NUMERICAL LINEAR ALGEBRA

HW – 3.1
HW - 3.2
HW - 3.3
HW – 3.4
HW – 3.5
HW – 3.6
HW – 3.7

Murat Çabuk

HW - 3.1

What you can say and prove about eigenvalues of a symmetric positive definite matrix?

If all eigenvector positive $\rightarrow$proof

If all eigenvector real $\rightarrow$proof

HW - 3.2

What condition of A makes A^TA nonsingular? If so, that is A^TA is non-singular than a unique solution exists.

Answer

A is an A_{mxn} matrix. After multiplication A^T and A, rank will be nxn, it is same as rank of (A).

A^TA is nonsingular if and only if the rank of A is n, and if A has linearly independent columns.

HW - 3.3

In QR Factorization, $||Qx||_2 = ||x||_2$ → Prove it

Answer

Q is an orthogonal matrix. It means that

$Q^TQ = QQ^T = I$ and $Q^T=Q^{-1}$

Orthogonal Matrices preserve the 2 norms of the vectors they multiply.

$||Qx||^2_2 = (Qx)^T (Qx) = x^TQ^TQx=x^Tx=||x||^2_2$

HW - 3.4

In QR Factorization, any H matrix has all the following properties

a. H is symmetric

b. H is Orthogonal

Please prove all.

Answer

H is symmetric : $H^T = H$

H is orthogonal : $H^{-1}=H^T$

Example If A is any matrix (square or not), then A^T square. A^TA is also symmetric

because $(A^TA)^T=A^TA^{TT}=A^TA$

In QR Factorization $H = I - \frac{2}{u^T u} uu^T = I + auu^T$

$H^T = (I + auu^T)^T = I^T + a(uu^T)^T = I + a(u^T)^T uT = I + auu^T = H$ It means H is symmetric.

HW - 3.5

Implement the power method to compute the largest eigenvalue of a given matrix w.r.t. a desired precision. That is, your function should take as input a matrix and a value for precision (ex. myPowerMethod(A, 1e-5)).

```
function [x,c1]= myPowerMethod(A,tolerance)

err=1;

x=transpose([1 2 3]);

while (err- tolerance >= 0)

y = A*x;

c1 = norm(y,inf);

y = y/c1;

err = norm(y-x);

x=y;

end
```

HW - 3.6

Make a summary of solving a set of linear equations $A_{mxn}\ x_{nx1} = b_{mx1}$

a. Existence of a unique solution, how (explain all of the methods one by one) for m = n

b. Existence of a unique solution, how (explain all of the methods one by one) for m > n

c. Existence of a unique solution, how (explain all of the methods one by one) for m < n

Linear Equations

Direct Methods

-Cramer Rule
-Gauss Elimination
-LU / LDU Factorization
-QR (m<>n)

Iterative Methods

Stationary Methods
-Jacobi
-Gauss-Seidel
-Relaxation Methods

Non-Stationary Methods
-CG (Symmetric Positive -Definite System)
-GMRES (For Non-positive Definite even indefinite System)

Direct Methods

Cramer Rule

Cramer Rule is one of the direct method.

$$Ax=b\, x=A^{-1}b$$

$$A^{-1}=\frac{1}{Det(A)}Adj(A)$$

$$\begin{cases} a_1x+b_1y &= c_1 \\ a_2x+b_2y &= c_2 \end{cases} \qquad \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2-b_1c_2}{a_1b_2-b_1a_2}$$

Determinant of [[c1 b1], [c2 b2]]

Determinant of A

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2-c_1a_2}{a_1b_2-b_1a_2}$$

Gauss Elimination

Gaussian elimination (also known as row reduction) is an algorithm for solving systems of linear equations. It is usually understood as a sequence of operations performed on the associated matrix of coefficients.

To perform row reduction on a matrix, one uses a sequence of elementary row operations to modify the matrix until the lower left-hand corner of the matrix is filled with zeros, as much as possible. There are three types of elementary row operations: 1) Swapping two rows, 2) Multiplying a row by a non-zero number, 3) Adding a multiple of one row to another row. Using these operations, a matrix can always be transformed into an upper triangular matrix, and in fact one that is in row echelon form. (Source: Wikipedia)

Example

System of equations	Row operations	Augmented matrix
$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$		$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ 2y + z &= 5 \end{aligned}$	$\begin{gathered} L_2 + \frac{3}{2}L_1 \to L_2 \\ L_3 + L_1 \to L_3 \end{gathered}$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ -z &= 1 \end{aligned}$	$L_3 + -4L_2 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$
The matrix is now in echelon form (also called triangular form)		
$\begin{aligned} 2x + y &= 7 \\ \frac{1}{2}y &= \frac{3}{2} \\ -z &= 1 \end{aligned}$	$\begin{gathered} L_2 + \frac{1}{2}L_3 \to L_2 \\ L_1 - L_3 \to L_1 \end{gathered}$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1/2 & 0 & 3/2 \\ 0 & 0 & -1 & 1 \end{array}\right]$
$\begin{aligned} 2x + y &= 7 \\ y &= 3 \\ z &= -1 \end{aligned}$	$\begin{gathered} 2L_2 \to L_2 \\ -L_3 \to L_3 \end{gathered}$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$
$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= -1 \end{aligned}$	$\begin{gathered} L_1 - L_2 \to L_1 \\ \frac{1}{2}L_1 \to L_1 \end{gathered}$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

LU / LDU Factorization

$$A = LU$$

L= Lower triangular matrix

U= Upper triangular matrix

$$Ax = b \qquad Lc = b \qquad Ux = c \qquad Ax = LUx = L(Ux) = Lc = b$$

How to find L and U?

Example

$$A = \begin{bmatrix} 5 & -1 & 2 \\ 10 & 3 & 7 \\ 15 & 17 & 19 \end{bmatrix}$$

Lets find L and U for this matrix.

$$\underset{\text{L3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}} \underset{\text{L2}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}} \underset{\text{L1}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\text{A}}{\begin{bmatrix} 5 & -1 & 2 \\ 10 & 3 & 7 \\ 15 & 17 & 19 \end{bmatrix}} = \underset{\text{U}}{\begin{bmatrix} 5 & -1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix}}.$$

So, what is L?

$$L1^{-1}.L2^{-1}.L3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} = L$$

Another full example

$$A=\begin{bmatrix} 6 & 0 & 2 \\ 24 & 1 & 8 \\ -12 & 1 & -3 \end{bmatrix} \quad X \quad \overset{[X]}{\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}} \quad = \quad \overset{[B]}{\begin{bmatrix} 4 \\ 9 \\ -16 \end{bmatrix}}$$

$$\begin{bmatrix} \boxed{6} & 0 & 2 \\ 24 & 1 & 8 \\ -12 & 1 & -3 \end{bmatrix} \xrightarrow{R2-4R1} \begin{bmatrix} \boxed{6} & 0 & 2 \\ 0 & 1 & 0 \\ -12 & 1 & -3 \end{bmatrix} \xrightarrow{R3+2R1} \begin{bmatrix} 6 & 0 & 2 \\ 0 & \boxed{1} & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R3-R2} \begin{bmatrix} 6 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}.$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Lc = b

$$\begin{bmatrix} 1 & & \\ 4 & 1 & \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 19 \\ -6 \end{bmatrix}$$

we can find c1, c2 and c3 using <u>forward substitution</u>

$$\begin{aligned} c_1 &= 4 \\ 4c_1 + c_2 &= 19 \implies 4(4) + c_2 = 19 \implies c_2 = 3 \\ -2c_1 + c_2 + c_3 &= -6 \implies -2(4) + (3) + c_3 = -6 \implies c_3 = -1. \end{aligned}$$

Ux = c

$$\begin{bmatrix} 6 & 0 & 2 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

we can find x1, x2 and x3 using <u>back substitution</u>

$$\begin{aligned} x_3 &= -1 \\ x_2 &= 3 \\ 6x_1 + 2x_3 &= 4 \implies 6x_1 + 2(-1) = 4 \implies x_1 = 1 \end{aligned}$$

For m > n (Overdetermined Systems)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \qquad (1) \\ a_{21}x_1 + a_{22}x_2 &= b_2 \qquad (2) \\ a_{31}x_1 + a_{32}x_2 &= b_3 \qquad (3) \end{aligned}$$

(1), (2) and (3) represent three straight lines. If these lines intersect each other at the same point, the system has one solution, otherwise it has no solutions in which case

we have to find a least squares approximation x so that ||b − A x|| is as small as possible.

Example:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

We therefore seek an approximate solution $x = [x_1 , x_2]^T$, so that ||b − Ax|| is a minimum.

$$\hat{\mathbf{b}} = \hat{x}_1 \mathbf{a}_1 + \hat{x}_2 \mathbf{a}_2$$

$$\mathbf{e} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \hat{x}_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \hat{x}_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

An alternative formulation is as follows: $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$

In this step we have two options:

– minimization by employing derivatives, or

– geometrical considerations and matrices.

Minimization by employing derivatives

We subsequently minimize $||e||^2$

$$||\mathbf{e}||^2 = (1 - \hat{x}_1 - \hat{x}_2)^2 + (2 - \hat{x}_1 - \hat{x}_2)^2 + (3 - \hat{x}_2)^2$$

$$\frac{\partial ||\mathbf{e}||^2}{\partial \hat{x}_1} = 0 \quad \text{and} \quad \frac{\partial ||\mathbf{e}||^2}{\partial \hat{x}_2} = 0$$

And result

$4x_1 + 4x_2 = 6$

$4x_1 + 6x_2 = 12$

and has the solution $[x_1 , x_2]^T = [-1.5, 3.0]$

Geometrical considerations and matrices

Example:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \hat{\mathbf{b}} = \hat{x}_1 \mathbf{a}_1 + \hat{x}_2 \mathbf{a}_2 \qquad \mathbf{e} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \hat{x}_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \hat{x}_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$A^T e = 0$ **or** $A^T (b - Ax) = 0$

$A^T Ax = A^t b$

For the above example we have that:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A^T \mathbf{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

In this step we can use many methods like Gauss,Cramer Rule, LU / LDU Factorization.. etc for this equations.

If we solve the problem we'll see this result

$$\hat{\mathbf{x}} = [-1.5, 3.0]^T$$

For m < n (Underdetermined Systems)

Least Squares Solution

we can write the solution as x = At for some vector t of m unknowns.

$A(A^Tt) = b$ $(A A^T)t = b$

After these steps we can solve equation for t

The value of t into x = A t to get the least-squares solution x of the original system.

The least squares solution can be written as x = At, and in fact the least squares solution is precisely the unique solution which can be written this way.

Example

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$(A A^T)t = b$

$$\left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} t_1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$[2]\,[t_1] = [2]$ $[t_1] = [1]$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

QR Factorization (m> n and m<n)

$H_n \ldots\ldots H_2.H_1.A=R$

$Q = H_n \ldots\ldots H_2.H_1$

$Q.A=R \qquad Q^T.Q.A= Q^T.R \qquad I.A= Q^T.R \qquad A= Q^T.R$

$A^T.Ax=A^T.b \qquad R^T.Rx=R^TQb \qquad Rx=Q^Tb$

Calculating Q and R, we learned Householder Transformation Method. There is also most used method is Gram-Schmidt.

Householder Method

Example

$$A=\begin{bmatrix}1 & -1 & 4\\ 1 & 4 & -2\\ 1 & 4 & 2\\ 1 & -1 & 0\end{bmatrix}$$

V1 = a1 – sign(a11)||a1||e1

$$v1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-2\begin{bmatrix}1\\ \boxed{0}\\ \boxed{0}\\ \boxed{0}\end{bmatrix}=\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$$

Taken as zero

$$H_1=I-2\frac{v1.v1^T}{v1^T.v1} \qquad H_1=\begin{bmatrix}1/2 & 1/2 & 1/2 & 1/2\\ 1/2 & 1/2 & -1/2 & -1/2\\ 1/2 & -1/2 & 1/2 & -1/2\\ 1/2 & -1/2 & -1/2 & 1/2\end{bmatrix}$$

||a1|| = norm first column of A

$$A^1=H1*A \qquad A^1=\begin{bmatrix}2 & 3 & 2\\ 0 & 0 & 0\\ 0 & 0 & 4\\ 0 & -5 & 2\end{bmatrix}$$

For next step we need submatrix of A1

$$A^1=\begin{bmatrix}0 & 0\\ 0 & 4\\ -5 & 2\end{bmatrix} \qquad v2=\begin{bmatrix}0\\0\\-5\end{bmatrix}-\begin{bmatrix}5\\0\\0\end{bmatrix}=\begin{bmatrix}-5\\0\\-5\end{bmatrix}$$

$v2 = a1^{(1)} – \mathrm{sign}(A^{(1)}_{(2,2)})||a1^{(1)}||e1$

$$H_2=\begin{bmatrix} 1 & 0 \\ 0 & I-2\dfrac{v2.v2^T}{v2^T.v2} \end{bmatrix} \qquad H_2=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$Q=(H_2 * H_1)^{-1} \;=\; (H_2 * H_1)^T \;=\; H_1^T * H_2^T \;=\; H_1 * H_2$$

$$Q=H_1 * H_2=\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

$$H_2 * H_1 * A=\left(\frac{R}{0}\right)=\begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow R \qquad R=\begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

Iterative Methods

Jacobi Method

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\ &\vdots \\ x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}) \end{aligned}$$

We can use the vextor x as initial x = $[x_1,x_2,x_3,........x_n]$

Example

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2 \end{aligned}$$

We can use $x_1 = 0$, $x_2= 0$ and $x_3=0$ as initial approximation because we don't know actual solution.

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200$$
$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}(0) - \tfrac{1}{9}(0) \approx 0.222$$
$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}(0) - \tfrac{1}{7}(0) \approx -0.429$$

We can use new x_1,x_2 and x_3 as new initials.

We will continue until convergence.

n	0	1	2	3	4	5	6	7
x_1	0.000	−0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	−0.429	−0.517	−0.416	−0.421	−0.424	−0.423	−0.423

As we can see in 7^{th} step we had a convergence.

$x_1 = 0.186$, $x_1 = 0.331$, $x_1 = -0.423$

The Gauss-Seidel Method

With the Gauss- Seidel method, on the other hand, you use the new values of each x_i as soon as they are known. That is, once you have determined x_1 from the first equation, its value is then used in the second equation to obtain the new x_2 . Similarly, the new x_1 and x_2 are used in the third equation to obtain the new x_3 , and so on.

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200$$

$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}(-0.200) - \tfrac{1}{9}(0) \approx 0.156.$$

$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}(-0.200) - \tfrac{1}{7}(0.156) \approx -0.508.$$

n	0	1	2	3	4	5
x_1	0.000	−0.200	0.167	0.191	0.186	0.186
x_2	0.000	0.156	0.334	0.333	0.331	0.331
x_3	0.000	−0.508	−0.429	−0.422	−0.423	−0.423

As we can see in 5^{th} step we had a convergence.
$x_1 = 0.186$, $x_2 = 0.331$, $x_1 = -0.423$

Successive Over Relaxation

Method, is a generalization of and improvement on the Gauss-Seidel Method. In this method we use relaxation parameter (w) for improvement. Relaxation parameter should be close to 1.

$$x_i^{(k+1)} = x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(k)}\right)$$

Comparison

$$x_i^{(k+1)} = x_i^{(k)} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i}^{n} a_{ij} x_j^{(k)}\right)$$ Jakobi

$$x_i^{(k+1)} = x_i^{(k)} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(k)}\right)$$ Gauss-Seidel

$$x_i^{(k+1)} = x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(k)}\right)$$ Successive Over Relaxation

HW - 3.7

Figure out how you can use symbolic computation in Matlab and write a method for fixed point iteration to solve f(x) = 0

There is a tool in Matlab for symbolic mathematic. Tool name is “Symbolic Math Toolbox”. We can use two ways to create symbolic variables, syms and sym.

Example

syms x

y = sym('y')

The first command creates a symbolic variable x in the MATLAB workspace with the value x assigned to the variable x. The second command creates a symbolic variable y with value y. Therefore, the commands are equivalent.

Symbolic Expression

for quadratic function like *f = ax 2 + bx + c*

we should symbolic variables a, b, c, and x:

syms a b c x

Then, assign the expression to f:

*f = a*x^2 + b*x + c;*

if we assign a value to a,b and c

a=1

b=3

c=4

and call the eval function

eval(f)

we will get following result

ans =

*x^2 + 3*x + 4*

Symbolic Functions

syms f(x, y)

for x^3y^3

*f(x, y) = x^3*y^3*

now we can take difference

diff(f)

and result

ans(x, y) =

*3*x^2*y^3* = $3x^2y^3$

Symbolic Matrices

$$A=\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

For matrix A

syms a b c

A = [a b c; c a b; b c a]

if run A*A, result is

ans =

*[a^2 + 2*b*c, c^2 + 2*a*b, b^2 + 2*a*c]*

*[b^2 + 2*a*c, a^2 + 2*b*c, c^2 + 2*a*b]*

*[c^2 + 2*a*b, b^2 + 2*a*c, a^2 + 2*b*c]*

if we assign a value to a,b and c

a=2

b=3

c=4

and call eval function

*eval(A*A)*

result is

ans =

28 28 25

25 28 28

28 25 28

Solving Equations (solve function)

syms f(x)

*f(x) = 2*x-2*

f(x) =

*2*x - 2*

eqn = f(x)==0

eqn =

*2*x - 2 == 0*

solve(f)

ans =

1