AP Physics C: Mechanics

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Before You Begin

This study guide will guide you through each of the topics covered on the AP Physics C: Mechanics exam, and cover core concepts, formulas, and other important info. However, it's important to note that there's a lot of stuff you should know before starting on E&M.

Firstly, you're going to need a decent grasp of basic calculus. Both Physics C exams are **calculus-based**, rather than algebra-based, like Physics 1 and 2 are. You will need to know how to do basic integration and differentiation, as well as solve separable differential equations. It's not Calculus BC type stuff: there won't be any integration by parts or anything, so don't fret about not being able to comprehend any super-advanced calculus concepts, because you won't need them here.

You should also have a decent grasp of regular high school curriculum-level physics concepts. If you've taken Physics 1 or 2, that works as well. The course and exam expand on some of the classical mechanics concepts covered there in greater detail, so if you already have a grasp of the basics, it'll really help you when trying to wrap your head mechanics concepts.

I will **BOLD** any variables in formulas that are vector quantities. That's how I'll be notating vectors in formulas in this study guide. Keep that in mind as you use this guide: some people bold variables instead, to mark them as vectors, but I'll be using the arrows.

Also: the AP Physics C: Mechanics exam and class specifically allow you to use **10 m/s^2** as the acceleration of gravity (when close to Earth's surface). This is what I will use in the guides, because it simplifies some math and is specifically allowed, and won't throw answers off that much from 9.81 m/s^2 anyways.

With all that said, let's get started!

Kinematics - Unit 1

There are 3 key quantities in kinematics and motion which we are interested in: position, velocity, and acceleration. This first unit only concerns motion in **one dimension**.

The Kinematic Equations

There are 3 main kinematic equations that you will need to use in the AP Physics course and exam.

$$x = x_0 + v_{x_0}t + \frac{1}{2}a_x t^2$$

$$v_x = v_{x_0} + a_x t$$

$$v_x^2 = v_{x_0}^2 + 2a_x(x - x_0)$$

Constant/Average Velocity and Acceleration Models

If velocity is constant, then velocity is given by the displacement and time elapsed.

$$v_x = \frac{\Delta x}{\Delta t}$$

There are also similar models for average velocity and average/constant acceleration:

$$v_{x(avg)} = \frac{\Delta x}{\Delta t}$$

$$a_{x(avg)} = \frac{\Delta v_x}{\Delta t}$$

Calculus-Based Relationships and Models

Now: the above kinematic equations only work in very, very specific situations: when acceleration is constant, and velocity/acceleration, etc. relationship isn't changing with time, position, or some other variable. When that happens, you'll need **calculus-based relationships**.

If you have functions for velocity, position, or acceleration, then you need these models to relate each quantity.

$$v_x = \frac{dx}{dt}$$

$$a_x = \frac{dv_x}{dt}$$

We can extend these using the fundamental theorem of calculus:

$$\Delta x = \int v_x dt$$

$$\Delta v = \int a_x dt$$

If you took AP Physics 1, your teacher may have said something about the "area under the curve" on a velocity-time or acceleration-time graph: these integrals represent that.

Acceleration and velocity can be nonuniform, and can follow any sort of function, really.

<u>Examples:</u>

Free-Fall

Let's imagine that you drop a basketball off a helicopter that is 500 meters above the ground on Earth. How long would it take to hit the ground?

Let's think:

- We know that it will start with no initial velocity (you are dropping it after holding it still)
- We know that it will have an acceleration downwards of g (which we can assume is 10 m/s^2 downwards in AP Physics-land)
- We know that it has to fall 50 meters to hit the ground
- We DON'T know how long it will take, and we want to figure that out.

Therefore, we want to use a kinematic equation that takes these variables into account. We will use the first one.

$$x = x_0 + v_{x_0}t + \frac{1}{2}a_x t^2$$

x_0 is 0 m (we take the helicopter as the start point), x = 500m, v_x = 0 m/s, and $a = 10\ m/s^2$.

$$x = \frac{1}{2}a_x t^2 \longrightarrow t = \sqrt{\frac{2x}{a}}$$

$$t = \sqrt{\frac{2(500)}{10}} \longrightarrow t = \sqrt{100} = 10s$$

Calculus-Based Example: Non-Uniform Velocity

Imagine that we are doing calculations with a very fast experimental vehicle. When it goes full throttle, its velocity can be described with relation to time as $v(t) = 2t^3$. Starting from rest, if the test driver floors the accelerator, how far would the vehicle travel in 5 seconds?

Let's think: velocity **isn't constant**, so we can't just multiply by time. However: we have a **function for velocity** that is **with respect to time**, and so we can just use a calculus-based relationship.

Recall:

$$\Delta x = \int_{t_1}^{t_2} v(t)dt$$

t_1 is 0, and t_2 is 5 (we start from 0 and go for 5 seconds). Let's integrate!

$$\Delta x = \int_{0}^{5} 2t^3 dt$$

$$\Delta x = \left(\frac{1}{2}t^4\right)_0^5$$

$$\Delta x = \frac{1}{2}\left(5^4\right)$$

$$\Delta x = 312.5\ meters$$

Newton's Laws of Motion - Unit 2

Newton's First Law

An object at rest or in uniform motion will remain at rest or in uniform motion unless acted on by a net external force.

Uniform motion means constant velocity. "At rest" just means that an object is in uniform motion, but with v = 0. As long as an object is moving in uniform motion, it must have 0 net external force acting on it ($\mathbf{F}_{net} = 0$). This means **all directions**.

For example:

- If you have a hockey puck sliding on smooth and slippery ice, the gravity and normal force with the ice will cancel to create zero net force, and the puck will keep on sliding with constant velocity.
- If you have a spaceship in deep space that isn't close to any stars or planets, there will be no (or negligible) forces acting on it, and it will just keep on going at a constant velocity.

Do note, however: **the mass of the object must be constant**. This year, I had a free response question with a moving cart that had **changing mass**: it's possible that you could get changing mass in an exam problem, too.

The Second Law of Motion

The sum of the forces acting on an object is proportional to its mass and its acceleration.

$F = \sum \qquad F = ma$ Note: this is actually a special case that assumes that **mass is constant**. If mass isn't constant, then you need to do something a little more clever: that's outside of this unit, however.

The Third Law of Motion

For every action, there is an equal and opposite reaction.

For every action force on an object B due to another object A, there is a reaction force which is equal in magnitude but opposite in direction on object A due to object B.

$$F_{A\ on\ B} = -F_{B\ on\ A}$$

Remember: the action and reaction forces are acting on **different objects**.

Forces

Forces are interactions between 2 objects: they are pushes or pulls. There are 2 categories of forces: **contact forces**, which act between objects that are in contact, and **non-contact** forces that act between objects without them touching.

Equilibrium

If the net force on an object is zero, then the object is in a state of equilibrium. There are 2 types of equilibrium:

- **Dynamic equilibrium** is when the object is moving relative to us (the observers)
- **Static equilibrium** is when the object is *not* moving relative to us.

Normal Force

Normal force is the reaction force that prevents objects from going into each other. When 2 objects are in contact and pressing into each other, there will be normal force preventing, say, you from falling through the floor, or your cup from falling through the table.

Friction Force

Friction is a force that opposes the sliding of two surfaces against one another. It always acts in a direction that opposes motion or attempted motion.

Frictional force is dependent on:

- Normal force *N*, or the force of the two surfaces pressed against each other
- Coefficient of friction, either static or kinetic, which is dependent on smoothness of the surfaces and any lubricants used.

Static Friction

Static friction between two surfaces is when there is **no relative motion** between them. It increases to match any applied force, just enough to prevent the two surfaces from moving relative to each other, and is at maximum when the object is just about to move.

$$f_s \leq \mu_s N$$

μ_s is the coefficient of static friction, and *N* is the magnitude of the normal force.

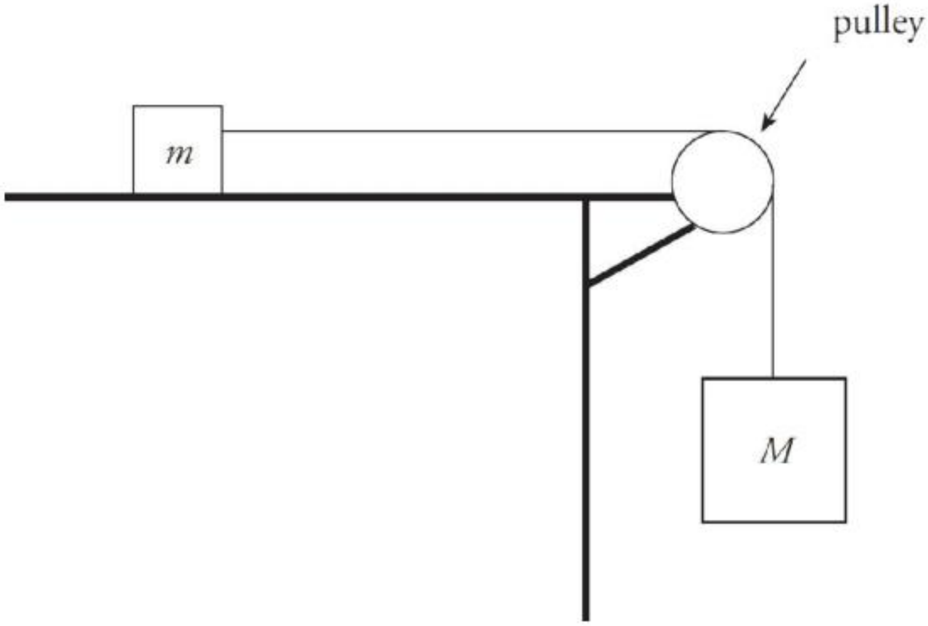
Kinetic Friction

Kinetic friction occurs between two surfaces when they are moving relative to each other. Kinetic friction is **constant** as long as the normal force stays constant.

$$f_k = \mu_k N$$

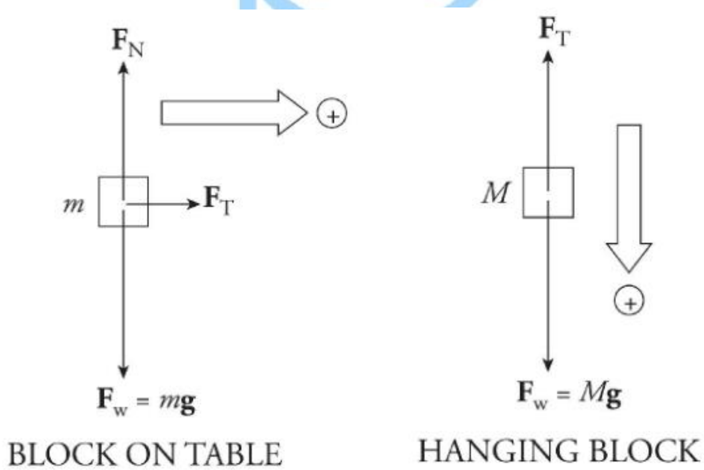
μ_k is the kinetic friction coefficient, and again, *N* is the magnitude of normal force.

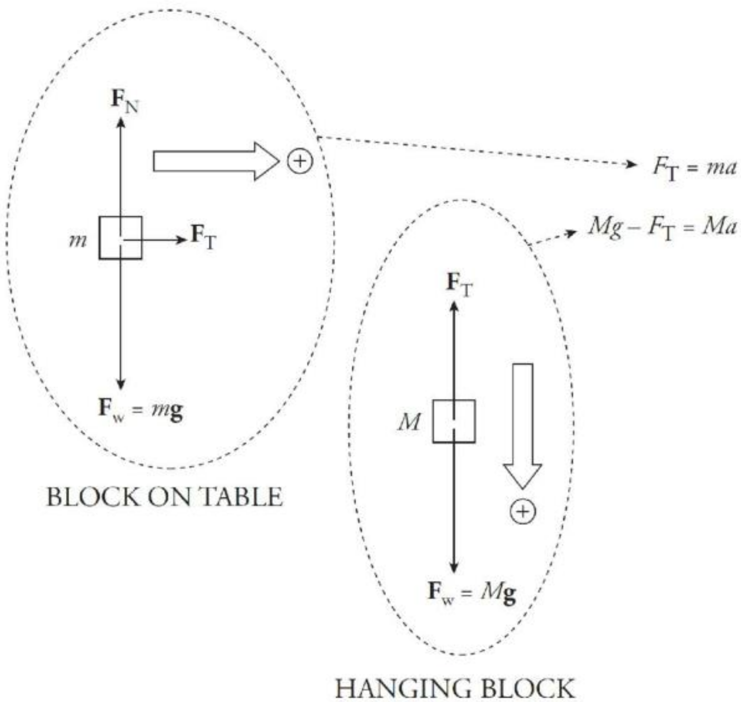
Applying the Laws: Pulleys and Atwood Machines



Consider this pulley system. Assuming that the tabletop is frictionless and that the weight of the pulley and the string is negligible, how do we determine the acceleration of the blocks?

We have 2 blocks, so we need to draw 2 free-body diagrams, and use the Second Law to find acceleration.





We get 2 different equations from these free-body diagrams + the Second Law, and we can add the equations to remove the tension force. We can then solve for acceleration.

$$Mg = Ma + ma$$

$$Mg = (M + m)a$$

$$a = \frac{Mg}{M + m}$$

Working out the relationships between the motion of different objects using Newton's Second and Third Laws is a key idea of this unit, and will come up many times throughout the course. Think about how different objects interact with each other: what forces they will apply on each other, if their accelerations or velocities are linked (like with the pulleys), etc.

Uniform Circular Motion

If an object is travelling in a circle around a center point, it needs some specific conditions. The object MUST be travelling at constant **speed** (NOT velocity) around the center, and so much acceleration acting on it that is pointing to the center of the circular motion. This acceleration is called the **centripetal acceleration**.

It is given by:

$$a_c = \frac{v^2}{r}$$

For example:

If there is an object moving in a circle at a constant speed of 10 m/s and the motion has a radius of 2 meters, what is the centripetal acceleration?

$$a_c = \frac{v^2}{r} = \frac{10^2}{2} = 50\, m/s^2$$

NOTE: Centripetal force **isn't an actual force**. Centripetal force is the **result of other forces**. Some other force like tension or gravity must provide a force towards the center of the circular motion.

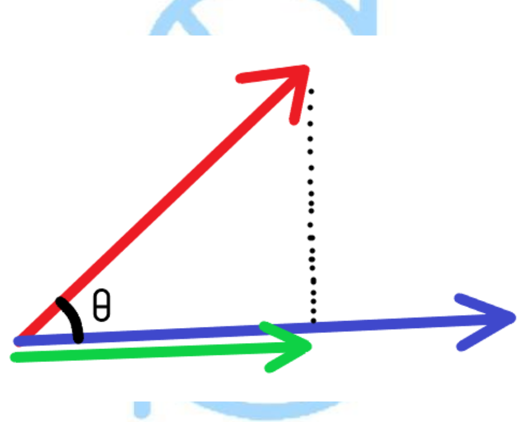
Work, Energy, and Power - Unit 3

When a force displaces an object, it does **work** on that object. Your calculator works using the amount of force that acts parallel to an object's displacement: in other words, only force that contributes to displacement actually does any work. If there is a constant force acting only in the direction of displacement, then the work done by a force *F* when it displaces something *d* meters is:

$$W = Fd$$

Work *W* is in units of joules (1 J = 1 N/m).

However: what happens when force isn't only acting in the direction of displacement? Take this diagram, where red is the applied force and blue is the displacement.



How do you calculate the work that the red force does? You need to consider **only the amount of force acting in the direction of displacement**.

If we have force ***F*** and displacement ***r*** (these are vector quantities), then we get the work done by force ***F*** with a **dot product**.

$$W = F \cdot r$$

However, if we only know the scalar magnitudes of these quantities along with the angle between them (which is theta here), then we can use the cosine definition of the dot product.

$$W = Frcos(\theta)$$

Work When Force Isn't Constant

Force isn't always constant, though. What if force varies over displacement, time or something else? We need calculus to figure out how that would work.

If we imagine taking the sum of all of the infinitesimal bits of force and displacement to find infinitesimal bits of work, we know that we will need an integral.

The work done on an object by a force is calculated with the following integral:

$$W = \int_{x_1}^{x_2} F(r) \cdot dr$$

If you still want to think in terms of "areas under graphs", then work is the area under a force vs. position graph.

Work and Energy

The net work done on a point mass is equal to the change in kinetic energy.

$$W_{net} = \Delta K$$

This is the **work-energy theorem**. You can model objects or systems in this way when objects can be described as point-like particles that don't deform or otherwise internally store energy.

Kinetic Energy Definition

The definition of kinetic energy is:

$$K = \frac{1}{2}mv^2$$

Note: net work is the sum of all of the individual work provided by each of the forces acting on an object.

Conservative Forces

A force is a conservative force if the work done on objects by the force depends only on the initial and final positions of the objects.

Therefore: the (net) work done by a conservative force is **zero** if the object takes a path that goes back to its original starting position.

Forces that don’t conform to this definition are known as **dissipative forces**. Forces like friction, for example, are generally dissipative.

Potential Energy with Conservative Forces

Conservative forces that are internal to a system relate to the potential energy of the system through an integral relationship.

$$\Delta U = -\int_a^b F_{cons} \cdot dr$$

You can also use the one-dimensional differential relationship.

$$F_x = -\frac{dU(x)}{dx}$$

Now, let’s examine some common conservative forces and how they relate to potential energy.

Elastic Potential Energy (Springs)

An ideal spring that acts on an object exerts a conservative force within a system of the object and the spring. Recall Hooke's Law, which gives us the general relationship between spring compression/stretch and the force they exert, given a particular spring constant.

$$F_s = -k\Delta x$$

Using the integral relationship on the previous page, we can find the potential energy in an ideal spring by integrating the force function.

$$U_s = \frac{1}{2}k(\Delta x)^2$$

Note that this is for linear springs. You may encounter springs that don't follow this law on the exam. For example, I once had an exam question with a spring that had a different compression-force relationship.

$$F_s = -k\Delta x^2$$

However, the process for solving the problem was the same as with a linear spring: I just had to integrate this new relationship to find the potential energy in relation to the compression.

$$U_s = -\int -k\Delta x^2 dx$$

$$U_s = \frac{1}{3}k\Delta x^3$$

Gravitational Potential Energy (near-Earth)

Recall that when you are close to Earth's surface and your elevation change is small relative to Earth's radius, force from gravity is constant.

$$F_g = mg$$

Gravitational potential energy near Earth's surface in the system of the Earth and the object in question is therefore:

$$U_g = mgh$$

And the change in gravitational potential energy is:

$$\Delta U_g = mg\Delta h$$

Gravitational Potential Energy (non-trivial distances)

When the distance between, say, Earth and a satellite are non-negligible in comparison to the radius of the Earth, we cannot approximate the force from gravity as constant anymore. We must use the actual force from gravity.

$$F_G = \frac{Gm_1m_2}{r^2}$$

Using this equation, we can get the potential energy of an object-Earth system.

$$U_G = -\frac{Gm_1m_2}{r}$$

The potential energy of this system is zero at an infinite distance between the object and Earth.

The Conservation of Energy

This isn't about saving energy, no no. This is about the idea that the energy in a system can change from one type of energy to another, without changing the total amount of energy.

If the only forces acting on a system are internal to the system, and no external work is being done, then the total change in mechanical energy of the system MUST be ZERO. This is something called a **conservative system**.

In essence: the sum of any potential energy and any kinetic energy at any time must ALWAYS be the same. For example, if we had a system of the Earth, a mass and a spring that the mass is resting on, then the sum of all the mechanical energy in the system would be

$$E = U_g + U_s + K$$

No matter what happens, this sum will be constant. If, for example, the object is oscillating on the spring, compressing the spring a little, then whatever potential energy that would be gained from the compression is matched by a loss in kinetic energy and gravitational potential energy.

However, not everything is lost when external forces act on the system: if any nonconservative forces act on the system, then the work they do will change the total amount of energy in the system.

$$W_{noncons} = \Delta E$$

You can use this to your advantage when solving problems, where you may be able to still apply conservation of energy, if you account for any external, nonconservative work.

Power

Power is the rate of change of the energy in a system or object. It can be defined in 2 ways.

Firstly, the differential relation:

$$P = \frac{dE}{dt}$$

There is also another relationship.

Power is the rate of change of energy in a system, and work is the change in energy of a system. Therefore, the average power of a force is the amount of work it does divided by the time it takes.

$$\underline{P} = \frac{W}{t} \longrightarrow \underline{P} = \frac{Fd}{t}$$

$$\frac{d}{t} = \underline{v} \longrightarrow \underline{P} = F\underline{v}$$

It also can be expressed with vector quantities:

$$P = F \cdot v$$

Systems of Particles & Linear Momentum - Unit 4

Linear motion of an object is usually thought of as displacement, velocity and acceleration of its center of mass. When you draw a free-body diagram, for example, you often have all of your forces represented as acting on a single point, which is the center of mass.

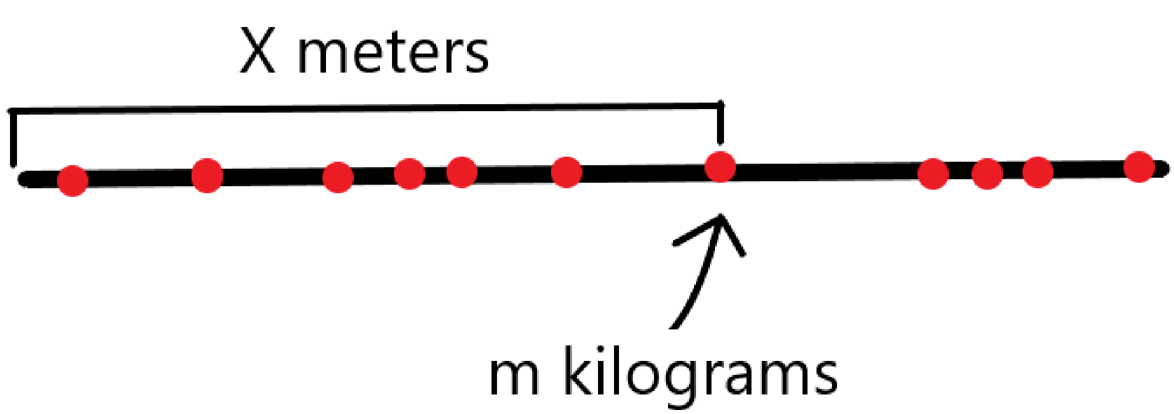
Where is the Center of Mass?

The center of mass in a symmetrical, regular solid of uniform mass density is at its geometric center. For a non-uniform solid, we have to think a little deeper.

If we have a collection of point masses, then the center of mass of that collection of masses is:

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i}$$

Essentially, it is the sum of the products of the masses and positions of each particle, divided by the sum of the masses of the particles.



Center of Mass (Calculus Definition)

The calculus definition of the center of mass is as follows:

$$x_{cm} = \frac{\int xdm}{\int dm}$$

Movement of the Center of Mass

If there is no net force acting on an object or system, then the center of mass **won't accelerate**. This means that if, for example, an object explodes while moving (an internal interaction and internal force) and splits into 2 objects, then the center of mass of the 2 fragments will move at the same velocity it was going at. And there is a special case for this concept: if there is a force providing the same acceleration to every single fragment, like gravity, then if you, for example, launch an object that exploded in mid-air, the center of mass of the fragments would continue in the same path that the object originally would have taken.

Linear Momentum

Any object moving with velocity has momentum defined by:

$$p = mv$$

The total momentum of a system of objects is the vector sum of the momenta of each object.

The rate of change of the momentum of a system is equal to the net external force.

$$F = \frac{dp}{dt}$$

We can derive the Second Law of Motion from this with a little work.

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} = ma$$

Impulse

Impulse is the average force acting on an object over a time interval.

$$J = F_{avg}\Delta t$$

Impulse is also equivalent to the change in momentum of the object receiving impulse.

$$J = \int Fdt = \Delta p$$

As you can see from the relationship, impulse (and therefore change in momentum) can be calculated as the area underneath a force-time graph.

Collisions

When a system of objects isn’t acted upon by any external forces, linear momentum is conserved. You can use this conservation of momentum in collision situations, sometimes alongside conservation of energy, to find out more about how things are moving.

This is a result of the fact that forces internal to a system won’t change the momentum of the center of mass of the system (the sum of the momenta of each component).

You can use this to your advantage. For example: if you have 2 pool balls going at a certain velocity that collide, and you know the direction + velocity that one of the balls after the collision, then you can find the second ball’s velocity and direction by equating the sum of the momentums before and after.

Elastic Collisions

Elastic collisions are a special type of collision, where both momentum and kinetic energy are conserved. Regular collisions are usually not elastic because they lose some energy in the collision: the sound that things make when they collide, for example, is energy that is emitted through sound.

Elastic collisions occur usually when there is something that stores energy and then releases it during the collision, such that the energy isn’t lost in the collision. For example, springs on 2 carts that compress when they touch, eventually fully expanding to release all kinetic energy back into their motion, rather than losing some energy.

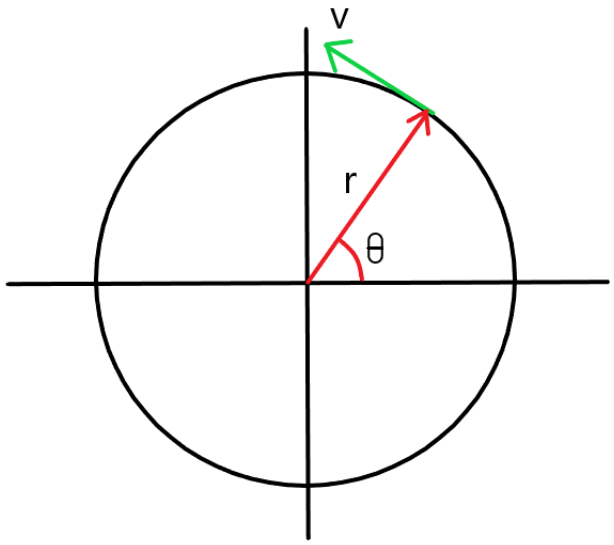
Completely Inelastic Collisions

There is one last type of special collision: this is the **completely inelastic collision**, where the **maximum** amount of kinetic energy is lost. This is usually when the 2 objects in the collisions stick together and move with the same velocity, resulting in the most energy lost.

Rotation - Unit 5

Rotational motion is based on completely different quantities for certain things. We use θ as the rotational equivalent for position (in radians), ω as the rotational equivalent for velocity (angular frequency/velocity in radians per second) and α as the rotational equivalent for acceleration.

If you have a spinning object and its tangential velocity is ***v***, then the angular velocity is related to the cross product with position vector ***r***.



$$v = \omega \times r$$

There is also the scalar form.

$$v = r\omega$$

Similarly, we can also find the relation between angular acceleration and tangential acceleration.

$$a_\theta = r\alpha$$

Relations Between Each Quantity

Each quantity is related in very similar ways to how rectilinear quantities (position, velocity and acceleration) are related.

$$\alpha = \dot{\omega} = \ddot{\theta}$$

Rotational Kinematics

With constant α, just like with rectilinear motion, we can have kinematic equations that are basically the same.

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \theta_0 + \frac{\omega_0 + \omega}{2} t$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Remember: this is **only** for **constant** α. Without a constant angular acceleration, you need to integrate. However, where it *is* constant, you're golden! For example: an object moves in a circle with angular acceleration of 8.0 rad/s^2. It starts from rest. How long does it take for the object to go in a circle? We can just use a kinematics equation and substitute in what we know.

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta_0 = 0,\ \omega_0 = 0, \theta = \pi, \alpha = 4.0\pi$$

$$t = \sqrt{\frac{2\theta}{\alpha}} = \frac{1}{2}\ seconds$$

Torque

Torque is the tendency for a force to change the rotational motion of a body. A force acting at distance ***r*** from a fulcrum or pivot that applies at least some force tangentially will provide torque.

When applying a force F_a at a distance r and at an angle ϕ to the body, then we can find the torque it provides.

$$\tau = rF_a \sin \sin \phi$$

There is also the vector form of the equation.

$$\tau = r \times F_a$$

Note:

- A force applied only towards the pivot (no tangential force) will **not** provide a torque.
- A force applied at the pivot (r = 0) will not provide any torque, either. It shows in the equation: if r = 0, then the torque will always be zero.

We can also relate torque to angular acceleration. Just like F = ma, we have a relation for the rotational equivalent.

$$\tau = I\alpha$$

I is the rotational inertia of the body. Don't worry, we cover this soon.

Rotational Inertia/Moment of Inertia

The moment of inertia of a body is basically the rotational equivalent of mass. If you have a single point of mass at a distance *r* from a pivot, then the rotational inertia is given by:

$$I = r^2 m$$

For a collection of particles, you just add them up.

$$I = \sum r_i^2 m_i$$

For a continuous distribution of mass, this extends to an integral.

$$I = \int r^2 dm$$

It's used absolutely everywhere when dealing with rotational problems. You can see how it acts as an equivalent to mass. For example:

$$F = ma \longrightarrow \tau = I\alpha$$

To calculate moment of inertia for continuous distributions of mass, you need to set up integrals for the mass and shape. However, usually, if the College Board doesn't want you to specifically calculate the moment of inertia, they'll just give it to you.

Rotational Momentum

Rotational momentum just describes the overall rotational state of a system, just like how linear momentum describes the overall translational state of the system. Just like linear momentum, rotational momentum is related to rotational inertia and angular velocity, and its rate of change is torque.

Angular momentum is described most basically as the cross product of the radius and momentum.

$$L = r \times p$$

Let's derive the relationship with angular velocity!

Given a particle of mass *m*, travelling at speed *v* at a distance *r* from a center:

$$L = r \times (mv) = mr \times (\omega \times r) = mr^2\omega = I\omega$$

We can really see the equivalence with linear concepts here.

$$p = mv \longrightarrow L = I\omega$$

$$F = \frac{dp}{dt} \longrightarrow \tau = \frac{dL}{dt}$$

Rotational Kinetic Energy

Rotational kinetic energy is derived either by summing (or integrating) the kinetic energy of the individual particles in the rotating physical system.

You get the expression for rotational kinetic energy.

$$K = \frac{1}{2} I\omega^2$$

We also have expressions for work done by torque.

For constant torque:

$$W = Fd \longrightarrow W = \tau\theta$$

For torque that isn't constant:

$$W = \int Fdx \longrightarrow W = \int \tau d\theta$$

The total amount of kinetic energy in a rotating system of particles is given by the sum of the rotational kinetic energy about the system's **center of mass** and any translational (linear) kinetic energy.

Rolling

Rolling without slipping is a very special type of rolling: it means that the point of contact of the rolling object with the ground isn't moving. Generally, it means that there is static friction between the object and the ground. This means that $v = r\omega$, and that $a = r\alpha$. You can use this to your advantage. For example:

If you have a cylinder that rolls without slipping down a slope with a known change in elevation, you can find both the translational and rotational velocity using the mechanical energy: you know the exact relation between the linear and rotational velocity, so you can make an expression for the energy.

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2$$

However, do note: **not all rolling is rolling without slipping**. You can ONLY consider something to be rolling without slipping if the problem **says** that it is rolling without slipping, or you can determine through other means that $v = r\omega$. Otherwise, you **can not** rely on this relationship to determine angular velocity from linear velocity, and must rely on angular acceleration, torque, etc.

Oscillation - Unit 6

Simple harmonic motion, or oscillation, can be described with the behaviour of a mass on a spring. If you put a mass on a spring and then compress or stretch it a little, it will expand or contract back into its equilibrium position: however, it will have some velocity because of that. It will then continue its motion until the force of the spring fully stops it. However, at that point, the spring would be slightly compressed or stretched again. It will keep on oscillating forever in ideal situations.

The relationship for the position of the mass in SHM is given by the following equation:

$$x(t) = A\ cos\ cos\ (\omega t - \phi)$$

A is the amplitude of the SHM, and ω is the angular frequency of the SHM, while ϕ is the phase shift.

The relationships for velocity and acceleration are:

$$v(t) = -A\omega\ sin\ sin\ (\omega t - \phi)$$

$$a(t) = -A\omega^2\ cos\ cos\ (\omega t - \phi)\ = -\omega^2 x$$

Now, why is this? We'll need to think with some calculus. You won't need to fully understand the exact calculus as it's a little beyond the scope of the course, but try to follow the general reasoning.

First, we start with Hooke's Law.

$$F_s = -kx$$

We also know that

$$F = ma$$

And that

$$a = \frac{d^2x}{dt^2}$$

We can now equate them.

$$-kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This is called a *second-order ordinary differential equation with constant coefficients*, in standard form. To solve this ODE, we want to look for a function of x(t) where the second derivative with respect to time will look like x, but with a negative sign. The most likely candidates are the trigonometric functions sin(t) and cos(t). We start with cos(t) because cos(0) is 1, and most oscillations begin at maximum amplitude A (you pull and hold the mass on a spring before letting it go, for example). You can always alter the phase shift to get equivalent functions, so it doesn't really matter. We start with the general form of it:

$$x(t) = A\, cos\, cos\, (\omega t - \phi)$$

When we take the derivatives of this, we get:

$$v(t) = -A\omega\, sin\, sin\, (\omega t - \phi)$$

$$a(t) = -A\omega^2\, cos\, cos\, (\omega t - \phi) = -\omega^2 x$$

We can substitute this value of a(t) back into our original equation to find the value of ω.

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$-\omega^2 x + \frac{k}{m}x = 0 \longrightarrow \omega^2 x = \frac{k}{m}x$$

$$\omega = \sqrt{\frac{k}{m}}$$

Other Types of SHM: the Pendulum

You can do this same rigmarole with all sorts of different physical situations. *Any system creating a linear restoring force will exhibit SHM characteristics*.

For example, you can examine a **simple pendulum** at small angles and find that there is essentially a linearly-related restoring force, because of the small-angle approximation. The angular frequency is described in terms of properties of the pendulum (at small angles).

For a simple pendulum:

$$\omega = \sqrt{\frac{g}{l}}$$

Here, l is the length of the pendulum.

You can also do this for **physical pendulums**, which are pendulums that are objects with rotational inertia rotating on a pivot.

$$\omega = \sqrt{\frac{mgL_{cm}}{I}}$$

Here, *I* is the rotational inertia of the swinging object, while L_{cm} is the distance from the center of mass that the pivot point is at. Note that if you pivot it through the center of mass, there cannot be oscillation.

Period

The period of the SHM can be found if you look at how cos(t) behaves.

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

We can think about the period for a few different types of SHM. For a mass on a spring:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

For a simple pendulum (mass on a string):

$$T = 2\pi\sqrt{\frac{l}{g}}$$

For a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{mgL_{cm}}}$$

The frequency for each of these will just be the reciprocal.

Energy in Oscillation

The total energy in a spring-mass system is:

$$E_{total} = \frac{1}{2}kA^2$$

It is the sum of the 2 sources of mechanical energy: $E_{total} = K + U_s$

Gravitation - Unit 7

The magnitude of gravitational force between 2 masses is determined by **Newton's Universal Law of Gravitation**.

Given 2 masses m_1 and m_2, and the distance r_{12} between them, the force that m_1 would exert on m_2 is:

$$F_{12} = G\frac{m_1 m_2}{|r_{12}|^2}\hat{r}_{12}$$

If we are only worried about the magnitude of force, then we can use this form of the law.

$$|F_G| = \frac{Gm_1 m_2}{r^2}$$

G in these equations is the **gravitational constant**, which is about 6.674×10^{-11} Nm^2/kg^2.

Non-Point Masses

If you have objects that cannot be treated as point masses (like symmetrically spherical planets) and take up space (have *spatial extent*), you need to think a little differently to work with them.

Firstly: when a mass *M* is subjected to forces from many other points, the net gravitational force is simply the vector sum of all of the forces that are acting on it.

$$F = \sum_i F_i = GM\left(\sum_{i=1}^{N} \frac{m_i}{r_i^2}\hat{r}_i\right)$$

As $N \longrightarrow \infty$, this summation becomes an integral that we can use to describe objects that take up space.

$$F = \int \quad dF = GM \int \quad \frac{dm}{r^2}\hat{r}$$

Gravitational Field

Generally we describe gravitational force as:

$$F = mg$$

To find ***g***, we need to group the variables from Newton's Law of Universal Gravitation.

$$F = m\left[-\frac{Gm_1}{|r|^2}\hat{r}\right] = mg$$

g is a *vector field function*, and is also the acceleration due to gravity. Like the electric field ***E*** and magnetic field ***B*** in from E&M, ***g*** is also known as the *gravitational field* in field theory.

When we are close to Earth's surface and our distance from Earth's center ***r*** doesn't change very much, ***g*** can be treated as a constant. If $m_1 = m_{earth} = 5.972 \times 10^{24} kg$ and $r = r_{earth} = 6.371 \times 10^6 m$, we can compute a value of 9.81 m/s^2 for ***g***.

In general, the gravitational field ***g*** generated by a point mass *m* shows how it influences the gravitational forces on other masses.

$$g(m, r) = -\frac{Gm}{|r|^2}\hat{r}$$

The net gravitational field at a position ***r*** is given by the vector sum of all the gravitational fields acting there, just like gravitational force.

$$g = \sum_i g_i = G\left(\sum_i \frac{m_i}{r_i^2}\hat{r}_i\right)$$

We can also describe this with an integral for objects with *spatial extent*.

$$g = \int dg = G\int \frac{dm}{r^2}\hat{r}$$

Gravitational Potential Energy

Gravity is a conservative force, and that means that masses in a gravitational field have **potential energy**.

Gravitational potential energy of something in Earth's gravitational influence is given by the equation:

$$U_g = -\frac{Gm_{earth}m_{object}}{r}$$

Obviously you can replace the mass of the Earth with some other body to examine other orbits.

You can see that gravitational potential energy is always negative, unless you are an infinite distance away from Earth (or some other planet), where potential energy would be zero.

Circular Orbit Velocity

If we have a small mass *m* in a circular orbit around a larger mass *M*, then the centripetal force required for the circular motion is supplied by gravitational force.

$$F_g = F_c \longrightarrow \frac{GMm}{r^2} = \frac{mv^2}{r}$$

We solve for *v* and get the **orbital speed** (or orbital velocity), which does NOT depend on the mass of the object that is in orbit.

$$v_{orbit} = \sqrt{\frac{GM}{r}}$$

Note that this ONLY works for objects that are **very small** in comparison to what they orbit.

Orbital Period: Kepler's Third Law

Kepler's Third Law states that the square of the orbital period of something orbiting something else is directly proportional to the cube of the semi-major axis of its orbit (a circle is a special ellipse: the semi-major axis is just the radius). We can derive the orbital period of a circular orbit to see this law in action.

$$T = \frac{distance}{velocity}$$

$$T = \frac{2\pi r}{\sqrt{\frac{GM}{r}}} \longrightarrow T^2 = \frac{4\pi^2 r^2}{\frac{GM}{r}} \longrightarrow T^2 = \frac{4\pi^2 r^3}{GM}$$

Conservation of Angular Momentum and Mechanical Energy

Gravity is always **centripetal** (pointing to the center), and is a conservative force. There are 2 important consequences of orbits that come from this. Firstly: **ANGULAR MOMENTUM IS ALWAYS CONSERVED**. Angular momentum is only affected by forces acting tangentially: centripetal forces never affect angular momentum. Angular momentum is therefore conserved in any type of orbit (even in elliptical orbits, where it will be important. Secondly: **MECHANICAL ENERGY IS ALWAYS CONSERVED**. Gravity is a conservative force, so the sum of the kinetic energy of the orbiting object and the potential energy in the system will always stay the same.

These 2 properties of orbits mean that even if an orbit isn't perfectly circular (like an elliptical orbit), you can determine things about the orbit based on velocity, distance from the central body, etc.