Review of Supersymmetry

Arif Er

Abstract A review and revision of my study of Supersymmetry I studied in the summer of 2020. This review will cover the topics covered in *Supersymmetry DeMYSTiFied* by Labelle [1]

Arif Er
NUS, e-mail: e0032348@u.nus.edu

Chapter 2
Introduction to Weyl spinors

We begin the revision by recapping on the physics of the Weyl spinors and how they are relevant in our study of the Quantum Field Theories. We will then attempt to formulate possible Lorentz invariants from the Dirac spinors so that they can be used in the Lagrangian formalism. Lastly, we will look at the Van der Waarden notation, a more compact and useful notation for Weyl spinors especially in the context of Supersymmetry.

1 The Dirac equation

Our starting point is the Dirac equation. It relates shows how one can obtain the eigenvalue of the momentum operator of a quantum particle.

$$\gamma^\mu P_\mu \psi = m\psi \quad , P_\mu \equiv i\partial_\mu \tag{1}$$

Using the Dirac slash, it is identically

$$\not{P}\psi = m\psi \tag{2}$$

The Lagrangian for a Dirac particle is thus

$$\mathcal{L}_{Dirac} = \bar{\psi}(\gamma^\mu P_\mu - m)\psi \tag{3}$$

The γ^μ used above are the Dirac matrices, 4×4 matrices that are built off the Pauli matrices.

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{4}$$

Using the mostly negative signature metric (i.e. $\eta_\mu\nu = (+,-,-,-)$), the Dirac matrices are:

$$\gamma^\mu = (\gamma^0, \gamma^i) \quad , \quad \gamma_\mu = (\gamma^0, -\gamma^i) \tag{5}$$

We were also introduced another new Dirac matrix, for the fact that it simplifies a large deal of work in the later part of our journey.

$$\gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{6}$$

From the properties of Pauli matrices, we see some interesting results that would turn out to be central to the formulation of the framework.

$$\begin{aligned} \sigma^2(\sigma^i)^T &= -(\sigma^i)\sigma^2 \\ \sigma^2(\sigma^i)^* &= -(\sigma^i)\sigma^2 \end{aligned} \tag{7}$$

$$\begin{aligned} \sigma^2(\sigma^i)\sigma^2 = -(\sigma^i)^T &= -(\sigma^i)^* \\ \therefore \sigma^2(\sigma^i)^T\sigma^2 &= -(\sigma^i) \end{aligned} \tag{8}$$

We thus have the following representation of a vector weighted matrix

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{\sigma}\sigma^j &= A^i\sigma^i\sigma^j \\ &= A^i(\sigma^j\sigma^i - [\sigma^i,\sigma^j]) \\ &= A^i\sigma^j\sigma^i - 2i\varepsilon^{ijk}A^i\sigma^k \\ &= \sigma^j\boldsymbol{A}\cdot\boldsymbol{\sigma} - 2i(\boldsymbol{A}\times\boldsymbol{\sigma}) \end{aligned} \tag{9}$$

2 Dirac spinors

These are reducible 4 component spinors. Their lowest representation is a 2 component spinor, a ‘left-chiral’ and a ‘right-chiral’ spinor.

$$\psi = \begin{pmatrix} \eta \\ \chi \end{pmatrix} \tag{10}$$

Extracting the irreducible spinors is simple with a projection operator. Clearly, if

$$P_R\psi = \begin{pmatrix} \eta \\ 0 \end{pmatrix} \quad , \quad P_L\psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$$

The projection operators have to be:

$$P_R = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad P_L = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

which relates back to the γ_5 matrix in Eqn 6 as

$$\gamma_5 = P_R - P_L \tag{11}$$

We will use this to expand the Dirac equation into its irreducibles.

$$\gamma^\mu P_\mu \psi = m\psi \implies \begin{cases} (E\mathbb{1} + \boldsymbol{\sigma} \cdot \boldsymbol{p})\chi = m\eta \\ (E\mathbb{1} - \boldsymbol{\sigma} \cdot \boldsymbol{p})\eta = m\chi \end{cases} \tag{12}$$

$$= \begin{cases} \bar{\sigma}^\mu P_\mu \chi = m\eta \\ \sigma^\mu P_\mu \eta = m\chi \end{cases} \tag{13}$$

where we are introduced to $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$ to get to Equation 13 from Equation 12

The Dirac Lagrangian in its irreducible forms is thus

$$\mathcal{L}_{Dirac} = \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + \eta^\dagger i\sigma^\mu \partial_\mu \eta - m(\chi^\dagger \eta + \eta^\dagger \chi) \tag{14}$$

This reveals the coupled nature of the Dirac spinor irreducibles, or at least if they are massive. The special case of the massless Dirac spinor are known Weyl spinors. Being massless, we get the Weyl equations that tell us the eigenvalues of the spinors with respect to the $\boldsymbol{\sigma} \cdot \boldsymbol{p}$ operator:

$$\begin{cases} E\eta = \boldsymbol{\sigma} \cdot \boldsymbol{p}\eta \\ E\chi = -\boldsymbol{\sigma} \cdot \boldsymbol{p}\chi \end{cases} \implies \begin{cases} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|p|}\eta = \eta \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|p|}\chi = \chi \end{cases} \tag{15}$$

And very conveniently, making use of the fact that $\boldsymbol{S} \cdot \hat{p}$ is the helicity operator, we see why η and χ are called chiral spinors!

$$\boldsymbol{S} \cdot \hat{p}\eta = \frac{\hbar}{2}\eta \quad , \quad \boldsymbol{S} \cdot \hat{p}\chi = -\frac{\hbar}{2}\chi \tag{16}$$

3 Lorentz invariances

We want to build Lorentz invariants made of Weyl spinors, so that we might add them as interactions when building Lagrangians of any theory that might come along. For this, we need to know how they transform. From the fact that the Dirac Lagrangian is necessarily Lorentz invariant, the terms of Equation 14 also has to be. Let us look into each term carefully.

3.1 Pure terms

The 'pure' terms are $\chi^\dagger \eta$ and $\eta^\dagger \chi$. We know how each spinor transforms under the Lorentz group:

$$\eta \rightarrow \left(\mathbb{1} + \frac{1}{2}i\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma} - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)\eta \tag{17}$$

$$\chi \rightarrow \left(\mathbb{1} + \frac{1}{2}i\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)\chi \tag{18}$$

The difference between the transformation of the left and right chiral spinors is very subtle. But this difference allows us to understand why the 'pure' terms are invariant. Moving on to the remainder of the terms,

3.2 Mixed terms

These are the terms with σ^μ in them: $\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi$ and $\eta^\dagger i\sigma^\mu\partial_\mu\eta$. We know from earlier that $\chi^\dagger\eta$ forms an invariant. Therefore the term that couples with $\chi^\dagger$ has to transform like a right chiral spinor. This means that $i\bar{\sigma^\mu}\partial_\mu\chi$ transforms like a right chiral spinor. This is trivial to prove. One way is to recall the coupled irreducibles in Equation 13. m is an invariant, so the remaining terms have to transform in the same manner as each other. The other way is to explicitly find the transformation rule of $i\bar{\sigma}^\mu\partial_\mu\chi$ to find out that it does indeed transform as a right chiral spinor. It is the same as $i\sigma^\mu\partial_\mu\eta$, which transforms neatly as a left chiral spinor.

Moreover, if we were to do an integration by parts (IbP) on these mixed terms, we will find that:

$$-\partial_\mu(\chi^\dagger i\bar{\sigma}^\mu)\chi = (i\bar{\sigma}^\mu\partial_\mu\chi)^\dagger\chi \tag{19}$$

is also in invariant! Clearly for Equation 19 to make sense with the results in the Section 3.1 $i\bar{\sigma}^\mu\partial_\mu\chi$ has to transform as a right chiral, which again agrees with what we did earlier.

3.3 Using only left chirals

Using the properties of σ^2, we can create Lorentz invariants using only left chirals, without the need of any vector matrices. The transformation of $\chi^{\dagger^T}$ in the contraction $\chi^\dagger\eta$ is

$$\begin{aligned}\chi^{\dagger^T} &\rightarrow \left(\mathbb{1} + \frac{1}{2}i\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)^* \chi^{\dagger^T} \\ &= \left(\mathbb{1} - \frac{1}{2}i\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\right)\chi^{\dagger^T}\end{aligned} \tag{20}$$

To pair this with the left chiral χ, we need to get it to transform as a right chiral, for which is Equation 20 but with the opposite sign. Multiplying throughout by $i\sigma^2$

does exactly that, as we did in Equation 7. Therefore, $i\sigma^2\chi^{\dagger^T}$ transforms like a right chiral!

The invariant we will get out of this combination is done by taking the hermitian conjugate of it and pairing it with χ.

$$\left(i\sigma^2\chi^{\dagger^T}\right)^\dagger = \chi^T\left(-i\sigma^2\right)\chi \tag{21}$$

3.4 Method

From what we have seen so far, Lorentz invariants made of Weyl spinors all require a combination of one left and right chiral spinor, with either of them being a hermitian conjugate. We can extend this further by making use of the fact that $\chi^\dagger\bar{\sigma}^\mu\chi$ is a (1,0) tensor, since P_μ necessarily transforms as a vector under the Lorentz group.

? Creating a rank 2 covariant tensor from left chirals, without derivatives

This is a rather trivial exercise if we work out how the rank 1 covariant tensor $\chi^\dagger\bar{\sigma}^\mu\chi$ transforms. Let us define the transformation of χ as $\chi \to A^{-1}\chi$, where A is naturally a transformation matrix.

$$\begin{aligned}\chi^\dagger\bar{\sigma}^\mu\chi \to \chi'^\dagger\bar{\sigma}'^\mu\chi' &= \chi^\dagger A^{-1^\dagger}\bar{\sigma}^\mu A^{-1}\chi \\ &= \chi^\dagger\Lambda^\mu_\nu\bar{\sigma}^\nu\chi\end{aligned}$$

From this, we know that the transformation of $\bar{\sigma}^\mu$ is

$$\bar{\sigma}^\mu \to A^\dagger\Lambda^\mu_\nu\bar{\sigma}^\nu A$$

Likewise for σ^μ, since $\eta^\dagger\sigma^\mu\eta$ is a rank 1 covariant tensor and $\eta^\dagger\chi$ is invariant (i.e. $\eta \to A^\dagger\eta$

$$\sigma^\mu \to A^{-1}\Lambda^\mu_\nu\sigma^\nu A^{-1^\dagger}$$

Putting them together, we have

$$\sigma^\mu\bar{\sigma}^\nu \to A^{-1}\Lambda^\mu_\alpha\Lambda^\nu_\beta\sigma^\alpha\sigma^\beta A$$

Fitting this with a something that transforms as $? \to A^\dagger$ (i.e. a right chiral spinor) on the left and $? \to A^{-1}$ (i.e. a left chiral spinor) on the right, we will get a rank 2 covariant tensor! Since we are looking to populate these positions with only left chirals, the options are obvious.

$$\chi^T(-i\sigma^2)\sigma^\mu\bar{\sigma}^\nu\chi \to \Lambda^\mu_\alpha\Lambda^\nu_\beta\chi^T\left(-i\sigma^2\right)\sigma^\alpha\bar{\sigma}^\beta\chi$$

4 Van der Waerden notation

Instead of the cumbersome $\pm i\sigma^2$ in between the chiral spinors, the Van der Waerden notation defines a new kind of dot product between chiral spinors. We saw how $\chi^T(-i\sigma^2)\chi$ and $\eta^\dagger(i\sigma^2)\eta^{\dagger^T}$ are invariants, so we shall define 2 dot products as such:

$$\chi \cdot \chi \equiv \chi^T(-i\sigma^2)\chi$$
$$\bar{\chi} \cdot \bar{\chi} \equiv \chi^\dagger(i\sigma^2)\chi^{\dagger^T}$$

Expanding the components,

$$\chi \cdot \chi = \chi_2\chi_1 - \chi_1\chi_2$$
$$\bar{\chi} \cdot \bar{\chi} = \chi_1^\dagger\chi_2^\dagger - \chi_2^\dagger\chi_1^\dagger \tag{22}$$

The good thing about this notation is that it shows that the dot product is now hermitian! So we can employ this directly in the Lagrangian fully knowing that we need not worry about any real-ness violations.

Chapter 3
A new notation

Chapter 4
Weyl, Majorana, and Dirac spinors

In this chapter, we look at close relations between Weyl, Majorana, and Dirac spinors and how we can jump between one to the other (and when we should not). The advantages of understanding this is that it helps paint a clearer picture behind the interpretation of these rather 'abstract' representations of what particles are.

1 Particle-antiparticle

We will begin this chapter by looking at the intricacies of particle-antiparticle existence. The source of their coexistence is by the natural imposition of charge-parity-time-reversal (CPT) invariance on the theory of particles. The particle-antiparticle pair would ensure that total charge, parity, and time-reversal is upheld. In the following text, the particle will be denoted by the subscript p whereas the antiparticle will be denoted by the subscript $\bar{p}$. Their representations as Dirac spinors (in terms of the irreducible Weyl spinors) are

$$\psi_p = \begin{pmatrix} \eta_p \\ \chi_p \end{pmatrix} \quad , \quad \psi_{\bar{p}} = \begin{pmatrix} \eta_{\bar{p}} \\ \chi_{\bar{p}} \end{pmatrix}$$

It should be necessary to mention that it is not the case that $\eta_p = \eta_{\bar{p}}$ and $\chi_p = \chi_{\bar{p}}$.

The relation between the conjugate Dirac spinor and its barred transpose used in the Lagrangian is

$$\psi_{\bar{p}} = \psi_p^C = C\bar{\psi}_p^T \quad , \quad C = -i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}$$

With this,

$$\psi_p^C = \begin{pmatrix} i\sigma^2 \chi_p^{\dagger T} \\ -i\sigma^2 \eta_p^{\dagger T} \end{pmatrix} \implies \begin{cases} \eta_{\bar{p}} = i\sigma^2 \chi_p^{\dagger T} \\ \chi_{\bar{p}} = -i\sigma^2 \eta_p^{\dagger T} \end{cases} \tag{23}$$

We now see how intricately related the Weyl spinors of the particle and antiparticle are. One very important thing to point out is that from Equation 23, we can see that our discussion in Sec 3.3 agrees that $i\sigma^2 {\chi_p^\dagger}^T$ behaves as a right chiral and $-i\sigma^2 {\eta_p^\dagger}^T$ as a left chiral! Using this knowledge, we can get rid of any explicit right chiral representations in the Dirac spinor and simply express it as

$$\psi = \begin{pmatrix} i\sigma^2 {\chi_{\bar{p}}^\dagger}^T \\ \chi_p \end{pmatrix}$$

As mentioned at the start, the particle-antiparticle relations only exists because of the CPT invariance imposed on the Lagrangian. The other necessary constraint is that the Lagrangian needs to be real. (i.e. $\mathcal{L}^\dagger = \mathcal{L}$) This constraint tells us that the Lagrangian should either have both the hermitian conjugates of any chiral spinors, or none at all. The interpretation of this in QFT is very physical. There has to either have both the annihilation and creation operator of a particle, or none at all. Number operators will thus be a conserved operation.

The simplest contributor to the Lagrangian that has both left chiral spinors and ensures CPT and reality invariance is

$$\mathcal{L} = \chi^\dagger i\sigma^\mu \partial_\mu \chi$$

There is both a left chiral particle spinor creation and annihilation field operator in this Lagrangian.

However, using the relation in Equation 23, the same Lagrangian then becomes

$$\mathcal{L} = \eta_{\bar{p}}^T (i\sigma^2) i\sigma^\mu \partial_\mu \chi$$

which is a left chiral particle spinor and right chiral antiparticle spinor creation field operator! A very thought-provoking interpretation of the particle-antiparticle relationship.

Insofar as we have used the term Weyl spinor, we have used it to identify particles that are both eigenstates of the helicity operator and the chiralty operator. However, there is a very subtle difference between the two that paints very different pictures of what a Weyl spinor really is. As eigenstates of the helicity operator, Weyl spinors are necessarily massless as shown in 16. However, as eigenstates of the chiralty operator, they are simply eigenstates with fixed transformation rules under the $SU(2) \times SU(2)$ Lorentz group as in Equations 17 and 18. Thus, there is no constraint on them being massless. They can be as massive as they need be, as long as they are eigenstates that of the Lorentz group. However, for the sake of continuing the discussion regarding massive particles using the Weyl spinor representation, we shall adopt the convention of the latter, while duly keeping in mind that actual Weyl spinors are necessarily massless.

Returning to the CPT invariance, we now see that we have a scheme that relates η_p to $\chi_{\bar{p}}$ and its conjugates. Through the Lorentz transformation, it is also possible (**for massive particles**) for the chiralty of the particle to change, i.e from η_p to χ_p

and vice versa. These 4 particles are thus related to each other as a multiplet that must exist as a collective state. It is because of this fact that we are allowed to express the right chiral particle as the left chiral antiparticle with impunity. This is evident in how the mass term of the Lagrangian can be expressed in either of the following representations:

$$\begin{aligned} m\bar{\psi}\psi &= m(\chi^\dagger \eta + \eta^\dagger \chi) \\ &= m(\chi \cdot \chi + \bar{\chi} \cdot \bar{\chi}) \end{aligned} \tag{24}$$

2 Majorana spinors

The Majorana is a special subset of (massive) Dirac spinors. Its antiparticle state is the same as its particle state, i.e. $\eta_P = \eta_{\bar{P}}$ and $\chi_P = \chi_{\bar{\chi}}$. Unlike the general Dirac spinor, we now have 2 degrees of freedom instead of 4. The Majorana spinor in left chiral representation is

$$\psi_M = \begin{pmatrix} i\sigma^2 \chi_P^{\dagger T} \\ \chi_P \end{pmatrix} \tag{25}$$

As good as the Majorana and Weyl representations are, it is not possible to build actual theories using them only as parity is not conserved. In the Lagrangian formalism of strictly Weyl or Majorana spinors, the mass terms will only be mass terms of left chirals, with no way of satisfying the parity between left and right chirals.

Looking at the two from another angle, we see that

$$\begin{cases} \bar{\psi}_M \psi_M = \chi \cdot \chi + \bar{\chi} \cdot \bar{\chi} \\ \bar{\psi}_M \gamma_5 \psi_M = -\chi \cdot \chi + \bar{\chi} \cdot \bar{\chi} \end{cases}$$

which with some simple manipulation and generalisation, simply gives us

$$\begin{cases} \lambda \cdot \chi = \bar{\Lambda}_M P_L \psi_M \\ \bar{\lambda} \cdot \bar{\chi} = \bar{\Lambda}_M P_R \psi_M \end{cases}$$

Lastly, making use of the fact that γ^μ can be represented off-diagonally as

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \tag{26}$$

we have

$$\begin{cases} \bar{\psi}_M \gamma^\mu \Lambda_M = \chi^\dagger \bar{\sigma}^\mu \lambda - \lambda^\dagger \bar{\sigma}^\mu \chi \\ \bar{\psi}_M \gamma_5 \gamma^\mu \Lambda_M = \chi^\dagger \bar{\sigma}^\mu \lambda + \lambda^\dagger \bar{\sigma}^\mu \chi \end{cases}$$

$$\implies \begin{cases} \chi^\dagger \bar{\sigma}^\mu \lambda = \bar{\psi}_M P_R \gamma^\mu \Lambda_M \\ \lambda^\dagger \bar{\sigma}^\mu \chi = -\bar{\psi}_M P_L \gamma^\mu \Lambda_M \end{cases}$$

a very neat relation between the Weyl and Majorana representations of the **massive** Majorana spinor.

Chapter 5
Building the Lagrangian

We will build attempt at building the most basic Lagrangians with the invariants and constraints from the previous chapters. Several constraints on the Lagrangian will have to be imposed to ground our discussion in renormalisable theories.

1 Dimensionfull Lagrangians

In building the Lagrangian, we shall keep to renormalisable theories where $D = 4$ is maximally the further we will go in dimensions. The dimensions are defined in terms of powers of energy and as should be, the natural units are 1 (thus being dimensionless). With this, scalar fields are of dimension 1, derivatives are of dimension 1, fermion fields are of dimension $3/2$. Simple dimensional analysis will give us these.

2 The simplest Lagrangian

Let us consider the simplest toy model we can make – a single free massless fermionic pair and a single free massless bosonic pair. They do not interact with each other (this will be introduced in Chapter 6). The Lagrangian is simply

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi \tag{27}$$

The transformation of these fields are

$$\begin{aligned} \phi &\to \phi + \delta\phi \\ \chi &\to \chi + \xi\chi \end{aligned}$$

Let us bring in the postulate of supersymmetry – that bosons will transform into fermions and vice versa.

$$\delta\phi \propto \xi\chi \quad , \quad \xi \ll 1 \tag{28}$$

For Equation 28 to satisfy the dimensionality of both sides of the equation, we see that ξ has to be a Grassmann spinor of dimension $-1/2$. To actually determine the proportionality of the relationship in Equation 28, we have to impose the Lorentz invariance of the Lagrangian to obtain any more information.

Since ξ is spinor, we have the freedom to pick a left chiral spinor, so we can have

$$\delta\phi = \xi \cdot \chi \tag{29}$$

which is fully Lorentz invariant and a valid term in a Lagrangian.

We move on to the transformation of the fermion.

$$\delta\chi = -i(\partial\phi)\sigma^{\mu}(i\sigma^{2})\xi^{*} \tag{30}$$

where we obtained this the same way, by imposing the equality of dimensions on both sides of the equation, Lorentz invariablity, and the reality of the Lagrangian.

Chapter 6
SUSY Charges

We will start off with a short review on the necessary elements we need to know about symmetry chargers. We will look at how to derive and interpret the charges that make up supersymmetry. Once we have the charges, we will attempt to obtain the supersymmetry transformation rules of fermions, bosons, and auxiliary fields.

1 Quick review

We define a unitary transformation as $U \equiv \exp[\pm i\varepsilon \cdot Q]$ where ε is an infinitesimal factor and Q is the charge of the symmetry. The dot product implies that there may be more than 1 charge to the symmetry. The transformation of a state in the symmetry is defined as $\phi'(x) \equiv U\phi(x)U^\dagger$

The transformation of a state in the symmetry may be defined in 2 manners: through the unitary transformations; or through a varaince.

$$\begin{aligned} \phi'(x) &\equiv U\phi(x)U^\dagger \\ &\equiv \phi(x) + \delta\phi(x) \end{aligned}$$

Making use of the fact that ε is infinitesimal, we may, to leading order of ε, relate the variance of $\phi(x)$ to the commutator relation of Q and ϕ.

$$\delta\phi(x) = \pm i[\varepsilon \cdot Q, \phi(x)] \tag{31}$$

The explicit representations of the charges may be obtained from either of 2 way: through the symmetry currents in Equation 32; or as differential operators in Equation 33. We will obtain the algebra of the symmetry if we work out all the commutator relations of the charges in the symmetry.

$$Q^i = \int \mathrm{d}^3 x J_0^i(\boldsymbol{x}, t) \tag{32}$$

$$\phi(x') \equiv \exp[\pm i\varepsilon \cdot \hat{Q}]\phi(x) \tag{33}$$

where here we emphasise $\hat{Q}$ is a differential operator through the hat notation.

Instead of finding the explicit representations of the charges to determine the algebra of the symmetry, we may consider the charges as quantum field operators and work it out. Consider 2 consecutive transformations, with infinitesimal factors α and β:

$$\begin{aligned} U_\beta U_\alpha \phi U_\alpha^\dagger U_\beta^\dagger &\approx \phi + i[\alpha \cdot Q, \phi] + i[\beta \cdot Q, \phi] - [\beta \cdot Q, [\alpha \cdot Q, \phi]] + \ldots \\ &= \delta_\beta \delta_\alpha \phi \end{aligned} \tag{34}$$

Working out the opposite order,

$$[\delta_\beta, \delta_\alpha]\phi = \big[[\alpha \cdot Q, \beta \cdot Q], \phi\big] \tag{35}$$

2 Deriving the supersymmetric charges

We have 2 charges to consider, since there are 4 degrees of freedom that are be grouped as 2 pairs of spinors (i.e. ξ, ξ^*). Using the transformation rules from the previous chapter, we can now express them in terms of the SUSY charge commutator relations as

$$\big[iQ \cdot \xi + i\bar{Q} \cdot \bar{\xi}\big] = -i\xi \cdot \chi \tag{36}$$

$$\big[iQ \cdot \xi + i\bar{Q} \cdot \bar{\xi}\big] = -i(\partial_\mu \phi)\sigma^\mu \sigma^2 \xi^* \tag{37}$$

Since ξ and ξ^* are independent, we see that the only non-vanishing terms are:

$$[\xi \cdot Q, \phi] = -i\xi \cdot \chi \tag{38}$$

$$\big[\bar{\xi} \cdot \bar{Q}, \chi\big] = -i(\partial_\mu \phi)\sigma^\mu \sigma^2 \xi^* \tag{39}$$

Matching the charges with each other in a commutator relation, we get the following:

$$\begin{aligned} [Q \cdot \xi, Q \cdot \beta] &= (\sigma^2)^{ab}(\sigma^2)^{cd}\xi_b \beta_d \{Q_a, Q_c\} \\ \big[Q \cdot \xi, \bar{Q} \cdot \bar{\beta}\big] &= -(\sigma^2)^{ab}(\sigma^2)^{cd}\xi_b \beta_d^* \{Q_a, Q_c^\dagger\} \\ \big[\bar{Q} \cdot \bar{\xi}, Q \cdot \beta\big] &= (\sigma^2)^{ab}(\sigma^2)^{cd}\xi_b^* \beta_d \{Q_a^\dagger, Q_c\} \\ \big[\bar{Q} \cdot \bar{\xi}, \bar{Q} \cdot \bar{\beta}\big] &= (\sigma^2)^{ab}(\sigma^2)^{cd}\xi_b^* \beta_d^* \{Q_a^\dagger, Q_c^\dagger\} \end{aligned} \tag{40}$$

The algebra of the symmetry is embedded in the anti-commutator relations in these equations.

With these, we know how to get

$$[\delta_\beta, \delta_\xi]\phi = \Big[\,\big[Q \cdot \xi + \bar{Q} \cdot \bar{\xi}, Q \cdot \beta + \bar{Q} \cdot \bar{\beta}\big], \phi\Big] \equiv [O, \phi] \tag{41}$$

Expanding the LHS of Equation 41 for a boson,

$$\begin{aligned}
\left[\delta_\beta, \delta_\xi\right] \phi &= -i(\xi^\dagger \bar{\sigma}^\mu \beta - \beta^\dagger \xi)\partial_\mu \phi \\
&= (\xi^T \sigma^2 \sigma^\mu \sigma^2 \beta^* - \beta^T \sigma^2 \sigma^\mu \sigma^2 \xi^*)[P_\mu, \phi] \\
\therefore O &= (\xi^T \sigma^2 \sigma^\mu \sigma^2 \beta^* - \beta^T \sigma^2 \sigma^\mu \sigma^2 \xi^*) P_\mu \\
&= -(\sigma^2)^{ab}(\sigma^2)^{cd}(\xi_b \beta^*_d \sigma^\mu_a c + \xi^*_b \beta_d \sigma^\mu_{ca}) P_\mu
\end{aligned} \tag{42}$$

Comparing against the coefficients in Equations 31, we will arrive at the following anti-commutator relations:

$$\begin{aligned}
\{Q_a, Q_c\} &= \{Q^\dagger_a, Q^\dagger_c\} = 0 \\
\{Q_a, Q^\dagger_c\} &= \sigma^\mu_{ac} P_\mu \\
\{Q^\dagger_a, Q_c\} &= \sigma^\mu_{ca} P_\mu
\end{aligned}$$

and by normalising the charges, $Q \rightarrow Q/\sqrt{2}$, the non-vanishing anti-commutator relations are

$$\{Q_a, Q^\dagger_c\} = 2\sigma^\mu_{ac} P_\mu \tag{43}$$

$$\{Q^\dagger_a, Q_c\} = 2\sigma^\mu_{ca} P_\mu \tag{44}$$

Note that since Q and $\bar{Q}$ are spacetime independent, the algebra between the momentum Poincaré charges and supersymmetric charges necessarily vanish.

$$[Q, P_\mu] = [Q^\dagger, P_\mu] = 0 \tag{45}$$

However, the angular Poincaeé charges and supersymmetric charges do not vanish.

$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu})^b_a Q_b \quad , \quad \sigma_{\mu\nu} \equiv \frac{i}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu) \tag{46}$$

If we were to conclude that the algebra for SUSY is complete with this, we would be sorely mistaken as it does not close for the spinor fields as they are now. This is simply because the spinor fields we have are on-shell spinors with a total of 2 degrees of freedom, 2 short of the bosonic degrees of freedom. To handle this, we will have to introduce auxiliary fields that will vanish on-shell while accounting for the missing 2 degrees of freedom. We will allow the auxiliary fields to be bosonic. Since they must vanish on-shell, the simplest form they can take is $F^\dagger F$. Naturally, the dimension for the auxiliary field has to be 2 in the Lagrangians we have been working in. The free field Lagrangian is now:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F \tag{47}$$

The explicit transformation rule of F needs to be linear in the infinitesimal ξ and one other field, all while ensuring its dimension and Lorentz invariance. The right choice of δF is

$$\delta F = K\xi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\chi \tag{48}$$

To ensure that this addition of the auxiliary field to the Lagrangian will not interfere with the overall invariance, we have to apply a variance on the fields.

$$\delta(F^{\dagger}F) = (K^{*}\xi F)^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\chi - \chi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}(K^{*}\xi F) \tag{49}$$

Noticing that this is similar in structure to the variance of the free spinor fields in Equation 50, we can define a new spinor field as in Equation 51.

$$\delta(\chi^{\dagger}i\bar{\sigma}^{\mu}\partial_{\mu}\chi) = (\delta\chi)^{\dagger}i\bar{\sigma}^{\mu}\partial_{\mu}\chi + \chi^{\dagger}i\bar{\sigma}^{\mu}\partial_{\mu}(\delta\chi) \tag{50}$$

$$\delta\tilde{\chi} \equiv \delta\chi - iK^{*}\xi F \tag{51}$$

Because of the freedom we have for K, we can conveniently set it to i. This way, our new Lagrangian in Equation 47 will be closed under the SUSY algebra in Equations 44 and 44 with the following field super-transformations:

$$\begin{aligned} \delta\phi &= \xi\cdot\chi \\ \delta\chi &= -i\sigma^{\mu}(i\sigma^{2}\xi^{*})\partial_{\mu}\phi + F\xi \\ \delta F &= -i\xi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\chi \end{aligned} \tag{52}$$

Chapter 7
Applications of SUSY algebra

Continuing off the previous chapter, we will work on the SUSY algebra to create the SUSY multiplets. We will also formulate the algebra again, but in the Majorana form to see the significance of its interpretation over the Weyl spinor representation. We will then attempt to find the explicit forms of the supercharges both as symmetry currents and quantum filed operators. Lastly, we will discuss about the extension of the algebra into areas outside of SUSY.

1 Casimir Operators

The Casimir operators are operators which commute with every generator of a group. Because of that, its eigenvalues can be used to classify the group representations. For example, $P^\mu P_\mu$ is a Casimir operator of the Poincaré group with an eigenvalue m^2. Another (more useful) Casimir operator of the Poincaré group is the Pauli-Lubonski operator W^μ.

$$W^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho} M_{\rho\sigma} P_\nu \tag{53}$$

For a massive particle, the Pauli-Lubonski operator gives the total angular momentum of a particle as its eigenvalue.

$$W^i \left|p\right\rangle = m(L^i + S^i)\left|p\right\rangle \tag{54}$$

Contracting it with itself and applying onto a particle at rest,

$$W_\mu W^\mu \left|p\right\rangle = -m^3 s(s+1)\left|p\right\rangle \tag{55}$$

On the other hand, on a massless particle, the eigenvalue of the Pauli-Lubonski operator is the helicity of the particle. Take for example a massless particle in with an angle of rotation in the z-axis (i.e. $P^\mu \left|p\right\rangle = (E, 0, 0, E)\left|p\right\rangle$).

$$W^\mu \left|p\right\rangle = (Es_z, 0, 0, Es_z)\left|p\right\rangle \tag{56}$$

2 Applying onto supercharges

Applying the Pauli-Lubonski operator to the supercharges in a commutator relation,

$$[Q_a, W^0] = -\frac{1}{2}(\sigma^3)^b_a Q_b P_3 \tag{57}$$

The only non-zero terms of σ^3 are the diagonal terms so what we essentially have is:

$$[Q_1, W_0] = -\frac{1}{2}Q_1 P_3 \tag{58}$$

$$[Q_2, W_0] = \frac{1}{2}Q_2 P_3 \tag{59}$$

We can use this to derive what the supercharges do onto a particle state.

$$\begin{aligned} W_0(Q_1 |p, h\rangle) &= [W_0, Q_1] |p, h\rangle + Q_1 W_0 |p, h\rangle \\ &= E(h + \frac{1}{2})Q_1 |p, h\rangle \\ \implies Q_1 |p, h\rangle &= |p, h + \frac{1}{2}\rangle \end{aligned} \tag{60}$$

Likewise for Q_2:

$$Q_2 |p, h\rangle = |p, h - \frac{1}{2}\rangle \tag{61}$$

From Equations 59 and 60, we now know that Q_1 raises the helicity of the particle by $\frac{1}{2}$ whereas Q_2 lowers the helicity of the particle by $\frac{1}{2}$.

3 Building the SUSY multiplets

Let us look at the massless, rest particle. Recalling the SUSY algebra in Equation 43, the algebra all depend on $\sigma^\mu_{ab} P_\mu$.

$$\sigma^\mu_{ab} P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2P^0 \end{pmatrix} |p, h\rangle \tag{62}$$

Thus, the only non-vanishing algebra is

$$\{Q_2, Q_2^\dagger\} = 2E |p, h\rangle \tag{63}$$

The vanishing algebra also sheds some insight into the inner workings of the multiplet.

$$\{Q_1, Q_1^\dagger\} = 0 \implies Q_1 |p, h\rangle = Q_1^\dagger |p, h\rangle = 0 \tag{64}$$

This means that in SUSY multiplet, there is a minimum helicity to consider!

Let us define the minimum helicity h_{min}

$$Q_2 \left|p, h_{min}\right\rangle = 0 \quad , \quad Q_2^\dagger \left|p, h_{min}\right\rangle = \left|p, h_{min} + \frac{1}{2}\right\rangle \tag{65}$$

Moreover,

$$\{Q_2^\dagger, Q_2^\dagger\} = 0 \implies Q_2^\dagger Q_2^\dagger \left|p, h_{min}\right\rangle = 0 \tag{66}$$

this implies that the multiplet has only 2 states of the same momentum, but a helical difference of $1/2$.

To make the duet CPT invariant, we need to add the CPT conjugates of each of the 2 states. Thus, there needs to be 4 states to a SUSY multiplet. For example, if $h_{min} = 0$, we will have a scalar multiplet with helicities $0, 0, \frac{1}{2}, -\frac{1}{2}$; if $h_{min} = \frac{1}{2}$, we will have a vector multiplet with helicities $\frac{1}{2}, 1, -\frac{1}{2}, -1$.

4 Supercharges through symmetry currents

The general Lagrangian made of complex scalar fields is:

$$\mathcal{L} = \mathcal{L}(\phi, \phi^\dagger, \partial_\mu\phi, \partial_\mu\phi^\dagger) \tag{67}$$

The variance of the Lagrangian is:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi^\dagger}\delta\phi^\dagger + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} \tag{68}$$

where on-shell,

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial(\partial_\mu\phi)} \tag{69}$$

giving us

$$\partial_\mu\mathcal{K}^\mu \equiv \delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\delta\phi^\dagger\right] \tag{70}$$

where $\mathcal{K}^\mu$ is introduced as we know that in general, the variance of the Lagrangian can total differential as it will disappear in the integral. The terms in the bracket are the Noether's current, denoted by j^μ. The conserved current J^μ is thus defined as:

$$J^\mu = j^\mu - K^\mu \tag{71}$$

The supercharges are derived using the 0-th index of J^μ as in Equation 32.

5 VEV of the Hamiltonian

From

$$\{Q_a, Q_b^\dagger\} = \sigma^\mu P_\mu \implies \langle \{Q_1, Q_1^\dagger\} + \{Q_2, Q_2^\dagger\} \rangle = 2 \langle \mathcal{H} \rangle \tag{72}$$

The positive-definitivity of the LHS implies that $\langle \mathcal{H} \rangle \geq 0$. The equality is achieved when both supercharges annihilate the vacuum state, and by extension, a strict inequality is enforced when the supercharges do not annihiliate the vacuum state – spontaneous supersymmetry breaking.

6 SUSY in the Majorana Form

Recall that the right chiral spinor of the Majorana spinor is related to its left chiral spinor as $\eta = i\sigma^2 \chi^{\dagger T}$. This means that all 4 components of the Majorana supercharge can be expressed as

$$Q_M \equiv \begin{pmatrix} i\sigma^2 Q^{\dagger T} \\ Q \end{pmatrix} = \begin{pmatrix} -Q_2^\dagger \\ -Q_1^\dagger \\ Q_1 \\ Q_2 \end{pmatrix} \tag{73}$$

7 Explicit supercharges

In Equations 32, we saw how to obtain the supercharges explicitly using the conserved symmetry current. To get the expression for the conserved symmetry current, we need the Noether's current (j^μ) and the surface differential terms that might have been 'discarded' in the derivation of the Lagrangian ($\partial_\mu K^\mu$).

For example, in the free supersymmetric Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F \tag{74}$$

its conserved current is

$$\mathcal{J}^\mu_{SUSY} = (\partial_\nu \phi) \chi^\dagger \bar{\sigma}^\mu \sigma^\nu (i\sigma^2) \xi^* - (\partial_\nu \phi^\dagger) \xi^T (i\sigma^2) \sigma^\nu \bar{\sigma}^\mu \chi \tag{75}$$

Putting Equation 74 into Equation 32, we get

$$\xi \cdot Q + \bar{\xi} \cdot \bar{Q} = \int \mathrm{d}^3 x (\partial_\nu \phi) \chi^\dagger \bar{\sigma}^\mu \sigma^\nu (i\sigma^2) \xi^* - (\partial_\nu \phi^\dagger) \xi^T (i\sigma^2) \sigma^\nu \bar{\sigma}^\mu \chi \tag{76}$$

Comparing the coefficients of ξ and ξ^*,

$$Q = \int \mathrm{d}^3 x \partial_\nu \phi^\dagger \sigma^\nu \bar{\sigma} \mu \chi \tag{77}$$

$$Q^\dagger = \int \mathrm{d}^3 x \chi^\dagger \bar{\sigma}^\mu \sigma^\nu \partial_\nu \phi^\dagger \tag{78}$$

Two more identities that need to be included when using the explicit charges are

$$\left[\phi(x,t), \dot{\phi}^\dagger(y,t)\right] = i\delta^3(x-y) \tag{79}$$

$$\left\{\chi_a(x,t), \chi_b^\dagger(y,t)\right\} = \delta_{ab}\delta^3(x-y) \tag{80}$$

With these 4 equations, the explicit supercharges may be used freely in applications such as verifying the field transformations.

References

1. P. LaBelle, *Supersymmetry DeMYSTiFied* (McGraw-Hill Education, 2010). URL https://books.google.co.jp/books?id=SWHhdGbxukkC