Calculus And Analytical Geometry
MTH 101

Virtual University

Virtual University of Pakistan

Knowledge beyond the boundaries

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Lecture 1

Coordinates, Graphs and Lines

What is Calculus??

Well, it is the study of the continuous rates of the change of quantities. It is the study of how various quantities change with respect to other quantities. For example, one would like to know how distance changes with respect to (from now onwards we will use the abbreviation w.r.t) time, or how time changes w.r.t speed, or how water flow changes w.r.t time etc. You want to know how this happens continuously. We will see what continuously means as well.

In this lecture, we will talk about the following topics:

-Real Numbers

-Set Theory

-Intervals

-Inequalities

-Order Properties of Real Numbers

Let's start talking about Real Numbers. We will not talk about the COMPLEX or IMANGINARY numbers, although your text has something about them which you can read on your own. We will go through the history of REAL numbers and how they popped into the realm of human intellect. We will look at the various types of REALS - as we will now call them. So Let's START.

The simplest numbers are the ***natural numbers***

<u>Natural Numbers</u>

1, 2, 3, 4, 5,…

They are called the natural numbers because they are the first to have crossed paths with human intellect. Think about it: these are the numbers we count things with. So our ancestors used these numbers first to count, and they came to us naturally! Hence the name

NATURAL!!!

The natural numbers form a subset of a larger class of numbers called the ***integers.*** I have used the word SUBSET. From now onwards we will just think of SET as a COLLECTION OF THINGS.

This could be a collection of oranges, apples, cars, or politicians. For example, if I have the SET of politicians then a SUBSET will be just a part of the COLLECTION. In mathematical notation we say A is subset of B if $\forall x \in A \Rightarrow x \in B$ .Then we write $A \subseteq B$.

Set

The collection of well defined objects is called a set. For example

{George Bush, Toney Blair, Ronald Reagoan}

Subset

A portion of a set B is a subset of *A* iff every member of *B* is a member of *A*. e.g. one subset of above set is

{George Bush, Tony Blair}

The curly brackets are always used for denoting SETS. We will get into the basic notations and ideas of sets later. Going back to the Integers. These are

…, -4, -3, -2, -1, 0, 1, 2, 3, 4,…

So these are just the natural numbers, plus a 0, and the NEGATIVES of the natural numbers.

The reason we didn't have 0 in the natural numbers is that this number itself has an interesting story, from being labeled as the concept of the DEVIL in ancient Greece, to being easily accepted in the Indian philosophy, to being promoted in the use of commerce and science by the Arabs and the Europeans. But here, we accept it with an open heart into the SET of INTEGERS.

What about these NEGATIVE Naturals??? Well, they are an artificial construction. They also have a history of their own. For a long time, they would creep up in the solutions of simple equations like

x+2 = 0. The solution is x = - 2

So now we have the Integers plus the naturals giving us things we will call REAL numbers. But that's not all. There is more. The integers in turn are a subset of a still larger class of numbers called the **rational numbers**. With the exception that division by zero is ruled out, the rational numbers are formed by taking ratios of integers.

Examples are

2/3, 7/5, 6/1, -5/2

Observe that every integer is also a rational number because an integer *p* can be written as a ratio. So every integer is also a rational. Why not divide by 0? Well here is why:

If *x* is different from zero, this equation is contradictory; and if *x* is equal to zero, this equation is satisfied by any number *y,* so the ratio does not have a unique value a situation that is mathematically unsatisfactory.

$$x/0 = y \Rightarrow x = 0.y \Rightarrow x = 0$$

For these reasons such symbols are not assigned a value; they are **said to be undefined.**

So we have some logical inconsistencies that we would like to avoid. I hope you see that!! Hence, no division by 0 allowed! Now we come to a very interesting story in the history of the development of Real numbers. The discovery of IRRATIONAL numbers.

Pythagoras was an ancient Greek philosopher and mathematician. He studied the properties of numbers for its own sake, not necessarily for any applied problems. This was a major change in mathematical thinking as math now took on a personality of its own. Now Pythagoras got carried away a little, and developed an almost religious thought based on math. He concluded that the size of a physical quantity must consist of a certain whole number of units plus some fraction m / n of an additional unit. Now rational numbers have a unique property that if you convert them to decimal notation, the numbers following the decimal either end quickly, or repeat in a pattern forever.

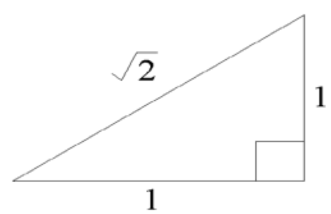
Example:

1/2 =0.500000… =0.5

1/3= 0.33333…

This fit in well with Pythagoras' beliefs. All is well. But this idea was shattered in the fifth century B.C. by Hippasus of Metapontum who demonstrated the existence of *irrational numbers,* that is, numbers that cannot be expressed as the ratio of integers.

Using geometric methods, he showed that the hypotenuse of the right triangle with base and opposite side equal to 1 cannot be expressed as the ratio of integers, thereby proving that $\sqrt{2}$ is an IRRATIONAL number. The hypotenuse of this right triangle can be expressed as the ratio of integers.

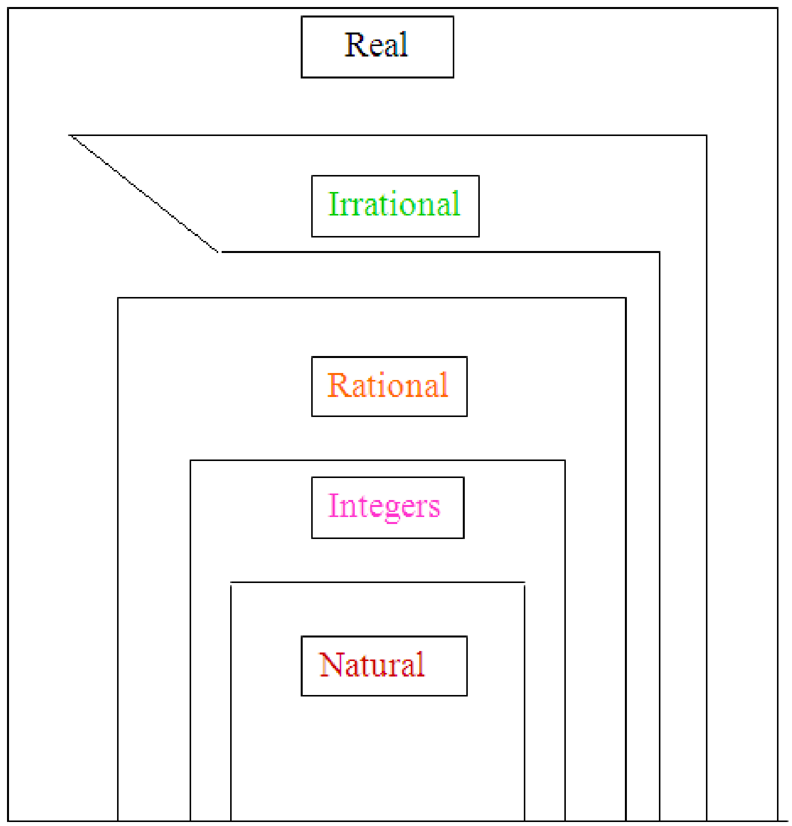


Other examples of irrational numbers are

$$Cos\, 19^0, 1+\sqrt{2}$$

The rational and irrational numbers together comprise a larger class of numbers, called REAL NUMBERS *or* sometimes the REAL NUMBER SYSTEM.**So here is a pictorial summary of the hierarchy of REAL NUMBNERS.**

Pictorial summary of the hierarchy of REAL NUMBNERS

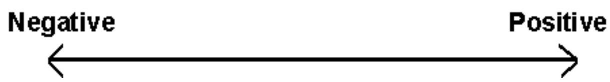


COORDINATE Line

In the 1600's, *analytic geometry* was "developed". It gave a way of describing algebraic formulas by geometric curves and, conversely, geometric curves by algebraic formulas. So basically you could DRAW PICTURES OF THE EQUATIONS YOU WOULD COME ACROSS, AND WRITE DOWN EQUATIONS OF THE PICTURES YOU RAN INTO!

The developer of this idea was the French mathematician, Descartes .The story goes that he wanted to find out as to what Made humans HUMANS?? Well, he is said to have seated himself in a 17th century furnace (it was not burning at the time!) and cut himself from the rest of the world. In this world of cold and darkness, he felt all his senses useless. But he could still think!!!! So he concluded that his ability to think is what made him human, and then he uttered the famous line : " I THINK, THEREFORE I AM" .In analytic geometry , the key step is to establish a correspondence between real numbers and points on a line. We do this by arbitrarily designating one of the two directions along the line as the positive direction and the other as the negative direction.

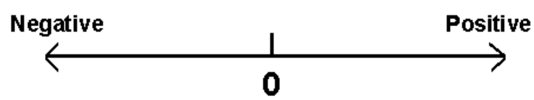
So we draw a line, and call the RIGHT HAND SIDE as POSITIVE DIRECTION, and the LEFT HAND SIDE as NEGATIVE DIRECTION. We could have done it the other way around too. But, since what we just did is a cultural phenomenon where right is + and left is -, we do it this way. Moreover, this has now become



a standard in doing math, so anything else will be awkward to deal with. The positive direction is usually marked with an arrowhead so we do that too. Then we choose an arbitrary point and take that as our point of reference.

We call this the ORIGIN, and mark it with the number 0. So we have made our first correspondence between a real number and a point on the Line. Now we choose a unit of measurement, say 1 cm. It can be anything really. We use this unit of measurement to mark of the rest of the numbers on the line. Now this line, the origin, the positive direction, and the unit of measurement define what is called a coordinate line or sometimes a real line.

With each real number we can now associate a point on the line as fo1lows:



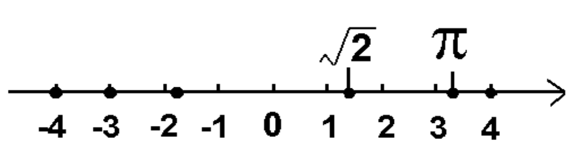
- Associate the origin with the number 0.
- Associate with each positive number *r* the point that is a distance of *r* units (this is the unit we chose, say 1 cm) in the positive direction from the origin.
- Associate with each negative number '- *r*' the point that is a distance of *r* units in the negative direction from the origin.

The real number corresponding to a point on the line is called the coordinate of the point.

Example 1:

In Figure we have marked the locations of the points with coordinates -4, -3, -1.75, -0.5, π , $\sqrt{2}$ and 4. The locations of π and $\sqrt{2}$ which are approximate, were obtained from their decimal approximations,

π = 3.14 and $\sqrt{2}$ = 1.41



It is evident from the way in which real numbers and points on a coordinate line are related that each real number corresponds to a single point and each point corresponds to a single real number. To describe this fact we say that the real numbers and the points on a coordinate line are in one-to-one correspondence.

Order Properties

In mathematics, there is an idea of ORDER of a SET. We won't go into the general concept, since that involves SET THEORY and other high level stuff. But we will define the ORDER of the real number set as follows:

For any two real numbers a and b, if b-a is positive, then we say that b > a or that a < b.

Here I will assume that we are all comfortable working with the symbol "<" which is read as "less than" and the symbol ">" which is read as "greater than." I am assuming this because this stuff was covered in algebra before Calculus. So with this in mind we can write the above statement as

<	>
less than	greater than

If b - a is positive, then we say that $b > a$ or that $a < b$.*A* statement involving < or > are called an INEQUALITY .Note that the inequality $a < b$ can also be expressed as $b > a$.

So ORDER of the real number set in a sense defines the SIZE of a real number relative to another real number in the set. The SIZE of a real number *a* makes sense only when it is compared with another real *b*. So **the ORDER tells you how to "ORDER" the numbers in the SET and also on the COORDINATE LINE!**

A little more about inequalities. The inequality $a \le b$ is defined to mean that either $a < b$ or $a = b$.

So there are two conditions here. For example, the inequality $2 \le 6$ would be read as 2 is less than or it is equal to 6. We know that it's less than 6, so the inequality is true. SO IF ONE OF THE CONDITIONS IS TRUE, THEN THE INEQUALITY WILL BE TRUE, We can say a similar thing about. The expression $a < b < c$ is defined to mean that $a < b$ and $b < c$. It is also read as "b is between a and c".

As one moves along the coordinate line in the positive direction, the real numbers increase in size. In other words, the real numbers are ordered in an ascending manner on the number line, just as they are in the SET of REAL NUMBERS. So that on a horizontal coordinate line the inequality $a < b$ implies that *a* is to the left of *b,* and the inequality $a < b < c$ implies that *a* is to the left of *b* and *b* is to the left of c.

The symbol $a < b < c$ means $a < b$ and $b < c$. I will leave it to the reader to deduce the meanings of such symbols as $\le$ and $\ge$.

Here is an example of INEQUALITIES.

$$a \le b < c$$
$$a \le b \le c$$
$$a < b < c < d$$

Example:

Correct Inequalities

$$3 < 8, \quad -7 < 1.5, \quad -12 \le x, \quad 5 \le 5$$
$$0 \le 2 \le 4, \quad 8 \ge 3, \quad 1.5 > -7$$
$$-\pi > -12, \quad 5 \ge 5, \quad 3 > 0 > -1 > -3$$

Some incorrect inequalities are:

$$2 \ge 4, \quad \pi \le 0, \quad 5 < -3$$

REMARK: To distinguish verbally between numbers that satisfy $a \ge 0$ and those that satisfy a > 0, we shall call *a nonnegative* if $a \ge 0$ and *positive* if $a > 0$.

Thus, a nonnegative number is either positive or zero.

The following properties of inequalities are frequently used in calculus. We omit the proofs, but will look at some examples that will make the point.

THEOREM 1.1.1

$a)$ *If* $a < b$ *and* $b < c$, *then* $a < c$

$b)$ *If* $a < b$ *and* $a + c < b + c$, *then* $a - c < b - c$

$c)$ *If* $a < b$ *and* $ac < bc$, *when* c *is positive*
and $ac > bc$ *when* c *is negative.*

$d)$ *If* $a < b$ *and* $c < d$, *then* $a + c < b + d$

$e)$ *If* a *and* b *are both positive or both negative*
and $a < b$ *then* $\dfrac{1}{a} > \dfrac{1}{b}$

REMARK These five properties remain true if < and > are replaced by $\le$ and $\ge$

INTERVALS

We saw a bit about sets earlier. Now we shall assume in this text that you are familiar with the concept of a set and fully understand the meaning of the following symbols. However, we will give a short explanation of each.

Given two sets A and B

$a \in A$: *a is an element of the set A,*

$$2 \in \{1, 2, 3, 4\}$$

$a \notin A : a \text{ is NOT an element of the set } A$

$$5 \notin \{1, 2, 3, 4\}$$

$\varnothing$ represents the Empty set, or the set that contains nothing.

$A \cup B$ represents the SET of all the elements of the Set A and the Set B taken together.

Example:

A = {1,2,3,4}, B = {1,2,3,4,5,6,7}, then, A $\cup$ B = {1,2,3,4,5,6,7}

$A \cap B$ represents the SET of all those elements that are in Set A AND in Set B.

Example:

A = {1,2,3,4}, B = {1,2,3,4,5,6,7}, then A $\cap$ B = {1,2,3,4}

A = B means the A is exactly the same set as B

Example:

A = {1,2,3,4} and B = {1,2,3,4}, then A = C

and $A \subset B$ means that the Set A is contained in the Set B. Recall the example we did of the Set of all politicians!

{George Bush, Tony Blair} $\subset$ {George Bush, Toney Blair, Ronald Reagoan}

One way to specify the idea of a set is to list its members between braces. Thus, the set of all positive integers less than 5 can be written as

{ 1, 2, 3, 4}

and the set of all positive even integers can be written as

{ 2, 4, 6, …}

where the dots are used to indicate that only some of the members are explicitly and the rest can be obtained by continuing the pattern. So here the pattern is that the set consists of the even numbers, and the next element must be 8, then 10, and then so on. When it is inconvenient or impossible to list the members of a set, as would be if the set is infinite, then one can use the set- builder notation. This is written as

{ x : __________ }

which is read as **"the set of all *x* such that ______"** , In place of the line, one would state a property that specifies the set, Thus,

{ x : x is real number and 2 < x < 3}

is read**, "the set of all *x* such that *x* is a real number and 2 < *x* < 3,"** Now we know by now that

$2 < x < 3$ means that all the x between 2 and 3.
This specifies the "description of the elements of the set" This notation describes the set, without actually writing down all its elements.
When it is clear that the members of a set are real numbers, we will omit the reference to this fact. So we will write the above set as Intervals.
We have had a short introduction of Sets. Now we look particular kind of sets that play a crucial role in Calculus and higher math. These sets are sets of real numbers called ***intervals.*** What is an interval?

$$\{ x: 2< x< 3\}$$

Well, *geometrically*, an interval is a line segment on the co-ordinate line. S if a and b are real numbers such that $a < b$, then an interval will be just the line segment joining a and b.

But if things were only this simple! Intervals are of various types. For example, the question might be raised whether a and b are part of the interval? Or if a is, but b is not?? Or maybe both are?

Well, this is where we have to be technical and define the following:

The closed interval from a to b is denoted by [a, b] and is defined as

$$[a,b] = \{x : a \le x \le b\}$$

Geometrically this is the line segment

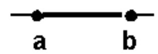
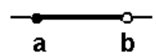
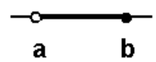
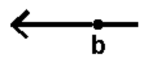
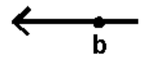
So this includes the numbers a and b, a and b a are called the END- POINTS of the interval.

The *open* interval from a to b is denoted by and is defined by

$$(a, b) = \{ x: a < x < b\}$$

This excludes the numbers a and b. The square brackets indicate that the end points are included in the interval and the parentheses indicate that they are not.

Here are various sorts of intervals that one finds in mathematics. In this picture, the geometric pictures use solid dots to denote endpoints that are included in the interval and open dots to denote endpoints that are not.

INTERVAL NOTATION	SET NOTATION	GEOMETRIC PICTURE	CLASSIFICATION
(a,b)	{x : a< x< b}		Finite ; open
[a,b]	{x : a ≤ x≤ b}		Finite ; closed
[a,b)	{x : a ≤ x< b}		Finite ; half-open
(a,b]	{x : a< x ≤ b}		Finite ; half open
(- ∞,b]	{x : x ≤ b}		Infinite ; closed
(-∞,b)	{x : x < b}		Infinite ; open

As shown in the table, an interval can extend indefinitely in either the positive direction, the negative direction, or both. The symbols $-\infty$ (read "negative infinity") and $+\infty$ (read , 'positive infinity' ') do not represent numbers: the $+\infty$ indicates that the interval extends indefinitely in the positive direction, and the $-\infty$ indicates that it extends indefinitely in the negative direction.

An interval that goes on forever in either the positive or the negative directions, or both, on the coordinate line or in the set of real numbers is called an INFINITE interval. Such intervals have the symbol for infinity at either end points or both, as is shown in the table

[a, +∞)	{ x:x ≥ a}		Infinite; closed
(a, +∞)	{ x:x > a}		Infinite; open
(+∞, +∞)	{ x:x is a real number}		Infinite; open and closed

An interval that has finite real numbers as end points are called finite intervals.

A finite interval that includes one endpoint but not the other is called *half-open* (or sometimes *half-closed).*

$[a,+\infty), (a,-\infty), (-\infty,b], (-\infty,b)$

Infinite intervals of the form $[a,+\infty)$ and $(-\infty,b]$ are considered to be closed because they contain their endpoint. Those of the form $(a,-\infty)$ and $(-\infty,b)$ has no endpoints; it is regarded to be both open and closed. As one of my Topology Instructors used to say:

“A set is not a DOOR! It can be OPEN, it can be CLOSED, and it can be OPEN and CLOSED!!

Let's remember this fact for good!” Let's look at the picture again for a few moments and digest the information. PAUSE 10 seconds.

SOLVING INEQUALITIES

We have talked about Inequalities before. Let's talk some more. First Let's look at an inequality involving and unknown quantity, namely *x*. Here is one: $x < 5, x = 1$, is a solution of this inequality as 1 makes it true, but $x = 7$ is not. So the set of all solutions of an inequality is called its solution set. The solution set of $x < 5$ will be

It is a fact, though we wont prove this that if one does not multiply both sides of an inequality by zero or an expression involving an unknown, then the operations in Theorem 1.1.1 will not change the solution set of the inequality. The process of finding the solution set of an inequality is called *solving* the Inequality.

Let's do some fun stuff, like some concrete example to make things a bit more focused

Example 4.

Solve

$$3+7x \le 2x-9$$

Solution.

We shall use the operations of Theorem 1.1.1 to isolate *x* on one side of the inequality

$$7x \le 2x-12 \qquad \textit{Subtracting } 3 \textit{ from both sides}$$
$$5x \le -12 \qquad \textit{Subtracting } 2x \textit{ from both sides}$$
$$x \le -12/5 \qquad \textit{Dividing both sides by } 5$$

Because we have not multiplied by any expressions involving the unknown *x,* the last inequality has the same solution set as the first. Thus, the solution set is the interval shown in Figure 1.1.6.

Example

Solve $7 \le 2-5x < 9$

Solution ; The given inequality is actually a combination of the two inequalities

$$7 \le 2-5x \textit{ and } 2-5x < 9$$

We could solve the two inequalities separately, then determine the value of x that satisfy both by taking the intersection of the solution sets , however, it is possible to work with the combined inequality in this problem

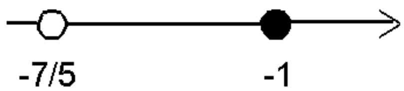
$5 \le -5x < 7$ *Subtracting 2 from both sides*

$-1 \ge x < -7/5$ *Dividing by* -5 *inequality symbols reversed*

$-7/5 < x \le -1$ Re*writing with smaller number first*

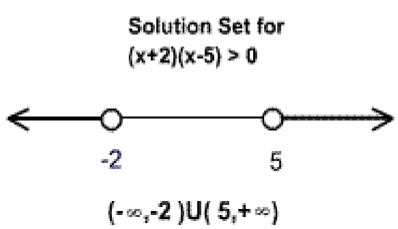
$(-7/5, -1]$

Thus the solution set is interval shown the figure



Example

Similarly, you can find



Lecture # 2

Absolute Value

In this lecture we shall discuss the notation of Absolute Value. This concept plays an important role in algebraic computations involving radicals and in determining the distance between points on a coordinate line.

Definition

The absolute vale or magnitude of a real number **a** is denoted by **|a|** and is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \textit{that is, a is non} - \textit{negative} \\ -a & \text{if } a < 0, \textit{that is, a is positive.} \end{cases}$$

Technically, 0 is considered neither positive, nor negative in Mathematics. It is called a non-negative number. Hence whenever we want to talk about a real number a such that **a** $\geq$ 0, we call **a** non-negative, and positive if **a >** 0.

Example

$|5| = 5$, $\left|-\frac{4}{7}\right| = -(-\frac{4}{7}) = \frac{4}{7}$, $|0| = 0$

since 5>0 since -4/7 < 0 since 0 $\geq$ 0

Note that the effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is non-negative. Thus, |a| is a non-negative number for all values of a and $-|a| \leq a \leq |a|$, if 'a' is itself is negative, then '–a' is positive and '+a' is negative.

$$a + b \geq 0 \; or \; a + b < 0$$

$$a + b = |\; a + b \;|$$
$$|a + b| \leq |a| + |b|$$

$$|a + b| = -(a + b)$$

Caution: Symbols such as +a and –a are deceptive, since it is tempting to conclude that +a is positive and –a is negative. However this need not be so, since a itself can represent either a positive or negative number. In fact , if a itself is negative, then –a is positive and +a is negative.

Example:

Solve $|x-3|=4$

Solution:

Depending on whether x-3 is positive or negative , the equation $|x-3| = 4$ can be written as

$x-3 = 4$ or $x-3 = -4$

Solving these two equations give

$x=7$ and $x=-1$

Example

Solve $|3x-2|=|5x+4|$

Because two numbers with the same absolute value are either equal or differ only in sign, the given equation will be satisfied if either

$$3x-2=5x+4$$
$$3x-5x=4+2$$
$$-2x=6$$
$$x=-3$$

Or

$$3x-2=-(5x+4)$$
$$3x-2=-5x-4$$
$$3x+5x=-4+2$$
$$x=-\frac{1}{4}$$

Relationship between Square Roots and Absolute Values :

Recall that a number whose square is **a** is called a square root of **a**.

In algebra it is learned that every positive real number **a** has two real square roots, one positive and one negative. The positive square root is denoted by $\sqrt{a}$ For example, the number 9 has two square roots, -3 and 3. Since 3 is the positive square root, we have $\sqrt{9} = 3$.

In addition, we define $\sqrt{0} = 0$.

It is common error to write $\sqrt{a^2} = a$. Although this equality is correct when a is nonnegative, it is false for negative a. For example, if a=-4, then $\sqrt{a^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq a$

The positive square root of the square of a number is equal to that number.

A result that is correct for all a is given in the following theorem.

Theorem:

For any real number **a**, $\sqrt{a^2} = |a|$

Proof :

Since $a^2 = (+a)^2 = (-a)^2$, the number +a and –a are square roots of a^2. If $a \geq 0$, then +a is nonnegative square root of a2, and if $a < 0$, then -a is nonnegative square root of a^2. Since $\sqrt{a^2}$ denotes the nonnegative square root of a^2, we have

$$\text{if} \quad \begin{array}{ll} \sqrt{a^2} = +a & \textit{if } a \geq 0 \\ \sqrt{a^2} = -a & \textit{if } a < 0 \end{array}$$

That is, $\sqrt{a^2} = |a|$

Properties of Absolute Value

Theorem

If **a** and **b** are real numbers, then

(a) |-a| = |a|, a number and its negative have the same absolute value.

(b) |ab| = |a| |b|, the absolute value of a product is the product of absolute values.

(c) |a/b| = |a|/|b|, the absolute value of the ratio is the ratio of the absolute values

Proof (a) :

$$|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$$

Proof (b) :

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a||b|$$

This result can be extended to three or more factors. More precisely, for any n real numbers, a1,a2,a3,……an, it follows that

$$|a_1a_2\ldots a_n| = |a_1|\ |a_2|\ \ldots|a_n|$$

In special case where $a_1, a_2, \ldots, a_n$ have the same value, a, it follows from above equation that

$$|a^n| = |a|^n$$

Example

(a)

|-4|=|4|=4

(b)

|(2)(-3)|=|-6|=6=|2| |-3|=(2)(3)=6

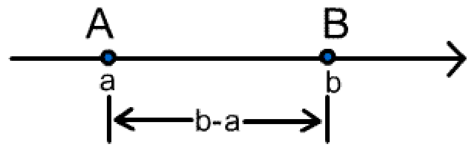
(c)

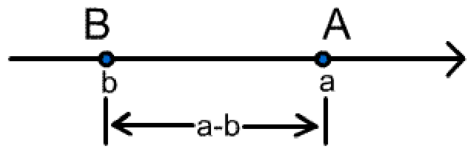
|5/4|=5/4=|5|/|4|=5/4

Geometric Interpretation Of Absolute Value

The notation of absolute value arises naturally in distance problems, since distance is always nonnegative. On a coordinate line, let A and B be points with coordinates a and b, the distance d between A and B is

$$d = \begin{cases} b-a & if\ a<b \\ a-b & if\ a>b \\ 0 & if\ a=b \end{cases}$$





As shown in figure b-a is positive, so b-a=|b-a| ; in the second case b-a is negative, so

$$a-b = -(b-a) = |b-a| .$$

Thus, in all cases we have the following result:

Theorem

(Distance Formula)

If A and B are points on a coordinate line with coordinates a and b, respectively, then the distance d between A and B is

$$\mathbf{d = |b-a|}$$

This formula provides a useful geometric interpretation

Of some common mathematical expressions given in table here

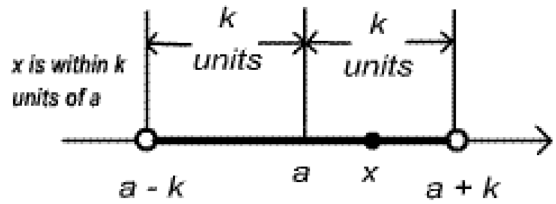
Table

EXPRESSION	GEOMETRIC INTERPRETATION ON A COORDINATE LINE
$\lvert x-a \rvert$	The distance between x and a
$\lvert x+a \rvert$	The distance between x and -a
$\lvert x \rvert$	The distance between x and origin

Inequalities of the form $|x-a|<k$ and $|x-a|>k$ arise often, so we have summarized the key facts about them here in following table

TABLE 1.2.2(a)

$|x - a| < k \quad (k > 0)$

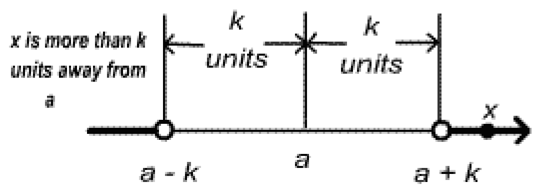


Alternative Form $-k < x - a < k$

Solution Set $(a - k, a + k)$

TABLE 1.2.2(a)

$|x - a| > k \quad (k > 0)$



Alternative Form $x-a<-k$ or $x-a>k$

Solution Set $(-\infty, a - k) \cup (a + k, +\infty)$

Example

Solve $|x-3| < 4$

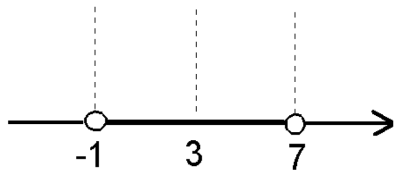
Solution: This inequality can be written as

$-4 < x-3 < 4$

adding 3 throughout we get

$-1 < x < 7$

This can be written in interval notation as $(-1,7)$



Example

Solve $\left|x+4\right| \geq 2$

Solution: The given inequality can be written as

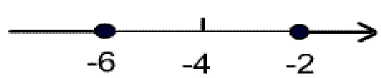
$$\begin{cases} x+4 \leq -2 \\ or \\ x+4 \geq 2 \end{cases}$$

or simply

$$\begin{cases} x \leq -6 \\ or \\ x \geq -2 \end{cases}$$

Which can be written in set notation as

$$(-\infty, -6] U [-2, +\infty)$$



(-∞, -6] U [-2, +∞)

The Triangle Inequality

It is not generally true that $|a+b| = |a| + |b|$

For example , if $a = 2$ *and* $b = -3$, *then* $a + b = -1$

so that $|a+b| = |-1| = 1$

whereas

$|a| = |b| = |2| + |-3| = 2 + 3 = 5$

so $|a+b| \neq |a| + |b|$

It is true, however, that the absolute value of a sum is always less than or equal to the sum of the absolute values. This is the content of the following very important theorem, known as the triangle inequality . This TRIANGLE INEQUALITY is the essence of the famous HISENBURG UNCERTAINITY PRINCIPLE IN QUANTUM PHYSICS, so make sure you understand it fully.

THEOREM 1.2.5

(Triangle Inequality)

If a and b are any real numbers, then

$|a+b| \leq |a| + |b|$

PROOF

Remember the following inequalities we saw earlier .

$-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$

Let's add these two together. We get

$$-|a| \leq a \leq |a| \quad + \quad -|b| \leq b \leq |b|$$
$$= \; (-|a|) + (-|b|) \leq a + b \leq |a| + |b| \qquad \text{(B)}$$

Since a and b are real numbers, adding them will also result in a real number. Well, there are two types of real numbers. What are they?? Remember!!!!! They are either > = 0, or they are < 0! Ok!!

SO we have

$a + b \geq 0 \; or \; a + b < 0$

In the first of these cases where $a + b \geq 0$ certainly $a + b = |\, a + b \,|$

by definition of absolute value. so the right-hand inequality in (B) gives

$|a+b| \leq |a| + |b|$

In the second case

$$|a+b| = -(a+b)$$

But this is the same as

$$a+b = -|a+b|$$

So the left-hand inequality in (B) can be written as

$$-(\ |\ a\ |+|\ b\ |\ ) \leq\ -|\ a+b\ |$$

Multiplying both sides of this inequality by - 1 give

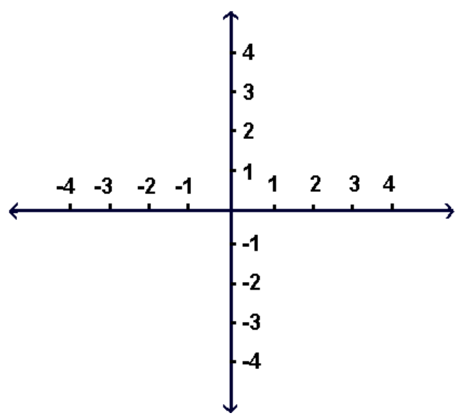
$$|a+b| \leq |a|+|b|$$

Lecture # 3

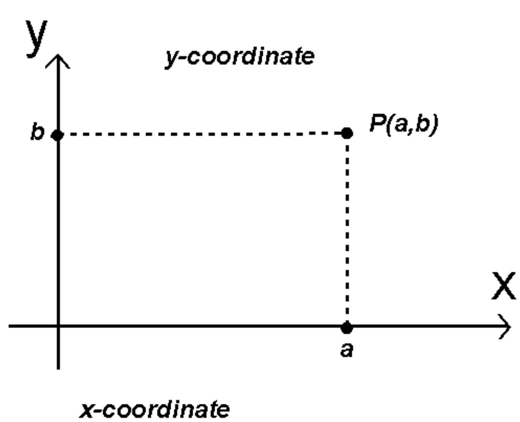
In this lecture we will discuss

- **Graphs in the coordinate plane.**
- **Intercepts.**
- **Symmetry Plane.**

We begin with the Coordinate plane. Just as points on a line can be placed in one-to-one correspondence with the real numbers, so points in the PLANE can be placed in one-to-one correspondence with pairs of real numbers. What is a plane?

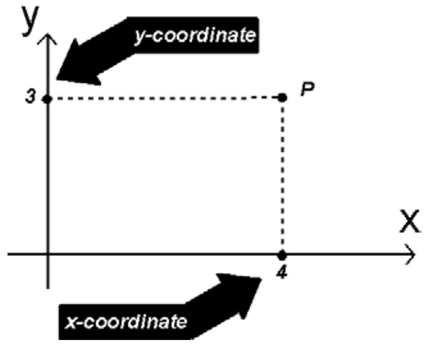


A PLANE is just the intersection of two COORDINATE lines at 90 degrees. It is technically called the COORDINATE PLANE, but we will call it the plane also whenever it is convenient. Each line is a line with numbers on it, so to define a point in the PLANE, we just read of the corresponding points on each line. For example I pick a point in the plane

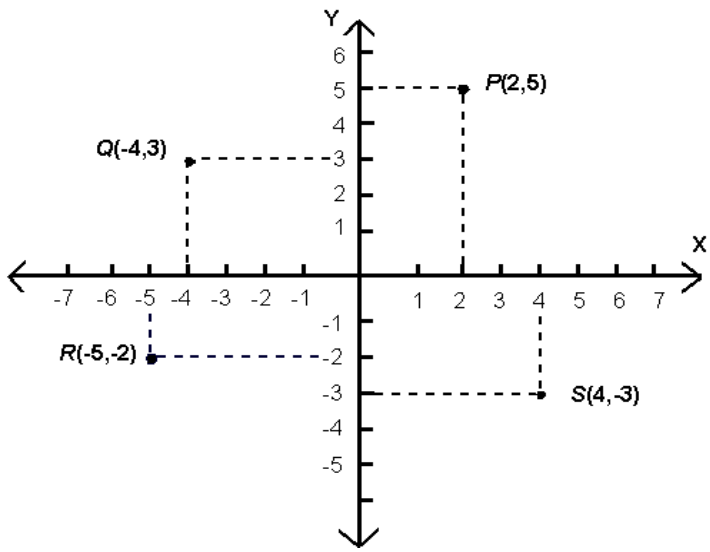


By an ordered pair of real numbers we mean two real numbers in an assigned order. Every point P in a coordinate plane can be associated with a unique ordered pair of real numbers by drawing two lines through P, one perpendicular to the x-axis and the other to the y-axis.

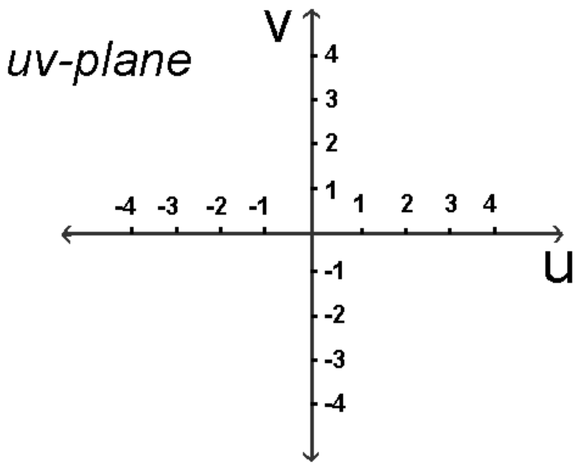
For example if we take (a,b)=(4,3), then on coordinate plane

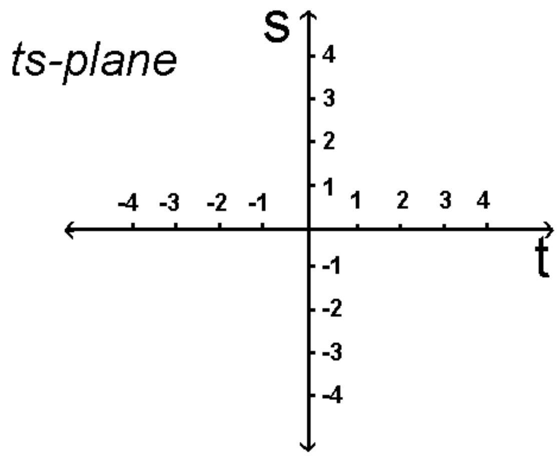


To plot a point *P(a, b)* means to locate the point with coordinates *(a, b)* in a coordinate plane. For example, In the figure below we have plotted the points P(2,5), Q(-4,3), R(-5,-2), and S(4,-3).Now this idea will enable us to visualise algebraic equations as geometric curves and, conversely, to represent geometric curves by algebraic equations.

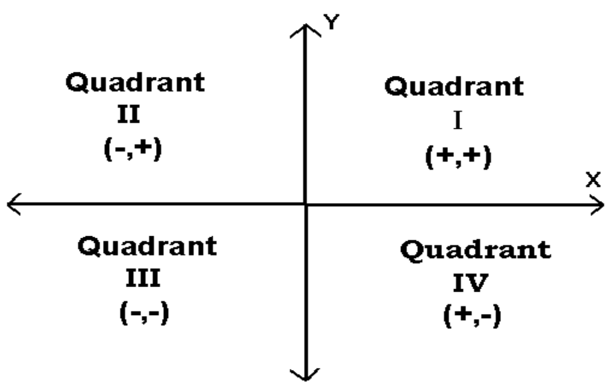


Labelling the axes with letters *x* and *y* is a common convention, but any letters may be used. If the letters x and y are used to label the coordinate axes, then the resulting plane is also called an *xy-plane.* In applications it is common to use letters other than *x* and *y* to label coordinate axes. Figure below shows a uv-plane and a ts-plane. The first letter in the name of the plane refers to the horizontal axis and the second to the vertical axis.





Here is another terminology. The COORDINATE PLANE and the ordered pairs we just discussed is together known as the RECTANGULAR COORDINATE SYSTEM. In a rectangular coordinate system the coordinate axes divide the plane into four regions called *quadrants*. These are numbered counter clockwise with Roman numerals as shown in the Figure below.



Consider the equations

$$5xy = 2$$
$$x^2 + 2y^2 = 7$$
$$y = x^3 - 7$$

We define a solution of such an equation to be an ordered pair of real numbers(a,b) so that the equation is satisfactory when we substitute x=a and y=b.

Example 1

The pair (3,2) is a solution of

$$6x - 4y = 10$$

since this equation is satisfied when we substitute x = 3 and y = 2. That is

$$6(3)-4(2)=10$$

which is true!!

However, the pair (2,0) is not a solution, since

$$6(2)-4(0)=18\neq 10$$

We make the following definition in order to start seeing algebraic objects geometrically.

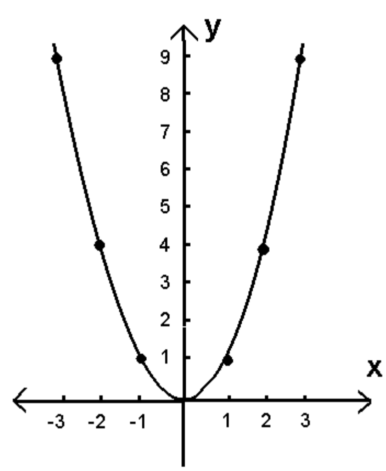
Definition. The GRAPH of an equation in two variables x and y is the set of all points in the xy-plane whose coordinates are members of the solution set of the equation.

Example 2

Sketch the graph of $y=x^2$

x	$y = x^2$	(x , y)
0	0	(0 , 0)
1	1	(1 , 1)
2	4	(2 , 4)
3	9	(3 , 9)
-1	1	(-1 , 1)
-2	4	(-2 , 4)
-3	9	(-3 , 9)

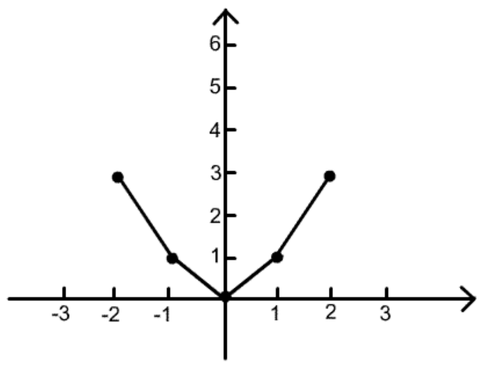
When we plot these on the xy-plane and connect them, we get this picture of the graph



IMPORTANT REMARK.

It should be kept in mind that the curve in above is only an approximation to the graph of $y = x^2$.

When a graph is obtained by plotting points, whether by hand, calculator, or computer, there is no guarantee that the resulting curve has the correct shape. For example, the curve in the Figure here pass through the points tabulated in above table.

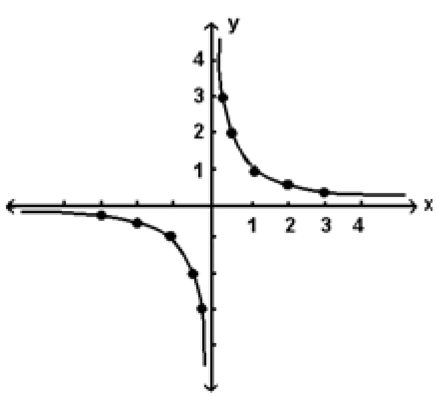


Example : Sketch the graph of

y = 1/x

x	y = 1/x	(x , y)
1/3	3	(1/3 , 3)
1/2	2	(1/2 , 2)
1	1	(1 , 1)
2	1/2	(2 , 1/2)
3	1/3	(3 , 1/3)
-1/3	-3	(-1/3 , -3)
-1/2	-2	(-1/2 , -2)
-1	-1	(-1 , -1)
-2	-1/2	(-2, -1/2)
-3	-1/3	(-3 , -1/3)

Because 1/x is undefined when x=0, we can plot only points for which x=0



INTERCEPTS

Points where a graph intersects the coordinate axes are of special interest in many problems. As illustrated before, intersections of a graph with the x-axis have the form (a, 0) and intersections with the y-axis have the form (0, b). The number a is called an x-intercept of the graph and the number b a y-intercept.

Example: Find all intercepts of

(a) $3x + 2y = 6$

(b) $x = y^2 - 2y$

(c) $y = 1/x$

Solution

$3x + 2y = 6$

x-intercepts

Set $y = 0$ and solve for x

$3x = 6$ or **$x = 2$**

is the required x-intercept

$3x + 2y = 6$

y-intercepts

Set $x = 0$ and solve for y

$2y = 6$ or **$y = 3$**

is the required y-intercept

Similarly you can solve part (b), the part (c) is solved here

$y = 1/x$

x-intercepts

Set $y = 0$

$1/x = 0 \Rightarrow$ **x** is undefined

No x-intercept

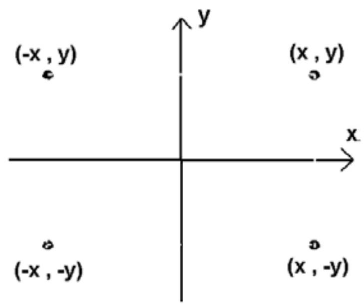
y-intercepts

Set $x = 0$

$y = 1/0 \Rightarrow$ **y** is undefined

No y-intercept

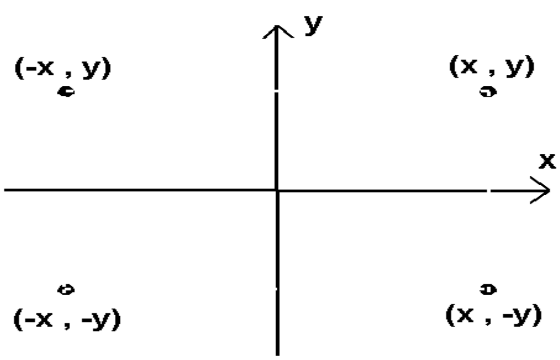
In the following figure, the points (x,y),(-x,y),(x,-y) and (-x,-y) form the corners of a rectangle.

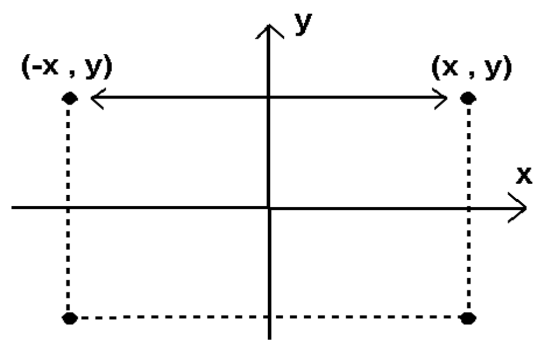


SYMMETRY

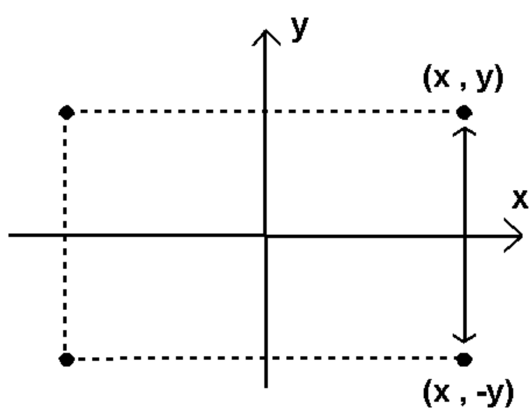
Symmetry is at the heart of many mathematical arguments concerning the structure of the universe, and certainly symmetry plays an important role in applied mathematics and engineering fields. Here is what it is.

As illustrated in Figure the points





Symmetric about the y-axis



Symmetric about the x-axis

(x, y), $(-x, y)$, $(x, -y)$ *and* $(-x, -y)$ form the corners of a rectangle.

For obvious reasons, the points (x,y) and (x,-y) are said to be symmetric about the x-axis and the points

(x, y) and (-x, y) are symmetric about the y-axis and the points (x, y) and (-x, -y) symmetric about the origin.

SYMMETRY AS A TOOL FOR GRAPHING

By taking advantage of symmetries when they exist, the work required to obtain a graph can be reduced considerably.

Example 9

Sketch the graph of the equation

$$y = \frac{1}{8}x^4 - x^2$$

Solution. The graph is symmetric about the y-axis since substituting $-x$ for x yields $y = \frac{1}{8}(-x)^4 - (-x)^2$

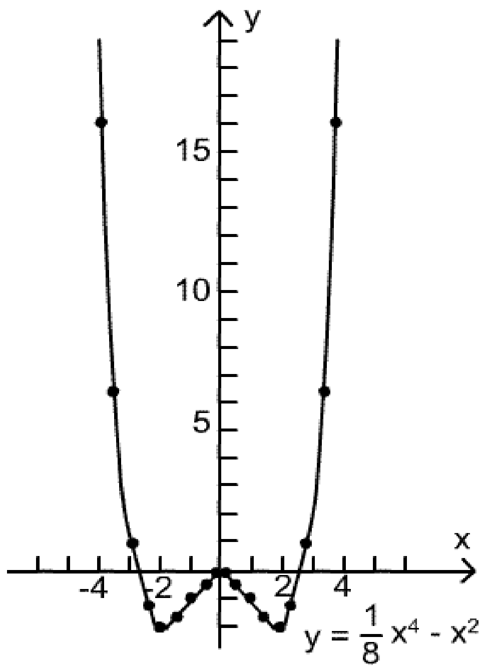
which simplifies to the original equation.

As a consequence of this symmetry, we need only calculate points on the graph that lies in the right half of the xy-plane (x >= 0).

The corresponding points in the left half of the xy-plane (x <= 0).

can be obtained with no additional computation by using the symmetry. So put only positive x-values in given equation and evaluate corresponding y-values.

Since graph is symmetric about y-axis, we will just put negative signs with the x-values taken before and take the same y-values as evaluated before for positive x-values.

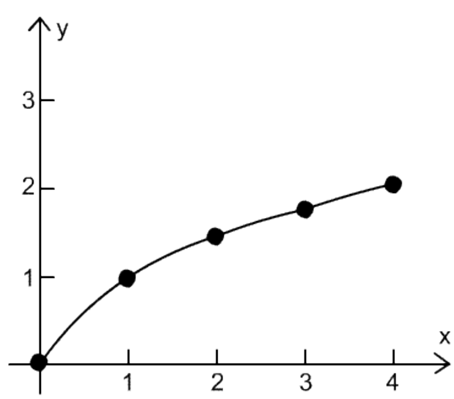


Example 10 Sketch the graph of $x = y^2$

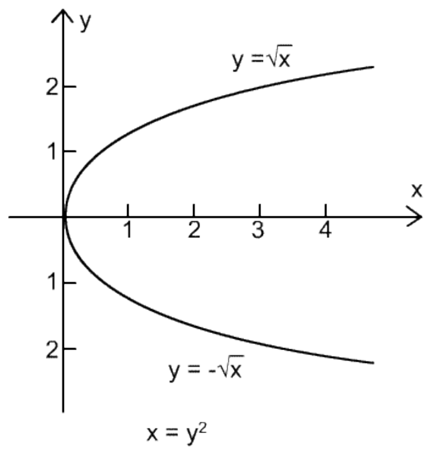
Solution. If we solve $x = y^2$ for y in terms of x, we obtain two solutions,

$y = \sqrt{x}$ and $y = -\sqrt{x}$

The graph of $y = \sqrt{x}$ is the portion of the curve $x = y^2$ that lies above or touches the x-axis (since $y = \sqrt{x} \geq 0$), and the graph of $y = -\sqrt{x}$ is the portion that lies below or touches the x-axis (since $y = -\sqrt{x} \leq 0$). However, the curve $x = y^2$ is symmetric about the x-axis because substituting $-y$ for *y* yields $x = (-y)^2$ which is equivalent to the original equation. Thus, we need only graph $y = \sqrt{x}$ and then reflect it about the x-axis to complete the graph $x = y^2$.



is the graph of the function. $y = \sqrt{x}$



is the required graph of the function.

Lecture # 4

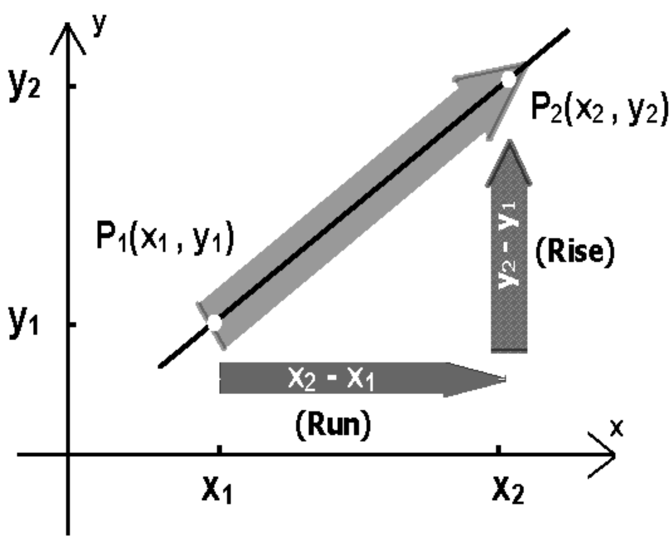
Lines

In this section we shall discuss ways to measure the "steepness" or "slope" of a line in the plane. The ideas we develop here will be important when we discuss equations and graphs of straight lines. We will assume that you have sufficient understanding of trigonometry.

Slope

In surveying, slope of a hill is defined to be the ratio of its rise to its run. We shall now show how the surveyor's notion of slope can be adapted to measure the steepness of a line in the xy-plane.

Consider a particle moving left to right along a non vertical line segment from a point P1(x1,y1) to a point P2(x2,y2). As shown in the figure below,



the particle moves y2-y1 units in the y-direction as it travels x2-x1 units in the positive x-direction. The vertical change y2-y1 is called the rise, and the horizontal change x2-x1 the run. By analogy with the surveyor's notion of slope we make the following definition.

Definition 1.4.1

If P1(x1,y1) and P2(x2,y2) are points on a non-vertical line, then the slope m of the line is defined by

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

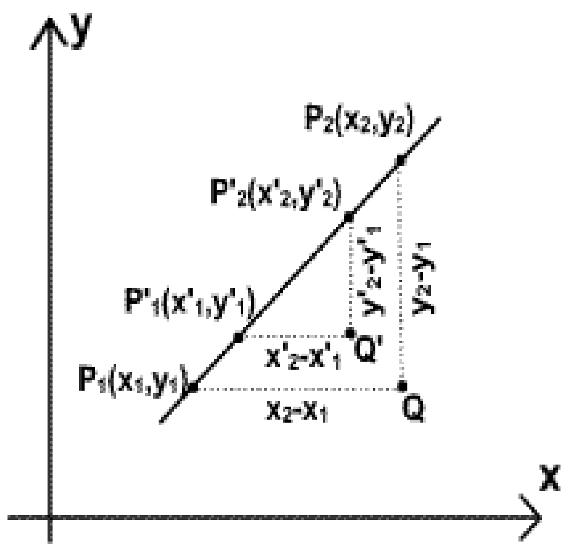
So the slope is the ratio of the vertical distance and the horizontal distance between two points on a line. We make several observations about Definition 1.4.1.

Definition 1.4.1 does not apply to vertical lines. For such lines we would have

$x_2 = x_1$

so (1) would involve a division by zero. The slope of a vertical line is UNDEFINED. Speaking informally, some people say that a vertical line has infinite slope . When using formula in the definition to calculate the slope of a line through two points, it does not matter which point is called P1 and which one is called P2, since reversing the points reverses the sign of both the numerator and denominator of (1), and hence has no effect on the ratio. Any two distinct points on a non-vertical line can be used to calculate the slope of the line that is, the slope m computed from any other pair of distinct points P1 and P2 on the line will be the same as the slope m' computed from any other pair of distinct points P'1 and P'2 on the line. All this is shown in figure below

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'} = m'$$



Example

In each part find the slope of the line through

(a) the points (6,2) and (9,8) (b) the points (2,9) and (4,3)

(c) the points (-2,7) and (5,7)

Solution:

a) $m = \frac{8-2}{9-6} = \frac{6}{3} = 2$

b) $m = \frac{3-9}{4-2} = \frac{-6}{2} = -3$

c) $m = \frac{7-7}{5-(-2)} = \frac{0}{7} = 0$

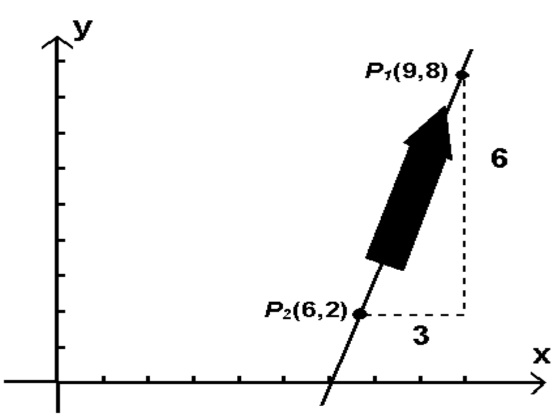
d) **Interpretation of slope**

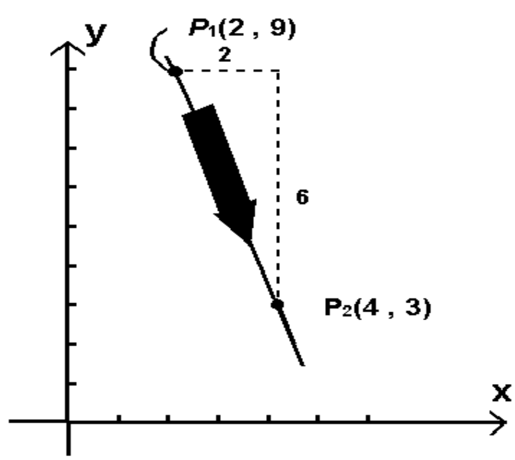
e) Since the slope m of a line is the rise divided by the run, it follows that

f) rise = m . run

g) so that as a point travels left to right along the line, there are m units of rise for each unit of run. But the rise is the change in y value of the point and the run is the change in the x value, so that the slope m is sometimes called the **rate of change of y with respect to x** along the line.

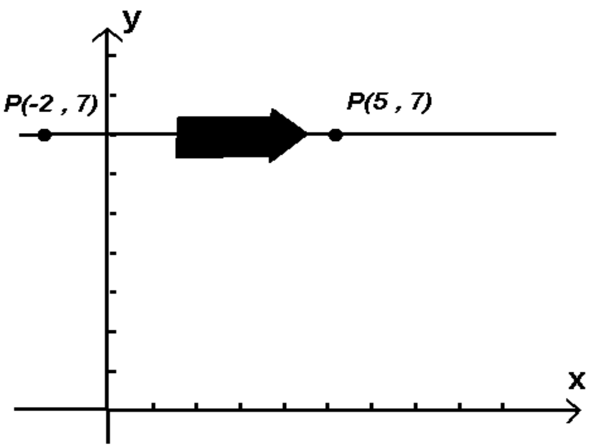
h) As illustrated in the last example, the slope of a line can be positive, negative or zero.





m=2
Traveling left to right, a point on the line rises two units for each unit it moves in the positive x-direction.

m = -3
Traveling left to right, a point on the line falls three units for each unit it moves in the positive x-direction.



m = 0
Traveling left to right, a point on the line neither rises nor falls

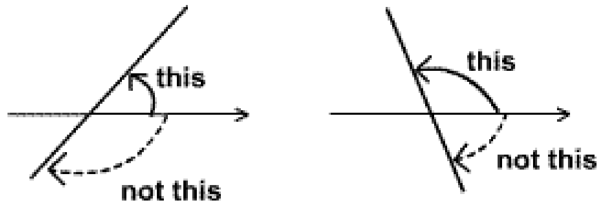
A positive slope means that the line is inclined upward to the right, a negative slope means that it is inclined downward to the right, and a zero slope means that the line is horizontal.

Angle of Inclination

If equal scales are used on the coordinate axis, then the slope of a line is related to the angle the line makes with the positive x-axis.

Definition 1.4.2

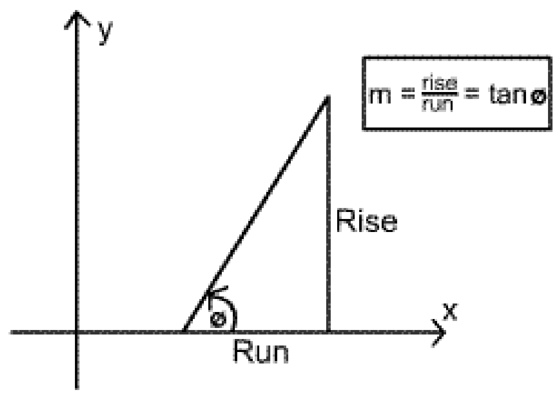
For a line L not parallel to the x-axis, the angle of inclination is the smallest angle measured counterclockwise from the direction of the positive x-axis to L (shown in figure below). For a line parallel to the x-axis, we take $\varnothing = 0$.



Angles of inclination are measured counterclockwise from the x-axis

In degree measure the angle of inclination satisfies $0^0 \leq \phi \leq 180^0$ and in radian measure it satisfies $0 \leq \phi \leq \pi$.

The following theorem, suggested



by the figure at right, relates the

Slope of a line to its angle of Inclination.

Theorem 1.4.3

For a nonvertical line, the slope m and angle of inclination $\varnothing$ are related by

$$m = \tan\varnothing \qquad (2)$$

If the line L is parallel to the *y*-axis, then $\varnothing = \frac{1}{2}\pi$

so $\tan\varnothing$ is undefined. This agrees with the fact that the slope m is undefined for vertical lines.

Example

Find the angle of inclination for a line of slope m = 1 and also for a line of slope m = -1.

Solution : If m = 1, then from (2)

$$\tan \varnothing = 1$$

$$\varnothing = \frac{1}{4}\pi$$

If m = -1, then from (2)

$\tan \varnothing = -1$, from this equality and the fact that $0 \le \varnothing \le \pi$ we obtain

$$\varnothing = \frac{3\pi}{4}.$$

Parallel and Perpendicular Lines

Theorem 1.4.4

Let L1 and L2 be non-vertical lines with slopes m1 and m2, respectively

(a) The lines are parallel if and only if

$$m_1 = m_2 \quad \ldots\ldots \quad (3)$$

(b) The lines are perpendicular if and only if

$$m_1 m_2 = -1 \quad \ldots\ldots\ldots \quad (4)$$

Basically, if two lines are parallel, then they have the same slope, and if they are perpendicular, then the product of there slopes is -1.

Formula (4) can be rewritten in the form

$$m_2 = -\frac{1}{m_1}$$

In words, this tells us that two non-vertical lines are perpendicular if and only if their slopes are NEGATIVE RECIPROCALS OF ONE ANOTHER

Example

Use slopes to show that the points A(I, 3), B(3, 7), and C(7, 5) are vertices of a right triangle.

Solution:

Slope through A and B = m1 = (7-3)/(3-1) = 2

Slope through B and C = m2 = (5-7)/(7-3) = - 1/2

Since m1m2 = (2)(-1/2)=-1

The line through A and B is perpendicular to the line through B and C; thus, ABC is a right triangle.

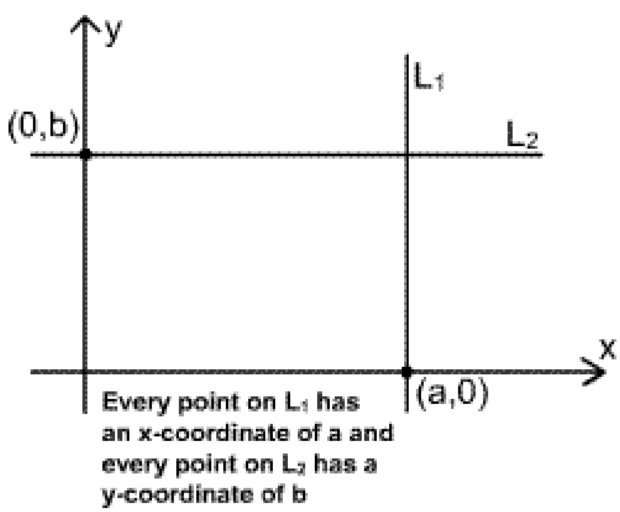
Equations of Lines

Lines Parallel to the Coordinate axes

We now turn to the problem of finding equations of lines that satisfy specified conditions.

The simplest cases are lines PARALLEL TO THE COORDINATE AXES: A line parallel to the *y*-axis intersects the *x*-axis at some point (a, 0).

This line consists precisely of those points whose *x*-coordinate is equal to a. Similarly, a line parallel to the x-axis intersects the y-axis at some point (0, b). This line consists precisely of those points whose y-coordinate is equal to b.



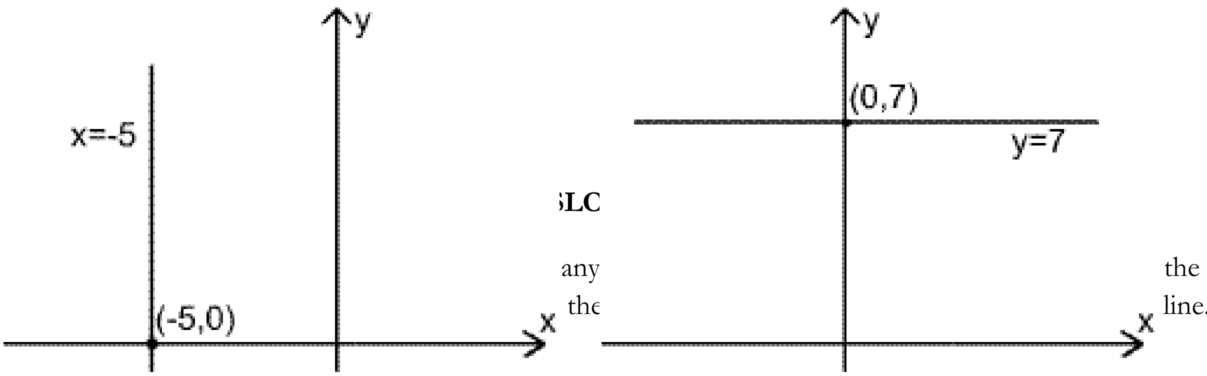
Theorem 1.4.5

The vertical line through (a,0) and the horizontal line through (0,b) are represented, respectively, by the equations

$x = a$ and $y = b$

Example

The graph of x = -5 is the vertical line through (-5,0) and the graph of y = 7 is the horizontal line through (0,7).



Let us see how we can find the equation of a non-vertical line L that passes through a point P1(x1,y1) and has slope m. If P(x,y) is any point on L, different from P1, then the slope m can be obtained from the points P(x,y) and P1(x1,y1); this gives

$$m = \frac{y - y_1}{x - x_1}$$

which gives

$$y - y_1 = m(x - x_1)$$

In summary, we have the following theorem.

Theorem 1.4.6

The line passing through P1(x1,y1) and having slope m is given by the equation

$$y - y_1 = m(x - x_1)$$

This is called the **point-slope** form of line.

Example

Write an equation for the line through the point (2,3) with slope

-3/2.

Solution:

We substitute x1=2, y1=3 and m=-3/2 into the point-slope equation and obtain

y-3 = -3/2 (x-2)   on simplification   y = -3/2 x + 6

Example

Write an equation for the line through the point (-2,-1) and (3,4).

Solution: The line's slope is

$$m = (-1-4) / (-2-3) = -5 / -5 = 1$$

We can use this slope with either of the two given points in the point-slope equation

With $(x1,y1) = (-2,-1)$	**With** $(x1,y1) = (3,4)$
$y = -1 + 1(x-(-2))$	$y = 4 + 1(x-3)$
$y = -1+x+2$	$y = 4 + x -3$
$y = x+1$	$y = x+1$

Lines Determined by Slope and y-Intercept

A nonvertical line crosses the y-axis at some point (0,b). If we use this point in the point slope form of its equation, we obtain

$$y-b = m(x-0)$$

Which we can rewrite as $y = mx+b$

Theorem 1.4.7

The line with y-intercept b and slope m is given by the equation

$$y = mx+b$$

This is called the **slope-intercept form** of the line.

Example

The line $y = 2x-5$ has slope 2 and y-intercept -5.

Example

Find the slope-intercept form of the equation of line with slope -9 and that crosses the y-axis at (0,-4)

Solution : We are given with $m=-9$ and $b=-4$, so slope-intercept form of line is

$$y = -9x-4$$

Example

Find slope-intercept form of the equation of line that passes through (3,4) and (-2,-1).

Solution :

The line's slope is

$$m = (-1-4) / (-2-3) = -5 / -5 = 1$$

We can use this slope with either of the two given points in the point-slope equation

With $(x1,y1) = (-2,-1)$

$$y - (-1) = 1 (x- (-2))$$

$$y + 1 = x+2$$

$$y = x+1$$

Which is required slope-intercept form.

The General Equation of a Line

An equation that is expressible in the form

$$Ax + By + C =0$$

Where A, B and C are constants and A and B are not both zero, is called a first-degree equation in x and y. For example

$$4x + 6y -5 = 0$$

is a first-degree equation in x and y.

Theorem 1.4.8

Every first degree equation in x and y has a straight line as its graph and, conversely, every straight line can be represented by a first-degree equation in x and y.

Example

Find the slope and y-intercept of the line $8x + 5y = 20$

Solution :

Solve the equation for y to put it in the slope-intercept form, then read the slope and y-intercept from equation

$$5y = -8x + 20$$

$$y = - 8/5 x + 4$$

The slope is $m = - 8/5$ and the y-intercept is $b = 4$

Applications

The Importance of Lines and Slopes

Light travel along with lines, as do bodies falling from rest in a planet's gravitational field or coasting under their own momentum (like hockey puck gliding across the ice). We often use the equations of lines (called **Linear equations**) to study such motions

Many important quantities are related by linear equations. One we know that a relationship between two variables is linear, we can find it from the any two pairs of corresponding values just as we find the equation of a line from the coordinates of two points.

Slope is important because it gives us way to say how steep something is (roadbeds, roofs, stairs). The notion of slope also enables us to describe how rapidly things are changing. For this reason it will play an important role in calculus.

Example

Fahrenheit temperature (F) and Celsius temperature (F) are related by a linear equation of the form

$$F = mC + b$$

The freezing point of water is $F = 32$ or $C = 0$,

while the boiling point is

$$F = 212 \quad \text{or} \quad C = 100 \; .$$

Thus

$$32 = 0m + b \;\; \text{and} \;\; 212 = 100m + b$$

Solving these two equation we get

$$b = 32 \quad \text{and} \quad m = 9/5 \text{ , therefore}$$

$$F = 9/5\, C + 32$$

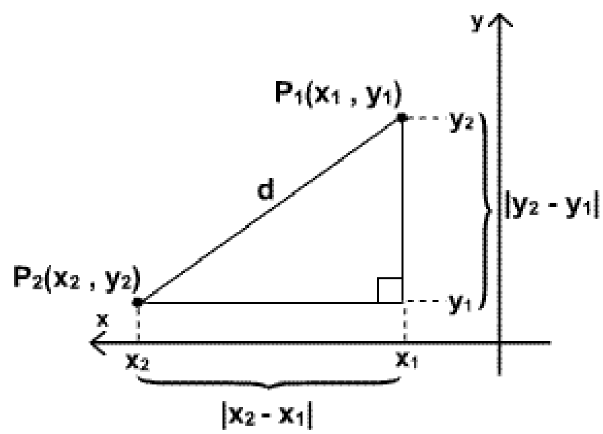
Lecture # 5

Distance; Circles; Equations of the form y = ax2+bx+c

In this lecture we shall derive a formula for the distance between two points in a coordinate plane, and we shall use that formula to study equations and graphs of circles. We shall also study equations of the form $y = ax^2+bx+c$ and their graphs.

Distance between two points in the plane

As we know that if **A** and **B** are points on a coordinate line with coordinates **a** and **b**, respectively, then the distance between **A** and **B** is **|b-a|.** We shall use this result to find the distance d between two arbitrary points P1(x_1,y_1) and P2(x_2,y_2) in the plane.



If, as shown in figure, we form a right triangle With P1 and P2 as vertices, then length of the horizontal side is $|x_2-x_1|$ and the length of the vertical side is $|y_2-y_1|$, so it follows from the Pythagoras Theorem that

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

But for every real number a we have $|a|^2 = |a^2|$, thus

$$|x_2-x_1|^2 = (x_2-x_1)^2 \quad \text{and} \quad |y_2-y_1|^2=(y_2-y_1)^2$$

Theorem 1.5.1

The distance d between two points (x_1,y_1) and (x_2,y_2) in a coordinate plane is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example

Find the distance between the points (-2,3) and (1,7).

Solution: If we let (x1,y1) be (-2,3) and let (x2,y2) be (1,7) then by distance formula we get

$$d = \sqrt{(1-(-2))^2 + (7-3)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

When using distance formula, it does not matter which point is labeled (x1,y1) and which is labeled as (x2,y2). Thus in the above example, if we had let (x1,y1) be the point (1,7) and (x2,y2) the point (-2,3) we would have obtained

$$d = \sqrt{(-2-1)^2 + (3-7)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

which is the same result we obtained with the opposite labeling.

The distance between two points P1 and P2 in a coordinate plane is commonly denoted by d (P1,P2) or d (P2,P1).

Example

Show that the points A(4,6), B(1,-3),C(7,5) are vertices of a right triangle.

Solution: The lengths of the sides of the triangles are

$$d(A,B) = \sqrt{(1-4)^2 + (-3-6)^2} = \sqrt{9+81} = \sqrt{90}$$
$$d(A,C) = \sqrt{(7-4)^2 + (5-6)^2} = \sqrt{9+1} = \sqrt{10}$$
$$d(B,C) = \sqrt{(7-1)^2 + (5-(-3))^2} = \sqrt{36+64} = \sqrt{100} = 10$$

Since

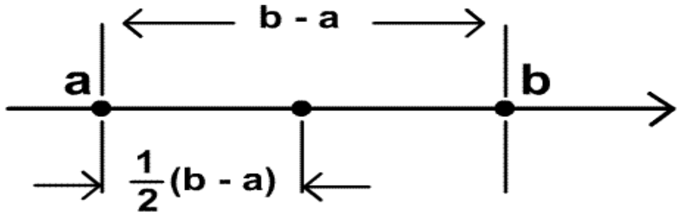
$$\left[d(A,B)\right]^2 + \left[d(A,C)\right]^2 = \left[d(B,C)\right]^2$$

It follows that ABC is a right triangle with hypotenuse BC

The Midpoint Formula

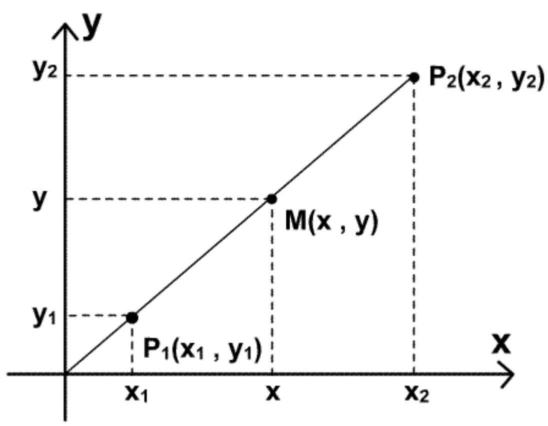
It is often necessary to find the coordinates of the midpoint of a line segment joining two points in the plane. To derive the midpoint formula, we shall start with two points on a coordinate line. If we assume that the points have coordinates **a** and **b** and that **a** **b,** then, as shown in the following figure, the distance between **a** and **b** is **b-a**, and the coordinate of the midpoint between **a** and **b** is

$$a + \frac{1}{2}(b-a) = \frac{1}{2}a + \frac{1}{2}b = \frac{1}{2}(a+b)$$



Which is the arithmetic average of **a** and **b**. Had the points been labelled with **b** $\le$ **a**, the same formula would have resulted. Therefore, the midpoint of two points on a coordinate line is the arithmetic average of their coordinates, regardless of their relative positions. If we now let P1(x1,y1) and (x2,y2) be any two points in the plane and M(x,y) the midpoint of the line segment joining them (as shown in figure) then it can be shown using similar triangles that x is the midpoint of x1 and x2 on the x-axis and y is the midpoint of y1 and y2 on the y-axis, so

$$x = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad y = \frac{1}{2}(y_1 + y_2)$$



Thus, we have the following result.

Theorem1.5.2

(The Midpoint Formula)

The midpoint of the line segment joining two points (x1,y1) and (x2,y2) in a coordinate plane is

$$mid\ po\text{int} \quad (x, y) = (\frac{1}{2}(x_1 + x_2),\ \frac{1}{2}(y_1 + y_2))$$

Example

Find the midpoint of the line segment joining (3,-4) and (7,2).

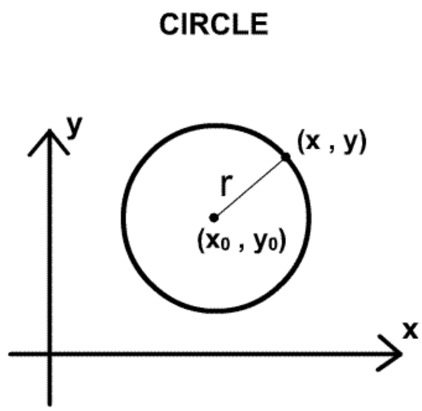
Solution: The midpoint is

$$\left(\frac{1}{2}(3+7), \frac{1}{2}(-4+2)\right) = (5,-1)$$

Circles

If (x0,y0) is a fixed point in the plane, then the circle of radius r centered at (x0,y0) is the set of all points in the plane whose distance from (x0,y0) is r (as shown in figure). Thus, a point (x,y) will lie on this circle if and only if

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = r$$



or equivalently

$$(x-x_0)^2+(y-y_0)^2 = r^2$$

This is called the standard form of the equation of circle.

Example

Find an equation for the circle of radius 4 centered at (-5,3).

Solution: Here x0=-5, y0=3 and r=4

Substituting these values in standard equation of circle we get

$$(x-(-5))^2+(y-3)^2 = 4^2$$
$$(x+5)^2+(y-3)^2 = 16$$

If desired, it can be written in expanded form as

$$(x^2+10x+25)+(y^2-6y+9)-16=0$$
$$x^2+y^2+10x-6y+18=0$$

Example

Find an equation for circle with center (1,-2) that passes through (4,2)

Solution: The radius r of the circle is the distance between (4,2) and (1,-2), so

$$r=\sqrt{(1-4)^2+(-2-2)^2}=\sqrt{(-3)^2+(-4)^2}=\sqrt{9+16}=5$$

We now know the center and radius, so we can write

$$(x-1)^2+(y+2)^2=25 \quad \text{or} \quad x^2+y^2-2x+4y-20=0$$

Finding the center and radius of a circle

When you encounter an equation of the form

$$(x-x_0)^2+(y-y_0)^2=r^2$$

You will know immediately that its graph is a circle; its center and radius can then be found from the constants that appear in the above equation as:

$$(x-x_0)^2 \qquad + \qquad (y-y_0)^2 \qquad = \qquad r^2$$

X-coordinate of the center is x_0	y-coordinate of the center is y_0	radius squared

Equation of a circle	center	radius
$(x-2)^2+(y-5)^2=9$	(2,5)	3
$(x+7)^2+(y+1)^2=16$	(-7,-1)	4
$x^2+y^2=25$	(0,0)	5
$(x-4)^2+y^2=5$	(4,0)	$\sqrt{5}$

The circle $x^2+y^2=1$, which is centered at the origin and has radius 1, is of special importance; it is called the **unit circle**.

Other forms for the equation of a circle

By squaring and simplifying the standard form of equation of circle, we get an equation of the form

$$x^2+y^2+dx+ey+f=0$$

where d, e, and f are constants.

Another version of the equation of circle can be obtained by multiplying both sides of above equation by a nonzero constant A. This yields an equation of the form

$$Ax^2+By^2+Dx+Ey+F=0$$

Where A, D, E and F are constants and $A \neq 0$ & $B \neq 0$

If the equation of a circle is given in any one of the above forms, then the center and radius can be found by first rewriting the equation in standard form, then reading off the center and radius from that equation.

Example

Find the center and radius of the circle with equation

(a) $x^2+y^2-8x+2y+8=0$ (b) $2x^2+2y^2+24x-81=0$

Solution: (a)

First, group x-terms, group y-terms, and take the constant to the right side:

$$(x^2-8x) + (y^2+2y) = -8$$

we use completing square method to solve it as

$$(x^2-8x+16) + (y^2+2y+1) = -8+16+1$$

or

$$(x-4)^2 + (y+1)^2 = 9$$

This is standard form of equation of circle with center (4,-1) and radius 3

Solution: (b)

Dividing equation through by 2 we get

$$x^2+y^2+12x-81/2 = 0$$

$$(x^2+12x)+y^2 = 81/2$$

$$(x^2+12x+36)+y^2 = 81/2+36$$

$(x+6)^2 + y^2 = 153/2$

This is standard form of equation of circle, the circle has center

(-6,0) and radius $\sqrt{153/2}$

Degenerate Cases of a Circle

There is no guarantee that an equation of the form represents a circle. For example, suppose that we divide both sides of this equation by A, then complete the squares to obtain

$$(x-x_0)^2 + (y-y_0)^2 = k$$

Depending on the value of k, the following situations occur:

- ($k > 0$) The graph is a circle with center (x_0,y_0) and
- radius k
- ($k = 0$) The only solution of the equation is $x=x_0$, $y=y_0$,
- so the graph is the single point (x_0,y_0).
- ($k < 0$) The equation has no real solutions and

  consequently no graph

Example

Describe the graphs of

(a) $(x-1)^2 + (y+4)^2 = -9$ (b) $(x-1)^2 + (y+4)^2 = 0$

Solution: (a)

There are no real values of x and y that will make the left side of the equation negative. Thus, the solution set of the equation is empty, and the equation has no graph.

Solution: (b)

The only values of x and y that will make the left side of the equation 0 are $x=1$, $y=-4$. Thus, the graph of the equation is the single point (1,-4).

Theorem

An equation of the form

$$Ax^2+By^2+Dx+Ey+F = 0$$

where A = 0, represents a circle, or a point, or else has no graph.

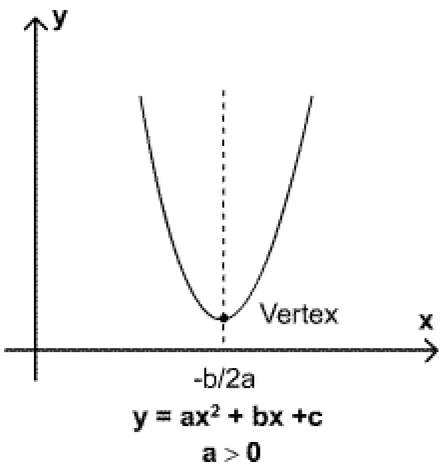
The last two cases in this theorem are called degenerate cases. In spite of the fact these degenerate cases can occur, above equation is often called the general equation of circle.

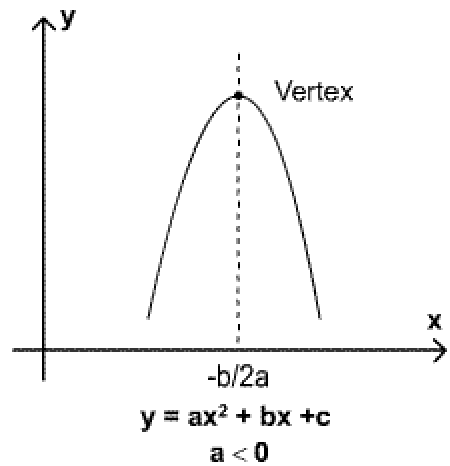
The graph of $y = ax^2+bx+c$

An equation of the form

$$y = ax^2+bx+c \qquad (a \neq 0)$$

Is called a quadratic equation in x. Depending on whether a is positive or negative, the graph, which is called a parabola, has one of the two forms shown below





In both cases the parabola is symmetric about a vertical line parallel to the y-axis. This line of symmetry cuts the parabola at a point called the vertex. The vertex is the low point on the curve if $a>0$ and the high point if $a<0$.

Here is an important fact. The x-coordinate of the vertex of the parabola can be found by the following formula $\qquad x = - b/2a$

Once you have the x-coordinate of the vertex, you can find the y-coordinate easily by substituting the value of x into the equation corresponding to the graph.

With the aid of this formula, a reasonably accurate graph of a quadratic equation in *x* can be obtained by plotting the vertex and two points on each side of it.

Example

Sketch the graph of

a) $y = x^2 - 2x - 2$

b) $y = -x^2 + 4x - 5$

Solution:

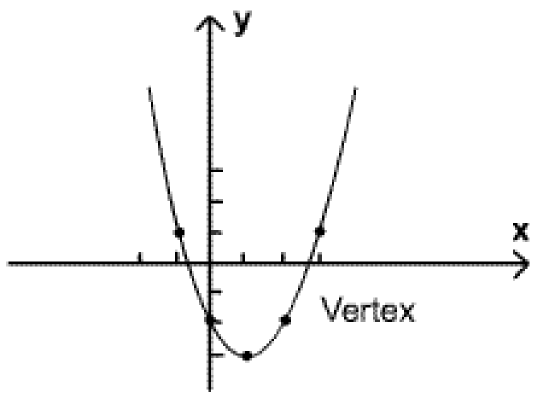
(a) This is quadratic equation with a=1, b=-2 and c=-2.

So x-coordinate of vertex is

$$x = -b/2a = 1$$

Using this value and two additional values on each side as shown here

x	$y = x^2 - 2x - 2$
-1	1
0	-2
1	-3
2	-2
3	1



$y = x^2 - 2x - 2$

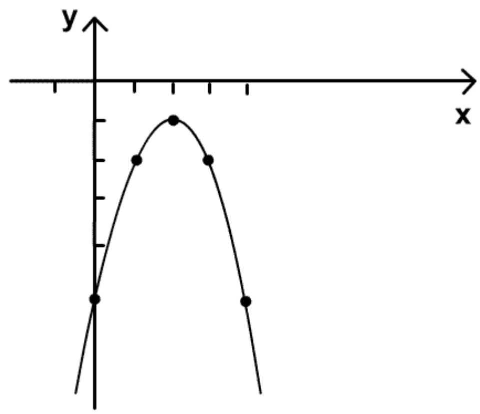
$a > 0$

Solution: (b) This is also a quadratic equation with a=-1, b=4, and c=-5. So x-coordinate of vertex is

$$x = -b/2a = 2$$

Using this value and two additional values on each side, we obtain the table and following graph

x	$y = -x^2 + 4x - 5$
0	-5
1	-2
2	-1
3	-2
4	-5



$y = -x^2 + 4x - 5$

$a < 0$

Often, the intercepts of parabola y=ax2+bx+c are important to know. You can find y-intercept by setting x=0 and x-intercepts by setting y=0 .

When we put y=0, then we have to solve quadratic equation

$$ax^2+bx+c=0$$

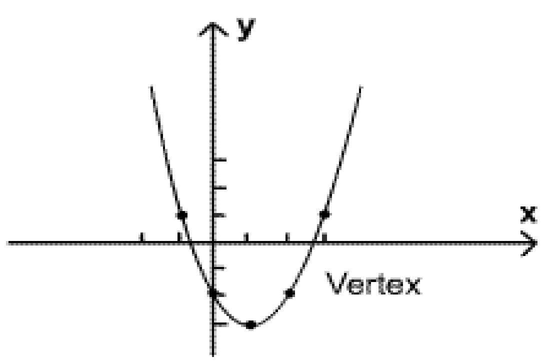
Example

Solve the inequality

$$x^2-2x-2 > 0$$

Solution: Because the left side of the inequality does not have discernible factors, the test point method is not convenient to use.

Instead we shall give a graphical solution. The given inequality is satisfied for those values of x where the graph of $y = x^2-2x-2$ is above the x-axis.



$y = x^2 - 2x - 2$

$a > 0$

From the figure those are the values of x to the left of the smaller x-intercept or to the right of larger intercept.
To find these intercepts we set y=0 to obtain

$x^2-2x-2=0$

Solving by the quadratic formula
Gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{12}}{2}$$

$$= 1 \pm \sqrt{3}$$

Thus, the x-intercepts are

$x = 1 + \sqrt{3}$ and $x = 1 - \sqrt{3}$

and the solution set of the inequality is

$(-\infty, 1 - \sqrt{3}) \cup (1 + \sqrt{3}, +\infty)$

Example

A ball is thrown straight up from the surface of the earth at time $t = 0$ sec with an initial velocity of 24.5 m/sec. If air resistance is ignored, it can be shown that the distance *s* (in meters) of the ball above the ground after *t* sec is given by

$$s = 24.5t - 4.9t^2$$

(a) Graph *s* versus *t,* making the *t*-axis horizontal and the *s*-axis vertical
(b) How high does the ball rise above the ground?

Solution (a):

The given equation is a quadratic equation with a=-4.9, b=24.5 and c=0, so t-coordinate of vertex is

t = - b/2a = - 24.5/2(-4.9) = 2.5 sec

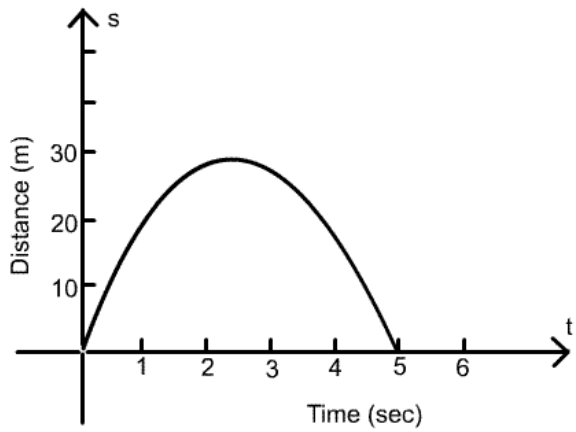
And consequently the s-coordinate of the vertex is

s = 24.5(2.5)-4.9(2.5)2 = 30.626 m

The given equation can factorize as

s = 4.9t(5-t)

so the graph has t-intercepts t=0 and t=5. From the vertex and the intercepts we obtain the graph as shown here



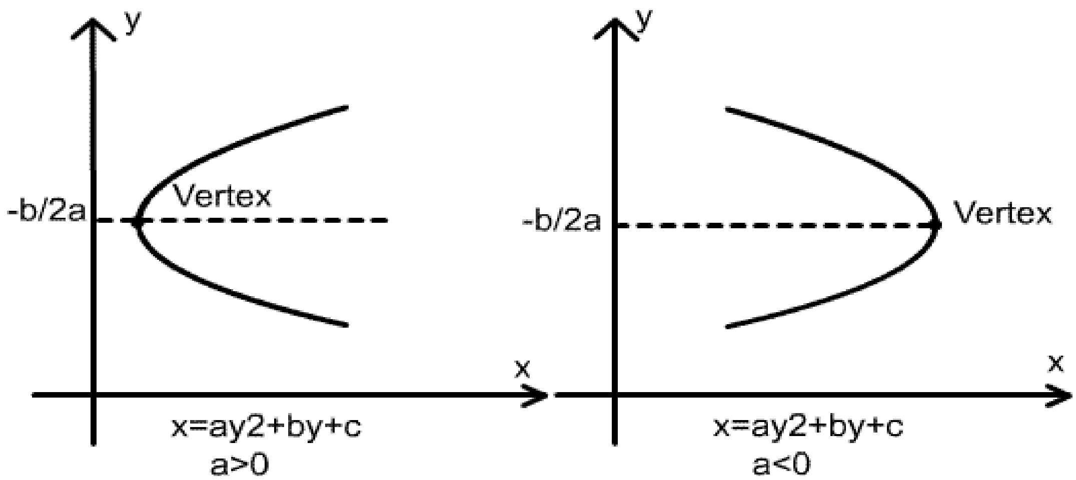
Solution (b) : remember that the height of the graph of a quadratic is maximum or minimum, depending on whether the graph opens UP or DOWN. From the s-coordinate of the vertex we deduce the ball rises 30.625m above the ground

Graph of $x = ay^2+by+c$

If x and y are interchanged in general quadratic equation, we get

$$x = ay^2+by+c$$

is called a quadratic in y. The graph of such an equation is a parabola with its line of symmetry parallel to the x-axis and its vertex at the point with y-coordinate y = -b



Lecture # 6

Functions

In this lecture we will discuss

- **What a function is**
- **Notation for functions**
- **Domain of a function**
- **Range of a function**

Term function was used first by the French mathematician Leibiniz. He used it to denote the DEPENDENCE of one quantity on another.

EXAMPLE

The area A of a circle depends on its radius r by the formula

$$A = \pi r^2$$

so we say " Area is a FUNCTION of radius "

The velocity v of a ball falling freely in the earth's gravitational field increases as time t passes by. So

"velocity is a FUNCTION of time "

Function

If a quantity y depends on another quantity x in such a way that each value of x determines exactly one value of y, we say that y is a function of x.

Example

$y = 4x+1$ is a function. Here is a table

x	Value of y = 4x + 1
2	9
1	5
0	1
-1/4	0
$\sqrt{3}$	$4\sqrt{3} + 1$

Above table shows that each value assigned to x determines a unique value of y.

• We saw many function in the first chapter.

• All equations of lines determine a functional relationship between x and y

NOT A FUNCTION

$y = \pm\sqrt{x}$

$If\ \ x = 4\ then\ y = \pm\sqrt{4}$

thus y = +2 and y = -2

So a single value of x does not lead to exactly one value of y here. So this equation does not describe a function.

NOTATIONS FOR FUNCTIONS

In the 1700's, Swiss mathematician Euler introduced the notation which we mean y = f(x).

This is read as " y equals f of x " and it indicate that y is a function of x.

This tells right away which variable is independent and dependent.

- The one alongside the f is INDEPENDENT (usually x)
- The other one is DEPENDENT (usually y)
- f (x) is read as "y function of x" NOT AS "f multiplied by x"
- f does not represent a number in anyway. It is just for expressing functional relationship

Functions are used to describe physical phenomenon and theoretical ideas concretely.

- The idea of $A = \pi r^2$ gives us a way to express and do calculations concerning circles.
- Nice thing about this notation is that it shows which values of x is assigned to which y value.

Example

$y = f(x) = x^2$

Then $f(3) = (3)^2 = 9$

$f(-2) = (-2)^2 = 4$

Any letter can be used instead of f

y = f(x), y = g(x), y = h(x)

- Also, any other combination of letters can be used for Independent and dependant variables instead of x and y

For example s = f (t) states that the dependent variable s is a function of the independent variable t .

Example:

$$If\ \phi(x)=\frac{1}{x^3-1}$$

Then

$$\phi(5^{1/6})=\frac{1}{(5^{1/6})^3-1}=\frac{1}{5^{3/6}-1}=\frac{1}{\sqrt{5}-1}$$

$$\phi(1)=\frac{1}{1-1}=\frac{1}{0}=undefined$$

So far we have used numerical values for the x variable to get an output for the y as a number. We can also replace x with another variable representing number. Here is example

If $F(x) = 2x^2-1$

Then $F(d) = 2d^2-1$

and $F(t-1) = 2(t-1)^2-1 = 2t^2-4t+2-1 = 2t^2-4t+1$

- If two functions look alike in all aspects other than a difference in g variables, then they are the SAME

  $g(c) = c^2 -4c$ and $g(x) = x^2 -4x$

These two are the SAME function

You can substitute a few values for c and x in the two functions and notice that the results are the same.

Formula structure matters, not the variables used.

DOMAIN OF A FUNCTION

- The independent variable is not always allowed to take on any value in functions
- It may be restricted to take on values from some set.
- This set is called the DOMAIN of the function.
- It is the set consisting of all allowable values for the independent variable.
- DOMAIN is determined usually by physical constraints on the phenomenon being represented by functions.

EXAMPLE

Suppose that a square with a side of length x cm is cut from four corners of a piece of cardboard that is 10 cm square, and let y be the area of the cardboard that remain. By subtracting the areas of four corners squares from the original area, it follows that

$$y=100-4x^2$$

Here x can not be negative, because it denote length and its value can not exceed 5.
Thus x must satisfy the restriction $0 \le x \le 5$. Therefore, even though it is not stated explicitly, the underlying physical meaning of x dictates that the domain of function is the set $\{0 \le x \le 5\}$ = [0,5]

We have two types of domain

1 NATURAL DOMAIN

2 RESTRICTED DOMAIN

NATURAL DOMAIN

Natural domain comes out as a result of the formula of the function. Many functions have no physical or geometric restrictions on the independent variable. However, restrictions may arise from formulas used to define such functions.

Example

$$h(x) = \frac{1}{(x-1)(x-3)}$$

- If x = 1, bottom becomes 0
- If x = 3, bottom becomes 0

So 1 and 3 is not part of the domain Thus the domain is

$$(-\infty,1)U(1,3)U(3,+\infty)$$

If a function is defined by a formula and there is no domain explicitly stated, then it is understood that the domain consists of all real numbers for which the formula makes sense, and the function has a real value. This is called the **natural domain** of the function.

RESTRICTED DOMAINS

Sometimes domains can be altered by restricting them for various reasons. It is common procedure in algebra to simplify functions by canceling common factors in the numerator and denominator. However, the following example shows that this operation can alter the domain of a function.

Example

$$h(x) = \frac{x^2-4}{x-2}$$

This function has a real value everywhere except at x=2, where we we have a division by zero. Thus the domain of **h** consists of all x except x=2. However if we rewrite it as

$$h(x) = \frac{(x-2)(x+2)}{x-2}$$
$$h(x) = (x+2)$$

Now h(x) is defined at x=2, since h(2)=2+2=4

Thus our algebraic simplification has altered the domain of the function. In order to cancel the factor and not alter the domain of h(x), we must restrict the domain and wrlte

h(x) = x+2 , x = 2

Range of a Function

- For every values given to the independent variable from the domain of a function, we get a corresponding y value .
- The set of all such y values is called the RANGE of the Function
- In other words, Range of a function is the set of all possible values for f(x) as x varies over the domain.

Techniques For Finding Range

- **By Inspection**

Example

Find the range of $f(x) = x^2$

Solution: Rewrite it as $y = x^2$

Then as x varies over the reals, y is all positive reals.

Example

Find the range of $g(x) = 2+\sqrt{x-1}$

Solution: Since no domain is stated explicitly, the domain of g is the natural domain $[1,+\infty)$. To determine the range of the function g, let $y = 2+\sqrt{x-1}$

As x varies over the interval $[1,+\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0,+\infty)$, so the value of $y = 2+\sqrt{x-1}$ varies over the interval $[2,+\infty)$. This is the range of g.

- **By some algebra**

Example:

Find the range of the function $y = \frac{x+1}{x-1}$

Solution: The natural domain of x is all real numbers except 1. The set of all possible y values is not at all evident from this equation. However solving this equation for x in terms of y yields

$$x = \frac{y+1}{y-1}$$

It is now evident that y=1 is not in the range. So that range of the function is $\{y : y \neq 1\} = (-\infty, 1)U(1, +\infty)$.

Functions Defined Piecewise

Sometime the functions need to be defined by formulas that have been "pieced together".

Example

The cost of a taxicab ride in a certain metropolitan area is 1.75 rupees for any ride up to and including one mile. After one mile the rider pays an additional amount at the rate of 50 paisa per mile. If f(x) is the total cost in dollars for a ride of x miles, then the value of f(x) is

$$f(x) = \begin{cases} 1.75 & 0 < x \leq 1 \\ 1.75 + 0.50(x-1) & 1 < x \end{cases}$$

pieces have different domains

Reversing the Roles of x and y

- Usually x is independent and y dependent
- But can always reverse roles for convenience sake or other reasons whatever they maybe. For example

$$x = 4y^5 - 2y^3 + 7y - 5$$

is of the form x = g(y): that is , x is expressed as a function of y. Since it is complicated to solve it for y in terms of x, it may be desirable to leave it in this form, treating y as the independent variable and x as the dependent variable. Sometimes an equation can be solved for y as a function of x or for x as a function of y with equal simplicity. For example, the equation

$$3x + 2y = 6$$

can be written as $y = -3/2\ x + 3$ or $x = -2/3\ y + 2$

The choice of forms depends on how the equation will be used.

Lecture # 7

Operations on Functions

- Like numbers, functions can be OPERATED upon
- Functions can be added
- Functions can be subtracted
- Functions can be multiplied
- Functions can be divided
- Functions can be COMPOSED with each other

Arithmetic Operations on Functions

Just like numbers can be added etc, so can be functions

Example

$f(x) = x^2$ and $g(x) = x$

then

$f(x) + g(x) = x^2 + x$

This process defines a new function called the SUM of *f* and *g* functions We denote this SUM as follows So formally we say

$(f+g)(x) = f(x) + g(x)$

Definitions for Operations on Functions

Given functions f and g, then we define their sum, difference, product, and quotient as follows

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f.g)(x) = f(x).g(x)$$
$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$$

- For the function $f+g$, $f-g$, $f.g$, the domains are defined as the intersection of the domains of *f* and *g*

- For *f* / *g*, the domain is the intersection of the domains of *f* and *g* except for the points where g(x)=0

Example

$f(x) = 1 + \sqrt{x-2} \qquad g(x) = x - 1$

$(f+g)(x) = f(x) + g(x)$

$= (1 + \sqrt{x-2}) + (x-1)$

$= x + \sqrt{x-2}$

- Domain of f is $[2, +\infty)$
- Domain of g is $(-\infty, +\infty)$
- Domain of f +g is $[2, +\infty) \cap (-\infty, +\infty) = [2, +\infty)$

Example

$f(x) = 3\sqrt{x}$ and $g(x) = \sqrt{x}$

Find (f .g)(x)

$$(f.g)(x) = f(x).g(x) = (3\sqrt{x})(\sqrt{x})$$
$$= 3x$$

- The natural domain of 3x is (-inf, +inf).
- But this would be wrong in light of the definition of (f .g)
- By definition, the domain should be the intersection of f and g, which is [0, +inf).
- So we need to clarify that this 3x is got from a product and is different b y virtue of its domain from the standard 3x
- We do this by writing

$$(f.g)(x) = 3x \quad x \geq 0$$

NOTATION

f multiplied by itself twice

$$f^2(x) = f(x) \cdot f(x)$$

f multiplied by itself n times

$$f^n(x) = f(x) \cdot f(x) \ldots \cdot f(x)$$

e.g. $(\sin x)^2 = \sin^2(x)$

Composition of Functions

- A new operation called COMPOSITION
- Has no analog with the arithmetic operations we saw
- Remember that the independent variable usually *x* can be given a numerical values from the domain of the function
- When two functions are composed, ONE is assigned as a VALUE to the independent variable of the other.

$f(x) = x^3$ and $g(x) = x + 4$

Compose f with g is written and defined as

$$(fog)(x) = f(g(x))$$

So

$$\begin{aligned}(fog)(x) &= f(g(x)) \\ &= (g(x))^3 \\ &= (x+4)^3\end{aligned}$$

Domain of the new function $(f \; o \; g)$ consists of all x in the domain of *g* for which *g(x)* is in the domain of *f*.

In order to compute $f(g(x))$ one needs to FIRST compute g(x) for an x from the domain of g, then

one needs g(x) in the domain of *f* to compute $f(g(x))$

Put the Sock on first, then the show

$$f(x)=x^2+3 \quad g(x)=\sqrt{x}$$
$$(f \ o \ g)(x)=f(g(x))$$
$$=(g(x))^2+3$$
$$=(\sqrt{x})^2+3=x+3$$

- Domain of g is $[0,+\infty)$
- Domain of f is $(-\infty,+\infty)$
- Domain of $(f \ o \ g)$ consists of all x in $[0,+\infty)$ such that g (x) lies in $(-\infty,+\infty)$
- So its domain is $[0,+\infty)$

$$(fog) \neq (gof)$$

Generally. Try on the last example

Put Sock on then Shoe ≠ Put Shoe on first then Sock.

- Expressing functions as a decomposition
- Sometimes want to break up functions into simpler ones
- This is like DECOMPOSING them into composition of simpler ones

$$h(x)=(x+1)^2$$

- First we add 1 to x
- Then we square x + 1
- So we can break up the function as

$$f(x)=x+1$$
$$g(x)=x^2$$
$$h(x)=g(f(x))$$

There are more than one way to decompose a function

Example

$$(x^2+1)^{10} = [(x^2+1)^2]^5 = f(g(x)) \qquad g(x) = \left(x^2+1\right)^2 \quad f(x) = x^5$$

AND

$$(x^2+1)^{10} = [(x^2+1)^3]^{\frac{10}{3}} = f(g(x)) \qquad g(x) = \left(x^2+1\right)^5 \quad f(x) = x^{\frac{10}{3}}$$

$$T(x) = \sqrt{(\frac{x}{3})^3} = f(g(h(x)))$$

$$f(x) = \sqrt{x}$$

$$g(x) = x^3$$

$$h(x) = \frac{x}{3}$$

Here is a tables of some functions decomposed as compositions of other functions.

Function	g(x) Inside	f(x) Outside	composition
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10}=f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x=f(g(x))$
$1/(x+1)$	$x+1$	$1/x$	$1/(x+1) = f(g(x))$
$\tan(x^5)$	x^5	$\tan x$	$\tan(x^5)=f(g(x))$

Classification of Functions

- Constant Functions

  These assign the same NUMBER to every x in the domain f(x) = 2 SO f(1) = 2 f(-7) = 2 etc

- Monomial in x

Anything that looks like cx^n with c a constant and n any NONEGATIVE INTEGER e.g

$$2x^5, \sqrt{3}x^{55}$$

$4x^{-4}, 5x^{\frac{3}{2}}$ Not MONOMIAL as powers are not NONEGATIVE INTEGERS.

Polynomial in x Things like

$$4x^4 + 3x^2 + 1, \quad 17 - \frac{4}{3}x^2$$

In general anything like

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Lecture # 8

Graphs of Functions

- Represent Functions by graphs
- Visualize behavior of Functions through graphs
- How to use graphs of simple functions to create graphs of complicated functions

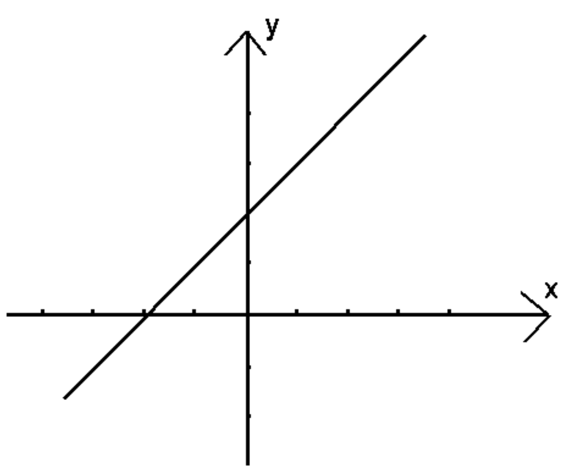
Definition of Graph of a function

- We saw earlier the relationship between graph and its equation
- A graph of an equation is just the points on the xy- plane that satisfy the equation
- Similarly, the graph of a Function *f* in the xy-plane is the GRAPH of the equation y = f (x)

Example

Sketch the graph of $f(x) = x+2$

By definition, the graph of *f* is the graph of $y = x+2$. This is just a line with y-intercept 2 and slope 1. We saw how to plot lines in a previous lecture



graph of f(x) = x + 2

Example

Sketch the graph of $f(x) = |x|$

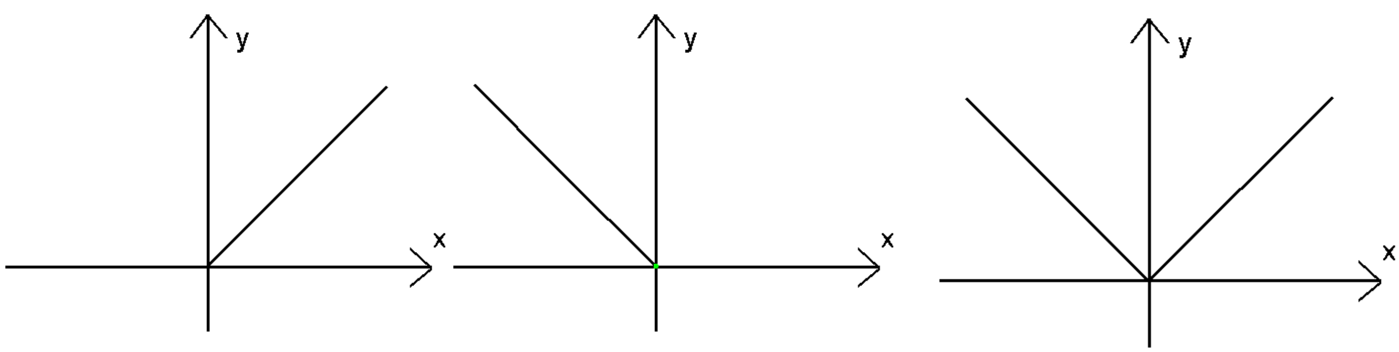
The graph will be that of $y = |x|$

Remember that absolute value is defined as

$$y = |x| = \begin{cases} x & if \ x \geq 0 \\ -x & if \ x < 0 \end{cases}$$

Remember that absolute value function is PIECEWISE defined. The top part is the function y = f(x) = x, the bottom is the function y = f(x) = -x

y = x is just a straight line through the origin with slope 1, but is only defined for x > = 0.
y = -x is a straight line through the origin with slope –1 but defined only for x < 0.
Here is the GRAPH



graph of f(x) = x　　graph of f(x) = -x　　graph of f(x) = |x|

EXAMPLE

$$t(x) = \frac{x^2 - 4}{x - 2}$$

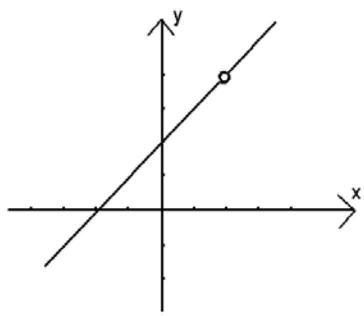
This is the same as $t(x) = x + 2 \quad x \neq 2$

Graph will be of $y = x + 2, \quad x \neq 2$

Remember from Lecture 6??

t(x) is the same as in example 1, except that 2 is not part of the domain, which means there is no y value corresponding to x = 2. So there is a HOLE in the graph.

Picture



graph of y = x + 2 x ≠ 2

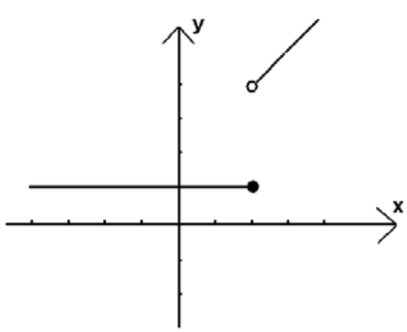
EXAMPLE

$$g(x)=\begin{cases}1 & \text{if } x\le 2\\ x+2 & \text{if } x>2\end{cases}$$

Graph will be of

$$y=\begin{cases}1 & \text{if } x\le 2\\ x+2 & \text{if } x>2\end{cases}$$

For x is less than or equal to 2, the graph is just at y = 1. This is a straight line with slope 0. For x is greater than 2, the graph is the line x+2



Graphing functions by Translations

Suppose the graph of *f(x)* is known.
Then we can find the graphs of $y=f(x)+c,\ \ y=f(x)-c,\ \ y=f(x+c),\ \ y=f(x-c)$.

c is any POSITIVE constant.

- If a positive constant *c* is added to *f (x)* ,the geometric effect is the translation of the graph of *y=f (x)* UP by *c* units.
- If a positive constant *c* is subtracted from *f (x)* the geometric effect is the translation of the graph of *y=f (x)* DOWN by *c* units.
- If a positive constant *c* is added to the independent variable *x* of *f (x)* ,the geometric effect is the translation of the graph of *y=f (x)* LEFT by *c* units.
- If a positive constant *c* is subtracted from the independent variable *x* of *f (x),* the geometric effect is the translation of the graph of *y=f (x)* RIGHT by *c* units.

Here is a table summarizing what we just talked about in terms of translations of function

y = f(x) + c	graph of f(x) translates *UP* by c units
y = f(x) - c	graph of f(x) translates *DOWN* by c units
y = f(x + c)	graph of f(x) translates *LEFT* by c units
y = f(x - c)	graph of f(x) translates *RIGHT* by c units

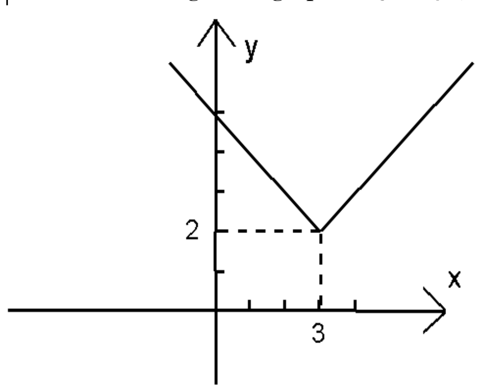
Example

Sketch the graph of $y = f(x) = |x-3| + 2$

This graph can be obtained by two translations

Translate the graph of $y = |x|$ 3 units to the RIGHT to get the graph of $y = |x-3|$.

Translate the graph of $y = |x-3|$ 2 units UP to get the graph of $y = f(x) = |x-3| + 2$.



graph of f(x) = |x - 3| + 2

Example

Sketch graph of $y = x^2 - 4x + 5$

Complete the square

Divide the co-efficient of x by 2

Square this result and add to the both sides of your equation

$y + 4 = (x^2 - 4x + 5) + 4$

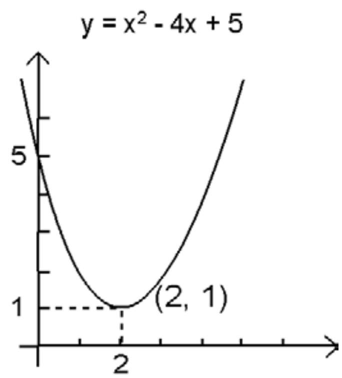
$y = (x^2 - 4x + 4) + 5 - 4$

$y = (x - 2)^2 + 1$

This equation is the same as the original

Graph

$y = x^2$



Shift it RIGHT by 2 units to get graph of $y = (x-2)^2$

Shift this UP by 1 Unit to get $y = (x-2)^2 + 1$

Reflections

From Lecture 3

- (-x, y) is the reflection of (x, y) about the y-axis
- (x, -y) is the reflection of (x, y) about the x-axis

Graphs of y = f(x) and y = f(-x) are reflections of one another about the y-axis.

Graphs of y = f(x) and y = -f(x) are reflections of one another about the x-axis.

Example

Sketch the graph of $y = \sqrt[3]{2 - x}$

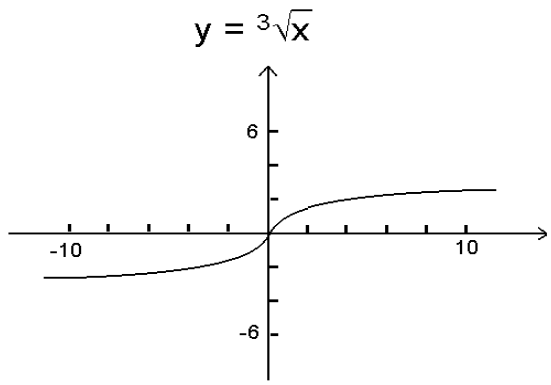
We can always plot points by choosing x-values and getting the corresponding y-values. Better if we get the graph by REFELCTION and TRANSLATION

Here is HOW

- First graph $y = \sqrt[3]{x}$
- Reflect it about the y-axis to get graph of $y = \sqrt[3]{-x}$
  Remember that negative numbers HAVE cube roots
- Translate this graph RIGHT by 2 units to get graph of
  $y = \sqrt[3]{2-x} = \sqrt[3]{-x+2} = \sqrt[3]{-(x-2)}$ scaling.

If *f (x)* is MULTIPLIED by a POSITIVE constant *c*, then the following geometric effects take place

- The graph of *f (x)* is COMPRESSED vertically if $0 < c < 1$
- The graph of *f (x)* is STRETCHED vertically if $c > 1$
- This is called VERTICAL Scaling by a factor of *c*.
-



Example

$$y = 2\sin(x)$$
$$y = \sin(x)$$
$$y = \frac{1}{2}\sin(x)$$

$c = 2$ and $c=1/2$ is used and here are the corresponding graphs with appropriate VERTICAL SCALINGS

Vertical Line Test

- So far we have started with a function equation, and drawn its graph
- What if we have a graph given. Must it be the graph of a function??
- Not every curve or graph in the xy-plane is that of a function

Example

Here is a graph which is not the graph of a function. Figure 2.3.12

Its not a graph because if you draw a VERTICAL line through the point x = a, then the line crosses the graph in two points with y values y = b, y = c.

This gives you two points on the graph namely

(a,b) and (a,c)

But this cannot be a function by the definition of a function.

VERTICAL LINE TEST

A graph in the plane is the graph of a function if and only if NO VERTICAL line intersects the graph more than once.

Example

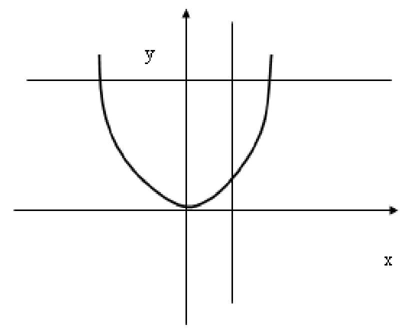
$x^2 + y^2 = 25$

The graph of this equation is a CIRLCE. Various vertical lines cross the graph in more than 2 places. So the graph is not that of a function which means that equations of Circles are not functions x as a function of y.
A given graph can be a function with y independent and x dependent. That is why, it could be the graph of an equation like x = g(y).
This would happen if the graph passes the HORIZONTAL LINE test.
This is so because for each y, there can be only one x by definition of FUNCTION.
Also,

$y = x^2$ Gives

$g(y) = x = \pm\sqrt{y}$

SO for each x, two y's and its not a function in y as clear from the graph of the function



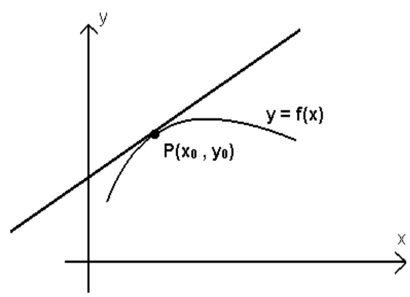
LECTURE # 9

LIMITS

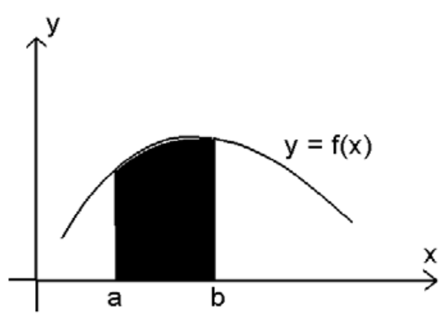
Calculus was motivated by the problem of finding areas of plane regions and finding tangent lines to curves. In this section we will see both these ideas
We will see how these give rise to the idea of LIMIT. We will look at it intuitively, without any mathematical proofs. These will come later.

THE TANGENT PROBLEM

Given a function $f(x)$ and a point $P(x_0, y_0)$ on its graph, find an equation of the line tangent to the graph at P



AREA PROBLEM
Given a function f, find the area between the graph of f and the interval $[a, b]$ on the x-axis



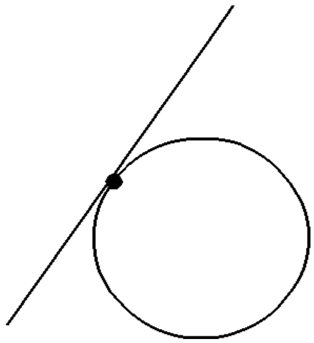
- Traditionally, the Calculus that comes out of the tangent problem is called DIFFERENTIAL CALCULUS .
- Calculus that comes out of the area problem is called INTEGRAL CALCULUS.

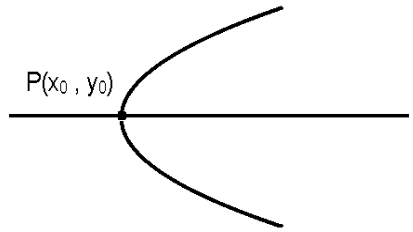
Both are closely related

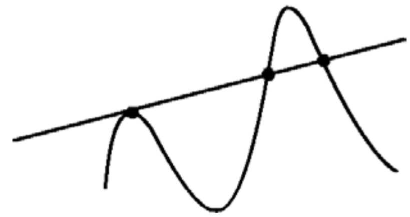
The PRECISE definition of “tangent” and “Area” depend on a more fundamental notion of LIMIT.

Tangent Lines and LIMITS

In geometry, a line is called tangent to a circle if it meets the circle at exactly one point. Figure 2.4.3a. We would like something like this to be our definition of a tangent line.
But this is not the case if you look at another curve like in figure 2.4.3b. This is a sideways parabola with a line meeting it at exactly one point. But this is not what we want as a tangent.



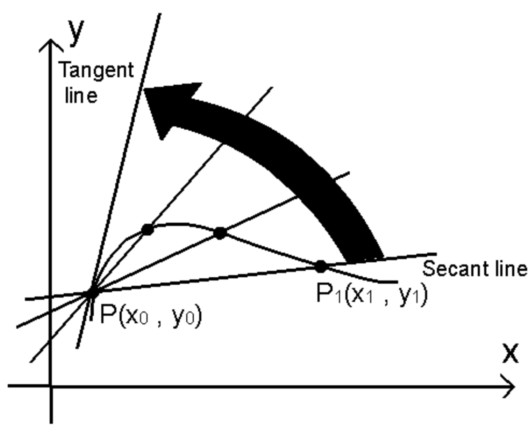




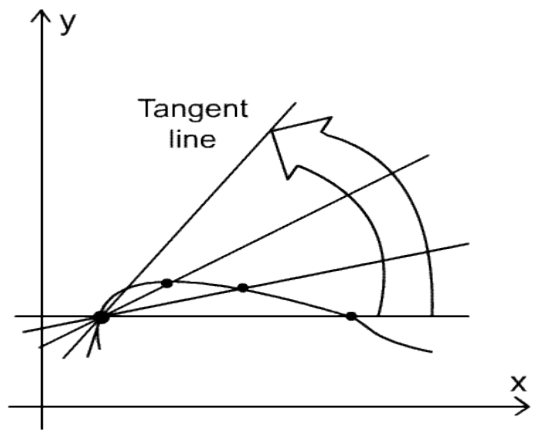
In figure 2.4.3c, we have the picture of a line that is tangent as we would like it to be, but it meets the curve more than once, and we want it to meet the curve only once.

We need to make a definition for tangent that works for all curves besides circles

Consider a point *P* on a curve in the xy-plane. Let Q be another point other than P on the curve. Draw a line through P and Q to get what is called the SECANT line for the curve. Now move the Point Q toward P. The Secant line will rotate to a "limiting" position as Q gets closer and closer to P. The line that will occupy this limiting position will be called the TANGENT line at P



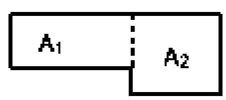
This definition works on circles too as you can see here

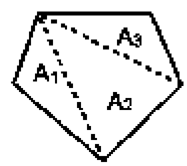


So we see how Tangents or Tangent Lines are defined using the idea of a LIMIT.

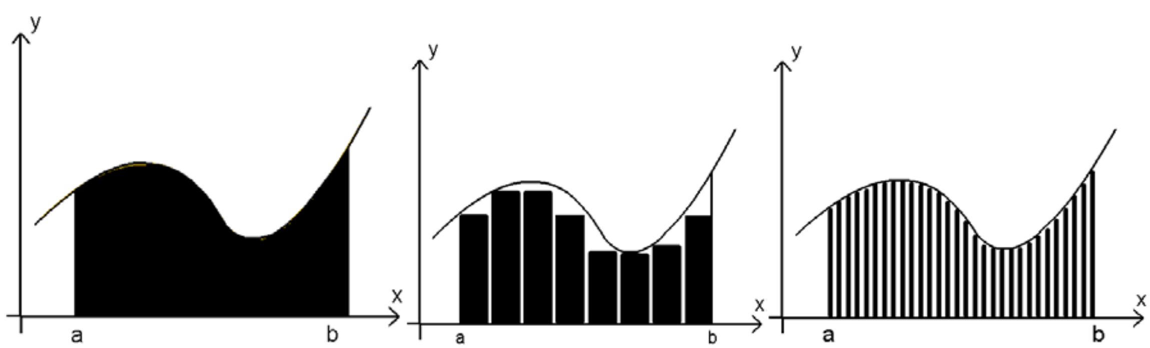
Area as a LIMIT

For most geometric shapes, the area enclosed by them can be get by subdividing the shape into finitely many rectangles and triangles. These FILL UP the shape





Some time this is not possible. Here is a regions defined in the xy-plane.



Can't be broken into rectangles and triangles that will FILL UP the area btw the curve and the interval [a,b] on the x-axis. Instead we use rectangles to APPROXIMANTE the area. Same width rects and we add their areas. If we let our rectangles increase in number, then the approx will be better and the result will be get as a

LIMITING value on the number of rects. If we let our rectangles increase in number, then the approx will be better and the result will be getting as a LIMITING value on the number of rects.

LIMITS

Let's discuss LIMIT in detail. Limits are basically a way to study the behavior of the y-values of a function in response to the x-values as they approach some number or go to infinity.

EXAMPLE

Consider

$$f(x)=\frac{\sin(x)}{x}$$ where x is in radians.

Remember that *PI* radians = 180 degrees.

f (x) is not defined at x = 0.
What happens if you get very close to x = 0??
We can get close to 0 from the left of 0, and from the right of 0.
x can approach 0 along the negative x-axis means from the left.
x can approach 0 along the positive x-axis means from the right.
From both sides we get REALLY close to 0, but not equal to it.
This getting really close is called the LIMITING process.
We write

$$\lim_{x\to 0^+}\frac{\sin(x)}{x}$$

To mean
"The limit of f (x) as x approaches 0 from the right" , the plus on the 0 stands for "from the right"
This is called the RIGHT HAND LIMIT.

$$\lim_{x\to 0^-}\frac{\sin(x)}{x}$$

To mean
"The limit of f (x) as x approaches 0 from the left" , the minus on the 0 stands for "from the left"
This is called the LEFT HAND LIMIT.
Let us see what happens to f (x) as x gets close to 0 from both right and left

X	$f(x)=\frac{\sin(x)}{x}$
1.0	0.84147
0.8	0.89670
0.6	0.94107
0.4	0.97355
0.2	0.99335
0.01	0.99998

X	$f(x)=\frac{\sin(x)}{x}$
-1.0	0.84147
-0.8	0.89670
-0.6	0.94107
-0.4	0.97355
-0.2	0.99335
-0.01	0.99998

The tables show that as x approaches 0 from both sides f(x) approaches 1, We write this as

$$\lim_{x\to 0^+}\frac{\sin(x)}{x}=1 \qquad \lim_{x\to 0^-}\frac{\sin(x)}{x}=1$$

When both the left hand and right hand limits match, we say that the LIMIT exists
We write this as

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

0 was a special point. In general its limit as x approaches x_0, Write this as $x \to x_0$

TABLE of Limit Notations and situations

NOTATION	HOW TO READ THE NOTATION
$\lim_{x \to x_0^+} f(x) = L_1$	The limit of f(x) as x approaches x_0 from the right is equal to L_1
$\lim_{x \to x_0^-} f(x) = L_2$	The limit of f(x) as x approaches x_0 from the left is equal to L_2
$\lim_{x \to x_0} f(x) = L$	The limit of f(x) as x approaches x_0 is equal to L

Sometimes Numerical evidence for calculating limits can mislead.

EXAMPLE

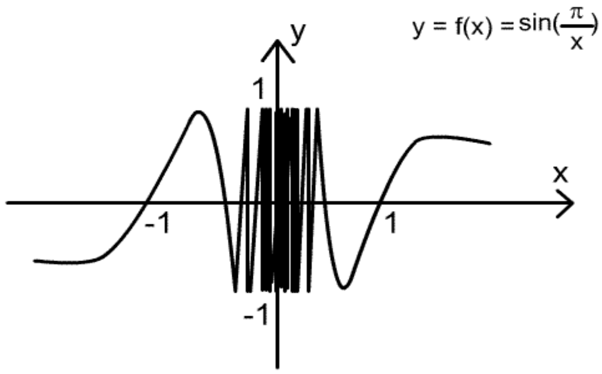
Find

$$\lim_{x\to 0}\ \sin(\frac{\pi}{x}) = \lim_{x\to 0^+}\ \sin(\frac{\pi}{x}) = \lim_{x\to 0^-}\ \sin(\frac{\pi}{x})$$

Table Showing values of f (x) for various x

x	f(x)	x	f(x)
1	0	-1	0
0.1	0	-0.1	0
0.01	0	-0.01	0
0.001	0	-0.001	0
0.0001	0	-0.0001	0

Table suggests Limit is 0



Graph has NO LIMITING value as it OSCILLATES btw 1 and -1

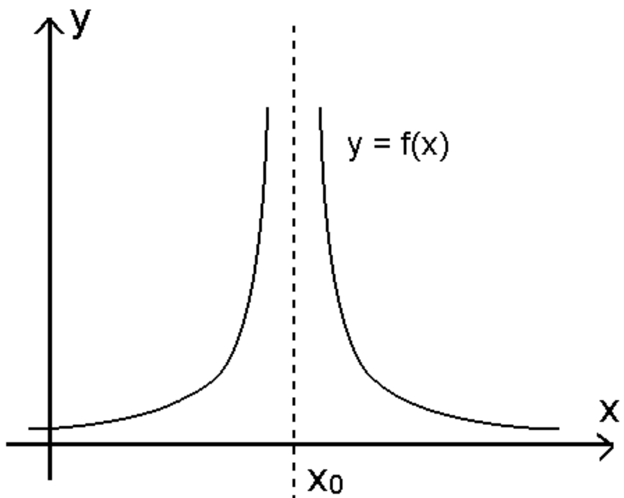
Existence of Limits

Functions don't always have a limit as the x values approach some number.
If this is the case, we say, LIMIT DOES NOT EXIST OR DNE!
Limits fail for many reason, but usual culprits are

- Oscillations
- unbounded Increase or decrease

Example

The graph of a function f(x) is given here



$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} f(x) = +\infty$$

Note that the values of f(x) = y increase without bound as from both the left and the right. We say that

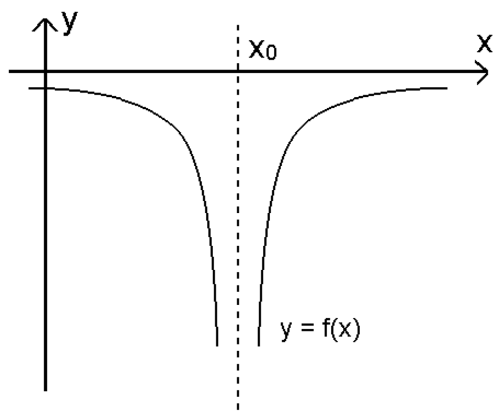
$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0} f(x) = +\infty$$

This is a case where the LIMIT FAILS to EXIST because of unbounded-ness.

The +infinity is there to classify the DNE as caused by unbounded-ness towards +infinity. It is not a NUMBER!!

Example

The graph of a function f (x) is given here



$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} f(x) = -\infty$$

Note that the values of f (x) = y DECREASE without bound as $x \to x_0$ from both the left and the right . We say

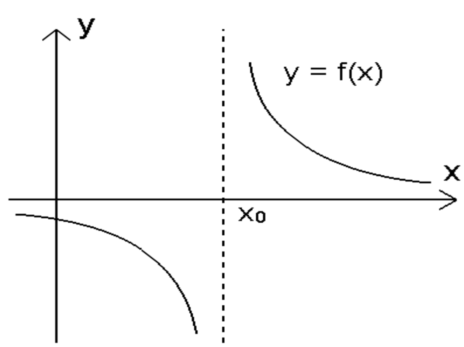
$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) = -\infty$$

This is a case where the LIMIT FAILS to EXIST because of unbounded-ness .

The - infinity is there to classify the DNE as caused by unbounded-ness towards – infinity. It is not a NUMBER!!

Example

Let f be a function whose graph is shown in the picture and let x approach x0, then from the picture



$$\lim_{x \to x_0^-} f(x) = -\infty \qquad \lim_{x \to x_0^+} f(x) = +\infty$$

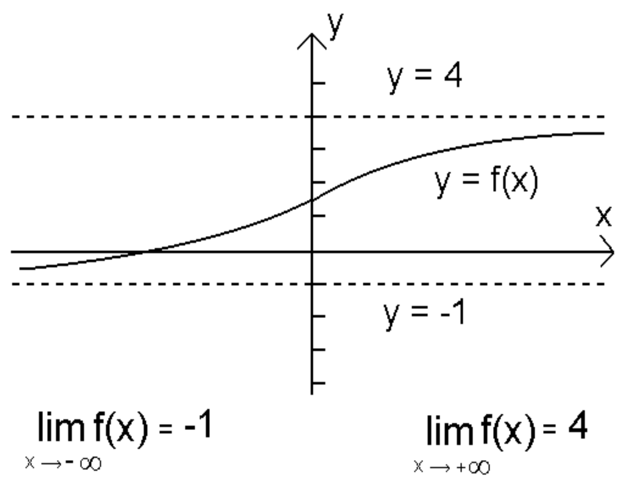
$$\lim_{x \to x_0^+} f(x) = +\infty$$

$$\lim_{x \to x_0^-} f(x) = -\infty$$

Here the two sided limits don't match up! Thus, the limit does not exist.
So far we saw limits as x approached some point x0 Now we see some limits as x goes to +inf or –inf.

Example

The graph of y = f (x) is given here We can see from it that

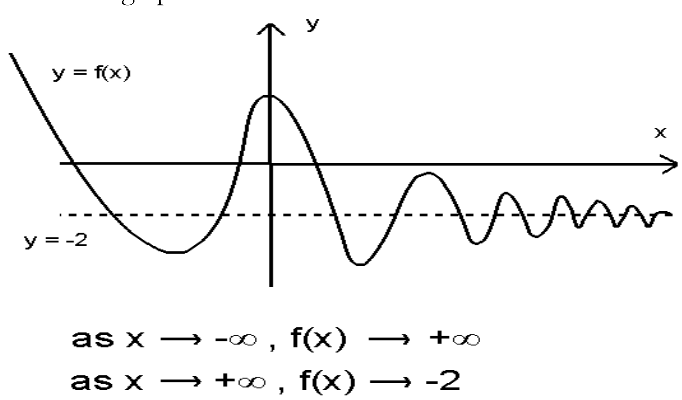


$$\lim_{x \to x_0^+} f(x) = 4$$
$$\lim_{x \to x_0^-} f(x) = -1$$

Note that when we find limits at infinity, we only do it from one side. The reason is that you can approach infinity from only one side!! x goes to infinity means that x gets bigger and bigger, and x an do that only from one side depending on whether it goes to +inf or –inf.

EXAMPLE

For this function we have this graph



Although the graph oscillates as x goes to +inf, the oscillations decreases and settle down on y = -2

Lecture # 10

LIMITS AND COMPUTATIONAL TECHNIQUES

Previous lecture was about graphical view of Limits. This lecture will focus on algebraic techniques for finding Limits. Results will be intuitive again. Proofs will come later after we define **LIMIT** Mathematically.
We will see how to use limits of basic functions to compute limits of complicated functions.
In this section, if I write down Lim x->a f(x), I will assume that f(x) will have a limit that matches from both sides and so the LIMIT EXISTS for f (x). So I won't distinguish between left and right hand limits.
We begin with a table of LIMITS of two basic functions
The functions are

$$f(x) = k$$
$$g(x) = x$$

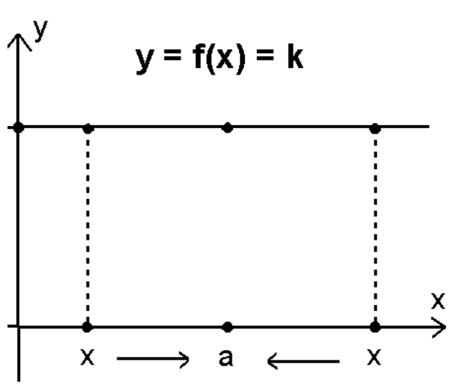
Here is the table of the limits and the same information from the graph

$f(x) = k$

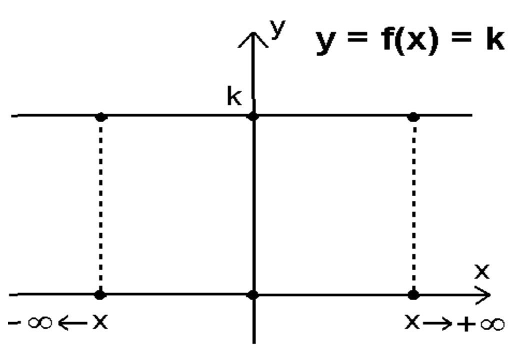
Limit	Example
$\lim_{x \to a} k = k$	$\lim_{x \to 2} 3 = 3,\quad \lim_{x \to -2} 3 = 3$
$\lim_{x \to +\infty} k = k$	$\lim_{x \to +\infty} 3 = 3,\quad \lim_{x \to +\infty} 0 = 0$
$\lim_{x \to -\infty} k = k$	$\lim_{x \to -\infty} 3 = 3,\quad \lim_{x \to -\infty} 0 = 0$

$g(x) = x$

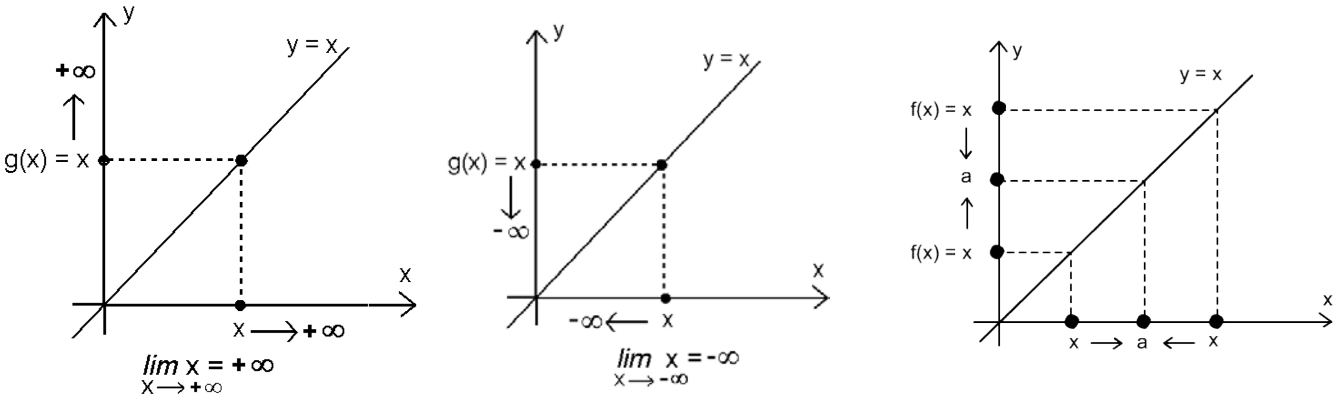
Limit
$\lim_{x \to a} x = a$
$\lim_{x \to +\infty} x = +\infty$
$\lim_{x \to -\infty} x = -\infty$



$\lim_{x \to a} k = k$



$\lim_{x \to +\infty} k = k \qquad \lim_{x \to -\infty} k = k$



Here we have a theorem that will help with computing limits. Won't prove this theorem, but some of the parts of this theorem are proved in Appendix C of your text book.

THEOREM 2.5.1

Let Lim stand for one of the limits

$$\lim_{x\to a}, \quad \lim_{x\to a^-}, \quad \lim_{x\to a^+}, \quad \lim_{x\to +\infty}, \quad \lim_{x\to -\infty}$$

$$if\ L_1 = \lim f(x)\ and\ L_2 = \lim g(x)\ both\ exists, then$$

$$\begin{aligned}
a) \lim[\,f(x)+g(x)] &= \lim\, f(x) + \lim g(x) \\
&= L_1 + L_2 \\
b) \lim[\,f(x)-g(x)] &= \lim\, f(x) - \lim g(x) \\
&= L_1 - L_2 \\
c) \lim[\,f(x)\,.\,g(x)] &= \lim\, f(x)\,.\lim g(x) \\
&= L_1 . L_2 \\
d) \lim[\frac{f(x)}{g(x)}\,] &= \frac{\lim\, f(x)}{\lim g(x)} \\
&= \frac{L_1}{L_2} \quad (L_2 \neq 0)
\end{aligned}$$

For the Last theorem, say things like "Limit of the SUM is the SUM of the LIMITS etc.
Parts a) and c) of the theorem apply to as many functions as you want
Part a) gives

$$\begin{aligned}
&\lim[\,f_1(x) + f_2(x) + ... + f(x_n)] \\
&= \lim\, f_1(x) + \lim\, f_2(x) + ... \lim\, f(x_n)
\end{aligned}$$

Part c) gives

$$\lim[f_1(x)\cdot f_2(x)\cdot\ldots\cdot f_n(x)]$$
$$=\lim f_1(x)\cdot\lim f_2(x)\cdot\ldots\cdot\lim f_n(x)$$

Also if

$$f_1 = f_2 = \ldots = f_n \text{ then } \lim[f(x)]^n = [\lim f(x)]^n$$

From this last result we can say that

$$\lim_{x\to a}(x^n) = [\lim_{x\to a} x^n] = a^n$$

This is a useful result and we can use it later.
Another useful result follows from part c) of the theorem. Let f(x) = k in part c), where k is a constant (number).

$$\lim[kg(x)] = \lim(k)\cdot\lim g(x) = k\cdot\lim g(x)$$

So a constant factor can be moved through a limit sign

LIMITS OF POLYNOMIAL

Polynomials are functions of the form

$$f(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots. + b_1 x + b_0$$

Where the a's are all real numbers Let's find the Limits of polynomials and x approaches a numbers *a*

Example

$$\lim_{x\to 5}(x^2 - 4x + 3)$$
$$=\lim_{x\to 5} x^2 - \lim_{x\to 5} 4x + \lim_{x\to 5} 3$$
$$=\lim_{x\to 5} x^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3$$
$$=(5)^2 - 4(5) + 3 = 8$$

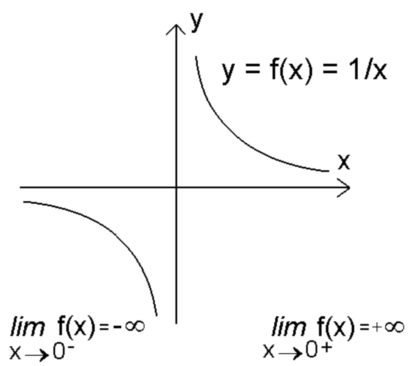
Theorem 2.5.2

Proof

$$\lim_{x\to a} p(x) = \lim_{x\to a}(c_0 + c_1 x + \ldots + c_n x^n)$$
$$=\lim_{x\to a} c_0 + \lim_{x\to a} c_1 x + \ldots + \lim_{x\to a} c_n x^n$$
$$=\lim_{x\to a} c_0 + c_1\lim_{x\to a} x + \ldots + c_n\lim_{x\to a} x^n$$
$$=c_0 + c_1 a + \ldots + c_n a^n = p(a)$$

Limits Involving $\frac{1}{x}$

Let's look at the graph of $f(x) = \frac{1}{x}$



Then by looking at the graph AND by looking at the TABLE of values we get the following Results

	Values	Conclusion
x 1/x	1 .. .01 .. .001 ... 1 .. 100 .. 1000 ...	$x \longrightarrow 0^+$ $1/x \longrightarrow +\infty$
x 1/x	-1 .. -.01 .. -.001 ... -1 .. -100 .. -1000 ...	$x \longrightarrow 0^-$ $1/x \longrightarrow -\infty$
x 1/x	1 .. 100 .. 1000 ... 1 .. .01 .. .001 ...	$x \longrightarrow +\infty$ 1/x decreases towards 0
x 1/x	-1 .. -100 .. -1000 ... -1 .. -.01 .. -.001 ...	$x \longrightarrow -\infty$ 1/x increases towards 0

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

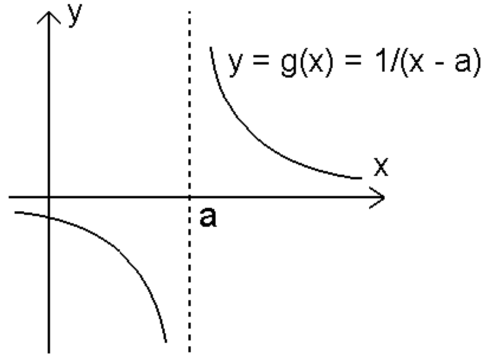
$$\lim_{x\to +\infty} \frac{1}{x} = 0$$

$$\lim_{x\to -\infty} \frac{1}{x} = 0$$

For every real number *a,* the function

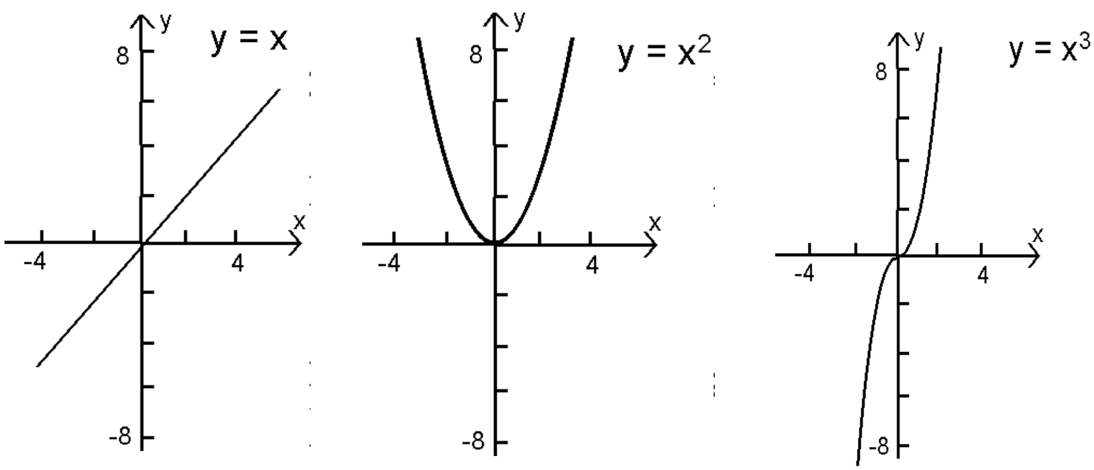
$$g(x) = \frac{1}{x-a} \quad \text{is a translation of } f(x) = \frac{1}{x}.$$

So we an say the following about this function



LIMITS OF POLYNOMIALS AS X GOES TO +INF AND –INF

From the graphs given here we can say the following about polynomials of the form



$$\lim_{x\to+\infty} x^n = +\infty \quad n = 1, 2, 3, \ldots$$

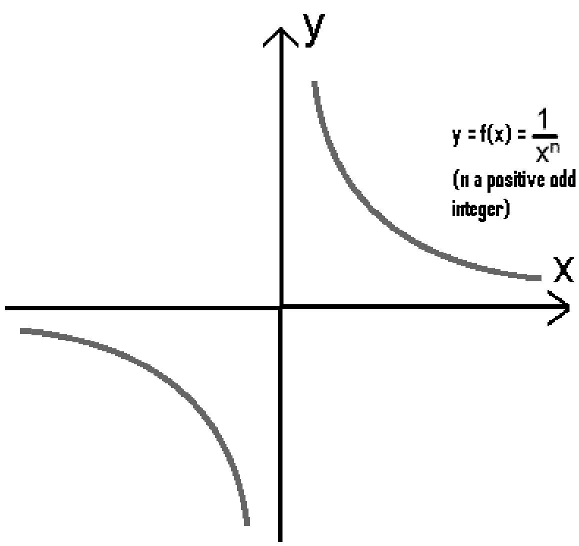
$$\lim_{x\to-\infty} x^n = \begin{cases} +\infty & n = 2, 4, 6, \ldots \\ -\infty & n = 1, 3, 5, \ldots \end{cases}$$

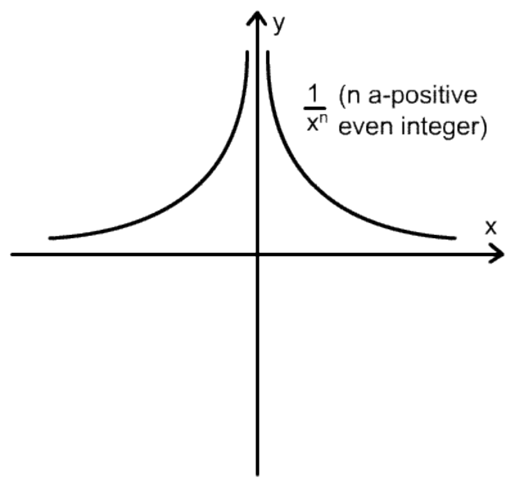
EXAMPLE

$$\lim_{x \to +\infty} 2x^5 = +\infty$$
$$\lim_{x \to +\infty} -7x^6 = -\infty$$

$$\lim_{x \to +\infty} \frac{1}{x^n} = \left(\lim_{x \to +\infty} \frac{1}{x} \right)^n = 0$$
$$\lim_{x \to -\infty} \frac{1}{x^n} = \left(\lim_{x \to -\infty} \frac{1}{x} \right)^n = 0$$

HERE are the graphs of the functions





$y = f(x) = 1/x^n$ (n *is positive* integer)

Limit as x goes to +inf or –inf of a polynomial is like the Limits of the highest power of x

$$\lim_{x \to +\infty} (c_0 + c_1 x + \ldots + c_n x^n) = \lim_{x \to +\infty} c_n x^n$$

Motivation

$$(c_0 + c_1 x + ... + c_n x^n) = x^n (\frac{c_0}{x^n} + \frac{c_1}{x^{n-1}} + ... + c_n)$$

Factor out x^n , and then from what we just saw about the limit of $\frac{1}{x^n}$, everything goes to 0 as $x \to +\infty$ or $x \to -\infty$ except c_n

Limits of Rational Functions as x goes to *a*

A rational function is a function defined by the ratio of two polynomials

Example

Find $\lim_{x \to 2} \frac{5x^3 + 4}{x - 3}$

Sol:

$$= \frac{\lim_{x \to 2} 5x^3 + 4}{\lim_{x \to 2} x - 3}$$

$$= \frac{5(2)^3 + 4}{2 - 3} = -44$$

We used d) of theorem 2.5.1 to evaluate this limit. We would not be able to use it if the denominator turned out to be 0 as that is not allowed in Mathematics. If both top and bottom approach 0 as x approaches a, then the top and bottom will have a common factor of x – a. In this case the factors can be cancelled and the limit works out.

Example

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x + 2)(x - 2)}{(x - 2)}$$

$$= \lim_{x \to 2} (x + 2) = 4$$

Note that x is not equal to two after Simplification for the two functions to be the same. Nonetheless, we calculated the limit as if we were substituting x = 2 using rule for polynomials That's ok since REALLY LIMIT means you are getting close to 2, but not equaling it!!

What happens if in a rational functions, the bottom limit is 0, but top is not?? It's like the limit as x goes to 0 of f(x) = 1/x.

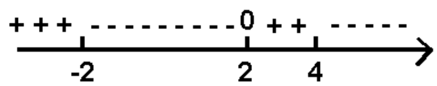
- The limit may be + inf
- The limits may be –inf
- +inf from one side and –inf from another

Example

Find $\lim\limits_{x \to 4^+} \dfrac{2-x}{(x-4)(x+2)}$

The top is –2 as x goes to 4 from right side. The bottom goes to 0, so the limit will be inf of some type. To get the sign on inf, Let's analyze the sign of the bottom for various values of real numbers

Break the number line into 4 intervals as in



The important numbers are the ones that make the top and bottom zero. As x approaches 4 from the right, the ratio stays negative and the result is –inf. You can say something about what happens from the left. Check yourselves by looking at the pic.

So $\lim\limits_{x \to 4^+} \dfrac{2-x}{(x-4)(x+2)} = -\infty$

LIMITS of Rational Functions as x goes to +inf and -inf

Algebraic manipulations simplify finding limits in rational functions involving +inf and –inf.

Example

$$\lim_{x \to -\infty} \frac{4x^2 - x}{2x^3 - 5}$$

Divide the top and the bottom by the highest power of x

$$\lim_{x \to -\infty} \frac{\dfrac{4x^2}{x^3} - \dfrac{x}{x^3}}{\dfrac{2x^3}{x^3} - \dfrac{5}{x^3}} = \lim_{x \to -\infty} \frac{\dfrac{4}{x} - \dfrac{1}{x^2}}{2 - \dfrac{5}{x^3}} = \frac{\lim\limits_{x \to -\infty} (\dfrac{4}{x} - \dfrac{1}{x^2})}{\lim\limits_{x \to -\infty} (2 - \dfrac{5}{x^3})}$$

$$= \frac{4 \lim\limits_{x \to -\infty} \dfrac{1}{x} - \lim\limits_{x \to -\infty} \dfrac{1}{x^2}}{2 - 5 \lim\limits_{x \to -\infty} \dfrac{1}{x^3}}$$

$$= \frac{4(0)-0}{2-5(0)} = 0$$

Quick Rule for finding Limits of Rational Functions as x goes to +inf or –inf

$$\lim_{x\to+\infty} \frac{c_0 + c_1 x + \ldots + c_n x^n}{d_0 + d_1 x + \ldots + d_n x^n} = \lim_{x\to+\infty} \frac{c_n x^n}{d_n x^n}$$

$$\lim_{x\to-\infty} \frac{c_0 + c_1 x + \ldots + c_n x^n}{d_0 + d_1 x + \ldots + d_n x^n} = \lim_{x\to-\infty} \frac{c_n x^n}{d_n x^n}$$

Not true if x goes to a finite number *a*.

Example

$$\lim_{x\to-\infty} \frac{4x^2 - x}{2x^3 - 5} = \lim_{x\to-\infty} \frac{4x^2}{2x^3} = \lim_{x\to-\infty} \frac{2}{x} = 0$$

Same answer as the one we got earlier from algebraic manipulations

Lecture # 11

Limits: A Rigorous Approach

In this section we will talk about

-**Formal Definition of Limit**

- Left-hand and Right-hand Limits

So far we have been talking about limits informally. We haven't given FORMAL mathematical definitions of limit yet. We will give a formal definition of a limit. It will include the idea of left hand and right hand limits. We intuitively said that

$$\lim_{x \to a} f(x) = L$$

means that as x approaches a, f (x) approaches L. The concept of "approaches" is intuitive.
The concept of "approaches" is intuitive so far, and does not use any of the concepts and theory of Real numbers we have been using so far.
So let's formalize LIMIT
Note that when we talked about "f (x) approaches L" as "x approaches a" from left and right, we are saying that we want f(x) to get as close to L as we want provided we can get x as close to *a* as we want as well, but maybe not equal to a since f (*a*) maybe undefined and f (*a*) may not equal L. So naturally we see the idea of INTERVALS involved here.
I will rephrase the statement above in intervals as

For any number $\varepsilon > 0$ *if we can find an open* interval (x_0, x_1)
on the x – axis containing a point a such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

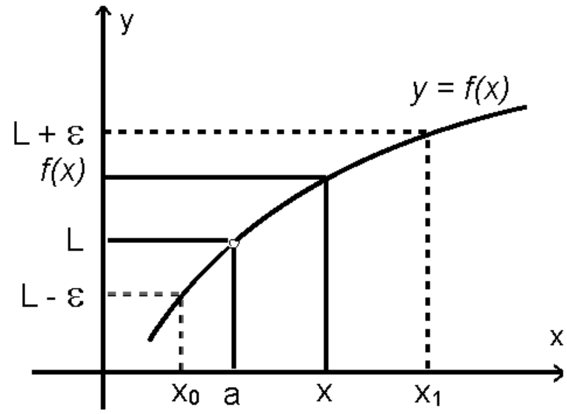
for each x in (x_0, x_1) *except possibly* $x = a$. *Then*
we say

$$\lim_{x \to a} f(x) = L$$

So, $f(x)$ *is in the* interval $(L - \varepsilon, L + \varepsilon)$

Now you may ask, what is this ε all about??
Well, it is the number that signifies the idea of "f (x) being as close to L as we want to be" could be a very small positive number, and that why it Let's us get as close to f (x) as we want. Imagine it to be something like the number at the bottom is called a GOOGOLPLEX!!

$$\frac{1}{10^{10^{100}}}$$



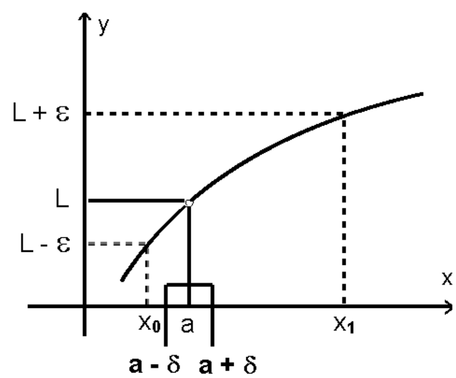
So $\varepsilon > 0$ but very close to it, and for ANY such ε we can find an interval on the x-axis that can confine *a*.
Let's pin down some details. Notice that
When we said

$L - \varepsilon < f(x) < L + \varepsilon$

holds for every x in the interval (x0, x1) (except possibly at x= a), it is the same as saying that the same inequality hold of all x in the interval set $(x_0, a) \cup (a, x_1)$.

But then the inequality $L - \varepsilon < f(x) < L + \varepsilon$ holds in any subset of this interval, namely $(x_0, a) \cup (a, x_1)$

ε is any positive real number smaller than $a - x_0$ and $x_1 - a$.
Look at figure below



$L - \varepsilon < f(x) < L + \varepsilon$

can be written as

$| f(x) - L | < \varepsilon$ *and* $(a - \delta, a) \cup (a, a + \delta)$ as $0 < | x - a | < \delta$.

Are same sets by picking numbers close to a and a-delta. Have them look at definition again and talk.
Let's use this definition to justify some GUESSES we made about limits in the previous lecture.

Example

Find

$\lim\limits_{x \to 2}(3x - 5) = 1$

Given any positive number ε we can find an δ such that

$\left|(3x-5)-1\right| < \varepsilon \quad \text{if } x \text{ satisfies} \quad 0 < \left|x-2\right| < \delta$

In this example we have

$f(x) = 3x - 5 \qquad L = 1 \qquad a = 2$

So our task is to find out the ε for which will work for any δ we can say the following

$$|(3x-5)-1|<\varepsilon \quad \text{if} \quad 0<|x-2|<\delta$$

$$\Rightarrow |3x-6|<\varepsilon \quad \text{if} \quad 0<|x-2|<\delta$$

$$\Rightarrow 3|x-2|<\varepsilon \quad \text{if} \quad 0<|x-2|<\delta$$

$$\Rightarrow |x-2|<\frac{\varepsilon}{3} \quad \text{if} \quad 0<|x-2|<\delta$$

Now we find our δ that makes our statement true. Note that the first part of the statement depends on the second part for being true. So our CHOICE of δ will determine the trueness of the first part.

I let $\frac{\varepsilon}{3}$ in the second part which makes the first part true. So we have

$$|x-2|<\frac{\varepsilon}{3} \quad \text{if} \quad 0<|x-2|<\frac{\varepsilon}{3}$$

Hence we have proved that

$$\lim_{x\to 2}(3x-5)=1$$

Example

$$f(x)=\begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

Show that $\lim_{x\to 0} f(x)$ does not exit.

Suppose that the Limit exists and its L. So $\lim_{x\to 0} f(x)=L$. Then for any $\varepsilon>0$ we can find $\delta>0$ such that

$$|f(x)-L|<\varepsilon \quad \text{if} \quad 0<|x-0|<\delta$$

In particular, if we take $\varepsilon=1$ there is a $\varepsilon>0$ such that

$$|f(x)-L|<1 \quad \text{if} \quad 0<|x-0|<\delta$$

But $x=\frac{\delta}{2}$ and $x=-\frac{\delta}{2}$ both satisfy requirement above, so

$\left| f(\frac{\delta}{2}) - L \right| < 1$ and $\left| f(-\frac{\delta}{2}) - L \right| < 1$

But $\frac{\delta}{2}$ is positive and $-\frac{\delta}{2}$ is negative, so

$f(\frac{\delta}{2}) = 1$ and $f(-\frac{\delta}{2}) = -1$

So we get

$|1 - L| < 1$ and $|-1 - L| < 1$

$\Rightarrow 0 < L < 2$ and $-2 < L < 0$

But this is a contradiction since L cannot be between these two bounds at the same time.

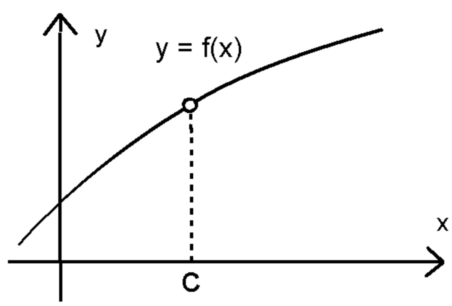
LECTURE # 12

Continuity

- Develop the concept of CONTINUITY by examples
- Give a mathematical definition of continuity of functions
- Properties of continuous functions
- Continuity of polynomials and rational functions
- Continuity of compositions of functions
- The Intermediate values theorem

CONTINUITY of a function becomes obvious from its graph at certain points in the plane .We will say CONTINUITY of a function or graph of a function interchangeably.

DISCONTINUITY



The above given curve is discontinuous at point c since f(x) is not defined there.
when the following things happens then there is a break or discontinuity in the graph of a function f(x) at x = c

- f is undefined at c
- The $\lim_{x \to c} f(x)$ does not exist.
- The function is defined at c and the $\lim_{x \to c} f(x)$ exists, but the values of f(x) and the values of the limit differ at the point c
- So we get the following definition for continuity

Definition2.7.1

(a) f(c) is defined

(b) $\lim_{x\to c} f(x)$ exists

(c) $\lim_{x\to c} f(x) = f(c)$

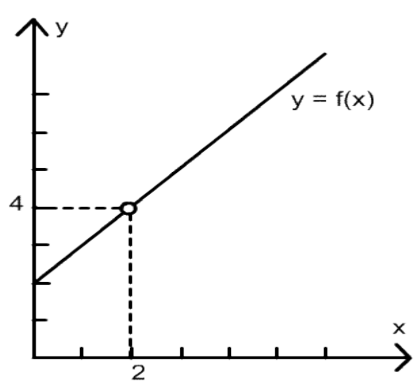
If any of these conditions in this definition fail to hold for a function f(x) at a point c, then f is called discontinuous at c

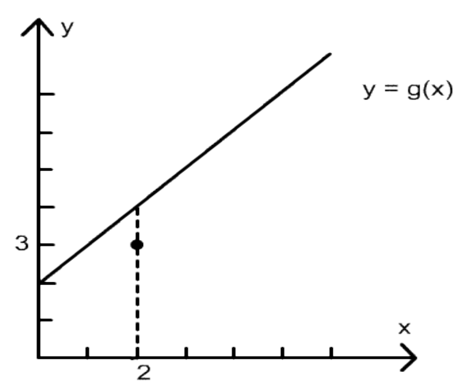
- c is called the point of discontinuity
- If f(x) is continuous at all points in an interval (a, b), then we say that f is **continuous on (a,b)**
- A function continuous on the interval $(-\infty, +\infty)$ is called a **continuous function**

Example

$$f(x) = \frac{x^2-4}{x-2}$$

$$g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$





f is discontinuous at x = 2 because f(2) is undefined.

g is discontinuous because g(2)=3 and

$$\lim_{x\to 2} g(x) = \lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4$$

So

$$\lim_{x\to 2} g(x) \neq g(2)$$

The last equation does not satisfy the condition of continuity. Condition (3) of the definition is enough to determine whether a function is continuous or not. This is so because if (3) is true, then (1) and (2) have to be true

Example

Show that $f(x) = x^2 - 2x + 1$ is a continuous function.

CONTINUOUS means continuous at all real numbers. Show that part (3) of definition is met for all real number *c*. By what we know about polynomials so far, we have

$$\lim_{x \to c} f(x) = f(c)$$

So

$$\lim_{x \to c}(x^2 - 2x + 1) = c^2 - 2c + 1$$

Part (3) is met and f(x) is continuous

Theorem 2.7.2

Polynomials are continuous functions.

Proof:

If P is polynomial and c is any real number then by theorem 2.5.2

$$\lim_{x \to c} p(x) = p(c)$$

Where *p* is a polynomial, and *c* is any real number. Since *c* is any real number, it follows that *p(x)* is continuous.

Example

Show that $f(x) = |x|$ is continuous

Rewrite f(x) as

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Show $\lim_{x \to c} f(x) = f(c)$ for any real number *c*.

Let $c \geq 0$. *Then* $f(c) = c$ *by definition of f (x).*

Also $\lim_{x \to c} f(x) = \lim_{x \to c} |x| = \lim_{x \to c} x = c$ Since $c \geq 0$

x may be negative to begin with, but since it approaches c which is positive or 0, we use the first part of the definition of f(x) to evaluate the limit

That is just f(x) = x which is a polynomial and hence we get the desired result.

Now let $c < 0$. Then again $f(c) = -c$ by definition of f (x) and

$$\lim_{x \to c} f(x) = \lim_{x \to c} |x| = \lim_{x \to c} x = -c$$

x may be Positive or 0 to begin with, but since it approaches c which is negative, we use the Second part of the definition of f(x) to evaluate the limit. That is just f(x) = -x which is a polynomial and hence we get the desired result.

Properties of Continuous Functions

Theorem 2.7.3

If the function f and g are continuous at c, then

a) f + g is continuous at c;

b) f – g is continuous at c;

c) f . g is continuous at c;

d) f/g is continuous at c if $g(c) \neq 0$ and is discontinuous at c if g(c)=0

PROOF

Let f and g be continuous function at the number c

Then

$$\lim_{x \to c} f(x) = f(c)$$

$$\lim_{x \to c} g(x) = g(c)$$

So

$$\lim_{x \to c} f(x).g(x) = \lim_{x \to c} f(x).\lim_{x \to c} g(x) \quad \text{by Limit Rules}$$

$$= f(c).g(c) \quad \text{by continuity of } f \text{ and } g$$

Continuity of Rational Functions

Example

Where is $h(x) = \dfrac{x^2 - 9}{x^2 - 5x + 6}$ continuous?

Since the top and the bottom functions in h are polynomials, they are continuous everywhere Hence, by property (d) of theorem 2.7.3, h will be continuous at all points c as long as $g(c) \neq 0$.

$$x^2 - 5x + 6 = 0$$

Will give us all the x values where h will be discontinuous. These are $x = 2$ $x = 3$ which you get after solving the above equation for x.

Continuity of Composition of functions

Theorem 2.7.5

Let limit stand for one of the limits $\lim\limits_{x\to c}$, $\lim\limits_{x\to c^+}$, $\lim\limits_{x\to c^-}$, $\lim\limits_{x\to +\infty}$, or $\lim\limits_{x\to -\infty}$. If $\lim g(x) = L$ and if the function f is continuous at L, Then $\lim f(g(x)) = f(L)$. that is $\lim f(g(x)) = f(\lim g(x))$.

Example

$f(x) = \left|5 - x^2\right|$

Here, $f(x) = |x|$, $g(x) = 5 - x^2$

SO by theorem 2.7.5

$$\lim_{x\to 3}\left|5 - x^2\right| = \left|\lim_{x\to 3} 5 - x^2\right| = |-4| = 4$$

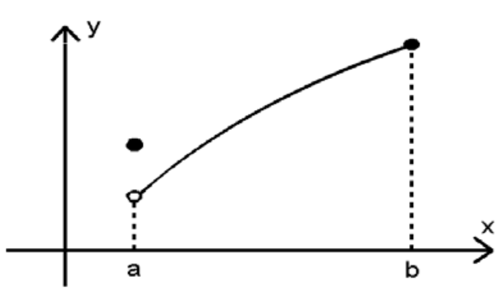
Theorem 2.7.6

If the function g is continuous at the point c and the function f is continuous at the point g (c), then the composition $f \circ g$ is continuous at c.

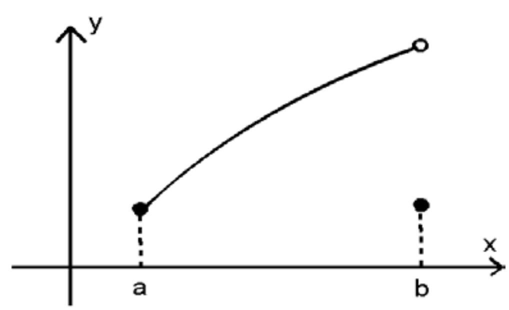
Continuity from the left and right

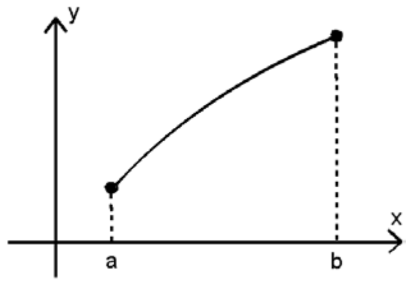
Definition we use does not incorporate end points as at end points only left hand or right hand limits make sense

(a)



(b)





(c)

- Graph of function in a) shows that *f* is discontinuous at *a*
- Graph of function in b) shows that *f* is discontinuous at *b*
- Graph of function in c) shows that *f* is continuous at *a* and *b*

Definition 2.7.7

A function f is called **continuous from the left at point c** if the conditions in the left column below are satisfied, and is called **continuous from the right at the point c** if the conditions in the right column are satisfied.

1. f(c) is defined.	1`. f(c) is defined
2. $\lim\limits_{x \to c^-} f(x)$ *exists.*	2`. $\lim\limits_{x \to c^+} f(x)$ *exists.*
3. $\lim\limits_{x \to c^-} f(x) = f(c).$	3`. $\lim\limits_{x \to c^+} f(x) = f(c).$

Definition 2.7.8

A function f is said to be continuous on a closed interval [a, b] if the following conditions are satisfied:

1. *f* is continuous on (a, b).
2. *f* is continuous from the right at a.
3. *f* is continuous from the left at b.

EXAMPLE

Show that $f(x) = \sqrt{9 - x^2}$ is continuous on the interval [3,-3].By definition 2.7.8 and theorem 2.5.1(e), for *c* in (3,-3)

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c}(9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

So *f* is continuous on (3,-3) Also

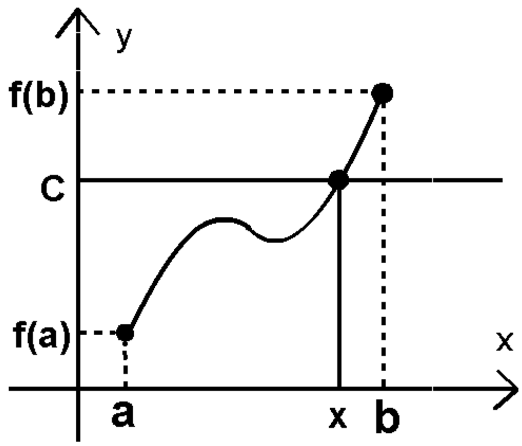
$$\lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-}(9 - x^2)} = f(3) = 0$$

$$\lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = f(3) = 0$$

Why approach 3 from the left and –3 from the right? Well, draw the graph of this function and you will see WHY!?? So f is continuous on [-3, 3].

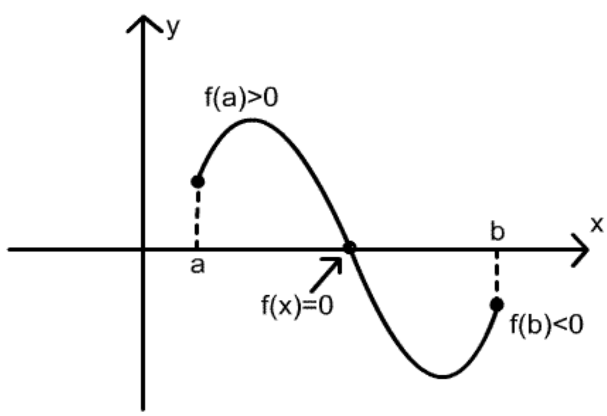
Intermediate Value Theorem(Theorem 2.7.9)

If *f* is continuous on a closed interval [a, b] and C is any number between *f(a)* and *f(b)*, inclusive, then there is at least one number *x* in the interval [a, b] such that *f(x)* = C.



Theorem 2.7.10

If *f* is continuous on [a, b], and if *f(a)* and *f(b)* have opposite signs, then there is at least one solution of the equation *f(x)* = 0 in the interval (a, b).



Example

$$x^3 - x - 1 = 0$$

Cannot be solved easily by factoring. However, by the MVT, f(1) = -1 and f(2) = 5 implies that the equation has one solution in the interval (1,2).

LECTURE # 13

Limits and continuity of Trigonometric functions

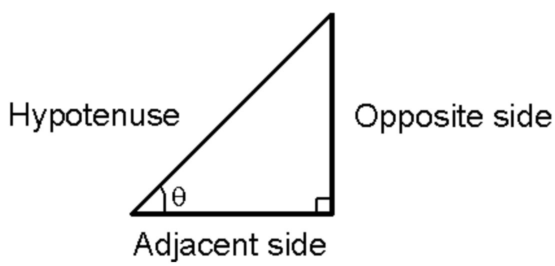
- Continuity of Sine and Cosine functions
- Continuity of other trigonometric functions
- Squeeze Theorem
- Limits of Sine and Cosine as x goes to +- infinity

You will have to recall some trigonometry. Refer to Appendix B of your textbook.

Continuity of Sine and Cosine

Sin and Cos are ratios defined in terms of the acute angle of a right angle triangle and the sides of the triangle. Namely,

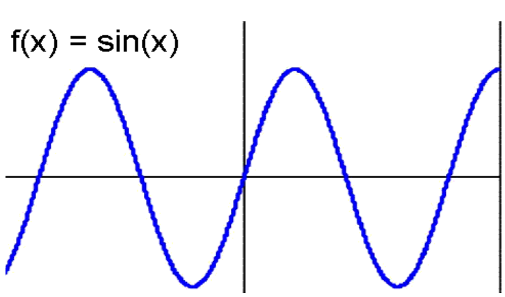
$$\cos\theta = \frac{adjacent\ side}{Hypoteneous} \qquad \sin\theta = \frac{Opposite\ side}{Hypoteneous}$$

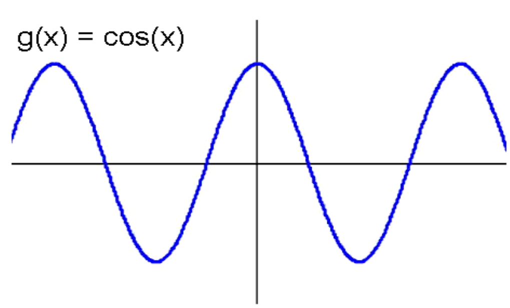


We look at these ratios now as functions. We consider our angles in radians

- Instead of θ we will use x

Here is a picture that shows the graph of f(x) = sin (x). Put the circle picture here, and then unravel it and get the standard picture.



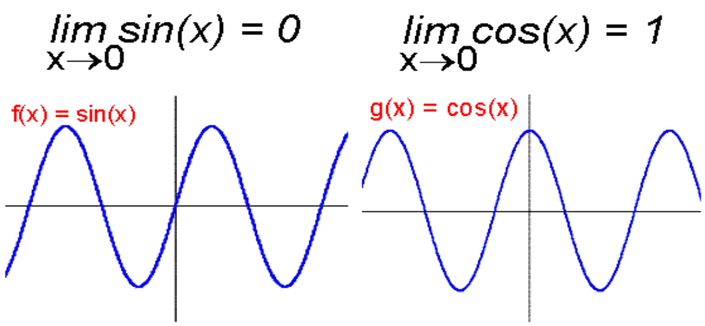


From the graph of Sin and cosine, its obvious that

$$\lim_{x \to 0} \sin(x) = 0 \qquad\qquad \lim_{x \to 0} \cos(x) = 1$$

This is the intuitive approach. Prove this using the Delta Epsilon definitions!!
Note that sin(0) = 0 and cos(0) = 1
Well, the values of the functions match with those of the limits as x goes to 0!! So we have this theorem
THEOREM 2.8.1

The functions sin(x) and cos(x) are continuous. As clear from figure



Here is the definition of continuity we saw earlier.
A function *f* is said to be continuous at *c* if the following are satisfied

(a) f(c) is defined
(b) $\lim_{x \to c} f(x)$ exists
(c) $\lim_{x \to c} f(x) = f(c)$

Let $h = x - c$. So $x = h + c$. Then $x \to c$ is equivalent to the requirement that $h \to 0$. So we have

Definition

A function is continuous at *c* if the following are met

(a) f(c) is defined
(b) $\lim_{h \to 0} f(h+c)$ exists
(c) $\lim_{h \to 0} f(h+c) = f(c)$

We will use this new definition of Continuity to prove

Theorem 2.8.1
The functions sin(x) and cos(x) are continuous.

Proof

We will assume that $\lim_{x \to 0} \sin(x) = 0$ and $\lim_{x \to 0} \cos(x) = 1$

From the above, we see that the first two conditions of our continuity definition are met. So just have to show by part 3) that

$$\lim_{h \to 0} \sin(c+h) = \sin(c)$$

$$\begin{aligned}\lim_{h \to 0} \sin(c+h) &= \lim_{h \to 0} [\sin(c)\cos(h) + \cos(c)\sin(h)] \\ &= \lim_{h \to 0} \sin(c)\cos(h) + \lim_{h \to 0} \cos(c)\sin(h) \\ &= \sin(c)\lim_{h \to 0} \cos(h) + \cos(c)\lim_{h \to 0} \sin(h) \\ &= \sin(c)(1) + \cos(c)(0) = \sin(c)\end{aligned}$$

The continuity of cos(x) is also proved in a similar way, and I invite you to try do that!

Continuity of other trigonometric functions

Remember by theorem 2.7.3 that if f(x) and g(x) are continuous , then so is $h(x) = \dfrac{f(x)}{g(x)}$. Except where g(x) = 0. So tan(x) is continuous everywhere except at cos(x) = 0 which gives

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$x = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, \ldots$$

Likewise, since

$$\cot(x) = \frac{\cos(x)}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)} \quad \operatorname{cosec}(x) = \frac{1}{\sin(x)}$$

We can see that they are all continuous on appropriate intervals using the continuity of sin(x) and cos(x) and theorem 2.7.3

Squeeze Theorem for finding Limits

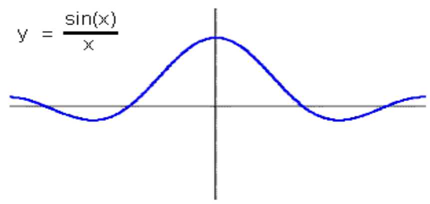
We will show that .These is important results which will be used later. If you remember, the very first example of limits we saw was

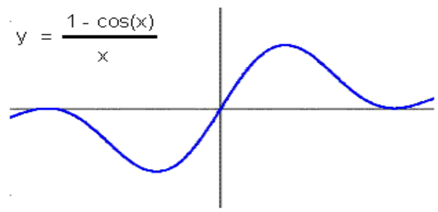
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$$

Now we prove this

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

Here are the graphs of the functions.





They suggest that the limits are what we want them to be! We need to prove this PROBLEM. As x goes to 0, both the top and the bottom functions go to 0. Sin(x) goes to 0 means that the fraction as a whole goes to 0.
x goes to zero means that the fraction as a whole goes to +inf! . There is a tug of war between the Dark Side and the Good Side of the Force.
So there is a tug-of-war between top and bottom.
To find the limit we confine our function between two simpler functions, and then use their limits to get the one we want.

SQUEZZING THEOREM

Let fm g and h be functions satisfying $g(x) \le f(x) \le h(x)$ for all x in some open interval containing the point a, with the possible exception that the inequality need not to hold at a.

We g and h have the same limits as x approaches to a, say

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

Then f also has this limit as x approaches to a, that is

$$\lim_{x \to a} f(x) = L .$$

Example

$$\lim_{x \to 0} x^2 \sin^2(\frac{1}{x})$$

Remember that the $0 \le \sin(x) \le 1$.

So certainly $0 \le \sin^2(x) \le 1$.

And so $0 \le \sin^2(\frac{1}{x}) \le 1$.

Multiply throughout this last inequality by x^2.
We get

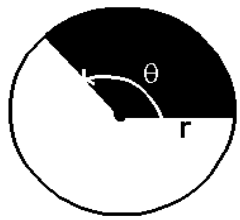
$$0 \le x^2 \sin^2(\frac{1}{x}) \le x^2 ,$$

But $\lim_{x\to 0} 0 = \lim_{x\to 0} x^2 = 0$

So by the Squeezing theorem

$$\lim_{x\to 0} x^2 \sin^2(\frac{1}{x}) = 0$$

Now Let's use this theorem to prove our original claims. The proof will use basic facts about circles and areas of SECTORS with center angle of θ radians and radius *r*.



The area of a sector is given by $A = \frac{1}{2} r^2 \theta$.

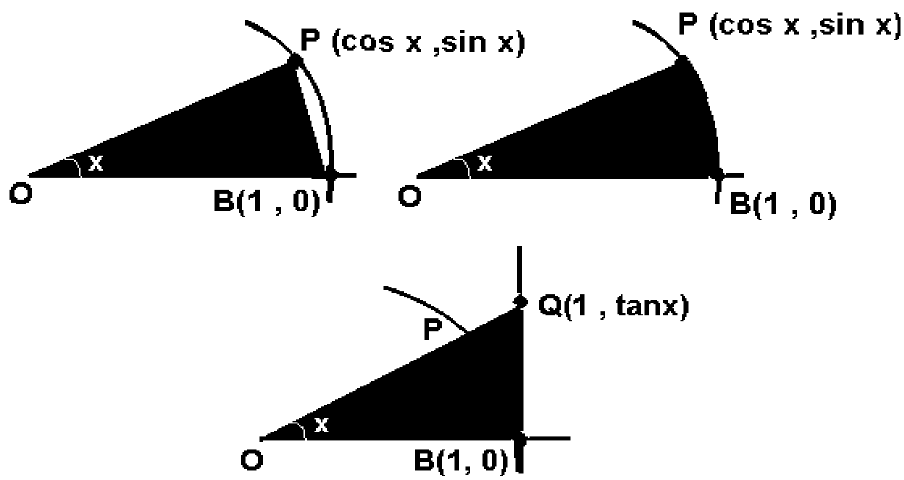
Theorem 2.8.3

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Proof

Let *x* be such that $0 < x < \frac{\pi}{2}$. Construct the angle *x* in the standard position starting from the center of a unit circle.

We have the following scenario



From the figure we have

$0 <$ area of $\Delta\ OBP <$ area of sector $OBP <$ area of ΔOBQ

Now

$$\text{area of } \Delta\ OBP = \frac{1}{2} base.height = \frac{1}{2}(1).\sin(x) = \frac{1}{2}\sin(x)$$

$$\text{area of sector } OBP = \frac{1}{2}(1)^2.x = \frac{1}{2}x$$

$$\text{area of } \Delta\ OBQ = \frac{1}{2} base.height = \frac{1}{2}(1)\tan(x) = \frac{1}{2}\tan(x)$$

So

$$0 < \frac{1}{2}\sin(x) < \frac{1}{2}x < \frac{1}{2}\tan(x)$$

Multiplying through by $\frac{2}{\sin(x)}$ gives

$$1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$$

Taking reciprocals gives

$$\cos(x) < \frac{\sin(x)}{x} < 1$$

We had made the assumption that $0 < x < \frac{\pi}{2}$.

Also works when $-\frac{\pi}{2} < x < 0$ You can check when you do exercise 4.9 So our last equation holds for all angles x except for $x = 0$.

Remember that $\lim_{x\to 0}\cos(x) = 1$ and $\lim_{x\to 0} 1 = 1$

Taking limit now and using squeezing theorem gives

$$\lim_{x\to 0}\cos(x) < \lim_{x\to 0}\frac{\sin(x)}{x} < \lim_{x\to 0} 1$$

$$= 1 < \lim_{x\to 0}\frac{\sin(x)}{x} < 1$$

Since the middle thing is between 1 and 1, it must be 1!!

Prove yourself that

$$\lim_{x\to 0}\frac{1-\cos(x)}{x} = 0$$

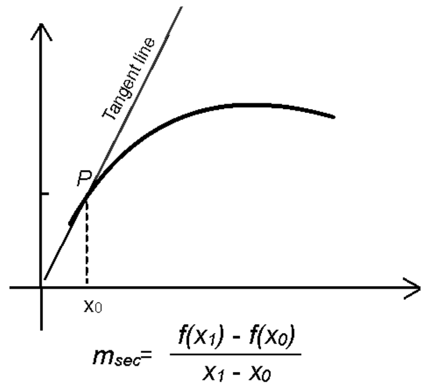
Limits of sin(x) and cos(x) as x goes to +inf or –inf

By looking at the graphs of these two functions its obvious that the y-values oscillate btw 1 and –1 as x goes to +inf or –inf and so the limits DNE!!

LECTURE # 14

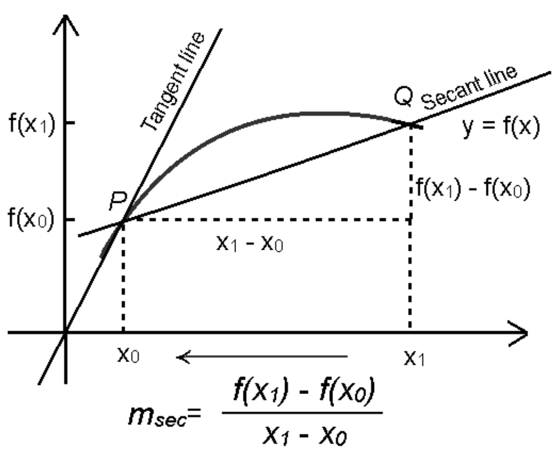
Tangent Lines and Rates of Change

In lecture 9, we saw that a Secant line between two points was turned into a tangent line. This was done by moving one of the points towards the other one. The secant line rotated into a LIMITING position which we regarded as a TANGENT line.



For now just consider Secant lines joining two points on a curve (graph) if a function of the form y = f (x).
If $P(x_0, y_0)$ and $Q(x_1, y_1)$ are distinct points on a curve y = f(x), then secant line connecting them has slope

$$m_{\sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



If we let $x_1 \to x_0$ then Q will approach P along the graph of the function y = f(x), and the secant line will approach the tangent line at P.

This will mean that the slope of the Secant line will approach that of the Tangent line at P as $x_1 \to x_0$, So we have the following

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

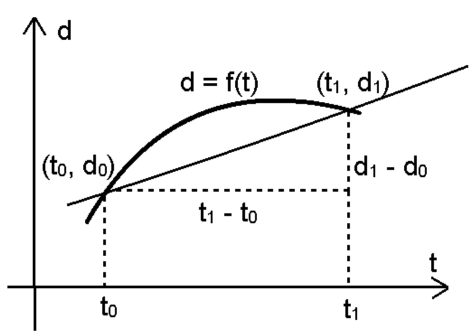
We just saw how to find the slope of a tangent line.

This was a geometric problem. In the 17th century, mathematicians wanted to define the idea of Instantaneous velocity. This was a theoretical idea. But they realized that this could be defined using the geometric idea of tangents.

Let's define Average velocity formally

$$\text{Avergae Velocity} = \frac{\text{distance travelled}}{\text{Time Elapsed}}$$

This formula tells us that the average velocity is the velocity at which one travels on average during some interval of time!!
More interesting than Average Velocity is the idea of Instantaneous velocity. This is the velocity that an object is traveling at a given INSTANT in time. When a car hits a tree, the damage is determined by the INSTANTANEOUS velocity at the moment of impact, not on the average speed during some time interval before the impact.
To define the concept of instant velocity, we will first look at distance as a function of time,
d = f(t). After all, distance covered is a physical phenomenon which is always measured with respect to time.
Going from New York to San Francisco (km) takes about 6 hours, if your average speed is 800 km/h. This will give us a way to plot the position versus time curve for motion.
Now we will give a geometric meaning to the concept of Average Velocity.
Average velocity is defined as the distance traveled over a given time of period. So if your curve for f(t) looks like as given below



then the average velocity over the time interval $[t_0, t_1]$ is defined as

$$\text{Avergae Velocity} = \frac{\text{distance traveled during the interval}}{\text{Time Elapsed}}$$

$$\frac{d_1 - d_0}{t_1 - t_0} = v_{ave} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

$d_1 - d_0$ is the distance traveled in the interval.

So average velocity is just the slope of the Secant line joining the points (t_0, d_0) and (t_1, d_1).

Say we want to know the instantaneous velocity at the point t_0. We can find this by letting t1 approach t_0. When this happens, the interval over which the average velocity is measured shrinks and we can approximate instant velocity.

As t_1 gets very close to t_0, our approximate instantaneous velocity will get better and better. As this continues, we can see that the average velocity over the interval gets closer to instantaneous velocity

at t_0. So we can say

$$v_{inst} = \lim_{t_1 \to t_0} v_{ave} = \lim_{t_1 \to t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

But this is just the slope of the tangent line at the point (t_0, d_0) Remember that the limit here means that the two sided limits exist.

Average and Instantaneous rates of change

Let's make the idea of average and instantaneous velocity more general. Velocity is the rate of change of position with respect to time. Algebraically we could say:
Rate of change of **d** with respect to **t** .Where d = f(t).
Rate of change of bacteria w.r.t time.
Rate of change a length of a metal rod w.r.t to temperature
Rate of change of production cost w.r.t quantity produced.
All of these have the idea of the rate of change of one quantity w.r.t another quantity.

We will look at quantities related by a functional relationship y = f(x)
So we consider the rate of change of ***y*** w.r.t ***x*** or in other words, the rate of change of the dependant variable (quantity) w.r.t the Independent variable (quantity) .
Average rate of change will be represented by the slope of a certain Secant Line.
Instantaneous rate of change will be represented by the slope of a certain tangent Line.

Definition 3.1.1

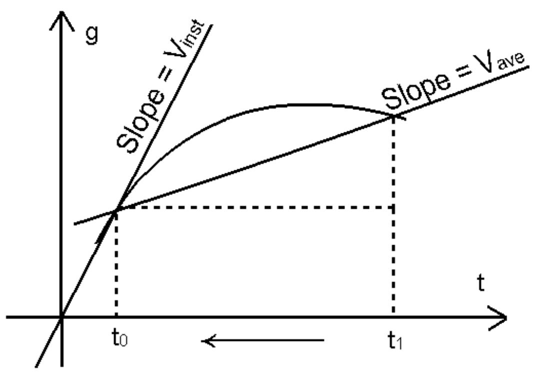
If $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is the slope $m_{\sec}$ of the secant line joining the points (x_0, $f(x_0)$) and (x_1, $f(x_1)$) on the graph of f

$$m_{\sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

If $y = f(x)$, then the **Average rate of Change of *y* with respect to *x* over the interval** $[x_0, x_1]$ is the slope of the secant line joining the points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$. That is

$$m_{\sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

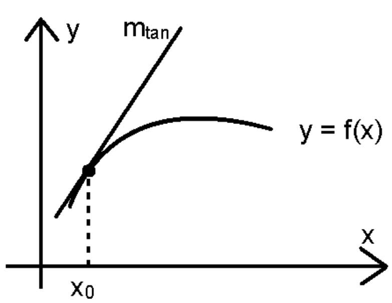
And on the graph of f.



Definition 3.1.1

If $y = f(x)$, then the **instantaneous rate of Change of *y* with respect to *x*** at the point x_0 is the slope $m_{\tan}$ of the tangent line to graph of f at the point x_0, that is

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Example

Let $y = f(x) = x^2 + 1$

a) Find the average rate of *y* w..r.t to *x* over the interval [3,5]

b) Find the instantaneous rate of change of y w.r.t *x* at the $x = x_0$ point x_0 = -4

c) Find the instantaneous rate of change of ***y*** w.r.t ***x*** at a general point

Solution:

We use the formula in definition of Average rate with

$y = f(x) = x^2 + 1,\ \ x_0 = 3$ and $x_1 = 5$

$$m_{\sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5-3} = \frac{26-10}{5-3} = 8$$

So *y* increase 8 units for each unit increases in ***x*** over the interval [3,5]

b) Applying the formula with $y = f(x) = x^2 + 1$ and $x_0 = -4$ gives

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to -4} \frac{(x_1^{\ 2} + 1) - 17}{x_1 + 4}$$
$$= \lim_{x_1 \to -4} \frac{x_1^{\ 2} - 16}{x_1 + 4} = \lim_{x_1 \to -4} (x_1 - 4) = -8$$

Negative inst rate of change means its DECREASING

c) Here we have

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{(x_1^{\ 2} + 1) - (x_0^{\ 2} + 1)}{x_1 - x_0}$$
$$= \lim_{x_1 \to x_0} \frac{x_1^{\ 2} - x_0^{\ 2}}{x_1 - x_0} = \lim_{x_1 \to x_0} (x_1 + x_0) = 2x_0$$

The result of part b) can be obtained from this general result by letting.

Lecture # 15

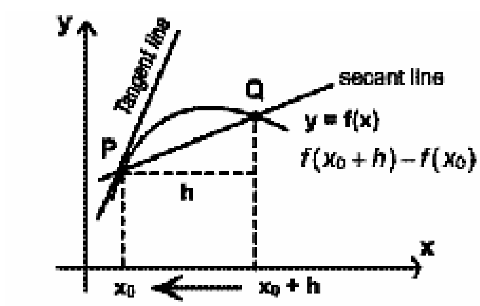
The Derivative

In the previous lecture we saw that the slope of a tangent line to the graph of y = f (x) is given by

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Let's do some algebraic manipulations. Let
$h = x_1 - x_0$ so that $x_1 = x_0 + h$ and $h \to 0$ as $x_1 \to x_0$.
So we can rewrite the above tangent formula as

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



Definition 3.2.1

If P(x_0, y_0) is a point on the graph of a function f then the tangent line to the graph of f at P is defined to be the line through P with slope

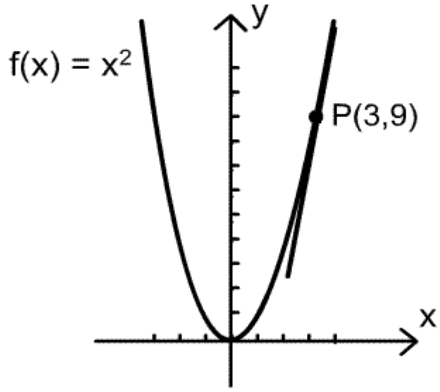
$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Tangent line at $P(x_0, y_0)$ is just called the tangent line at x_0 for brevity. Also a point $P(x_0, y_0)$ make here that the Equation. We make this definition provided that the LIMIT in the definition exists! Equation of the tangent line at the point $P(x_0, y_0)$ is

$$y - y_0 = m_{\tan}(x - x_0)$$

Example

Find the slope and an equation of the tangent line to the graph of $f(x) = x^2$ at the point P(3,9). Here is the



We use the formula given in the above definition with $x_0 = 3$ and $y_0 = 9$.

First we find the slope of the tangent line at $x_0 = 3$

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{(3+h)^2-9}{h} \\ &= \lim_{h\to 0}\frac{(9+6h+h^2)-9}{h} = \lim_{h\to 0}\frac{6h+h^2}{h} \\ &= \lim_{h\to 0}\frac{h(6+h)}{h} = \lim_{h\to 0}(6+h) = 6 \end{aligned}$$

Now we find the equation of the tangent line

$$\begin{aligned} &y-9 = 6(x-3) \\ &\Rightarrow y = 6x-9 \end{aligned}$$

Now notice that $m_{\tan}$ is a function of x_0 because since it depends on where along the curve is being computed. Also, from the formula for it, it should be clear that h eventually shrinks to 0 and whatever is left will be in terms of x_0. This can be further modified by saying that we will call x_0 is x . Then we have $m_{\tan}$ as a function of x and this is nice.

Since now we can say that we have associated a new function $m_{\tan}$ to any given function. We can rewrite the formula for $m_{\tan}$ as

$$m_{\tan} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

This is a function of *x* and its very important. Its called the Derivative function with respect to *x* for the function *y = f (x)*

Definition 3.2.2

The function f defined by the formula

$$f' = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is called the derivative with respect to x of the function f. The domain of f' consists of all x for which limit exists.

We can interpret this derivative in 2 ways **Geometric interpretation of the Derivative** *f'* is the function whose value at *x* is the slope of the tangent line to the graph of the function *f* at *x* **Rate of Change is an interpretation of Derivative.** If y = f(x), then *f'* is the function whose value at *x* is the instantaneous rate of change of y with respect to *x* at the point *x*.

Example

Let $f(x) = x^2 + 1$

Find *f' (x)*.

Use the result from part a) to find the slope of the tangent line to

$y = f(x) = x^2 + 1$

$$\begin{aligned} f(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2+1]-[x^2+1]}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2+1-x^2-1}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &= \lim_{h\to 0}(2x+h) = 2x \end{aligned}$$

we show that the slope of the tangent line at ANY point *x* is $f'(x) = 2x$, So at point *x* = 2 we have slope $f'(2) = 2(2) = 4$ at point *x* = 0 we have slope $f'(0) = 2(0) = 0$ at point *x* = -2 we have slope $f'(-2) = 2(-2) = -4$

Example 3

It should be clear that at each point on a straight line $y = mx + b$ the tangent line coincides with the line itself. So the slope of the tangent line must be the same as that of the original line, namely *m*. We can prove this here

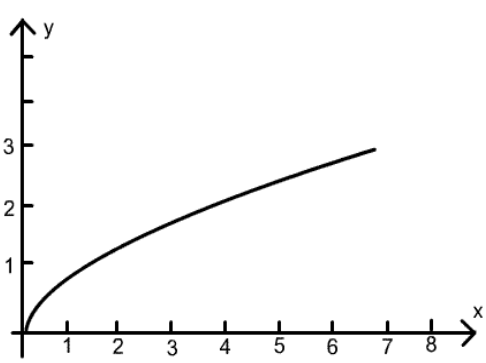
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[m(x+h)+b]-(mx+b)}{h}$$
$$= \lim_{h\to 0}\frac{mx+mh+b-mx-b}{h} = \lim_{h\to 0}\frac{mh}{h} = \lim_{h\to 0} m = m$$

Find the derivative with respect to *x* of $f(x) = \sqrt{x}$

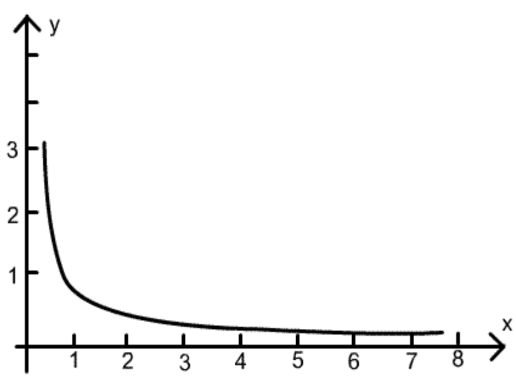
$$f(x) = \sqrt{x}$$
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Here are the graphs of f (x) and its derivative we just found. Note that

$$\lim_{x\to 0^+}\frac{1}{2\sqrt{x}} = +\infty$$



y=f(x)=$\sqrt{x}$



y=f '(x)= 1/2$\sqrt{x}$

the derivative of graph shows that as x goes to 0 from the right side, the slopes of the tangent lines to the graph of y = f (x) approach +inf, meaning that the tangent lines start getting VERTICAL!! Can you see this??!

Derivative Notation

The process of finding the derivative is called DIFFERENTIATION.

It is useful often to think of differentiation as an OPERATION that is applied to a given function to get a new one *f'*. Much like an arithmetical operation +

In case where the independent variable is *x* the differentiation operation is written as

This is read as "the derivative of *f* with respect to *x* $\frac{d}{dx}[f(x)]$

So we are just giving a new notation for the same idea but this will help us when we want to think of Derivative or Differentiation from a different point of view $\frac{d}{dx}[f(x)] = f'(x)$.

With this notation we can say about the previous example

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}[\sqrt{x}]\Big|_{x=x_0} = \frac{1}{2\sqrt{x}}\Bigg|_{x=x_0} = \frac{1}{2\sqrt{x_0}}$$

If we write *y*= f(x), then we can say

$$\frac{d}{dx}[y] = f'(x)$$

$$\frac{dy}{dx} = f'(x)$$

So we could say for the last example

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

This looks like a RATIO, and later we will see how this is true in a certain sense. But for now $\frac{dy}{dx}$ should be regarded as a single SYMBOL for the derivative of a function y = f(x).

If the independent variable is not *x* but some other variable, then we can make appropriate adjustments. If it is *u,* then

$$\frac{dy}{du} = f'(u) \quad \text{and} \quad \frac{d}{du}[f(u)] = f'(u)$$

One more notation can be used when one wants to know the value of the derivative a a certain point $x = x_0$

$$\frac{d}{dx}[f(x)]\Big|_{x=x_0} = f'(x_0)$$

For example

$$\frac{d}{dx}[\sqrt{x}]\Big|_{x=x_0} = \frac{1}{2\sqrt{x}}\Big|_{x=x_0} = \frac{1}{2\sqrt{x_0}}$$

Existence of Derivatives

From the definition of the derivative, it is clear that the derivative exists only at the points where the limit exists.

If x_0 is such a point, then we say that *f* **is differentiable at** x_0 OR ***f*** **had a derivative at** x_0. This basically defines the domain of *f'* as those points *x* at which *f* is differentiable

f is **differentiable on an open interval (a,b)** if it is differentiable at EACH point in (a,b). *f* is **differentiable function** if its differentiable on the interval. The points at which *f* is not differentiable, we will say **the derivative of *f* does not exist at those points.**

Non differentiability usually occurs when the graph of f(x) has

- **corners**
- **Vertical tangents**
- **Points of discontinuity**

Let's look at each case and get a feel for why this happens

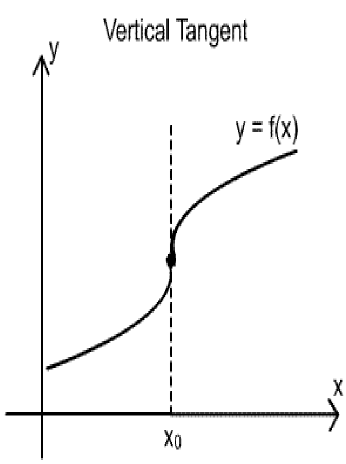


Fig 3.2.6(b)

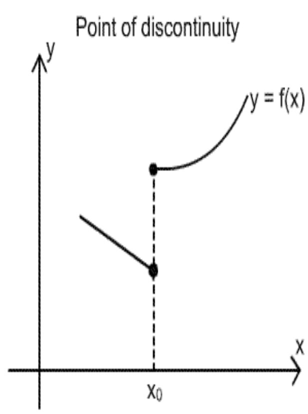


Fig 3.2.6(c)

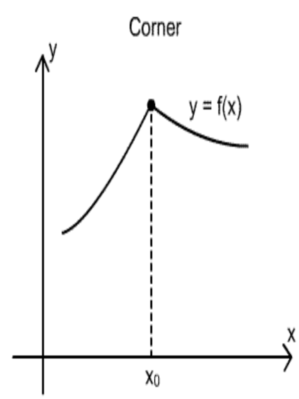
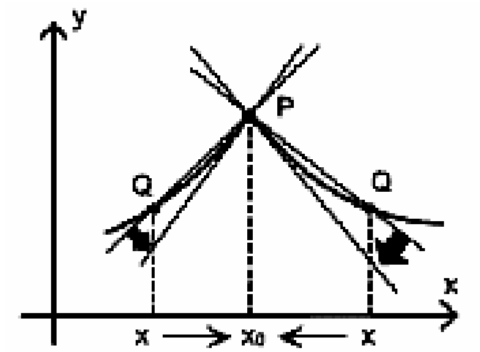


Fig 3.2.6(a)

At corners, the two sided limits don't match up when we take the limit of the secant lines to get the slope of the tangents



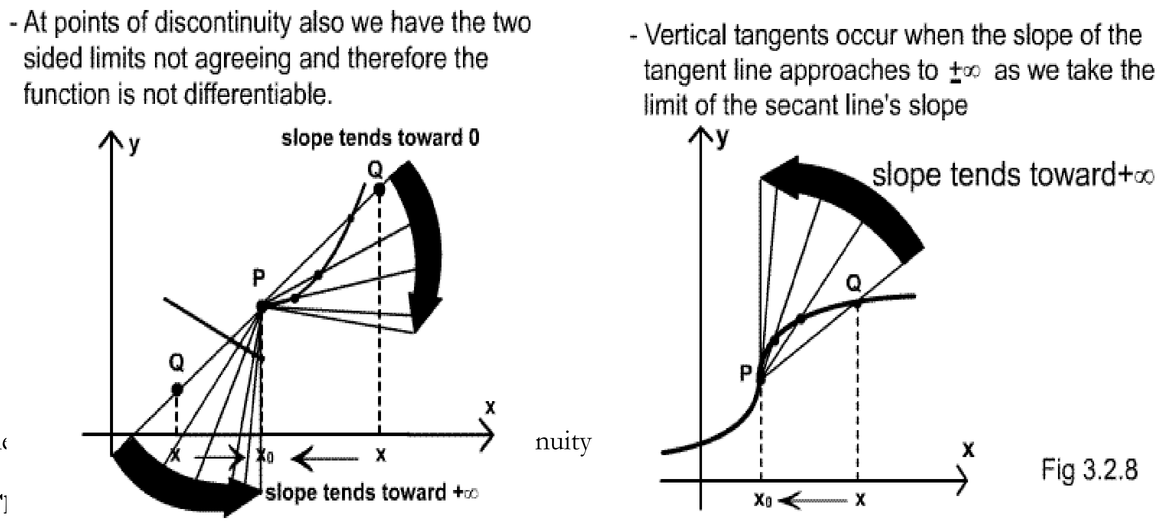


Fig 3.2.8

Rc nuity

Tl

If f is differentiable at a point x_0 then f is also continuous at x_0.

Proof

We will use this definition we saw earlier of continuity $\lim_{h\to 0} f(h+x_0) = f(x_0)$ Where x_0 is any point. So we will show that

$$\lim_{h\to 0} f(x_0+h) = f(x_0) \text{ or equivalently,}$$

$$\lim_{h\to 0}[f(x_0+h) - f(x_0)] = 0$$

$$\lim_{h\to 0}[f(x_0+h)-f(x_0)] = \lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}\cdot h\right]$$

$$= \lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}\right]\cdot \lim_{h\to 0} h$$

$$= f'(x_0).0 = 0$$

So this theorem says that a function cannot be differentiable at a point of discontinuity

Example

$$f(x) = |x|$$

Find *f'* (x).

Remember that $|x| = \sqrt{x^2}$ .Can you use this to differentiate? Yes, but for this we need more theory and we will see how to do this later

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$f'(x) = \frac{d}{dx}|x| = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Lecture # 16

Techniques Of Differentiation

- In the lectures so far, we obtained some derivatives directly by definition.
- In this sections, we develop theorems which will give us short cuts for calculation derivatives of special functions
- Derivatives of Constant Functions
- Derivatives of Power functions
- Derivative of a constant multiple of a function Etc!!

Derivatives of Constant Functions
Theorem 3.3.1
If "f" is a constant function f(x)=c for all x, then

$$f'(x)=\frac{d}{dx}[f(x)]=\frac{d}{dx}[c]=0$$

Proof

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Since f(x) = c so $\lim_{h\to 0} f(x+h) = c$

$$=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}$$

$$=\lim_{h\to 0} 0$$

$$=0$$

This result is also obvious geometrically since the function y = c is a horizontal line with slope 0. And we saw earlier that a line function has tangent line slope equal to its own slope which is 0.

Example
f(x)=5 so $f'(x)=0$

Theorem 3.3.2 (Power Rule)
If n is a positive integer, then

$$\frac{d}{dx}[x^n]=n.x^{n-1}$$

Proof
Let $f(x)=x^n$, n is a positive integer. Then

$$\frac{d}{dx}[x^n]=f`(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

Using the binomial theorem on $(x + h)^n$

$$f'(x) = \lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h\to 0}\frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n}{h}$$

Take *h* common from numerator and cancel with the denominator to get

$$= \lim_{h\to 0}\,[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \ldots + nxh^{n-2} + h^{n-1}]$$

Distributing the limit over the sum give all the terms equal to zero except the first one.
So

$$f`(x) = nx^{n-1}$$

Example

$$\frac{d}{dx}[x^5] = 5x^4,\quad \frac{d}{dx}[x] = 1x^{1-1} = 1x^0 = 1\cdot 1 = 1$$

Theorem 3.3.3
Let c be a constant and f be a function differentiable at x, then so is the function c.f and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Proof

$$\frac{d}{dx}[cf(x)] = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h\to 0} c\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= c\lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= c\frac{d}{dx}[f(x)]$$

Example

$$\frac{d}{dx}[3x^8]$$

$$= 3\frac{d}{dx}[x^8]$$

$$= 3(8x^7) \quad 1x^0 = 1\cdot 1 = 1$$

$$= 24x^7 \quad nx^{n-1}$$

Derivative of Sums and Differences of Functions

If f and g are differentiable functions at x, then so is f+g, and

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Proof

$$\frac{d}{dx}[[f(x)+g(x)] = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$== \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Similarly for Difference. Left as an exercise.

Example

$$\frac{d}{dx}[x^4 + x^3]$$

$$= \frac{d}{dx}[x^4] + \frac{d}{dx}[x^3]$$

$$= 4x^3 + 3x^2$$

In general

$$\frac{d}{dx}[f_1(x) + f_2(x) + \ldots + f_n(x)]$$

$$= \frac{d}{dx}[f_1(x)] + \frac{d}{dx}[f_2(x)] + \ldots + \frac{d}{dx}[f_n(x)]$$

Derivative of a Product

Theorem 3.3.5

If f and g are differentiable functions at x, then so is f.g and

$$\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Proof

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

If we add and subtract f(x + h)g(x) in the numerator

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h\to 0} f(x+h).\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$+ \lim_{h\to 0} g(x).\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} f(x+h).\frac{d}{dx}[g(x)] + \lim_{h\to 0} g(x).\frac{d}{dx}[f(x)]$$

Since $\lim_{h\to 0} g(x) = g(x)$ becasue there is no h involved

and $\lim_{h\to 0} f(x+h) = f(x)$, we have the desired result

$$= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$(f.g)' = f.g' + g.f'$$

Example

$$\frac{d}{dx}[(4x^2)(3x)]$$

$$= 4x^2\frac{d}{dx}(3x) + 3x\frac{d}{dx}(4x^2)$$

$$= 4x^2(3) + 3x(8x) = 12x^2 + 24x^2 = 36x^2$$

Derivative of Quotient

Theorem 3.3.6

If f and g are differentiable functions at x, and $g(x) \neq 0$. Then f/g is differentiable at x and

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Prove Yourself!

$$\left(\frac{f}{g}\right)' = \frac{g.f' - f.g'}{g^2}$$

Example

$$\frac{d}{dx}\left(\frac{3x}{5x^2}\right)$$

$$= \frac{5x^2 \frac{d}{dx}(3x) - 3x\frac{d}{dx}(5x^2)}{(5x^2)^2}$$

$$= \frac{5x^2(3) - 3x(10x)}{25x^4}$$

$$= \frac{15x^2 - 30x^2}{25x^4} = \frac{-15x^2}{25x^4} = \frac{-3}{5x^2}$$

Derivative of a Reciprocal
Theorem 3.3.7

If g is differentiable at x, and $g(x) \neq 0$,the $\frac{1}{g(x)}$ is differentiable at x and

$$\frac{d}{dx}[\frac{1}{g(x)}] = -\frac{\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Student can prove this using the quotient rule!!
Using the Reciprocal Theorem we can generalize Power Rule (Theorem3.3.1) for all integers (negative or non-negative)

Theorem 3.3.8

If n is any integer, then $\frac{d}{dx}[x^n] = x^{n-1}$

Lecture # 17

Derivatives of Trigonometric functions

- Derivative of f(x) =sin(x)
- Derivative of f(x) =cos(x)
- Derivative of f(x) =tan(x)
- Derivative of f(x) =sec(x)
- Derivative of f(x) =cosec(x)
- Derivative of f(x) =cot(x)
- Derivative of the functions made of above functions

Derivative of f (x) =sin (x)

- We want to find the derivative of sin (x) or to differentiate sin (x).
- By definition of derivative we have the following calculations

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h}$$

$$= \lim_{h\to 0}\left[\sin(x)\left(\frac{\cos(h)-1}{h}\right)+\cos(x)\left(\frac{\sin(h)}{h}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos(x)\left(\frac{\sin(h)}{h}\right)-\sin(x)\left(\frac{1-\cos(h)}{h}\right)\right]$$

In sin (x) and cos (x) don't involve h, they are constant as $h \to 0$

$$\lim_{h\to 0}\sin(x)=\sin(x)$$

$$\lim_{h\to 0}\cos(x)=\cos(x)$$

And so

$$\frac{d}{dx}\sin(x)=\cos(x)\lim_{h\to 0}\left(\frac{\sin h}{h}\right)-\sin(x)\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)$$

$$=\cos(x)(1)-\sin(x)(0)=\cos(x)$$

So we have proved that

$$\mathbf{\frac{d}{dx}\sin(x)=\cos(x)}$$

Derivative of f (x) = cos (x)

In the same way we can find the derivative of the cos function

$$\frac{d}{dx}\cos(x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

This is what we get from the definition of the derivative. The student can work out the details of the calculations here!

Derivative of f (x) =tan (x)

We can use the definition of derivative to get

$$\frac{d}{dx}\tan(x) = \lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

I don't recall the expansion for tan (x +h)!! However, we can use the identity

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

And expand it

$$\tan(x+h) = \frac{\sin(x+h)}{\cos(x+h)} = \frac{\sin(x)\cos(h)+\sin(h)\cos(x)}{\cos(x)\cos(h)-\sin(x)\sin(h)}$$

So we get

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}\\ &= \lim_{h\to 0}\frac{\dfrac{\sin(x+h)}{\cos(x+h)}-\dfrac{\sin(x)}{\cos(x)}}{h}\\ &= \lim_{h\to 0}\frac{\left[\dfrac{\sin(x)\cos(h)+\sin(h)\cos(x)}{\cos(x)\cos(h)-\sin(x)\sin(h)}\right]-\dfrac{\sin(x)}{\cos(x)}}{h}\end{aligned}$$

BIG Formula!!!

I will leave to the student to solve this and get the derivative. But here is what I will do. A simpler way of finding the derivative of tan(x).

Remember the Quotient Rule from precious lectures?? Well, we can use it here instead of the definition of Derivative for tan(x).

Here is how

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\dfrac{d}{dx}\sin(x)-\sin(x)\dfrac{d}{dx}\cos(x)}{\cos^2(x)}\\ &= \frac{\cos(x)\cos(x)-\sin(x)[-\sin(x)]}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)} = \sec^2(x)\end{aligned}$$

We used the quotient rule and the derivatives of sin (x) and cos(x)

Derivative of f(x) =sec(x)

$$\frac{d}{dx}\sec(x)=\frac{d}{dx}\left(\frac{1}{\cos(x)}\right)=\frac{\cos(x)(0)-(1)[-\sin(x)]}{\cos^2(x)}$$

$$=\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos(x)}=\sec(x)\tan(x)$$

Derivative of f(x) =cosec(x)

$$\frac{d}{dx}\operatorname{cosec}(x)=\frac{d}{dx}\left(\frac{1}{\sin(x)}\right)=\frac{\sin(x)(0)-(1)[\cos(x)]}{\sin^2(x)}$$

$$=\frac{-\cos(x)}{\sin(x)}\cdot\frac{1}{\sin(x)}=-\operatorname{cosec}(x)\cot(x)$$

Derivative of f(x) =cot(x)

$$\frac{d}{dx}\cot(x)=\frac{d}{dx}\left(\frac{1}{\tan(x)}\right)=\frac{\tan(x)(0)-(1)[\sec^2(x)]}{\tan^2(x)}$$

$$=\frac{-\sec^2(x)}{\tan^2(x)}=-\cos ec^2(x)$$

Example

Suppose that the rising sun passes directly over a building that is 100 feet high and let θ be the angle of elevation of the sun. Find the rate at which the length x of the building's shadow is changing with respect to θ

When $\theta=45^\circ$. Express the answer in units of feet/degree.

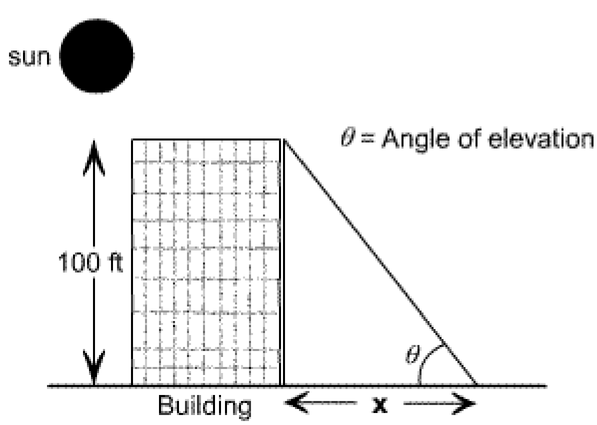
Solution

From the figure, we see that the variable θ and x are related by the equation

$$\tan\theta=\frac{100}{x}\Rightarrow x=100\cot$$

We want to find the Rate of Change of x wrt θ or in other words

$$\frac{dx}{d\theta}=?$$



$$\tan\theta=\frac{100}{x}$$

$$x=100\cot\theta$$

I would like to use the fact we got earlier that

$$\frac{d}{dx}\cot(\theta) = -\cos ec^2(\theta)$$

This will work only if theta is defined in RADIANS. WHY, because we want cot to be a function which is defined in terms of radians.
We can do that here and instead of degrees, use radians to measure theta. So 45 deg will become radians. $\frac{\pi}{4}$

So we get

$$\frac{dx}{d\theta} = -100\cos ec^2\theta$$

This is the rate of change of the length x of shadow wrt to the elevation angle theta in units of feet/radian. When theta is $\frac{\pi}{4}$ radians, then

$$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{4}} = -100\cos ec^2(\frac{\pi}{4}) = -200\, feet/radian$$

Now we want to go back to degrees because we were asked to answer the question with the angle in degrees. We have the relationship

$$180 \text{ degrees} = \pi \text{ radians}$$

$$1 \text{ degree} = \frac{\pi}{180}\text{radian} \Rightarrow \text{ There are } \frac{\pi}{180}\text{radian/degree}$$

This Gives

$$\text{-200 feet/radian } \cdot \frac{\pi}{180}\text{radians/degree} = -\frac{10}{9}\pi \text{ feet/degree}.$$

Lecture # 18

The Chain Rule

- Derivative of Composition of Functions (Chain Rule)
- Generalized Derivative formula
- More Generalized Derivative formula
- An Alternative approach to using Chain Rule

Derivative of Composition of Functions (Chain Rule)

Suppose we have two functions *f* and *g* and we know their derivatives. Can we use this information to find the derivative of the composition

$$(f \circ g)(x) = f(g(x))$$

It turns out that we can by a rule call the CHAIN RULE for differentiation . Look at

$y = (f \circ g)(x) = f(g(x))$

Let us introduce the equation u = g(x). Then the first one becomes

$$y = (f \circ g)(x) = f(g(x)) = f(u)$$

We want to use the known things:

$\dfrac{dy}{du} = f'(u) \quad \text{and} \quad \dfrac{du}{dx} = g'(u)$

To find the derivative

$\dfrac{dy}{dx} = \dfrac{d}{dx} f(g(x))$

Here is the way to do it

Theorem 3.5.2 Chain Rule

If g is differentiable at the point x and f is differentiable at the point g(x), then the composition f (g (x)) is differentiable at the point x. Moreover, if

$y = f(g(x)) \quad \text{and} \quad u = g(x), \text{ then } y = f(u)$

and

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

So should we prove this? Well, Let's leave this as an exercise for the students. Its not too difficult, but may be a bit lengthy. It is given in Section III of Appendix C of the textbook.

Example

Find

$\dfrac{dy}{dx}$ if $y = 4\cos(x^3)$

Let $u = x^3$ so that

$y = 4\cos(u)$

By the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{d}{du}[4\cos(u)].\frac{d}{dx}[x^3]$$
$$= (-4\sin(u))(3x^2) = (-4\sin(x^3))(3x^2)$$
$$= -12x^2\sin(x^3)$$

This formula for finding the derivative of a composition of function is easy to remember if you think of canceling the *du* on the top and the bottom resulting in dy / dx!! This is only a technique to remember, this does not
actually happen.

Generalized Derivative formula

The formula we saw for finding the derivative of composition of functions is a little cumbersome. Here is a simpler one.

The chain rule is $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

Now y = f(u) gives upon differentiation w..r.t u

$$\frac{dy}{du} = f'(u)$$

Using this in the equation of the chain rule gives

$$\frac{dy}{dx} = \frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$

Powerful formula: Simple and effective

Example

$$f(x) = (x^2 - x + 1)^{23}$$

Let $u = x^2 - x + 1,$ so $f(x)$ becomes

$$f(u) = u^{23}$$

Now we apply the new formual we just got to $f(u)$ to get

$$\frac{d}{dx}[(x^2 - x + 1)^{23}] = \frac{d}{dx}[u^{23}] = 23u^{22}\cdot\frac{du}{dx}$$
$$= 23(x^2 - x + 1)^{23}.\frac{d}{dx}(x^2 - x + 1)$$
$$= 23(x^2 - x + 1)^{23}.(2x - 1)$$

Note that this formula involves derivative of functions which have a different independent variable than the variable we are "differentiating with respect to!"

Note that we had in our last example

$$\frac{d}{dx}[u^{23}] = 23u^{22}\frac{du}{dx}$$

Let $u = x$. Then we get

$$\frac{d}{dx}[x^{23}] = 23x^{22}\frac{d}{dx}(x) = 23x^{22}$$

This matches up with what we have seen before. So this formula is a generalization of our differentiation ideas from previous lectures

Here is a table for your reference

Table 3.5.1

$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$	$\frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx}$
$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$	$\frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u\tan u\frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u\cot u\frac{du}{dx}$

Example

$$\frac{d}{dx}[\sin(2x)]$$

Let $u = 2x$. Then using formula A, we have

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin(u)] = \cos(u)\frac{du}{dx} = \cos(2x).2 = 2\cos(2x)$$

Example

$$\frac{d}{dx}[\tan(x^2+1)]$$

Let $u = x^2+1$ in formula A, then we get

$$\frac{d}{dx}[\tan(x^2+1)] = \frac{d}{dx}\tan(u) = \sec^2(u)\frac{du}{dx} = \sec^2(x^2+1)\cdot 2x$$

$$= 2x\cdot\sec^2(x^2+1)$$

How do we know what to let u equal?

Well, you make your substitution so that the result comes out to be a function that you already know how to differentiate. Like in the last one, we made it so that we got tan(u) which is easy to differentiate.

Example

$$\frac{d}{dx}[\sqrt{x^3+\cos ec(x)}]$$

Let $u = x^3 + \cos ec(x)$. Then we get from A

$$\frac{d}{dx}[\sqrt{x^3+\cos ec(x)}] = \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$

$$= \frac{1}{2\sqrt{x^3 + \cos ec(x)}} \cdot \frac{d}{dx}[x^3 + \cos ec(x)]$$

$$= \frac{1}{2\sqrt{x^3 + \cos ec(x)}}(3x^2 - \cos ec(x)\cot(x)]$$

An alternative approach to using Chain Rule

Remember that we started with f(g(x)) and then labeled g(x) as u by u = g(x). If we don't do this then we get

$$\frac{d}{dx}[f(g(x))] = f'(g(x)).g'(x)$$

With this notation, we can say informally that the chain rule says
"Derivative of the OUTER function *f*, then Derivative of the INNER function *g*, and multiply the two together"

Example

$$\frac{d}{dx}[\cos(3x+1)]$$

Here, $f(x) = \cos(x),\ g(x) = 3x + 1$. So

$$\frac{d}{dx}[\cos(3x+1)] = [\cos(3x+1)]' \cdot (3x+1)' = -\sin(3x+1).3 = -3\sin(3x+1)$$

Lecture # 19

Implicit Differentiation

- The method of Implicit Differentiation
- Derivatives of Rational Powers of x
- Differentiability of Implicit functions

Implicit differentiation

Consider this equation. We want to find its derivative or in other words. But how will we find it if y is not alone on one side of the equation?
Take x y = 1
One way is to solve this equation first to get y
y = 1/x
Differentiating on both sides

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x)^{-1}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

We know that this is the derivative because we used the POWER Rule to differentiate
In this example it was possible to solve the equation for y. What if we cant in some example? Let's see if we can find the derivative in this example without solving for y.

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = 0$$

$$x\frac{dy}{dx} + y(1) = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Here i am treating y as an unknown function of x

$$\frac{dy}{dx} = -\frac{y}{x} \qquad \text{Remeber that } xy = 1 \Rightarrow y = \frac{1}{x}$$

$$\text{So, } \frac{dy}{dx} = -\frac{1}{x^2} \text{ same as in the first case}$$

So here was a different way of finding the derivative
In this example, we found dy/dx without solving for y first. This is called **IMPLICIT DIFFERENTIATION**
This is used mostly when it is inconvenient or impossible to separate the y or the dependent variable on one side.

Example

$$\text{Find } \frac{dy}{dx} \text{ if } 5y^2 + \sin y = x^2$$

Hard to separate the y variable on one side in this case in order to find the derivative of this function.
Use Implicit differentiation

$$\frac{d}{dx}(5y^2 + \sin y) = \frac{d}{dx}(x^2)$$

$$5\frac{d}{dx}\left(y^2\right) + \frac{d}{dx}\left(\sin y\right) = 2x$$

$$5\left(2y\frac{dy}{dx}\right) + \cos y.\frac{dy}{dx} = 2x$$

Chain rule here because *y* is to be treated as an unknown function of *x*.

$$\frac{dy}{dx}\left(10y + \cos y\right) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

The formula for the derivative involves both *x* and *y* and they cannot be separated by using algebraic rules.

Since the original equation cannot be solved for *y* either, the derivative formula must be left like this

Example

Find the slope of the tangent line at the point (4,0) on the graph of

$$7y^4 + x^3y + x = 4$$

To find the slope, we must find dy/dx.

We will use implicit differentiation because the original equation is hard to solve for *y*.

$$\frac{d}{dx}\left[7y^4 + x^3y + x\right] = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(7y^4) + \frac{d}{dx}(x^3y) + \frac{d}{dx}(x) = 0$$

$$28y^3\frac{dy}{dx} + \left(x^3\frac{dy}{dx} + y\frac{d}{dx}\left(x^3\right)\right) + 1 = 0 \qquad \text{Using Product Rule and the Chain Rule}$$

$$28y^3\frac{dy}{dx} + x^3\frac{dy}{dx} + 3yx^2 + 1 = 0$$

$$\frac{dy}{dx} = -\frac{3yx^2 + 1}{28y^3 + x^3}$$

We want to find the slope of the tangent line at the point (4,0) So we have *x* = 4, We want to find the slope of the tangent line at the point (4,0)

So we have x = 4, *y* = 0, so

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=4\\y=0}} = -\frac{1}{64}$$

Example

$$\text{Find } \frac{d^2y}{dx^2} \text{ if } 4x^2 - 2y^2 = 9$$

Differentiating both sides implicitly gives

$$8x - 4y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{y}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{2x}{y}\right)$$

$$\frac{d^2y}{dx^2} = \frac{y(2) - (2x)\frac{dy}{dx}}{y^2} = \frac{2y - 2x\left(\frac{2x}{y}\right)}{y^2}$$

$$= \frac{2y^2 - 4x^2}{y^3}$$

From the original equation we get finally

$$\frac{d^2y}{dx^2} = -\frac{9}{y^3}$$

Derivatives of Rational Powers of x

We saw earlier that the power rule for differentiation holds for ALL Integers

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

Now we want to expand it to powers that involve Rational numbers

$$\frac{d}{dx}\left[x^r\right] = rx^{r-1}$$

Where r is a rational number

Lecture # 20

Derivatives of Logarithmic and Exponential Functions and Inverse functions and their derivatives

- Derivative of the logarithmic function $f(x) = \log_b(x)$
- Derivative of the Natural log functions $f(x) = \ln(x)$
- Logarithmic Differentiation
- Derivatives of Irrational powers of x
- Derivatives of Exponential functions $f(x) = a^x$
- Inverse functions
- Derivatives of Inverse Functions

Derivative of the logarithmic function

$y = f(x) = \log_b(x)$

Recall the Logarithm of a real number. Let x, b be real, and an unknown real number y.
What should y be so that if you raise y to b, you get x.

That is $b^y = x$

This is denoted as $y = f(x) = \log_b(x)$

b is called the *base.*

The equation above is read as “log *base b* of x is y .We want to find the derivative of the log function. Before we do that, here are some formulas we will need that follow from the properties of log functions. You can refer to them in section 7.1 of your text.

Theorem 7.1.1

(a) $\log_b 1 = 0$

(b) $\log_b b = 1$

(c) $\log_b ac = \log_b a + \log_b c$

(d) $\log_b \dfrac{a}{c} = \log_b a - \log_b c$

(e) $\log_b a^r = r \log_b a$

(f) $\log_b \dfrac{1}{c} = -\log_b c$

Let's apply the derivative on both sides and use its definition to get

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} \qquad \therefore\ e \approx 2.71$$

$$\frac{dy}{dx} = \frac{d}{dx}[\log_b(x)] = \lim_{h\to 0}\frac{\log_b(x+h) - \log_b(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\log_b\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\log_b\left(1+\frac{h}{x}\right)$$

$$= \lim_{v\to 0}\frac{1}{vx}\log_b(1+v)$$

Last step comes from letting

$v = \frac{h}{x}$ and then $v \to 0$ as $h \to 0$

$$= \frac{1}{x} \lim_{v \to 0} \frac{1}{v} \log_b (1+v)$$

$$= \frac{1}{x} \lim_{v \to 0} \log_b (1+v)^{\frac{1}{v}}$$

Using Theorem 7.1.1 part (e)

$$= \frac{1}{x} \log_b \left[\lim_{v \to 0} (1+v)^{\frac{1}{v}} \right]$$

Using the fact that log is continuous

$$= \frac{1}{x} \log_b e$$

Thus

$$\frac{d}{dx}[\log_b (x)] = \frac{1}{x} \log_b e, \; x > 0$$

There exists a "change of base" formula which Let's you convert your given log to a certain base to another. This is helpful because modern calculators have two log button, one with base 10, written as only log (x) and the other for the Natural log to the base *e* written as ln (x). To keep calculations easy, we can rewrite the above formula for the derivative as

$$\frac{d}{dx}[\log_b (x)] = \frac{1}{x \ln(b)}, \; x > 0$$

If the base happens to be the number *e,* then we get from this formula

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}, \; x > 0$$

So you see that the the simplest formula for the derivative comes from base $b = e$. That's why ln (x) is a very important function in terms of Calculus

Derivative of the Natural log functions

$f(x) = \ln(x)$

We will generalize the derivative formula we just found for the log or ln function to include composition of functions where one is the ln function, and the other is some function u(x) > 0 and differentiable at x.

$$\frac{d}{dx}[\log_b (u)] = \frac{1}{u \ln(b)} \cdot \frac{du}{dx}$$

Using the chain rule here

$$\frac{d}{dx}[\ln(u)] = \frac{1}{u} \cdot \frac{du}{dx}$$

Example

Find $\frac{d}{dx}\left[\ln(x^2+1)\right]$

$$\frac{d}{dx}\left[\ln(x^2+1)\right]=\frac{1}{(x^2+1)}\cdot\frac{d}{dx}[x^2+1]$$

$$=\frac{1}{(x^2+1)}\cdot 2x=\frac{2x}{(x^2+1)}$$

Example

$$\frac{d}{dx}\left[\ln\left(\frac{x^2\sin(x)}{\sqrt{1+x}}\right)\right]$$

$$=\frac{d}{dx}\left[\ln[x^2\sin(x)]-\ln\sqrt{1+x}\right]=\frac{d}{dx}\left[2\ln(x)+\ln(\sin x)-\frac{1}{2}\ln(1+x)\right]$$

$$=\frac{2}{x}+\frac{\cos(x)}{\sin(x)}-\frac{1}{2(1+x)}$$

Logarithmic Differentiation Using log function properties, we can simplify differentiation of messy functions.

Example

$$y=\frac{x^2\,.\,\sqrt[3]{7x-14}}{(1+x^2)^4}$$

Apply ln to both sides and simplify the right side using ln properties

$$\ln y=\ln\left(\frac{x^2\,.\,\sqrt[3]{7x-14}}{(1+x^2)^4}\right)=2\ln x+\frac{1}{3}\ln(7x-14)-4\ln(1+x^2)$$

Differentiating both sides w. r. t. x gives

$$\frac{1}{y}\frac{dy}{dx}=\frac{2}{x}+\frac{\frac{7}{3}}{7x-14}-\frac{8x}{1+x^2}$$

$$\Rightarrow\frac{dy}{dx}=\left(\frac{2}{x}+\frac{\frac{7}{3}}{7x-14}-\frac{8x}{1+x^2}\right)\cdot y$$

$$\frac{dy}{dx}=\left(\frac{2}{x}+\frac{\frac{7}{3}}{7x-14}-\frac{8x}{1+x^2}\right)\cdot\frac{x^2\,.\,\sqrt[3]{7x-14}}{(1+x^2)^4}$$

Derivatives of Irrational powers of *x*

Remember that we proved the power rule for positive integers, then for all integers, then for rational numbers?

Now we prove it for all REAL numbers by proving it for the remaining type of real numbers, namely the irrationals.

Power Rule

$$\frac{d}{dx}\left[x^r\right] = r \cdot x^{r-1}$$

We will use ln to prove that the power rule holds for any real number r

Let $y = x^r$, r any real number

Proceed by differentiating this function using ln

$$\ln y = \ln x^r = r \ln x$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[r \ln x]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x} \cdot y = \frac{r}{x} \cdot x^r = r \cdot x^{r-1}$$

Derivatives of Exponential functions $f(x) = b^x$

Now we want to find the derivative of the function $y = f(x) = b^x$

To do so, we will use logarithmic differentiation on $y = b^x$

$$\ln y = \ln b^x = x \ln b$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \ln b]$$

$$\frac{1}{y}\frac{dy}{dx} = \ln b$$

$$\frac{dy}{dx} = y \cdot \ln b = b^x \cdot \ln b$$

$$So \quad \frac{d}{dx}b^x = b^x \cdot \ln b$$

In general

$$\frac{d}{dx}[b^u] = b^u \cdot \ln b \cdot \frac{du}{dx}$$

If $b = e$

$$\frac{d}{dx}e^x = e^x \cdot \ln e = e^x$$

In general

$$\frac{d}{dx}[e^u] = e^u \cdot \ln e \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx}$$

Inverse Functions

We have talked about functions.

One way of thinking about functions was to think of a function as a process that does something to an input and then throws out an output.
So this way, when we put in a real number *x*, it comes out as some new number *y,* after some ACTION has taken place.
Now that question is, that is there another ACTION, that undoes the first action?
If *f* is a function that performs a certain action on *x,* is there a function *g* that undoes what *f* does?
Sometimes there is and sometime not. When there is a function *g* undoes what *f* does, then we say that *g* is an inverse function of *f.*
Here is how to determine if two given function are inverses of each other or not.

Definition 7.4.1

If the functions *f* and *g* satisfy the two conditions

f(g(x)) = *x* for every *x* in the domain of *g*
g(f(x)) = *x* for every *x* in the domain of *f*

then we say , that *f* is an inverse of *g* and *g* is an inverse of *f* or, alternatively that *f* and *g* are inverse functions.

Example

$f(x) = 2x$ and $g(x) = \frac{1}{2}x$

are inverse functions. This is obvious, but we can use def 7.4.1

$$f(g(x)) = 2(\frac{1}{2}x) = x$$

$$g(f(x)) = \frac{1}{2}(2x) = x$$

Inverses don't always exists for a given function.
There are conditions for it.
Most important one is that a function must be ONE to ONE to have an inverse.
Do you recall what One to One is??
Its when a function does not have two values for the same *x* value in the domain. Review this.
The inverse function to a function *f* is usually denoted as f^{-1} read as "*f* inverse" and

$$y = f^{-1}(x) \quad \Rightarrow \quad x = f(y)$$

Derivatives of Inverse Functions
Theorem 7.4.7

Suppose that the function f has an inverse and that the value of $f^{-1}(x)$ varies over an interval on which f has a nonzero derivative as x varies over an interval I. Then f^{-1} is differentaible on I and derivative of f^{-1} is given by the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

The formula in this theorem can be simplified if you write

$$y = f^{-1}(x) \quad \Rightarrow \quad x = f(y)$$

Then $\dfrac{dy}{dx} = (f^{-1})'(x) \quad \Rightarrow \quad \dfrac{dx}{dy} = f'(y) = f'(f^{-1}(x))$

Plugging these values into the theorem we get a simpler formula

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Lecture # 21

Applications of Differentiation

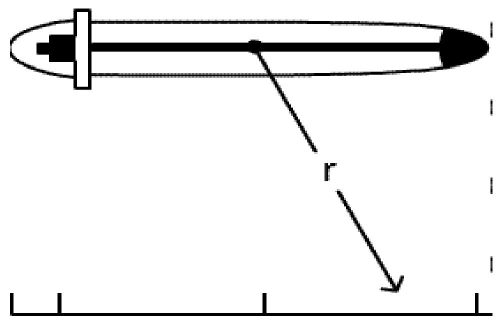
- Related Rates
- Increasing Functions
- Decreasing Functions
- Concavity of functions

Related Rates

Related Rates are real life problems.
These involve finding the rate at which one quantity changes w.r.t another quantity.
For example, we may be interested in finding out how fast the polar ice caps are melting w.r.t the changes in temperature.
We may want to know how fast a satellite is changing altitude w.r.t changes in time, or w.r.t changes in gravity.
To solve problems involving related rates, we use the idea of derivatives, which measure the rate of change.

Example

Assume that oil spilled from a ruptured tanker spread in a circular pattern whose radius increases at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60ft?
Let t = number of seconds elapsed from the time of the spill
Let r = radius of the spill in feet after t seconds
Let A = area of the spill in square feet after t seconds



We want to find $\left.\dfrac{dA}{dt}\right|_{r=60}$ given that $\dfrac{dr}{dt} = 2\,ft/\sec$

The spill is circular in shape so $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \quad \Rightarrow \quad \left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60)(2) = 240\pi\, ft^2/\sec$$

The following steps are helpful in solving related rate problems

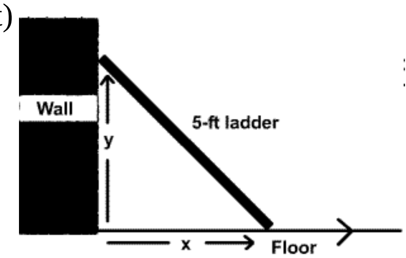
1. Draw a figure and label the quantities that change
2. Identify the rates of change that are known and those that are to be found
3. Find an equation that relates the quantity whose rate of change is to be found to those quantities whose rates of change are known.
4. Differentiate the equation w.r.t the variable that quantities are changing in respect to. Usually Time.
5. Evaluate the derivative at appropriate points.

Example

A five foot ladder is leaning against a wall. It slips in such a way that its base is moving away from the wall at a rate of 2 ft/sec at the instant when the base is 4ft from the wall. How fast is the top of the ladder moving down the wall at that instant?
Let t = number of seconds after the ladder starts to slip
Let x = distance in feet from the base of the ladder
Let y = distance in feet from the top of the ladder to the floor.

$\frac{dx}{dt} =$ rate of change of the base of the ladder (horizontal movement)

$\frac{dy}{dt} =$ rate of change of the top of the ladder (vertical movement)



Using Pythagoras theorem

$x^2 + y^2 = 5^2 = 25$

We want $\left.\frac{dy}{dt}\right|_{x=4}$ Given That $\left.\frac{dx}{dt}\right|_{x=4} = 2\, ft/\sec$

How do we relate the x and the y?
Look at the picture again I see Pythagoras's theorem in here!

$x^2 + y^2 = 25$

Differentiating wrt t and using chain rule gives

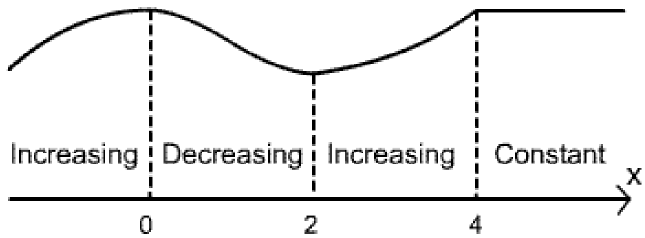
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$\left.\frac{dy}{dt}\right|_{x=4} = -\frac{4}{3}(2) = -\frac{8}{3}\, ft/\sec$$

When x = 4, use the above equation to find corresponding y.

Increasing and decreasing functions
We saw earlier in the lectures that we can get an idea of the graph of a function by plotting a few values. But remember that we also said that this graph was an approximation as a few points may not give all the info.
Now we will see that we can use derivatives to get accurate info about the behavior of the graph in an interval when we move from left to right.
Increasing function on an interval means that as we move from left to right in the x-direction, the y-values increase in magnitude.
Decreasing function on an interval means that as we move from left to right in the x-direction, the y-values decrease in magnitude.
An interval means that as we move from left to right in the x-direction, the y-values decrease in magnitude.
The graph of the function in this figure shows that the function is increasing on the intervals and $(-\infty, 0)$ and $[2, 4]$ decreasing on the interval [0,2].



Let's make this idea concrete.

Definition 4.2.1

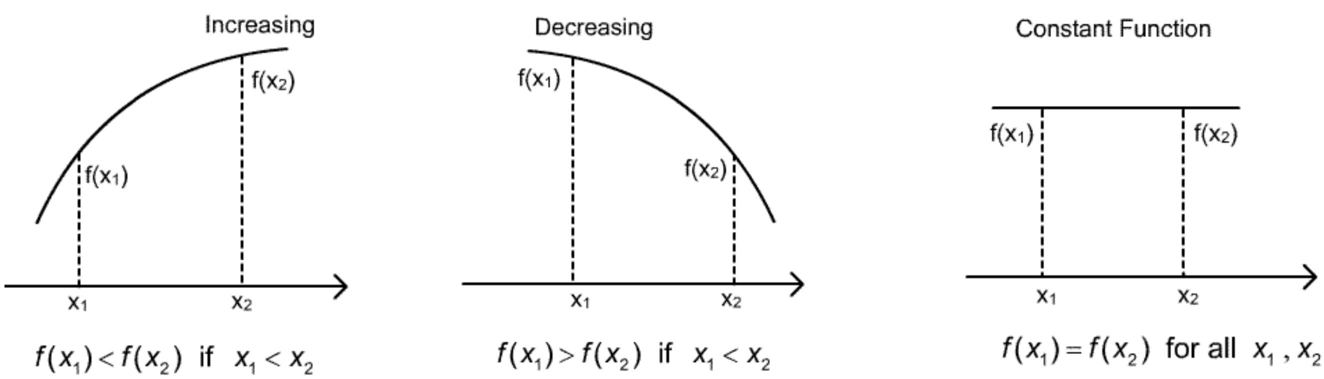
Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$

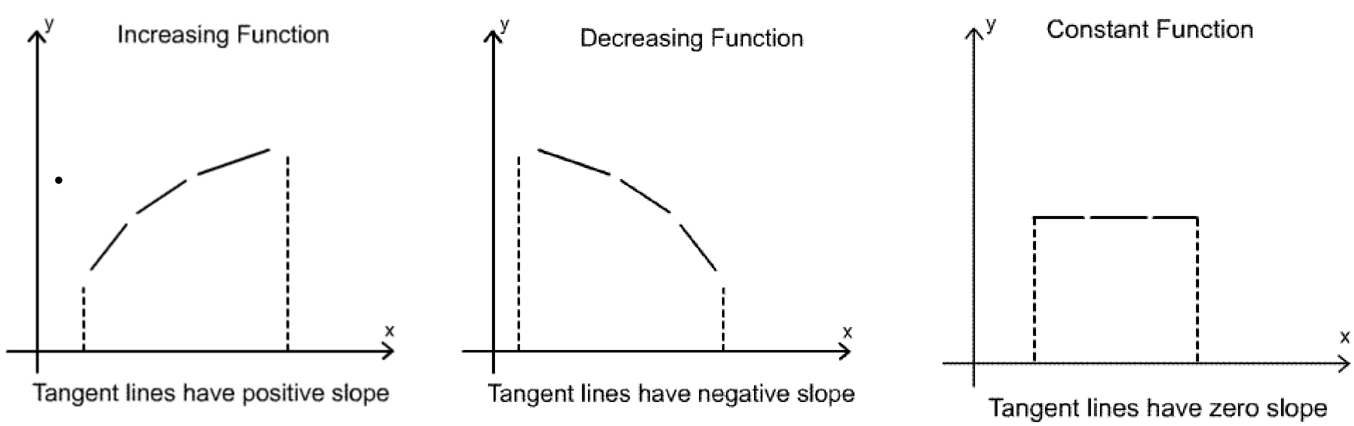
(b) f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$

(c) f is constant on the interval if $f(x_1) = f(x_2)$ for all x_1 and x_2

As shown in the figures below



Let's take a few points on the 3 graphs in above figures and make tangent lines on these points. This gives



Note that incase where the graph was increasing, we get tangent line with positive slopes, decreasing we get negative slope, and constant gives 0 slope.

Let's formalize this idea

Theorem 4.2.1

Let f be a function that is continuous on a closed interval [a,b] and differentiable on the open interval (a,b)

(a) If $f'(x) > 0$ for every value of x in (a,b), then f is increasing on [a,b].

(b) If $f'(x) < 0$ for every value of x in (a,b), then f is decreasing on [a,b].

(c) If $f'(x) = 0$ for every value of x in (a,b), then f is constant on [a,b].

Example

Find the intervals on which the function is increasing and those on which its decreasing.

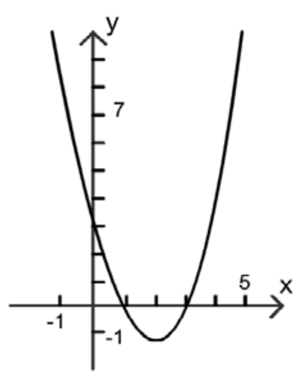
$$f(x) = x^2 - 4x + 3$$

Differentiating f gives

$$f'(x) = 2x - 4 = 2(x-2)$$

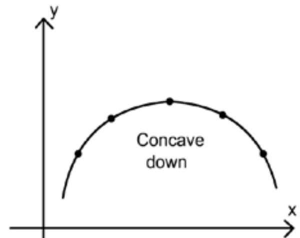
$$f'(x) < 0 \;\Rightarrow\; 2(x-2) < 0 \;\Rightarrow\; -\infty < x < 2$$

$$f'(x) > 0 \;\Rightarrow\; 2(x-2) > 0 \;\Rightarrow\; 2 < x < +\infty$$

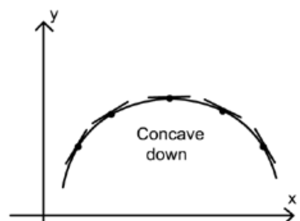


$f(x) = x^2 - 4x + 3$

Since f is continuous on (2,+∞), the function is actually increasing on the interval [2,+∞)
Similarly it is decreasing on the interval (-∞,2]
The derivative is 0 at the point x=2
Since this is the only point not the interval we think of the point x=2 as the point where the transition occurs from decreasing to increasing in f.

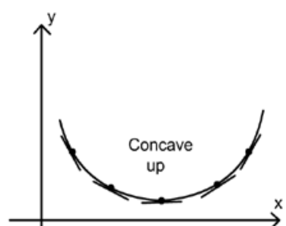


(a)



(a)

Slope of tangent lines decreasing



(b)

Slope of tangent lines increasing

Definition 4.2.3

Let f be differentiable on an interval

(a) f is called concave up on the interval if f is increasing on the interval.

(b) f is called concave down on the interval if f is decreasing on the interval.

Example

Find the open intervals on which the given function is concave up and on those on which it is concave down.

$$f(x) = x^2 - 4x + 3$$

$$f'(x) = 2x - 4$$

$$f''(x) = 2$$

THEOREM 4.2.2

(a) If $f''(x) > 0$ on a open interval (a,b) then f is concave up on (a,b)

(b) If $f''(x) < 0$ on a open interval (a,b) then f is concave down on (a,b)

$$f''(x) = 2$$

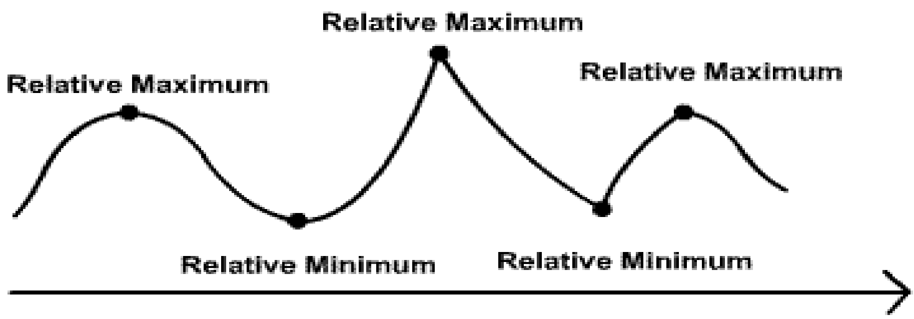
Since f'' is greater than 0 for all x , the graph of this function is concave up on the interval $(-\infty, +\infty)$

Lecture # 22

<u>Relative Extrema</u>

- Relative Maxima
- Relative Minima
- Critical Points
- First Derivative test
- Second Derivative test
- Graphs of Polynomials
- Graphs of Rational functions

Relative Maxima



Most of the graphs we have seen have ups and downs, much like Hills and valleys on earth.
The Ups or the Hills are called relative Maxima.
The downs or the Valleys are called relative Minima
The reason we use the word relative is that just like a given Hill in a mountain range need not necessarily be the Highest point in the range. Similarly a given maxima in a graph need not be the maximum possible value in the graph.
Same goes for the relative minima.
In general, we may say that a given Hill is the highest one in some area. Look at relative maxima in a given interval.
Again, same is true for valleys and relative minima.
So when we talk about relative maxima and relative minima, we talk about them in the context of some interval.

Definition 4.3.1

A function f is said to have a relative maximum at x_0 if $f(x_0) \geq f(x)$ for all x in some open interval containing x_0

Definition 4.3.2

A function is said to have a relative minimum at x_0 if $f(x_0) \leq f(x)$ for all x in some open interval containing x_0.

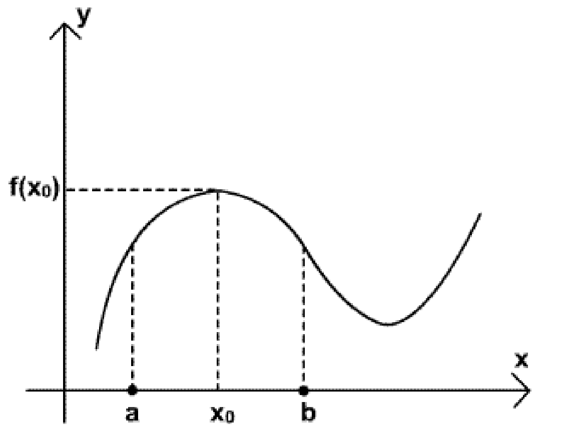
Definition 4.3.3

A function f is said to have a relative extremum at x_0 if it has either a relative maximum or relative minimum at x_0

Example

Here is a graph of a function *f*. This has a relative maximum in the interval (a, b) because from the graph its obvious that

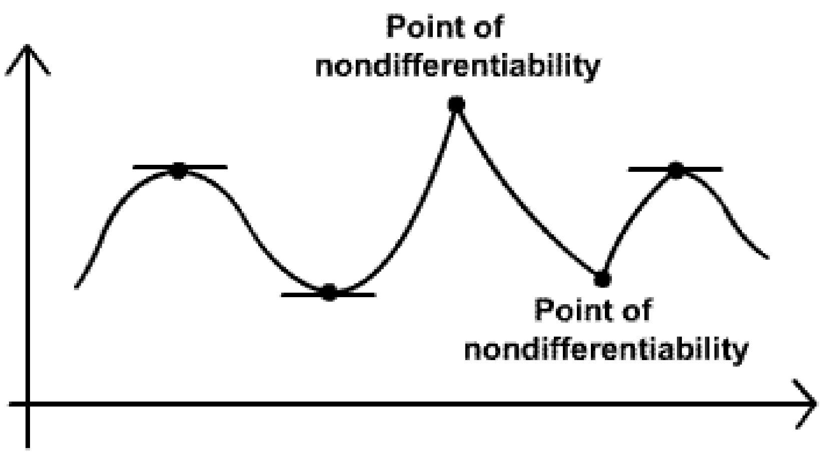
$f(x_0) \geq f(x)$



Function has a relative maximum in the interval (a,b)

Critical Points

It so happens that relative extrema can be viewed as transition points that separate the regions where a graph of a function is increasing from those where a graph is decreasing.



Here is a figure. This shows that relative extrema of a function occur at points where *f* has a horizontal tangent, or where the function is not differentiable.
Horizontal tangent means derivative = 0.
Non-differentiable means corners.

Theorem 4.3.4

If f has a relative extremum at x_0, then either $f'(x_0) = 0$ or f is not differentiable at x_0

Definition 4.3.5

A critical point for a function f is any value of x in the domain of f at which $f'(x) = 0$ or at which f is not differentiable; the critical points where $f'(x) = 0$ are called stationary points of f.

So theorem 4.3.4 can be read as now with this new terminology As "The relative extrema of a function, if any, occur at critical points."

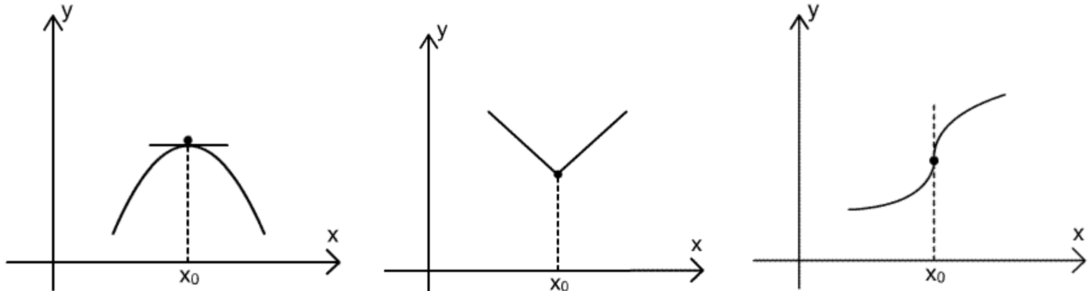
Example

a) x_0 here is a critical and stationary point as tangent line has slope 0

f) x_0 here is a critical point and it has minimum value at that point but the tangent line is not defined at that point.

g) x_0 is a critical point but not stationary as derivative does not exist

Here are the figures of the situations.



4.3.4 (a) 4.3.4 (f) 4.3.4 (g)

First Derivative test and Second Derivative test

Note that in (g) of the last figure, x_o was a critical point, but there was not relative extrema there! This can happen.
So how do we know at which critical point a relative extrema occurs or not?
Here is a theorem for that.

THEOREM 4.3.6
(First Derivative Test)

a) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0 from x_0 and $f'(x) > 0$ on an open interval

b) If $f'(x) < 0$ on an open interval extending left

extending right from x_0, then f has a rel. minimum at x_0

(c) If $f'(x)$ has the same sign [either $f'(x) > 0$ or $f'(x) < 0$] on an open interval extending left from x_0 and on an open interval extending right from x_0, then f does not have a relative extremum at x_0

In short: " The relative extrema, if any, on an open interval where a function f is continuous and not constant occurs at those critical points where f' changes sign"

Example

Locate the relative extrema of

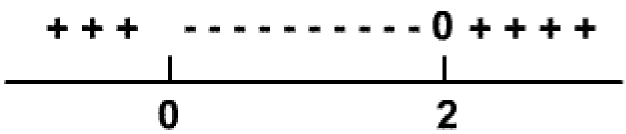
$$f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$$

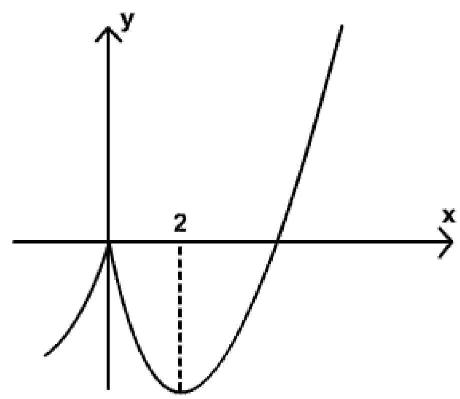
$$f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = 5x^{-\frac{1}{3}}(x-2) = \frac{5}{x^{\frac{1}{3}}}(x-2)$$

Note that there are two critical points, namely x = 0 and x = 2 Because at x = 2, the derivative f' = 0, and at x = 0, the derivative does not exists.

Now we need to know where there is a relative extrema by checking for the changing sign of f' at the 2 critical points.

We use theorem 4.3.6 and draw a number line test





We see that there is relative maximum at 0
We see that there is relative minimum at 2

There is another test for finding extrema easier than the first derivative test.

Theorem 4.3.7 (Second Derivative Test)

Suppose that f is twice differentiable at a stationary point x_0

(a) If $f''(x_0) > 0$, then f has relative minimum at x_0.

(b) If $f''(x_0) < 0$, then f has relative maximum at x_0.

EXAMPLE

Locate the relative extrema of $f(x) = x^4 - 2x^2$

$$f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$$

$$f''(x) = 12x^2 - 4$$

Setting $f'(x) = 0$ gives stationary points $x = 0$ and $x = \pm 1$

Also,

$$f''(0) = -4 < 0$$

$$f''(1) = 8 > 0$$

$$f''(-1) = 8 > 0$$

There is a relative maximum at x = 0, and relative minima at x = 1 and x = -1

Graphs of Polynomials

In applied sciences and engineering, it is required many times to understand the behavior of a function.
Graphs are a good way to understand function behavior. But many times it is hard to graph the function.
So it is often necessary to understand the behavior in terms of maxima and minima and concavity etc.
We will look at how the stuff from the last lecture helps us in graphing polynomial and rational functions
In applied sciences and engineering, it is required many times to understand the behavior of a function.
Graphs are a good way to understand function behavior. But many times it is hard to graph the function.
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We will look at how the stuff from the last lecture helps us in graphing polynomial and rational functions

Graphs of Polynomials

Let P(x) be a polynomial function

- Calculate $P'(x)$ and $P''(x)$
- Using P', determine the stationary points and the intervals of increase and decrease
- Use P'' to determine the inflection points and interval where P is concave up and concave down
- Plot all of the above and the x and y intercepts

Example

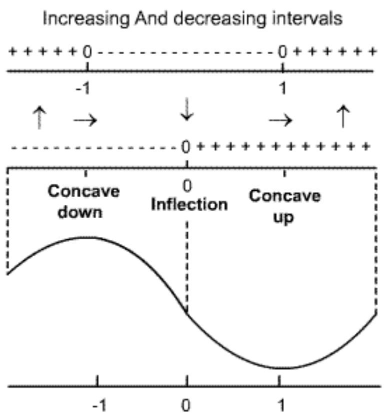
Sketch the graph of $P(x) = y = x^3 - 3x + 2$

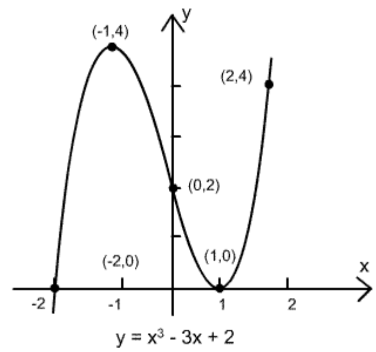
$$\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1)$$

$$\frac{d^2y}{dx^2} = 6x$$

You find the stationary points, inflection points.

x	$y = x^3 - 3x + 2$
−2	0
−1	4
0	2
1	0
2	4





Above figure Shows the intervals of increase/decrease and of concavity Y intercept at (0,2) Inflection point is at x = 0

Graph of Rational Functions

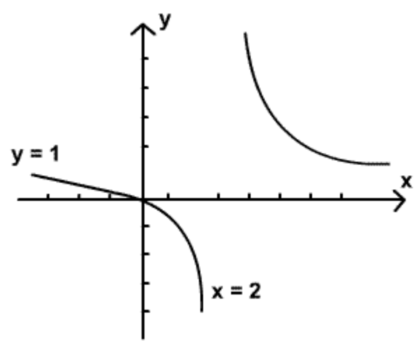
Rational function, remember, is a function defined by the ratio of two polynomials

$$R(x) = \frac{P(x)}{Q(x)}$$

Its obvious that if Q(x) = 0, then R(x) has discontinuity at those values of *x* where Q(x) = 0

Consider the following graph of

$$f(x) = \frac{x}{x-2}$$



A line $x = x_0$ is called a vertical asymptots for the graph of a function f if $f(x) \to +\infty$ or $f(x) \to -\infty$ as x approaches x_0 from the right or from the left .A line $y = y_0$ is called a horizental asymptote for the graph of f if

$$\lim_{x \to +\infty} f(x) = y_0 \quad \text{or} \quad \lim_{x \to -\infty} f(x) = y_0$$

Vertical asymptotes occur where the denominator is 0

Example
Find horizontal and vertical asymptotes of

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

The vertical asymptotes occur at the points where $x^2 - 1 = 0$; these are the points $x = -1$ and $x = 1$, Since

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^2 + 2x}{x^2 - 1} = \lim_{x \to +\infty} \frac{x^2}{x^2} = 1$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2 + 2x}{x^2 - 1} = \lim_{x \to -\infty} \frac{x^2}{x^2} = 1$$

It follows that $y = 1$ is a horizental asymptote.

Lecture # 23

Maximum and Minimum Values of Functions

- Absolute Extrema
- Finding Absolute Extrema for a continuous function
- Summary of extreme behaviors of functions over
- Applied maximum and minimum problems
- Problems involving finite closed intervals
- Problems involving intervals that are non-finite and closed

Absolute Extrema

Previously we talked about relative maxima, relative minima of functions.
These were like the highest mountain and the deepest valley in a given vicinity or neighborhood.
Now we will talk about absolute maximum and minimum values of functions.
These are like the highest peak in a mountain range, and the deepest valley.

DEFINITION 4.6.1

If $f(x_0) \geq f(x)$ for all x in the domain of f, then $f(x_0)$ is called the absolute maximum value or simply the maximum value of f.

DEFINITION 4.6.2

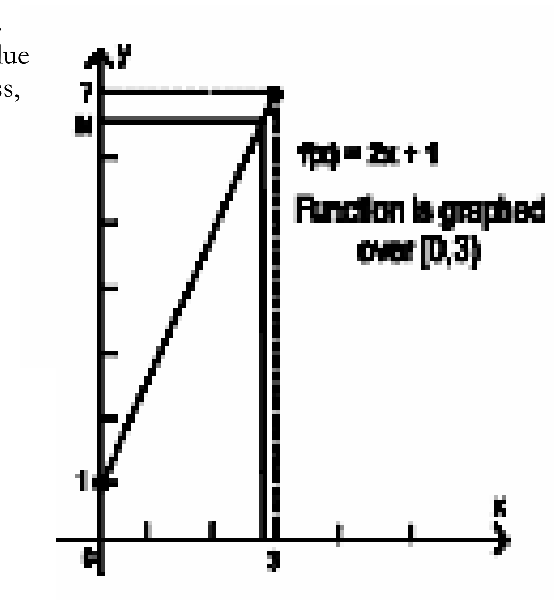
If $f(x_0) \leq f(x)$ for all x in the domain of f, then $f(x_0)$ is called the absolute minimum value or simply the minimum value of f.

Absolute maximum means that the value is the maximum one over the entire domain of the function.
Absolute minimum means that the value is the minimum one over the entire domain of the function.
If we think of the Earth's surface as defining some function, then its absolute maximum will be Mt. Everest, and absolute minimum will be the Marianna Trench in the Pacific Ocean near Hawaii.

Example

Consider the following picture of the graph of f (x) = 2x+1 on the interval [0,3).
The minimum value is 1 at
x = 0. But is there a maximum?

No. Because the function is defined on the interval [0,3) which excludes the point x = 3. So note that you can get very close to 7 as the maximum value as you get very close to x = 3, but this is in a limiting process, and you can always get more closer to 7, and yet never EQUAL 7! So 7 looks like a max, but its NOT!



The question of interest given a function f(x) is:
Does f (x) has a maximum (minimum) value?
If f (x) has a maximum (minimum) , what is it?
If f (x) has a maximum value, where does it occur?

THEOREM 4.6.4
(Extreme-Value Theorem)

If a function f is continuous on a closed interval [a,b], then f has both a maximum value and a minimum value on [a,b]

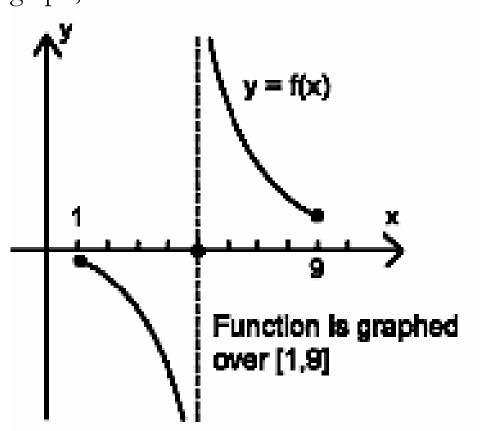
Won't prove this as its difficult. Just use it. This theorem doesn't tell us what the max and the min are, just the conditions on a function which will make it have a max or min

Example

Note that in the previous example we saw a function f (x) = 2x+1 which was defined over the interval [0, 3). This one is a continuous function on that interval, but had no maximum because the interval was not closed!

Example

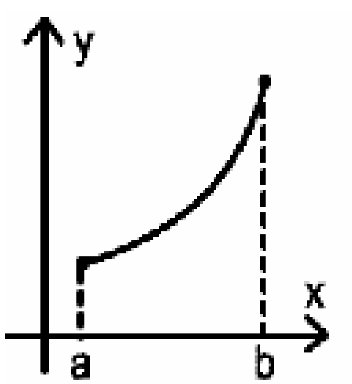
Here is another function graphed over the interval [1,9]. Although the interval is closed, the function not continuous on this interval as we can see from the graph, and so has no maximum or minimum values on that interval .

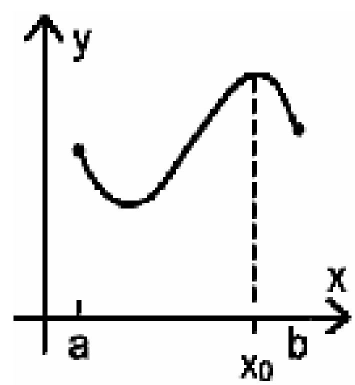


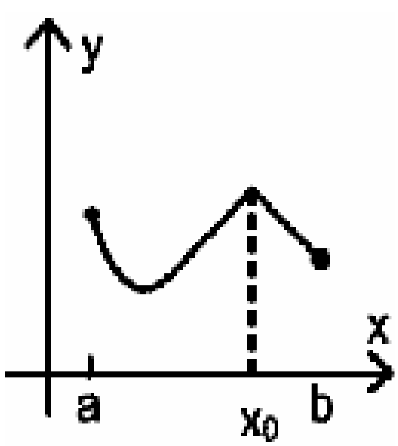
THEOREM 4.6.5

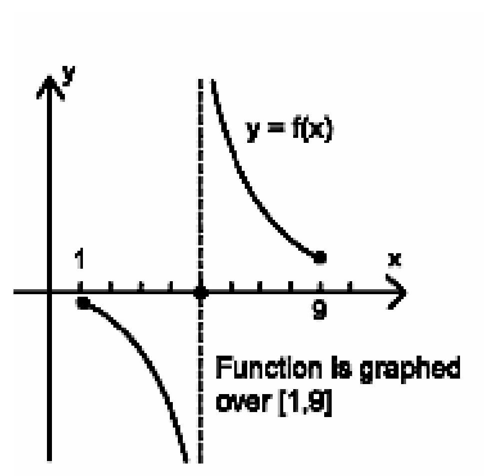
If a function f has an extreme value (either a maximum or a minimum) on an open interval(a,b), then the extreme value occurs at a critical point of f.

Step 1: Find the critical points of *f* in (a,b)
Step 2: Evaluate *f* at all the critical points and the endpoint a and b
Step 3: The largest of the values in Step 2 is the maximum value of *f* on [a,b] and the smallest is the minimum.







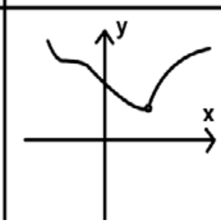
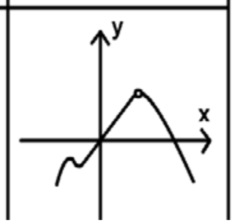


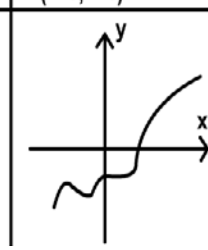
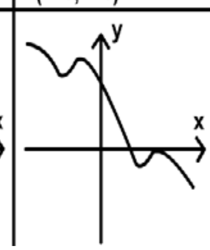
Example

Find the maximum and the minimum values of $f(x) = 2x^3 - 15x^2 + 36x$ on the interval [1, 5]
Since *f* is a polynomial, it's continuous and differentiable on the interval (1, 5)

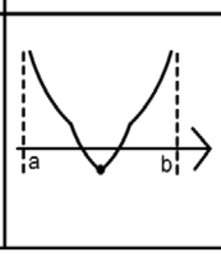
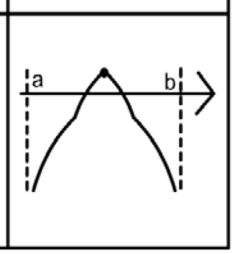
$$f'(x) = 6x^2 - 30x + 36 = 0$$
$$f'(x) = (x-3)(x-2) = 0$$

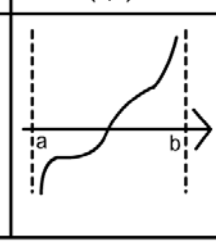
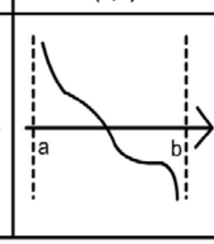
So f' is 0 at x = 2 and x = 3. Max or min will occur at these two points or at the end points.
Evaluate the function at the critical points and the endpoints and we see that max is 55 at x = 5 and min is 23 at x = 1
We want to know the max and min over $(-\infty, +\infty)$
Here is how to find them for continuous functions

Limits	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
Conclusion if f is continuous	**f has a minimum but no maximum on $(-\infty,+\infty)$**	**f has a maximum but no minimum on $(-\infty,+\infty)$**
Graph		

Limits	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
Conclusion if f is continuous	**f has neither a maximum nor a minimum on $(-\infty,+\infty)$**	**f has neither a maximum nor a minimum on $(-\infty,+\infty)$**
Graph		

Summary of extreme behaviors of functions over (a,b)

Limits	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = -\infty$
Conclusion if f is continuous on (a,b)	**f has a minimum but no maximum on (a,b)**	**f has a maximum but no minimum on (a,b)**
Graph		

Limits	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = -\infty$
Conclusion if f is continuous on (a,b)	**f has neither a maximum nor a minimum on (a,b)**	**f has neither a maximum nor a minimum on (a,b)**
Graph		

Find the max and min values if any, of the function $f(x) = x^4 + 2x^3 - 1$ on the interval $(-\infty, +\infty)$
This is a continuous function on the given interval and

$$\lim_{x\to+\infty}(x^4 + 2x^3 - 1) = +\infty \quad \text{and} \quad \lim_{x\to-\infty}(x^4 + 2x^3 - 1) = +\infty$$

So *f* has a minimum but no maximum on (-inf, +inf). By Theorem 4.6.5, the min must occur at a critical point. So

$$f'(x) = 4x^3 + 6x^2 = 2x^2(2x+3) = 0$$

This gives x = 0, and x = -3/2 as the critical points. Evaluating gives min = -43/16 at x = -3/2

Applied maximum and minimum problems

We will use what we have learnt so far to do some applied problems in OPTIMIZATION. Optimization is the way efficiency is got in business, machines and even in nature in terms of animals competing for resources.

Problems involving continuous functions and Finite closed intervals

These are problems where the function is defined over a closed interval. These problems always have a solution because it is guaranteed by the extreme values theorem.

Example

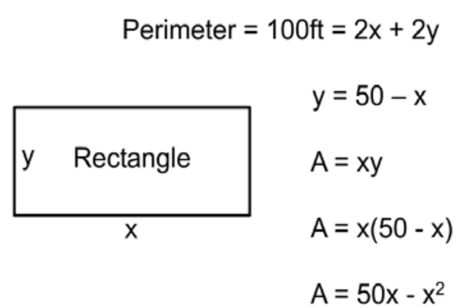
Find the dimensions of a rectangle with perimeter 100 ft whose area is as large as possible.

Let

x = length of the rectangle in feet

y = width of the rectangle in feet

A = area of the rectangle



Then A = xy

We want to maximize the area

$$A = x(50 - x) = 50x - x^2$$

Perimeter = 100ft = 2x + 2y or y = 5 – x

Use this values of y in the equation A = xy to get A function of *x*

Because *x* represents a length, it cannot be negative and it cannot be a value that exceeds the perimeter of 100 ft. So we have the following constraints on *x*

$$0 \le x \le 50$$

So the question is of finding the max of $A = 50x - x^2$ on the interval [0,50]. By what we have seen so far, that max must occur at the end points of this interval or at a critical point

Critical points:

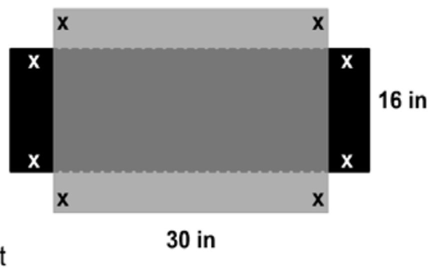
$$\frac{dA}{dx} = 50 - 2x = 0 \quad \Rightarrow \quad x = 25$$

So now we substitute x = 0, x = 25, and x = 50 into the function A to get the max 625 at the point x = 25.

Note that y = 25 also for x = 25. So the rectangle with perimeter 100 with the greatest area is a square with sides 25ft.

Example

An open box is to be made from a 16 inch by 30 inch piece of cardboard by cutting out squares of equal size from the 4 corners and bending up the sides. What size should the squares be to obtain a box with largest possible volume?



Let

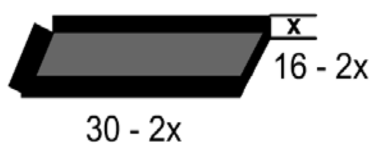
x = length of the sides of the squares to be cut out

V = volume of the resulting box.

We want to maximize the Volume V.

If we cut out the squares from the 4 corners of the cardboard, the resulting BOX will have dimensions

(16 – 2x) by (30 – 2x)



$$\begin{aligned} V &= (\text{length})(\text{width})(\text{Height}) \\ &= (16-2x)(30-2x)\,x \\ &= 480x-92x^2+4x^3 \end{aligned}$$

$$0 \le x \le 8$$

s

So want to find max of the function on [0,8].

$$\frac{dV}{dx} = 480 - 184x + 12x^2 = 0 \quad \Rightarrow \quad x = \frac{10}{3} \ \text{ and } \ x = 12$$

Because x = 12 is out of [0, 8], ignore it. Check V at the end points and at x = 10/3. We see then that V = 19600/27 is max when x = 10/3.

Lecture # 24

Newton's Method, Rolle's Theorem, and the Mean Value Theorem

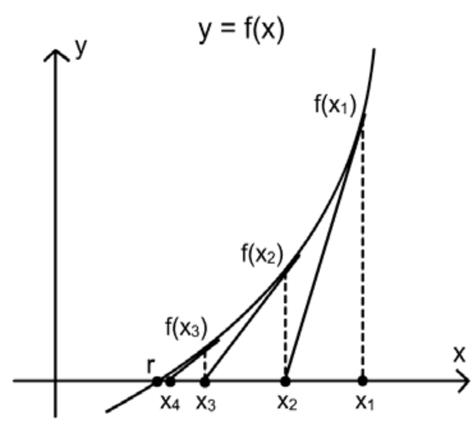
- Newton's method for approximating solutions to f(x)=0
- Some difficulties with Newton's method
- Rolle's theorem
- Mean Value Theorem

Newton's method for approximating solutions to f (x) =0

We have seen in algebra that the solution for the equation $ax+b=0$ is $x=-\dfrac{b}{a}$.

Similarly we have algebraic formulas for polynomial equation up to degree 5.
But there is no algebraic solution to the equation of the kind
$x-\cos(x)=0$
For this equation and many like it, we settle for approximate solutions. How do we approximate the solutions? There are many methods, and one of them is Newton's method. Here is how Newton's method works to find approximate solutions to equations.
What does it mean for an equation to have a solution? or that f (x) = 0 ?
It means that we are looking for those x values, for which the corresponding y value or f (x) is 0.
This means that the solutions are those points where the graph of the function crosses the x axis.
Suppose that x = r is the solution we are looking for.
Let's approximate it by an initial guess called.
We draw a line tangent to the graph of the given function at the point.
If the tangent line is not parallel to the x-axis, then it will x_2 eventually intersect the x-axis at some point x_1 which will generally be closer to *r* than We will repeat the process, with a tangent line at x2 that meets x axis at x3, and so on…
This is Newton's method.

We need a formula for this method.
Note that the tangent line at x_1 has the equation
$y-f(x_1)=f'(x_1)(x_2-x_1)$
If $f'(x_1)\neq 0$, then the line meets the x axis at $(x_2,0)$



Plug this coordinate into above equation, we get
$-f(x_1)=f'(x_1)(x_2-x_1)$
$$\Rightarrow x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$
Repeating this process for a third point $(x_3,0)$ gives
$-f(x_2)=f'(x_2)(x_3-x_2)$
$$\Rightarrow x_3=x_2-\frac{f(x_2)}{f'(x_2)}$$

In general, then we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

There is limiting process involved here for finding the solutions. We get as close as we like to the solution.

Example

The equation $x = \cos(x)$ has a solution between 0 and 1. Approximate it using Newton's method.

Rewrite as $x = \cos(x)$ so our function is $f(x) = x - \cos(x)$. The derivative is

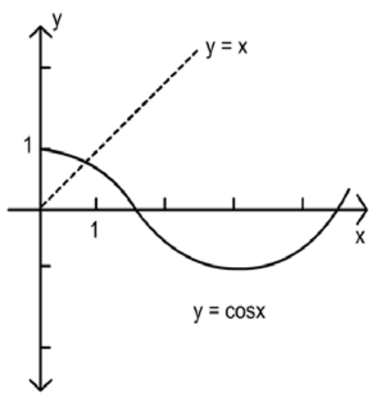
$$f'(x) = 1 + \sin(x)$$

So we have

$$x_{n+1} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)}$$

As our approximation formula

Here is a graph of the situation. From the graph it looks like the solution is closer to 1 than 0. So we will use $x_1 = 1$.So we get the following approximations.



$$x_2 = x_1 - \frac{x_1 - \cos(x_1)}{1 + \sin(x_1)} = 1 - \frac{1 - \cos(1)}{1 + \sin(1)} = 0.7503$$

$$x_3 = x_2 - \frac{x_2 - \cos(x_2)}{1 + \sin(x_2)} = 0.7503 - \frac{0.7503 - \cos(0.7503)}{1 + \sin(0.7503)} = 0.7391$$

You may continue if you will, but we will say that the solution is approximately $x \approx 0.7391$

Some difficulties with Newton's method

- Newton's method does not always work.
- If for some values of *n* $f'(x_n)$ then the formula for Newton's method involves division by 0 and we are out of business.

- Such a case will occur if the tangent line for some approximation has slope 0 or is parallel to the x axis.
- Sometimes the approximations don't converge to a solution.

Consider the equation $x^{\frac{1}{3}} = 0$
The only solution is x = 0. Let's approximate it by Newton's Method with initial approx $x_1 = 1$. We get the following Formula

$$x_{n+1} = x_n - \frac{(x_n)^{\frac{1}{3}}}{\frac{1}{3}(x_n)^{-\frac{2}{3}}} = -2x_n$$

Plug in $x_1 = 1$ and then the following approx to see that the values do not converge

Rolle's theorem
Rolle's Theorem says essentially that for a certain kind of function, if it crosses the x-axis at two point, then there is one point between those two points where the derivative of *f* is 0.

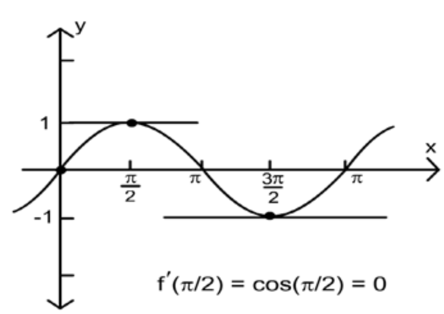
THEOREM 4.9.1
(Rolle's Theorem)

Let f be differentiable on (a,b) and continuous on [a,b] . If f(a) = f(b) = 0, then there is at least one point c in (a,b) where f′(c) = 0.

Example
The function $f(x) = \sin(x)$ is continuous and differentiable everywhere, hence continuous on [0 , 2π] and differentiable on (0, 2π). Also, f (0) = sin (0) = 0 and
f(2π)=sin (2π) = 0
So the function satisfies the hypotheses of Rolle's Theorem. So there exists a point *c* in the interval (0 , 2π) such that

$$f'(c) = \cos(c) = 0$$

$$y - f(a) = \frac{f(b) - f(a)}{b-a}(x-a)$$
$$\Rightarrow$$
$$y = \frac{f(b) - f(a)}{b-a}(x-a) + f(a)$$



Here is a more tangible way to think of Roll's theorem

I leave from Lahore to Islamabad.
When i start driving from Lahore, my velocity is 0, and when i reach Islamabad, my velocity is 0 as well.
Velocity is a continuous function on the interval [0 , 376].
Also, velocity is differentiable on (0 , 376) as its derivative acceleration is defined at each point on the velocity curve.
Hence during my drive from Lahore to ISB, there is some point on the motorway where the acceleration of the car was 0.
One could argue: What if you keep accelerating on the motorway!

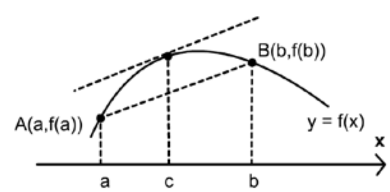
Mean Value Theorem
This says basically that under the right conditions, a function will have the same slope for the tangent line at a point as that of a certain secant line.

THEOREM 4.9.2
(Mean-Value Theorem)

Let f be differentiable on (a,b) and continuous on [a,b]. Then there is at least one point c in (a,b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof of MVT
From this figure we have the following:
Slope of Secant line joining A and B:



v(x) = vertical distance between curve f(x) and secant line through A and B

$$v(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]$$

Since f(x) is continuous [a , b] and differentiable on (a , b), so is *v*(x) by its formula involving f(x).
Also note that

$$v(a) = 0 \text{ and } v(b) = 0$$

So v(x) satisfies the assumptions of Roll's theorem on the interval [a , b]. So there is a point *c* in (a , b) such that $v'(c)=0$. But note that

$$v'(x) = f'(x) - \left[\frac{f(b) - f(a)}{b - a}\right]$$

$$\Rightarrow v'(c) = f'(c) - \left[\frac{f(b) - f(a)}{b - a}\right]$$

So by this last formula, at the point where $v'(c) = 0$, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

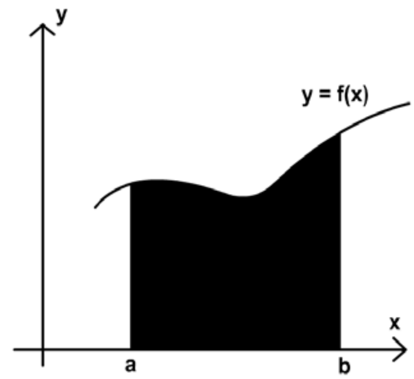
Lecture # 25

Integrations

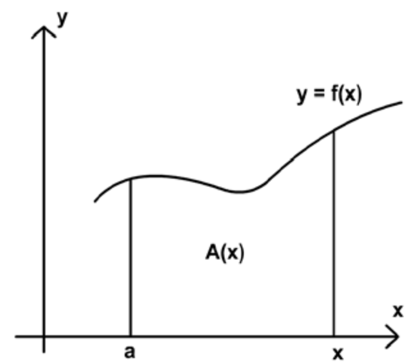
In this lecture we will look at the beginnings of the other major Calculus problem.

- The Area Problem
- Anti-derivatives (Integration)
- Integration formulas
- Indefinite Integral
- Properties of Indefinite Integral

The Area Problem

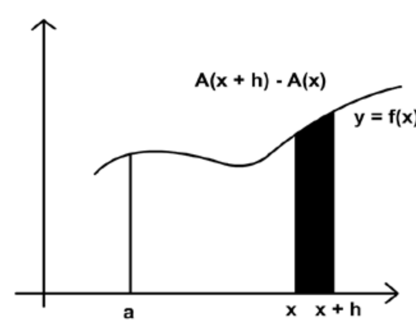


Given a continuous and non negative function on an interval [a , b], find the area between the graph of f and the interval [a , b] on the x-axis.
Instead of trying to solve a particular case like the one in the picture we just saw, we will generalize to solve this problem where the right end point will be any number x greater than or equal to b instead of just b.
We will denote the area we are trying to find as A(x) because this will be a function of x now as it depends on how far away x is from a.



It was the idea of Newton and Leibniz that to find the unknown area A(x), first find it derivative $A'(x)$ and use this derivative to determine what A (x) is ! Interesting approach.
So we want to find out first.

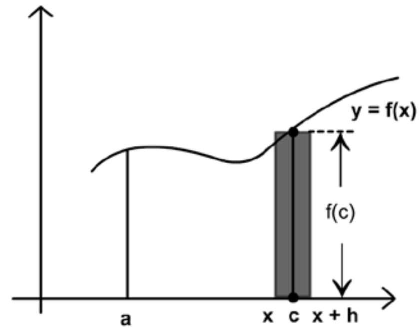
$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$



Let's assume for now that h > 0
The top of the derivative quotient is the difference of the two areas A(x) and A(x + h)
Let c be the midpoint of between x and x +h. Then the difference of areas can be approximated by the area of the rectangle with base length h and height f(c).

So we have

$$\frac{A(x+h)-A(x)}{h} \approx \frac{f(c)\cdot h}{h} = f(c)$$



Note that the error in the approximation from this rectangle in **A** will approach 0 as h goes to 0.
Then we have

$$A^{/}(x) = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} = \lim_{h\to 0} f(c)$$

As h goes to 0, c approaches x. Also, *f* is assumed to be a continuous function, so we have that f(c) goes to f (x) as c goes to *x*. Thus

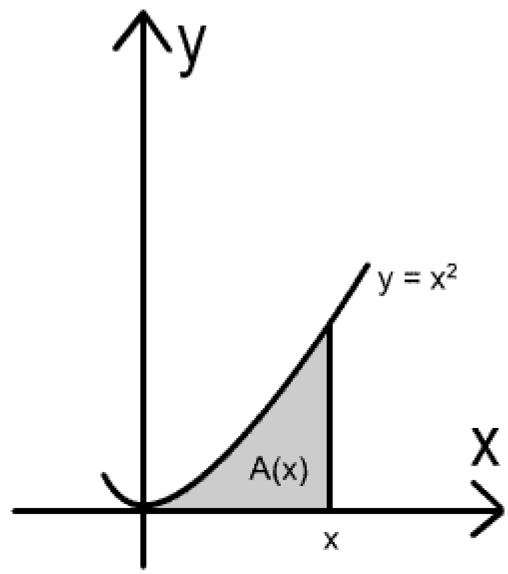
$$\lim_{h\to 0} f(c) = f(x) \quad \Rightarrow \quad A^{/}(x) = f(x)$$

So: The derivative of the area function A (x) is the function whose graph forms the upper boundary of the region under which the area is to be found

Example

Find the area of the region under the graph of $y = f(x) = x^2$ over the interval [0 , 1]
Look at the situation over the interval [0 , x]. Then we have from the discussion that $A'(x) = x^2$
To find A(x) we look for a function whose derivative is x^2
This is called an antidifferentiation problem as we are trying to find A(x) by undoing a differentiation.
A guess is the function

$$A(x) = \frac{1}{3}x^3$$



This is a formula for the areas function. So on the interval [0 , 1], we have x = 1 and our result is
A(1) = 1/3 units

Anti-derivatives
Definition 5.2.1
A function F is called antiderivative of a function *f* on a given interval if $F^{/}(x) = f(x)$ for all x in the interval.

Example

The functions $\frac{1}{3}x^3,\ \ \frac{1}{3}x^3-\pi,\ \ \frac{1}{3}x^3+C$ are all anti-derivatives of

$f(x)=x^2$ on the interval $(-\infty,+\infty)$

As the derivative of each is $f(x)=x^2$

If F (x) is any anti-derivative of *f* (x), then so is F (x) +C where C is a constant.

Here is a theorem

THEOREM 5.2.2

If F(x) is any antiderivative of f(x) on a given intreval,then for any value of C the function F(x) + C is also an antiderivative of f(x) on that interval; moreover every antiderivative of f(x) on the interval is expressible in the form F(x) + C. where C is constant.

Indefinite Integral

The process of finding anti-derivatives is called anti-differentiation or Integrations.

If there is some function F such that $\frac{d}{dx}[F(x)]=f(x)$

Then function of the form F (x) +C are anti-derivatives of f (x).

We denote this by $\int f(x)dx=F(x)+C$

The symbol $\int$ is called the integral sign and f (x) is called the integrand.

It is read as the "Indefinite integral of f (x) equals F (x)" $\int f(x)dx=F(x)+C$

The right side of the above equation is not a specific function but a whole set of possible functions.
That's why we call it the Indefinite integral.
C is called the constant of integrations.

Example

As we saw earlier, the anti-derivatives of $f(x)=x^2$ are functions of the form So we can write

$F(x)=\frac{1}{3}x^3+C$

The dx serves to identify the independent variable in the function involved in the integration.

Examples:

DERIVATIVE FORMULA	EQUIVALENT INTEGRATION FORMULA
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2 dx = x^3 + C$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t dt = \tan t + C$
$\frac{d}{dx}[u^{3/2}] = 3/2u^{1/2}$	$\int 3/2u^{1/2} du = u^{3/2} + C$

Example
From the table we just saw, we obtain the following results.

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int x^3 dx = \frac{x^4}{4} + C$$

$$\int \frac{1}{x^5} dx = \int x^{-5} dx = -\frac{1}{4x^4} + C$$

Properties of Indefinite Integral

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

THEOREM 5.2.3

a) A constant can be moved through an integral sign; that is,

$$\int cf(x)dx = c\int f(x)dx$$

b) An antiderivative of a sum is the sum of the antiderivative; that is

$$\int [f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

c) An antiderivative of a difference is the difference of the antiderivative; that is

$$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$$

Example

Evaluate $\int 4\cos(x)dx$

$$\int 4\cos(x)dx = 4\int \cos(x)dx = 4[\sin(x)+C] = 4\sin(x)+K$$

Where 4C = K

Example

$$\int (x^2+x)dx$$

$$\int (x^2+x)dx = \int x^2 dx + \int x\cdot dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Generalized version of the Theorem 5.2.3 b and c

$$\int [c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x)]dx$$

$$= c_1\int f_1(x)dx + c_2\int f_2(x)dx + \ldots + c_n\int f_n(x)dx$$

Example

$$\int (3x^6 - 2x^2 + 7x + 1)dx$$

$$= 3\int x^6 dx - 2\int x^2 dx + 7\int x dx + \int dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C$$

Example

$$\int \frac{\cos(x)}{\sin^2(x)}dx$$

$$= \int \frac{1}{\sin(x)}\frac{\cos(x)}{\sin(x)}dx$$

$$= \int \cos ec(x)\cot(x)dx = -\cos ec(x) + c$$

Lecture # 26

Integration by substitution

This is like the Chain Rule we saw for differentiation. The idea is to integrate functions that are composition of different functions.

$$\frac{d}{du}[G(u)] = f(u)$$

Also this means that

$$\int f(u)du = \int \frac{d}{du}[G(u)]du = G(u) + C$$

Let U be a function of x. Then we have

$$\frac{d}{dx}[G(u)] = \frac{d}{du}[G(u)] \cdot \frac{du}{dx} = f(u)\frac{du}{dx}$$

Now if we apply the integral on both sides w . r . t x we get

$$\int \left(f(u)\frac{du}{dx} \right) dx = \int \frac{d}{dx}[G(u)]\ dx = G(u) + C$$

$\Rightarrow$

$$\int \left(f(u)\frac{du}{dx} \right) dx = \int f(u)du$$

Example

$$\int (x^2+1)^{50}.2x\ dx$$

Let $u = x^2 + 1$. Then $\dfrac{du}{dx} = 2x$.

Now we can rewrite the given problem as

$$\int (x^2+1)^{50}.2x\ dx = \int \left[u^{50}\frac{du}{dx} \right] dx = \int u^{50} du$$

$$\uparrow \qquad\qquad \uparrow$$

$$\int \left[f(u)\frac{du}{dx} \right] dx = \int f(u)du$$

$$\int u^{50}du = \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C \quad , \text{ where } u = x^2 + 1$$

Caution: Don't feel tempted to just add 1 to the power 50 in the original problem!! That will be incorrect.

The reason is that the "Power Rule" for the integrals is applicable to function which are not a composition of others.

In this example, $(x^2+1)^{50}$ is a composition of two functions.

Here is a summary of the **general procedure** we need to follow to do integration by ***u*-substitution**.

Step 1: Make the choice for *u*, say *u* = g (x)

Step 2: Compute du/dx = g'(x)

Step 3: Make the substitution u = g (x), du = g'(x) dx in the original integral.

By this point, the whole original integral should be in terms of u and there should be no x's in it.

Step 4: Evaluate the resulting integral

Step 5: Replace u by g (x) so the final answer is in *x*.

- No hard and fast rule for choosing *u*.
- The choice of u should be such that the resulting calculus
- Results in a simplified integral.
- There may be more than one choice of *u* in some cases.

Only practice and experience makes things easier.

Think of this choice of *u* like playing chess. You need to choose it so that the future looks bright for problem solution!

Integrand is the derivative of a known function with a constant added or subtracted from the independent variable.

If the integrand is the derivative of a known function with a constant added or subtracted from the independent variable, then the substitution is the easiest.

Here is table 5.2.1 that we saw earlier. This will help recall BASIC integration formulas.

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
$\frac{d}{dx}[x] = 1$	$\int dx = x + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r$	$\int x^r dx = \frac{x^{r+1}}{r+1} + C (r \neq -1)$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x dx = -\cos x + C$

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$

We want to integrate a function in which "The integrand is the derivative of a known function with a constant added or subtracted from the independent variable"

Example

$$\int \sin(x+9)dx = \int \sin(u)du = -\cos(u) + C = -\cos(x+9) + C$$

$$u = x+9$$

$$\frac{du}{dx} = 1 \quad \Rightarrow \quad du = 1.dx$$

The integrand sine is the derivative of cosine function

Example

$$\int (x-8)^{23}dx = \int u^{23}du = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C$$

$$u = x-8$$

$$du = 1.dx = dx$$

Integrand is the derivative of a known function and a constant multiplies the independent variable

Example

$$\int \cos(5x)dx = \int \cos(u)\frac{du}{5} = \frac{1}{5}\int \cos(u)du = \frac{1}{5}\sin(u) + C = \frac{1}{5}\sin(5x) + C$$

$$u = 5x$$

$$du = 5dx \quad \Rightarrow \quad \frac{du}{5} = dx$$

Example

$$\int \sin^2(x)\cos(x)dx$$

Let $u = \sin(x)$, then

$$du = \cos(x)dx$$

So

$$\int \sin^2(x)\cos(x)dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sin^3(x)}{3} + C$$

Example

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$$

Let $u = \sqrt{x},$ then

$$du = \frac{1}{2\sqrt{x}} dx$$

So

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx = \int 2\cos(u)du = 2\int \cos(u)du = 2\sin(u) + C = 2\sin(\sqrt{x}) + C$$

Complicated **example**

$$\int t^4\sqrt{3-5t^5}dt \; \int (x^2+1)^{50}.2x\; dx = \int \left[u^{50}\frac{du}{dx} \right] dx = \int u^{50} du$$

What should the substitution be?

Well the idea is to get an expression to equal *u* so that when we differentiate it, we get a formula that involves a *du* and everything that was left over in *x* after the substitution.

$$\int t^4\sqrt{3-5t^5}dt$$

$$u = 3 - 5t^5$$

$$du = -25t^4 dt \quad \Rightarrow \quad -\frac{1}{25}du = t^4 dt$$

So

$$\int t^4\sqrt{3-5t^5}dt = -\frac{1}{25}\int \sqrt[3]{u}\; du = -\frac{1}{25}\int u^{\frac{1}{3}}\; du = -\frac{1}{25}\frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C$$

$$= -\frac{3}{100}(3-5t^5)^{\frac{4}{3}} + C$$

Lecture #. 27

Sigma Notation

- Sigma notation is used to write lengthy sums in compact form.
- Sigma or Σ is an Upper case letter in Greek.
- This symbol is called Sigma or summation as it is used to represent lengthy sums.

Here is an example of how this notation works.

Consider the sum

$$1^2+2^2+3^2+4^2+5^2$$

Note that every term in this sum is the square of an integer from 1 to 5.

Let's assign these integers a variable *k* and keep it in mind that this *k* can take on values from 1 to 5.

Then we can say that k^2 represent each of the elements in the sum. So we can write this as $\sum_{k=1}^{5} k^2$

"Summation of k^2 where k goes from 1 to 5

Example

$$\sum_{k=4}^{8} k^3 = 4^3+5^3+6^3+7^3+8^3$$

$$\sum_{k=1}^{5} 2k = 2(1)+2(2)+2(3)+2(4)+2(5) = 2+4+6+8+10$$

$$\sum_{k=0}^{5} (-1)^k (2k+1) = 1-3+5-7+9-11$$

The number on the top are called the upper limits of the summation and the numbers at the bottom are called lower limits of the summation.

The letter *k* is called the index of the summation

It is not necessary that the letter *k* represents the index of summation. We could use *i or j or m etc.*

Example

$$\sum_{i=1}^{4}\frac{1}{i},\ \sum_{n=1}^{4}\frac{1}{n},\ \sum_{j=1}^{4}\frac{1}{j},$$

All denote the sum

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

If the upper and lower limits are the same, then the summation reduces to just one

term $\sum_{k=2}^{2} k^3 = 2^3$

If the expression to the right of the summation does not involve the index of the summation, then do the following

$$\sum_{i=1}^{5} 2 = 2+2+2+2+2$$

$$\sum_{k=3}^{6} x^3 = x^3 + x^3 + x^3 + x^3$$

A sum can be written in more than one way with the Sigma notation if we change the limits of the summation

Example

The following summations all represent the sum of the first five positive integers

$$\sum_{k=1}^{5} 2k = 2+4+6+8+10$$

$$\sum_{k=0}^{4} (2k+2) = 2+4+6+8+10$$

$$\sum_{k=2}^{6} (2k-2) = 2+4+6+8+10$$

Changing the index of the summation

It is often necessary and useful to change a given Sigma notation for a sum to another sigma notation with different limits of summation

Example

Express $\sum_{k=3}^{7} 5^{k-2}$ in sigma notation so that the lower limit is 0 rather than 3.

Define a new summation index *j* by the following formula

$j = k-3 \Rightarrow k = j+3$

Then as *k* runs from 3 to 7, *j* runs from 0 to 4. So

$$\sum_{k=3}^{7} 5^{k-2} = \sum_{j=0}^{4} j^{(j+3)-2} = \sum_{j=0}^{4} j^{j+1}$$

You should check that the two actually represent the same sum by putting values into the index in both notations.

To represent a general sum, we will use letters with subscripts.

Example

$a_1 + a_2 + a_3$ represent the general sum with three terms

This can also be written as in sigma notation

$$\sum_{k=1}^{3} a_k = a_1 + a_2 + a_3$$

General sum with *n* terms can be written as

$$\sum_{k=1}^{n} b_k = b_1 + b_2 + b_3 + \ldots + b_n$$

Properties of Sigma notation

Here are a few properties of sigma notation that will be helpful later on

THEOREM 5.4.1

a) $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

b) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

c) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

Here are some sum formulas written in sigma notation that will be helpful later.

THEOREM 5.4.2

a) $\sum_{k=1}^{n} k = 1+2+3+\ldots\ldots+n = \frac{n(n+1)}{2}$

b) $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

c) $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example

$$\sum_{k=1}^{30} k(k+1) = \sum_{k=1}^{30} (k^2+k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$

$$= \frac{(30)(31)(61)}{6} + \frac{30(31)}{2} = 9920$$

Theorem 5.4.2 a) and b)

In a formula involving summation like this one

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+2)}{6}$$

$$\Rightarrow 1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+2)}{6}$$

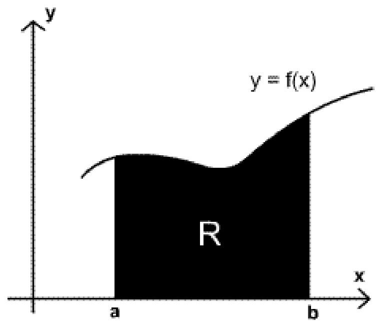
The left part is called the open form of the sum.

The right part is called the closed form of the sum.

Lecture # 28

Area as Limits

- Definition of Area
- Some technical considerations
- Numerical approx of area
- Look at the figure below



In this figure, there is a region bounded below by the x-axis, on the sides by the lines x= a and x = b, and above by a curve or the graph of a continuous function y = f(x) which is also non-negative on the interval [a, b].

Earlier we saw that such an area can be computed using anti-derivatives.

Let's make the concept precise.

Recall that the slope of the tangent line was defined in terms of the limit of the slopes of secant lines.

Similarly, we will define area of a region R as limits of the areas of simpler regions whose areas are known.

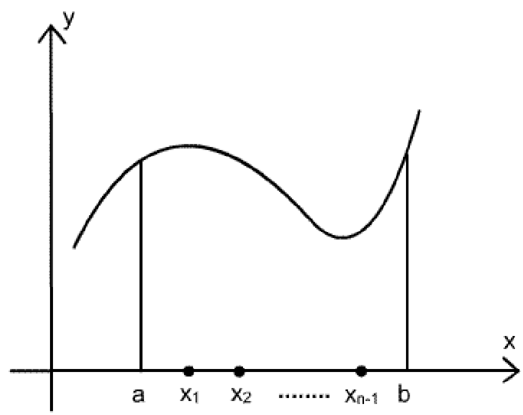
We will break up R into rectangles, and then find the area of each rectangle in R and then add all these areas up.

The result will be an approximation to the region R. Let's call it

If we let n go to infinity, the resulting will give a better and better approximation to R as the rectangle in R will get thinner and thinner, and the gaps will be filled in.

Here is the formal idea

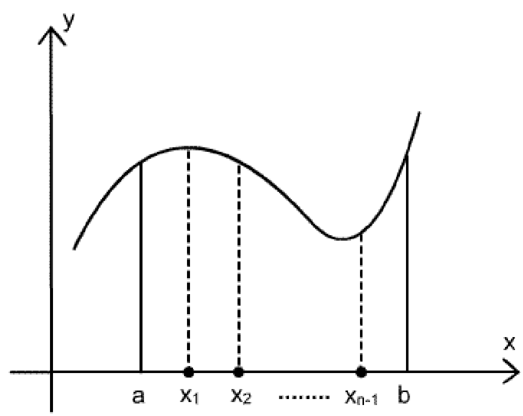
Choose an arbitrary positive integer *n*, and divide the interval [a,b] into *n* subintervals of width $\frac{b-a}{n}$ by inserting *n*-1 equally spaced points between *a* and *b* say $x_1, x_2, \ldots x_{n-1}$



These points of subdivision form a regular partition of [a,b] $a, x_1, x_2, \ldots x_{n-1}, b$

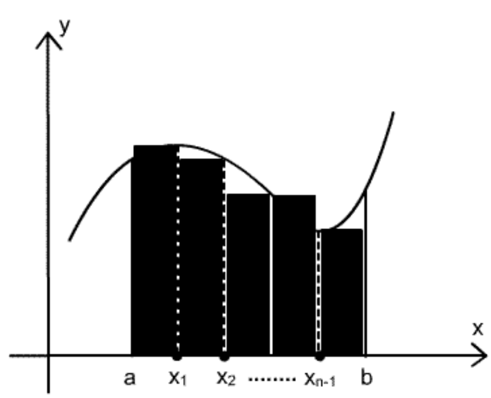
Next draw a vertical line through the points $x_1^*, x_2^*, \ldots x_n^*$

This will divide the region R into *n* strips of uniform width



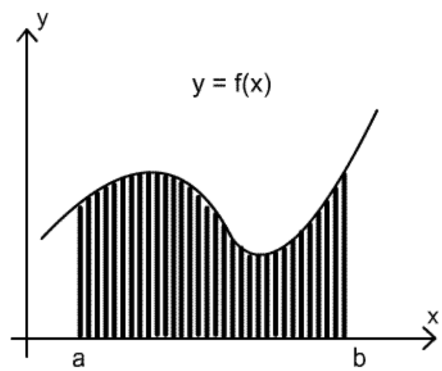
Now we want to approximate the area of each strip by the area of a rectangle. For this, choose an arbitrary point in each subinterval ,

Over each subinterval construct a rectangle whose height is the values of the function *f* at the selected arbitrary point.



The union of these rectangles form the region R_n that we can regard as a reasonable approx to the region R. The area of R_n can be got by adding the areas of all the rectangles forming it

If now we let n get big, the number of rectangles gets big, and the gaps btw the curve and the rectangles get filled in.



As n goes to infinity, the approx gets as good as the real thing.

So we define $A = area(R) = \lim_{n \to +\infty} [area(R_n)]$

For all the following work and computation, we will treat *n* as a POSITIVE integer

For computational purposes, we can write **A** in a different form as follows

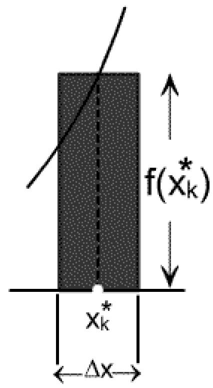
In the interval [a,b], each of the approximation rectangles has width $\dfrac{b-a}{n}$

We call this delta x or $\Delta x = \dfrac{b-a}{n}$

The heights of the approximating rectangles are at the points $x_1^*, x_2^*, \ldots x_n^*$

So the approximating rectangles making up the region have areas

$$f(x_1^*)\Delta x, f(x_2^*)\Delta x, \ldots, f(x_n^*)\Delta x$$



Area of k[th] rectangle = $f(x_k^*)\,.\,\Delta x$

So the total area of R_n is given by

$$area(R_n) = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

$$OR$$

$$area(R_n) = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

So now **A** can be written as

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

This we will take to be the PRECISE def of the area of the region R

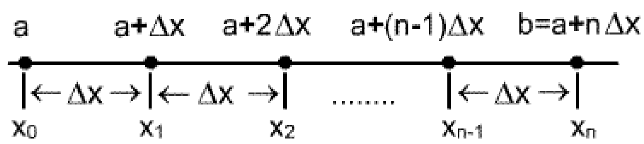
Some Technical Considerations

The points $x_1^*, x_2^*, \ldots x_n^*$ were chosen arbitrarily.

What if some other points were chosen? Would the resulting values for f(x) at these points be different? And if so then the definition of area will not be well-defined?
It is proved in advanced courses that since *f* is continuous it doesn't matter what points are taken in a subinterval.

Usually the point x_k^* in a subinterval is chosen so that it is the left end point of the interval, the right end point of the interval or the midpoint of the interval.

Note that we divided the interval [a,b] by the points $x_1, x_2, \ldots, x_{n-1}$ with $x_0 = a$ and $x_n = b$ into equal parts of width Δx



This figure shows that

$$x_k = a + k\Delta x \text{ for } k = 0,1,2 \ldots n$$

SO

$$x_k^{\ *} = x_{k-1} = a + (k-1)\Delta x \qquad \text{Left end point}$$

$$x_k^{\ *} = x_k = a + k\Delta x \qquad \text{Right end point}$$

$$x_k^{\ *} = \frac{1}{2}(x_{k-1} + x_k) = a + (k - \frac{1}{2})\Delta x \qquad \text{Midpoint}$$

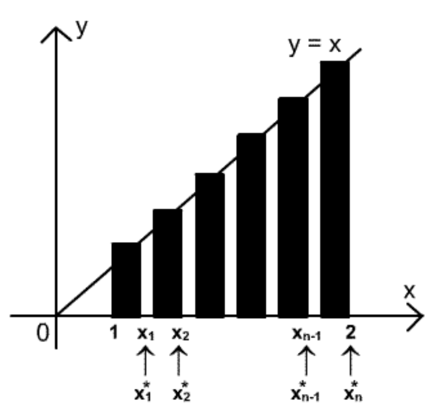
Example

Use the definition of the Area with $x_k^{\ *}$ as the right end point of each subinterval to find the area under the line y = x over the interval [1,2].

Subdivide [1,2] into *n* equal parts, then each part will have length

$\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{n} = \dfrac{1}{n}$ We are taking x_k^* as the right end point so we

Have $x_k^* = a + k\Delta x = 1 + \dfrac{k}{n}$



Thus, the kth rectangle has area

$$f(x_k^*)\Delta x = x_k^*\Delta x = \left(1+\frac{k}{n}\right)\Delta x = \left(1+\frac{k}{n}\right)\frac{1}{n}$$

And the sum of the areas of the *n* rectangles is

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty}\left(\frac{3}{2}+\frac{1}{2n}\right) = \frac{3}{2}$$

Note that the region we have computed the area of is a trapezoid with height h = 1 and bases $b_1 = 1$ and $b_2 = 2$. From basic geometry we have that

Area of trapezoid = $A = \frac{1}{2}h(b_1+b_2) = \frac{1}{2}(1)(1+2) = \frac{3}{2}$

Example

Same problem as before but with left end points.[1,2].

Use the definition of the Area with x_k^* as the left end point of each subinterval to find the area under the line y = x over the interval [1,2].

Subdivide [1,2] into *n* equal parts, then each part will have length

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

We are taking x_k^* as the left end point so we

Have $x_k^* = a + k\Delta x = 1 + \frac{k}{n}$

Thus, the kth rectangle has area

$$f(x_k^*)\Delta x = x_k^*\Delta x = \left(1+\frac{k}{n}\right)\Delta x = \left(1+\frac{k}{n}\right)\frac{1}{n}$$

And the sum of the areas of the *n* rectangles is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[\left(1+\frac{k}{n}\right)\frac{1}{n}\right] = \sum_{k=1}^{n}\left[\left(\frac{1}{n}+\frac{k}{n^2}\right)\right] = \frac{1}{n}\sum_{k=1}^{n}1 + \frac{1}{n^2}\sum_{k=1}^{n}k$$

$$= \frac{1}{n}\cdot n + \frac{1}{n^2}\left[\frac{1}{2}n(n+1)\right]$$

Note that the region we have computed the area of is a trapezoid with height h = 1 and bases $b_1 = 1$ and $b_2 = 2$. From basic geometry we have that

Area of trapezoid = A = $\frac{1}{2}h(b_1+b_2)=\frac{1}{2}(1)(1+2)=\frac{3}{2}$

Example

Use the area definition with right endpoints of each subinterval to find the area under the parabola $y=9-x^2$ over the interval [0,3].

Do calculations from book on page 272 (end) and page 273 for example.

Numerical Approximations of Area

As we have all seen so far, the computations involved in computing the limits are tedious and long. In some cases, it is even impossible to carry out the computations by the definition of the area. In such cases, it is easier to get a GOOD approx for the area using large values of *n* and using a computer or a calculator.

Example

Use a computer of a calculator to find the area under the curve y = $9-x^2$ over the interval [0,3] and n = 10, 20 and 50.

Left end point approximation			Right end point approximation		Mid point approximation	
K	x_k^*	$9-(x_k)^2$	x_k^*	$9-(x_k)^2$	x_k^*	$9-(x_k)^2$
1	0.0	9.000000	0.3	8.910000	0.15	8.977500
2	0.3	8.910000	0.6	8.640000	0.45	8.797500
3	0.6	8.640000	0.9	8.190000	0.75	8.437500
4	0.9	8.190000	1.2	7.560000	1.05	7.897500
5	1.2	7.560000	1.5	6.750000	1.35	7.177500
6	1.5	6.750000	1.8	5.760000	1.65	6.277500
7	1.8	5.760000	2.1	4.590000	1.95	5.197500
8	2.1	4.590000	2.4	3.240000	2.25	3.937500
9	2.4	3.240000	2.7	1.710000	2.55	2.497500
10	2.7	1.710000	3.0	0.00000	2.85	0.877500
$\Delta x\sum_{k=1}^{n} f(x_k^*)$ = 19.30500			=16.605000		=18.022500	

And for different values of n we have the following approximations.

n	**Left end point approximation**	**Right end point approximation**	**Mid point approximation**
10	19.305000	16.605000	18.02250
20	18.663750	17.313750	18.005625
50	18.268200	17/728200	18.000900

Lecture # 29

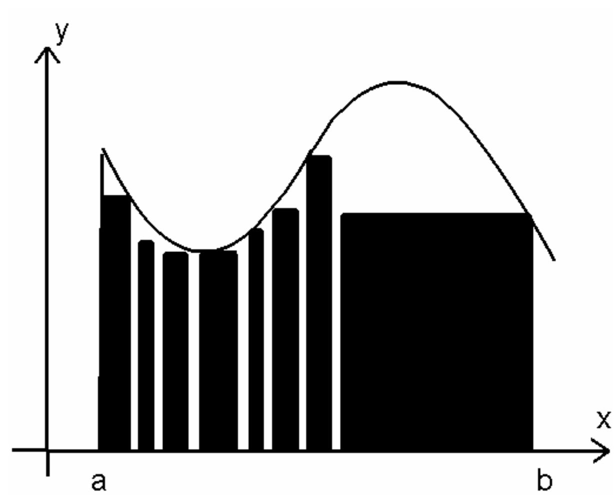
The Definite Integral

- Definition of Definite Integral
- Definite Integral of continuous functions with nonnegative values
- Definite Integral of continuous functions with negative and positive values
- Definite Integral of functions with discontinuities
- Properties of the Definite Integral
- Inequalities involving Definite Integral

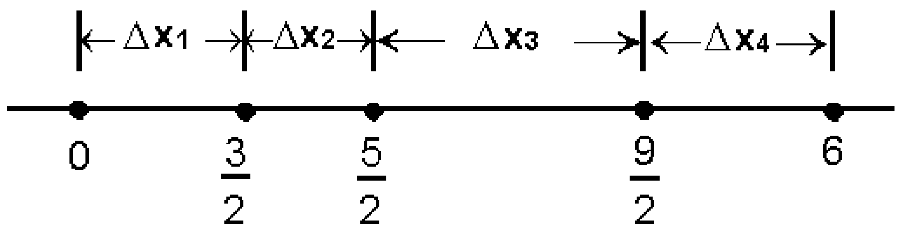
We have so far developed a definition of Area as limit of a bunch of rectangles with equal widths. The question arises that why choose widths? The answer is that we don't have to.
We want now to have a definition of Area that includes the general case where the widths of the rectangles are not necessarily equal.
When the rectangles widths were equal note that the formula for the width was defined as

$$\Delta x = \frac{b-a}{n}$$

In this formula you can see that as *n* goes to infinity, the width delta x goes to zero
If the widths are not the same for all rectangles, then this is not necessarily the case. Suppose that we have a rectangle construction in which we divide [a,b] so that one half is continually subdivided into smaller Rectangle, while the right is left as one big rectangle.



In this case, when n goes to inf, left part goes to 0, not the Right half.
Let's fix this problem.
Subdivide [a,b] into *n* subinterval whose widths are $\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n$
The subintervals are said to partition the interval, and the largest subinterval width is called the mesh size of the partition. This is denoted by $\max \Delta x_k$



$$\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = 2$$

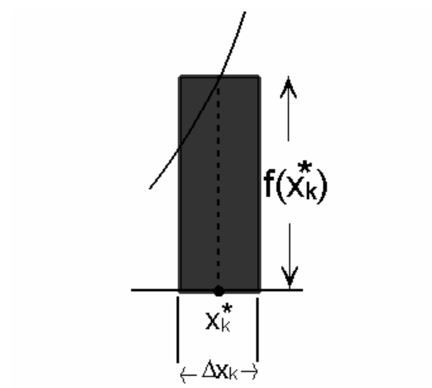
Read as "maximum of the Δx_k."

In this figure

$$\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = 2$$

We see that

If [a,b] is partitioned into n subintervals, and if x_k is an arbitrary point in kth subinterval, then $f(x_k^*)\Delta x_k$ is the area of the rectangle with height f(x_k) and width



Area of k[th] rectangle = f(x*k) . Δxk

delta x_k , and $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ is the sum of the shaded rectangular areas

Now if we let $\max \Delta x_k \to 0$ the width of EVERY rectangle tends to 0 because none of them exceeds the max. So we have the area under the curve now defined more generally as

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

DEFINITION 5.6.1
(Area under a curve)

If the function f is continuous an [a, b] and if $f(x) \geq 0$ for all x in [a, b], then the area under the curve $y = f(x)$ over the interval [a, b] is defined by $A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$

Definite Integral of continuous functions with nonnegative values

The limit in the definition we just saw is VERY important. It has special notation. We write it as

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

The expression on the left is called the definite integral of f from a to b. a and b are called the upper and lower limits of integration respectively.

There is a relationship btw this definite integral and the indefinite integral we talked about earlier.

We will see it later.

So with this notation we can say that

$\int_a^b f(x)dx =$ Area under the curve $y = f(x)$ over [a, b].

Our goal will be to evaluate the definite integral efficiently instead of using the definition all the time.

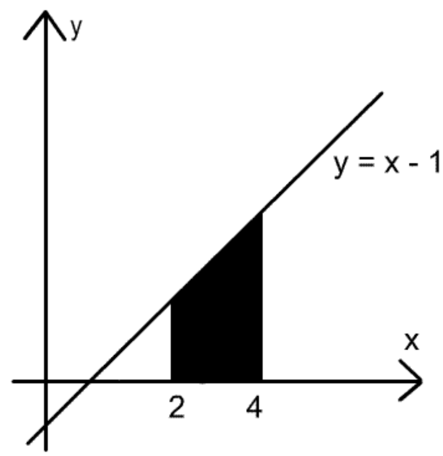
Example

$$\int_2^4 (x-1)dx$$

This represents the area under the curve y = (x – 1) over [2,4].

Here is a picture of it.

Note that the region described



here is just a trapezoid with h = 2, and $b_1 = 1$ and $b_2 = 3$. So we can evaluate this integral from basic geometry as $\int_2^4 (x-1)dx = \frac{1}{2}(2)(1+3) = 4$

The sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ is called the Riemann Sum in honor of the German mathematician Bernhard Riemann.

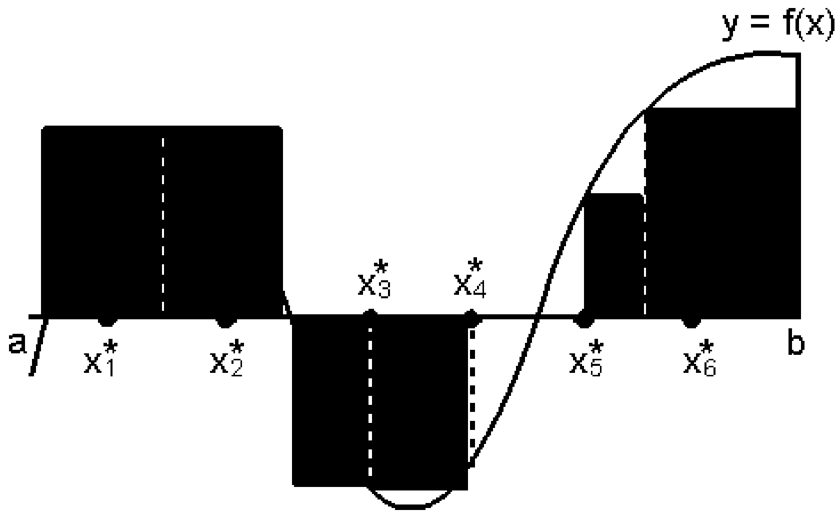
The formula for the area we just saw was for continuous and nonnegative functions. It involves a limit. The question is: does the limit always exist, so that we can always find the area ?

It is proved in advanced levels that for continuous and nonnegative functions, this limit always exists.

Definite Integral of continuous functions with negative and positive values

Now we want to extend our area definition to include continuous functions on [a,b] that have both positive and negative values on [a,b].

Look at the following figure

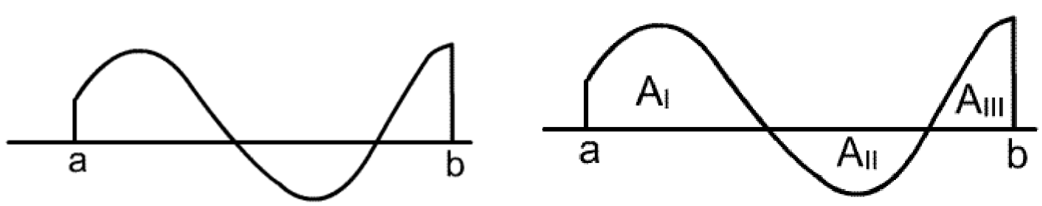


The rectangles below [a,b] but above the curve f(x) have some area just the way their counterparts above [a,b] but below curve f(x) do. The difference is that we can view the number representing the area of the rectangles below as negative values of f(x) for some x_k^*. So the Riemann sum gives

$$\sum_{k=1}^{6} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \ldots f(x_6^*)\Delta x_6$$
$$= A_1 + A_2 - A_3 - A_4 + A_5 + A_6$$
$$= (A_1 + A_2 + A_5 + A_6) - (A_3 + A_4)$$

This says that the Riemann sum is the difference of two areas: the total area of rectangle above the x-axis, minus the total areas of rectangles below the x-axis.

Now as max delta x goes to 0, the situation still works out nicely with the large # of rectangles filling in any left over space between the earlier rectangles and the curve.



So we have the following definition

Definition 5.6.2

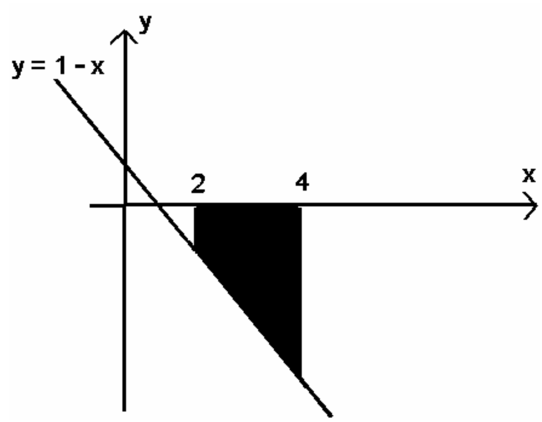
If the function f is continuous on [a,b], and can assume both positive and negative values ,then the **net signed area** A between y= f (x) and the interval [a,b] is defined by

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Net signed area means that the total difference of the two areas above and below may be negative. If that happens then we just treat the negative as representing the fact that the area below is larger than the area above.

EXAMPLE

$$\int_2^4 (1-x)dx$$



Geometrically, this region lies below the interval [2,4]. The region is a trapezoid and we can find the area as 4 using its dimensions. Since its below the x-axis we write

$$\int_2^4 (1-x)dx = -4$$

Definite Integral of functions with discontinuities

For the two definitions we have seen so far, *f* was a continuous function. This guaranteed that the are defined as a limits could always be found since the limits existed for continuous *f*.

If *f* is not continuous, then the area may or may not be found as the limit of the Sum may or may not exist.

Here is a definition for such a function for the area.

DEFINITION 5.6.3
(Area under a curve)

If the function f is defined on the close interval [a, b] then, f is called Riemann integrable on [a, b] or more simply integrable on [a, b] if the limit exists

$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ If f is integrable on [a, b], then we define The definite integral of f from a to b by

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

So far we assumed that the upper limit of integration was greater than the lower limit.
Here is a definition that allows for the limits to be the same, and the case where the upper may be less than the lower.

DEFINITION 5.6.4

(a) If a is in the domain of f , we define

$$\int_a^a f(x)dx = 0$$

If f is integrable on [a, b], then we define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Properties of the definite Integral

Here are some properties of the definite integral

Theorem 5.6.5

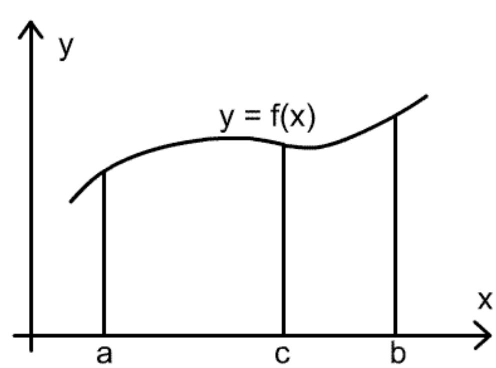
If f and g are integrable on [a, b] and if c is a constant, then cf, f + g, and f – g are integrable on [a, b] and

(a) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(b) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(c) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Now look at this figure
It is clear from the picture
that the area under the curve
over [a,b] can be split into
a sum of two areas, one area
from a to c, and the other area from c to b. Formally



Theorem 5.6.5

If f is integrable on a closed interval containing the three points a, b and c then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

no matter how the points are ordered.

Example

$$\int_1^5 f(x)dx = -1, \quad \int_3^5 f(x)dx = 3, \quad \int_3^5 g(x)dx = 4$$

Find

$$\int_1^3 f(x)dx$$

From thm5.6.6 with a = 1, b = 5, c = 3

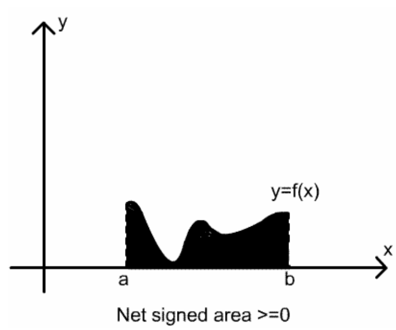
$$\int_1^5 f(x)dx = \int_1^3 f(x)dx + \int_3^5 f(x)dx$$

So

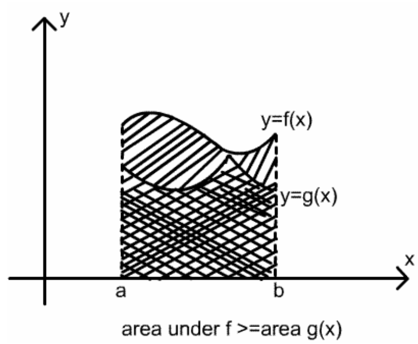
$$\int_1^3 f(x)dx = \int_1^5 f(x)dx - \int_3^5 f(x)dx = -1 - 3 = -4$$

Inequalities involving definite integral

Look at this figure



This says that if a function is nonnegative (the graph does not go below x-axis) on [a,b], then the net area between the curve and x-axis must be greater than or equal to 0. Look at this figure



The graph of f does not go below that of g,

or in other words f(x) >=g(x) over the interval and f and g are nonnegative, then area under f must be >= area under g over the interval

Theorem 5.6.7

(a) If *f* is integrable on [a,b] and $f(x) \geq 0$ for all x in [a,b],then

$$\int_a^b f(x) \geq 0$$

(b) If f and g are integrable functions on[a,b] and $f(x) \geq g(x)$ for all x in

[a, b] then $\int_a^b f(x) \geq \int_a^b g(x)$

Works with <, > <= also

Example

Show that $\int_0^1 \frac{\cos(x)}{2x^3-5}dx$ is negative.

On the interval [0,1], cos(x) > 0, but $2x^3-5<0$. So f(x) as a whole is < 0. So from theorem 5.6.7a), with < instead of >= the integral is negative

Definition 5.6.8

A function f is said to be bounded on interval [a, b] if there is a positive number M such that

$$-M \leq f(x) \leq M$$

for all x in [a, b]. Geometrically, this means that the graph of f on the interval [a, b] lies between the lines y = -M and y = M.

$y = x^2$ on the interval [-2, 2] is bounded since its graph lies between the lines y = 0 and y = 5

Theorem 5.6.9

Let f be a function that is defined at all the points in the interval [a,b].

(a) If f is continuous on [a,b] ,then f is integrable on [a,b].

(b) If f is bounded on [a,b] and has only finite many points of discontinuity on [a,b] then f is integrable on [a,b].

(c) If f is not bounded on [a,b] ,then f is not integrable on [a,b]

Lecture # 30

First Fundamental Theorem of Calculus

- 1st fundamental theorem of calculus.
- Relationship between definite and indefinite integrals.
- Mean Values theorem for Integrals.
- Average Values of a function.

This is the 1st fundamental theorem of calculus.

Theorem 5.7.1(The first fundamental theorem of calculus)

If f is continuous on [a, b] and if F is an antiderivative of f on [a, b] , then

$$\int_a^b f(x)dx = F(b) - F(a)$$

This theorem tells us how to evaluate EASILY the definite integral.
It says that find the anti-derivative of f(x)first, call it F(x), and then evaluate this function on the limits of the integration. Let's prove this.

Proof
We will use the Mean Value Theorem involving derivatives to prove the first fundamental theorem of calculus.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Let us subdivide the given interval of integration namely [a, b] into n subintervals using the points $x_1, x_2, ..., x_{n-1}$ in the interval [a, b] such that
$a < x_1 < x_2 < ... < x_{n-1} < b$.
So now we have n subinterval of [a, b]. We can denote the widths of these intervals as $\Delta x_1, \Delta x_2, ..., \Delta x_n$. Where for example $\Delta x_2 = x_2 - x_1$.
Since $F'(x) = f(x)$ for all x in [a, b], it is obvious that F(x) satisfies the requirements of the Mean Value Theorem involving derivatives on each of the subintervals. So by MVT, we can find points $x_1^*, x_2^*, ..., x_n^*$
In each of the respective subintervals $[a, x_1], [x_1, x_2], ..., [x_{n-1}, b]$.

So we now make the following equations

$$F(x_1) - F(a) = F'(x_1^*)(x_1 - a) = f(x_1^*)\Delta x_1$$
$$F(x_2) - F(x_1) = F'(x_2^*)(x_2 - x_1) = f(x_2^*)\Delta x_2$$
$$F(x_3) - F(x_2) = F'(x_3^*)(x_3 - x_2) = f(x_3^*)\Delta x_3$$
.
.
.
$$F(b) - F(x_{n-1}) = F'(x_n^*)(b - x_{n-1}) = f(x_n^*)\Delta x_n$$

Adding up these equations we get

$$F(b) - F(a) = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Now increase n in such a way that

$$\max \Delta x_k \to 0$$

Since f is assumed to be continuous, we have the following result

$$F(b) - F(a) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \underset{\substack{\uparrow \\ \text{THEOREM 5.6.9(a)}}}{\int_a^b f(x)dx}$$

I did not apply the limits on the left because even if I did, the expression does not involve *n* and therefore nothing happens.

$F(b) - F(a)$ can also be written as $F(x)]_a^b$ so we have

$$\int_a^b f(x)dx = F(x)]_a^b$$

Example

Evaluate $\int_1^2 x dx$

$\int_1^2 x dx$ .The function $F(x) = \frac{1}{2}x^2$ is and antiderivative of $f(x) = x$ .So we have

$$\int_1^2 x dx = \frac{1}{2}x^2 \Big]_1^2 = \frac{1}{2}(4) - \frac{1}{2}(1) = 2 - \frac{1}{2} = \frac{3}{2}$$

Here are a few properties of the bracket notation we just saw.
Prove these yourself

Properties

$$\left[cF(x)\right]_a^b = c\left[F(x)\right]_a^b$$
$$\left[F(x) + G(x)\right]_a^b = \left[F(x)\right]_a^b + \left[G(x)\right]_a^b$$
$$\left[F(x) - G(x)\right]_a^b = \left[F(x)\right]_a^b - \left[G(x)\right]_a^b$$

These are easy if you remember that this notation and the definite integral are the same thing!!
Relationship between definite and indefinite integrals
In applying the 1st theorem of Calc, it does not matter WHICH anti-derivative of f is used.
If F is any anti-derivative of f then all the others have form
F(x) + C by theorem 5.2.2
Hence we have the following:

$$\left[F(x)+C\right]_a^b = \left[F(b)+C\right]-\left[F(a)+C\right]$$

$$= F(b)-F(a) = F(x)\Big]_a^b = \int_a^b f(x)dx \qquad \ldots(A)$$

This shows that all anti-derivatives of f on [a,b] give the same values for the definite integral
Now since

$$\int f(x)dx = F(x)+C$$

It follows from (A) that

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$$

So we can first evaluate the indefinite integral and use the limits as well to evaluate the definite integral
This relates the definite and the indefinite integrals.

Example
Using the 1st theorem of calculus, find the area under the curve
y = cos(x) over the interval [0, 2π]
Since $\cos(x) \geq 0$ for $0 \leq x \leq \pi$, the area is

$$A = \int_0^{\frac{\pi}{2}} \cos(x)dx = \left[\int \cos(x)dx\right]_0^{\frac{\pi}{2}}$$

$$= \sin(x)\Big]_0^{\frac{\pi}{2}} = \sin(\frac{\pi}{2}) - \sin(0) = 1$$

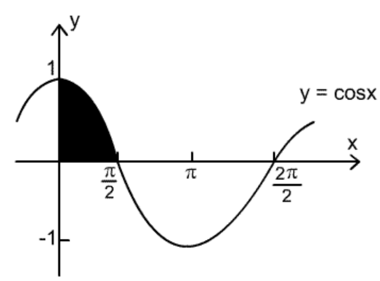


Figure shows that cosx ≥ 0 for 0 ≤ x ≤ π/2

I deliberately chose C = 0 since we just saw that the value of C=0 doesn't change the answer.

Example

Evaluate $\int_0^3 (x^3 - 4x + 1)dx$

$$\int_0^3 (x^3 - 4x + 1)dx = \left[\int (x^3 - 4x + 1)dx \right]_0^3$$

$$= \left[\int (x^3 dx - \int 4x.dx + \int 1.dx \right]_0^3$$

$$= \left[\frac{x^4}{4} - 4.\frac{x^2}{2} + x \right]_0^3 = (\frac{81}{4} - 18 + 3) - (0) = \frac{21}{4}$$

Example

Evaluate $\int_0^6 f(x)dx$ if $f(x) = \begin{cases} x^2 & x < 2 \\ 3x - 2 & x \geq 2 \end{cases}$

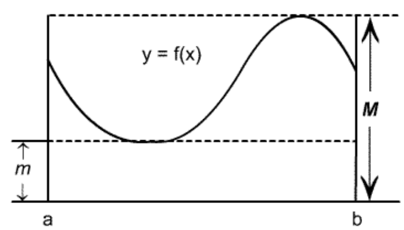
From theorem 5.6.6

$$\int_0^6 f(x)dx \text{ if } f(x) = \int_0^2 f(x)dx + \int_2^6 f(x)dx$$

$$= \int_0^2 x^2 dx + \int_2^6 (3x - 2)dx$$

$$= \left. \frac{x^3}{3} \right]_0^2 + \left[\frac{3x^2}{2} - 2x \right]_2^6 = \left(\frac{8}{3} - 0 \right) + (42 - 2) = \frac{128}{3}$$

Mean Values theorem for Integrals



Consider the picture.
This figure shows a continuous function *f* on [a,b].
Let M be the maximum values of *f* on [a,b], and m the minimum.
Then if we draw a rectangle of height M and one of height m, then its clear from the picture that the area under the curve of *f* is at least as large as the area of rectangle with height m AND no larger than the area of the rectangle with height M.
So we would like to say that there is some rectangle of a certain height for which the area under *f* = area of rectangle.

THEOREM 5.7.2(The Mean-Value Theorem for Integrals)

If f is continuous on a closed interval [a, b], then there is at least one number x^* in [a, b]such that

$$\int_a^b f(x)dx = f(x^*)(b - a)$$

Proof

By the extreme value theorem (theorem 4.6.4) *f* assumes a max M and a min m on [a,b]. So for all x in [a,b]

$$m \le f(x) \le M$$

$$\Rightarrow \int_a^b m\,dx \le \int_a^b f(x)dx \le \int_a^b M\,dx \qquad \text{Theorem 5.6.7(b)}$$

$$\Rightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$\Rightarrow m \le \frac{1}{b-a}\int_a^b f(x)dx \le M \qquad \text{Evaluating the end integral}$$

The last inequality states that $\frac{1}{b-a}\int_a^b f(x)dx$ is a number btw m and M on [a,b]

Since *f* is continuous and takes on all values in [a,b], we can say using the Intermediate Values theorem (theorem 2.7.9) that *f* must assume $\frac{1}{b-a}\int_a^b f(x)dx$ on [a,b] for some point x*

So we have

$$\frac{1}{b-a}\int_a^b f(x)dx = f(x^*)$$

$$\Rightarrow \int_a^b f(x)dx = f(x^*)(b-a)$$

Example

$f(x) = x^2$ is continuous on [1, 4], the MVT for Integrals guarantees that there exists a number x^* in [1, 4] such that

$$\int_1^4 x^2 dx = f(x^*)(4-1) = 3(x^*)^2$$

But

$$\int_1^4 x^2 dx = \left.\frac{x^3}{4}\right]_1^4 = 21$$

So

$$3(x^*)^2 = 21 \Rightarrow x^* = \sqrt{7}$$

Average Values of of a function

Definition 5.7.3

If f is integrable on [a, b] then the average value (or mean value) of f on [a, b] is defined to be

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

If $y = f(x)$, then f_{avg} is also called the average value of y with respect to x over [a, b].

Lecture # 31

Evaluating Definite Integral by Substitution.

Evaluating Definite Integrals.

- By substitution
- Approximation by Riemann Sums

Evaluating definite integral by substitution

We want to evaluate the following definite integral by substitution

$$\int_a^b f(x)dx$$

Method 1

Evaluate the indefinite integral $\int f(x)dx$ by substitution , and then use the relationship.

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$$

Method 2

First represent the definite integral in the form $\int_a^b f(x)dx = \int_a^b h(g(x))g'(x)dx$

And then make the substitution $u = g(x), \;\; du = g'(x)dx$ directly into the definite integral. But now we have to change integration limits from x-limits to u-limits as follows

$$u = g(a), \;\; \text{if } x = a$$
$$u = g(b), \;\; \text{if } x = b$$

This leaves us with a new integral in terms of u

$$\int_a^b f(x)dx = \int_{g(a)}^{g(b)} h(u)du$$

This will be simpler if the choice of u is good.

Example

Evaluate $\int_0^2 x(x^2+1)^3 dx$

Solution:

Method 1

$u = x^2 + 1$ so $du = 2xdx$

$$\int x(x^2+1)^3 dx = \frac{1}{2}\int u^3 du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C$$

So

$$\int_0^2 x(x^2+1)^3 dx = \left[\int x(x^2+1)^3 dx\right]_0^2 = \left.\frac{(x^2+1)^4}{8}\right]_0^2 = 78$$

Method 2

$u = 1$ if $x = 0$

$u = 5$ if $x = 2$

Thus

$$\int_0^2 x(x^2+1)^3 dx = \frac{1}{2}\int_1^5 u^3 du = \left.\frac{u^4}{8}\right]_1^5 = 78$$

Example

$$\int_0^{\frac{\pi}{4}} \cos(\pi - x)dx$$

Solution:

Let $u = \pi - x$ so that $du = -dx$

Also $u = \pi$ if $x = 0$, $u = \frac{3\pi}{4}$ if $x = \frac{\pi}{4}$

So,

$$\int_0^{\frac{\pi}{4}} \cos(\pi - x)dx = -\int_{\pi}^{\frac{3\pi}{4}} \cos(u)(du) = -\sin(u)\Big]_{\pi}^{\frac{3\pi}{4}} = -[\sin(\frac{3\pi}{4}) - \sin(\pi)]$$

$$= -[\frac{1}{\sqrt{2}} - 0] = -\frac{1}{\sqrt{2}}$$

Approximation by Riemann Sums

Recall that a Riemann Sum is the expression $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$

which occurs when we try to approximate the area under the curve.

If we take the limits of this sum, we get the definite integral.

If we don't take the limits, then we can get a good approximation to the definite integral if n is relatively LARGE.

So we can say that for large n $\int_a^b f(x)dx \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k$

We need to use this approx when it's impossible to evaluate exactly.

It is a good idea in this case to use a regular partition of the interval [a,b] which gives same width for each subinterval.

So we then write

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \Delta x_k \left[f(x_1^*) + f(x_2^*) + \ldots f(x_n^*) \right]$$

This formula produces a left endpoint approximation, a right end point approximation and midpoint approximation , depending on the choice of the point x_k^*

Example

Approximate $\int_0^1 \sqrt{1-x^2}\,dx$ using the left endpoint, right end point, and the

midpoint approx each with n=10, n=20, n=50 and n=100 subintervals.

Solution:

For n =10 we can do this easily, but for others we should use a computer. We will do for n=10 using the ideas from previous lectures. Here is a table of results

n	Left endpoint approximation	Right endpoint approximation	Midpoint approximation
10	0.826129582	0.726129582	0.788102858
20	0.807116220	0.757116220	0.786357647
50	0.794567128	0.774567128	0.785641388
100	0.790104258	0.780104258	0.785484214

The exact value of the integral is $\frac{\pi}{4}$. This is consistent with the preceding computations , since

$\frac{\pi}{4} \approx 0.785398163.$

Lecture # 32
Second Fundamental Theorem of Calculus

Second Fundamental Theorem of Calculus

- Dummy Variable
- Definite Integrals with variable upper limit
- Second fundamental theorem of Calculus
- Existence of Ant derivatives for continuous functions
- Functions defined by Integrals

Dummy Variable

Here is some notational stuff.

If we change the letter for the variable of integration but don't change the limits, then the values of the definite integral are unchanged.

That is to say

$$\int_a^b f(x)dx, \quad \int_a^b f(t)dt, \quad \int_a^b f(y)dy$$

All have the same value

For this reason, we call the letter used for the variable of integration DUMMY variable

Example

$$\int_1^2 x^2 dx = \left.\frac{x^3}{3}\right]_1^2 = \frac{26}{3}$$

$$\int_1^2 f(t)dt = \left.\frac{t^3}{3}\right]_1^2 = \frac{26}{3}$$

$$\int_1^2 f(y)dy = \left.\frac{y^3}{3}\right]_1^2 = \frac{26}{3}$$

Definite Integrals with variable upper limit of integration

We will now consider definite integrals of the form $\int_a^x --$ where the upper limit is a variable rather than a number.

In such integrals, we will use a different letter for the integration variable

This will distinguish between the limit of integration and the variable of integration.

Let's do an example

Example

Evaluate

$$\int_2^x t^2 dt$$

Solution:

$$\int_2^x t^2 dt = \frac{t^3}{3}\Big]_2^x = \frac{x^3}{3} - \frac{8}{3}$$

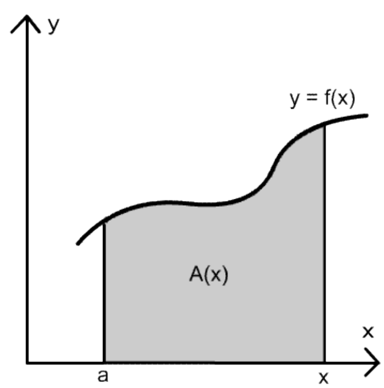
Note that the result in this case is a FUNCTION of *x*.

In all such integrals with an *x* as a limit, the final result is a function of *x*.

This is like when we said that instead of evaluating the derivative at a number, do it at a point *x* to get the derivative function.

Second fundamental theorem of Calculus

In lecture 25, we say that if *f* is a nonnegative continuous function over [a, b], and A (x) represents the area under the curve y = f (x) over the interval [a ,x] as a function of *x*, then $A'(x) = f(x)$



Now Let's write A (x) as a definite integral since that's how it is represented

$$A(x) = \int_a^x f(t)dt$$

If we now take the derivative w.r.t to *x* , we get

$$\frac{d}{dx}[A(x)] = A'(x) = \frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

This result holds for all continuous functions, and we can state it as a theorem

Theorem 5.9.1 (2nd fundamental theorem of Calculus)

Read as

1) If the integrand is continuous, then the derivative of a definite integral w.r.t its upper limit is equal to the integrand evaluated at the upper limit.
2) The derivative of a function representing the area under the curve of another continuous function f is equal to the function f on a given interval.

We usually write the theorem as

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Example

Since $f(x) = x^3$ is continuous everywhere, evaluate the integral $\int_1^x t^3 dt$ to check the validity of 2nd fundamental theorem of calculus.

Solution:

By 2nd fundamental theorem of calculus:

$$\frac{d}{dx}\left[\int_1^x t^3 dt\right] = x^3$$

We check it by evaluating the integral

$$\int_1^x t^3 dt = \left.\frac{t^4}{4}\right]_1^x = \frac{x^4}{4} - \frac{1}{4}$$

Differentiating this gives x^3.

Existence of Anti-derivatives of continuous functions:

If 'f is a continuous function on an interval 'I' and 'a' is any point in 'I' then this formula

$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ tells us that $F(x) = \int_a^x f(t)dt$ is an anti-derivative of f on I. So now with the 2nd theorem of calculus, we can say that EVERY continuous function on an interval has an anti-derivative on that interval.

Functions defined by Integrals

So far we have been able to determine the anti-derivative of a given function by integration techniques.

But its not always possible to find the anti-derivative of a given function f which is continuous on an interval and thus is guaranteed to have an anti-derivative on that interval.
If this is the case, we get functions (anti-derivatives) that are defined in terms of integrals and nothing simpler that we have seen so far.

Example

$$\int_1^3 \frac{1}{x}dx$$

Solution:
This function is continuous on the interval [1, 3] so it is integrable on [1, 3]. By the 2nd theorem of calculus, an antiderivative is

$$F(x) = \int_1^x \frac{1}{t}dt$$

Using the 1st theorem of calculus we can attempt to evaluate this integral

$$\int_1^x \frac{1}{t}dt = F(3) - F(1) = \int_1^3 \frac{1}{t}dt - \int_1^1 \frac{1}{t}dt = \int_1^3 \frac{1}{t}dt$$

But this is not a function like any we have seen so far.
In fact, 1/t cannot be integrated using polynomials, rational functions, or any other function we have seen so far.
But we can approximate this function by using numerical methods like the Riemann sums. Now this formula $\int_1^x \frac{1}{t}dt$ defines a FUNCTION in terms of integrals.

Lecture # 33

Application to the Definite Integral

Application of the Definite Integral

- Area problem: Area Between two Curves
- Area between y = f(x) and y = g(x)
- Area between x = v(y) and x = w(y)

First Area problem

Area between two curves:

Area between y=f(x) and y=g(x)

Suppose that *f* and *g* are continuous functions on an interval [a, b] and

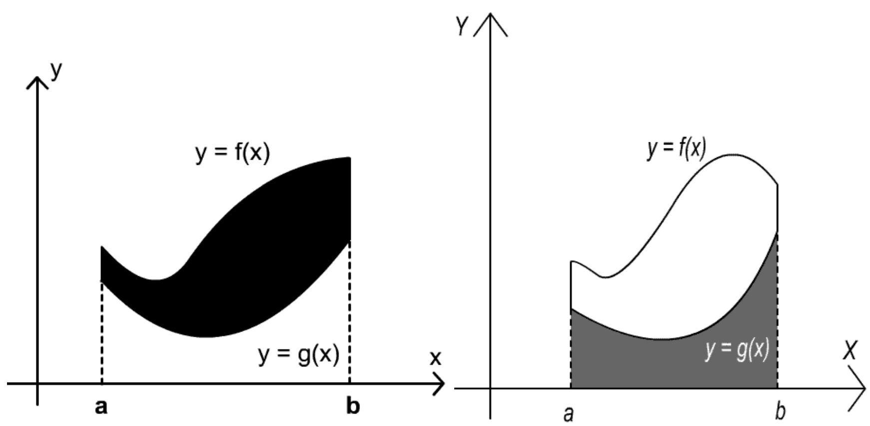
$f(x) \geq g(x)$ for $a \leq x \leq b$

(This means that the curve y = f (x) is above y = g (x) and that the two can touch but not cross).

Find the area A of the region bounded above by y= f (x), below by y = g(x), and on the sides by the lines x = a and x = b

If f and g are nonnegative on [a,b] then we have

A = [area under f] – [area under g]



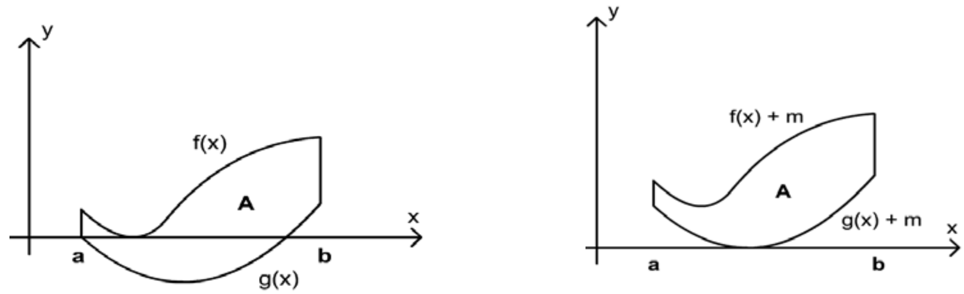
In terms of integrals we can say this as $A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$

What if f and g take on negative values too?

This means that their graphs are below the x-axis for some values in [a,b].

This can be remedied if we translate the two graphs by a constant so big that it shifts both f and g above the x-axis.

This shift does not affect the area between the two curves



$$A = \int_a^b [f(x) + m]dx - \int_a^b [g(x) + m]dx = \int_a^b [f(x) - g(x)]dx$$

The constants cancel each other in the calculations.

Here is the formal definition

DEFINITION 6.1. (Area Formula)

If f and g are continuous functions on the interval [a, b], and $f(x) \geq g(x)$ for all x in [a, b], then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line x=a, and on the right of the line x=b is

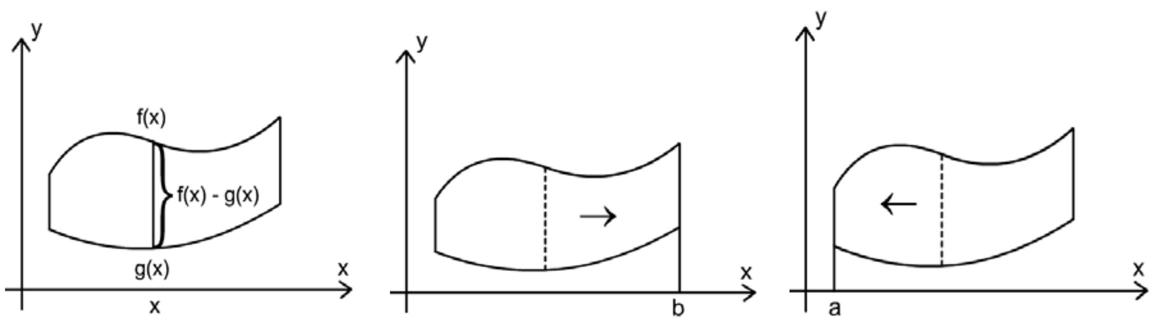
$$A = \int_a^b [f(x) - g(x)]dx$$

A few things to keep in mind.

If the region confined by two curves is complicated, then it may require some careful thought to determine the integrand.

It may be hard to find the limits of integration.

Here is a procedure we can follow



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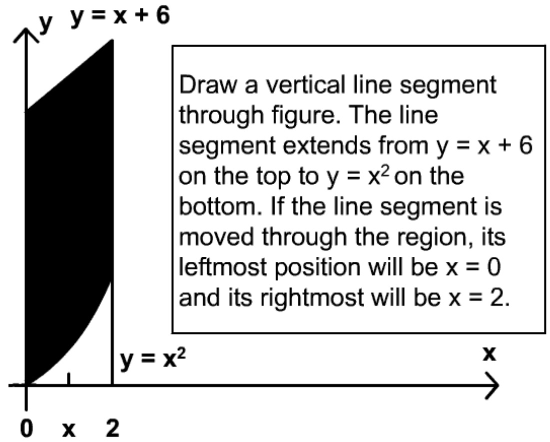
Example

Find the area of the region bounded above by y = x + 6, and below by

$y = x^2$ and on the sides by lines x = 0 and x = 2.

Solution:

The region and a vertical line through it are shown in this figure.



The line extends from f(x) = x + 6 on the top to $y = x^2$ at the bottom.

As the line moves left to right, its leftmost position will be x = 0 and right most will be x = 2

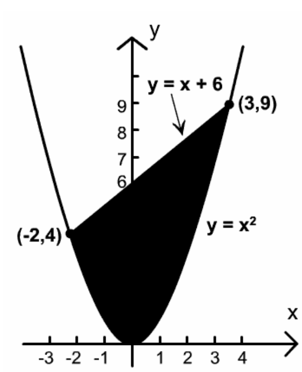
So we have

$$A = \int_0^2 [(x+6) - x^2]dx = \left.\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3}$$

Example

Find the area of the region that is enclosed between $y = x^2$ and y = x + 6.

solution:



There are no lines at the right or the left. So how do we find the limits of integration? Well here is a sketch of the region.

Note that the integration limits are defined by the points where the two curves intersect.

We can find these points by equating the two equations and solving for x

We get

$$x^2 = x + 6$$
$$\Rightarrow \; x^2 - x - 6 = 0$$
$$\Rightarrow \; (x+2)(x-3) = 0$$
$$\Rightarrow \; x = -2 \quad \text{and} \quad x = 3$$

So now we have

$$A = \int_{-2}^{3} [(x+6) - x^2]dx = \frac{x^2}{2} + 6x - \frac{x^3}{3}\Bigg]_{-2}^{3} = \frac{125}{6}$$

Example

Find the area of the region enclosed by $x = y^2$ and y = x –2

Solution:

First we need to know where the two curves intersect.

Let's write down y = x – 2 as x = y + 2 and then equate

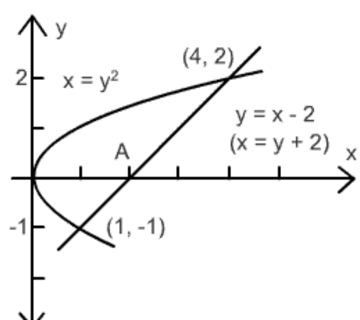
$$x = y^2 \text{ and } x = y + 2$$

This gives

$$y^2 = y + 2 \;\; \Rightarrow \; y^2 - y - 2 = 0 \; \Rightarrow \; y = -1 \text{ and } y = 2$$

Substituting these into the original equations will give us the desired x-values, which are x = 1 and x = 4 respectively

Here is the graph of the situation



(a)

From this figure, we see that the upper boundary or curve is defined by the equation y = +√(x).

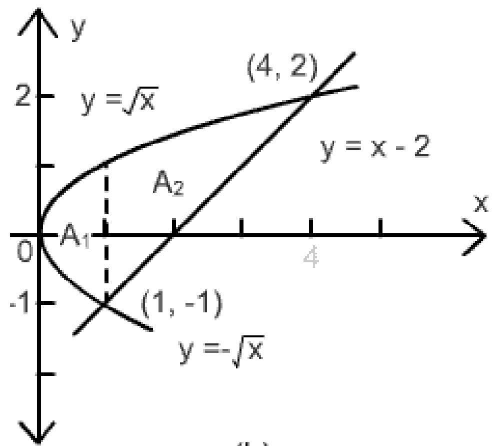
The lower boundary has two pieces

1) y= -√(x) for x between –1 and 0
2) y= x-2 for x between1 and 4

This causes us problems when we try to determine f(x) – g(x) for the integration.

Instead we will divide the region into two parts and evaluate the area of each region and then add together.

Here is how we will divide it



(b)

f(x) = +√(x), g(x) = -√(x), a = 0, b = 1

$$A_1 = \int_0^1 \left[\sqrt{x} - (-\sqrt{x})\right]dx = 2\int_0^1 \sqrt{x}dx$$

$$= 2\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{4}{3}$$

f(x) = +√ (x), g(x) = x – 2, a = 1, b = 4

$$A_2 = \int_1^4 \left[\sqrt{x} - (x-2)\right]dx = \int_1^4 (\sqrt{x} - x + 2)dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right]_1^4 = \frac{19}{6}$$

Total area or region is 19/6 + 4/3 = 9/2

Area between x = v(y) and x = w(y)

We can avoid splitting the region if we integrate w.r.t y instead of x.

Second Area problem

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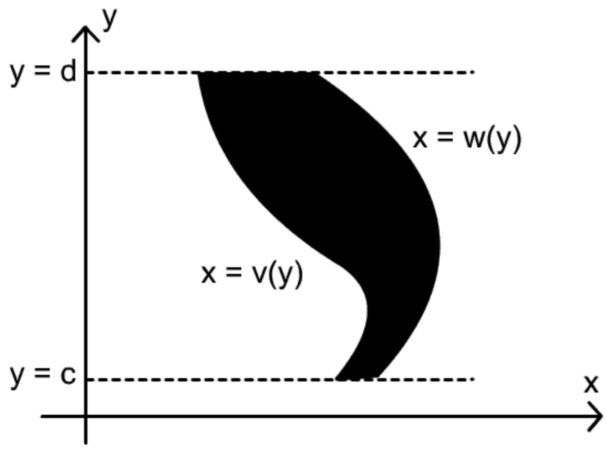


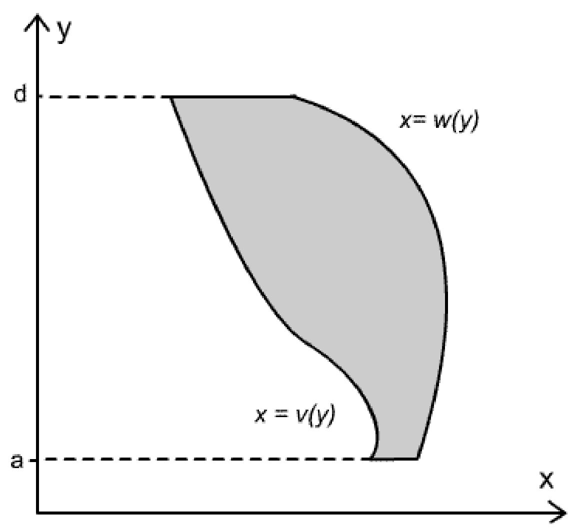
Just as we derived an answer earlier in the lecture for the first area problem for regions confined by two curves, we get the following definition

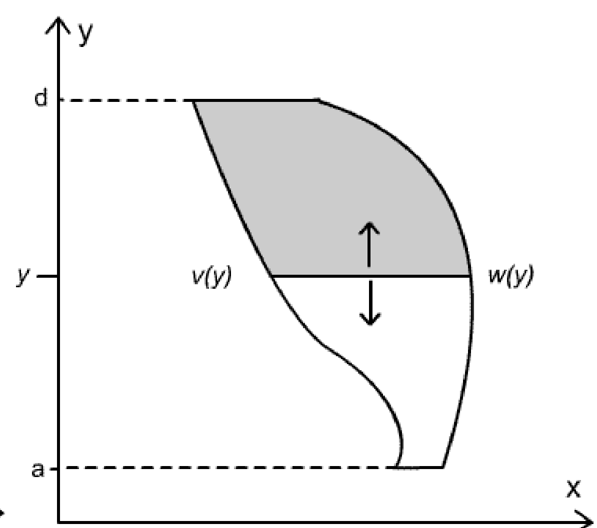
DEFINITION 6.1.2 (Area Formula)

If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in [c, d], then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by y = c, and above by y = d is

$$A = \int_c^d [w(x) - v(x)]dy$$

The procedure for finding the integrand and limits of integration is the same as earlier. Here are two figures to help you see this





Example

Find the area of the region confined by $x = y^2$ and y=x-2 or x=y+2

Solution:

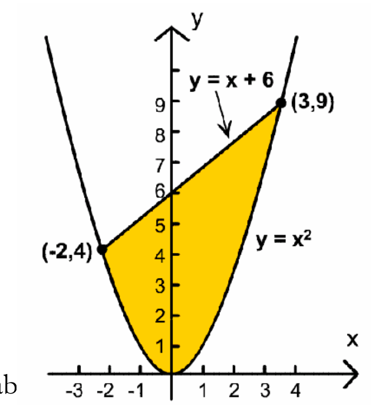
Left boundary is $x = y^2$

Right boundary is x = y + 2

Limits are from –1 to 2

$$A = \int_{-1}^{2} [(y+2) - y^2] dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^{3} = \frac{9}{2}$$

The area is shown in the figure ab



Lecture # 34

Volume by Slicing; Disks and Washers

- Definite Integrals to find Volumes of three dimensional solids
- Cylinders
- The method of Slicing
- Volumes by cross sections perpendicular to the x axis
- Volumes by cross sections perpendicular to the y axis
- Volumes of solids of revolution by:
    i. Volumes by Disks perpendicular to x axis
    ii. Volumes by Disks perpendicular to y axis
    iii. Volumes by washers perpendicular to x axis
    iv. Volumes by washers perpendicular to y axis

Cylinders

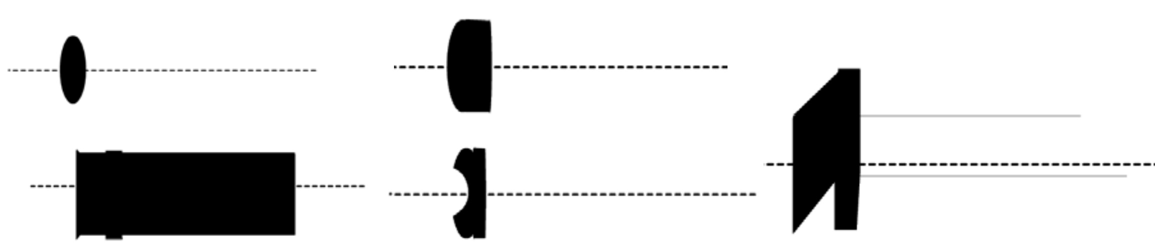
How to go from 0-d to 3-d and beyond in terms of a cube.

We can get other 3d solids by moving 2d planes in a certain fashion in 3d space.

If I move a 2d circle along a line that is perpendicular to the circle, I will get a cylinder.

If I move a circle with a hole in it, or a washer, along a line perpendicular to the washer, then I will get a cylinder with a hole in it.

In general, if I move a 2d plane in a direction along a line perpendicular to the region, then I get a RIGHT CYLINDER.

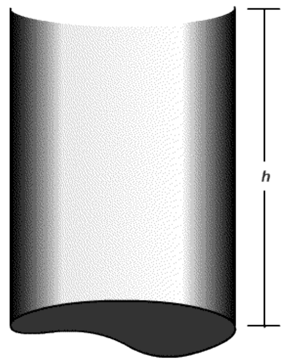


Every solid object has Volume, Volume is the 3d area!

We can find the volume of a right cylinder generated by moving a 2d plane by the following formula

$$V = A.h$$

Where A is the area of the 2d object, and h is the height of the cylinder



Volume = A. *h*

So the Volume of a cylinder is the area of a cross section of the cylinder multiplied by the height of the cylinder.

But there are 3d solids that are not right cylinders and are neither made up of finitely many right cylinders.

So we cannot use the formulas for the area of the right cylinders to find the area of such an object.

In this case we use the technique of slicing.

The method of Slicing

Here is such a 3d solid

This is not made up of finitely many right cylinders.

We find the volume of this object by slicing.

Impose an x axis on the solid.

Then we can imagine that the solid is bounded to the left by a plane perpendicular to the x- axis at x = a and to the right by a plane at x = b

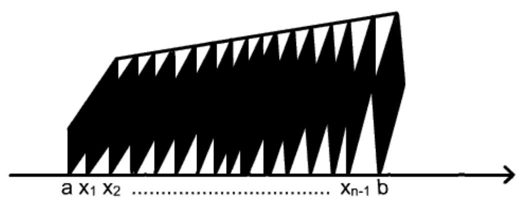
The problem with finding the volume of this non-cylindrical solid is clear now: at any point along the x-axis, the cross sections perpendicular to the x-axis will have different areas!

Let's call the area of a cross section at some arbitrary point x A (x)

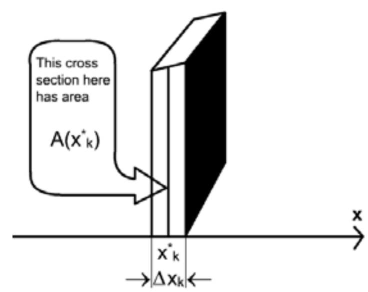
Now Let's divide the interval [a,b] into *n* subintervals of width $\Delta x_1, \Delta x_2, ..., \Delta x_n$ By inserting the points $x_1, x_2, ..., x_n$ between a and b.

We can now pass a plane perpendicular to the x-axis through each of these points.

These planes will subdivide the solid into SLICES $S_1, S_2, ..., S_n$.



Consider a typical slice S_k as shown in the figure



If this slice is very thin, then the cross section of this slice S_k will not vary too much in the interval Δx_k which defines the thickness of the slice S_k . That is, the cross section will have the same shape in the interval if Δx_k is thin.

So let's choose a point x_k^* in the interval Δx_k and take a cross section of the slice at this point. Then the area of this cross section will be A(Δx_k)

The volume of the slice S_k then will be approximately

$$V_k \approx A(x_k^*)\Delta x_k$$

We can use this formula for all the THIN slices we make to get their respective volumes.

The total volume of the solid can be approximated by

$$V = V_1 + ... + V_n \approx \sum_{i=1}^{n} A(x_k^*)\Delta x_k$$

Now increase the number of slices. Then the slices will become VERY THIN and the approx will get better and better. So we can take the limit to get

$$\max \Delta x_k \to 0$$

$$V = \lim_{\max \Delta x_k \to 0} \sum_{i=1}^{n} A(x_k^{*})\Delta x_k = \int_a^b A(x)dx$$

Volumes by cross sections perpendicular to the x axis

VOLUME FORMULA

Let S be a solid bounded by two parallel planes perpendicular to the x-axis at x = a and x = b. If , for each x in [a,b], the cross-sectional area of S perpendicular to the x-axis is A(x), then the volume of the solid is

$$V = \int_a^b A(x)dx$$

Volumes by cross sections perpendicular to the y axis

VOLUME FORMULA

Let S be a solid bounded by two parallel planes perpendicular to the y-axis at y = c and y = d. If , for each y in [c,d], the cross-sectional area of S perpendicular to the y-axis is A(y), then the volume of the solid is

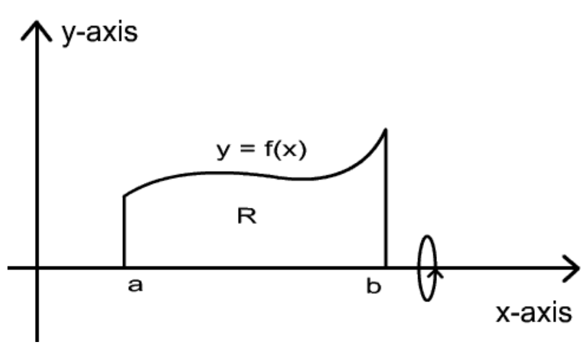
$$V = \int_c^d A(y)dy$$

Volumes of solids of revolution by:

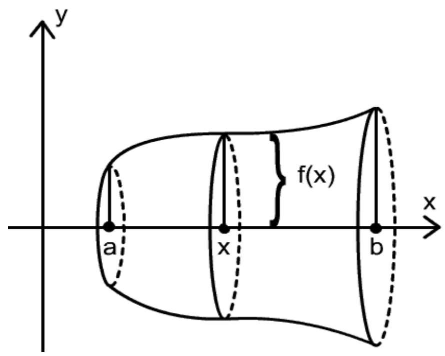
I. Disks perpendicular to x-axis

Let *f* be a nonnegative and continuous function on [a,b]

Let R be the region bounded above by the graph of *f* and on the sides by lines x = a and x = b



If this solid is revolved around the x-axis, it generates a solid with a circular cross sections.



The cross section at x has radius f(x) so the area of this cross section is

$$A(x) = \pi\left[f(x)\right]^2$$

So from the formulas we just saw we can write for the volume of this solid:

$$V = \int_a^b \pi\left[f(x)\right]^2 dx$$

Example 2

Find the volume of solid that is obtained when the region under the curve $y = \sqrt{x}$ over the [1,4] is revolved about the x-axis .

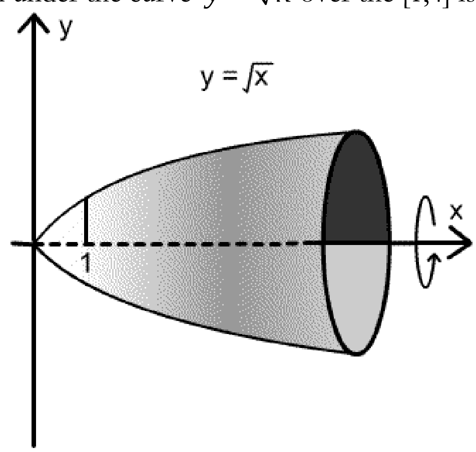
Solution:

From $V = \int_a^b \pi\left[f\left(x\right)\right]^2 dx$,the volume is

$$V = \int_a^b \pi\left[f\left(x\right)\right]^2 dx = \int_1^4 \pi x dx$$

on evaluating

$$V = \frac{15\pi}{2}$$



Example 3

Derive the formula for the volume of a sphere of radius r.

Solution:

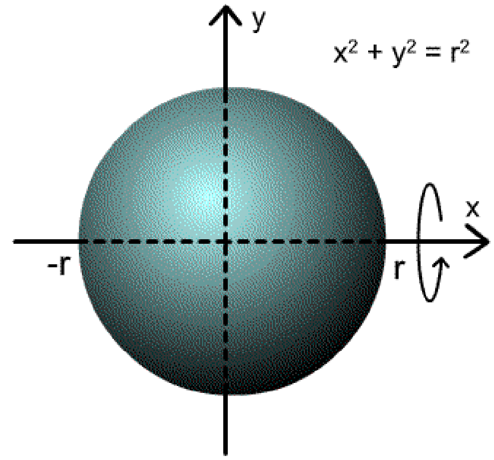
As indicated in the figure , a sphere of radius r can be generated by revolving the upper half of the circle

$$x^2 + y^2 = r^2$$

About the x-axis. Since upper half of this circle is graph of

$$y = f(x) = \sqrt{r^2 - x^2}$$

it follows from above formula that the volume of the sphere is



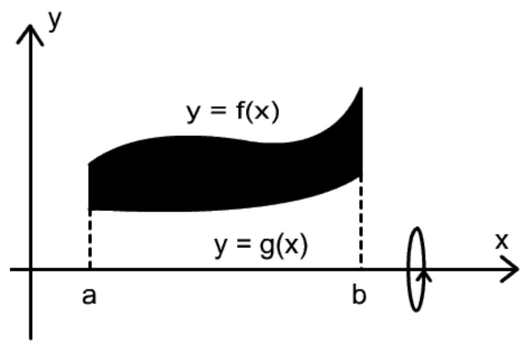
$$V = \int_a^b \left[f(x) \right]^2 dx = \int_{-r}^{r} \pi \left(r^2 - x^2 \right) dx$$

On evaluating

$$V = \frac{4}{3}\pi r^3$$

Volumes by washers perpendicular to x axis

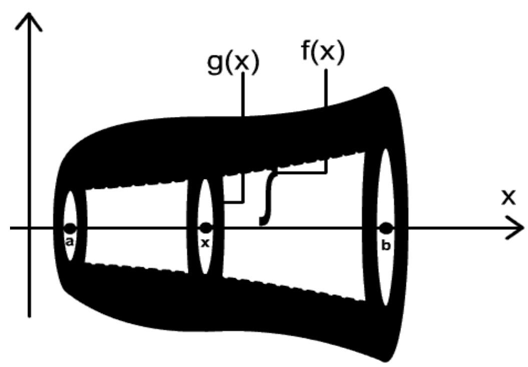
Now consider this picture



Here R is a region confined by the graph of two non-negative continuous functions *f* and *g* with f greater than or equal to g on the interval [a,b]

When this region is revolved around the x-axis, we get a solid with a hole in the center

This solid has a cross section that looks like a washer.



This washer cross section has two radii defined by the function f and g as is in the picture.

So the area of this cross section will be

$$A(x) = \pi \left[f(x) \right]^2 - \pi \left[g(x) \right]^2 = \pi \left(\left[f(x) \right]^2 - \left[g(x) \right]^2 \right)$$

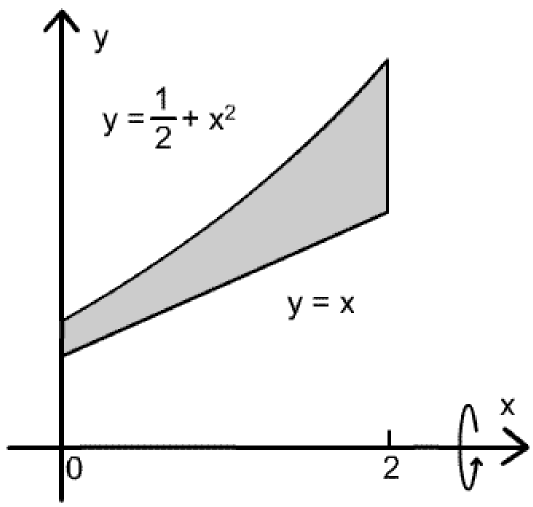
And the volume of the solid now can be expressed

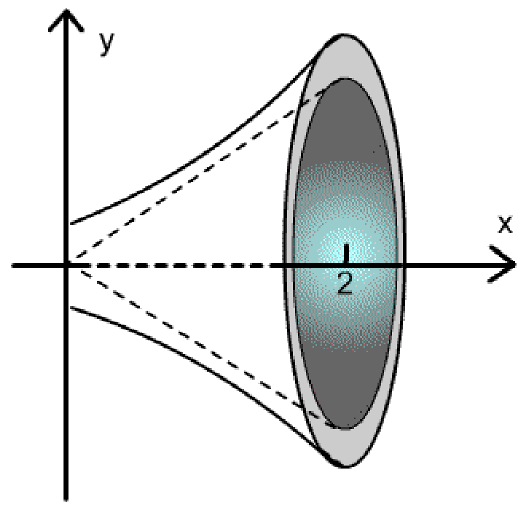
$$V = \int_a^b \pi \left(\left[f(x) \right]^2 - \left[g(x) \right]^2 \right)$$

Example 4

Find the volume of the solid generated when the region between the graph of

$f(x) = \frac{1}{2} + x^2$ *and* $g(x) = x$ over the interval [0,2] Is revolved about the x-axis .





Solution;

From

$$V = \int_a^b \pi\{\left[f(x) \right]^2 - \left[g(x) \right]^2\}dx$$

the volume is

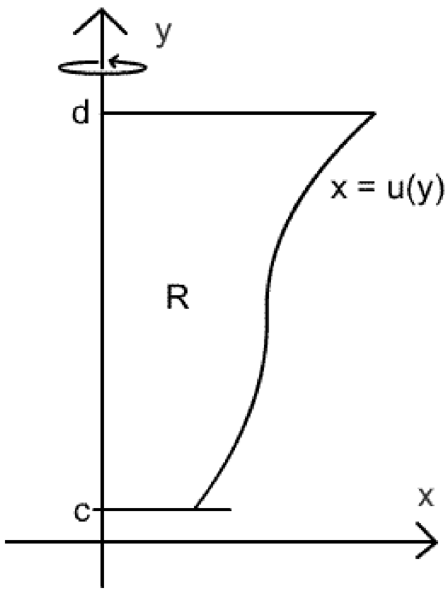
$$V = \int_a^b \pi\{\left[f(x) \right]^2 - \left[g(x) \right]^2\}dx = \int_0^2 \pi \left(\left[\frac{1}{2} + x^2 \right] - x^2 \right) dx$$

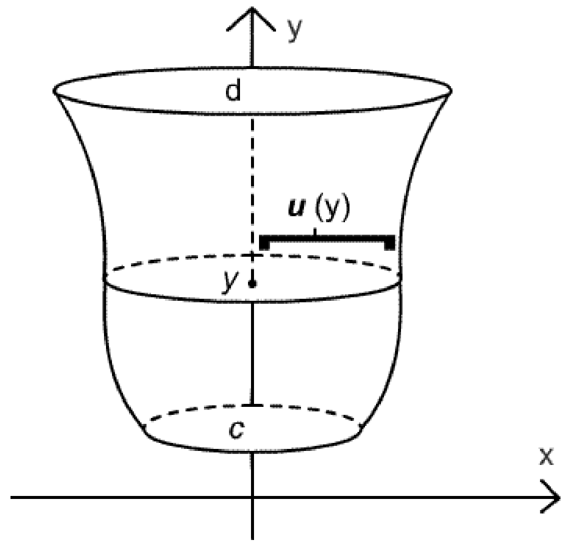
$$= \int_0^2 \pi \left(\frac{1}{4} + x^4 \right) dx = \frac{69\pi}{10} \qquad (on\ evaluating)$$

Volume by Disks perpendicular to y axis

Just as we found the volumes of solids generated by revolving a region around the x-axis, we can find the volume of solids generated by revolving a region around the y-axis in which the cross section is a disk.

Here is how in a picture



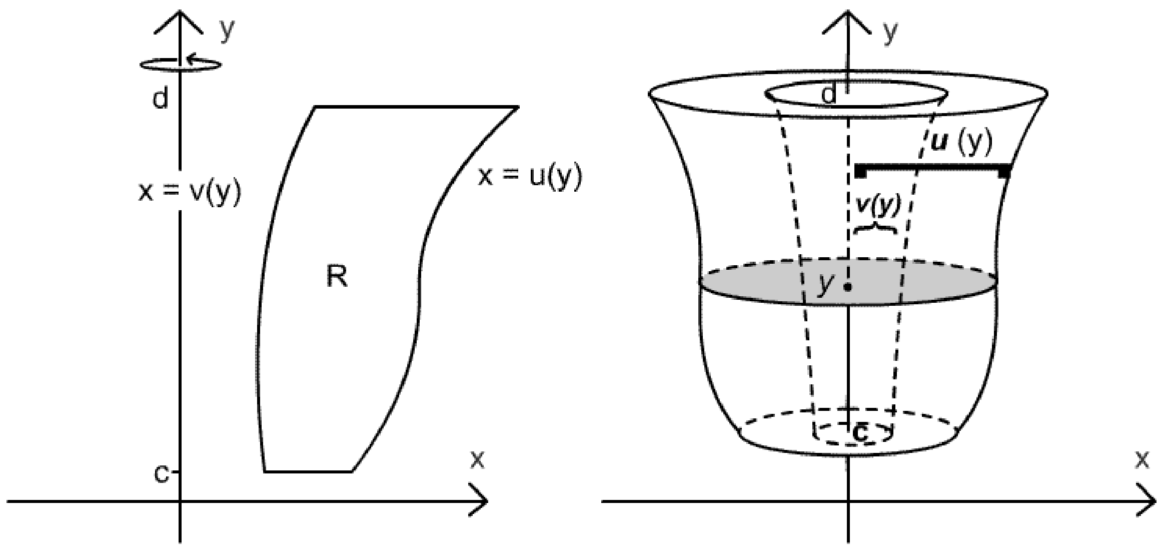


The formula in this case for volume is the following

$$V = \int_c^d \pi \, [u(y)]^2 dy$$

Volume by Washers perpendicular to y axis

Just as we found the volumes of solids generated by revolving a region around the x-axis, we can find the volume of solids generated by revolving a region around the y-axis in which the cross section is a washer



The formula expressing it is $V = \int_c^d \pi\ (\ [u(y)]^2 - [v(y)]^2) dy$

Lecture # 35

Volume by Cylindrical Shells

- Cylindrical Shells
- Volume of a Cylindrical Shell
- Cylindrical Shells centered on the y-axis
- Volume through the surface area of a surface created from a cylindrical shell

Cylindrical Shells

If we take a washer – a disk with a hole in it – and extend it UP, we generate a solid called a cylindrical shell.

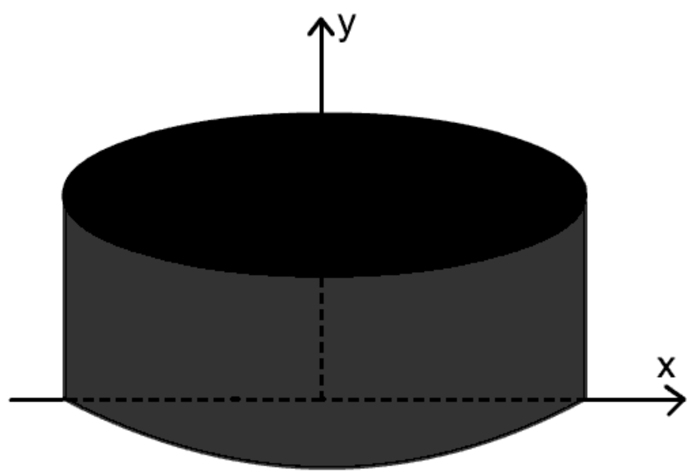
This is a solid confined by two concentric right circular cylinders

The Volume of a cylindrical shell can be expressed as

$$V = \text{(area of cross section).(hieght)} = (\pi\, r_2^{\,2} - \pi\, r_1^{\,2})h$$

$$= \pi\,(r_2 + r_1)(r_2 - r_1)h$$

$$= 2\pi\left[\frac{1}{2}(r_2 + r_1)\right].h.(r_2 - r_1)$$



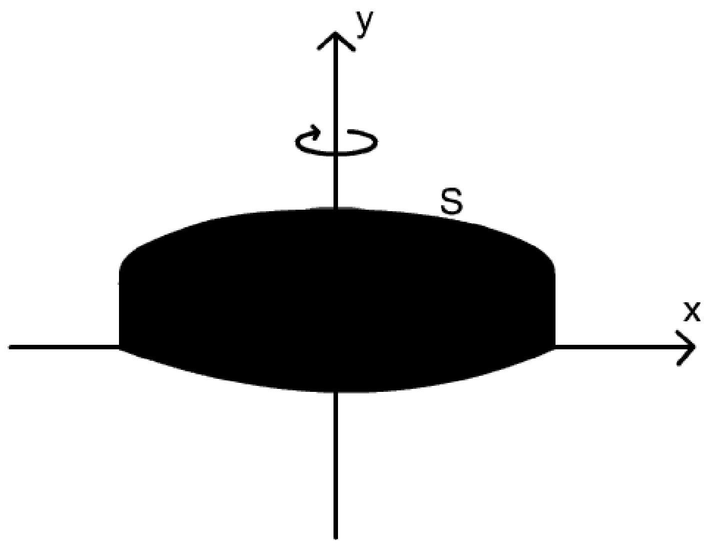
Let's rewrite this rearrangement of the Volume of a cylindrical shell as

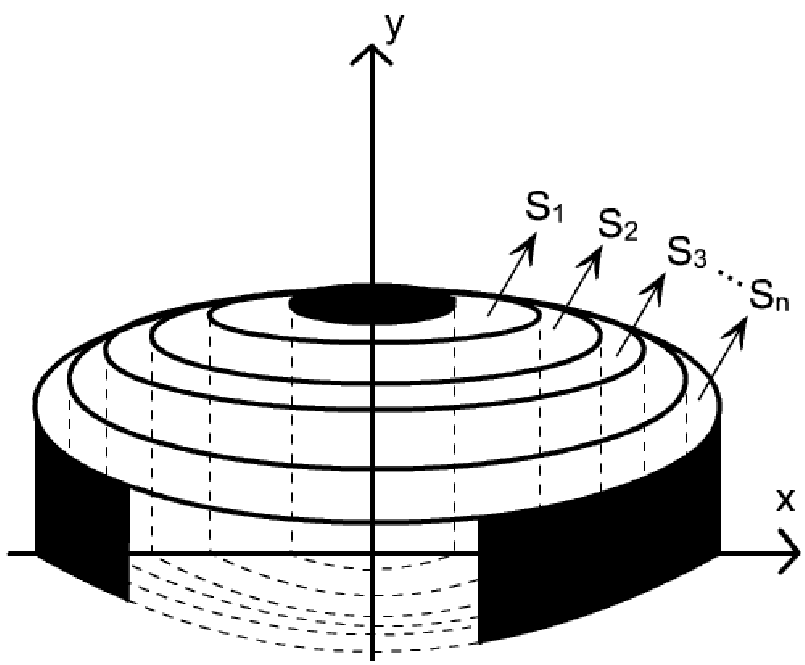
V = 2π (average radius) .(height) .(thickness)

Now we can use this formula to compute the Volume of a Solid generated by revolution of a surface around an axis.

Consider the following

R is a region bounded by the graph of f (x) on top, below by the x-axis, and to the left by x = a, and right by x = b





When we revolve R around the x-axis , we get the solid S as in the picture.

To find the volume of S, here is what we do.

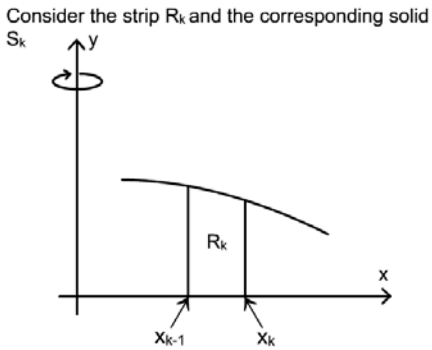
Subdivide [a,b] into *n* subintervals with widths $\Delta x_1 ..., \Delta x_n$ by inserting the points $x_1 ..., x_n$ between a and b.

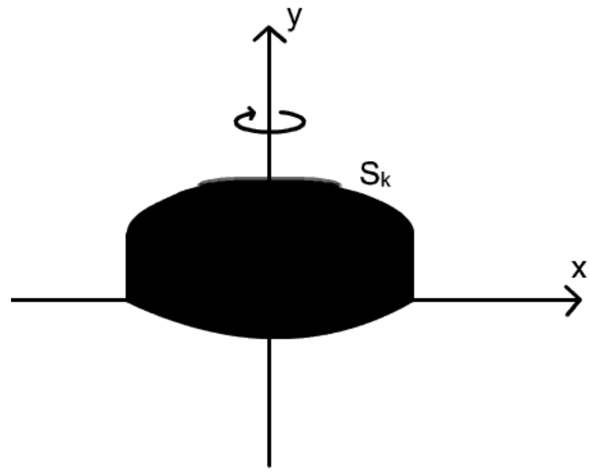
If we draw vertical lines from these points to the graph of *f*, we get subdivisions $R_1, ..., R_n$ of the region R.

Revolving these strips around the y-axis gives solids $S_1, ..., S_n$

So we can do the following to get the volume of S

$$V(S) = V(S_1)+...+V(S_n)$$





Consider the strip R_k and the corresponding solid S_k.

The solid S_k is not necessarily a cylindrical shell as it may have a curve upper surface depending on the graph of f.

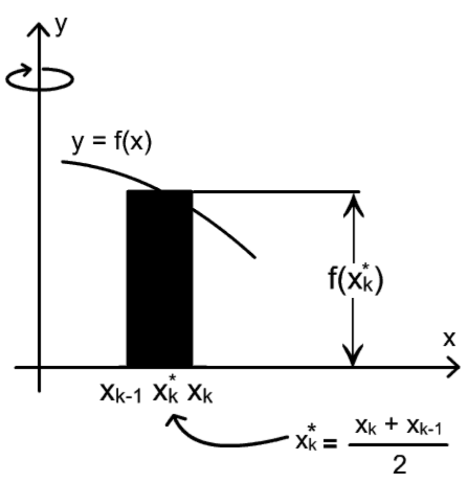
However, if the width of Δx_k is very small, then we can get a good approx to the region R_k by the rectangle of width Δx_k and height

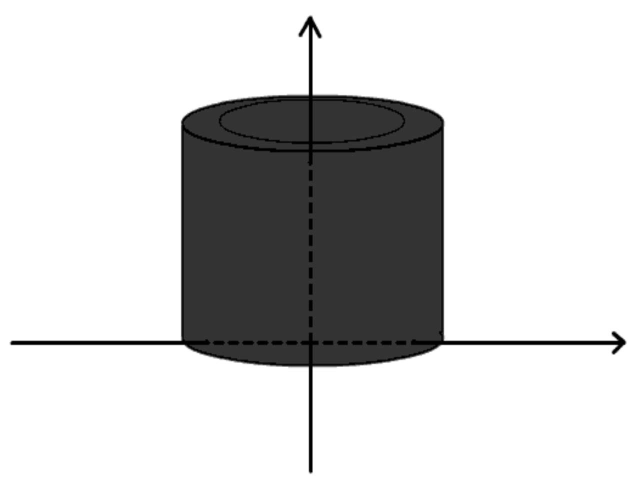
f(x_k^*) where

$$x_k^* = \frac{x_k + x_{k+1}}{2}$$

Is the midpoint of the interval $\left[x_{k-1}^*, x_k^*\right]$

This in turn will give us a cylindrical shell when we revolve it around y-axis. which will be a good approx to the solid.





The volume of S_k can now be approximated using the width height f(x_k^*), average radius x_k^* of the cylindrical shell as Δx_k

$$V(S_k) \approx \text{volume of approximating cylinderical shell } = 2\pi\ x_k^* f(x_k^*)\Delta x_k$$

The volume of the whole solid S will now be just the sum

$$V(S) = \sum_{k=1}^{n} 2\pi\ x_k^{\ *} f(x_k^{\ *})\Delta x_k$$

We can take the limit and get the exact Volume of S as

$$V(S) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi\ x_k^{\ *} f(x_k^{\ *})\Delta x_k = \int_a^b 2\pi\ xf(x)dx$$

In this process, we see that we have a formula to computed volume of a solid which is got from revolving a region around the Y-AXIS

Cylindrical Shells centered on the y-axis

VOLUME FORMULA

Let R be a plane region bounded above by a continuous curve y = f(x) below by x-axis, and on the left and right , respectively, by the lines x = a and x = b. Then the volume of the solid generated by revolving R about the y-axis is given by

$$V = \int_a^b 2\pi x f(x) dx$$

Example 1

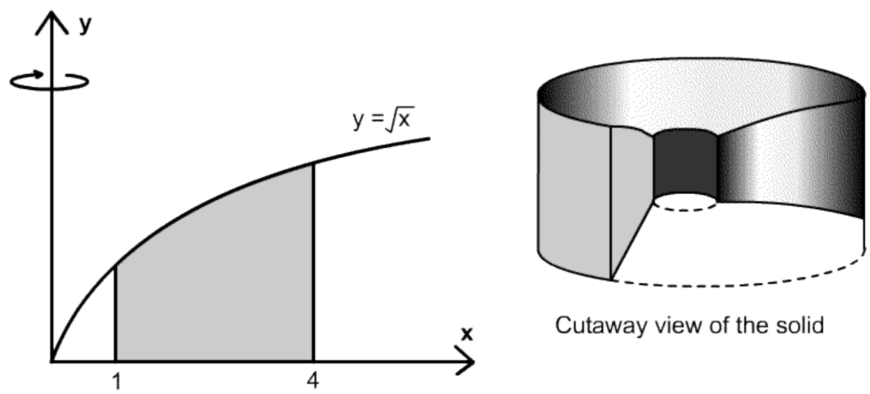
Use cylindrical shells to find the volume of the solid generated by the region enclosed between $y = \sqrt{x}, x = 1, x = 4$ and the x-axis is revolved about the y-axis.

Solution:

Since $f(x) = \sqrt{x}, a = 1, and\ b = 4$ and the above volume formula yields

$$V = \int_1^4 2\pi x\sqrt{x}dx = 2\pi\int_1^4 x^{\frac{3}{2}}dx$$

$$V = 2\pi\left[\frac{2}{5}x^{\frac{5}{2}}\right]_1^4 = \frac{124\pi}{5}$$



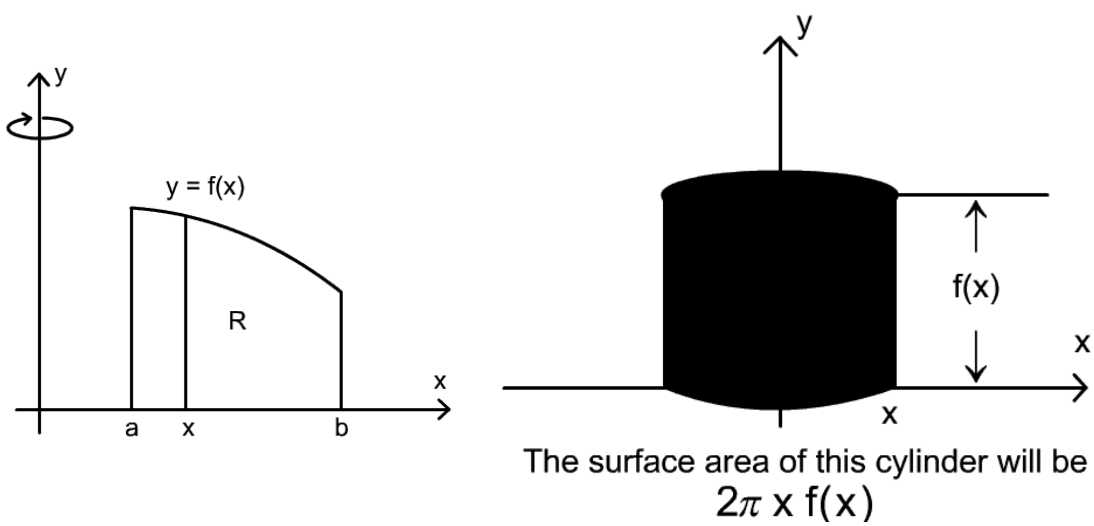
Volume through the surface area of a surface created from a cylindrical shell

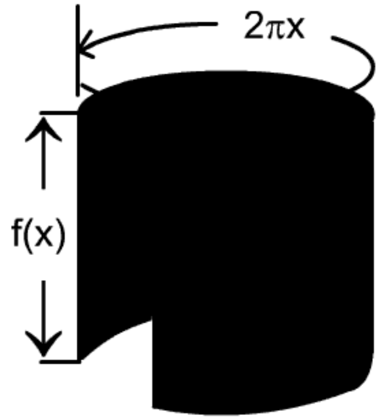
A bit more general and perhaps easier to conceptualize method can be worked out for cylindrical shells

Instead of thinking of the solid S_k as having a width Δx_k suppose that it has no width

This would mean that S_k, will be a straight forward right circular cylinder having radius x, height f (x).

The surface area of this cylinder will be $2\pi\, xf(x)$.





The surface area of this cylinder will be

$$2\pi \text{ x } f(x)$$

This is exactly the integrand in the Volume formula we saw earlier. So this really means that :

VOLUME **V** BY CYLIDRICAL SHELLS IS THE INTEGRAL OF THE SURFACE AREA GENERATED BY AN ARBITRARY SECTION OF THE REGION **R** TAKEN PARALLEL TO THE AXIS ABOUT WHICH R IS REVOLVED.

This view of volume by cylindrical shells helps us do calculation in more general setting where for example the lower boundary may not be interval

Example 2

Use cylindrical shells to find the volume of the solid when the region R in the first quadrant enclosed between $y = x, and\ y = x^2$

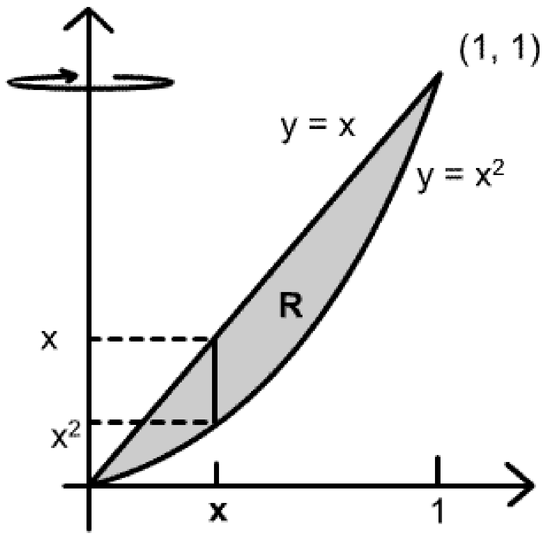
Is revolved about the y-axis

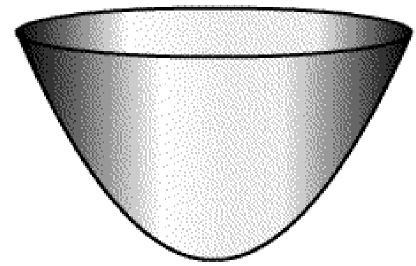
Solution:

At each x in [0,1] the cross section of R parallel to the y-axis generates a cylindrical surface of height $x - x^2$ and radius x. Since the area of the surface is $2\pi x(x - x^2)$ .Thus the volume of the solid is given by the following formula and the figure is also below

$$V = \int_0^1 2\pi x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx$$

$$V = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\pi}{6}$$


(1, 1)
y = x
y = x²
R
x
x²
x
1



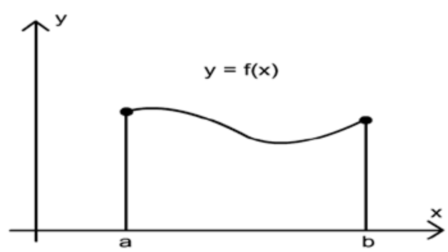
Lecture # 36
Length of a Plane Curve

Arc Length:

We all know how to find the length of a line. In this section, we will develop the ways of finding the length of curves, or lines that twist and turn. The curves we look at will be graphs of functions. The functions we look at will be such that will be continuous on a given interval. Such functions are called **smooth** functions, and their graphs **smooth curves**.
So in this lecture, we are concerned with the ARC LENGTH PROBLEM

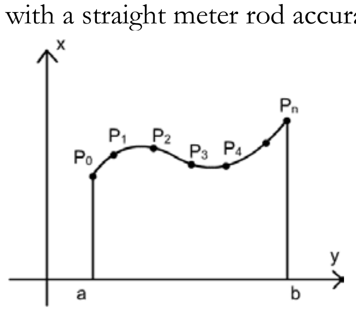
Arc Length Problem:

Suppose that f is a smooth function on the interval $[a,b]$. Find the arc length L of the curve $y = f(x)$ over the interval $[a,b]$.

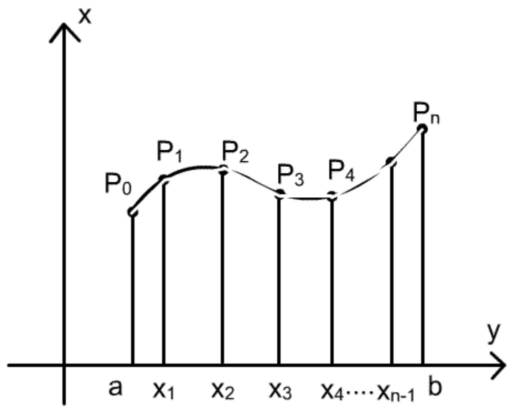


Here question arises that **How can we measure the arc length?**

Arc length is just the length of a line that is not straight, but can be rounded in various places. Sometimes we want to measure something with a measuring tape, something that is straight. But often the tape hangs loosely away from the straight object. In such cases our result is not accurate, since if the tape is loose then it will show some extra length. Similarly, if we have a curve surface, we can not measure its length with a straight meter rod accurately. Consider this curve in the figure

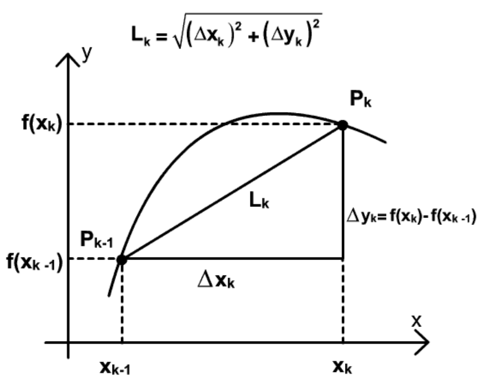


Now say I want to measure the length from P_0 to P_1. Let's say I use a straight stick. The stick will be rigid and straight, and so I will miss some length that is curved between the points P_0 and P_1. If P_0 and P_1 are so close that distance between them become straight then straight stick can be used. Now I can do the same for all the lengths between all the points in the figure.



The line segments joining the points $P_i \;\; i = (1 \text{ to } n)$ form a polygonal path and it's clear that this path is a good approximation of the length of the curve.

Well, it should also be clear that just as we have done so far, if we let the number of these segments joining all points between P_0 to P_n increase, we will get a better approximation to the curve.
We can do it in the following way
Note in the figure that to each point P_i , there corresponds a point x_i on the x-axis in the interval [a, b]
Let's assign the distance between the x_i 's as $\Delta x_1, \Delta x_2, ... \Delta x_n$.
Lets look at the k_{th} line segment. We will call it L_k. Here is what it looks like.
Figure 6.4.3



We want to find its length. Using the distance formula, we get

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

From the figure 6.4.3, it's clear that $\Delta y_k = f(x_k) - f(x_{k-1})$

So we get $L_k = \sqrt{(\Delta x_k)^2 + (f(x_k) - f(x_{k-1}))^2}$

The Mean Value Theorem (Theorem 4.9.2) can be applied here.

$$\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}} = f'(x_k^*) \text{ for some point } x_k^* \text{ between } x_k \text{ and } x_{k-1}$$

$$\Rightarrow f(x_k)-f(x_{k-1}) = f'(x_k^*)(x_k-x_{k-1}) = f'(x_k^*)\Delta x_k$$

So, we can write our first equation as

$$L_k = \sqrt{(\Delta x_k)^2 + \left(f'(x_k^*)\right)^2 (\Delta x_k)^2}$$

$$L_k = \sqrt{1+\left(f'(x_k^*)\right)^2}\ \Delta x_k$$

So, this is the length of one segment.
The length of the WHOLE polygonal path will be

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n}\sqrt{1+\left(f'(x_k^*)\right)^2}\ \Delta x_k$$

Just as we have done before, if we increase the number of divisions of the interval [a, b] in such a way that the widest of the subinterval goes to 0, we get

$L = \lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n}\sqrt{1+\left(f'(x_k^*)\right)^2}\ \Delta x_k$ which we will call arc length of the curve given by the function f(x).

Of course, the above equation defines an integral $L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\ dx$

Arc Length Formulas:
If f is a smooth function on $[a,b]$, then the arc length L of the curve $y = f(x)$ from $x = a$ to $x = b$ is defined by

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

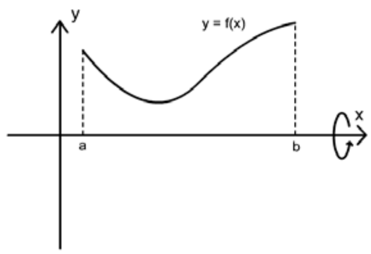
Lecture # 37
Area of a surface of Revolution

We will study in this lecture

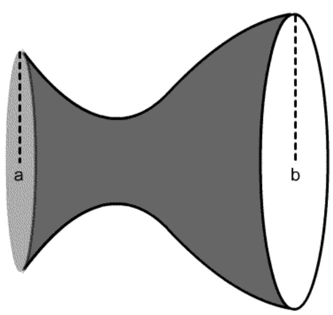
- Definition of Surface Area
- Surface Area Formulas

Surface Area Problem:

Let f be a smooth, non negative function on $[a,b]$. Find the area of the surface generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis.

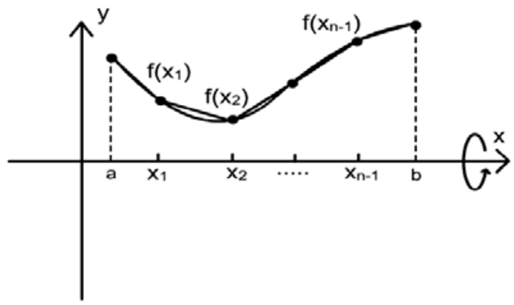


By revolving this curve we get a 3-dimensional solid as



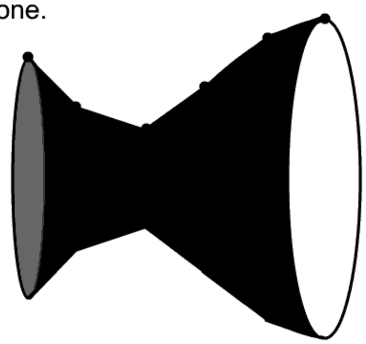
Surface area is roughly the area covered by the surface of a solid in 3-dimensional space.
But we need a more precise definition.
Say we have the graph of the function **f** as stated earlier in 6.5.1 area problem.
As usual, divide the interval [a, b] into subinterval with widths $\Delta x_1, \Delta x_2, \ldots \Delta x_n$ having x-coordinates $a, x_1, \ldots x_{n-1}, b$.
This in turn subdivides the curve into "sub-curves" which can be approximated by a polygonal path made up of line segments joining the end points of the "sub-curves"

Figure 6.5.2 a)



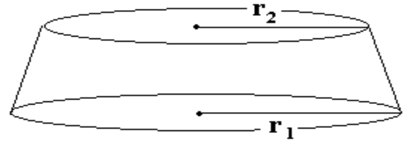
If we now rotate this polygonal path around the x-axis, we get a solid that is approximately the same in properties to the one we had by rotating the graph of f around the x-axis.

Surface generated by the polygonal path is made of parts, each of which is a frustum of a cone.



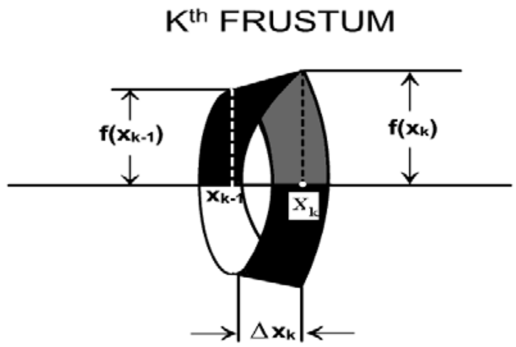
This approximated solid is made up of frustums of a cone.
A frustum of a cone is the solid that you get if you chop of the pointed top of a cone.

FRUSTUM



It has two radii and a slanted length.
Its surface area is measured by $S = \pi\ (r_1 + r_2)l$

In our case of the solid, the radii correspond to the height of the points of subdivision on the curve in an interval, and *l* is the length of the line segment (arc length) in a given interval.
So we can use this formula to find the surface area of each frustum on the approx. solid, and then add them all up.
To get the surface area of the original solid, if we let the number of our subdivisions increase without bound, then we get the surface area of the solid. Here is how it is done.
Lets consider the k-th frustum of our approx. solid.



Its surface area can be found using the formula for the frustum

$$S_k = \pi\,[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2}$$

as $r_1 = f(x_{k-1})$

$r_2 = f(x_k)$

$l = \sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2}$

The formula for l is just that of arc length so we can write it as $\sqrt{1+[f'(x_k^*)]^2}\,\Delta x_k$

So

$$S_k = \pi\,[f(x_{k-1}) + f(x_k)]\sqrt{1+[f'(x_k^*)]^2}\,\Delta x_k$$

By the Intermediate Value Theorem, there exists a point x_k^{**} in the interval $[x_{k-1}, x_k]$ such that

$$\frac{1}{2}[f(x_{k-1}) + f(x_k)] = f(x_k^{**})$$

So, we can re-write our equation for S_k as $S_k = 2\pi\, f(x_k^{**})\sqrt{1+[f'(x_k^*)]^2}\,\Delta x_k$

The area of the approximated solid is then $\sum_{k=1}^{n} S_k = \sum_{k=1}^{n} 2\pi\, f(x_k^{**})\sqrt{1+[f'(x_k^{*})]^2}\,\Delta x_k$

As we have done before, let the largest width on the x-axis approach 0 to get

$$S = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi\, f(x_k^{**})\sqrt{1+[f'(x_k^{*})]^2}\,\Delta x_k$$

$$S = \int_a^b 2\pi\, f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

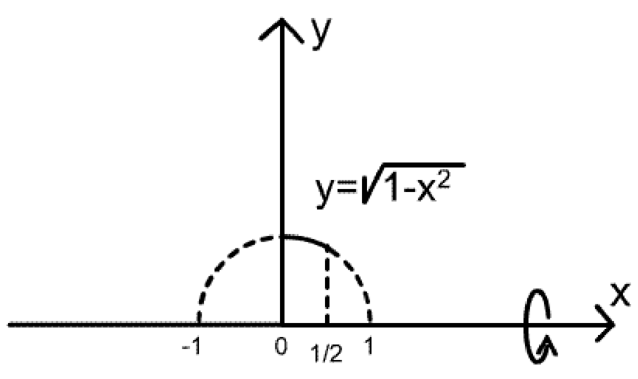
Surface Area Formulas:

Let f be a smooth, nonnegative function on $[a,b]$. Then the surface area S generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is

$$S = \int_a^b 2\pi\, f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

Example:
Find the surface area of the portion of the sphere generated by revolving the curve
$y = \sqrt{1-x^2} \quad 0 \le x \le \frac{1}{2}$ about the x-axis

Solution: Since



$f(x) = \sqrt{1-x^2}$

$f'(x) = -\dfrac{x}{\sqrt{1-x^2}}$

Thus from surface area formula

$$S = \int_0^{1/2} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}dx = \int_0^{1/2} 2\pi dx = 2\pi x]_0^{1/2} = \pi$$

Lecture # 38

Work and Definite Integral

In this lecture we will discuss

- Work done by a constant force
- Work done by a variable force
- Fluid Pressure
- Pascal's Principle

Work done by a constant force

If an object moves a distance *d* along a line while a CONSTANT force F is acting on it, then work W done on the object by the force F is defined as

W = F. ***d => Work = Force*** **x** ***distance***

Distance unit is meter represented by m
Force units are Pounds (lbs), Dynes or Newton (N).

One Dyne = the force needed to give a mass of 1 gram an acceleration of 1 cm/s^2

1 Newton = the force needed to give a mass of 1 Kg an acceleration of 1 m/s^2

The most common units of work are

- foot-pounds(ft lb)

- dyne-centimeters(dyne cm)

- newton-meter(N m)

One newton-meter is also called a joule(j)

Example
An object moves 5 ft along a line while subjected to a force of 100 lbs in its direction of motion. The work done is

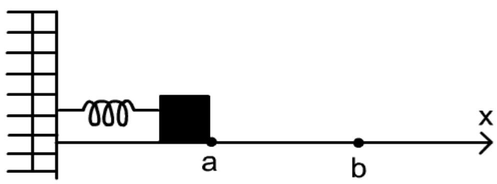
W = F.d = 100(5) = 500 ft.lbs.

Work done by a variable force

- So far the force was a constant force.
- What if it's changing constantly?
- The equation of Work we just saw will not work.
- We need Calculus

6.7.1 PROBLEM

Suppose that an object moves in the positive direction along a coordinate line while subject to a force F(x), in the direction of motion , whose magnitude depends on the coordinate x. Find the work done by the force when the object moves over an interval [a,b].



In this figure, we have a block subjected to the force of a compressed spring.
As the block moves from a to b, the spring gets un-compressed and the force it applied diminishes.
So here is a case where the force varies with the position x of the spring. So the force F(x) is a function of the position of the block on the x-axis.
We need to define work done by a variable force. This, in turn, will give us the answer to calculate it as well through **definite integral.**

As before we subdivide the interval [a, b] into subintervals with coordinates
$a, x_1, ...x_{n-1}, b$ and widths Δx_1 etc

Work and definite integral
Let's consider the kth interval and the force F over it.

If the interval $[x_{k-1}, x_k]$ is small, then the force F will be almost constant on it.

We can approx F on this interval by $F(x_k^*)$ where x_k^* is a point in the kth interval. The width of this interval will be Δx_k

So the work done on this interval is $W_k = F(x_k^*)\Delta x_k$

So the work done over the whole interval is $\sum_{k=1}^{n} W_k = \sum_{k=1}^{n} F(x_k^*)\Delta x_k$

Let the largest subinterval go to 0 to get $W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} F(x_k^*)\Delta x_k = \int_a^b F(x)dx$

DEFINITION 6.7.2

If an object moves in the positive direction over the interval [a,b] while subjected to a variable force F(x) in the direction of motion ,then the work done by the force is

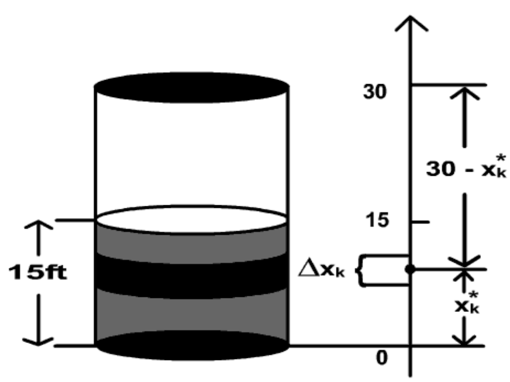
$$W = \int_a^b F(x)dx$$

Example

A cylindrical water tank of radius 10 ft and height 30 ft is half filled with water. How much force is needed to pump all the water over the upper rim of the tank?

Solution

Introduce a coordinate line as shown in figure, imagine the water to be divided into n thin layers with thicknesses $\Delta x_1, \Delta x_2, ..., \Delta x_n$



($30 - x_k^*$) represent approximate distance covered by the k^{th} layer when it moves above the rim

How much force is required to move the Kth layer of water above the rim?

The force required to move the Kth layer equals the weight of the layer, which can be found by multiplying its volume by weight density of water

$$\begin{bmatrix}\text{force to} \\ \text{move the} \\ \text{k}^{\text{th}} \text{ layer}\end{bmatrix} = \left(\pi r^2 \Delta x_k\right)\begin{bmatrix}\text{weight density} \\ \text{of water}\end{bmatrix}$$

$$= \left(\pi (10)^2 \Delta x_k\right)(62.4)$$

$$= 6240\pi \Delta x_k$$

$$W_k \approx \left(30 - x_k^*\right)6420\pi\Delta x_k$$

and the work W required to pump all layers will be approximately

$$W = \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n}\left(30 - x_k^*\right)\left(6420\pi\right)\Delta x_k$$

To find the exact value of the work we take the limit as $\max \Delta x_k \to 0$ .This yields

$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n}\left(30 - x_k^*\right)\left(6420\pi\right)\Delta x_k$$

$$= \int_0^{15}(30 - x)(6420\pi)dx$$

$$= 6420\left(30x - \frac{X^2}{2}\right)_0^{15}$$

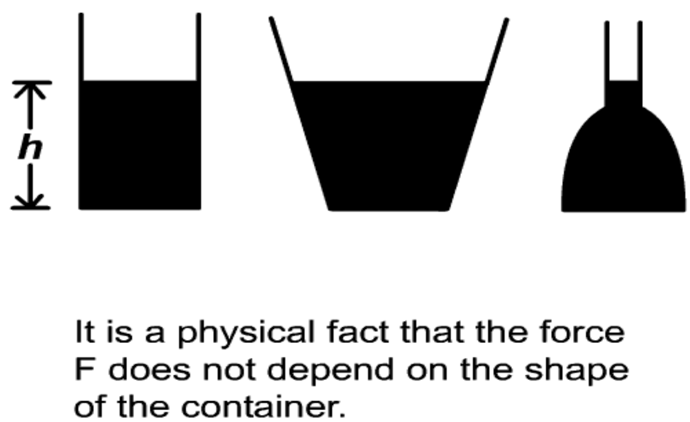
$$= 2{,}106{,}000\pi \text{ft} / \text{lb} \approx 6{,}616{,}194 \text{ ft} / \text{lb}$$

Pressure is defined as Force per unit Area.
P = F / A = ρh

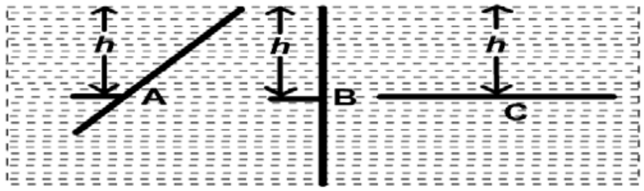
Where p is weight density and h is depth below surface of fluid.

Fluid Pressure
If a flat surface of area A is submerged horizontally in a fluid at a depth h, then the fluid exerts a force F perpendicular to the surface
This is given by $F = phA$
p is the weight density of the fluid.
p for water is 62.4 lbs/ft^3
It is a fact from Physics that the shape of the container containing the fluid does not in any way effect F.
If the three containers here have the same area for their bases and the fluid has the same height, then the F on the bases will be equal.

Figure 6.8.1

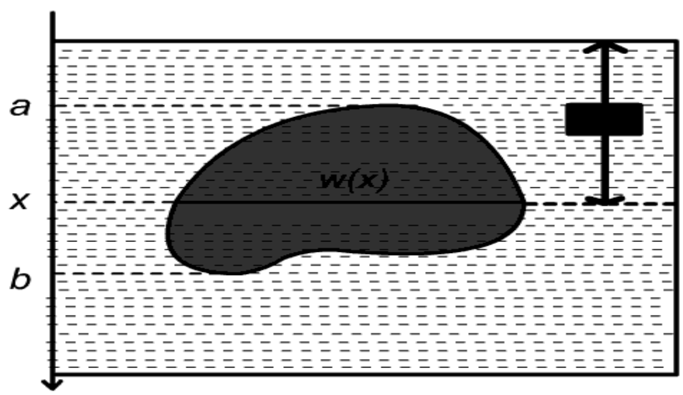


Pascal's Principle
Fluid pressure is the same in all directions at a given height.
Figure 6.8.2



By Pascal's Principle the fluid pressure at point A ,B and C is the same

If a flat surface is submerged horizontally, then the total force on its face can be measured easily since the pressure is the same at all the points since the height of these points does not vary.
Suppose we submerge a flat surface in a fluid VERTICALLY.
Then at each point along the height of the surface, the pressure will be different and therefore Force will be different.
Here is where we need calculus
Look at the picture

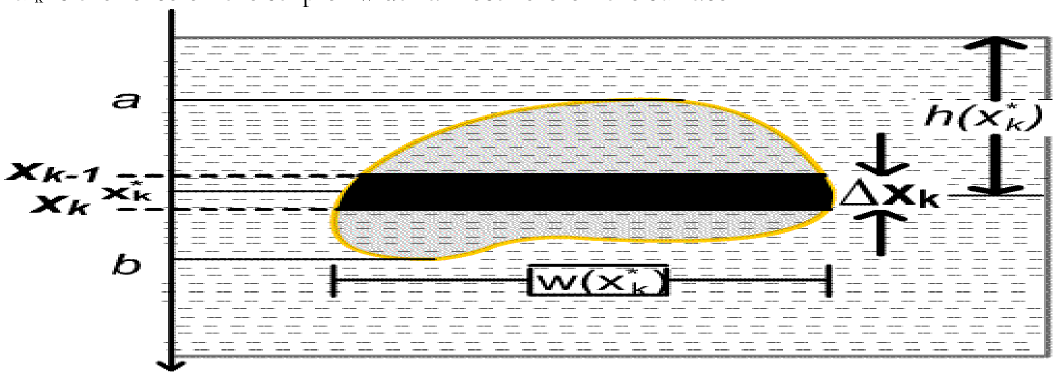


We have a surface submerged in a tank.
There is VERTICAL axis on the tank
The surface is confined by **x = a** and **x = b**
There is a height function of x that measures the depth of a point on the surface from the top of the tank.
There is a width function of x that measures the width w (x•) of the surface at the height h (x).
At different height (depth) Force on a section of the surface will be different.
We want to know the total force on the flat surface.
Subdivide the interval [a,b] into subintervals of various widths.

Using the kth interval, which form a rectangle that will be used to approximate the force on that interval on the surface?
The smaller the interval, the more accurate the approximation.
The approximation will be made by $F_k \approx ph(x_k^*)w(x_k^*)\Delta x_k$
p = fluid density

$h(x_k^*) = depth$

$w(x_k^*)\Delta x_k = area\ of\ rectangle$
• F_k is the force on the strip of width almost zero on the surface



In the figure, imagine the rectangle to shrink to the dashed line.
Hence, the total force in this fashion will be $F = \sum_{k=1}^{n} F_k \approx \sum_{k=1}^{n} ph(x_k^*)w(x_k^*)\Delta x_k$

Make the largest interval go to 0 to get $F = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} ph(x_k^*)w(x_k^*)\Delta x_k = \int_a^b ph(x)w(x)dx$

Formula For Fluid Force
Assume that a flat surface is immersed vertically in a liquid of weight density P and that the submerged portion extends from x=a to x=b on a vertical x-axis.
For a<=x<=b, let w(x) be the width of the surface at x and let h(x) be the depth of the point x. Then the total **fluid pressure** on the surface is $F = \int_a^b ph(x)w(x)dx$

Lecture # 39

Improper Integral

In this lecture we will study

- Integrals over Infinite Interval
- Integrals who's Integrands become Infinite

Integrals over Infinite Interval

As we saw before, for a given continuous function *f*, the definite integral is $\int_a^b f(x)dx$

It is assumed that the interval [a, b] is finite.

What if we look at [a, +∞) and the corresponding integral $\int_a^{+\infty} f(x)dx$

In this case, we define what is called an improper integral over an infinite interval.

What does it mean to integrate all the way to + ∞ ?

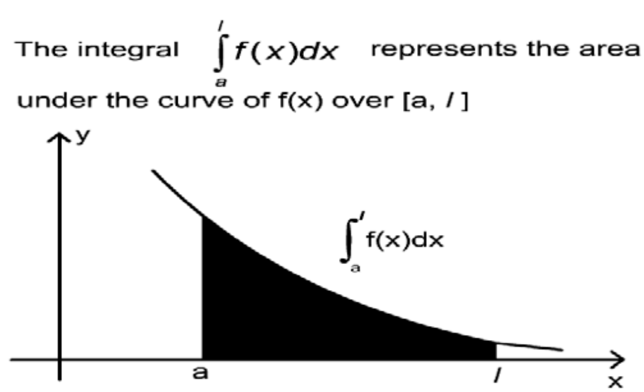
The answer will be clear if we define this integral as a limit in the following way.

$$\int_a^{+\infty} f(x)dx = \lim_{l \to +\infty} \int_a^{l} f(x)dx$$

What this does is to first turn the integral into more familiar form of over a finite interval, and then we let the upper limits of the interval approach 0 and see what happen to the answer we had got earlier.

If this limits exists, then we say that the Improper Integral **Converges**, and the value of the limit is assigned to the integral.
If the limit does not exist, then we say that the Improper Integral **Diverges**, and no finite value is assigned.

Lets get some geometric ideas to understand things.



Evaluate $\int_1^{+\infty} \frac{dx}{x^2} = \int_1^{+\infty} \frac{1}{x^2}\, dx$

We begin by replacing the infinite upper limit with a finite upper limit l

$$\int_1^{l} \frac{dx}{x^2} = \left.\frac{-1}{x}\right]_1^{l}$$

$$= -\frac{1}{l} - (-1) = 1 - \frac{1}{l}$$

Thus

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{l \to +\infty} \int_1^{l} \frac{dx}{x^2} = \lim_{l \to +\infty} \left(1 - \frac{1}{l}\right)$$

$$= 1 - \frac{1}{\infty} = 1 - 0 = 1$$

EXAMPLE

Evaluate $\int_1^{+\infty} \frac{dx}{x} = \int_1^{+\infty} \frac{1}{x}\, dx$

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{l \to +\infty} \int_1^{l} \frac{dx}{x}$$

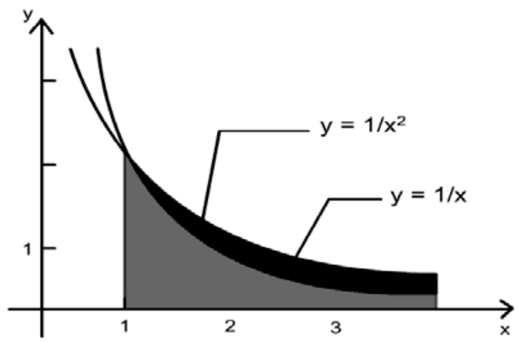
$$= \lim_{l \to +\infty} \left[\ln|x|\right]_1^{l}$$

$$= \lim_{l \to +\infty} \ln|l| = +\infty$$

What's happening here in these examples?

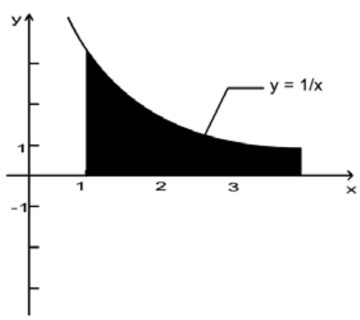
In the first case with $f(x) = 1/x^2$, we get finite answer over the same interval?
In the second case we get a divergent limit and so we were unable to calculate the area under graph of $1/x$ over $[1,+\infty)$

Look at the graphs of the two functions

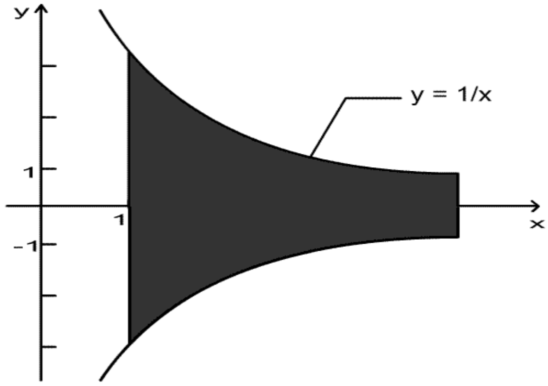


We can see geometrically that the graph of $1/x^2$ is approaching $y = 0$ much faster than that of $1/x$. Algebraically also, if you divide 1 by the square of a number, the result is much smaller than if you divide by the number itself.

For example $\frac{1}{2} > \frac{1}{4}$ and $1/8 > 1/64$
So the idea is that as x goes to $+\infty$, $1/x^2$ goes to 0 much faster than $1/x$, so much so that when we attempt to find the area under the graph over the infinite interval $[1, +\infty)$ the first is convergent, and the other is divergent.
Lets think about Volume of second case.
Lets rotate the graphs of $1/x$ over $[1, +\infty)$ around the x-axis.



We get solid of revolution that look like funnels with no lower point as shown in figure below.



We would like to find the volume of this solid.
The cross section is a disk with radius f(x).
For f(x) = 1/x, we get for volume

$$\lim_{l\to+\infty}\int_1^l \pi\ [f(x)]^2 dx$$

$$= \lim_{l\to+\infty}\int_1^l \pi\ \left[\frac{1}{x}\right]^2 dx \ = \lim_{l\to+\infty}\int_1^l \frac{\pi}{x^2}\ dx$$

$$= \lim_{l\to+\infty}\left[-\frac{\pi}{l}+\frac{\pi}{1}\right] \quad = \ \pi$$

So we can find out how much paint can be held in this solid, but we cannot paint the inside of the solid!!!

We can also have an **improper integral** of this type: $\int_{-\infty}^{b} f(x)dx = \lim_{l\to-\infty}\int_{l}^{b} f(x)dx$

Integrals over Infinite Interval

We saw earlier that if a function *f* is not bounded on an interval [a, b], then *f* is not integrable on [a,b]

The integral

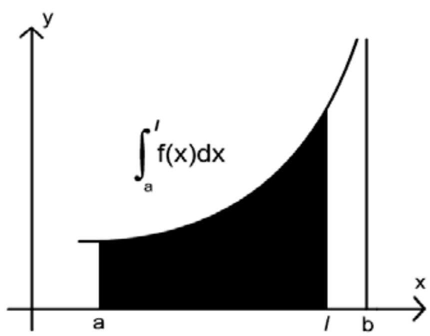
$\int_0^3 \frac{1}{(x-2)^{\frac{2}{3}}}\,dx$ is unbounded at x = 2 in [0, 3].

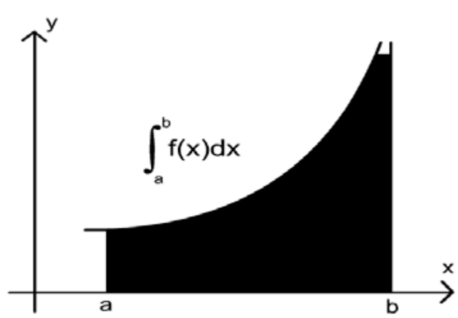
We can get around this problem by doing the following

If f is continous on [a,b) but does not have a limit from the left then we define the improper integral as a limit in this way:

$$\int_a^b f(x)dx = \lim_{l \to b^-} \int_a^l f(x)dx$$

Geometrically it can be represented as





If f is continuous on (a,b] but fails to have a limit as x approaxhes a from the right, then we define the improper integral as:

$$\int_a^b f(x)dx = \lim_{l\to a^+}\int_l^b f(x)dx$$

If f is continuous on [a, b] except that at some point c such that a<c<b, f(x) becomes infinite as x goes to c from left or right.

If both improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ **Converge** then we say that the improper integral $\int_a^b f(x)dx$ Converges as we define $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

EVALUATE: $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

Solution: The integrand approaches $+\infty \ as \ x \to 2$ so we solve it as

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \lim_{l\to 2^-}\int_1^l \frac{dx}{(x-2)^{2/3}} = \lim_{l\to 2^-}\left[3(l-2)^{1/3} - 3(1-2)^{1/3}\right] = 3$$

$$\lim_{l\to 2^+}\int_l^4 \frac{dx}{(x-2)^{2/3}} = \lim_{l\to 2^+}\left[3(4-2)^{1/3} - 3(l-2)^{1/3}\right] = 3\sqrt[3]{2}$$

Lecture # 40

L'Hopital's Rule and Indeterminate forms

- L'Hopital's Rule and 0 / 0
- Indeterminate form of type ∞ / ∞
- Indeterminate form of type 0. ∞
- Indeterminate forms of type 0^0, ∞^0 and 1^∞ and ∞-∞

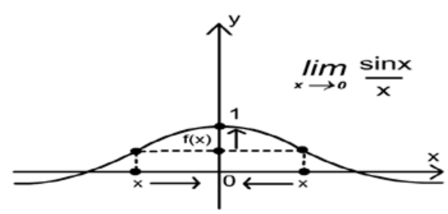
If we look at $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$ and $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

The top and the bottom both approach 0 and so these types of limits are called indeterminate forms of type 0/0.
This kind of limit can converge in which case it will have finite real values, or it can diverge.
The value, if it converges, is not obvious right away.
The first expression can be factored and bottom cancelled to get a finite value and the second expression requires geometrical observation as shown below

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}$$

$$=\lim_{x\to 2}(x+2)=4$$



We need a more general concept that work almost always.
L'Hopital's Rule provide us solution that we are looking for

THEOREM 10.2.1
(L'Hopital's Rule for Form 0/0)

Let *lim* stand for one of the *limits*

$$\lim_{x\to a}\quad \lim_{x\to a^+}\quad \lim_{x\to a^-}\quad \lim_{x\to +\infty}\quad \lim_{x\to -\infty}$$

and suppose that *lim* f(x) = 0 and *lim* g(x) = 0.If *lim*[f′(x) / g′(x)] has a finite value L, or if this *limit* is +∞ or -∞ , then

$$\lim\frac{f(x)}{g(x)}=\lim\frac{f'(x)}{g'(x)}$$

How we can use this theorem, here are some steps to be followed

STEP1: Check that *lim* [f(x)/g(x)] is an intermediate form. If it is not , then L'Hopital's rule cannot be used.

STEP2: Differentiate f and g seperately.

STEP3: Find *lim* [f'(x)/g'(x)]. If this limit is finite, +∞ or -∞ , then it is equal to *lim* [f(x)/g(X)]

EXAMPLE

Use L'Hopital's rule to evaluate

a) $\lim\limits_{x\to 2}\dfrac{x^2 - 4}{x - 2}$ b) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

Since

$$\lim_{x\to 2}(x^2 - 4) = 0 \text{ and } \lim_{x\to 2}(x - 2) = 0$$

the given limit is an intermediate form of type 0/0. Thus,L'Hopital's rule applies and we can write

$$\lim_{x\to 2}\frac{(x^2 - 4)}{x - 2} = \lim_{x\to 2}\frac{\frac{d}{dx}\left[x^2 - 4\right]}{\frac{d}{dx}\left[x - 2\right]}$$

$$= \lim_{x\to 2}\frac{2x}{1} = 4$$

b) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

Since

$$\lim_{x\to 0}\sin 2x = 0 \quad \text{and} \quad \lim_{x\to 2} x = 0$$

the given limit is an intermediate form of type 0/0. Thus,L'Hopital's rule applies and we can write

$$\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}[\sin 2x]}{\frac{d}{dx}[x]}$$

$$= \lim_{x\to 0}\frac{2\cos 2x}{1} = 2$$

EXAMPLE

Evaluate

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}$$

$$\lim_{x\to \pi/2}(1-\sin x) = 0 \qquad \lim_{x\to \pi/2}\cos x = 0$$

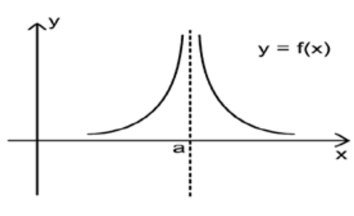
the given limit is an intermediate form of type 0/0. Thus by , L'Hopital's rule

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x} = \lim_{x\to \pi/2}\frac{\frac{d}{dx}[1-\sin x]}{\frac{d}{dx}[\cos x]}$$

$$= \lim_{x\to \pi/2}\frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0$$

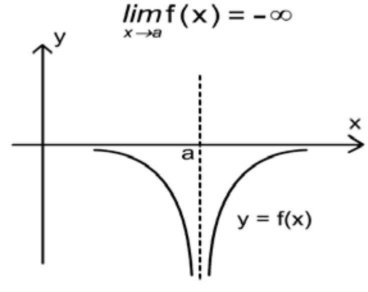
Some Notation

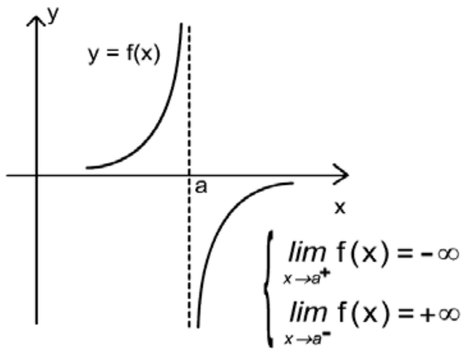
In the four parts of figure below,

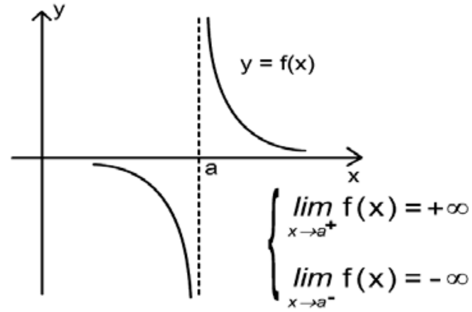


$\lim_{x\to a} f(x) = +\infty$

$\lim_{x\to a} f(x) = -\infty$







The one-sided limits are either $+\infty$ or $-\infty$. When we want to indicate that one of these four situations occurs without specifying which one, we shall write

Indeterminate form of type ∞ / ∞
An indeterminate form of type ∞/∞ is a limit i.e. lim f(x)/g(x) in which
lim f(x) = ∞ and lim g(x) =∞ . Some examples are

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x} \quad \boxed{\begin{array}{l} \text{Numerator} \rightarrow -\infty \\ \text{Denominator} \rightarrow +\infty \end{array}}$$

The following version of L'Hopital Rule is used for problems like this

THEOREM 10.3.1
(L'Hopital's Rule for Form ∞/∞)

Let lim stand for one of the limits

$$\lim_{x \to a} \qquad \lim_{x \to a^+} \qquad \lim_{x \to a^-} \qquad \lim_{x \to +\infty} \qquad \lim_{x \to -\infty}$$

and suppose that *lim* f(x) = ∞ and *lim* g(x) = ∞ .If *lim*[$f'(x) / g'(x)$] has a finite value L, or if this limit is $+\infty$ or $-\infty$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

EXAMPLE

Evaluate $\lim\limits_{x \to +\infty} \frac{x}{e^x}$

$$\lim_{x \to +\infty} x = +\infty \text{ and } \lim_{x \to +\infty} e^x = +\infty$$

so that the given *limit* is an indeterminite form of type ∞/∞ .Thus , by L'Hopital's Rule

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{\frac{d}{dx}[x]}{\frac{d}{dx}\left[e^x\right]} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

If $\lim f(x) = 0$ *and* $g(x) = \infty$, *then a product* limit *like* $\lim f(x)g(x)$ *is of the type* $0.\infty$

Limit problems of this type can be converted to the form of 0/0 by writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)}$$

EXAMPLE

Evaluate $\lim\limits_{x \to 0^+} x \ln x$

Since the given problem is an indeterminite form of type $0.\infty$. We shall convert the problem to the form ∞/∞ and apply L'Hopital rule as follows:

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

Indeterminate forms of type 0^0, ∞^0 and 1^∞ and ∞-∞

Limits of the form $\lim_{x \to a} f(x)^{g(x)}$ give rise to indeterminate form of type $\mathbf{0^0}$, $\mathbf{\infty^0}$ **and** $\mathbf{1^\infty}$.

All these types are treated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then calculating

$$\lim_{x \to a} \ln y = \lim_{x \to a} \left[\ln\left(f(x)^{g(x)} \right) \right]$$

$$= \lim_{x \to a} \left[g(x) \ln f(x) \right]$$

once the value of $\lim_{x \to a} \ln y$ is known , it is simple to determine

$$\lim_{x \to a} y = \lim_{x \to a} f(x)^{g(x)}$$

EXAMPLE

Show that $\lim_{x \to 0} (1+x)^{1/x} = e$

Since

$\lim_{x \to 0} (1+x)^{1/x} = 1$ and $\lim_{x \to 0} 1/x = \infty$

the given limit is an indeterminite form of type 1^∞.
As discussed above , we introduce a dependent variable

$$y = (1+x)^{1/x}$$

and take the natural logarithm of both sides

$$\ln y = \ln(1+x)^{1/x} = \frac{1}{x} \ln(1+x) = \frac{\ln(1+x)}{x}$$

The limit

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

is an indeterminite form of type 0/0, so by L'Hopital's Rule,

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

$$= \lim_{x \to 0} \frac{1/(1+x)}{1} = 1$$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

$$= \lim_{x \to 0} \frac{1/(1+x)}{1} = 1$$

Since $\ln y \to 1$ as $x \to 0$, it follows from the continuity of the natural exponential function that $e^{\ln y} \to e^1$ or equivalent , $y \to e$ as $x \to 0$
Therefore

$$\lim_{x \to 0} (1+x)^{1/x} = e$$

Lecture # 41

Sequences and Monotone Sequences

- Definition of a sequence
- Graphs of sequences
- Limit of a sequence
- Recursive sequence
- Testing for monotonicity
- Eventually monotonic sequences
- Convergence of monotonic sequence
- More on convergence intuitively

Definition of a sequence

A sequence in math is a succession of numbers e.g 2,4,6,8,……. & 1,2,3,4,…….

The numbers in a sequence are called terms of a sequence
Each term has a positional name, like 1st term, 2nd term etc and we can write them as a_1, a_2 etc.
It is convenient to write a sequence as a formula
2, 4, 6, 8,… can be written as a formula as $\{2n\}_{n=1}^{+\infty}$

EXAMPLE

List the first five terms of the sequence

$$\{2^n\}_{n=1}^{+\infty}$$

Substituting n = 1,2,3,4,5 into the formula we get

$$2^1, 2^2, 2^3, 2^4, 2^5$$

or equivalently,

$$2,4,8,16,32$$

We will write a sequences like a_1 , a_2,a_3….. as $\{a_n\}_{n=1}^{+\infty}$

Here is a formal definition of a sequence

DEFINITION 11.1.1

A sequence or infinite sequence is a function whose domain is the set of positive integers; that is ,

$$\{a_n\}_{n=1}^{+\infty}$$

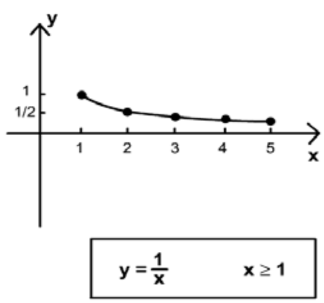
which is an alternative notation for the function

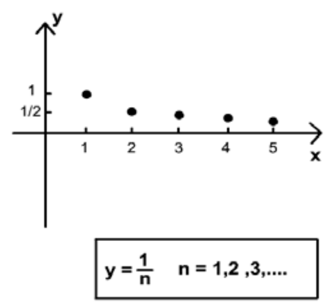
$$f(n) = a_n \qquad \text{where} \quad n = 1,2,3 \ldots\ldots$$

Graphs of Sequences

As we see that sequences are functions, we can talk about the graphs of them.

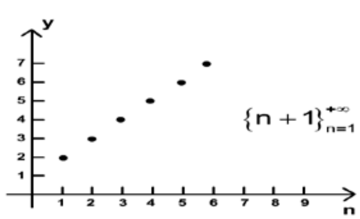
The graph of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{+\infty}$ is the graph of the function equation y = 1/n for n = 1, 2,3….

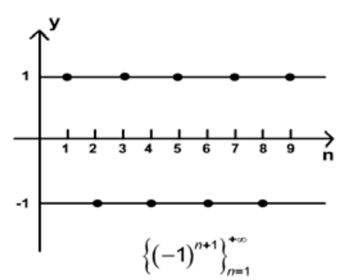


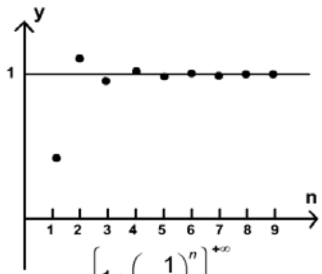


Limit of a sequence

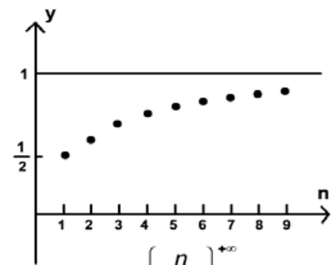
Here are graphs of four sequences, each of which behaves differently as n increases







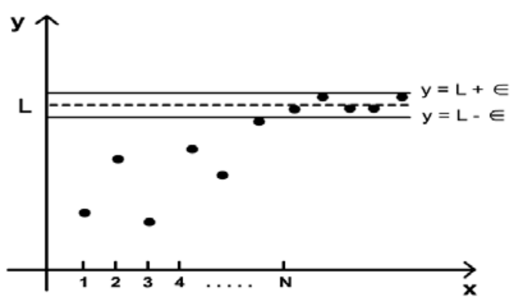
$$\left\{1+\left(-\frac{1}{2}\right)^n\right\}_{n=1}^{+\infty}$$



$$\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$$

Some of these graphs have the concept of a limit in them.

A sequence an converges to a limit L if for any positive number E there is a point in the sequences after which all terms lie btw lines L+E and L-E



From point N on the terms in the sequence are all within $\in$ units of L.

DEFINITION 11.1.2

A sequence $\{a_n\}$ is said to *converge* to the limit L if given any $\in > 0$, there is poitive integer N such that $|a_n - L| < \in$ for $n \geq N$. In this case we write

$$\lim_{n \to +\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to *diverge.*

As we apply limit the following two sequences converge

$$\lim_{n \to +\infty} \frac{n}{(n+1)} = 1$$

and

$$\lim_{n \to +\infty} \left(1 + \left(-\frac{1}{2}\right)^n\right) = 1$$

The following theorem shows that the familiar properties of limits apply to sequence

THEOREM 11.1.3

Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2 respectively, and c is a contant . Then

a) $\lim\limits_{n \to +\infty} c = c$

b) $\lim\limits_{n \to +\infty} ca_n = c \lim\limits_{n \to +\infty} a_n = cL_1$

c) $\lim\limits_{n \to +\infty} (a_n + b_n) = \lim\limits_{n \to +\infty} a_n + \lim\limits_{n \to +\infty} b_n = L_1 + L_2$

d) $\lim\limits_{n \to +\infty} (a_n - b_n) = \lim\limits_{n \to +\infty} a_n - \lim\limits_{n \to +\infty} b_n = L_1 - L_2$

e) $\lim\limits_{n \to +\infty} (a_n b_n) = \lim\limits_{n \to +\infty} a_n \lim\limits_{n \to +\infty} b_n = L_1 L_2$

f) $\lim\limits_{n \to +\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n \to +\infty} a_n}{\lim\limits_{n \to +\infty} b_n} = \dfrac{L_1}{L_2}$ (If $L_2 \neq 0$)

Determine whether the sequence converges or diverges

$$\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$$

Dividing numerator and denomenator by n yields

$$\lim_{n\to+\infty}\frac{n}{2n+1}=\lim_{n\to+\infty}\frac{1}{2+1/n}$$

$$=\frac{\lim\limits_{n\to+\infty}1}{\lim\limits_{n\to+\infty}(2+1/n)}$$

$$=\frac{\lim\limits_{n\to+\infty}1}{\lim\limits_{n\to+\infty}2+\lim\limits_{n\to+\infty}1/n}$$

$$=\frac{1}{2+0}=\frac{1}{2}$$

Thus the sequence converges to 1/2

Recursive Sequence

Some sequences are defined by specifying one or more initial terms and giving a formula that relates each subsequent term to the term that precedes it , such sequences are said to be defined recursively .

EXAMPLE

Let $\{a_n\}$ be the sequence defined by $a_1 = 1$ and the recursion formula

$$a_{n+1}=\frac{1}{2}(a_n+3/a_n) \quad \text{for } n\geq 1$$

$$a_2=\frac{1}{2}(a_1+3/a_1)=\frac{1}{2}(1+3)=2 \qquad \boxed{n=1}$$

$$a_3=\frac{1}{2}(a_2+3/a_2)=\frac{1}{2}(2+\frac{3}{2})=\frac{7}{4} \qquad \boxed{n=2}$$

$$a_4=\frac{1}{2}(a_3+3/a_3)=\frac{1}{2}(\frac{7}{4}+\frac{12}{7})=\frac{97}{56} \qquad \boxed{n=3}$$

Monotonicity and testing for monotonicity

DEFINITION 11.2.1

A sequence $\{a_n\}$ is called

Increasing if

$$a_1 < a_2 < a_3 < \ldots\ldots < a_n < \ldots.$$

Nondecreasing if

$$a_1 \le a_2 \le a_3 \le \ldots\ldots \le a_n \le \ldots.$$

Decreasing if

$$a_1 > a_2 > a_3 > \ldots\ldots > a_n > \ldots.$$

Nonincreasing if

$$a_1 \ge a_2 \ge a_3 \ge \ldots\ldots \ge a_n \ge \ldots.$$

A sequence that is either nondecreasing or nonincreasing is called monotone, and a sequence that is increasing or decreasing is called strictly monotone .Observe that a strictly monotone is monotone , but not conversely.

EXAMPLE

$\frac{1}{2},\frac{2}{3},\frac{3}{2},\ldots\ldots,\frac{n}{n+1},\ldots..$ is increasing

$1,\frac{1}{2},\frac{1}{3},\ldots..,\frac{1}{n},\ldots\ldots.$ is decreasing

$1,1,2,2,3,3,\ldots\ldots..$ is nondecreasing

$1,1,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3},\ldots\ldots$ is nonincreasing

All four of these sequences are monotone, but the sequence

$$1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\ldots\ldots..,(-1)^{n+1}\frac{1}{n},\ldots\ldots$$

is not monotone. The first and second sequences are strictly monotone.

Testing for monotonicity

Monotone sequences are classified as follows

Difference between successive terms	Classification
$a_{n+1} - a_n > 0$	Increasing
$a_{n+1} - a_n < 0$	Decreasing
$a_{n+1} - a_n \geq 0$	Nondecreasing
$a_{n+1} - a_n \leq 0$	Nonincreasing

Monotone Sequence with positive terms are classified as follows

Ratio of successive terms	Classification
$a_{n+1} / a_n > 1$	Increasing
$a_{n+1} / a_n < 1$	Decreasing
$a_{n+1} / a_n \geq 1$	Nondecreasing
$a_{n+1} / a_n \leq 1$	Nonincreasing

If f(n) = a_n is the nth term of a sequence, and if f is differentiable for x>=1 then we have the following results

Derivative of f for $x \geq 1$	Classification of the sequence with f(x) = a_x
$f'(x) > 0$	Increasing
$f'(x) < 0$	Decreasing
$f'(x) \geq 0$	Nondecreasing
$f'(x) \leq 0$	Nonincreasing

EXAMPLE

Show that $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$

is an increasing sequence

It is intuitively clear that the sequence is increasing .To prove that this is so, let

$$a_n = \frac{n}{n+1}$$

We can obtain a_{n+1} by replacing n by n+1 in this formula. This yields

$$a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$$

Thus for $n \geq 1$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$= \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)} > 0$$

which proves that the sequence is increasing, so sequence is strictly monotone.

EXAMPLE

Show that the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

is increasing by examining the ratio of successive terms.

As shown earlier

$$a_n = \frac{n}{n+1} \quad \text{and} \quad a_{n+1} = \frac{n+1}{n+2}$$

Thus

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/(n+2)}{n/(n+1)}$$

$$= \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n}$$

Since the numerator exceeds the denominator, the ratio exceeds 1, that is $a_{n+1}/a_n > 1$ for $n \geq 1$. This proves that the sequence is increasing.

Eventually monotonic sequences

A sequence an is eventually monotone if there is some integer N such that the sequence is monotone for *n* >= N

EXAMPLE

Show that the sequence $\left\{\frac{10^n}{n!}\right\}_{n=1}^{+\infty}$ is eventually decreasing.

We have

$$a_n = \frac{10^n}{n!} \qquad \text{and} \qquad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$$

so

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}/(n+1)!}{10^n/n!}$$

$$= \frac{10^{n+1}n!}{10^n(n+1)!} = 10\frac{n!}{(n+1)n!}$$

$$= \frac{10}{n+1}$$

This is clear that $a_{n+1}/a_n < 1$ for all $n \geq 10$, so the sequence is eventually decreasing.

Convergence of monotonic sequence:

THEOREM 11.2.2

If $a_1 \le a_2 \le a_3 \le ..... \le a_n \le .....$ is a nondecreasing sequence, then there are two possibilities:

a) There is a constant M, called a upper bound for the sequence, such that $a_n \le M$ for all n, in which case the sequence converges to limit L satisfying $L \le M$

b) No upper bound exists, in which case

$$\lim_{n \to +\infty} a_n = +\infty$$

THEOREM 11.2.3

If $a_1 \ge a_2 \ge a_3 \ge ...... \ge a_n \ge .. ..$ is a nonincreasing sequence, then there are two possibilities:

a) There is a constant M, called a lower bound for the sequence, such that $a_n \ge M$ for all n, in which case the sequence converges to limit L satisfying $L \ge M$

b) No lower bound exists, in which case

$$\lim_{n \to +\infty} a_n = -\infty$$

EXAMPLE

Show that the sequence $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{+\infty}$ converges.

We showed earlier that the given sequence is increasing (hence nondecreasing).It is evident that the number M=1 is an upper bound for the sequence since

$$an = \frac{n}{n+1} < 1 \qquad n = 1,2,....$$

Thus, by Theorem 11.2.2 the sequence converges to some limit L such that $L \leq M = 1$.This is indeed the case since

$$\lim_{n\to+\infty}\frac{n}{n+1}=\lim_{n\to+\infty}\frac{1}{1+1/n}=1$$

An Intuitive View of Convergence:

Informally stated, the convergence or divergence of a sequence does not depend on the behavior of the "initial terms" of the sequence, but rather on the behavior of the "tail end". Thus for sequence { an } to converge to a limit L, it does not matter if the initial terms are for from L, just so the terms in the sequence are eventually arbitrarily close to L. This being the case , one can add, delete, or alter finitely many terms without affecting the convergence, divergence, or the limit (if it exists).

Lecture # 42

Infinite Series

Definition 11.3.1
An Infinite series is an expansion that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \ldots + u_k + \ldots$$

The numbers u_1, u_2, … are called the terms of the series.

What does it mean to add infinitely many terms?
Can we add 1+2+3+4+…. All the way to infinity?

This is physically and mentally impossible!
But we can compute an infinite sum using the idea of limits
Look at this decimal expansion

0.333333….

We can rewrite this as an infinite series

0.3+0.03+0.003+0.0003+…
Or as

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots$$

Note that we already know that 0.333…. is really 1/3

So our sum of the above series should also be 1/3

Let's work out a definition for the sum of an infinite series. Consider the following sequences of (finite) sums

$$s_1 = \frac{3}{10} = 0.3$$

$$s_2 = \frac{3}{10} + \frac{3}{10^2} = 0.33$$

$$s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333$$

The sequence of numbers s_1,s_2, etc. can be viewed as a succession of approximations to the "sum " of the infinite series. We know that the answer is 1/3, so it is better be that the approximations are close to this at the start, and get closer and closer to this number as more and more terms are added together. This suggests that if we take the limit of the sequence of approximations, we should get 1/3
Note that the nth term of the approximating series is

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots \frac{3}{10^n}$$

Taking its limit gives (limit of the nth partial sum of the series)

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\ldots\frac{3}{10^n}\right)$$

Now we do some algebraic manipulation here

$$S_n = \frac{3}{10}+\frac{3}{10^2}+\ldots+\frac{3}{10^n} \quad \text{......(A)}$$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(\frac{3}{10}+\frac{3}{10^2}+\ldots+\frac{3}{10^n}\right)$$

$$\frac{1}{10}S_n = \frac{3}{10^2}+\frac{3}{10^3}+\ldots\ldots\ldots+\frac{3}{10^n}+\frac{3}{10^{n+1}} \quad \text{......(B)}$$

Subtracting (B) from (A)

$$S_n - \frac{1}{10}S_n = \frac{3}{10}-\frac{3}{10^{n+1}}$$

$$\frac{9}{10}S_n = \frac{3}{10}\left(1-\frac{1}{10^n}\right)$$

$$S_n = \frac{1}{3}\left(1-\frac{1}{10^n}\right)$$

Since $1/10^n \to 0$ as $n \to +\infty$, it follows that

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\frac{1}{3}\left(1-\frac{1}{10^n}\right) = \frac{1}{3}$$

Now we can formalize the idea of a sum of an infinite series

DEFINITION 11.3.2

Let $[S_n]$ be the sequence of partial sums of the series $u_1 + u_2 + u_3 + \dots + u_k + \dots$. If the sequence $[S_n]$ converges to a limit S, then the series is said to converge, and S is called the **sum** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to diverge: A divergent series has no sum.

Example

Determine whether the series

$$1 - 1 + 1 - 1 + 1 - 1 + \dots$$

converges or diverges. If it converges, find the sum.

$$s_1 = 1$$
$$s_2 = 1 - 1 = 0$$
$$s_3 = 1 - 1 + 1 = 1$$
$$s_4 = 1 - 1 + 1 - 1 = 0$$

and so forth. Thus the sequence of partial sum is

$$1, 0, 1, 0, 1, 0, \dots$$

Since this is a divergent sequence, the given series diverges and consequently has no sum.

Geometric Series

A geometric series is one of the form
$a+ar+ar^2+...+ar^{k-1}+..$
Each term is obtained by multiplying the previous one by a constant number, *r*, and this number is called the ratio for the series.
Examples are
1+2+4+8+…
1+1+1+1+…

THEOREM 11.3.3

A geometric series

$$a + ar + ar^2 + ... + ar^{k-1} + ... \qquad (a \neq 0)$$

converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series converges, then the sum is

$$\frac{a}{1-r} = a + ar + ar^2 + ... + ar^{k-1} + ...$$

Example

The series

$$5 + \frac{5}{4} + \frac{5}{4^2} + ... + \frac{5}{4^{k-1}} + ...$$

is a geometric series with $a = 5$ and $r = 1/4$. Since $|r| = 1/4 < 1$, the series converges and the sum is

$$\frac{a}{1-r} = \frac{5}{1-1/4} = \frac{20}{3}$$

Harmonic Series

A Harmonic series is of the type

$$\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\ldots$$

It may look like this series converges as each successive term is smaller than the first, but in fact this series diverges!

Convergence Tests

Here are some tests that are used to determine if a series converges or diverges. These tests are applied on the kth term of a series (general term), and not on the nth partial sum as we did earlier to derive the definition of the sum of an infinite series.

THEOREM 11.4.1
(The Divergence Test)

a) If $\lim\limits_{k\to+\infty} u_k \neq 0$, then the series $\sum u_k$ ***diverges***

b) If $\lim\limits_{k\to+\infty} u_k = 0$, then the series $\sum u_k$ may either ***converge*** or ***diverge***.

This theorem tells us when a series diverges
The alternative form of part (a) is sufficiently important that we state it separately

THEOREM 11.4.2

If the series $\sum u_k$ converges, then

$$\lim_{k\to+\infty} u_k = 0$$

Example

The series

$$\sum_{k=1}^{\infty}\frac{k}{k+1}=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+.......+\frac{k}{k+1}+....$$

Diverges since

$$\lim_{k\to\infty}\frac{k}{k+1}=\lim_{k\to\infty}\frac{1}{1+\frac{1}{k}}=1\neq 0$$

Algebraic Properties of Infinite Series

THEOREM 11.4.3

a) If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum(u_k+v_k)$ and $\sum(u_k-v_k)$ are convergent series and the sums of these series are related by

$$\sum_{k=1}^{\infty}(u_k+v_k)=\sum_{k=1}^{\infty}u_k+\sum_{k=1}^{\infty}v_k$$

$$\sum_{k=1}^{\infty}(u_k-v_k)=\sum_{k=1}^{\infty}u_k-\sum_{k=1}^{\infty}v_k$$

b) If c is a nonzero constant , then the series $\sum u_k$ and $\sum cu_k$ both *converge* or both *diverge*. In this case of *convergence*, the sums are related by

$$\sum_{k=1}^{\infty}cu_k=c\sum_{k=1}^{\infty}u_k$$

c) Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer K, the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \ldots\ldots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \ldots\ldots$$

both converge or both diverge.

Example

Find the sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right)$$

Solution: The series

$$\sum_{k=1}^{\infty}\frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \ldots\ldots\ldots\ldots$$

is a convergent series (a = ¾, r = ¼) and the series

$$\sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \ldots\ldots\ldots\ldots$$

is also a convergent series (a = 2, r = 1/5).

Thus, from Theorem 11.4.3(a) and 11.3.3 the given series converges and

$$\sum_{k=1}^{\infty}\frac{3}{4^k} - \sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = \frac{3/4}{1-1/4} - \frac{2}{1-1/5} = -\frac{3}{2}$$

THEOREM 11.4.4

(The Integral Test)

Let $\sum u_k$ be a series with positive terms, and let f(x) be the function that results when k is replaced by x in the formula for u_k.

If f is decreasing and continuous on the interval $[a, +\infty]$, then

$$\sum_{k=1}^{\infty} u_k \text{ and } \int_a^{\infty} f(x)dx$$

both converge or both diverge.

This theorem allows us to study the convergence of a series by studying a related integral.

This integral is the improper integral over the interval [1, +∞) and the function is the kth term of the series.

It is interesting that the integral represents a continuous phenomenon, while the summation a discrete one!

EXAMPLE

Use the integral test to determine whether the following series converge or diverge

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

If we replace k by x in the general term 1/k, we obtain the function f(x) = 1/x.

Since

$$\int_1^{+\infty} \frac{1}{x} dx = \lim_{l \to +\infty} \int_1^{l} \frac{1}{x} dx$$

$$= \lim_{l \to +\infty} \left[\ln l - \ln 1 \right] = +\infty$$

the integral diverges and consequently so does the series

p-Series

***p*-series** or **hyper-harmonic series** is an infinite series of the form.

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \qquad where\ p = 1$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots \qquad where\ p = \frac{1}{2}$$

The following theorem tells us whether a p-series converges or diverges

THEOREM 11.4.5
(Convergence of p-series)

$$\sum_{k=1}^{\infty}\frac{1}{k^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+......+\frac{1}{k^p}+......$$

converges if $p>1$ and diverges if $0<p\leq 1$

EXAMPLE

$$1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+.......+\frac{1}{\sqrt[3]{k}}+.......$$

Since $p=\frac{1}{3}<1$ so p-series diverges

Lecture # 43

ADDITIONAL CONVERGENCE TESTS

In this section we shall develop some additional convergence tests for series with positive terms.

THEOREM 11.5.1
(The Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with nonnegative terms and suppose that

$a_1 \le b_1, a_2 \le b_2, a_3 \le b_3, ..., a_k \le b_k$

(a) If the "bigger series" $\sum b_k$ converges than the "smaller series" $\sum a_k$ also converges.

(b) If the "smaller series" $\sum a_k$ diverges, then the "bigger series" $\sum b_k$ also diverges.

This test basically compares two series with each other, and if the series in which each term is bigger than that of the other series converges, then so does the smaller one.

If the series in which each term is smaller than that of the other series diverges, then so does the bigger one.

We will see more of this later. For now, we shall use the comparison test to develop some other tests that are easier to apply.

THEOREM 11.5.2
(The Ratio Test)

Let $\sum u_k$ be a series with positive terms and suppose that $\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$

(a) If ρ <1, the series converges.
(b) If ρ >1 or ρ =+ ∞ , the series diverge.
(c) If ρ =1, the series may converge or diverge, so that another test must be tried.

Use the ratio test to determine whether the following series converge or diverge.

a) $\sum_{k=1}^{\infty} \frac{1}{k!}$ b) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

c) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ d) $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$

a)

$$\rho = \lim_{k\to+\infty} \frac{u_k + 1}{u_k} = \lim_{k\to+\infty} \frac{1/(k+1)!}{1/k!}$$

$$= \lim_{k\to+\infty} \frac{k!}{(k+1)!} = \lim_{k\to+\infty} \frac{k!}{(k+1)k!}$$

$$= \lim_{k\to+\infty} \frac{1}{(k+1)} = 0 < 1$$

The series converges.

b)

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k}$$

$$= \frac{1}{2} \lim_{k\to+\infty} \frac{k+1}{k} = \frac{1}{2} < 1$$

The series converges

c)

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k}$$

$$= \lim_{k\to+\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to+\infty} \left(1 + \frac{1}{k}\right)^k$$

$$= e > 1$$

The series diverges

d)

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{2k!}$$

$$= \lim_{k\to+\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4} \right)$$

$$= \frac{1}{4} \lim_{k\to+\infty} (2k+2)(2k+1) = +\infty$$

The series diverges

Determine whether the series

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots\ldots\ldots+\frac{1}{2k-1}+\ldots\ldots$$

converges or diverges.

The ratio test is of no help since

$$\rho=\lim_{k\to+\infty}\frac{u_{k+1}}{u_k}=\lim_{k\to+\infty}\frac{1}{2(k+1)-1}\cdot\frac{2k-1}{1}$$
$$=\lim_{k\to+\infty}\frac{2k-1}{2k+1}=1$$

However, the integral test proves that the series diverges since

$$\int_1^{+\infty}\frac{dx}{2x-1}=\lim_{l\to+\infty}\int_1^{l}\frac{dx}{2x-1}$$
$$=\lim_{l\to+\infty}\frac{1}{2}\ln(2x-1)\Big]_1^{l}=+\infty$$

So, this series diverges.

Sometimes the following result is easier to apply than ratio test

Theorem 11.5.3
(The Root Test)
Let $\sum u_k$ be a series with positive terms and suppose that

(a) If ρ <1, the series converges.
(d) If ρ >1 or ρ =+ ∞ , the series diverge.
(e) If ρ =1, the series may converge or diverge, so that another test must be tried.

Example

$$\sum_{k=2}^{\infty}\left(\frac{4k-5}{2k+1}\right)^k$$

Use the root test to determine whether the following series converge or diverge.

Solution: The series diverges, since $\rho=\lim_{k\to\infty}\left(u_k\right)^{1/k}=\lim_{k\to\infty}\frac{4k-5}{2k+1}=2>1$

We have not really talked about the comparison test so far. To apply comparison test, we use an alternative version of this test that is easier to work with.

INFORMAL PRINCIPLE 11.6.1
Constant terms in the denominator of u_k can usually be deleted without affecting the convergence or divergence of the series.

Example
Use the above informal principle to help guess whether the following series converge or diverge

$$\sum_{k=1}^{\infty} \frac{1}{2^k + 1}$$

Solution: Deleting the constant 1 suggests that $\sum_{k=1}^{\infty} \frac{1}{2^k + 1}$ & $\sum_{k=1}^{\infty} \frac{1}{2^k}$ behaves alike.

The modified series is a convergent geometric series, so the given series is likely to converge.

INFORMAL PRINCIPLE 11.6.2
If a polynomial in k appears as a factor in the numerator or denominator of u_k, all but the highest power of k in the polynomial may usually be deleted without affecting the convergence or divergence of the series.

Example

Use the above principle to help guess whether the following series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 + 2k}}$ converge or diverge.

Solution: Deleting the term 2k suggests that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 + 2k}}$ & $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

behaves alike
Since the modified series is a convergent p-series (p=3/2) the given series is likely to converge.

The Limit Comparison Test
The following result can be used to establish convergence or divergence by examining the limit of the ratio of the general term of the series in question with general term of a series whose convergence properties are known.

THEOREM 11.6.3
(The Limit Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that $\rho = \lim_{k \to \infty} \frac{a_k}{b_k}$

If ρ is finite and ρ >0, then the series both converge or both diverge.

Example

Use the limit comparison test to determine whether the following series $\sum_{k=1}^{\infty} \frac{1}{2k^2 - k}$ converge or diverge.

Solution: The given series behaves like the series $\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2}$

$a_k = \frac{1}{2k^2 - k}$ $b_k = \frac{1}{2k^2}$ which is constant times a convergent p-series.
Thus the given series is likely to converge.
By applying theorem 11.6.3 with $\rho = \lim_{k \to +\infty} \frac{a_k}{b_k} = \lim_{k \to +\infty} \frac{2k^2}{2k^2 - k} = \lim_{k \to +\infty} \frac{2}{2 - 1/k} = 1$

Since ρ is finite and positive, it follows from Theorem 11.6.3 that the given series converge.

The Comparison Test
We shall now discuss some techniques for applying the comparison test (Theorem 11.5.1).
There are two basic steps required to apply the comparison test to a series $\sum u_k$ of positive terms:

- Guess whether the series $\sum u_k$ converges or diverges.

- Find a series that proves the guess to be correct. Thus if the guess is divergence, we must find a divergent series whose terms are "smaller" than the corresponding terms of $\sum u_k$, and if the guess is convergence,

we must find a convergent series whose terms are "bigger" than the corresponding terms of $\sum u_k$.

Example
$$\sum_{k=1}^{\infty} \frac{1}{k - 1/4}$$
Use the comparison test to determine whether the following series converge or diverge.

Solution:
As $\frac{1}{k - 1/4} > \frac{1}{k}$
Note that the series behaves like the divergent harmonic series, and hence is likely to diverge. Thus our goal is to find a divergent series that is "smaller" than the given series. We can do this by dropping the constant -1/4 in the denominator, thereby decreasing the size of the general term:

for k=1, 2, ……..

Thus the given series diverges by the comparison test, since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Lecture # 44

Alternating Series; Conditional convergence

- So far we have seen infinite series that have positive terms only.
- We have also defined the Limit and the Sum of such infinite series
- Now we look at series that have terms which have alternating signs, known as **Alternating series**

Example

1) 1-1+1-1+…
2) 1+2-3+4….

More generally, an alternating series has one of the two following forms

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \dots \qquad (1)$$

$$\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + a_4 - \dots \qquad (2)$$

Note that all the terms a_k are to be taken as being positive.
The following theorem is the key result on convergence of alternating series.

Theorem11.7.1
(Alternating Series Test)
An alternating series of either form (1) or (2) converges if the following two conditions are satisfied:

(a) $a_1 > a_2 > a_3 \dots\dots > a_k > \dots\dots$
(b) $\lim_{k \to +\infty} a_k = 0$

This theorem tells us when an alternating series converges.

Example

Use the alternating series test to show that the following series converge

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$$

Solution:

The two conditions in the alternating series test are satisfied since $a_k = \frac{1}{k} > \frac{1}{k+1} = a_{k+1}$

and $\lim_{k \to +\infty} a_k = \lim_{k \to +\infty} \frac{1}{k} = 0$

Note that this series is the Harmonic series, but alternates. It's called the Alternating Harmonic Series.
The harmonic series diverges. The alternating Harmonic series converges.
Now we will look at the errors involved in approximating an alternating series with a partial sum.

Theorem 11.7.2
If an alternating series satisfies the conditions of the alternating series test, and if the sum S of the series is approximated by the nth partial sum S_n , thereby resulting in an error of $S- S_n$,then

$$| S- S_n | < a_{n+1}$$

Moreover, the sign of error is the same as that of the coefficient of a_{n+1} in the series.

Example
The alternating series

$$1 – 1/2 + 1/3 – 1/4 +\dots\dots + (-1)^{k+1} 1/k +\dots\dots$$

satisfies the condition of the alternating series test ; hence the series has a sum S, which we know must lie between any two successive partial sums. In particular, it must lie between

$$S_7 = 1 – 1/2 + 1/3 – 1/4 + 1/5 – 1/6 + 1/7 = 319/420$$

And

$$S_7 = 1 – 1/2 + 1/3 – 1/4 + 1/5 – 1/6 + 1/7 – 1/8 = 533/840$$

So

$$533/840 < S < 319/420$$

If we take S = ln2 then

$$533/840 < \ln2 < 319/420$$
$$0.6345 < \ln2 < 0.7596$$

The value of ln2, rounded to four decimal places, is .6931, which is consistent with these inequalities. It follows from Theorem 11.7.2 that

$$| \ln2 – S_7 | = | \ln2 – 319/420 | < a_8 =1/8$$

And

$$| \ln2 – S_8 | = | \ln2 – 533/840 | < a_9 =1/9$$

Absolute and Conditional Convergence

The series

$$1 – 1/2 - 1/2^2 +1/2^3 + 1/2^4 – 1/2^5 – 1/2^6 +\dots\dots\dots\dots$$

does not fit in any of the categories studied so far, it has mixed signs , but is not alternating. We shall now develop some convergence tests that can be applied to such series.

Definition 11.7.3

A series $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \text{..........} + u_k + \text{.......}$

is said to converge absolutely, if the series of absolutes values

$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \text{..........} + |u_k| + \text{.......}$ converges.

Example

The series

$1 - 1/2 - 1/2^2 + 1/2^3 + 1/2^4 - 1/2^5 - 1/2^6 + \ldots\ldots\ldots$

Converges absolutely since the series of absolute values

$1 + 1/2 + 1/2^2 + 1/2^3 + 1/2^4 + 1/2^5 + 1/2^6 + \ldots\ldots\ldots$

is a convergent geometric series.

On the other hand, the alternating harmonic series

$1 - 1/2 + 1/3 - 1/4 + 1/5 - \ldots\ldots\ldots$

does not converge absolutely since the series of absolute values

$1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots\ldots\ldots$

diverges.

Absolute convergence is of importance because of the following theorem.

Theorem 11.7.4

If the series $\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \text{..........} + |u_k| + \text{.......}$

Converges, then so does the series $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \text{..........} + u_k + \text{.......}$

In other words, if a series converges absolutely, then it converges.

Since the series

$1 - 1/2 - 1/2^2 + 1/2^3 + 1/2^4 - 1/2^5 - 1/2^6 + \ldots\ldots\ldots$

Converges absolutely. It follows from Theorem 11.7.4 that the given series converges.

Example

Show that the series $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converges.

Solution: Since $|\cos k| \leq 1$ for all k,

$$\left| \frac{\cos k}{k^2} \right| \leq \frac{1}{k^2}$$

thus

$$\left|\sum_{k=1}^{\infty}\frac{\cos k}{k^2}\right| \sum_{k=1}^{\infty}\frac{\cos k}{k^2}$$

converges by the comparison test, and consequently

$$\sum_{k=1}^{\infty}\frac{\cos k}{k^2} \quad \text{converges.}$$

If $\sum |u_k|$ diverges, no conclusion can be drawn about the convergence or divergence of $\sum u_k$

For example, consider the two series

1 - 1/2 + 1/3 - 1/4 +…….+ $(-1)^{k+1}$ 1/k +……. (A)

- 1 - 1/2 - 1/3 - 1/4 -…….- 1/k -……. (B)

Series (A), the alternating harmonic series, converges, whereas series (B), being a constant times the harmonic series, diverges.

Yet in each case the series of absolute values is

1 + 1/2 + 1/3 +…… + 1/k + ……..

which diverges. A series such as (A), which is convergent, but not absolutely convergent, is called **conditionally convergent**.

Theorem 11.7.5

(Ratio Test for Absolute Convergence)

Let $\sum u_k$ be a series with nonzero terms and suppose that $\rho = \lim_{k\to\infty}\frac{|u_{k+1}|}{|u_k|}$

(a) If ρ <1, the series $\sum u_k$ converges absolutely and therefore converges.

(b) If ρ >1 or $\rho = +\infty$, then the series $\sum u_k$ diverges.

(c) If ρ =1, no conclusion about convergence or absolute convergence can be drawn from this test.

EXAMPLE

The series

$$\sum_{k=1}^{\infty}(-1)^k\frac{2^k}{k!}$$

converges absolutely since

$$\rho = \lim_{k\to+\infty}\frac{|u_{k+1}|}{|u_k|} = \lim_{k\to+\infty}\frac{2^{k+1}}{(k+1)!}\cdot\frac{k!}{2^k}$$

$$= \lim_{k\to+\infty}\frac{2}{k+1} = 0 < 1$$

Power Series in x

If c_0, c_1, c_2, ………. are constants and x is a variable, then a series of the form

$\sum_{k=o}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + ... + c_k x^k + ...$ is called a power series in x.

Some examples of power series in x are

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + ...$$

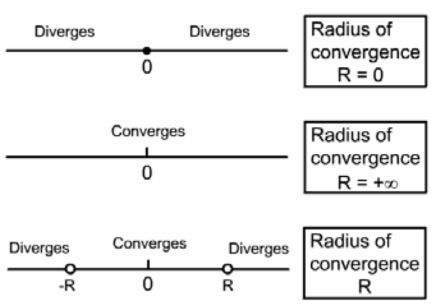
$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Theorem 11.8.1
For any power series in x, exactly one of the following is true:

(a) The series converges only for x=0
(b) The series converges absolutely (and hence converges) for all real values of x.
(c) The series converges absolutely (and hence converges) for all x in some finite open interval (-R, R), and diverges if x<-R or x>R. At either of the points x = R or x = -R, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

Radius and Interval of Convergence
Theorem 11.8.1 states that the set of values for which a power series in x converges is always an interval centered at 0; we call this the interval of convergence, corresponding to this interval series has radius called radius of convergence.



Example:
Find the interval of convergence and radius of convergence of the following power series.

$$\sum_{k=1}^{\infty} x^k$$

Solution: We shall apply the ratio test for absolute convergence. We have

$$\rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\right| = \lim_{k\to\infty}|x| = |x|$$

So the ratio test for absolute convergence implies that the series converges absolutely if $\rho = |x| < 1$ and diverges if $\rho = |x| > 1$ .The test is inconclusive if $|x| = 1$ (i.e. x = 1 or x = -1), so convergence at these points must be investigated separately . At these points the series becomes

$$\sum_{k=0}^{\infty} 1^k = 1+1+1+1..... \qquad x=1$$

$$\sum_{k=0}^{\infty} (-1)^k = 1-1+1-1+...... \qquad x=-1$$

Both of which diverge; thus, the interval of convergence for the given power series is (-1,1), and the radius of convergence is R = 1.

Power series in x-a

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + ...+ c_k(x-a)^k + ...$$

This series is called power series in x-a. Some examples are

$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k+1} = 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + ... \qquad a=1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x+3)}{k!} = 1 - (x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + ... \qquad a=-3$$

Theorem 11.8.1

For any power series in $\sum c_k(x-a)^k$, exactly one of the following is true:

(a) The series converges only for x=a
(b) The series converges absolutely (and hence converges) for all real values of x.
(c) The series converges absolutely (and hence converges) for all x in some finite open interval (a-R,a+R), and diverges if x<a-R or x>a+R. At either of the points x =a-R or x =a+R, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

It follows from this theorem that now interval of convergence is centered at x=a.

Lecture # 45

Taylor and Maclaurin Series

One of the early applications of calculus was the computation of approximate numerical values for functions such as sin x, ln x, and ex. One common method for obtaining such values is to approximate the function by polynomial, then use that polynomial to compute the desired numerical values.

Problem

Given a function **f** and a point **a** on the x-axis, find a polynomial of specified degree that best approximates the function **f** in the "vicinity" of the point **a**.
Suppose that we are interested in approximating a function f in the vicinity of the point a=0 by a polynomial

$$P(x) = c_0 + c_1 x + c_2 x^2 + \dots\dots\dots\dots + c_n x_n \qquad (1)$$

Because P(x) has n+1 coefficients, it seems reasonable that we should be able to impose n+1 conditions on this polynomial to achieve a good approximation to f(x). Because the point a=0 is the center of interest, our strategy will be to choose the coefficients of P(x) so that the value of P and its first n derivatives are the same as the value of f and its first n derivatives at a=0. By forcing this high degree of "match" at a=0, it is reasonable to hope that f(x) and P(x) will remain close over some interval (possibly quite small) centered at a=0. Thus, we shall assume that f can be differentiated n times at 0, and we shall try to find the coefficients in (1) such that

$$f(0) = p(0),\ f'(0) = p'(0),\ f''(0) = p''(0), \dots\dots\dots,\ f^n(0) = p^n(0) \qquad (2)$$

We have

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots\dots\dots + c_n x^n$$
$$p'(x) = c_1 x + 2c_2 x + 3c_3 x^2 + \dots\dots\dots + nc_n x^{n-1}$$
$$p''(x) = 2c_2 + 3\cdot 2c_3 x + \dots\dots\dots + n(n-1)c_n x^{n-2}$$
$$p'''(x) = 3.2c_3 + \dots\dots\dots + n(n-1)(n-2)c_n x^{n-3}$$

.

.

$$.p^n(x) = n(n-1)(n-2)\dots\dots(1)c_n$$

Thus, to satisfy (2) we must have

$$f(0) = p(0) = c_0$$
$$f'(0) = p'(0) = c_1$$
$$f''(0) = p''(0) = 2c_2 = 2!c_2$$
$$f'''(0) = p'''(0) = 3.2c_3 = 3!c_3$$

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$$f^n(0) = p^n(0) = n(n-1)(n-2)\dots\dots(1)c_n = n!c_n$$

Which yields the following values for the coefficients of P(x)?

$$c_0 = f(0), c_1 = f'(0), c_2 = \frac{f''(0)}{2!}, c_3 = \frac{f'''(0)}{3!}, ..., c_n = \frac{f^n(0)}{n!}$$

MACLAURIN POLYNOMAILS

If f can be differentiated n times at 0 , then we define the *nth* Maclaurin Polynomial for f to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n$$

This polynomial has the property that its value and the values of its first n derivatives match the value of $f(x)$ and its first n derivatives when $x = 0$.

Example

Find the Maclaurin polynomials P_o, P_1, P_2, P_3, and P_n for e^x .

Solution: Let $f(x) = e^x$, Thus

$$f'(x) = f''(x) = f'''(x) = ... = f^n(x) = e^x$$

and

$$f(0) = f'(0) = f''(0) = f'''(0) = ... = f^n(0) = e^0 = 1$$

Therefore,

$$p_0(x) = f(0) = 1$$

$$p_1(x) = f(0) + f'(0)x = 1 + x$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2!}$$

.

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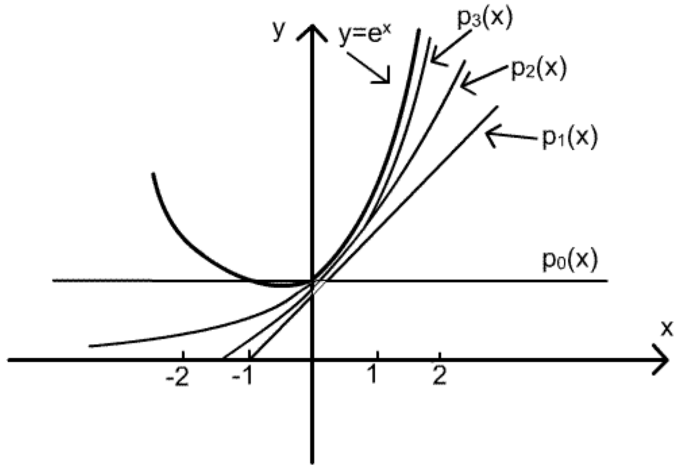
$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^n(0)}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$$



Graphs of e^x and first four Maclaurin polynomials are shown here. Note that the graphs of $P_1(x)$,P_2 (x), P_3 (x) are virtually indistinguishable from the graph of e^x near the origin, so these polynomials are good approximations of e^x near the origin.
But away from origin it does not give good approximation.
To obtain polynomial approximations of f(x) that have their best accuracy near a general point x=a, it will be convenient to express polynomials in powers of x-a, so that they have the form

$$P(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots\ldots\ldots\ldots.. + cn(x-a)^n$$

DEFINITION 11.9.2

If *f* can be differentiated *n* times at 0, then we define the *nth* Taylor polynomial for *f* about *x* = *a* to be

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots\ldots.. + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor and Maclaurin Series

For a fixed value of x near a, one would expect that the approximation of f(x) by its Taylor polynomial P_n (x) about x=a should improve as n increases, since increasing n has the effect of matching higher and higher derivatives of f(x) with those of P_n (x) at x=a. Indeed, it seems plausible that one might be able to achieve any desired degree of accuracy by choosing n sufficiently large; that is, the value of $P_n(x)$ might actually converge to f(x) as $n \to \infty$

DEFINITION 11.9.3

If *f* has derivatives of all orders at *a* , then we define the Taylor Series for *f* about $x = a$ to be

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{2}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \ldots\ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \ldots\ldots$$

In the special case where a=0, the **Taylor series** for **f** is called the **Maclaurin series** for **f**.

Find the Maclaurin Series for

a) e^x b) sinx

a) The nth Maclaurin polynomial for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!}$$

Thus, the Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\ldots + \frac{x^k}{k!} + \ldots\ldots$$

b) Let $f(x) = \sin x$,

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$

Since $f''''(x) = \sin x = f(x)$, the pattern 0,1,0,-1 will repeat over and over as we evaluate successive derivatives at 0.

Therefore , the successive Maclaurin polynomials for sinx are

$p_0(x) = 0$

$p_1(x) = 0 + x$

$p_2(x) = 0 + x + 0$

$p_3(x) = 0 + x + 0 - x^3/3!$

$p_4(x) = 0 + x + 0 - x^3/3! + 0$

$p_5(x) = 0 + x + 0 - x^3/3! + 0 + x^5/5!$

$p_6(x) = 0 + x + 0 - x^3/3! + 0 + x^5/5! + 0$

$p_7(x) = 0 + x + 0 - x^3/3! + 0 + x^5/5! + 0 - x^7/7!$

Because of the zero terms, each even-numbered Maclaurin polynomial [after $p_0(x)$] is the same as the odd-number Maclaurin polynomial; that is

$$p_{2n+1}(x) = p_{2n+2}(x)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(n = 0,1,2,3,.......)

Thus, the Maclaurin series for sinx is

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + ...$$