NUMERICAL
LINEAR ALGEBRA

HW - 2

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Question 1

Derive the formula for number of operations (addition, subtraction, multiplication, and division) for Gaussian Elimination based LU factorization and indicate the computational complexity.

Answer

We have nxn matrix

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

.

.

$a_{n1}x1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$

Ax = LUx = b

Ux = y

Ly = b

L = $L_1^{-1} L_2^{-1} \ldots L_{n-1}^{-1}$

U=$L_{n-1} \ldots L_2 L_1 A$

Matlab Script

Example:

```
A = [1 2 3; 5 6 7; 6 8 9]

[n, m] = size(A);

%Initialize U = A, L = I

L = eye(n);

U = A;

for k = 1 : m - 1
        for j = k + 1 : m
                L(j, k) = U(j, k)/U(k, k)

                U(j, k : m) = U(j, k : m) - L(j, k)*U(k, k : m)
        end
end
```

Loop

Its operations count $\frac{2}{3}n^3$, Complexity is n^3

Cost of LU Factorization

factor A as A = LU ($\frac{2}{3}n^3$ flops)

We get this work expression by converting loops in Matlab loops into sums and counting the number of operations in each loop. There is one division inside the loop over j, and there are two operations (a multiply and an add) inside the loop over k. There are far more multiplies and adds than divisions, so let us count only these operations:

$$Operation \;\; Count = \sum_{i=1}^{n-1} \sum_{j=i}^{n} \sum_{k=i}^{n} 2$$

Question 2

The computational complexity of LU decomposition seems to be bad. There is a claim that there exists a slightly better version in the sense of fewer number of FLOPS (FLoating point OPerationS). The algorithm is called Do Little method. Explain with an example how it works and drive a formula for FLOPS.

Answer

First Row of U $1u_{11} = a_{11} \rightarrow u_{11} = a_{11}$ $1u_{12} = a_{12} \rightarrow u_{12} = a_{12}$ . . . . . $1u_{1n} = a_{1n} \rightarrow u_{1n} = a_{1n}$	First Column of L $l_{21}u_{11} = a_{21} \rightarrow l_{21} = a_{21} / u_{11}$ $l_{31}u_{11} = a_{31} \rightarrow l_{31} = a_{31} / u_{11}$ . . . . . . $l_{n1}u_{n1} = a_{n1} \rightarrow l_{n1} = a_{n1} / u_{n1}$
Second Row of U $l_{21}u_{12} + 1u_{22} = a_{22} \rightarrow u_{22} = a_{22} - l_{21}u_{12}$ $l_{21}u_{13} + 1u_{23} = a_{23} \rightarrow u_{23} = a_{23} - l_{21}u_{13}$ . . . . . $l_{21}u_{1n} + 1u_{2n} = a_{2n} \rightarrow u_{2n} = a_{2n} - l_{21}u_{1n}$	**Second Column of L** $l_{31}u_{12} + l_{32}u_{22} = a_{32} \rightarrow l_{32} = (a_{32} - l_{31}u_{12}) / u_{22}$ $l_{41}u_{12} + l_{42}u_{22} = a_{42} \rightarrow l_{42} = (a_{42} - l_{41}u_{12}) / u_{22}$ . . . . . $l_{n1}u_{12} + l_{n2}u_{22} = a_{n2} \rightarrow l_{n2} = (a_{n2} - l_{n1}u_{12}) / u_{22}$
...	...
Thirth Row of U $l_{31}u_{13} + l_{32}u_{23} + 1u_{33} = a_{33}$ $\rightarrow u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$ . . . . . $l_{31}u_{1n} + l_{32}u_{2n} + 1u_{3n} = a_{3n}$ $\rightarrow u_{3n} = a_{3n} - l_{31}u_{1n} - l_{32}u_{2n}$	**Thirth of L** $l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} = a_{43}$ $\rightarrow l_{43} = (a_{43} - l_{41}u_{13} - l_{42}u_{23}) / u_{33}$ . . . . . . $l_{n1}u_{13} + l_{n2}u_{23} + l_{n3}u_{33} = a_{n3}$ $\rightarrow l_{n3} = (a_{n3} - l_{n1}u_{13} - l_{n2}u_{23}) / u_{33}$
Nth Row of U $l_{n1}u_{1n} + l_{n2}u_{2n} + l_{n3}u_{3n} + \ldots + 1u_{nn} = a_{nn}$ $\rightarrow u_{nn} = a_{nn} - l_{n1}u_{1n} - l_{n2}u_{2n} - l_{n3}u_{n3} - \ldots$	**Nth Column of L** $l_{nn} = 1$

i th row of U : $u_{ij}=a_{ij}-\sum_{k=1}^{i-1} l_{ik}u_{kj}, \quad j=i,i+1,\ldots,n$

i th column of L : $l_{ii}=1, l_{ji}=(a_{ji}-\sum_{k=1}^{i-1} l_{jk}u_{ki})/u_{ii}, \quad j=i,i+1,i+2,\ldots,n, \quad u_{ii}\neq 0$

$$\begin{array}{ccccc} 1 & 2 & 3 & \ldots & 5 \end{array} \rightarrow \textit{column number if pivoting is needed}$$

$$\begin{array}{ccccc} a_{11} & u_{12} & u_{13} & \ldots & u_{1n} \\ l_{21} & a_{22} & u_{23} & \ldots & u_{2n} \\ l_{31} & a_{32} & u_{33} & \ldots & u_{3n} \\ . & . & . & . & . \\ l_{n1} & l_{n2} & l_{n3} & \ldots & l_{nn} \end{array}$$

Example

$$[L] \quad X \quad [U] \quad = \quad [A]$$

$$\begin{bmatrix} 1 & & \\ . & 1 & \\ . & . & 1 \end{bmatrix} X \begin{bmatrix} 1 & . & . \\ & 1 & . \\ & & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{bmatrix}$$

We don't need pivoting

First row of U is equal to first row of A

$$\begin{bmatrix} 1 & & \\ . & 1 & \\ . & . & 1 \end{bmatrix} X \begin{bmatrix} 2 & -1 & -2 \\ & 1 & . \\ & & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{bmatrix}$$

Dot Rows 2-3 of L with Column 1 of U to obtain:

$$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -2 & . & 1 \end{bmatrix} X \begin{bmatrix} 2 & -1 & -2 \\ & 1 & . \\ & & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{bmatrix}$$

Dot Row 2 of L with Columns 2-3 of U to obtain:

$$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -2 & . & 1 \end{bmatrix} X \begin{bmatrix} 2 & -1 & -2 \\ & 1 & -1 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{bmatrix}$$

Dot Row 3 of L with Column 2 of U to obtain:

$$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -2 & -1 & 1 \end{bmatrix} X \begin{bmatrix} 2 & -1 & -2 \\ & 1 & -1 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{bmatrix}$$

Dot Row 3 of L with Column 3 of U to obtain:

$$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -2 & -1 & 1 \end{bmatrix} X \begin{bmatrix} 2 & -1 & -2 \\ & 4 & -1 \\ & & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{bmatrix}$$

An important characteristic of the algorithm is that as each entry into either [L] or [U] is made, the corresponding entry in [A] is no longer needed for the remainder of the algorithm. Hence, the space used for [A] can be used to store both the [L] and [U] matrices. For our example, if this were done, the [A] matrix would end up as

$$LU=\begin{bmatrix} 2 & -1 & -2 \\ -2 & 4 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

Cost of Doolittle

$$u_{1,j} = a_{1,j},\ \ j = 1, \cdots, n$$

for $k = 1 \cdots n-1$

$$u_{k,k} = (a_{k,k} - \sum_{j=1}^{k-1} l_{k,j} u_{j,k})/l_{k,k}$$

$$l_{j,k} = (a_{j,k} - \sum_{i=1}^{k-1} l_{j,i} u_{i,k})/u_{k,k},\ \ j = k+1, \cdots n$$

$$u_{k,j} = (a_{k,j} - \sum_{i=1}^{k-1} l_{k,i} u_{i,j})/l_{k,k}, j = k+1, \cdots n$$

Question 3

Implement both Gaussian Elimination based LU decomposition and Do Little method in Matlab. Choose an appropriate matrix and show in detail all the FLOP counts. That is, in detail provide the number of additions, subtractions, multiplications, and division for both method side by side (Use graphics if you like).

Answer

Mathlab Scripts

LU Decomposition	**DooLittle Method**

Example:
 	
A = [1 2 3; 5 6 7; 6 8 9]
[n, m] = size(A);
%Initialize U = A, L = I
L = eye(n);
U = A;
 	
for k = 1 : m - 1
        for j = k + 1 : m
            L(j, k) = U(j, k)/U(k, k)
 	
            U(j, k : m) = U(j, k : m) - L(j, k)*U(k, k : m)
        end
end
 	
Loop
 	
% Its operations count $\frac{2}{3}m^3$ but LU Factorization needs more memory than Doolittle
	Example:
	
	A = [1 2 3;5 6 7;6 8 9]
	
	[n m] = size(A)
	
	L=eye(n)
	
	U=zeros(n) %initialize
	
	for i=1:n
	    U(i,i)=A(i,i) - (L(i,1:i-1) * U(1:i-1,i))
	    for j=i+1:n
	      U(i,j) = A (i,j)- L(i,1:i-1) * U(1:i-1,j)
	      L(j,i) = (A(j,i)-L(j,1:i-1)*U(1:i-1,i)) / U(i,i)
	     end
	end
	
	Loop
	
	% approximately Its operations count $\frac{1}{3}n^3$ also Doolittle needs less memory than LU Factorization

Doolittle

$$u_{1,j} = a_{1,j}, \quad j = 1, \cdots, n$$

for $k = 1 \cdots n-1$

$$u_{k,k} = (a_{k,k} - \sum_{j=1}^{k-1} l_{k,j} u_{j,k}) / l_{k,k}$$

$$l_{j,k} = (a_{j,k} - \sum_{i=1}^{k-1} l_{j,i} u_{i,k}) / u_{k,k}, \quad j = k+1, \cdots n$$

$$u_{k,j} = (a_{k,j} - \sum_{i=1}^{k-1} l_{k,i} u_{i,j}) / l_{k,k}, \, j = k+1, \cdots n$$

Question 4

It is said that a unique solution for the linear sets of equations of Ax = b exists. Indicate at least 5 equivalent statement for having this unique solution and prove each one of them.

Answer

Ax=b has a unique solution.

and A is nxn matrix. The following statements are equivalent

1. $Det(A) \neq 0$
2. A is invertible
3. A has rank n
4. The column vectors of A are linearly independent.
5. The row vectors of A span R^n
6. The row vectors of A form a basis for R^n
7. The reduced row-echelon form of A is I
8. The homogeneous linear system always has the trivial solution x = 0

$AA^{-1}=A^{-1}A=I$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ it means $Det(A) \neq 0$

if $Det(A) \neq 0$, A invertible and rank(A) = n

if rank(A) = n, it means the column vectors of A are linearly independent.

If he column vectors of A are linearly independent, for every x in V it is possible that $x = a_1v_1 + \ldots + a_nv_n$.

Let V be a space of vector, and

A={v1, v2, v3 ….. vm} ⊂ V , thus A is subset of V

we can say that A is basis of V

- span(A) = R^n and

we have n independent vectors in an space of dimension n. hence, they must also span and form a basis.

Question 5

Provide definition for the cases below. Use examples where you find useful.

– Linear independence (of a set of vectors)

– Null space (of a matrix)

– Base (of a space)

– Norm(s) (of a vector)

– Norm(s) (of a matrix)

– Equivalency of p-norms for p=1, 2, and infinity

– Rank (of a matrix)

– Determinant (of a matrix)

– Trace (of a matrix)

Answers

Linear independence :

A set of vectors is said to be linearly dependent if one of the vectors in the set can be defined as a linear combination of the other vectors. If no vector in the set can be written in this way, then the vectors are said to be linearly independent.

The set $S = \{v_1, v_2, v_3\}$ of vectors in $\mathbb{R}^3$ is linearly independent if the only solution of

$$a_1v_1 + a_2v_2 + a_3v_3 = 0$$

is $a_1, a_2, a_3 = 0$

Otherwise S is linearly dependent.

Example

In the vector space of two-wide row vectors, the two-element set

$$\{\begin{pmatrix}40 & 15\end{pmatrix}, \begin{pmatrix}-50 & 25\end{pmatrix}\}$$

is linearly independent. To check this, set

$$c_1 \cdot \begin{pmatrix}40 & 15\end{pmatrix} + c_2 \cdot \begin{pmatrix}-50 & 25\end{pmatrix} = \begin{pmatrix}0 & 0\end{pmatrix}$$

and solving the resulting system

$$\begin{array}{rcrcl} 40c_1 & - & 50c_2 & = & 0 \\ 15c_1 & + & 25c_2 & = & 0 \end{array} \xrightarrow{-(15/40)\rho_1+\rho_2} \begin{array}{rcrcl} 40c_1 & - & 50c_2 & = & 0 \\ & & (175/4)c_2 & = & 0 \end{array}$$

shows that both C_1 and C_2 are zero. So the only linear relationship between the two given row vectors is the trivial relationship. In the same vector space, result is linearly dependent.

$$\{\begin{pmatrix}40 & 15\end{pmatrix}, \begin{pmatrix}20 & 7.5\end{pmatrix}\}$$

$$c_1\begin{pmatrix}40 & 15\end{pmatrix} + c_2 \cdot \begin{pmatrix}20 & 7.5\end{pmatrix} = \begin{pmatrix}0 & 0\end{pmatrix}$$

$$c_1 = 1 \quad c_2 = -2$$

Null space (of a matrix) :

First of all we should know vector space. A vector space is a collection of objects called vectors. A linear subspace (or vector subspace) is a vector space that is a subset of some other (higher- dimension) vector space.

$\mathbb{R}^2 \rightarrow \mathbb{R}^3$

The null space of A ∈ C $^{m \times n}$, written as null(A), is the set of vectors x that satisfy Ax = 0. A vector is multiply with matrix if the result is zero matrix, these matrix are null space.

$$A\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0. \end{array}$$

Example,

A = [1 2; 2 4]

B = [-2

1]

A * B = [0 0]

Base (of a space)

If W subset of the Vector space V has following two properties It is a base vector of V.

- the *linear independence* property,

if $a_1v_1 + \ldots + a_nv_n = 0$, then necessarily $a_1 = \ldots = a_n = 0$; and

- the *spanning* property,

for every x in V it is possible that $x = a_1v_1 + \ldots + a_nv_n$.

If V is a vector space and S = $\{x_1, x_2, \ldots, x_n\}$ is a set of vectors in V such that

- S is a linearly independent set
- A set of vectors is linearly independent if none of them can be written as a linear combination of finitely many other vectors in the set. And
- S spans V
- A span is a set of vectors ∈ V for which all other vectors ∈ V can be written as a linear combination of the vectors in the span.
- then S is a basis for V .
- The dimension of V is n.

Example,

Consider W the subset of the Vector space V where V is all 2x2 matrices:

W = [a a; a b] $a,b \in \mathbb{R}$

Basis for W

a[1 1 ; 1 0] + b[0 0 ; 0 1]

Let V be a space of vector, and

B={v1, v2, v3 ….. vm} ⊂ V , thus B is subset of V

we can say that B is basis of V if

- span(B) = V and

- the vectors v1,v2,v3,...vm are linearly independent

Norm(s) (of a vector)

Norms are valuable tools in arguing about the extent of error. The solution of a nonlinear equation f(X) = 0, the error $e = x_1 - x$ is a single number. The absolute value |e| gives us a good idea of the error.

For example In linear equation Ax=b, the exact solution and approximation are vector. Vector norms are alternative ways to measure accuracy of approximation.

A function $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ is called a vector norm if it has the following properties:

1. $\|x\| \geq 0$ for any vector $x \in \mathbb{R}^n$, and $\|x\| = 0$ if and only if $x = 0$

2. $\| ax\| = \|a\| \|x\|$ for any vector $x \in \mathbb{R}^n$ and any scalar $a \in \mathbb{R}$

3. $\|x + y\| \leq \|x\| + \|y\|$ for any vectors $x, y \in \mathbb{R}^n$.

The last property is called the triangle inequality. It should be noted that when n = 1, the absolute value function is a vector norm.

Example

X = [1 2 3 4]

1-Norm	2-Norm (Euclidean)	P - Norm	∞ - Norm
$\|x\|_1 = \sum_{i=1}^{n} \lvert x_i \rvert$	$\|x\|_2 = \left(\sum_{i=1}^{n} \lvert x_i \rvert^2 \right)^{\frac{1}{2}}$	$\|x\|_p = \left(\sum_{i=1}^{n} \lvert x_i \rvert^p \right)^{\frac{1}{p}}$	$\|x\|_\infty = \max_i \lvert x_i \rvert$
norm(X,1) = 10	norm(X,2) = 5.4772		norm(B,inf) = 4

Norm(s) (of a matrix)

a matrix norm to be a function $\| \cdot \| : \mathbb{R}^{mxn} \rightarrow \mathbb{R}$ that has the following properties:

$\|A\| \geq 0$ for any $A \in \mathbb{R}^{mxn}$, and $\|A\| = 0$ if and only if $A = 0$

$\|aA\| = |a|\|A\|$ for any mxn matrix A and scalar a

$\|A + B\| \leq \|A\| + \|B\|$ for any mxn matrices A and B

1-Norm : maxsimum number in sum of each column

∞ - Norm : maximum number in sum of each rows

X = [1 2 ; 3 4]

1-Norm (Column Based)	2-Norm (Frobenius)	∞ - Norm (Row Based
$\|\|A\|\|_1 = \max_j \sum_{i=1}^{n} \|a_{ij}\|.$	$\|\|A\|\|_F \equiv \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\|a_{ij}\|^2}$	$\|\|A\|\|_\infty = \max_i \sum_{j=1}^{n} \|a_{ij}\|.$
norm(X,1) = 6	norm(X,'fro') = 5.4772	norm(B,inf) = 7

Equivalency of p-norms for p=1, 2, and infinity

Two vector norms $\| \cdot \|_a$ and $\| \cdot \|_b$ are equivalent if there exists constants C_1 and C_2, that are independent of x, such that for any vector $x \in \mathbb{R}^n$,

$C_1\|x\|_a \leq \|x\|_b \leq C_2\|x\|_a$

if two norms are equivalent, then a sequence of vectors that converges to a limit with respect to one norm will converge to the same limit in the other. It can be shown that all l_p-norms are equivalent. In particular, if $x \in \mathbb{R}^n$, then

$$\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2,$$
$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty,$$
$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty.$$

Like vector norms, matrix norms are equivalent. For example, if A is an mxn matrix, we have

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2,$$
$$\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty,$$
$$\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_\infty.$$

Rank (of a matrix)

The rank of a matrix is defined as

- the maximum number of linearly independent column vectors in the matrix or - the maximum number of linearly independent row vectors in the matrix. Both definitions are equivalent.

For an r x c matrix,

-If r is less than c, then the maximum rank of the matrix is r.

-If r is greater than c, then the maximum rank of the matrix is c.

The rank of a matrix would be zero only if the matrix had no non-zero elements. If a matrix had even one non-zero element, its minimum rank would be one.

The maximum number of linearly independent vectors in a matrix is equal to the number of non-zero rows in its row echelon matrix. Therefore, to find the rank of a matrix, we simply transform the matrix to its row echelon form and count the number of non-zero rows.

Example,

A=[0 1 2;1 2 1;2 7 8]   echelon form of $A_{echelon}$ = [1 2 1;0 1 2;0 0 0]

A has two independent row vectors; and the rank of matrix A is 2.

Determinant (of a matrix)

Determinants are mathematical objects that are very useful in the analysis and solution of systems of linear equations.

The determinant of a matrix A,

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv a\,d - b\,c.$$

A k×k determinant can be expanded "by minors" to obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k2} & a_{k3} & \cdots & a_{kk} \end{vmatrix}$$

$$-a_{12} \begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k3} & \cdots & a_{kk} \end{vmatrix} + \ldots \pm a_{1k} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{k(k-1)} \end{vmatrix}.$$

A general determinant for a matrix A has a value

$$|\mathrm{A}| = \sum_{i=1}^{k} a_{ij}\, C_{ij},$$

the cofactor of a_{ij} defined by

$$C_{ij} \equiv (-1)^{i+j} M_{ij}.$$

Laplace's formula and the adjugate matrix

Laplace's formula expresses the determinant of a matrix in terms of its minors. The minor Mi,j is defined to be the determinant of the (n−1) × (n−1)-matrix that results from A by removing the ith row and the jth column. The expression (−1)i+jMi,j is known as cofactor. The determinant of A is given by

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j} = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j}.$$

Example,

$$A = \begin{bmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix},$$

$$\det(A) = (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} + (-1)^{2+2} \cdot 1 \cdot \begin{vmatrix} -2 & -3 \\ 2 & -1 \end{vmatrix} + (-1)^{3+2} \cdot 0 \cdot \begin{vmatrix} -2 & -3 \\ -1 & 3 \end{vmatrix}$$

$$= (-2) \cdot ((-1) \cdot (-1) - 2 \cdot 3) + 1 \cdot ((-2) \cdot (-1) - 2 \cdot (-3))$$

$$= (-2) \cdot (-5) + 8 = 18.$$

Determinant is used for calculating inverse of matrix.

a square matrix A, which is non-singular (det(A) does not equal zero), then there exists an n×n matrix A^{-1} which is called the inverse of A, such that this property holds:

$AA^{-1} = A^{-1}A = I$, where I is the identity matrix.

Trace (of a matrix)

The trace of the matrix is the sum of the main diagonal:

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

Example,

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \mathrm{tr}(A) = a + e + i$$

Basic properties

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$$

$$\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$$

A matrix and its transpose have the same trace:

$$\mathrm{tr}(A) = \mathrm{tr}(A^{\mathrm{T}})$$

Matlab Script

trace(A)

or

sum(diag(A))