

Lecture 12 Tangent Planes To surfaces

Different forms of equation of a circle:-

⇒ Slope-intercept form:-

$$y = mx + c$$

where m is slope & c is y-intercept.

⇒ Point Slope form:-

$$y - y_0 = m(x - x_0)$$ where m is slope

& $P(x_0, y_0)$ is point of line.

⇒ General Equation of Straight Line:-

$$Ax + By + C = 0$$

As $m = \text{slope} = \dfrac{\text{rise}}{\text{run}} = \dfrac{b}{a}$ then using point slope form

$$\boxed{y - y_0 = \frac{b}{a}(x - x_0)}$$

Parametric Equations of a line:-

$x = x_0 + at$, $y = y_0 + bt$

$x - x_0 = at$, $y - y_0 = bt$

$\dfrac{x - x_0}{a} = t$, $\dfrac{y - y_0}{b} = t$

Comparing both equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{or} \quad \boxed{\frac{b}{a}(x - x_0) = y - y_0}$$

Equation of line in Three dimensional:-

$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

$x - x_0 = at$, $y - y_0 = bt$, $z - z_0 = ct$

$\frac{x - x_0}{a} = t$, $\frac{y - y_0}{b} = t$, $\frac{z - z_0}{c} = t$

Comparing all equations

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

Example:- Find Parametric equations of line Through $A(2,4,3)$ & parallel to vector $\vec{v} = 4i - 7k$

here $x_0 = 2$, $y_0 = 4$, $z_0 = 3$, $a = 4$, $b = 0$, $c = -7$

So required parametric equations are

$x = 2 + 4t$, $y = 4 + 0t$, $z = 3 - 7t$

Different form of equation of curve:-

(i) Explicit Form:-

$y = f(x)$ single

Example:-

$y = \sqrt{9 - x^2}$

$-3 \leq x \leq 3$

(ii) Implicit form:-

$F(x, y) = 0$ multiple

Example:-

$x^2 + y^2 = 9$

$-3 \leq x \leq 3$

$0 \leq y \leq 3$

(iii) Parametric Form:-

$$x = f(t) \quad \& \quad y = g(t)$$

Example:-

$$x = 3\cos\theta \quad , \quad y = 3\sin\theta \qquad 0 \leq \theta \leq \pi$$

Take square on b/s

$$x^2 = 9\cos^2\theta \quad , \quad y^2 = 9\sin^2\theta$$

Now adding both equations

$$x^2 + y^2 = 9\cos^2\theta + 9\sin^2\theta$$
$$= 9(\cos^2\theta + \sin^2\theta)$$
$$= 9(1)$$
$$x^2 + y^2 = 9$$

(iv) Parametric vector Form:-

$$r(t) = f(t)\,i + g(t)\,j \qquad a \leq t \leq b$$

Example:-

$$r(t) = 3\cos\theta\, i + 3\sin\theta\, j \qquad 0 \leq \theta \leq \pi$$

Equation of a plane:-

A plane can be completely determined if we know its one point & direction of perpendicular to it.

Point Normal Form of Equation of plane:-

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

This equation can also be written as

$$ax-ax_0+by-by_0+cz-cz_0=0$$

$$ax+by+cz-ax_0-by_0-cz_0=0$$

$$ax+by+cz+d=0 \quad (d=-ax_0-by_0-cz_0)$$

Example:- Find eq of plane through $(3,-1,7)$ & perpendicular to vector $n = 4i+2j-5k$

Point normal form of equation = ?

here $x_0=3$, $y_0=-1$, $z_0=7$, $a=4$, $b=2$, $c=-5$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$4(x-3)+2(y+1)-5(z-7)=0$$

$$4x-12+2y+2-5z+35=0$$

$$4x+2y-5z+25=0$$

General equ. of straight line is $ax+by+c=0$

let two equations be $ax_1+by_1+c=0$ & ax_2+by_2+c

Subtracting both equations (1 from 2)

$$ax_2+by_2+c-(ax_1+by_1+c)=0-0$$

$$ax_2+by_2+c-ax_1-by_1-c=0$$

$$ax_2-ax_1+by_2-by_1+c-c=0$$

$$a(x_2-x_1)+b(y_2-y_1)=0$$

Vector Form:-

$$V=(x_2-x_1)i+(y_2-y_1)j$$

if $\phi(x,y) = ax + by$

Then $\phi_x = a$ & $\phi_y = b$

& $\nabla\phi = ai + bj = n$

$\nabla\phi \cdot \vec{V} = 0$ (because n & v are perpendicular)

Now General Equ of plane is $ax+by+cz+d=0$

let two equations be

$ax_1 + by_1 + cz_1 + d = 0$ & $ax_2 + by_2 + cz_2 + d = 0$

Subtracting equ 1 from 2.

$ax_2 + by_2 - cz_2 + d - ax_1 - by_1 - cz_1 - d = 0$

$ax_2 - ax_1 + by_2 - by_1 + cz_2 - cz_1 + d - d = 0$

$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$

Vector Form:-

$\vec{V} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k = 0$

if $\phi(x,y,z) = ax + by + cz$

Then $\phi_x = a$, $\phi_y = b$, $\phi_z = c$

& $\nabla\phi = ai + bj + ck$

$\nabla\phi \cdot \vec{V} = 0$ (because $\nabla\phi$ is normal to plane)

Gradients & Tangents to Surfaces:-

$f(x,y) = c$ —— ①

if $z = f(x,y)$ Then $z = c$

& if $x = g(t)$ & $y = h(t)$, $\vec{r} = g(t)i + h(t)j$

Then using eq ① $f(g(t), h(t)) = c$

differentiating b/s

$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x} \cdot \frac{dg}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dh}{dt} = 0 \quad \text{(chain rule)}$$

$$\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j\right) \cdot \left(\frac{dg}{dt} i + \frac{dh}{dt} j\right) = 0$$

$$\nabla f \cdot \frac{dr}{dt} = 0$$

So ∇f is normal to Tangent vector $\frac{dr}{dt}$

Tangent Plane:-

Example:- $9x^2 + 4y^2 - z^2 = 36$, $P(2, 3, 6)$

here $f(x, y, z) = 9x^2 + 4y^2 - z^2 - 36$

$f_x = 9(2x)$, $f_y = 8y$, $f_z = -2z$

$f_x = 18x$, $f_y = 8y$,

also
$a = f_x(P) = 18(2) = 36$

$b = f_y(P) = 8(3) = 24$

$c = f_z(P) = -2(6) = -12$

The Equation is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$$36(x-2) + 24(y-3) - 12(z-6) = 0$$

$$36x - 72 + 24y - 72 - 12z + 72 = 0$$

$$36x + 24y - 12z - 72 = 0$$

$$12(3x + 2y - z - 6) = 0$$

$$3x + 2y - z - 6 = \frac{0}{12}$$

$$\boxed{3x + 2y - z - 6 = 0}$$

Example:- $z = x\cos y - ye^{x}$ $(0,0,0)$

$$f(x,y,z) = x\cos y - ye^{x} - z$$

$$f_x = \cos y - ye^{x}, \quad f_y = -x\sin y - e^{x}, \quad f_z = -1$$

also

$$f_x(P) = \cos 0 - 0.e^{0} = 1 - 0 = 1$$

$$f_y(P) = -0\sin 0 - e^{0} = 0 - 1 = -1$$

$$f_z(P) = -1$$

The Tangent plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$f_x(0,0,0)(x-0) + f_y(0,0,0)(y-0) + f_z(0,0,0)(z-0) = 0$$

$$(1)(x) + (-1)(y) + (-1)(z) = 0$$

$$\boxed{x - y - z = 0}$$

Lecture 13 Orthogonal Surface

Example:- Find equ. of Tangent plane & normal of surface $f(x,y,z) = x^2+y^2+z^2-4$ at point $P(1,-2,+3)$.

$$f(x,y,z) = x^2+y^2+z^2-4$$

$$f_x = 2x \quad , \quad f_y = 2y \quad , \quad f_z = (2z)$$

$$f_x(P) = 2(1) \quad , \quad f_y(P) = 2(-2) \quad , \quad f_z(P) = 2(+3)$$

$$f_x(P) = \underline{2} \quad , \quad f_y(P) = \underline{-4} \quad , \quad f_z(P) = \underline{6}$$

Equation of Tangent

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

or

$$f_x(P)(x-x_0) + f_y(p)[y-y_0] + f_z(P)[z-z_0] = 0$$

$$(2)(x-1) + (-4)(y-(-2)) + 6(z-3) = 0$$

$$2x - 2 - 4y - 8 + 6z - 18 = 0$$

$$2x - 4y + 6z - 2 - 8 - 18 = 0$$

$$2x - 4y + 6z - 28 = 0$$

$$2(x - 2y + 3z - 14) = 0$$

$$\boxed{x - 2y + 3z - 14 = 0}$$

Equations of normal line $\quad \dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

or $\quad \dfrac{x-x_0}{f_x(P)} = \dfrac{y-y_0}{f_y(P)} = \dfrac{z-z_0}{f_z(P)}$

$$\frac{x-1}{2} = \frac{y+2}{-4} = \frac{z-3}{6} \quad \text{or} \quad \boxed{\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3}}$$

Example:- Find equ of Tangent & Normal plane

$$4x^2-y^2+3z^2=10 \quad , \quad P(2,-3,1)$$

$$f(x,y,z)=4x^2-y^2+3z^2-10$$

$$f_x = 8x \quad , \quad f_y=-2y \quad , \quad f_z=6z$$

$$f_x(P)=8(2) \quad , \quad f_y(P)=-2(-3) \quad , \quad f_z(P)=6(1)$$

$$f_x(P)=\underline{16} \quad , \quad f_y(P)=\underline{6} \quad , \quad f_z(P)=\underline{6}$$

Equation of Tangent

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

or $$f_x(P)(x-x_0)+f_y(P)[y-y_0]+f_z(P)[z-z_0]=0$$

$$16(x-2)+6(y-(-3))+6(z-1)=0$$

$$16x-32+6y+18+6z-6=0$$

$$16x+6y+6z-32+18-6=0$$

$$16x+6y+6z-20=0$$

$$2(8x+3y+3z-10)=0$$

$$\boxed{8x+3y+3z=10}$$

Equations of Normal line

$$\frac{x-x_0}{f_x(P)}=\frac{y-y_0}{f_y(P)}=\frac{z-z_0}{f_z(P)}$$

$$\frac{x-2}{16}=\frac{y+3}{6}=\frac{z-1}{6}$$

divide by 2

$$\boxed{\frac{x-2}{8}=\frac{y+3}{3}=\frac{z-1}{3}}$$

Example:- $z = \frac{1}{2} x^7 y^{-2}$ $\quad (2,4,4)$

$$f(x,y,z) = \frac{1}{2} x^7 y^{-2} - z$$

$$f_x = \frac{7}{2} x^6 y^{-2}, \quad f_y = -\frac{2}{2} x^7 y^{-3} \Rightarrow -x^7 y^{-3}, \quad f_z = -1$$

$$f_x(P) = \frac{7}{2}(2)^6(4)^{-2}, \quad f_y(P) = -(2)^7(4)^{-3}, \quad f_z(P) = \underline{-1}$$

$$= \frac{7}{2} \times \frac{64}{16}, \quad f_y(P) = -\frac{128}{64}$$

$$f_x(P) = \underline{14}, \quad f_y(P) = \underline{-2}$$

Equation of Tangent

$$f_x(P)[x - x_0] + f_y(P)[y - y_0] + f_z(P)[z - z_0] = 0$$

$$14(x-2) + (-2)(y-4) + (-1)(z-4) = 0$$

$$14x - 28 - 2y + 8 - z + 4 = 0$$

$$14x - 2y - z - 28 + 8 + 4 = 0$$

$$\boxed{14x - 2y - z - 16 = 0}$$

Equations of normal line

$$\frac{x - x_0}{f_x(P)} = \frac{y - y_0}{f_y(P)} = \frac{z - z_0}{f_z(P)}$$

$$\boxed{\frac{x-2}{14} = \frac{y-4}{-2} = \frac{z-4}{-1}}$$

Parametric equations are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

$$\boxed{x = 2 + 14t}, \quad \boxed{y = 4 - 2t}, \quad \boxed{z = 4 - t}$$

Orthogonal Surfaces:-

Two surfaces are orthogonal at point of their intersection if their normals at that point are orthogonal. They are said to intersect orthogonally if they are orthogonal at every point common to them.

Condition

Two normal lines are orthogonal if & only if

$$f_x g_x + f_y g_y + f_z g_z = 0$$

if (x, y, z) is point of intersection of $f(x, y, z)$ & $g(x, y, z)$.

Example:- Show that surfaces are orthogonal or not?

$$f(x, y, z) = x^2 + y^2 + z - 16 \quad —(1)$$

&

$$g(x, y, z) = x^2 + y^2 - 63z \quad —(2)$$

$f_x = 2x$, $f_y = 2y$, $f_z = 1$

$g_x = 2x$, $g_y = 2y$, $g_z = -63$

put values in

$$f_x g_x + f_y g_y + f_z g_z$$

$$= (2x)(2x) + (2y)(2y) + (1)(-63)$$

$$= 4x^2 + 4y^2 - 63$$

$$= 4(x^2 + y^2) - 63$$

$$=$$

First find value of x^2+y^2 & then put 5

For this Purpose Add eq ① & ②.

$$x^2+y^2+z-16=0 \qquad ; \quad z=-x^2-y^2+16$$
$$x^2+y^2-63z=0$$

$$2x^2+2y^2-62z-16=0$$

$$2x^2+2y^2-62(-x^2-y^2+16)-16=0$$

$$2x^2+2y^2+62x^2+62y^2-992-16=0$$

$$64x^2+64y^2-1008=0$$

$$64(x^2+y^2)=1008$$

$$x^2+y^2=\frac{1008}{64}$$

$$x^2+y^2=\frac{63}{4}$$

Now put it in

$$4(x^2+y^2)-63$$

$$=4\left(\frac{63}{4}\right)-63$$

$$=63-63$$

$$=0$$

So proved that surfaces are orthogonal.

Differentials of a function:-

For $y = f(x)$

$dy = f'(x)\,dx$ is called differential of functions $f(x)$.

$\Rightarrow$ dx & ∇x are same (or equal) but dy & ∇y are not same (or equal). It is just approximate change.

$\Rightarrow$ If Tangent passes Through $(x_0, f(x_0))$ & has slope $f'(x_0)$ Then point slope form will be

As $y - y_0 = m(x - x_0)$

$y - f(x_0) = f'(x_0)[x - x_0]$

or $\boxed{y = f(x_0) + f'(x_0)(x - x_0)}$

Example:- $f(x) = \sqrt{x}$, $x = 4$, $dx = \Delta x = 3$, find dy & Δy

$y = \sqrt{x}$

$y + \Delta y = \sqrt{x + \Delta x}$

$\Delta y = \sqrt{x + \Delta x} - y$

$= \sqrt{x + \Delta x} - \sqrt{x}$

$= \sqrt{4 + 3} - \sqrt{4}$

$= \sqrt{7} - 2$

$\boxed{\Delta y = 0.65}$

$y = \sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$dy = \frac{1}{2\sqrt{x}}\,dx$

$dy = \frac{1}{2\sqrt{4}}(3) \Rightarrow \frac{1}{2(2)}(3) \Rightarrow \frac{3}{4}$

$\boxed{dy = 0.75}$

Example:- Use differential approximation for the value of $\cos 61°$.

let $y = \cos x$ & $x = 60$, $dx = 61 - 60$
$dx = 1°$

As $\Delta y \approx dy$

So
$$\Delta y = dy = -\sin x \, dx = -\sin 60° (1°)$$
$$\Delta y = -\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{180}\right)$$

Now
$$y = \cos x$$
$$y + \Delta y = \cos(x + \Delta x)$$
$$= \cos(60 + 1)$$
$$y + \Delta y = \cos 61°$$
$$\cos 61° = y + \Delta y$$
$$= \cos x + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{180}\right)$$
$$= \cos 60° - \frac{\sqrt{3}}{2} \cdot \frac{\pi}{180}$$
$$= \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\pi}{180}$$
$$= 0.5 - 0.01511 = 0.48489$$

So
$$\cos 61° \approx 0.48489.$$

Example:

A box with a square base has its height twice its width. If width of box is 8.5 inches with ±0.3 inches of possible error. Find possible error of volume.

Let width $= x$ & height $= h$

$$x = 8.5 \quad , \quad dx = \Delta x = \pm 0.3$$

As $V = (x)(x)(h) \Rightarrow x^2 h$

height is twice of its width, so

$$V = x^2(2x)$$

$$V = 2x^3$$

$$\frac{dV}{dx} = 6x^2$$

$$dV = 6x^2 dx$$

$$dV = 6(8.5)^2(\pm 0.3)$$

$$\boxed{dV = \pm 130.05}$$

The possible error in volume is ± 130.05.

Total differential:-

df is called total differential if

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

Find Exact change in A if $x = 10, y = 8$ & $x = 10.3, y = 8.02$

Area $= xy$

$x = 10, \; y = 8$

$A = (10)(8) = \underline{80}$

if $x = 10.3, \; y = 8.02$

$A = (10.3)(8.02) = \underline{80.4406}$

Exact change in Area $= 80.4406 - 80$

$= \boxed{0.4406}$

Example:- A Rectangular plate expands in such a way that its length changes from 10 to 10.03 & breath changes from 8 to 8.02

$A = xy$

Take derivative ([redacted]).

$dA = A_y\,dy + A_x\,dx$

$dA = x\,dy + y\,dx$

$x = 10,\ dx = +0.03,\quad y = 8,\ dy = 0.02$

$dA = (10)(0.02) + (8)(+0.03)$

$dA = 0.44$ = Exact change.

Example:- The volume of a rectangular parallelopiped is given by formula $V = xyz$. If this solid compressed from above so that z is decreased by 2% while x & y each is increased by 0.75% approximately. Find increase or decrease in volume.

$V = xyz$

$dV = V_x\,dx + V_y\,dy + V_z\,dz$

$= yz\,dx + xz\,dy + xy\,dz \quad — (1)$

$dx = 0.75\%$ of x	$dy = 0.75\%$ of y	$dz = -2\%$ of z
$dx = \frac{0.75}{100}x$	$dy = \frac{0.75}{100}y$	$dz = \frac{-2}{100}z$

put values of dx, dy & dz in ①

$$dV = yz\,dx + xz\,dy + xy\,dz$$

$$= yz\left(\frac{0.75}{100}x\right) + xz\left(\frac{0.75}{100}y\right) + xy\left(\frac{-2}{100}z\right)$$

$$= \frac{0.75}{100}xyz + \frac{0.75}{100}xyz - \frac{2}{100}xyz$$

$$= \frac{0.75xyz + 0.75xyz - 2xyz}{100}$$

$$dV = \frac{-0.5}{100}xyz \Rightarrow \frac{-0.5}{100}V$$

So 0.5 % decrease in volume.

<u>Example:-</u> A formula for the area Δ of a triangle is $\Delta = \frac{1}{2}ab\sin C$. Approximately what error is made in computing Δ if a is taken to be 9.1 instead of 9, b is taken to be 4.08 instead of 4 & C is taken to be $30^\circ 3'$ instead of 30°.

By given conditions $a = 9,\ b = 4,\ C = 30^\circ$

$da = 0.1,\ db = 0.08,\ dC = 3'$

$, dC = \left(\frac{3}{60}\right)^\circ$

$dC = \frac{3}{60} \times \frac{\pi}{180}$

$= \frac{\pi}{3600}$

Put all values in

$$\Delta = \frac{1}{2}ab\sin C$$

But we'll take partial derivatives first.

$$d\Delta = \frac{\partial}{\partial a}\left(\frac{1}{2}ab\sin C\right)da + \frac{\partial}{\partial b}\left(\frac{1}{2}ab\sin C\right)db + \frac{\partial}{\partial C}\left(\frac{1}{2}ab\sin C\right)dC$$

$$= \frac{1}{2}b\sin C\, da + \frac{1}{2}a\sin C\, db + \frac{1}{2}ab\cos C\, dC$$

$$= \frac{1}{2}(4)\sin(30)[0.1] + \frac{1}{2}(9)\sin 30(0.08) + \frac{1}{2}(4)(9)\cos 30^\circ\left(\frac{\pi}{3600}\right)$$

$$= 0.293$$

$$\% \text{ change in area} = \frac{\text{change}}{\text{original}} \times 100 = \frac{0.293}{\Delta} \times 100$$

$$= \frac{0.293}{9} \times 100$$

$$= 3.25\%$$

$\because \Delta = \frac{1}{2}ab\sin C$

$a=4,\ b=9,\ C=30$

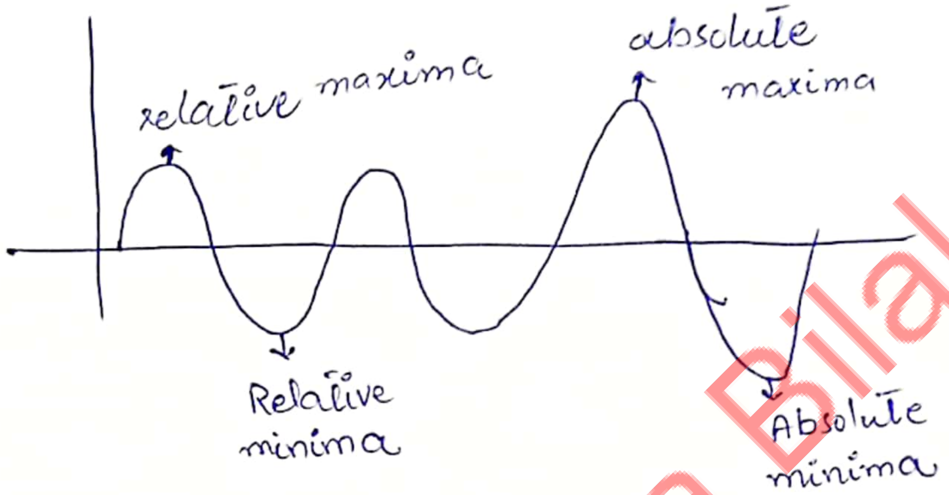
$\Delta = \frac{1}{2}(4)(9)\sin 30$

$\Delta = (2)(9)\left(\frac{1}{2}\right)$

$\Delta = 9$

Lecture 14.

Extrema of functions of Two variables



Absolute minimum:-

$f(x_0, y_0) \le f(x, y)$ for all $(x, y) \in D$.

Absolute maximum:-

$f(x_0, y_0) \ge f(x, y)$ for all $(x, y) \in D$

Absolute extremum

Relative minimum:-

$f(x_0, y_0) \le f(x, y)$ for all $(x, y) \in K$

Relative maximum:-

$f(x_0, y_0) \ge f(x, y)$ for all $(x, y) \in K$

Relative extremum

& $K \subset D$

Extreme value Theorm:-

If $f(x,y)$ is continuous on a closed & bounded set R, Then f has both an absolute maximum & absolute minimum on R.

Remarks:- If any one condition fails (i-e either function is discontinuous or it is not closed & bounded), Then There is need not to have absolute extrema.

Extrema values:-

The minimum & maximum values are called extreme values of extrema of f.

For example The function of an open disc is

$$K = \{(x,y) : (x-x_0)^2 + (y-y_0)^2 < r^2\}$$

let $f_x(x_0,y_0)$ & $f_y(x_0,y_0)$ both exist on K.

If This function has relative extrema at (x_0,y_0)

Then $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$

Saddle point:- A differentiable function has a saddle point (a,b) if $f(x,y) > f(a,b)$ & if $f(x,y) < f(a,b)$. The corresponding point $(a,b,f(a,b))$ on $z = f(x,y)$ is called saddle point.

The extreme values of a function occur at critical point.

Example:- Find critical points of

$$f(x,y) = x^3 + y^3 - 3axy \quad , \quad a > 0$$

for critical points we can use relation

$f_x = f_y = 0$

So find f_x & f_y first.

$$f_x = 3x^2 + 0 - 3a(1)y \Rightarrow 3x^2 - 3ay$$

$$f_y = 0 + 3y^2 - 3ax(1) \Rightarrow 3y^2 - 3ax$$

Keeping $f_x = f_y = 0$

$$3x^2 - 3ay = 3y^2 - 3ax = 0$$

$3x^2 - 3ay = 0$

$3(x^2 - ay) = 0$

$x^2 - ay = 0 \quad \text{---} \; (1)$

$3y^2 - 3ax = 0$

$3(y^2 - ax) = 0$

$y^2 - ax = 0 \quad \text{---} \; (2)$

$y^2 = ax$

$\dfrac{y^2}{a} = x \quad \text{---} \; (3)$

Put value of x from (2) in (1)

$$\left[\frac{y^2}{a}\right]^2 - ay = 0 \quad , \quad \frac{y^4}{a^2} - ay = 0$$

Taking LCM

$$\frac{y^4}{a^2} - \frac{ay}{1} = 0$$

$$\frac{y^4 - a^3 y}{a^2} = 0$$

$$y(y^3 - a^3) = 0 \times a^2$$

$$y(y^3 - a^3) = 0$$

$\boxed{y=0}$

Put $y=0$ in (3)

$$x = \frac{y^2}{a}$$

$$x = \frac{0}{a}$$

$\boxed{x=0}$

$$y^3 - a^3 = 0$$

$$y^3 = a^3$$

Take cube root on b/s

$\boxed{y=a}$

put it in (3)

$$x = \frac{y^2}{a}$$

$$x = \frac{a^2}{a}$$

$\boxed{x=a}$

So critical points are

$(0,0)$ & (a,a).

Lecture 15 Examples

$$f(x,y) = \sqrt{x^2+y^2}$$

Find critical point & absolute minimum value

First of all

$$f_x = \frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1} \cdot (2x+0)$$

$$= \frac{\cancel{2}x}{\cancel{2}(x^2+y^2)^{\frac{1}{2}}} \Rightarrow \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(0+2y)$$

$$= \frac{\cancel{2}y}{\cancel{2}(x^2+y^2)^{\frac{1}{2}}} \Rightarrow \frac{y}{\sqrt{x^2+y^2}}$$

As $f_x = f_y = 0$

$$\frac{x}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} = 0$$

$$\frac{x}{\sqrt{x^2+y^2}} = 0 \qquad \Bigg| \qquad \frac{y}{\sqrt{x^2+y^2}} = 0$$

$$x = 0\left(\sqrt{x^2+y^2}\right) \qquad \Bigg| \qquad y = 0\left(\sqrt{x^2+y^2}\right)$$

$$\boxed{x=0} \qquad \qquad \boxed{y=0}$$

The only critical point is $(0,0)$.

$f(0,0) = 0$ is absolute minimum value.

Example:-

Find critical point of $z = f(x,y) = x^2+y^2$

$f_x = 2x \quad , \quad f_y = 2y$

As $f_x = f_y = 0$

$2x = 2y = 0$

$2x = 0$

$x = 0/2$

$\boxed{x = 0}$

$2y = 0$

$y = 0/2$

$\boxed{y = 0}$

Critical point is $(0,0)$.

Find critical point of $z = h(x,y) = y^2 - x^2$

$h_x = -2x \quad , \quad h_y = 2y$

$h_x = h_y = 0$

$-2x = 2y = 0$

$-2x = 0$

$x = \frac{0}{-2}$

$\boxed{x = 0}$

$2y = 0$

$y = \frac{0}{2}$

$\boxed{y = 0}$

Critical point is $(0,0)$.

Find critical point of $z = g(x,y) = 1 - x^2 - y^2$

$g_x = -2x \quad , \quad g_y = -2y$

$g_x = g_y = 0$

$-2x = -2y = 0$

$-2x = 0$ | $-2y = 0$

$\boxed{x = 0}$ | $\boxed{y = 0}$

So The critical point is $(0,0)$.

Example:- $f(x,y) = \sqrt{x^2 + y^2}$

already solved. You can do it by above method also. So not solving it again.

The Second Partial Derivative Test:-

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2 (x_0, y_0)$$

$\Rightarrow$ if $D > 0$ & $f_{xx}(x_0, y_0) > 0$ Then f has relative minimum at (x_0, y_0)

$\Rightarrow$ if $D > 0$ & $f_{xx}(x_0, y_0) < 0$ Then f has relative maximum at (x_0, y_0).

$\Rightarrow$ if $D < 0$ Then f has a saddle point at (x_0, y_0)

$\Rightarrow$ if $D = 0$ Then no conclusion can be drawn.

If function has absolute extremum at any point of domain, Then its extremum occurs at critical point.

Example:-

$f(x,y) = 2x^2 - 4x + xy^2 - 1$

Find critical points & nature of each point by second partial derivative Test.

$f(x,y) = 2x^2 - 4x + xy^2 - 1$

$f_x = 4x - 4 + (1)y^2 - 0 \Rightarrow 4x - 4 + y^2$

$f_y = 0 - 0 + x(2y) - 0 \Rightarrow 2xy$

For critical points $f_x = f_y = 0$

$4x - 4 + y^2 = 2xy = 0$

$2xy = 0 \rightarrow (1)$

$x = \frac{0}{2y}$

$\underline{x = 0}$

$4x - 4 + y^2 = 0 \longrightarrow (2)$

put $x = 0$

$4(0) - 4 + y^2 = 0$

$-4 + y^2 = 0$

$y^2 = 4$

$\underline{y = \pm 2}$

again using (1)

if $2xy = 0$

$y = \frac{0}{2x}$

$\underline{y = 0}$

put $y = 0$ in (2)

$4x - 4 + 0 = 0$

$4x = 4$

$\underline{x = 1}$

So critical points are

$\boxed{(0,2), (0,-2), (1,0).}$

Now check nature of each point by

$$D = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0)$$

First we'll find f_{xx}, f_{yy} & f_{xy}

As $f_x = 4x - 4 + y^2$, $f_{xx} = 4 - 0 + 0 \Rightarrow \boxed{4}$

& $f_y = 2xy$, $f_{yy} = 2x(1) \Rightarrow \boxed{2x}$

$f_{xy} = f_{yx} = f_x(2xy) \Rightarrow \boxed{2y}$

using $D = f_{xx}(1,0) \cdot f_{yy}(1,0) - [f_{xy}(1,0)]^2$

$$\boxed{\begin{aligned} f_{xx}(1,0) &= 4 \\ f_{yy}(1,0) &= 2 \\ f_{xy}(1,0) &= 0 \end{aligned}}$$

$= (4)(2) - 0^2$

$= 8 > 0$

As $D > 0$ & $f_{xx}(1,0) > 0$, f has a relative minimum at $(1,0)$.

Now for second critical point $(0,-2)$.

$$\boxed{\begin{aligned} f_{xx}(0,-2) &= 4 \\ f_{yy}(0,-2) &= 0 \\ f_{xy}(0,-2) &= -4 \end{aligned}}$$

$D = f_{xx}(0,-2) \cdot f_{yy}(0,-2) - [f_{xy}(0,-2)]^2$

$= (4)(0) - (-4)^2$

$= 0 - 16$

$= -16 < 0$

also

$$\boxed{\begin{aligned} f_{xx}(0,2) &= 4 \\ f_{xx}(0,2) &= 0 \\ f_{xy}(0,2) &= 4 \end{aligned}}$$

$D = f_{xx}(0,2) f_{yy}(0,2) - [f_{xy}(0,2)]^2$

$= (4)(0) - (4)^2 \Rightarrow 0 - 16 \Rightarrow -16 < 0$

So f has saddle point at $(0,2)$ & $(0,-2)$.

Example:-

$$f(x,y) = e^{-(x^2+y^2+2x)}$$

$$f_x(x,y) = e^{-(x^2+y^2+2x)}(-2x-2)$$

$$= e^{-(x^2+y^2+2x)}[-2(x+1)]$$

$$f_y(x,y) = e^{-(x^2+y^2+2x)}(-2y)$$

For critical points

$$f_x(x,y) = 0$$

$$e^{-(x^2+y^2+2x)}[-2(x+1)] = 0$$

$$-2[x+1] = \frac{0}{e^{-(x^2+y^2+2x)}}$$

$$-2(x+1) = 0$$

$$x+1 = \frac{0}{-2}$$

$$x+1=0$$

$$\boxed{x=-1}$$

$$f_y(x,y) = 0$$

$$e^{-(x^2+y^2+2x)}(-2y) = 0$$

$$-2y = \frac{0}{e^{-(x^2+y^2+2x)}}$$

$$-2y = 0$$

$$y = \frac{0}{-2}$$

$$\boxed{y=0}$$

So the critical point is $(-1,0)$.

Now

$$f_{xx}(x,y) = \cancel{e^{-(x^2+y^2+2x)}(-2(x+1))(-2)(2)}$$

$$= e^{-(x^2+y^2+2x)}(-2x-2)$$

$$= e^{-(x^2+y^2+2x)}(-2) + (-2x-2)\,e^{-(x^2+y^2+2x)}(-2x-2)$$

$$= e^{-(x^2+y^2+2x)}\left[-2 + (-2x-2)^2\right]$$

$$f_{xx}(-1,0) = e^{-[(-1)^2+0^2+2(-1)]}\left[-2+[-2(-1)-2]^2\right]$$

$$= e^{-(1-2)}\left[-2+(2-2)^2\right]$$

$= e^{-1(-1)}(-2+0)$

$= e^{1}(-2) \Rightarrow -2e$

$f_{yy}(x,y) = e^{-(x^2+y^2+2x)}(-2y)$

$= e^{-(x^2+y^2+2x)}(-2) + (-2y)\left(e^{-(x^2+y^2+2x)}\right)(-2y)$

$= e^{-(x^2+y^2+2x)}\left[-2+(-2y)^2\right]$

$= e^{-(x^2+y^2+2x)}\left[-2+4y^2\right]$

$f_{yy}(-1,0) = e^{-((-1)^2+0^2+2(-1))}\left[-2+4(0)^2\right]$

$= e^{-(1-2)}(-2+0)$

$= e^{-(-1)}(-2)$

$= -2e$

$f_{xy}(x,y) = f_x\left(e^{-(x^2+y^2+2x)}(-2y)\right)$

$= -2y\left(e^{-(x^2+y^2+2x)}\right)(-2x-2)$

$f_{xy}(-1,0) = -2(0)\left[e^{-((-1)^2+0^2+2(-1))}\right.(-2(-1)-2)$

$= 0$

Now $D = f_{xx}(-1,0)f_{yy}(-1,0) - f^2_{xy}(-1,0)$

$= (-2e)(-2e) - 0$

$= 4e > 0$

So $f(x,y)$ is maximum at $(-1,0)$.

because $D>0$ & $f_{xx}(-1,0)$ is <0.

Example:-

$$f(x,y) = 2x^4 + y^2 - x^2 - 2y$$

$f_x(x,y) = 8x^3 - 2x$, $f_y(x,y) = 2y - 2$

For critical point

$f_x(x,y) = 0$

$8x^3 - 2x = 0$

$2x(4x^2 - 1) = 0$

$2x = 0$, $4x^2 - 1 = 0$

$\boxed{x = 0}$, $4x^2 = 1$

$\sqrt{x^2} = \sqrt{\frac{1}{4}}$

$\boxed{x = \pm \frac{1}{2}}$

$f_y(x,y) = 0$

$2y - 2 = 0$

$2y = 2$

$y = 2/2$

$\boxed{y = 1}$

So critical points are $(0,1)\left(\frac{1}{2},1\right)\left(\frac{-1}{2},1\right)$

$f_{xx}(x,y) = 24x^2 - 2$

$f_{xx}(0,1) = 24(0)^2 - 2$
$= -2$

$f_{yy}(x,y) = 2$

$f_{yy}(0,1) = 2$

$f_{xy} = \frac{\partial}{\partial x}(+2y - 2)$
$= 0$

$f_{xy}(0,1) = 0$

$D = f_{xx}(0,1)\, f_{yy}(0,1) - f_{xy}^2(0,1)$
$= (-2)(2) - 0$
$= -4 < 0$

So f has saddle point on $(-1,0)$.

Now for $(\frac{1}{2}, 1)$

$f_{xx} = 24x^2 - 2$
$f_{xx}(\frac{1}{2}, 1) = 24(\frac{1}{2})^2 - 2$
$= 24(\frac{1}{4}) - 2$
$= 6 - 2 = 4$

$f_{yy} = 2$
$f_{yy}(\frac{1}{2}, 1) = 2$

$f_{xy} = 0$
$f_{xy}(\frac{1}{2}, 1) = 0$

$D = f_{xx}(x,y) f_{yy}(x,y) - f^2_{xy}(x,y)$
$= (4)(2) - 0$
$= 8 > 0$

As $D > 0$ & $f_{xx}(x,y) > 0$, so f has relative minimum at $(\frac{1}{2}, 1)$.

The answers will be same for $(-\frac{1}{2}, 1)$.
So do yourself.

Example:- $f(x,y) = 4xy - x^4 - y^4$
Locate all relative extrema & saddle point.

$f_x(x,y) = 4y - 4x^3$, $f_y(x,y) = 4x - 4y^3$

For critical points.

$f_x(x,y) = 0$
$4y - 4x^3 = 0$

$f_y(x,y) = 0$
$4x - 4y^3 = 0$

$4(y-x^3)=0$

$y-x^3=0$

put $x=y^3$

$y-(y^3)^3=0$

$y-y^6=0$

$y(1-y^5)=0$

$\boxed{y=0}$, $1-y^5=0$

$\sqrt[5]{1}=\sqrt[5]{y^5}$

$\boxed{\pm 1=y}$

$4(x-y^3)=0$

$x-y^3=0$

Solve for x

$x=y^3$

put $y=0$

$x=0^3$

$x=0$

$(0,0)$

put $y=1$

$x=1^3$

$x=1$

$(1,1)$

put $y=-1$

$x=(-1)^3$

$x=-1$

$(-1,-1)$

So critical points are $(0,0)$, $(1,1)$ & $(-1,-1)$.

For (0,0)

Now $f_{xx}(x,y)=0-12x^2$

$=-12x^2$

$f_{yy}(x,y)=0-12y^2$

$=-12y^2$

$f_{xy}(x,y)=\frac{\partial}{\partial x}(4x-4y^3)$

$=4-0 \Rightarrow 4$

$f_{xx}(0,0)=-12(0)^2 \Rightarrow 0$

$f_{yy}(0,0)=-12(0)^2 \Rightarrow 0$

$f_{xy}(0,0)=4$

$D=f_{xx}(0,0)\,f_{yy}(0,0)-f_{xy}^2(0,0)$

$=(0)(0)-(4)^2$

$=0-4 \Rightarrow -4<0$

$(0,0)$ is saddle point.

For (1,1).

$f_{xx}(x,y) = -12x^2$, $f_{xx}(1,1) = -12(1)^2 \Rightarrow -12$

$f_{yy}(x,y) = -12y^2$, $f_{yy}(1,1) = -12(1)^2 \Rightarrow -12$

$f_{xy}(x,y) = 4$, $f_{xy}(1,1) = 4$

$D = f_{xx}(1,1)\, f_{yy}(1,1) - f^2_{xy}(1,1)$

$= (-12)(-12) - (4)^2$

$= 144 - 16 \Rightarrow 128 > 0$

As $D > 0$ & $f_{xx}(1,1) < 0$ So f has relative maxima at $(1,1)$.

For (−1,−1):-

$f_{xx}(x,y) = -12x^2$, $f_{xx}(-1,-1) = -12(-1)^2 \Rightarrow -12$

$f_{yy}(x,y) = -12y^2$, $f_{yy}(-1,-1) = -12(-1)^2 \Rightarrow -12$

$f_{xy}(x,y) = 4$, $f_{xy}(-1,-1) = 4$

$D = f_{xx}(-1,-1)\, f_{yy}(-1,-1) - f^2_{xy}(-1,-1)$

$= (-12)(-12) - (4)^2$

$= 144 - 16$

$= 128 > 0$

As $D > 0$ & $f_{xx}(-1,-1) < 0$ So f has relative maxima at $(-1,-1)$.

Lecture 16 Extreme valued Theorm

If the function f is continuous on closed interval $[a,b]$, then f has an absolute maximum value & an absolute minimum value on $[a,b]$.

Remarks:-

Necessary condition for a function to have relative extremum is critical point.

Absolute extremum value can be determined by following procedures.

1. Find critical points of f on $[a,b]$
2. Find function values at these critical points.
3. Find values of $f(a)$ & $f(b)$.
4. The largest value will be absolute maximum
5. The smallest value will be absolute minimum

Example:- Find absolute extrema of

$$f(x) = x^3 + x^2 - x + 1 \quad \text{at} \quad \left[-2, \tfrac{1}{2}\right].$$

First of all take derivative

$$f'(x) = 3x^2 + 2x - 1$$

put $f'(x) = 0$

$$3x^2+2x-1=0$$
$$3x^2+(3-1)x-1=0$$
$$3x^2+3x-1x-1=0$$
$$3x(x+1)-1(x+1)=0$$
$$(3x-1)(x+1)=0$$

$3x-1=0$ | $x+1=0$
$3x=1$ | $\boxed{x=-1}$
$\boxed{x=\frac{1}{3}}$

So critical points are (-1) & $\frac{1}{3}$.
Here we have four values of x (i-e)
$-1, \frac{1}{3}, -2, \frac{1}{2}$.

As $f(x)=x^3+x^2-x+1$

$f(-1)=(-1)^3+(-1)^2-(-1)+1=\cancel{-1}\cancel{+1}+1+1=2$

$f\left(\frac{1}{3}\right)=\left(\frac{1}{3}\right)^3+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)+1=\frac{1}{27}+\frac{1}{9}-\frac{1}{3}+\frac{1}{1}$

$=\frac{1+3-9+27}{27}=\frac{31-9}{27}=\frac{22}{27}=0.8148$

$f(-2)=(-2)^3+(-2)^2-(-2)+1=-8+4+2+1=-1$

$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^2-\frac{1}{2}+1=\frac{1}{8}+\frac{1}{4}-\frac{1}{2}+\frac{1}{1}$

$=\frac{1+2-4+8}{8}=\frac{11-4}{8}=\frac{7}{8}=0.875$

So absolute maximum value is 2 at $f(-1)$
& absolute minimum value is -1 at $f(-2)$.

Find absolute extrema of $f(x)=(x-2)^{2/3}$ on $[1,5]$

$f(x) = (x-2)^{2/3}$

$f'(x) = \frac{2}{3}(x-2)^{\frac{2}{3}-1} \Rightarrow \frac{2}{3}(x-2)^{-\frac{1}{3}}$

$= \dfrac{2}{3(x-2)^{\frac{1}{3}}}$

put $f'(x) = 0$

$\dfrac{2}{3(x-2)^{\frac{1}{3}}} = 0$

From this we cannot find value of x but here we can see that $f'(x)$ does not exist at $x=2$, because in this case denominator will be zero & answer will be undefined.

So critical point is $+2$

So we have three values of x $(2, 1, 5)$.

$f(1) = (1-2)^{2/3} = (-1)^{2/3} = [(-1)^2]^{1/3} = (1)^{\frac{1}{3}} = 1$

$f(2) = (2-2)^{2/3} = (0)^{2/3} = 0$

$f(5) = (5-2)^{2/3} = (3)^{2/3} = (3^2)^{\frac{1}{3}} = \sqrt[3]{9} = 2.08$

So absolute maximum value is $\sqrt[3]{9}$ at $x=5$

& absolute minimum value is 0 at $x=2$.

Find absolute extrema of $h(x) = x^{2/3}$ on $[-2, 3]$.

$$h(x) = x^{2/3}$$

$$h'(x) = \frac{2}{3} x^{\frac{2}{3}-1} \Rightarrow \frac{2}{3} x^{-1/3} \Rightarrow \frac{2}{3x^{1/3}}$$

Here again x is in denominator only. So we can't put $h'(x) = 0$. Instead we'll see that function is undefined at $x = 0$

So critical point is $x = 0$,

also $x = -2$ & $x = 3$ from given interval.

$$h(x) = x^{2/3}$$

$$h(0) = (0)^{2/3} = 0$$

$$h(-2) = (-2)^{2/3} = [(-2)^2]^{\frac{1}{3}} = (4)^{\frac{1}{3}} = 1.58$$

$$h(3) = (3)^{2/3} = [3^2]^{\frac{1}{3}} = (9)^{\frac{1}{3}} = 2.08$$

So $h(x)$ has maximum value at 3 (i-e) $9^{\frac{1}{3}}/2$.

& absolute minimum value at 0 (i-e) 0.

Absolute extrema of continuous function of two variables can be found by these steps.

1. Find critical points of f
2. Find all boundry points.
3. Evaluate $f(x,y)$ at all these points
4. The largest value will be absolute maxima.
5. The smallest value will be absolute minima.

Example:- Find absolute maximum & minimum values of $f(x,y) = 2+2x+2y-x^2-y^2$ while function is bounded by $x=0, y=0$ $y=9-x$. First of all find critical points:

$f_x = 0$ | $f_y = 0$

$2-2x = 0$ | $2-2y=0$

$2 = 2x$ | $2 = 2y$

$\boxed{1 = x}$ | $2/2 = y$, $\boxed{y=1}$

So critical point is $(1,1)$.

For Boundry points:

For $x = 0$

$f(0, y) = 2 + 2y - y^2$

$f'(0, y) = 2 - 2y$

put $f'(0, y) = 0$

$2 - 2y = 0$

$2 = 2y$

$\boxed{1 = y}$

$(0, 1)$

For $y = 0$

$f(x, 0) = 2 + 2x - x^2$

$f'(x, 0) = 2 - 2x$

put $2 - 2x = 0$

$2 = 2x$

$\boxed{1 = x}$

$(1, 0)$

For $y = 9 - x$

put $x = 0$

$y = 9 - 0 \Rightarrow 9 \quad (0, 9)$

put $y = 0$

$0 = 9 - x \Rightarrow x = 9 \quad (9, 0)$

Now all points to be put ~~are~~ in $f(x)$ are

$(0,0), (0,1), (1,0), (0,9), (9,0), (1,1).$

$f(x,y) = 2 + 2x + 2y - x^2 - y^2$

(x,y)	$(0,0)$	$(0,1)$	$(1,0)$	$(0,9)$	$(9,0)$	$(1,1)$
$f(x,y)$	2	3	3	-61	-61	4

So the absolute maximum value is 4 at $(1,1)$ &

the absolute minimum value is -61 at $(0,9)$ or $(9,0)$.

Example:- Find absolute maxima & absolute minima of $f(x,y) = 3xy - 6x - 3y + 7$ on closed triangular region r with vertices $(0,0)$, $(3,0)$ & $(0,5)$.

$$f(x,y) = 3xy - 6x - 3y + 7$$

$f_x = 3y - 6$

$f_x = 0$

$3y - 6 = 0$

$3y = 6$

$\boxed{y = 2}$

$f_y = 3x - 3$

$f_y = 0$

$3x - 3 = 0$

$3x = 3$

$\boxed{x = 1}$

critical point is $(1,2)$.

& other points are $(0,0)$, $(3,0)$, $(0,5)$.

(x,y)	$(0,0)$	$(3,0)$	$(0,5)$	$(1,2)$
$f(x,y)$	7	-11	-8	1

So absolute maximum value is 7 at $(0,0)$

& absolute minimum value is -11 at $(3,0)$

Lecture 18

Revision of Integration

Example. Simplify $\int_0^1 (xy+y^2)\,dx$

$$\int_0^1 (xy+y^2)\,dx = \int_0^1 xy\,dx + \int_0^1 y^2\,dx$$

$$= y\int_0^1 x\,dx + y^2\int_0^1 1\,dx$$

$$= y\left|\frac{x^2}{2}\right|_0^1 + y^2\left|x\right|_0^1 + c$$

$$= y\left[\frac{1^2}{2} - \frac{0^2}{2}\right] + y^2\left[1-0\right]$$

$$= y\left[\frac{1}{2} - 0\right] + y^2(1)$$

$$\int_0^1 (xy+y^2)\,dx = y\left(\frac{1}{2}\right) + y^2$$

Example:- Simplify $\int_0^1 (xy+y^2)\,dy.$

$$\int_0^1 (xy+y^2)\,dy = \int_0^1 xy\,dy + \int_0^1 y^2\,dy$$

$$= x\int_0^1 y\,dy + \int_0^1 y^2\,dy$$

$$= x\left|\frac{y^2}{2}\right|_0^1 + \left|\frac{y^3}{3}\right|_0^1 + c$$

$$= x\left[\frac{1^2}{2} - \frac{0^2}{2}\right] + \left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$= x\left(\frac{1}{2} - 0\right) + \left(\frac{1}{3} - 0\right)$$

$$= x\left(\frac{1}{2}\right) + \frac{1}{3}$$

$$\int_0^1 (xy+y^2)\,dy = \frac{x}{2} + \frac{1}{3}$$

Double integral

Double integral is represented by $\iint_R f(x,y)\,dx\,dy$

Example:- Use a double integral to find out the solid bounded above by the plane $z = 4-x-y$ & below by rectangle $R=\{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}$.

$$V = \iint_R (4-x-y)\,dA$$

$$V = \int_0^2 \int_0^1 (4-x-y)\,dx\,dy$$

$$= \int_0^2 \left[4\int_0^1 1\,dx - \int_0^1 x\,dx - y\int_0^1 1\,dx \right] dy$$

$$= \int_0^2 \left| 4x - \frac{x^2}{2} - xy \right|_0^1 dy$$

$$= \int_0^2 \left[\left(4(1) - \frac{1^2}{2} - (1)y\right) - \left(4(0) - \frac{0^2}{2} - (0)y\right) \right] dy$$

$$= \int_0^2 \left(4 - \frac{1}{2} - y - 0 + 0 + 0\right) dy$$

$$= \int_0^2 \left(\frac{7}{2} - y \right) dy$$

$$= \left[\frac{7}{2}\int_0^2 1\,dy - \int_0^2 y\,dy \right]$$

$$= \left| \frac{7}{2}y - \frac{y^2}{2} \right|_0^2 \Rightarrow \left(\frac{7}{2}(2) - \frac{2^2}{2}\right) - \left(\frac{7}{2}(0) - \frac{0^2}{2}\right)$$

$$= 7 - \frac{4}{2} \Rightarrow \frac{14-4}{2} \Rightarrow \frac{10}{2} \Rightarrow 5 \text{ Ans.}$$

Example:

Evaluate the double integral $\int_0^1\int_0^1 (xy+y^2)\,dx\,dy$

$$\int_0^1\int_0^1 (xy+y^2)\,dx\,dy$$

$$= \int_0^1 \left[\int_0^1 xy\,dx + \int_0^1 y^2\,dx \right] dy$$

$$= \int_0^1 \left[y\int_0^1 x\,dx + y^2\int_0^1 1\,dx \right] dy$$

$$= \int_0^1 \left[y\left(\frac{x^2}{2}\right) + y^2(x) \right]_0^1 dy$$

$$= \int_0^1 \left[\frac{x^2 y}{2} + xy^2 \right]_0^1 dy$$

$$= \int_0^1 \left[\left(\frac{(1)^2 y}{2} + (1)y^2\right) - \left(\frac{0^2 y}{2} + (0)y^2\right)\right] dy$$

$$= \int_0^1 \left(\frac{y}{2} + y^2\right) dy$$

$$= \frac{1}{2}\int_0^1 y + \int_0^1 y^2\,dy$$

$$= \left| \frac{1}{2}\cdot\frac{y^2}{2} + \frac{y^3}{3} \right|_0^1 + c$$

$$= \left| \frac{y^2}{4} + \frac{y^3}{3} \right|_0^1$$

$$= \left(\frac{1^2}{4} + \frac{1^3}{3}\right) - \left(\frac{0^2}{4} - \frac{0^3}{3}\right)$$

$$= \frac{1}{4} + \frac{1}{3} \Rightarrow \frac{3+4}{12} \Rightarrow \frac{7}{12} \quad \underline{\text{Ans.}}$$

Iterated or repeated integral

$\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$ is called iterated or repeated integral. It can also be written without brackets.

Example:- Evaluate $\int_0^1 \int_0^2 (x+3)\,dy\,dx$

$$\int_0^1 \int_0^2 (x+3)\,dy\,dx$$

$$= \int_0^1 (x+3) \int_0^2 1\,dy\,dx$$

$$= \int_0^1 (x+3) \left| y \right|_0^2 dx$$

$$= \int_0^1 (x+3)(2-0)\,dx$$

$$= \int_0^1 2(x+3)\,dx$$

$$= 2\left[\int_0^1 x\,dx + 3\int_0^1 1\,dx \right]$$

$$= 2\left[\left| \frac{x^2}{2} \right|_0^1 + 3\left| x \right|_0^1 \right]$$

$$= 2\left[\left(\frac{1^2}{2} - \frac{0^2}{2} \right) + 3(1-0) \right]$$

$$= 2\left[\frac{1}{2} + 3 \right]$$

$$= 2\left[\frac{1+6}{2} \right] \Rightarrow 2\left(\frac{7}{2} \right) \Rightarrow 7 \text{ Ans.}$$

Example:- Evaluate $\int_0^1 \int_0^1 (xy+y^2)\,dx\,dy$

$$\int_0^1 \left(\int_0^1 xy + y^2 \right) dx\,dy$$

$$= \int_0^1 \left| \frac{x^2 y}{2} + y^2 x \right|_0^1 dy$$

$$= \int_0^1 \left(\left[\frac{y(1)^2}{2} + y^2(1) \right] - \left[\frac{0^2}{2} y + y^2(0) \right] \right) dy$$

$$= \int_0^1 \left(\frac{y}{2} + y^2 \right) dy$$

$$= \left| \frac{y^2}{2\times 2} + \frac{y^3}{3} \right|_0^1$$

$$= \left(\frac{1^2}{4} + \frac{1^3}{3} \right) - \left(\frac{0^2}{4} + \frac{0^3}{3} \right)$$

$$= \frac{1}{4} + \frac{1}{3}$$

$$= \frac{3+4}{12} \Rightarrow \frac{7}{12} \quad \underline{\text{Ans}}.$$

Theorem: If $f(x,y)$ is continuous on a rectangle having $a \le x \le b$ & $c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example: Evaluate $\int_0^{\ln 2}\int_0^{\ln 3} e^{x+y}\,dx\,dy$

$$\int_0^{\ln 2}\int_0^{\ln 3} e^{x+y}\,dx\,dy$$

$$= \int_0^{\ln 2}\int_0^{\ln 3} (e^{x}\cdot e^{y})\,dx\,dy$$

$$= \int_0^{\ln 2} e^{y}\left(\int_0^{\ln 3} e^{x}\,dx\right)dy$$

$$= \int_0^{\ln 2} e^{y}\,\left|e^{x}\right|_0^{\ln 3}\,dy$$

$$= \int_0^{\ln 2} e^{y}\left(e^{\ln 3} - e^{0}\right)dy$$

$$= \int_0^{\ln 2} e^{y}\,(3-1)\,dy$$

$$= \int_0^{\ln 2} 2e^{y}\,dy \Rightarrow 2\left|e^{y}\right|_0^{\ln 2}$$

$$= 2\left(e^{\ln 2} - e^{0}\right) \Rightarrow 2(2-1) \Rightarrow 2 \quad \underline{\text{Ans.}}$$

Evaluate $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}\,dy\,dx$

$$\int_0^{\ln 3}\int_0^{\ln 2} e^{x}\cdot e^{y}\,dy\,dx$$

$$= \int_0^{\ln 3} e^{x}\left|e^{y}\right|_0^{\ln 2}\,dx$$

$$= \int_0^{\ln 3} e^{x}\left(e^{\ln 2} - e^{0}\right)dx$$

$$= \int_0^{\ln 3} e^{x}\,(2-1)\,dx$$

$$\int_0^{\ln 3} e^{x}\,dx$$

$$= \left|e^{x}\right|_0^{\ln 3}$$

$$= e^{\ln 3} - e^{0}$$

$$= 3-1$$

$$= 2 \quad \underline{\text{Ans.}}$$

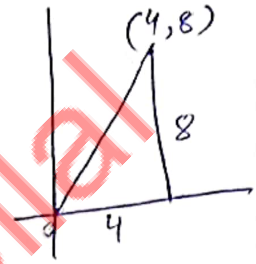
lecture 19 Use of integrals

Area as anti-derivative

(i) Find area of Triangle using.

$$\int_0^4 2x\,dx = \left|\frac{2x^2}{2}\right|_0^4 = \left|x^2\right|_0^4$$

$$= 4^2 - 0^2 = 16 \text{ Ans.}$$



OR geometrically

Area of Triangle $= \frac{1}{2}(\text{base} \times \text{height})$

$$= \frac{1}{2}(4)(8) = 16 \text{ Ans.}$$

volume as anti-derivative:-

$$\text{Volume} = \int_0^2 \int_0^3 5\,dy\,dx$$

$$= \int_0^2 \left|5y\right|_0^3 dx \Rightarrow \int_0^2 5(3) - 5(0)\,dx$$

$$= \int_0^2 15\,dx \Rightarrow 15\int_0^2 1\,dx$$

$$= \left|15x\right|_0^2 \Rightarrow 15(2) - 15(0) \Rightarrow 30$$

Theorm: $\iint_R c\, f(x,y)\,dx\,dy = c\iint_R f(x,y)\,dx\,dy$

$$\iint_R [f(x,y)+g(x,y)]\,dx\,dy = \iint_R f(x,y)\,dx\,dy + \iint_R g(x,y)\,dx\,dy$$

$$\iint_R [f(x,y)-g(x,y)]\,dx\,dy = \iint_R f(x,y)\,dx\,dy - \iint_R g(x,y)\,dx\,dy$$

Example

Use double integral to find volume under surface $z = 3x^3 + 3x^2y$ & the rectangle $\{(x,y) : 1 \le x \le 3,\ 0 \le y \le 2\}$

$$V = \int_0^2 \int_1^3 (3x^3 + 3x^2 y)\, dx\, dy$$

$$= \int_0^2 \left| \left(3\frac{x^4}{4} + 3y\frac{x^3}{3} \right) \right|_1^3 dy$$

$$= \int_0^2 \left[\left(\frac{3(3)^4}{4} + \frac{3y(3^3)}{3} \right) - \left(\frac{3(1)^4}{4} + \frac{3y(1)^3}{3} \right) \right] dy$$

$$= \int_0^2 \left(\frac{243}{4} + \frac{81}{3}y - \frac{3}{4} - \frac{3}{3}y \right) dy$$

$$= \int_0^2 \left(\frac{243}{4} - \frac{3}{4} + \frac{81}{3}y - \frac{3}{3}y \right) dy$$

$$= \int_0^2 \left(\frac{240}{4} + \frac{78}{3}y \right) dy$$

$$= \int_0^2 (60 + 26y)\, dy$$

$$= \left| 60y + \frac{26y^2}{2} \right|_0^2 \Rightarrow \left| (60y + 13y^2) \right|_0^2$$

$$= \left[60(2) + 13(2)^2 \right] - \left[60(0) + 13(0)^2 \right]$$

$$= 120 + 52 \Rightarrow 172 \quad \underline{Ans.}$$

Example:-

Use double integral to find volume of solid in first octant enclosed by surface $z = x^2$ & planes $x = 2$, $y = 3$ & $z = 0$

$$V = \int_0^3 \int_0^2 x^2 \, dx \, dy$$

$$= \int_0^3 \left| \frac{x^3}{3} \right|_0^2 dy$$

$$= \int_0^3 \left(\frac{2^3}{3} - \frac{0^3}{3} \right) dy$$

$$= \int_0^3 \left(\frac{8}{3} \right) dy \Rightarrow \frac{8}{3} \int_0^3 1 \, dy$$

$$= \frac{8}{3} |y|_0^3 \Rightarrow \frac{8}{3}(3-0) \Rightarrow \frac{8}{3}(3)$$

$$= 8 \text{ Ans}$$

Remarks:-

(i) $\iint_R f(x,y)\, dA \geq 0$ if $f(x,y) \geq 0$ on R.

(ii) $\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$

if $f(x,y) \geq g(x,y)$

(iii) $\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$

Volume of cross-section perpendicular to y-axis is

$$\text{Volume} = \int_c^d A(y)\, dy \quad \text{where } c \le y \le d$$

Volume of cross-section perpendicular to x-axis is

$$\text{Volume} = \int_a^b A(x)\, dx \quad \text{where } a \le x \le b$$

Double integral for non-rectangular region:-

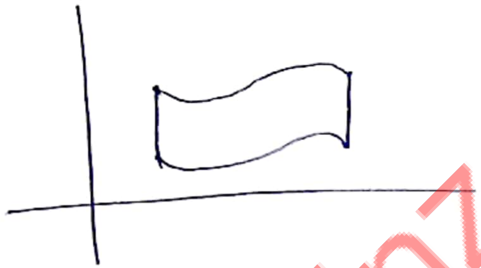
$$\text{Volume} = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

Lecture 20

Double integral for non-rectangular region:-

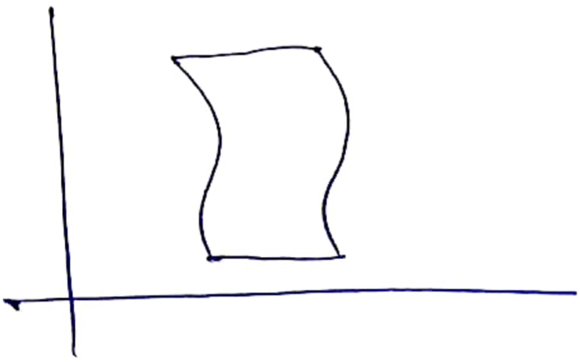
is bounded by left & right by vertical lines $x=a$ & $x=b$ & is bounded below & above by curves $y=g_1(x)$ & $y=g_2(x)$ where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$.

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$



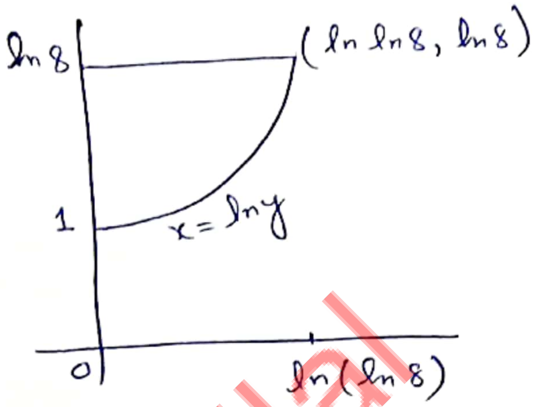
is bounded below & above by the horizontal lines $y=c$ & $y=d$ & is bounded on left & right by continuous curves $x=h_1(y)$ & $x=h_2(y)$ satisfying $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$.

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\, dx\, dy$$



Write down double integral of $f(x,y)$ on region whose sketch is given.

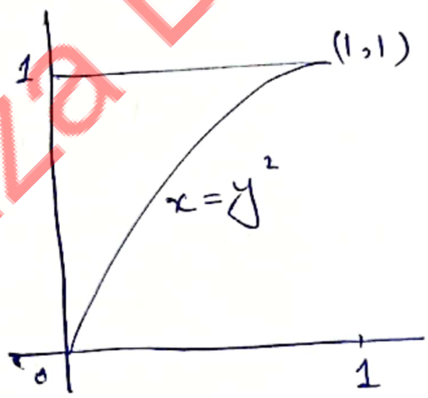
$$\int_0^{\ln 8} \int_0^{\ln y} f(x,y)\,dx\,dy$$



$$\int_0^{\ln(\ln 8)} \int_{e^x}^{\ln 8} f(x,y)\,dy\,dx$$

Write double integral for $f(x,y)$ whose sketch is shown.

$$\int_0^1 \int_0^{y^2} f(x,y)\,dx\,dy$$



$$\int_0^1 \int_{\sqrt{x}}^1 f(x,y)\,dy\,dx$$

Example:-

Evaluate an equivalent integral with the order of integration reversed.

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$$

here $x^2 \le y \le 2x$ & $0 \le x \le 2$

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy$$

<u>R.W</u>

$x^2 \le y \le 2x$ & $0 \le x \le 2$

$x^2 \le y$	$y \le 2x$
$\sqrt{x^2} \le \sqrt{y}$	$\frac{y}{2} < x$
$x \le \sqrt{y}$	put $x = 2$
put $x=0$	$y \le 2x$
$x^2 \le y$	$y \le 2(2)$
$0 \le y$	$y \le 4$

$$= \int_0^4 \left| 4\left(\frac{x^2}{2}\right) + 2(x) \right|_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 \left| 2x^2 + 2x \right|_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 \left(\left[2(\sqrt{y})^2 + 2(\sqrt{y})\right] - \left[2\left(\frac{y}{2}\right)^2 + 2\left(\frac{y}{2}\right)\right]\right) dy$$

$$= \int_0^4 \left(2y + 2\sqrt{y} - \frac{y^2}{2} \mp y\right) dy$$

$$= \left| 2\left(\frac{y^2}{2}\right) + 2\frac{y^{3/2}}{3/2} - \frac{1}{2}\left(\frac{y^3}{3}\right) \mp \frac{y^2}{2} \right|_0^4$$

$$= \left| y^2 + 2\cdot\frac{2}{3}\, y^{3/2} - \frac{y^3}{6} \mp \frac{y^2}{2} \right|_0^4$$

$$= \left[4^2 + \frac{4}{3}(4)^{3/2} - \frac{(4)^3}{6} \mp \frac{(4)^2}{2}\right] - \left[0 + \frac{4}{3}(0) - 0 \mp 0\right]$$

$$= 16 + \frac{4}{3}(2^2)^{\frac{3}{2}} - \frac{64}{6} \mp \frac{16}{2}$$

$$= \frac{16}{1} + \frac{32}{3} - \frac{64}{6} \mp \frac{16}{2}$$

$$= \cancel{\frac{16}{1} + 32} \quad \frac{96 + 64 - 64 \mp 48}{6} \Rightarrow \frac{48}{6} = 8$$

Example:- Evaluate $I = \int_0^4 \int_{\sqrt{y}}^2 y \cos x^5 \, dx\, dy$

Find The integration reversed.

here $0 \le y \le 4$ & $\sqrt{y} \le x \le 2$

$$I = \int_0^2 \int_0^{x^2} y \cos x^5 \, dy\, dx$$

$$= \int_0^2 \left[\frac{y^2}{2} \cos x^5 \right]_0^{x^2} dx$$

$$= \int_0^2 \left[\left(\frac{(x^2)^2}{2} \cos x^5 \right) - \left(\frac{0^2}{2} \cos x^5 \right) \right] dx$$

$$= \int_0^2 \left(\frac{x^4}{2} \cos x^5 \right) dx$$

$$= \frac{1}{2} \int_0^2 (x^4 \cos x^5) \, dx$$

$$= \frac{1}{2} \int_0^2 \cos x^5 \, \frac{(5x^4)}{5} \, dx$$

$$= \frac{1}{2 \times 5} \int_0^2 \cos x^5 \, (5x^4) \, dx$$

$$= \frac{1}{10} \left| \sin x^5 \right|_0^2 \Rightarrow \frac{1}{10} \left[\sin (2)^5 - \sin (0) \right]$$

$$= \frac{1}{10} (\sin 32 - 0) \Rightarrow \frac{1}{10} \sin 32$$

Example:- Evaluate $I = \int_0^{1/2}\int_{2x}^{1} e^{y^2}\,dy\,dx$

Here we'll change the order of integration because e^{y^2} has no antiderivative here.

Also $0 \le x \le \frac{1}{2}$ & $2x \le y \le 1$

R.W

$2x = y \quad , \quad y = 1$

$2(0) = y \quad , \quad y = 1$

$0 = y$

Now

$2x = y \quad , \quad y = 1$

$x = \frac{y}{2}$

$$I = \int_0^1 \int_0^{y/2} e^{y^2}\,dx\,dy$$

$$= \int_0^1 \left| e^{y^2} \cdot x \right|_0^{y/2} dy$$

$$= \int_0^1 e^{y^2}\left(\frac{y}{2} - 0\right) dy$$

$$= \int_0^1 \frac{y}{2} e^{y^2}\,dy$$

$$= \frac{1}{2}\int_0^1 e^{y^2}\,\frac{2y}{2}\,dy$$

$$= \frac{1}{2\times 2}\int_0^1 e^{y^2}(2y)\,dy$$

$$= \frac{1}{4}\left| e^{y^2} \right|_0^1$$

$$= \frac{1}{4}\left(e^{1^2} - e^{0^2}\right) \Rightarrow \frac{1}{4}\left(e^1 - e^0\right)$$

$$I = \frac{1}{4}(e-1) \quad \text{Ans.}$$

Evaluate $\int_1^3 \int_0^{\ln x} x \, dy \, dx$ by reversing the order.

here $1 \le x \le 3$, $0 \le y \le \ln x$

$$I = \int_0^{\ln 3} \int_{e^y}^{3} x \, dx \, dy$$

$$= \int_0^{\ln 3} \left| \frac{x^2}{2} \right|_{e^y}^{3} dy$$

$$= \int_0^{\ln 3} \left(-\frac{(e^y)^2}{2} + \frac{(3)^2}{2} \right) dy$$

$$= \int_0^{\ln 3} \left(-\frac{e^{2y}}{2} + \frac{9}{2} \right) dy$$

$$= \frac{1}{2} \int_0^{\ln 3} (9 - e^{2y}) \, dy$$

$$= \frac{1}{2} \left[\int_0^{\ln 3} 9 \, dy - \int_0^{\ln 3} \frac{e^{2y}}{2} (2) \, dy \right]$$

$$= \frac{1}{2} \left[\left| 9y - \frac{e^{2y}}{2} \right|_0^{\ln 3} \right]$$

$$= \frac{1}{2} \left[\left(9 \ln 3 - \frac{e^{2 \ln 3}}{2} \right) - \left(9(0) - \frac{e^{2(0)}}{2} \right) \right]$$

$$= \frac{1}{2} \left[9 \ln 3 - \frac{e^{\ln 3^2}}{2} - \frac{e^0}{2} \right] \Rightarrow \frac{1}{2} \left(9 \ln 3 - \frac{e^{\ln 9}}{2} - \frac{1}{2} \right)$$

$$= \frac{1}{2} \left(9 \ln 3 - \frac{9}{2} - \frac{1}{2} \right) = \frac{1}{2} (9 \ln 3 - 4) \Rightarrow \frac{9}{2} \ln 3 - 2$$