Review of Supersymmetry

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Abstract A review and revision of my study of Supersymmetry I studied in the summer of 2020. This review will cover the topics covered in *Supersymmetry DeMYSTiFied* by Labelle [1]

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Chapter 2
Introduction to Weyl spinors

We begin the revision by recapping on the physics of the Weyl spinors and how they are relevant in our study of the Quantum Field Theories. We will then attempt to formulate possible Lorentz invariants from the Dirac spinors so that they can be used in the Lagrangian formalism. Lastly, we will look at the Van der Waarden notation, a more compact and useful notation for Weyl spinors especially in the context of Supersymmetry.

1 The Dirac equation

Our starting point is the Dirac equation. It relates shows how one can obtain the eigenvalue of the momentum operator of a quantum particle.

$$\gamma^\mu P_\mu \psi = m\psi \quad , P_\mu \equiv i\partial_\mu \tag{2.1}$$

Using the Dirac slash, it is identically

$$\not{P}\psi = m\psi \tag{2.2}$$

The Lagrangian for a Dirac particle is thus

$$\mathcal{L}_{Dirac} = \bar{\psi}(\gamma^\mu P_\mu - m)\psi \tag{2.3}$$

The γ^μ used above are the Dirac matrices, 4×4 matrices that are built off the Pauli matrices.

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{2.4}$$

Using the mostly negative signature metric (i.e. $\eta_\mu\nu = (+,-,-,-)$), the Dirac matrices are:

$$\gamma^\mu = (\gamma^0, \gamma^i) \quad , \quad \gamma_\mu = (\gamma^0, -\gamma^i) \tag{2.5}$$

We were also introduced another new Dirac matrix, for the fact that it simplifies a large deal of work in the later part of our journey.

$$\gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{2.6}$$

From the properties of Pauli matrices, we see some interesting results that would turn out to be central to the formulation of the framework.

$$\begin{aligned} \sigma^2(\sigma^i)^T &= -(\sigma^i)\sigma^2 \\ \sigma^2(\sigma^i)^* &= -(\sigma^i)\sigma^2 \end{aligned} \tag{2.7}$$

$$\begin{aligned} \sigma^2(\sigma^i)\sigma^2 = -(\sigma^i)^T &= -(\sigma^i)^* \\ \therefore \sigma^2(\sigma^i)^T\sigma^2 &= -(\sigma^i) \end{aligned} \tag{2.8}$$

We thus have the following representation of a vector weighted matrix

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{\sigma}\sigma^j &= A^i\sigma^i\sigma^j \\ &= A^i(\sigma^j\sigma^i - [\sigma^i,\sigma^j]) \\ &= A^i\sigma^j\sigma^i - 2i\varepsilon^{ijk}A^i\sigma^k \\ &= \sigma^j\boldsymbol{A}\cdot\boldsymbol{\sigma} - 2i(\boldsymbol{A}\times\boldsymbol{\sigma}) \end{aligned} \tag{2.9}$$

2 Dirac spinors

These are reducible 4 component spinors. Their lowest representation is a 2 component spinor, a ‘left-chiral’ and a ‘right-chiral’ spinor.

$$\psi = \begin{pmatrix} \eta \\ \chi \end{pmatrix} \tag{2.10}$$

Extracting the irreducible spinors is simple with a projection operator. Clearly, if

$$P_R\psi = \begin{pmatrix} \eta \\ 0 \end{pmatrix} \quad , \quad P_L\psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$$

The projection operators have to be:

$$P_R = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad P_L = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

which relates back to the γ_5 matrix in Eqn 2.6 as

$$\gamma_5 = P_R - P_L \tag{2.11}$$

We will use this to expand the Dirac equation into its irreducibles.

$$\gamma^\mu P_\mu \psi = m\psi \implies \begin{cases} (E\mathbb{1} + \boldsymbol{\sigma} \cdot \boldsymbol{p})\chi = m\eta \\ (E\mathbb{1} - \boldsymbol{\sigma} \cdot \boldsymbol{p})\eta = m\chi \end{cases} \tag{2.12}$$

$$= \begin{cases} \bar{\sigma}^\mu P_\mu \chi = m\eta \\ \sigma^\mu P_\mu \eta = m\chi \end{cases} \tag{2.13}$$

where we are introduced to $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$ to get to Equation 2.13 from Equation 2.12

The Dirac Lagrangian in its irreducible forms is thus

$$\mathcal{L}_{Dirac} = \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + \eta^\dagger i\sigma^\mu \partial_\mu \eta - m(\chi^\dagger \eta + \eta^\dagger \chi) \tag{2.14}$$

This reveals the coupled nature of the Dirac spinor irreducibles, or at least if they are massive. The special case of the massless Dirac spinor are known Weyl spinors. Being massless, we get the Weyl equations that tell us the eigenvalues of the spinors with respect to the $\boldsymbol{\sigma} \cdot \boldsymbol{p}$ operator:

$$\begin{cases} E\eta = \boldsymbol{\sigma} \cdot \boldsymbol{p}\eta \\ E\chi = -\boldsymbol{\sigma} \cdot \boldsymbol{p}\chi \end{cases} \implies \begin{cases} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{|p|}\eta = \eta \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{|p|}\chi = \chi \end{cases} \tag{2.15}$$

And very conveniently, making use of the fact that $\boldsymbol{S} \cdot \hat{p}$ is the helicity operator, we see why η and χ are called chiral spinors!

$$\boldsymbol{S} \cdot \hat{p}\eta = \frac{\hbar}{2}\eta \quad , \quad \boldsymbol{S} \cdot \hat{p}\chi = -\frac{\hbar}{2}\chi \tag{2.16}$$

3 Lorentz invariances

We want to build Lorentz invariants made of Weyl spinors, so that we might add them as interactions when building Lagrangians of any theory that might come along. For this, we need to know how they transform. From the fact that the Dirac Lagrangian is necessarily Lorentz invariant, the terms of Equation 2.14 also has to be. Let us look into each term carefully.

3.1 Pure terms

The ‘pure’ terms are $\chi^\dagger \eta$ and $\eta^\dagger \chi$. We know how each spinor transforms under the Lorentz group:

$$\eta \rightarrow \left(\mathbb{1} + \frac{1}{2} i\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma} - \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\sigma}\right) \eta \tag{2.17}$$

$$\chi \rightarrow \left(\mathbb{1} + \frac{1}{2} i\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\sigma}\right) \chi \tag{2.18}$$

The difference between the transformation of the left and right chiral spinors is very subtle. But this difference allows us to understand why the 'pure' terms are invariant. Moving on to the remainder of the terms,

3.2 Mixed terms

These are the terms with σ^μ in them: $\chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi$ and $\eta^\dagger i\sigma^\mu \partial_\mu \eta$. We know from earlier that $\chi^\dagger \eta$ forms an invariant. Therefore the term that couples with $\chi^\dagger$ has to transform like a right chiral spinor. This means that $i\bar{\sigma^\mu} \partial_\mu \chi$ transforms like a right chiral spinor. This is trivial to prove. One way is to recall the coupled irreducibles in Equation 2.13. m is an invariant, so the remaining terms have to transform in the same manner as each other. The other way is to explicitly find the transformation rule of $i\bar{\sigma}^\mu \partial_\mu \chi$ to find out that it does indeed transform as a right chiral spinor. It is the same as $i\sigma^\mu \partial_\mu \eta$, which transforms neatly as a left chiral spinor.

Moreover, if we were to do an integration by parts (IbP) on these mixed terms, we will find that:

$$-\partial_\mu(\chi^\dagger i\bar{\sigma}^\mu)\chi = (i\bar{\sigma}^\mu \partial_\mu \chi)^\dagger \chi \tag{2.19}$$

is also in invariant! Clearly for Equation 2.19 to make sense with the results in the Section 3.1 $i\bar{\sigma}^\mu \partial_\mu \chi$ has to transform as a right chiral, which again agrees with what we did earlier.

3.3 Using only left chirals

Using the properties of σ^2, we can create Lorentz invariants using only left chirals, without the need of any vector matrices. The transformation of ${\chi^\dagger}^T$ in the contraction $\chi^\dagger \eta$ is

$$\begin{aligned} {\chi^\dagger}^T &\rightarrow \left(\mathbb{1} + \frac{1}{2} i\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\sigma}\right)^* {\chi^\dagger}^T \\ &= \left(\mathbb{1} - \frac{1}{2} i\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\sigma}\right) {\chi^\dagger}^T \end{aligned} \tag{2.20}$$

To pair this with the left chiral χ, we need to get it to transform as a right chiral, for which is Equation 2.20 but with the opposite sign. Multiplying throughout by

$i\sigma^2$ does exactly that, as we did in Equation 2.7. Therefore, $i\sigma^2\chi^{\dagger T}$ transforms like a right chiral!

The invariant we will get out of this combination is done by taking the hermitian conjugate of it and pairing it with χ.

$$\left(i\sigma^2\chi^{\dagger T}\right)^\dagger = \chi^T\left(-i\sigma^2\right)\chi \tag{2.21}$$

3.4 Method

From what we have seen so far, Lorentz invariants made of Weyl spinors all require a combination of one left and right chiral spinor, with either of them being a hermitian conjugate. We can extend this further by making use of the fact that $\chi^\dagger\bar{\sigma}^\mu\chi$ is a (1,0) tensor, since P_μ necessarily transforms as a vector under the Lorentz group.

? Creating a rank 2 covariant tensor from left chirals, without derivatives

This is a rather trivial exercise if we work out how the rank 1 covariant tensor $\chi^\dagger\bar{\sigma}^\mu\chi$ transforms. Let us define the transformation of χ as $\chi \to A^{-1}\chi$, where A is naturally a transformation matrix.

$$\begin{aligned}\chi^\dagger\bar{\sigma}^\mu\chi \to \chi'^\dagger\bar{\sigma}'^\mu\chi' &= \chi^\dagger {A^{-1}}^\dagger\bar{\sigma}^\mu A^{-1}\chi\\ &= \chi^\dagger\Lambda^\mu_\nu\bar{\sigma}^\nu\chi\end{aligned}$$

From this, we know that the transformation of $\bar{\sigma}^\mu$ is

$$\bar{\sigma}^\mu \to A^\dagger\Lambda^\mu_\nu\bar{\sigma}^\nu A$$

Likewise for σ^μ, since $\eta^\dagger\sigma^\mu\eta$ is a rank 1 covariant tensor and $\eta^\dagger\chi$ is invariant (i.e. $\eta \to A^\dagger\eta$

$$\sigma^\mu \to A^{-1}\Lambda^\mu_\nu\sigma^\nu {A^{-1}}^\dagger$$

Putting them together, we have

$$\sigma^\mu\bar{\sigma}^\nu \to A^{-1}\Lambda^\mu_\alpha\Lambda^\nu_\beta\sigma^\alpha\sigma^\beta A$$

Fitting this with a something that transforms as ? $\to A^\dagger$ (i.e. a right chiral spinor) on the left and ? $\to A^{-1}$ (i.e. a left chiral spinor) on the right, we will get a rank 2 covariant tensor! Since we are looking to populate these positions with only left chirals, the options are obvious.

$$\chi^T(-i\sigma^2)\sigma^\mu\bar{\sigma}^\nu\chi \to \Lambda^\mu_\alpha\Lambda^\nu_\beta\chi^T\left(-i\sigma^2\right)\sigma^\alpha\bar{\sigma}^\beta\chi$$

4 Van der Waerden notation

Instead of the cumbersome $\pm i\sigma^2$ in between the chiral spinors, the Van der Waerden notation defines a new kind of dot product between chiral spinors. We saw how $\chi^T(-i\sigma^2)\chi$ and $\eta^\dagger(i\sigma^2)\eta^{\dagger T}$ are invariants, so we shall define 2 dot products as such:

$$\begin{aligned}\chi \cdot \chi &\equiv \chi^T(-i\sigma^2)\chi \\ \bar{\chi} \cdot \bar{\chi} &\equiv \chi^\dagger(i\sigma^2)\chi^{\dagger T}\end{aligned}$$

Expanding the components,

$$\begin{aligned}\chi \cdot \chi &= \chi_2\chi_1 - \chi_1\chi_2 \\ \bar{\chi} \cdot \bar{\chi} &= \chi_1^\dagger\chi_2^\dagger - \chi_2^\dagger\chi_1^\dagger\end{aligned} \tag{2.22}$$

The good thing about this notation is that it shows that the dot product is now hermitian! So we can employ this directly in the Lagrangian fully knowing that we need not worry about any real-ness violations.

Chapter 3
A new notation

In this chapter, we will introduce a new notation for Weyl spinors that will be very useful in reading off the Lagrangian for interpretation and when working in the superfield approach.

1 Indices

We will begin off by defining the index notation. As we will mostly be working in the left-chiral basis, it will be the basis of which our notation will be built off from. We will begin from the invariant $\eta^\dagger \chi$. We will define as a contraction of

$$\eta^\dagger \chi \equiv \eta^\dagger \chi_a \tag{3.1}$$

The hermitian conjugate of the right chiral spinor is defined as having an upper un-dotted index. We will define the right chiral spinor as having an upper dotted index.

$$\eta \equiv \bar{\eta}^{\dot{a}} \implies \eta^a \equiv \left(\bar{\eta}^{\dot{a}}\right)^\dagger \tag{3.2}$$

With this, we can generalise the rest and see that the other fundamental invariant is

$$\chi^\dagger \eta \equiv \bar{\chi}_{\dot{a}} \bar{\eta}^{\dot{a}} \tag{3.3}$$

We will adopt a notation convention, that contractions between un-dotted indices are carried from top down, and for dotted indices from bottom up, as we see in the two definitions above.

2 Raising and lowering indices

We will make use of the fact that $(-i\sigma^2)_{ba}(i\sigma^2)^{ab} = \mathbb{1}$ allows us to define a metric to raise and lower the indices. To raise the indices,

$$\bar{\chi}^{\dot{a}} \equiv (i\sigma^2)^{\dot{a}b}\chi_b^\dagger = (i\sigma^2)^{\dot{a}\dot{b}}\bar{\chi}_{\dot{b}} \tag{3.4}$$

With this, we can see that explicitly,

$$\bar{\chi}^{\dot{1}} = \bar{\chi}_{\dot{2}} = \chi_2^\dagger \quad , \quad \bar{\chi}^{\dot{2}} = -\bar{\chi}_{\dot{1}} = -\chi_1^\dagger \tag{3.5}$$

Likewise, lowering the indices is just as similar:

$$\chi_b = (-i\sigma^2)_{ba}\chi^a \tag{3.6}$$

With this, the Van der Waerden dot product is

$$\eta \cdot \chi = \eta^1\chi_1 + \eta^2\chi_2 = \eta_2\chi_1 - \eta_1\chi_2 \tag{3.7}$$

3 The epsilon metric

To clean up the $(i\sigma^2)$ that is plaguing our notation, let us define

$$\begin{aligned}
\varepsilon^{ab} &\equiv (i\sigma^2)^{ab} \\
\varepsilon^{\dot{a}\dot{b}} &\equiv (i\sigma^2)^{\dot{a}\dot{b}} \\
\varepsilon_{ab} &\equiv (-i\sigma^2)_{ab} \\
\varepsilon_{\dot{a}\dot{b}} &\equiv (-i\sigma)^2_{\dot{a}\dot{b}}
\end{aligned} \tag{3.8}$$

As $i\sigma^2$ is completely anti-symmetric, so is ε. Moreover, the contraction of ε with itself is naturally

$$\varepsilon^{ab}\varepsilon_{bc} = -\varepsilon^{ba}\varepsilon_{bc} = -\varepsilon^{ab}\varepsilon_{cb} = \varepsilon^{ba}\varepsilon_{cb} = \delta^a_c \tag{3.9}$$

4 σ^μ and $\bar{\sigma}^\mu$ indices

We know that $i\sigma^\mu\eta$ is a left chiral spinor, so $i\sigma^\mu$ has to lower a dotted index to an un-dotted one. Likewise, $i\bar{\sigma}^\mu\chi$ is a right chiral spinor, so $i\bar{\sigma}^\mu$ raises an un-dotted index to a dotted one. We have:

$$\bar{\sigma}^\mu \equiv (\bar{\sigma}^\mu)^{\dot{a}b} \quad , \quad \sigma^\mu \equiv (\sigma^\mu)_{a\dot{b}} \tag{3.10}$$

We can see obtain this through ε

$$
\begin{aligned}
(i\sigma^2)\sigma^\mu(i\sigma^2) &= \varepsilon^{ca}(\sigma^\mu)_{a\dot{b}}\varepsilon^{\dot{b}\dot{d}} \\
&= (\bar{\sigma}^\mu)^{c\dot{d}} \\
&= -(\bar{\sigma}^\mu)^{\dot{d}c} \\
&= -(\bar{\sigma}^\mu)^T
\end{aligned}
\tag{3.11}
$$

With this, we can that supersymmetric invariants with σ^μ or $\bar{\sigma}^\mu$ needs to have spinors with both dotted and un-dotted indices, of the same generation:

$$
\bar{\chi}_{\dot{a}}\,(\bar{\sigma}^\mu)^{\dot{a}b}\,\lambda_b \quad , \quad \chi^a(\sigma^\mu)_{a\dot{b}}\bar{\lambda}^{\dot{b}} \tag{3.12}
$$

Chapter 4
Weyl, Majorana, and Dirac spinors

In this chapter, we look at close relations between Weyl, Majorana, and Dirac spinors and how we can jump between one to the other (and when we should not). The advantages of understanding this is that it helps paint a clearer picture behind the interpretation of these rather 'abstract' representations of what particles are.

1 Particle-antiparticle

We will begin this chapter by looking at the intricacies of particle-antiparticle existence. The source of their coexistence is by the natural imposition of charge-parity-time-reversal (CPT) invariance on the theory of particles. The particle-antiparticle pair would ensure that total charge, parity, and time-reversal is upheld. In the following text, the particle will be denoted by the subscript p whereas the antiparticle will be denoted by the subscript $\bar{p}$. Their representations as Dirac spinors (in terms of the irreducible Weyl spinors) are

$$\psi_p = \begin{pmatrix} \eta_p \\ \chi_p \end{pmatrix} \quad , \quad \psi_{\bar{p}} = \begin{pmatrix} \eta_{\bar{p}} \\ \chi_{\bar{p}} \end{pmatrix}$$

It should be necessary to mention that it is not the case that $\eta_p = \eta_{\bar{p}}$ and $\chi_p = \chi_{\bar{p}}$.

The relation between the conjugate Dirac spinor and its barred transpose used in the Lagrangian is

$$\psi_{\bar{p}} = \psi_p^C = C\bar{\psi}_p^T \quad , \quad C = -i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}$$

With this,

$$\psi_p^C = \begin{pmatrix} i\sigma^2 \chi_p^{\dagger T} \\ -i\sigma^2 \eta_p^{\dagger T} \end{pmatrix} \implies \begin{cases} \eta_{\bar{p}} = i\sigma^2 \chi_p^{\dagger T} \\ \chi_{\bar{p}} = -i\sigma^2 \eta_p^{\dagger T} \end{cases} \tag{4.1}$$

We now see how intricately related the Weyl spinors of the particle and antiparticle are. One very important thing to point out is that from Equation 4.1, we can see that our discussion in Sec 3.3 agrees that $i\sigma^2 \chi_p^{\dagger T}$ behaves as a right chiral and $-i\sigma^2 \eta_p^{\dagger T}$ as a left chiral! Using this knowledge, we can get rid of any explicit right chiral representations in the Dirac spinor and simply express it as

$$\psi = \begin{pmatrix} i\sigma^2 \chi_{\bar{p}}^{\dagger T} \\ \chi_p \end{pmatrix}$$

As mentioned at the start, the particle-antiparticle relations only exists because of the CPT invariance imposed on the Lagrangian. The other necessary constraint is that the Lagrangian needs to be real. (i.e. $\mathcal{L}^\dagger = \mathcal{L}$) This constraint tells us that the Lagrangian should either have both the hermitian conjugates of any chiral spinors, or none at all. The interpretation of this in QFT is very physical. There has to either have both the annihilation and creation operator of a particle, or none at all. Number operators will thus be a conserved operation.

The simplest contributor to the Lagrangian that has both left chiral spinors and ensures CPT and reality invariance is

$$\mathcal{L} = \chi^\dagger i\sigma^\mu \partial_\mu \chi$$

There is both a left chiral particle spinor creation and annihilation field operator in this Lagrangian.

However, using the relation in Equation 4.1, the same Lagrangian then becomes

$$\mathcal{L} = \eta_{\bar{p}}^T (i\sigma^2) i\sigma^\mu \partial_\mu \chi$$

which is a left chiral particle spinor and right chiral antiparticle spinor creation field operator! A very thought-provoking interpretation of the particle-antiparticle relationship.

Insofar as we have used the term Weyl spinor, we have used it to identify particles that are both eigenstates of the helicity operator and the chiralty operator. However, there is a very subtle difference between the two that paints very different pictures of what a Weyl spinor really is. As eigenstates of the helicity operator, Weyl spinors are necessarily massless as shown in 2.16. However, as eigenstates of the chiralty operator, they are simply eigenstates with fixed transformation rules under the $SU(2) \times SU(2)$ Lorentz group as in Equations 2.17 and 2.18. Thus, there is no constraint on them being massless. They can be as massive as they need be, as long as they are eigenstates that of the Lorentz group. However, for the sake of continuing the discussion regarding massive particles using the Weyl spinor representation, we shall adopt the convention of the latter, while duly keeping in mind that actual Weyl spinors are necessarily massless.

Returning to the CPT invariance, we now see that we have a scheme that relates η_p to $\chi_{\bar{p}}$ and its conjugates. Through the Lorentz transformation, it is also possible (**for massive particles**) for the chiralty of the particle to change, i.e from η_p to χ_p

and vice versa. These 4 particles are thus related to each other as a multiplet that must exist as a collective state. It is because of this fact that we are allowed to express the right chiral particle as the left chiral antiparticle with impunity. This is evident in how the mass term of the Lagrangian can be expressed in either of the following representations:

$$\begin{aligned} m\bar{\psi}\psi &= m(\chi^\dagger \eta + \eta^\dagger \chi) \\ &= m(\chi \cdot \chi + \bar{\chi} \cdot \bar{\chi}) \end{aligned} \tag{4.2}$$

2 Majorana spinors

The Majorana is a special subset of (massive) Dirac spinors. Its antiparticle state is the same as its particle state, i.e. $\eta_P = \eta_{\bar{P}}$ and $\chi_P = \chi_{\bar{\chi}}$. Unlike the general Dirac spinor, we now have 2 degrees of freedom instead of 4. The Majorana spinor in left chiral representation is

$$\psi_M = \begin{pmatrix} i\sigma^2 \chi_P^{\dagger T} \\ \chi_P \end{pmatrix} \tag{4.3}$$

As good as the Majorana and Weyl representations are, it is not possible to build actual theories using them only as parity is not conserved. In the Lagrangian formalism of strictly Weyl or Majorana spinors, the mass terms will only be mass terms of left chirals, with no way of satisfying the parity between left and right chirals.

Looking at the two from another angle, we see that

$$\begin{cases} \bar{\psi}_M \psi_M = \chi \cdot \chi + \bar{\chi} \cdot \bar{\chi} \\ \bar{\psi}_M \gamma_5 \psi_M = -\chi \cdot \chi + \bar{\chi} \cdot \bar{\chi} \end{cases}$$

which with some simple manipulation and generalisation, simply gives us

$$\begin{cases} \lambda \cdot \chi = \bar{\Lambda}_M P_L \psi_M \\ \bar{\lambda} \cdot \bar{\chi} = \bar{\Lambda}_M P_R \psi_M \end{cases}$$

Lastly, making use of the fact that γ^μ can be represented off-diagonally as

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \tag{4.4}$$

we have

$$\begin{cases} \bar{\psi}_M \gamma^\mu \Lambda_M = \chi^\dagger \bar{\sigma}^\mu \lambda - \lambda^\dagger \bar{\sigma}^\mu \chi \\ \bar{\psi}_M \gamma_5 \gamma^\mu \Lambda_M = \chi^\dagger \bar{\sigma}^\mu \lambda + \lambda^\dagger \bar{\sigma}^\mu \chi \end{cases}$$

$$\implies \begin{cases} \chi^\dagger \bar{\sigma}^\mu \lambda = \bar{\psi}_M P_R \gamma^\mu \Lambda_M \\ \lambda^\dagger \bar{\sigma}^\mu \chi = -\bar{\psi}_M P_L \gamma^\mu \Lambda_M \end{cases}$$

a very neat relation between the Weyl and Majorana representations of the **massive** Majorana spinor.

Chapter 5
Building the Lagrangian

We will build attempt at building the most basic Lagrangians with the invariants and constraints from the previous chapters. Several constraints on the Lagrangian will have to be imposed to ground our discussion in renormalisable theories.

1 Dimensionfull Lagrangians

In building the Lagrangian, we shall keep to renormalisable theories where $D = 4$ is maximally the further we will go in dimensions. The dimensions are defined in terms of powers of energy and as should be, the natural units are 1 (thus being dimensionless). With this, scalar fields are of dimension 1, derivatives are of dimension 1, fermion fields are of dimension $3/2$. Simple dimensional analysis will give us these.

2 The simplest Lagrangian

Let us consider the simplest toy model we can make – a single free massless fermionic pair and a single free massless bosonic pair. They do not interact with each other (this will be introduced in Chapter 6). The Lagrangian is simply

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi \tag{5.1}$$

The transformation of these fields are

$$\begin{aligned} \phi &\to \phi + \delta\phi \\ \chi &\to \chi + \xi\chi \end{aligned}$$

Let us bring in the postulate of supersymmetry – that bosons will transform into fermions and vice versa.

$$\delta\phi \propto \xi\chi \quad , \quad \xi \ll 1 \tag{5.2}$$

For Equation 5.2 to satisfy the dimensionality of both sides of the equation, we see that ξ has to be a Grassmann spinor of dimension $-1/2$. To actually determine the proportionality of the relationship in Equation 5.2, we have to impose the Lorentz invariance of the Lagrangian to obtain any more information.

Since ξ is spinor, we have the freedom to pick a left chiral spinor, so we can have

$$\delta\phi = \xi \cdot \chi \tag{5.3}$$

which is fully Lorentz invariant and a valid term in a Lagrangian.

We move on to the transformation of the fermion.

$$\delta\chi = -i(\partial\phi)\sigma^{\mu}(i\sigma^{2})\xi^{*} \tag{5.4}$$

where we obtained this the same way, by imposing the equality of dimensions on both sides of the equation, Lorentz invariablity, and the reality of the Lagrangian.

Chapter 6
SUSY Charges

We will start off with a short review on the necessary elements we need to know about symmetry chargers. We will look at how to derive and interpret the charges that make up supersymmetry. Once we have the charges, we will attempt to obtain the supersymmetry transformation rules of fermions, bosons, and auxiliary fields.

1 Quick review

We define a unitary transformation as $U \equiv \exp[\pm i\varepsilon \cdot Q]$ where ε is an infinitesimal factor and Q is the charge of the symmetry. The dot product implies that there may be more than 1 charge to the symmetry. The transformation of a state in the symmetry is defined as $\phi'(x) \equiv U\phi(x)U^\dagger$

The transformation of a state in the symmetry may be defined in 2 manners: through the unitary transformations; or through a varaince.

$$\begin{aligned} \phi'(x) &\equiv U\phi(x)U^\dagger \\ &\equiv \phi(x) + \delta\phi(x) \end{aligned}$$

Making use of the fact that ε is infinitesimal, we may, to leading order of ε, relate the variance of $\phi(x)$ to the commutator relation of Q and ϕ.

$$\delta\phi(x) = \pm i[\varepsilon \cdot Q, \phi(x)] \tag{6.1}$$

The explicit representations of the charges may be obtained from either of 2 way: through the symmetry currents in Equation 6.2; or as differential operators in Equation 6.3. We will obtain the algebra of the symmetry if we work out all the commutator relations of the charges in the symmetry.

$$Q^i = \int \mathrm{d}^3x J_0^i(\boldsymbol{x}, t) \tag{6.2}$$

$$\phi(x') \equiv \exp[\pm i\varepsilon \cdot \hat{Q}]\phi(x) \tag{6.3}$$

where here we emphasise $\hat{Q}$ is a differential operator through the hat notation.

Instead of finding the explicit representations of the charges to determine the algebra of the symmetry, we may consider the charges as quantum field operators and work it out. Consider 2 consecutive transformations, with infinitesimal factors α and β:

$$\begin{aligned} U_\beta U_\alpha \phi U_\alpha^\dagger U_\beta^\dagger &\approx \phi + i[\alpha \cdot Q, \phi] + i[\beta \cdot Q, \phi] - [\beta \cdot Q, [\alpha \cdot Q, \phi]] + \dots \\ &= \delta_\beta \delta_\alpha \phi \end{aligned} \tag{6.4}$$

Working out the opposite order,

$$[\delta_\beta, \delta_\alpha]\phi = \big[[\alpha \cdot Q, \beta \cdot Q], \phi\big] \tag{6.5}$$

2 Deriving the supersymmetric charges

We have 2 charges to consider, since there are 4 degrees of freedom that are be grouped as 2 pairs of spinors (i.e. ξ, ξ^*). Using the transformation rules from the previous chapter, we can now express them in terms of the SUSY charge commutator relations as

$$\big[iQ \cdot \xi + i\bar{Q} \cdot \bar{\xi}\big] = -i\xi \cdot \chi \tag{6.6}$$

$$\big[iQ \cdot \xi + i\bar{Q} \cdot \bar{\xi}\big] = -i(\partial_\mu \phi)\sigma^\mu \sigma^2 \xi^* \tag{6.7}$$

Since ξ and ξ^* are independent, we see that the only non-vanishing terms are:

$$[\xi \cdot Q, \phi] = -i\xi \cdot \chi \tag{6.8}$$

$$\big[\bar{\xi} \cdot \bar{Q}, \chi\big] = -i(\partial_\mu \phi)\sigma^\mu \sigma^2 \xi^* \tag{6.9}$$

Matching the charges with each other in a commutator relation, we get the following:

$$\begin{aligned} [Q \cdot \xi, Q \cdot \beta] &= (\sigma^2)^{ab}(\sigma^2)^{cd}\xi_b \beta_d \{Q_a, Q_c\} \\ \big[Q \cdot \xi, \bar{Q} \cdot \bar{\beta}\big] &= -(\sigma^2)^{ab}(\sigma^2)^{cd}\xi_b \beta_d^* \{Q_a, Q_c^\dagger\} \\ \big[\bar{Q} \cdot \bar{\xi}, Q \cdot \beta\big] &= (\sigma^2)^{ab}(\sigma^2)^{cd}\xi_b^* \beta_d \{Q_a^\dagger, Q_c\} \\ \big[\bar{Q} \cdot \bar{\xi}, \bar{Q} \cdot \bar{\beta}\big] &= (\sigma^2)^{ab}(\sigma^2)^{cd}\xi_b^* \beta_d^* \{Q_a^\dagger, Q_c^\dagger\} \end{aligned} \tag{6.10}$$

The algebra of the symmetry is embedded in the anti-commutator relations in these equations.

With these, we know how to get

$$[\delta_\beta, \delta_\xi]\phi = \Big[\big[Q \cdot \xi + \bar{Q} \cdot \bar{\xi}, Q \cdot \beta + \bar{Q} \cdot \bar{\beta}\big], \phi\Big] \equiv [O, \phi] \tag{6.11}$$

Expanding the LHS of Equation 6.11 for a boson,

$$
\begin{aligned}
\left[\delta_\beta, \delta_\xi\right] \phi &= -i(\xi^\dagger \bar{\sigma}^\mu \beta - \beta^\dagger \xi)\partial_\mu \phi \\
&= (\xi^T \sigma^2 \sigma^\mu \sigma^2 \beta^* - \beta^T \sigma^2 \sigma^\mu \sigma^2 \xi^*)[P_\mu, \phi] \\
\therefore O &= (\xi^T \sigma^2 \sigma^\mu \sigma^2 \beta^* - \beta^T \sigma^2 \sigma^\mu \sigma^2 \xi^*) P_\mu \\
&= -(\sigma^2)^{ab}(\sigma^2)^{cd}(\xi_b \beta^*_d \sigma^\mu_a c + \xi^*_b \beta_d \sigma^\mu_{ca}) P_\mu
\end{aligned}
\tag{6.12}
$$

Comparing against the coefficients in Equations 6.1, we will arrive at the following anti-commutator relations:

$$
\begin{aligned}
\{Q_a, Q_c\} &= \{Q^\dagger_a, Q^\dagger_c\} = 0 \\
\{Q_a, Q^\dagger_c\} &= \sigma^\mu_{ac} P_\mu \\
\{Q^\dagger_a, Q_c\} &= \sigma^\mu_{ca} P_\mu
\end{aligned}
$$

and by normalising the charges, $Q \to Q/\sqrt{2}$, the non-vanishing anti-commutator relations are

$$
\{Q_a, Q^\dagger_c\} = 2\sigma^\mu_{ac} P_\mu \tag{6.13}
$$

$$
\{Q^\dagger_a, Q_c\} = 2\sigma^\mu_{ca} P_\mu \tag{6.14}
$$

Note that since Q and $\bar{Q}$ are spacetime independent, the algebra between the momentum Poincaré charges and supersymmetric charges necessarily vanish.

$$
[Q, P_\mu] = [Q^\dagger, P_\mu] = 0 \tag{6.15}
$$

However, the angular Poincaeé charges and supersymmetric charges do not vanish.

$$
[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu})^b_a Q_b \quad , \quad \sigma_{\mu\nu} \equiv \frac{i}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu) \tag{6.16}
$$

If we were to conclude that the algebra for SUSY is complete with this, we would be sorely mistaken as it does not close for the spinor fields as they are now. This is simply because the spinor fields we have are on-shell spinors with a total of 2 degrees of freedom, 2 short of the bosonic degrees of freedom. To handle this, we will have to introduce auxiliary fields that will vanish on-shell while accounting for the missing 2 degrees of freedom. We will allow the auxiliary fields to be bosonic. Since they must vanish on-shell, the simplest form they can take is $F^\dagger F$. Naturally, the dimension for the auxiliary field has to be 2 in the Lagrangians we have been working in. The free field Lagrangian is now:

$$
\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F \tag{6.17}
$$

The explicit transformation rule of F needs to be linear in the infinitesimal ξ and one other field, all while ensuring its dimension and Lorentz invariance. The right choice of δF is

$$\delta F = K\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi \tag{6.18}$$

To ensure that this addition of the auxiliary field to the Lagrangian will not interfere with the overall invariance, we have to apply a variance on the fields.

$$\delta(F^\dagger F) = (K^*\xi F)^\dagger \bar{\sigma}^\mu \partial_\mu \chi - \chi^\dagger \bar{\sigma}^\mu \partial_\mu (K^*\xi F) \tag{6.19}$$

Noticing that this is similar in structure to the variance of the free spinor fields in Equation 6.20, we can define a new spinor field as in Equation 6.21.

$$\delta(\chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi) = (\delta\chi)^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu (\delta\chi) \tag{6.20}$$

$$\delta\tilde{\chi} \equiv \delta\chi - iK^*\xi F \tag{6.21}$$

Because of the freedom we have for K, we can conveniently set it to i. This way, our new Lagrangian in Equation 6.17 will be closed under the SUSY algebra in Equations 6.14 and 6.14 with the following field super-transformations:

$$\begin{aligned} \delta\phi &= \xi \cdot \chi \\ \delta\chi &= -i\sigma^\mu (i\sigma^2 \xi^*)\partial_\mu \phi + F\xi \\ \delta F &= -i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi \end{aligned} \tag{6.22}$$

Chapter 7
Applications of SUSY algebra

Continuing off the previous chapter, we will work on the SUSY algebra to create the SUSY multiplets. We will also formulate the algebra again, but in the Majorana form to see the significance of its interpretation over the Weyl spinor representation. We will then attempt to find the explicit forms of the supercharges both as symmetry currents and quantum filed operators. Lastly, we will discuss about the extension of the algebra into areas outside of SUSY.

1 Casimir Operators

The Casimir operators are operators which commute with every generator of a group. Because of that, its eigenvalues can be used to classify the group representations. For example, $P^\mu P_\mu$ is a Casimir operator of the Poincaré group with an eigenvalue m^2. Another (more useful) Casimir operator of the Poincaré group is the Pauli-Lubonski operator W^μ.

$$W^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho} M_{\rho\sigma} P_\nu \tag{7.1}$$

For a massive particle, the Pauli-Lubonski operator gives the total angular momentum of a particle as its eigenvalue.

$$W^i \left|p\right\rangle = m(L^i + S^i)\left|p\right\rangle \tag{7.2}$$

Contracting it with itself and applying onto a particle at rest,

$$W_\mu W^\mu \left|p\right\rangle = -m^3 s(s+1)\left|p\right\rangle \tag{7.3}$$

On the other hand, on a massless particle, the eigenvalue of the Pauli-Lubonski operator is the helicity of the particle. Take for example a massless particle in with an angle of rotation in the z-axis (i.e. $P^\mu \left|p\right\rangle = (E, 0, 0, E)\left|p\right\rangle$).

$$W^\mu \left|p\right\rangle = (Es_z, 0, 0, Es_z)\left|p\right\rangle \tag{7.4}$$

2 Applying onto supercharges

Applying the Pauli-Lubonski operator to the supercharges in a commutator relation,

$$[Q_a, W^0] = -\frac{1}{2}(\sigma^3)_a^b Q_b P_3 \tag{7.5}$$

The only non-zero terms of σ^3 are the diagonal terms so what we essentially have is:

$$[Q_1, W_0] = -\frac{1}{2}Q_1 P_3 \tag{7.6}$$

$$[Q_2, W_0] = \frac{1}{2}Q_2 P_3 \tag{7.7}$$

We can use this to derive what the supercharges do onto a particle state.

$$\begin{aligned} W_0(Q_1 |p, h\rangle) &= [W_0, Q_1] |p, h\rangle + Q_1 W_0 |p, h\rangle \\ &= E(h + \frac{1}{2})Q_1 |p, h\rangle \\ \implies Q_1 |p, h\rangle &= |p, h + \frac{1}{2}\rangle \end{aligned} \tag{7.8}$$

Likewise for Q_2:

$$Q_2 |p, h\rangle = |p, h - \frac{1}{2}\rangle \tag{7.9}$$

From Equations 7.8 and 7.9, we now know that Q_1 raises the helicity of the particle by $\frac{1}{2}$ whereas Q_2 lowers the helicity of the particle by $\frac{1}{2}$.

3 Building the SUSY multiplets

Let us look at the massless, rest particle. Recalling the SUSY algebra in Equation 6.13, the algebra all depend on $\sigma^\mu_{ab} P_\mu$.

$$\sigma^\mu_{ab} P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2P^0 \end{pmatrix} |p, h\rangle \tag{7.10}$$

Thus, the only non-vanishing algebra is

$$\{Q_2, Q_2^\dagger\} = 2E |p, h\rangle \tag{7.11}$$

The vanishing algebra also sheds some insight into the inner workings of the multiplet.

$$\{Q_1, Q_1^\dagger\} = 0 \implies Q_1 |p, h\rangle = Q_1^\dagger |p, h\rangle = 0 \tag{7.12}$$

This means that in SUSY multiplet, there is a minimum helicity to consider!

Let us define the minimum helicity h_{min}

$$Q_2 \left|p, h_{min}\right\rangle = 0 \quad , \quad Q_2^\dagger \left|p, h_{min}\right\rangle = \left|p, h_{min} + \frac{1}{2}\right\rangle \tag{7.13}$$

Moreover,

$$\{Q_2^\dagger, Q_2^\dagger\} = 0 \implies Q_2^\dagger Q_2^\dagger \left|p, h_{min}\right\rangle = 0 \tag{7.14}$$

this implies that the multiplet has only 2 states of the same momentum, but a helical difference of $1/2$.

To make the duet CPT invariant, we need to add the CPT conjugates of each of the 2 states. Thus, there needs to be 4 states to a SUSY multiplet. For example, if $h_{min} = 0$, we will have a scalar multiplet with helicities $0, 0, \frac{1}{2}, -\frac{1}{2}$; if $h_{min} = \frac{1}{2}$, we will have a vector multiplet with helicities $\frac{1}{2}, 1, -\frac{1}{2}, -1$.

4 Supercharges through symmetry currents

The general Lagrangian made of complex scalar fields is:

$$\mathcal{L} = \mathcal{L}(\phi, \phi^\dagger, \partial_\mu\phi, \partial_\mu\phi^\dagger) \tag{7.15}$$

The variance of the Lagrangian is:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi^\dagger}\delta\phi^\dagger + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} \tag{7.16}$$

where on-shell,

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial(\partial_\mu\phi)} \tag{7.17}$$

giving us

$$\partial_\mu\mathcal{K}^\mu \equiv \delta\mathcal{L} = \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\delta\phi^\dagger\right] \tag{7.18}$$

where $\mathcal{K}^\mu$ is introduced as we know that in general, the variance of the Lagrangian can total differential as it will disappear in the integral. The terms in the bracket are the Noether's current, denoted by j^μ. The conserved current J^μ is thus defined as:

$$J^\mu = j^\mu - K^\mu \tag{7.19}$$

The supercharges are derived using the 0-th index of J^μ as in Equation 6.2.

5 VEV of the Hamiltonian

From

$$\{Q_a, Q_b^\dagger\} = \sigma^\mu P_\mu \implies \langle \{Q_1, Q_1^\dagger\} + \{Q_2, Q_2^\dagger\} \rangle = 2\,\langle \mathcal{H} \rangle \tag{7.20}$$

The positive-definitivity of the LHS implies that $\langle \mathcal{H} \rangle \geq 0$. The equality is achieved when both supercharges annihilate the vacuum state, and by extension, a strict inequality is enforced when the supercharges do not annihiliate the vacuum state – spontaneous supersymmetry breaking.

6 SUSY in the Majorana Form

Recall that the right chiral spinor of the Majorana spinor is related to its left chiral spinor as $\eta = i\sigma^2 {\chi^\dagger}^T$. This means that all 4 components of the Majorana supercharge can be expressed as

$$Q_M \equiv \begin{pmatrix} i\sigma^2 {Q^\dagger}^T \\ Q \end{pmatrix} = \begin{pmatrix} -Q_2^\dagger \\ -Q_1^\dagger \\ Q_1 \\ Q_2 \end{pmatrix} \tag{7.21}$$

7 Explicit supercharges

In Equations 6.2, we saw how to obtain the supercharges explicitly using the conserved symmetry current. To get the expression for the conserved symmetry current, we need the Noether's current (j^μ) and the surface differential terms that might have been 'discarded' in the derivation of the Lagrangian ($\partial_\mu K^\mu$).

For example, in the free supersymmetric Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F \tag{7.22}$$

its conserved current is

$$\mathcal{J}^\mu_{SUSY} = (\partial_\nu \phi)\chi^\dagger \bar{\sigma}^\mu \sigma^\nu (i\sigma^2)\xi^* - (\partial_\nu \phi^\dagger)\xi^T (i\sigma^2)\sigma^\nu \bar{\sigma}^\mu \chi \tag{7.23}$$

Putting Equation 7.23 into Equation 6.2, we get

$$\xi \cdot Q + \bar{\xi} \cdot \bar{Q} = \int \mathrm{d}^3x (\partial_\nu \phi)\chi^\dagger \bar{\sigma}^\mu \sigma^\nu (i\sigma^2)\xi^* - (\partial_\nu \phi^\dagger)\xi^T (i\sigma^2)\sigma^\nu \bar{\sigma}^\mu \chi \tag{7.24}$$

Comparing the coefficients of ξ and ξ^*,

$$Q = \int d^3x \partial_\nu \phi^\dagger \sigma^\nu \bar{\sigma} \mu \chi \tag{7.25}$$

$$Q^\dagger = \int d^3x \chi^\dagger \bar{\sigma}^\mu \sigma^\nu \partial_\nu \phi^\dagger \tag{7.26}$$

Two more identities that need to be included when using the explicit charges are

$$\left[\phi(x,t), \dot{\phi}^\dagger(y,t)\right] = i\delta^3(x-y) \tag{7.27}$$

$$\left\{\chi_a(x,t), \chi_b^\dagger(y,t)\right\} = \delta_{ab}\delta^3(x-y) \tag{7.28}$$

With these 4 equations, the explicit supercharges may be used freely in applications such as verifying the field transformations.

Chapter 8
The Wess-Zumino Model

Up to this point, we have explicitly formulated the Lagrangian of free particles. We realised that for the SUSY algebra to be complete for both the boson and spinor fields, we needed to introduce a new field that disappears on-shell. In this chapter, we will look into adding some interactions between the fields so as to bring our discussions away from the toy model that it is right now. We will interactions between all 3 fields and their hermitian conjugates. Once our new Lagrangian is complete, we will be able to employ some tricks that is used widely in Lagrangian mechanics and identify an umbrella potential term that will be useful in our future attempt to break the symmetry. Lastly, as we did for the past few chapters, we will express the Lagrangian in the Majorana spinor representation.

1 Interactions to consider

We now have a total of 6 fields: $\phi, \phi^\dagger, F, F^\dagger, \chi, \chi^\dagger$. We will only be considering interactions that will satisfy the following constraints in the Lagrangian:

1. $D \leq 4$
2. Lorentz invariance
3. Hermitivity
4. Gauge invariance

We can exclude the 4th option for now since we are working without a gauge charge in this model, but in general these are the constraints in picking possible interactions for a super-renormalisable theory.

1.1 ϕ and $\phi^\dagger$ interactions only

Any arbitrary function $G(\phi, \phi^\dagger)$ will satisfy all 3 conditions.

1.2 $\phi, \phi^\dagger, F$, and $F^\dagger$ interactions

A function with the form $W_1(\phi, \phi^\dagger)F + h.c.$ will satisfy all 3 conditions. Note that W_1 is at most quadratic or bilinear in terms.

1.3 $\phi, \phi^\dagger, \chi$, and $\chi^\dagger$ interactions

To get Lorentz invariant terms of the spinors, we make use of what we have done in Chapter 2. $\chi \cdot \chi$ and $\bar{\chi} \cdot \bar{\chi}$ are Lorentz invariants. Possible interactions of this group come in the form $-\frac{1}{2}W_{11}(\phi, \phi^\dagger)\chi \cdot \chi + h.c.$. Note that W_{11} is at most linear in terms.

1.4 $\chi, \chi^\dagger, F$, and $F^\dagger$ interactions

Any Lorentz invariant combinations of these terms will necessarily violate the first condtion, and thus we need no consider any of these interactions for the Wess-Zumino model.

The indices of W_1 and W_{11} do not make sense now, but by the end we will see that they are indices of the fields they are attached to – F_i and $\chi_i \cdot \chi_j$.

2 The General Wess-Zumino Lagrangian

Let us put in the interactions we have guessed in the previous section.

$$\mathcal{L}_{WZ} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F + G + W_1 F + W_1^\dagger F^\dagger - \frac{1}{2} W_{11} \chi \cdot \chi - \frac{1}{2} W_{11}^\dagger \bar{\chi} \cdot \bar{\chi} \tag{8.1}$$

We now need $\mathcal{L}_{int}$ to transform supersymmetrically, as $\mathcal{L}_{free}$ did. We will do this by varying $\mathcal{L}_{int}$ and using the supersymmetric transformation rules as in Equations 6.22, ensure that the total variance either vanishes or gets swept away as a total derivative.

Doing the work, we will see that

$$\begin{aligned}\delta\mathcal{L}_{int} = {} & \frac{\partial G}{\partial \phi}\chi \cdot \xi + \frac{\partial W_1}{\partial \phi}\chi \cdot \xi F + \frac{\partial W_1}{\partial \phi^\dagger}\bar{\chi} \cdot \bar{\xi} F - iW_1 \xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi \\ & -\frac{1}{2}\frac{\partial W_{11}}{\partial \phi}\chi \cdot \xi \chi \cdot \chi - \frac{1}{2}\frac{\partial W_{11}}{\partial \phi^\dagger}\bar{\chi} \cdot \bar{\xi} \chi \cdot \chi \\ & -iW_{11}\chi^T i\sigma^2 \partial_\mu \phi \sigma^\mu i\sigma^2 \xi^* - W_{11} F \chi \cdot \xi + h.c.\end{aligned} \tag{8.2}$$

We see that for this to either vanish or gauge away as a total derivative, coefficients of the combinations of fields need to either vanish or gauge away as a total derivative. This gives us the following constraints:

$$\chi \cdot \xi \neq 0 \implies \frac{\partial G}{\partial \phi} = 0 \tag{8.3}$$

$$\bar{\chi} \cdot \bar{\xi} \chi \cdot \chi \neq 0 \implies \frac{\partial W_{11}}{\partial \phi^\dagger} = 0 \tag{8.4}$$

$$\bar{\chi} \cdot \bar{\xi} F \neq 0 \implies \frac{\partial W_1}{\partial \phi^\dagger} = 0 \tag{8.5}$$

$$\chi \cdot \xi \chi \cdot \chi = 0 \implies \frac{\partial W_{11}}{\partial \phi} \text{ has no constraints} \tag{8.6}$$

$$\chi \cdot \xi F \neq 0 \implies \frac{\partial W_1}{\partial \phi} - W_{11} = 0 \tag{8.7}$$

$$\chi^T (i\sigma^2)\sigma^\mu (i\sigma^2)\xi^* \neq 0 \implies \partial_\mu W_1 = W_{11}\partial_\mu \phi \tag{8.8}$$

and all their hermitian conjugates.

We have to note that G is real, so Equation 8.3 tells us that G is a constant that we can conveniently set to 0. Equations 8.4 and 8.5 tells us that W_1 and W_{11} are holomorphic in ϕ (i.e. $W_1 = W_1(\phi)$ and $W_{11} = W_{11}(\phi)$). The final 2 equations, Equations 8.7 and 8.8 are identical to each other – $W_{11} = \frac{\partial W_1}{\partial \phi}$. Equation 8.6 tells us that W_{11} is the only degree of freedom we have.

With these, we can now write the interaction Lagrangian as

$$\mathcal{L}_{int} = W_1(\phi)F - \frac{1}{2}\frac{\partial W_1}{\partial \phi}\chi \cdot \chi + h.c. \tag{8.9}$$

To have this Lagrangian satisfy $D \leq 4$, $[W_1] \leq 2$, which means that W_1 is at most quadratic in ϕ. Its most general form is

$$W_1 = m\phi + \frac{1}{2}y\phi^2 + C \tag{8.10}$$

where y is a dimensionless constant.

Here, we take a page off classical Lagrangian mechanics. The interaction Lagrangian may be written a a derivative the derivative of a potential term, for which we will label as $\mathcal{W}$.

$$\mathcal{W} = \frac{1}{2}m\phi^2 + \frac{1}{6}y\phi^3 + C\phi + f(\phi^\dagger) \tag{8.11}$$

The notation so far is complete for the 1 particle Wess-Zumino interaction, where we need not worry about particle indices and can leave all the indices at 1 or 11. If we were to extend this to n-particles however, we have to make a slight correction.

$$\mathcal{L}_{WZ} = \sum_i \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \chi_i^\dagger i\bar{\sigma}^\mu \partial_\mu \chi_i + F_i^\dagger F_i$$
$$+ \left(\sum_{i,j} W_i F_i - \frac{1}{2} W_{ij} \chi_i \cdot \chi_j + h.c. \right) \tag{8.12}$$

Going through the process again, we will see that W_{ij} has to be at most linear in ϕ. Its most general form has to be

$$W_{ij} = m_{ij} + y_{ijk}\phi_k \tag{8.13}$$

and for us to arrive at this, we had to impose that $\frac{\partial W_{ij}}{\partial \phi_k}$ is cyclic invariant, which then implies that y_{ijk} also needs to have cyclic symmetry. $\chi_i \cdot \chi_j$ is symmetric in indices so m_{ij} also has to be symmetric in indices. Therefore, we have found that W_{ij} is completely symmetric. One simply way to ensure its symmetricity is to have W_{ij} be a second order differential of a function.

$$W_{ij} = \frac{\partial^2 \mathcal{W}}{\partial \phi_i \partial \phi_j} \tag{8.14}$$

With the other constraints that W_i also having to be holomorphic in ϕ_i and that $W_i = \frac{\partial \mathcal{W}}{\partial \phi_i}$, the most general form $\mathcal{L}$ can take is:

$$\mathcal{W} = \frac{1}{2} m_{ij}\phi_i\phi_j + \frac{1}{6} y_{ijk}\phi_i\phi_j\phi_k + c_i\phi_i \tag{8.15}$$

Now, we can organise Equation 8.12 in terms of the superpotential $\mathcal{W}$ so that it would be easier to see the physics.

$$\mathcal{L}_{WZ} = \sum_i \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \chi_i^\dagger i\bar{\sigma}^\mu \partial_\mu \chi_i + F_i^\dagger F_i$$
$$+ \left(\frac{\partial \mathcal{W}}{\partial \phi_i} W_i F_i + \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial \phi_i \partial \phi_j} \chi_i \cdot \chi_j + h.c. \right) \tag{8.16}$$

Our Wess-Zumino Lagrangian is now completely supersymmetric! We can work a little more to remove the auxiliary fields from the Lagrangian solving their equations of motion (which we now for a fact is vanishing).

$$F_i^\dagger F_i^\dagger = - \left| \frac{\partial \mathcal{W}}{\partial \phi_i} \right|^2 \tag{8.17}$$

$$\therefore \mathcal{L}_{WZ} = \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \chi_i^\dagger i\bar{\sigma}^\mu \partial_\mu \chi_i - \left| \frac{\partial \mathcal{W}}{\partial \phi_i} \right|^2 - \frac{1}{2} \left(\frac{\partial^2 \mathcal{W}}{\partial \phi_i \partial_j} \chi_i \cdot \chi_j + h.c. \right) \tag{8.18}$$

This form is a much more insightful than Equation 8.16 as here, all our terms are built off the physical boson and spinor fields. It also shows us exactly where the potential terms are in the Lagrangian as it comes very neatly in the form $\mathcal{L} = T - V$.

Being explicit in Equation 8.18, the Lagrangian becomes

$$\mathcal{L}_{WZ} = \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \chi_i^\dagger i\bar{\sigma}^\mu \partial_\mu \chi_i - \left| m_{ij}\phi_j + \frac{1}{2} y_{ijk}\phi_j\phi_k + c_i \right|^2$$
$$-\frac{1}{2}(m_{ij}\chi_i \cdot \chi_j + y_{ijk}\phi_k \chi_i \cdot \chi_j + h.c.) \tag{8.19}$$

and we see that the masses of the bosons and spinors are the same!

3 The Wess-Zumino Lagrangian in Majorana Form

Let us consider a single particle and set $c = 0$ Equation 8.19 becomes

$$\mathcal{L}_{WZ} = \mathcal{L}_M + \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{1}{2} m y ({\phi^\dagger}^2 \phi + \phi^\dagger \phi^2)$$
$$-\frac{1}{4} y^2 (\phi^\dagger \phi)^2 - \frac{1}{2} y (\phi \chi \cdot \chi + \phi^\dagger \bar{\chi} \cdot \bar{\chi}) \tag{8.20}$$

We can decompose the complex scalar ϕ into its components $\phi = \frac{1}{\sqrt{2}}(A + iB)$ and express Equation 8.20 in terms of A, B, and Ψ_M. We will define $g \equiv \frac{1}{\sqrt{8}} y$ and the following terms will then be:

$$-\frac{1}{4} y^2 (\phi^\dagger \phi)^2 = -\frac{1}{2} g^2 (A^2 + B^2)^2 \equiv \mathcal{L}_1 \tag{8.21}$$

$$-\frac{1}{2} m y {\phi^\dagger}^2 \phi + h.c. = -mg(A^3 + AB^2) \equiv \mathcal{L}_2 \tag{8.22}$$

$$-\frac{1}{2} y (\phi \chi \cdot \chi + \phi^\dagger \bar{\chi} \cdot \bar{\chi}) = -g(A\bar{\Psi}_M \Psi_M + iB\bar{\Psi}_M \gamma^5 \Psi_M) \equiv \mathcal{L}_3 + \mathcal{L}_4 \tag{8.23}$$

We now have

$$\mathcal{L}_{WZ} = \mathcal{L}_{Free,WZ} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \tag{8.24}$$

where the numbered Lagrangians will come in very handy when we do diagram calculations in the next chapter.

Chapter 9
Some explicit calculations

Here, we will carry out some explicit calculations on the Wess-Zumino model and explicitly show the most desirable trait of supersymmetry – the vanishing quadratic divergences. The calculations carried out will be diagram calculations in QFT and will be briefly covered in the first section. Afterwards, we will apply those onto the interacting Lagrangians in Chapter 8 Section 3. Lastly, we will discuss the renormalisability of the theory to handle the logarithmic divergences that do not vanish like the quadratic divergences.

1 Quick overview of QFT process calculation

We will be working the calculations on n-point functions. This is where we will have calculations such as

$$\langle\Omega|T(\phi(x)\phi(y)..)|\Omega\rangle = \frac{\left\langle 0\middle| T\left(\phi_1(x)\phi_2(y)...\exp\left[i\int \mathrm{d}^4 z\mathcal{L}_{int}\phi_I(z)\right]\right)\middle|0\right\rangle}{\left\langle 0\middle| T\left(\exp\left[i\int \mathrm{d}^4 z\mathcal{L}_{int}\phi_I(z)\right]\right)\middle|0\right\rangle} \tag{9.1}$$

Several important results are the 2-point functions of scalar fields in the $\lambda\phi^4$ theory:

$$\langle 0\,|\,\phi(x)\phi(y)\,|\,0\rangle = D(x-y) + \mathcal{O}(\lambda) \tag{9.2}$$

where

$$\begin{aligned} D(x-y) &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\exp[-ik\cdot(x-y)]\frac{i}{k^2-m^2+i\varepsilon} \\ &\equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\exp[-ik\cdot(x-y)]D(k) \end{aligned} \tag{9.3}$$

To order λ of the same theory, we have another result:

$$
\begin{aligned}
D_1(x-y) &\equiv -i\lambda \left\langle 0 \middle| T\left(\phi(x)\phi(y)\int \mathrm{d}^4 z\phi^4(z)\right)\middle| 0\right\rangle \\
&= -12i\lambda \int \mathrm{d}^4 z \,\langle 0\,|\,[\phi(x)\sim\phi(z)][\phi(y)\sim\phi(z)][\phi(z)\sim\phi(z)]\,|\,0\rangle \\
&= -12i\lambda \int \mathrm{d}^4 z D(x-z)D(y-z)D(z-z) \qquad (9.4) \\
&= -12i\lambda \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot(x-y)} D(p)D(p) \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{i}{q^2-m^2+i\varepsilon} \qquad (9.5)
\end{aligned}
$$

The diagram of such an interaction visualised from Equation 9.4 is a loop at point z with ends at x and y. If we Fourier transform the LHS to the p-momentum space, we can match it to Equation 9.5 and see that

$$
\mathcal{F}\left\{\int \mathrm{d}^4 z D(x-z)D(y-z)D(z-z)\right\} = D(p)D(p)I_d \qquad (9.6)
$$

for

$$
\begin{aligned}
I_d &\equiv \int_0^\Lambda \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{i}{q^2-m^2+i\varepsilon} \\
&\approx \frac{1}{8\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda}{m}\right) - c\right] \qquad (9.7)
\end{aligned}
$$

where we see the quadratic and logarithmic terms all together, with a finite c. The common technique to work with these is to amputate $D(p)$ from Equation 9.6 to get the divergences

$$
D_1^{Am}(p) = -12i\lambda I_d \qquad (9.8)
$$

As for Majorana spinors,

$$
\left\langle 0 \middle| T\left(\Psi^M_\alpha(x)\bar{\Psi}^M_\beta(y)\right)\middle| 0\right\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot(x-y)} S_{\alpha\beta}(k) \qquad (9.9)
$$

where

$$
S_{\alpha\beta} \equiv i\frac{(\not{k}+m)_{\alpha\beta}}{k^2-m^2+i\varepsilon} \qquad (9.10)
$$

With $(\Psi^M)^C = C\bar{\Psi}^{M^T}$ and $C^2 = -\mathbb{1}$,

$$
\left\langle 0 \middle| T\left(\Psi^M_\alpha(x)\Psi^M_\beta(y)\right)\middle| 0\right\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot(x-y)} S_{\alpha\beta}(k)C^T_{\gamma\beta} \qquad (9.11)
$$

$$
\left\langle 0 \middle| T\left(\bar{\Psi}^M_\alpha(x)\bar{\Psi}^M_\beta(y)\right)\middle| 0\right\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot(x-y)} C^T_{\alpha\gamma}S_{\gamma\beta}(k) \qquad (9.12)
$$

2 Explicit calculations on the Wess-Zumino model

Now that we have covered the necessary, let us apply them onto the Wess-Zumino model. For the interacting Lagrangian in the exponential of the n-point function, we will use the interacting Lagrangian of the Wess-Zumino model, $\mathcal{L}_{int} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$. What we are interested in is to see that all the terms vanish except for the logarithmic divergence. For that, we will be grouping the coefficients of of the terms after the calculations and adding them together.

2.1 Single *A* field propagator

The non-vanishing contributions of having an *A* field particle propagate in the Wess-Zumino model is to the order of g.

$$\langle \Omega \,|\, T(A(x)) \,|\, \Omega \rangle = -ig \int \mathrm{d}^4 z \left\langle 0 \middle| T \left\{ A(x) \left(mA^3(z) + mA(z)B^2(z) + A(z)\bar{\Psi}\Psi \right) \right\} \middle| 0 \right\rangle \tag{9.13}$$

Carrying out the calculation (amputating the $D(p)$ terms to make it easier), we would see that the coefficient of I_d for the A field propagator vanishes!

2.2 Double *B* field propagator

There is no contribution with a 1st order of g because every term in $\mathcal{L}_{int}$ to order g has an odd number of A field, and the VEV will naturally vanish. So the terms to consider are the terms with g to the 0th order and 2nd order.

$$\begin{aligned} &\langle \Omega \,|\, T\,(B(x)B(y)) \,|\, \Omega \rangle \\ &= D_B(x-y) - \int \mathrm{d}^4 z \int \mathrm{d}^4 w \,\langle 0 \,|\, T\,(B(x)B(y)\mathcal{L}_1) \,|\, 0 \rangle \\ &\quad -\frac{1}{2} \int \mathrm{d}^4 z \int \mathrm{d}^4 w \,\langle 0 \,|\, T\,(B(x)B(y)(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4)(z)(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4)(w)) \,|\, 0 \rangle \end{aligned} \tag{9.14}$$

Carrying out the calculation (and amputating the $D(p)$ terms to make it easier), we yet again see that the coefficients of I_d vanishes! However, there is a remnant term that carries a logarithmic divergence. It is at this point where we should take a step back and allow the flow of things to work on as it can be shown that with a renormalisation of the fields, this logarithmic divergence will too, vanish.

This is the power of SUSY that appeals to the theoretical side of particle physicists. The simple and elegant removal of the ugly divergences that plagues the most successful theory to date.

Chapter 10
Supersymmetric Gauge Theories

We shall now move towards the Standard Model by looking at imposing supersymmetry to gauge theories. We shall first attempt to do so for a $U(1)$ gauge theory before moving on to generalise for an $SU(N)$ gauge theory. At the end, we will attempt to make QED supersymmetric.

1 $U(1)$ gauge theory

For this theory, a vector multiplet is sufficient to illustrate the effects. We will use the vector multiplet that we have built earlier in Chapter 7 Section 2, $h = \{\pm 1/2, \pm 1\}$. Let us consider a photon field given by A_μ and its superpartner λ. The free Lagrangian for this theory is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{\mu\nu} + \lambda^\dagger i\bar{\sigma}^\mu \partial_\mu \lambda \tag{10.1}$$

Since the field strength $F_{\mu\nu}$ is neutral in charge, so is λ. Because of that, there is no need for us to replace ∂_μ with a covariant derivative as the free Lagrangian is gauge invariant on its own.

Let us determine the supersymmetric transformation rule of the fields. Since we want this to be a supersymmetric, we want A_μ to transform into λ and back. Our first ansatz is

$$\delta A^\mu = \xi^\dagger \bar{\sigma}^\mu \lambda + \lambda^\dagger \bar{\sigma}^\mu \xi \tag{10.2}$$

where by dimensional analysis, $[\xi] = -1/2$.

For the transformation rule of the photino, we need to ensure gauge invariance so we shall make use of the gauge invariant $F_{\mu\nu}$. Our ansatz is

$$\delta\lambda = C F_{\mu\nu} \sigma^\mu \bar{\sigma}^\nu \xi \tag{10.3}$$

where the know that $\sigma^\mu \bar{\sigma}^\nu \xi$ is a proper rank 2 tensor. As λ is a left chiral (by choice), ξ also has to be left chiral.

To determine the value of the constant C, we impose the invariance of the Lagrangian:

$$\delta\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = -F_{\mu\nu}(\xi^{\dagger}\bar{\sigma}^{\nu}\partial^{\mu}\lambda + \partial^{\mu}\lambda^{\dagger}\bar{\sigma}^{\nu}\xi) \tag{10.4}$$

$$\delta(i\lambda^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\lambda) = iC`lambda^{\dagger}\bar{\sigma}^{\rho}\partial_{\rho}F_{\mu\nu}\sigma^{\mu}\bar{\sigma}^{\nu}\xi \tag{10.5}$$

Putting them together,

$$\delta\mathcal{L} = -F_{\mu\nu}\xi^{\dagger}\bar{\sigma}^{\nu}\partial^{\mu}\lambda(1 - 2iC^{*}) \tag{10.6}$$

which means that we should pick $C = i/2$.

Now as we move on to include the auxiliary fields, we need to hold ourselves back from adding the same F auxiliary fields that we had in the Wess-Zumino model. They are inherently different theories and so we need to treat them differently. For example, although A_{μ} also has 2 degrees of freedom on-shell, off-shell, it has 3. λ has 2 degrees of freedom on-shell and 4 degrees of freedom off-shell. We see here that for the photon-photino multiplet, there is an absence of 1 degree of freedom. Therefore, our auxiliary field has to be of 1 degree of freedom, which means it has to be a real scalar field. The Lagrangian contribution is

$$\mathcal{L}_{aux} = \frac{1}{2}D^2 \tag{10.7}$$

where we need to note that D is inherently charge-less as well because of the total gauge invariance.

Since $[D] = [F]$, we shall assume that it supersymmetrically transforms like F.

$$\delta D = \partial_{\mu}\left(\xi^{\dagger}(-i\bar{\sigma}^{\mu})\lambda + h.c.\right) \tag{10.8}$$

Since we see that D is real, gauge-invariant, and transforms as a total derivative, we can add a linear term to the Lagrangian and not have it affect the overall invariance. This will come in handy when spontaneously breaking supersymmetry.

$$\mathcal{L}_{FI} = \varsigma D \tag{10.9}$$

To complete the algebra, we have to fix the transformation rule of the photino as we did in the Wess-Zumino model.

In total, the free SUSY $U(1)$ gauge theory is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^{\dagger}\bar{\sigma}^{\mu}\partial_{\nu}\lambda + \frac{1}{2}D^2 + \varsigma D \tag{10.10}$$

with the following supersymmetric theories:

$$\begin{aligned}
\delta A^\mu &= \xi^\dagger \bar{\sigma}^\mu \lambda + \lambda^\dagger \bar{\sigma}^\mu \xi \\
\delta \lambda &= \frac{1}{2} i F_{\mu\nu} \sigma^\mu \bar{\sigma}^\nu \xi + D\varsigma \\
\delta D &= -i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \lambda + i\partial_\mu \lambda^\dagger \bar{\sigma}^\mu \xi
\end{aligned} \tag{10.11}$$

2 $SU(N)$ gauge theories

We shall now extend the work for $SU(N)$ gauge theories. Here, we shall seed the boson field A_μ (not necessarily the photon field from before) with a charge g. Because of that, our ∂_μ is now replaced by a covariant derivative.

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu^a T_F^a \tag{10.12}$$

where T_F^a is the fundamental representation of the gauge theory.

The A field transforms in the fundamental representation as

$$A_\mu \to A'_\mu = UA_\mu U^\dagger + \frac{1}{g}(\partial_\mu U)U^\dagger \tag{10.13}$$

and the covariant derivative transforms in the adjoint representation.

$$D_\mu \psi \to D'_\mu \psi' = UD_\mu \psi \tag{10.14}$$

To make the Dirac spinors gauge invariant, we use the covariant derivative instead of the normal partial derivative as well.

Since the field strength of the gauge field is given by the following expression:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{10.15}$$

where F_{μ_ν} is a matrix in the Lie algebra, we can find that $F_{\mu\nu}$ transforms in the fundamental representation.

$$F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^\dagger \tag{10.16}$$

3 QCD in Weyl spinors

Using the information from the previous section, we can express QCD using Weyl spinors. The Lagrangian of a quark is given by

$$\begin{aligned}
\mathcal{L}_{QCD} &= \bar{\Psi}_D(i\gamma^\mu D_\mu - m)\Psi_D \\
&= \bar{\Psi}_D(i\gamma^\mu \partial_\mu - g\gamma^\mu A_\mu - m)\Psi_D
\end{aligned} \tag{10.17}$$

where being $SU(3)$, $A_\mu = \frac{1}{2}\lambda^a A^a_\mu$.

Properly changing expanding the equation, we get

$$\mathcal{L}_{QCD} = i\chi^\dagger_{\bar{q}}\bar{\sigma}^\mu \left[\partial_\mu - \frac{1}{2}igA^a_\mu(\lambda^a)^*\right]\chi_{\bar{q}} + i\chi^\dagger_q\bar{\sigma}^\mu \left[\partial_\mu + \frac{1}{2}igA^a_\mu(\lambda^a)^*\right]\chi_q \\ -m(\chi_q \cdot \chi_{\bar{q}} + \bar{\chi}_{\bar{q}} \cdot \bar{\chi}_q) \tag{10.18}$$

where we see that the covariant derivative is different for q and $\bar{q}$, which means that they transform very differently and under nonequivalent representations. This is because the they have opposite charges!

4 Free Abelian Vector multiplet × Free chiral multiplet

Let us now try to combine what we have done for the free multiplets and introduce interactions to them. The chiral multiplet is the multiplet with χ, ϕ, and F. To couplet the multiplets together, we introduce a $U(1)$ charge to the chiral multiplet.

$$X = \exp[iq\Lambda(x)]X \tag{10.19}$$

where X is any of the particles in the chiral multiplet.

The free Lagrangian is now becomes

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) + i\chi^\dagger\bar{\sigma}^\mu D_\mu\chi + F^\dagger F - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}D^2 + \varsigma D \tag{10.20}$$

where some interaction terms come when we expand the covariant derivative. These terms are

$$(\mathcal{L}_{int})_1 = iq\phi A^\mu\partial_\mu\phi^\dagger - iq\phi^\dagger A^\mu\partial_\mu\phi - q\chi^\dagger\bar{\sigma}^\mu A_\mu\chi \tag{10.21}$$

In finding new interaction terms, we need to ensure the 4 constraints of Chapter 8 Section 1. The need for gauge invariance greatly narrows the options and the only remaining terms are

$$(\mathcal{L}_{int})_2 = c_1\left(\phi^\dagger\chi\cdot\lambda + h.c.\right) + c_2\phi^\dagger\phi D \tag{10.22}$$

Adding $(\mathcal{L}_{int})_2$ to Equation 10.20, we will have the general free abelian vector multiplet × free chiral multiplet Lagrangian. The transformation rules for each particle is that in their own theories in Equations 6.22 and 10.11, except that we cannot have both of them use the same infinitesimal parameter ξ. The vector multiplet will have an infinitesimal parameter $a\xi$, where a is a constant.

To determine the coefficients a, c_1, and c_2, we have to vary $(\mathcal{L}_{int})_1 + (\mathcal{L}_{int})_2$ and ensure that it either vanishes or gauges away as a total derivative, as what we did in Section 1. Going through the work, we will arrive at

$$a = -1/\sqrt{2} \quad , \quad c_1 = -\sqrt{2}q \quad , \quad c_2 = -q \tag{10.23}$$

However, now the auxiliary field does not close on its algebra. To fix this, we have to add some terms to the transformation rules of F and $F^\dagger$.

$$\delta F = -i\xi^\dagger \bar{\sigma}^\mu D_\mu \chi + \sqrt{2} q\phi \bar{\xi} \cdot \bar{\lambda} \tag{10.24}$$

where this means that the chiral auxiliary field transforms into spinors in the chiral and vector multiplets!

All in all, our Lagrangian of the free abelian vector multiplet × free chiral multiplet now reads

$$\begin{aligned}\mathcal{L} &= D_\mu \phi^\dagger D^\dagger \phi + i\chi^\dagger \bar{\sigma}^\mu D_\mu \chi + F^\dagger F - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda \\ &+ \frac{1}{2} D^2 + D\varsigma - \sqrt{2} q(\phi^\dagger \chi \cdot \lambda + h.c.) - q\phi^\dagger \phi D \end{aligned} \tag{10.25}$$

and the fields of this theory transforms supersymmetrically as

$$\begin{aligned}\delta A^\mu &= -\frac{1}{\sqrt{2}} (\xi^\dagger \bar{\sigma}^\mu \lambda + \lambda^\dagger \bar{\sigma}^\mu \xi) \\ \delta\lambda &= -\frac{1}{2\sqrt{2}} F_{\mu\nu} \sigma^\mu \bar{\sigma}^\nu \xi - \frac{1}{\sqrt{2}} D\xi \\ \delta D &= \frac{i}{\sqrt{2}} \xi^\dagger \bar{\sigma}^\mu \partial_\mu \lambda - \frac{i}{\sqrt{2}} (\partial_\mu \lambda)^\dagger \bar{\sigma}^\mu \xi \\ \delta\phi &= \xi \cdot \chi \\ \delta\chi &= -i(D_\mu \phi) \sigma^\mu (i\sigma^2) \xi^* + F\xi \\ \delta F &= -i\xi^\dagger \bar{\sigma}^\mu D_\mu \chi + \sqrt{2} q\phi \bar{\xi} \cdot \bar{\lambda} \end{aligned} \tag{10.26}$$

For there to be a superpotential for the single chiral multiplet, it is necessary to be holomorphic in ϕ, which means that the $U(1)$ charge of the chiral multiplet is $0, \implies \phi = \phi^\dagger$. However, if we have multiple chiral multiplets, we simply have to ensure that the combination of multiplets must have a vanishing overall gauge charge and we will have a superpotential that embeds all the information of their self interactions.

5 Nonabelian free vector multiplet × free chiral multiplet

To process is the same, where the only difference is that now we have to ensure that we keep track of the right indices for gauge fields and charges. After doing so, we will find that the interacting Lagrangian is

$$\mathcal{L}_{int} = c_1 \left([\phi^{\dagger^b} (T_F^a)^{bc} \chi^c] \cdot \lambda^a + h.c. \right) + c_2 [\phi^{\dagger^b} (T_F^a)^{bc} \phi^c] D^a \tag{10.27}$$

where we will yet again find that

$$c_1 = -\sqrt{2}g \quad , \quad c_2 = -g \tag{10.28}$$

And with that, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = &D_\mu\phi^\dagger D^\dagger\phi + i\chi^\dagger\bar{\sigma}^\mu D_\mu\chi + F^\dagger F - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda \\ &+\frac{1}{2}D^2 - \sqrt{2}g(\phi^\dagger\chi\cdot\lambda + h.c.) - g\phi^\dagger\phi D\end{aligned} \tag{10.29}$$

and just as with the abelian case, the transformation rule for F has to be modified to

$$\delta F = \ldots + \sqrt{2}g\phi^b(T_F^a)^{bc}\bar{\xi}\cdot\bar{\lambda}^a \tag{10.30}$$

Chapter 11
Superspace Formalism

As we have seen in Equation 6.13, the algebra of supersymmetric charges result in a translation in spacetime. This suggests that it should be embedded in some form in the explicit representation of Q and that we can treat the supercharges as differential operators. In this chapter, we will expand more on this idea and see how we can extend spacetime to include 4 additional dimensions on which the supercharges operate on. Afterwards, we will get the explicitly derive the differential operator representations of the supercharges and discuss the interpretation of it. Lastly, we will preemptively look at constraints that we can introduce to this formalism for our next chapter.

1 Supercharges in superspace

Let us consider the unitary operator of the Lorentz charge: $U(a) = \exp[ia^\mu P_\mu]$. a^μ is the infinitesimal spacetime displacement. Compare that to the unitary operator of the Supersymmetry charge: $U(\xi) = \exp[i\xi \cdot Q]$. If we were to make a direct comparison on the premise that Q here is a differential operator, this means that Q displaces the particle by an infinitesimal displacement ξ in whatever space it spans. We know too, that ξ is a spinor with Grassmann components, so it would make sense for us to assume that the space ξ spans is a two-component Grassmann space – what we will now call the superspace. Since there are 4 degrees of freedom, ξ and ξ^*, we have 4 'coordinates' in superspace. They are represented as left chiral spinors as well, for obvious reasons.

$$\theta \equiv \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \tag{11.1}$$

where we are using the new indexed notation introduced in Chapter 3.

Any supersymmetric transformation will therefore be a transformation of

$$\Phi(x, \theta, \bar{\theta}) \rightarrow \Phi(x'\theta', \bar{\theta}') = U(\alpha, \xi, \bar{\xi})\Phi(x, \theta, \bar{\theta})U^\dagger(\alpha, \xi, \bar{\xi}) \tag{11.2}$$

We will apply what we have done in Chapter 6 to derive the explicit differential operator representations of Q. We can decompose a unitary operator into a product of unitary operators

$$U(x', \theta', \bar{\theta}') = U(\alpha, \xi, \bar{\xi})U(x, \theta, \bar{\theta}) \tag{11.3}$$

if the arguments add up as

$$\begin{aligned} x' &= x + \alpha \\ \theta' &= \theta + \xi \\ \bar{\theta}' &= \bar{\theta} + \bar{\xi} \end{aligned}$$

However, this does not mean that the exponential arguments add up as they may be quantum field operators. Recall the Baker-Campbell-Hausdorff formula:

$$\exp[A]\exp[B] = \exp[A + B + \frac{1}{2}[A, B] + \dots] \tag{11.4}$$

The unitary operator will be represented as

$$U(x, \theta, \bar{\theta}) = \exp[ix \cdot P + i\theta \cdot Q, i\bar{\theta} \cdot \bar{Q}] \tag{11.5}$$

as its product with its hermitian conjugate to the first order gives us identity.

Expanding Equation 11.3 and the commutator relations that arise from the Baker-Campbell-Hausdorff formula, we get

$$\begin{aligned} \exp[ix' \cdot P + i\theta' \cdot Q + i\bar{\theta}' \cdot \bar{Q}] = \exp[&i(x + \alpha) \cdot P + i(\theta + \xi) \cdot Q + i(\bar{\theta} + \bar{\xi}) \cdot \bar{Q} \\ &-\frac{1}{2}\xi\sigma^\mu\bar{\theta}P_\mu + \frac{1}{2}\theta\sigma^\mu\bar{\xi}P_\mu] \end{aligned} \tag{11.6}$$

We can read off the coordinate transformations to be

$$\begin{aligned} x' &= x + \alpha - \frac{i}{2}\theta\sigma^\mu\bar{\xi} + \frac{i}{2}\xi\sigma^\mu\bar{\theta} \\ \theta' &= \theta + \xi \\ \bar{\theta}' &= \bar{\theta} + \bar{\xi} \end{aligned} \tag{11.7}$$

It shows exactly what we are hoping to see, the transformation of regular spacetime under supersymmetry!

2 Supercharges as explicit superspace differential operators

Let us now find the explicit differential forms of the supercharges. We will do this by operating the unitary transformation operator on a known state

$$
\begin{aligned}
S(x+\delta x, \theta+\delta\theta, \bar\theta+\delta\theta) &= \exp[-i\alpha^\mu P_\mu - i\xi^a Q_a - i\bar\xi_{\dot a}\bar Q^{\dot a}]S(x,\theta,\bar\theta) \\
&= S_0 + \left(\alpha^\mu + \frac{1}{2}\xi\sigma^\mu\bar\theta - \frac{1}{2}\theta\sigma^\mu\bar\xi\right)\partial_\mu S_0 + \xi^a\partial_a S_0 + \bar\xi_{\dot a}\bar\partial^{\dot a} \qquad (11.8)
\end{aligned}
$$

Once again, reading off the coefficients, we have:

$$\hat P_\mu = i\partial_\mu \tag{11.9}$$

$$Q_a = i\partial_a - \frac{1}{2}\sigma^\mu\bar\theta\partial_\mu \tag{11.10}$$

$$\bar Q^{\dot a} = i\bar\partial^{\dot a} - \frac{1}{2}\bar\sigma^\mu\theta_b\partial_\mu \tag{11.11}$$

We can lower the indices of Equation 11.11 to get

$$\bar Q_{\dot a} = -i\bar\partial_{\dot a} + \frac{1}{2}\theta\sigma^\mu\partial_\mu \tag{11.12}$$

We could use these explicit representations to verify the supersymmetry algebra in Equations 6.13 and find that they are consistent.

3 Adding constraints

If we were to use the superspace formalism as is to build the Wess-Zumino Lagrangian, the first thing we would notice is that there would be too many terms for a superfield $\mathcal{F} = \mathcal{F}(x,\theta,\bar\theta)$. However, $\mathcal{F} = \mathcal{F}(x,\theta)$ gives the right number of fields, but it will no longer be supersymmetric invariant as after transformation, there will be a $\bar\theta$ dependence. The approach to fix this is simple – add a constraint.

Let us see what happens when we add a constraint such that the superfield is independent of $\bar\theta$:

$$\frac{\partial}{\partial\bar\theta^{\dot a}}\mathcal{F}(x,\theta,\bar\theta) = 0 \tag{11.13}$$

This constraint should remain even after transformation.

$$\frac{\partial}{\partial\bar\theta^{\dot a}}\mathcal{F}'(x,\theta,\bar\theta) = -i\partial_{\dot a}\left[(\alpha^\mu\hat P_\mu + \xi\cdot\hat Q + \bar\xi\cdot\hat{\bar Q})\mathcal{F}\right] = 0 \tag{11.14}$$

$$\implies \left[\partial_{\dot a}, \xi\cdot\hat Q + \bar\xi\cdot\hat{\bar Q}\right] = 0 \tag{11.15}$$

$$\therefore \{\bar\partial_{\dot a}, \hat Q_b\} = \{\bar\partial_{\dot a}, \hat{\bar Q}^{\dot b}\} = 0 \tag{11.16}$$

From these, it can be seen that the second constraint is identically 0, and the first says that

$$\{\bar\partial_{\dot a}, \hat Q_b\} = \delta^{\dot c}_{\dot a} \neq 0 \tag{11.17}$$

is a contradiction. So on its own, Equation 11.13 is not sufficient enough as a constraint. To fix this, we will add some terms to the constraint to make get rid of the Dirac delta.

$$\bar{D}_{\dot{a}} = \bar{\partial}_{\dot{a}} + C_{\dot{a}c}\theta^c \tag{11.18}$$

Repeating the process from Equation 11.14, we will see that we will end up at

$$\{\bar{D}_{\dot{a}}, \hat{Q}_b\} = iC_{\dot{a}b} + \frac{i}{2}\sigma^{\mu}_{b\dot{a}}\hat{P}_{\mu} \tag{11.19}$$

which means that we can have a supersymmetrically consistent constraint:

$$\bar{D}_{\dot{a}} = \bar{\partial}_{\dot{a}} - \frac{i}{2}\theta^c\sigma^{\mu}_{c\dot{a}}\partial_{\mu} \tag{11.20}$$

Likewise, we can also have the un-barred flavour

$$D_a = \partial_a - \frac{i}{2}\sigma^{\mu}_{a\dot{b}}\bar{\theta}^{\dot{b}}\partial_{\mu} \tag{11.21}$$

Since these constraints are indexed, we can raise or lower them as we wish and we can also form supersymmetric invariants with them too. $D \cdot D$ and $\bar{D} \cdot \bar{D}$. These also work with the indexed constraints like $D_a D \cdot D\mathcal{S} = 0$.

Since constraints are built off Grassmann variables, they vanish under anti-commutators for all combinations except

$$\{D_a, \bar{D}_{\dot{b}}\} = -\sigma^{\mu}a\dot{b}\hat{P}_{\mu} \tag{11.22}$$

Chapter 12
Left chiral superfields

We will work off from where we have left off in 3 to build left chiral superfields. Afterwards, we will see how this left chiral superfield can be used to arrive at the Lagrangians we have derived in the Wess-Zumino model and supersymmetric QED.

1 Left chiralty

The term left chiral refers to the fact that the spinors that appear explicitly are all left chiral spinors. We will start from the constraint $\bar{D}_{\dot{a}}$ to see how we can extract the left chiral spinors.

As we know, we want our superfield to be dependent on only 2 types of coordinates, x^μ and θ. Identically, this means that we can introduce a new coordinate $y^\mu(x^\mu, \bar{\theta}$ that should naturally satisfy the constraint $\bar{D}_{\dot{a}}$, such that our superfield can be dependent on y^μ, θ.

An ansatz which fits the constraint is

$$y^\mu = x^\mu - \frac{i}{2}\theta\sigma^\mu\bar{\theta} \tag{12.1}$$

With this, the most general left chiral superfield is

$$\Phi_L(y, \theta) = \Phi(x^\mu - \frac{i}{2}\theta\sigma^\mu\bar{\theta}, \theta) \tag{12.2}$$

Likewise, the most general right chiral superfield (satisfying the constraint $D_a\Phi = 0$) is

$$\Phi_R(y, \theta) = \Phi(x^\mu + \frac{i}{2}\theta\sigma^\mu\bar{\theta}, \theta) \tag{12.3}$$

, for which we find is actually the hermitian conjugate of Φ_L! Therefore, for a left chiral superfield,

$$\bar{D}_{\dot{a}}\Phi = 0 \implies D_a\Phi^\dagger = 0 \tag{12.4}$$

Expanding the left chiral superfield,

$$\Phi(y,\theta) = \phi(y) + \theta \cdot \chi(y) + \frac{1}{2}\theta \cdot \theta F(y) \tag{12.5}$$

and we see why the left chiral superfield is called left chiral!

As we know that y is a function of x and $\bar{\theta}$, we can expand it further:

$$\begin{aligned}\Phi = {} & \phi(x) - \frac{i}{2}\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) - \frac{1}{16}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}\Box\phi(x) \\ & +\theta\cdot\chi(x) - \frac{i}{2}\theta\sigma^{\mu}\bar{\theta}\theta\cdot\partial_{\mu}\chi(x) + \frac{1}{2}\theta\cdot\theta F(x)\end{aligned} \tag{12.6}$$

2 Ensuring supersymmetry of the superfield

Now that we have expanded the superfield in the full coordinates, let us check if the superfield transforms with the same rules as supersymmetry.

We have on one side,

$$\begin{aligned}\Phi' = {} & \phi' - \frac{i}{2}\theta\sigma^{\mu}\bar{\theta}\theta\cdot\partial_{\mu}\phi'(x) - \frac{1}{16}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}\Box\phi'(x) \\ & +\theta\cdot\chi'(x) - \frac{i}{2}\theta\sigma^{\mu}\bar{\theta}\theta\cdot\partial_{\mu}\chi'(x) + \frac{1}{2}\theta\cdot\theta F'(x)\end{aligned} \tag{12.7}$$

and on the other,

$$\begin{aligned}\Phi' = {} & \Phi + \frac{i}{2}(\xi\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\xi})\partial_{\mu}\phi + \frac{1}{4}(\xi\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\xi})\theta\sigma^{\nu}\bar{\theta}\partial_{\nu}\partial_{\mu}\phi \\ & -\frac{i}{32}(\xi\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\xi})\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}\Box\partial_{\mu}\phi + \frac{i}{2}(\xi\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\xi})\theta\cdot\partial_{\mu}\chi \\ & +\frac{1}{4}(\xi\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\xi})\theta\sigma^{\nu}\bar{\theta}\theta\cdot\partial_{\nu}\partial_{\mu}\chi + \frac{i}{4}(\xi\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\xi})\theta\cdot\theta\partial_{\mu}F \\ & -\frac{i}{2}\xi\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi - \frac{1}{8}\xi\cdot\theta\bar{\theta}\cdot\bar{\theta}\Box\phi + \xi\cdot\chi - \frac{i}{2}\xi\sigma^{\mu}\bar{\theta}\theta\cdot\partial_{\mu}\chi \\ & -\frac{i}{2}\theta\sigma^{\mu}\bar{\theta}\xi\cdot\partial_{\mu}\chi + F\xi\cdot\theta + \frac{i}{2}\bar{\xi}\bar{\sigma}^{\mu}\theta\partial_{\mu}\phi - \frac{1}{8}\theta\cdot\theta\bar{\xi}\cdot\bar{\theta}\Box\phi + \frac{i}{2}\bar{\xi}\bar{\sigma}\theta\theta\cdot\partial_{\mu}\chi\end{aligned} \tag{12.8}$$

Comparing coefficients and remembering that any Grassmann terms of more than a quadratic relation is identically 0, we will arrive at the same SUSY algebra of Equations 6.13. Therefore, the superfield is supersymmetry invariant and we can use it to build our Lagrangians.

We can also find that any product of left chiral superfields will also be a left chiral superfield and remain invariant under supersymmetry. The same goes for right chiral superfields too, but that will not be of interest to us.

3 *F* terms

Let us recall that in the Wess-Zumino Lagrangian, the auxiliary field F transforms as a total derivative. The field can be extracted from the superfield through the Grassmann integration $\int d^2\theta^2\Phi(x,\theta,\bar{\theta})$. This means that $\int d^4 x d^2\theta^2\Phi(x,\theta,\bar{\theta})$ is an invariant under a supersymmetry transformation.

4 Building the Wess-Zumino Interactions

The Wess-Zumino model is a single multiplet with self-interactions. To bring out the interactions in the F terms of the superfield, we need to consider more than 1 superfield.

Let us look at the possible combinations of superfields we can have. We realise that $\Phi_i\Phi_j|_F$ has a dimension of 3, so we shall couple it to a dimension 1 coupling m_{ij}. The furthest we can then go for a renormalisable theory is to have Φ^3 in $y_{ijk}\Phi_i\Phi_j\Phi_k$

Put together, we have

$$\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{6}y_{ijk}\Phi_i\Phi_j\Phi_k + h.c.\bigg|_F \tag{12.9}$$

which is exactly the superpotential that we saw in the Wess-Zumino model. To show that it really does give us the right terms in the Wess-Zumino model, consider the superpotential with F_i:

$$\mathcal{W}(\Phi_1,\Phi_2\dots\Phi_n)|_{F_i} = \frac{\partial\mathcal{W}(\phi_1,\phi_2\dots\phi_n}{\partial\phi_i}F_i \tag{12.10}$$

which are the Wess-Zumino interactions that we wanted to see.

Besides those interactions, there were also the spinor interactions. The remaining terms of the $\big|_F$ operation on the superfield is

$$\begin{aligned}\mathcal{W}(\phi_1,\dots\theta\cdot\chi_i,\dots,\theta\cdot\chi_j\cdot\phi_n) &= \mathcal{W}(\phi_1,\dots 1,\dots,1\dots\phi_n)\chi_i\cdot\chi_j\theta\cdot\theta + h.c. \\ &= -\frac{1}{2}\theta\cdot\theta\frac{\partial^2\mathcal{W}(\phi_1,\phi_2,\dots\phi_n}{\partial\phi_i\partial(\phi_j}\chi_i\cdot\chi_j + h.c.\end{aligned} \tag{12.11}$$

Putting them together,

$$\oplus_\rangle,\dots\oplus_\backslash\big|_F = \frac{\partial\mathcal{W}(\phi_1,\phi_2\dots\phi_n}{\partial\phi_i}F_i - -\frac{1}{2}\frac{\partial^2\mathcal{W}(\phi_1,\phi_2,\dots\phi_n}{\partial\phi_i\partial(\phi_j}\chi_i\cdot\chi_j + h.c. \tag{12.12}$$

And we have exactly, the interactions of the Wess-Zumino model.

5 D terms

Besides the interaction terms, we also need a way to extract information about the kinetic terms of the Lagrangian from the superfield formalism. The most intuitive approach would be to use $\Phi^\dagger\Phi$, which is correct. However, instead of extracting out the F terms as we did for the interactions, we need to extract a different term this time. This is because unlike the interaction terms, $\Phi^\dagger\Phi$ is not a left chiral superfield. Remember that it was only because of the fact that $\Phi_i\Phi_j$ is left chiral that we were able to identify its corresponding auxiliary field F which we know transforms as a total derivative under supersymmetric transformations.

What we do now is to find a term which we know definitely is an invariant in the Lagrangian density and just ‘extract’ that term like we did for the F terms. These new set of terms are referred to as the D terms. If we were to expand $\Phi^\dagger\Phi$ in full, we would see that leading term is

$$\Phi^\dagger\Phi = \frac{1}{4}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}\partial_\mu\phi^\dagger\partial^\mu\phi + \dots \tag{12.13}$$

Since we already know that $\partial_\mu\phi^\dagger\partial^\mu\phi$ is an invariant (and one that we need), we shall extract this term. Taking the integral over $\int \mathrm{d}^2\theta^2\mathrm{d}^2\bar{\theta}^2$, we will get

$$\Phi^\dagger\Phi\Big|_D = \partial_\mu\phi^\dagger\partial^\mu\phi + i\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi + F^\dagger F \tag{12.14}$$

which when added to Equation 12.12, gives us the Wess-Zumino Lagrangian! In the next chapter, we will expand further, working on gauge field theories using the superspace formalism and we will see that this method is much more efficient than manually deriving the Lagrangians as what we have done in the earlier half of the book!

Chapter 13
supersymmetric Gauge Field Theories in the Superfield Approach

We have seen in the earlier chapter how we can extract out the Wess-Zumimno Lagrangian without any of the troublesome invariance checks just by using left chiral superfields. We will see in this chapter that the approach of using superfields is translatable even for more complex theories such as those with local gauges. In the first part of the chapter, we will be introduced to a new way of adding gauge to a superfield. Afterwards are just applying them to QED and QCD.

1 Adding a $U(1)$ gauge to left chiral superfields

Let us consider a $U(1)$ charge.

$$\Phi \to \exp[i2q\Lambda]\Phi \tag{13.1}$$

where for our current discussion, Λ is a spacetime dependent field. The factor of 2 is picked preemptively.

Since we need Φ to remain a left chiral superfield, Λ has to be a left chiral superfield as well. However, since Λ is an exponential argument, $[\Lambda] = 0$ which implies that $[\phi_\Lambda] = 0$, $[\chi_\Lambda] = 1/2$, and $[F_\Lambda] = 1$, different from the superfields of the previous chapter.

For a gauge invariant theory, we have to construct our fields such that the total gauge charges vanish. The kinetic terms are embedded in

$$\Phi^\dagger\Phi \to \exp[2iq(\Lambda - \Lambda^\dagger)]\Phi^\dagger\Phi \tag{13.2}$$

However, it is not definite that $\Phi^\dagger\Phi$ is gauge invariant as $\Lambda \in \mathbb{C}$. This can be solved by introducing a new real gauge field $\mathcal{V}$:

$$\Phi^\dagger\Phi \to \Phi^\dagger \exp[2q\mathcal{V}]\Phi \tag{13.3}$$

where

$$\mathcal{V} \to \mathcal{V} - i(\Lambda - \Lambda^\dagger) \tag{13.4}$$

The most general form $\mathcal{V}$ can take if it has to be a real field, and adding some scaling factors,

$$\begin{aligned}\mathcal{V} = C(x) &+ \frac{i}{\sqrt{2}}\theta\cdot\rho(x) - \frac{i}{\sqrt{2}}\bar{\theta}\cdot\bar{\rho}(x) + \frac{i}{4}\theta\cdot\theta(M(x)+iN(x)) \\ &- \frac{i}{4}\bar{\theta}\cdot\bar{\theta}\left(M^\dagger(x) - iN^\dagger(x)\right) + \frac{1}{2}\theta\sigma^\mu\bar{\theta}A_\mu(x) + \frac{1}{2\sqrt{2}}\theta\cdot\theta\left(\bar{\theta}\cdot\lambda + \frac{1}{2}\bar{\theta}\bar{\sigma}^\mu\partial_\mu\rho\right) \\ &+ \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\left(\theta\cdot\lambda - \frac{1}{2}\theta\sigma^\mu\partial_\mu\bar{\rho}\right) - \frac{1}{8}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}\left(D(x) + \frac{1}{2}\Box C(x)\right)\end{aligned} \tag{13.5}$$

Comparing the transformed $\mathcal{V}$, we find that the terms can be reduced to

$$\mathcal{V} = \frac{1}{2}\theta\sigma^\mu\bar{\theta}A_\mu + \frac{1}{2\sqrt{2}}\theta\cdot\theta\bar{\theta}\cdot\bar{\lambda} + \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\theta\cdot\lambda - \frac{1}{8}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}D \tag{13.6}$$

where λ and D are gauge invariant. So for abelian gauge theories, the Lagrangian can be retrieved from superfields as:

$$\mathcal{L} = (\mathcal{W}(\Phi_i)|_F + h.c.) + \sum_i \Phi_i^\dagger \exp[2q_i\mathcal{V}]\Phi_i|_D + \varpi D \tag{13.7}$$

where ϖD is the L.I. term.

Let us expand $\Phi^\dagger \exp[2q\mathcal{V}]\Phi|_D$ in full, noting that we only need to expand up to $\mathcal{V}^2$ since it is Grassmann.

$$\begin{aligned}\Phi^\dagger \exp[2q\mathcal{V}]\Phi|_D &= \mathcal{L}_{WZ,free} + 2q\Phi^\dagger\Phi\mathcal{V}|_D + 2q^2\Phi^\dagger\Phi\mathcal{V}^2|_D \\ &= D_\mu\phi^\dagger D^\mu\phi + i\bar{\chi}\bar{\sigma}^\mu D_\mu\chi + F^\dagger F - q\phi^\dagger\phi D \\ &\quad -\sqrt{2}q(\chi\cdot\lambda\phi^\dagger + h.c.)\end{aligned} \tag{13.8}$$

with $D_\mu \equiv \partial_\mu + iqA_\mu$ we see that these terms are what we see in the abelian gauge theory, without the field strength or photino kinetic terms, or $\frac{1}{2}D^2$.

To extract the field strength, we have to first understand mathematically where it could come from. Logically, the answer is through the gauge field $\mathcal{V}$, where it comes attached with 2 Grassmann variables. Applying the covariant derivative once on $\mathcal{V}$ will give us terms proportional to $\partial_\nu A_\mu$, but to extract it out for the Lagrangian, we need to add in 2 more Grassmann variables. This can be achieved by operating $\bar{D}\cdot\bar{D}$ on it! The super-field strength is thus

$$\mathcal{F}_a \equiv \bar{D}\cdot\bar{D}D_a\mathcal{V} \tag{13.9}$$

We can verify that $\mathcal{F}_a$ is a left chiral superfield by acting the constraint $\bar{D}_{\dot{b}}$ on it.

Let us assume that $\mathcal{F}_a$ transforms under the gauge as $\mathcal{V}$ does.

$$\mathcal{F}_a \to \mathcal{F}_a - i\bar{D}\cdot\bar{D}D_a\Lambda + i\bar{D}\cdot\bar{D}D_a\Lambda^\dagger \tag{13.10}$$

We will see that $\mathcal{F}_a$ is gauge invariant, as the field strength is in the abelian theory!

Applying the covariant derivatives explicitly, we find that the super-field strength is

$$\mathcal{F}_a = \sqrt{2}\lambda_a - \theta_a D - F_{\mu\nu}(\sigma^{\mu\nu})_a^b\theta_b + \frac{i}{\sqrt{2}}\theta\cdot\theta\sigma_a^\mu\partial_\mu\bar{\lambda} \tag{13.11}$$

Now, our super-field strength has an index and cannot be used as-is in a Lagrangian. To do so, we shall have to contract it and take the appropriate Grassmann integration. Since we built $\mathcal{F}_a$ completely out of left chiral superfields, we need to take the F terms of the contraction. Doing so, we arrive at

$$\frac{1}{4}\mathcal{F}^a\mathcal{F}_a\bigg|_F = \frac{1}{2}D^2 + i\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{13.12}$$

which are the missing terms that we were looking for!

So in total, the Lagrangian n-left chiral superfields coupled with a $U(1)$ gauge field is

$$\mathcal{L} = [\mathcal{W}(\Phi_i)|_F + h.c.] + \sum_i^n \Phi_i^\dagger \exp[2q_i\mathcal{V}]\Phi_i\bigg|_D + \frac{1}{4}\mathcal{F}\cdot\mathcal{F}\bigg|_F \tag{13.13}$$

2 The supersymmetric QED

Before we get ahead of ourselves and apply what we have done to the multiplet, we need to get our superfields in order. Unlike what we have done so far, QED has 2 left chiral and 2 right chiral particles. Applying what we did, we would have to deal with a mess of left and right chiral superfields at the same time. However, we can clean this up making use of the CPT invariance which brought us this problem, to sweep away all the right chiral fields.

Let us start with the electron. We can introduce a left chiral superfield for it as

$$\mathcal{E}_e = \tilde{\phi}_e + \theta\cdot\chi_e + \frac{1}{2}\theta\cdot\theta F_e \tag{13.14}$$

where the tides signify superpartners.

Because of CPT, this field is actually related to the right chiral positron's superfield! In that same vein, the right chiral superfield for the electron is related to the left chiral superfield for the positron. Our 4 superfields have reduced to 2 left chiral superfields! All we need to consider for the electrons (and the other lepton generations) are $\mathcal{E}_e$ and $\mathcal{E}_{\bar{e}}$. The other 2 fields are embedded in the hermitian conjugates of these fields.

Naturally, the $U(1)$ charges of these fields are $q = \pm e$, so under a gauge transformation, we have

$$\mathcal{E}_e \rightarrow \exp[-2ie\Lambda]\mathcal{E}_e$$
$$\mathcal{E}_{\bar{e}} \rightarrow \exp[-2ie\Lambda]\mathcal{E}_{\bar{e}}$$

Since the total charge of the 2 left chiral superfields vanish, we can have a superpotential with both superfields. The invariants we can build are

$$\mathcal{E}_e\mathcal{E}_{\bar{e}} \quad , \quad \mathcal{F}^a\mathcal{F}_a \quad , \quad \mathcal{E}_e^\dagger \exp[-2e\mathcal{V}]\mathcal{E}_e + h.c. \tag{13.15}$$

We can form more invariants by combining some of these terms too. Once we have all the combinations we need to consider (and adding the proper dimensions) we can apply what we did in the previous chapter to get the supersymmetric QED lagrangian:

$$\mathcal{L}_{sQED} = \frac{1}{4}\mathcal{F}^a\mathcal{F}_a\bigg|_F + \mathcal{E}_e^\dagger \exp[-2e\mathcal{V}]\mathcal{E}_e + \mathcal{E}_{\bar{e}}^\dagger \exp[2e\mathcal{V}]\mathcal{E}_{\bar{e}} + (\mathcal{W}|_F + h.c.) \tag{13.16}$$

3 Supersymmetric nonabelian gauge theory

As with what we done in Chapter 10 Section 5, we will start treat our gauge superfields as a representation of the gauge

$$\mathcal{V} \equiv \mathcal{V}^i T_F^i \quad , i \in [1, N^2 - 1] \tag{13.17}$$

for an $SU(N)$ nonabelian gauge theory. With this, our Lagrangian gets a little bit more complicated

$$\mathcal{L} = \frac{1}{2}Tr(\mathcal{F}_i^a \dashv\rangle)\bigg|_F + \Phi_i^\dagger \exp[2g\mathcal{V}]\Phi_i\bigg|_D + (\mathcal{W}(\Phi_i) + h.c.)|_F \tag{13.18}$$

showing once again how powerful the superspace formalism really is.

References

1. P. LaBelle, *Supersymmetry DeMYSTiFied* (McGraw-Hill Education, 2010). URL https://books.google.co.jp/books?id=SWHhdGbxukkC