# The Approximate Solution for A Triangular Fully Fuzzy Matrix Equation

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# Introduction

- Matrix is generally known as a rectangular array, arranged in rows and columns.
- Usually used to represent a linear system of equation, which can be solve analytically or numerically.
- Matrices have also been used independently in the form of matrix equations.

**Application of Control System Theory -** Used as an equation solver especially in designing and analyzing the feedback loop systems/state space representation (Zanoli & Pepe, 2018)

During designing & analyzing could involved with any uncertainty problems

- Conflicting requirements during system process
- Instability of environment/economic conditions/Distraction of any elements and noise

Example of Matrix Equations

$AX = B$

$AX \pm XB = C$

$AXB - X = C$

$AXB = C$

$AXA^T - X = C$

$AX + XA^T = C$

THE COEFFICIENTS OF THE MATRIX EQUATIONS WOULD BE CONSIDERED TO BE IN FUZZY NUMBERS.

# Objective

To construct a new method for solving a triangular Fully Fuzzy Matrix Equation (FFME)

$$\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$$

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \dots & \tilde{a}_{mn} \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \dots & \tilde{x}_{1m} \\ \tilde{x}_{21} & \tilde{x}_{22} & \dots & \tilde{x}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{n1} & \tilde{x}_{n2} & \dots & \tilde{x}_{nm} \end{pmatrix} \otimes \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \dots & \tilde{b}_{1n} \\ \tilde{b}_{21} & \tilde{b}_{22} & \dots & \tilde{b}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_{m1} & \tilde{b}_{m2} & \dots & \tilde{b}_{mn} \end{pmatrix} = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \dots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \dots & \tilde{c}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{m1} & \tilde{c}_{m2} & \dots & \tilde{c}_{mn} \end{pmatrix}$$

where $\tilde{A} = (a_{ij})$, $1 \le i \le m, 1 \le j \le n$, $\tilde{B} = (b_{ij})$, $1 \le i \le m, 1 \le j \le n$ and the right-hand side matrix $\tilde{C} = (c_{ij})$, $1 \le i \le m, 1 \le j \le n$ are the fuzzy matrices, and the solution $\tilde{X} = (x_{ij})$, $1 \le i \le n, 1 \le j \le m$ is an unknown fuzzy matrix.

# Literature Review

$\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$
(Guo & Shang, 2013)

-Incompatible for large matrices.
-Limited only for positive fuzzy coefficients.

$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$
(Shang et al., 2015)

$\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$
(Dookhitram et al., 2015)

$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$
(Malkawi et al., 2015b)

$\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$
(Daud et. al, 2018$b$)

$\tilde{X}\tilde{A} = \tilde{C}$
(Yang et. al, 2019)

$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$
(Kenyanpour et. al, 2018)
(Elsayed et. Al, 2020)

$\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$
(Daud et. al, 2021)

# Methodology

**Kronecker product & Vec-operator**

$(\tilde{B}^T \otimes_k \tilde{A})Vec\,(\tilde{X}) = Vec\,(\tilde{C})$

$$\left(\begin{array}{c|c|c} m^{\tilde{S}} & 0 & 0 \\ \hline -\beta^{\tilde{S}} & (m^{\tilde{S}}+\beta^{\tilde{S}})^{+} & -(m^{\tilde{S}}+\beta^{\tilde{S}})^{-} \\ \hline -\alpha^{\tilde{S}} & -(m^{\tilde{S}}-\alpha^{\tilde{S}})^{-} & (m^{\tilde{S}}-\alpha^{\tilde{S}})^{+} \end{array}\right)\left(\begin{array}{c} m^{\tilde{X}} \\ \hline \alpha^{\tilde{X}} \\ \hline \beta^{\tilde{X}} \end{array}\right) = \left(\begin{array}{c} m^{\tilde{C}} \\ \hline \alpha^{\tilde{C}} \\ \hline \beta^{\tilde{C}} \end{array}\right)$$

FFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$ → FFLS $\tilde{S}\tilde{X} = \tilde{C}$ → ALS $SX=C$ → Solution $X = S^{\dagger}C$

*$S^{\dagger}$ is the pseudoinverse of $S$

# Numerical Example

**Example 1** *Consider the following FFME of $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$*

$$\begin{pmatrix} (-3,1,7) \\ (-2,4,10) \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \end{pmatrix} \otimes \begin{pmatrix} (9,2,12) & (2,1,3) \\ (6,3,13) & (12,2,7) \\ (11,4,8) & (9,4,9) \end{pmatrix} = \begin{pmatrix} (420,2376,1536) & (327,1787,1133) \\ (280,4192,2654) & (218,3138,1972) \end{pmatrix}$$

*where the coefficients $\tilde{A}$ and $\tilde{B}$ are near-zero and positive TFN respectively, while $\tilde{X}$ is a fuzzy solution.*

**The Solution:**

Step 1: Convert the FFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$ to FFLS $\tilde{S}\tilde{X} = \tilde{C}$

$$\tilde{B}^T \otimes_k \tilde{A} = \begin{pmatrix} (9,2,12) & (6,3,13) & (11,4,8) \\ (2,1,3) & (12,2,7) & (9,4,9) \end{pmatrix} \otimes_k \begin{pmatrix} (-3,1,7) \\ (-2,4,10) \end{pmatrix}$$

$$= \begin{pmatrix} (-27,57,111) & (-18,58,94) & (-33,43,109) \\ (-18,108,186) & (-12,102,164) & (-22,92,174) \\ (-6,14,26) & (-36,40,112) & (-27,45,99) \\ (-4,26,44) & (-24,90,176) & (-18,90,162) \end{pmatrix}$$

From that, the FFLS of $\tilde{S}\tilde{X} = \tilde{C}$ is

$$\begin{pmatrix} (-27,57,111) & (-18,58,94) & (-33,43,109) \\ (-18,108,186) & (-12,102,164) & (-22,92,174) \\ (-6,14,26) & (-36,40,112) & (-27,45,99) \\ (-4,26,44) & (-24,90,176) & (-18,90,162) \end{pmatrix} \begin{pmatrix} (m_{11}^{\tilde{X}}, \alpha_{11}^{\tilde{X}}, \beta_{11}^{\tilde{X}}) \\ (m_{12}^{\tilde{X}}, \alpha_{12}^{\tilde{X}}, \beta_{12}^{\tilde{X}}) \\ (m_{13}^{\tilde{X}}, \alpha_{13}^{\tilde{X}}, \beta_{13}^{\tilde{X}}) \end{pmatrix} = \begin{pmatrix} (420,2376,1536) \\ (280,4192,2654) \\ (327,1787,1133) \\ (218,3138,1972) \end{pmatrix}$$

**Step 2: Convert FFLS $\tilde{S}\tilde{X} = \tilde{C}$ to an Associated Linear System (ALS) $SX = C$.**

$$m^{\tilde{S}} = \begin{pmatrix} -27 & -18 & -33 \\ -18 & -12 & -22 \\ -6 & -36 & -27 \\ -4 & -24 & -18 \end{pmatrix}, \quad \alpha^{\tilde{S}} = \begin{pmatrix} 57 & 58 & 43 \\ 108 & 102 & 92 \\ 14 & 40 & 45 \\ 26 & 90 & 90 \end{pmatrix}, \quad \beta^{\tilde{S}} = \begin{pmatrix} 111 & 94 & 109 \\ 186 & 164 & 174 \\ 26 & 112 & 99 \\ 44 & 176 & 162 \end{pmatrix}$$

$$m^{\tilde{C}} = \begin{pmatrix} 420 \\ 280 \\ 327 \\ 218 \end{pmatrix}, \alpha^{\tilde{C}} = \begin{pmatrix} 2376 \\ 4192 \\ 1787 \\ 3138 \end{pmatrix}, \beta^{\tilde{C}} = \begin{pmatrix} 1536 \\ 2654 \\ 1133 \\ 1972 \end{pmatrix}$$

$$(m^{\tilde{S}} - \alpha^{\tilde{S}}) = \begin{pmatrix} -84 & -76 & -76 \\ -126 & -114 & -114 \\ -20 & -76 & -72 \\ -30 & -114 & -108 \end{pmatrix},$$

$$(m^{\tilde{S}} - \alpha^{S})^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; (m^{\tilde{S}} - \alpha^{\tilde{S}})^{-} = \begin{pmatrix} -84 & -76 & -76 \\ -126 & -114 & -114 \\ -20 & -76 & -72 \\ -30 & -114 & -108 \end{pmatrix}$$

$$(m^{\tilde{S}} + \beta^{\tilde{S}}) = \begin{pmatrix} 84 & 76 & 76 \\ 168 & 152 & 152 \\ 20 & 76 & 72 \\ 40 & 152 & 144 \end{pmatrix},$$

$$(m^{\tilde{S}} + \beta^{\tilde{S}})^{+} = \begin{pmatrix} 84 & 76 & 76 \\ 168 & 152 & 152 \\ 20 & 76 & 72 \\ 40 & 152 & 1446 \end{pmatrix}; (m^{\tilde{S}} + \beta^{\tilde{S}})^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$m^{\tilde{S}} = \begin{pmatrix} -27 & -18 & -33 \\ -18 & -12 & -22 \\ -6 & -36 & -27 \\ -4 & -24 & -18 \end{pmatrix}, \quad \alpha^{\tilde{S}} = \begin{pmatrix} 57 & 58 & 43 \\ 108 & 102 & 92 \\ 14 & 40 & 45 \\ 26 & 90 & 90 \end{pmatrix}, \quad \beta^{\tilde{S}} = \begin{pmatrix} 111 & 94 & 109 \\ 186 & 164 & 174 \\ 26 & 112 & 99 \\ 44 & 176 & 162 \end{pmatrix}$$

$$m^{\tilde{C}} = \begin{pmatrix} 420 \\ 280 \\ 327 \\ 218 \end{pmatrix}, \alpha^{\tilde{C}} = \begin{pmatrix} 2376 \\ 4192 \\ 1787 \\ 3138 \end{pmatrix}, \beta^{\tilde{C}} = \begin{pmatrix} 1536 \\ 2654 \\ 1133 \\ 1972 \end{pmatrix}$$

$$(m^{\tilde{S}} - \alpha^{\tilde{S}}) = \begin{pmatrix} -84 & -76 & -76 \\ -126 & -114 & -114 \\ -20 & -76 & -72 \\ -30 & -114 & -108 \end{pmatrix},$$

$$(m^{\tilde{S}} - \alpha^{S})^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; (m^{\tilde{S}} - \alpha^{\tilde{S}})^{-} = \begin{pmatrix} -84 & -76 & -76 \\ -126 & -114 & -114 \\ -20 & -76 & -72 \\ -30 & -114 & -108 \end{pmatrix}$$

$$(m^{\tilde{S}} + \beta^{\tilde{S}}) = \begin{pmatrix} 84 & 76 & 76 \\ 168 & 152 & 152 \\ 20 & 76 & 72 \\ 40 & 152 & 144 \end{pmatrix},$$

$$(m^{\tilde{S}} + \beta^{\tilde{S}})^{+} = \begin{pmatrix} 84 & 76 & 76 \\ 168 & 152 & 152 \\ 20 & 76 & 72 \\ 40 & 152 & 1446 \end{pmatrix}; (m^{\tilde{S}} + \beta^{\tilde{S}})^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{c|c|c} m^{\tilde{S}} & 0 & 0 \\ \hline -\beta^{\tilde{S}} & (m^{\tilde{S}} + \beta^{\tilde{S}})^{+} & -(m^{\tilde{S}} + \beta^{\tilde{S}})^{-} \\ \hline -\alpha^{\tilde{S}} & -(m^{\tilde{S}} - \alpha^{\tilde{S}})^{-} & (m^{\tilde{S}} - \alpha^{\tilde{S}})^{+} \end{array}\right) \left(\begin{array}{c} m^{\tilde{X}} \\ \hline \alpha^{\tilde{X}} \\ \hline \beta^{\tilde{X}} \end{array}\right) = \left(\begin{array}{c} m^{\tilde{C}} \\ \hline \alpha^{\tilde{C}} \\ \hline \beta^{\tilde{C}} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc|ccc} -27 & -18 & -33 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -12 & -22 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -36 & -27 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -24 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -111 & -94 & -109 & 84 & 76 & 76 & 0 & 0 & 0 \\ -186 & -164 & -174 & 168 & 152 & 152 & 0 & 0 & 0 \\ -26 & -112 & -99 & 20 & 76 & 72 & 0 & 0 & 0 \\ -44 & -176 & -162 & 40 & 152 & 144 & 0 & 0 & 0 \\ \hline -57 & -58 & -43 & 84 & 76 & 76 & 0 & 0 & 0 \\ -108 & -102 & -92 & 126 & 114 & 114 & 0 & 0 & 0 \\ -14 & -40 & -45 & 20 & 76 & 72 & 0 & 0 & 0 \\ -26 & -90 & -90 & 30 & 114 & 108 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} m^{\tilde{X}}_{1,1} \\ m^{\tilde{X}}_{1,2} \\ m^{\tilde{X}}_{1,3} \\ \hline \alpha^{\tilde{X}}_{1,1} \\ \alpha^{\tilde{X}}_{1,2} \\ \alpha^{\tilde{X}}_{1,3} \\ \hline \beta^{\tilde{X}}_{1,1} \\ \beta^{\tilde{X}}_{1,2} \\ \beta^{\tilde{X}}_{1,3} \end{array}\right) = \left(\begin{array}{c} 420 \\ 280 \\ 327 \\ 218 \\ \hline 2376 \\ 4192 \\ 1787 \\ 3138 \\ \hline 1536 \\ 2654 \\ 1133 \\ 1972 \end{array}\right)$$

## Step 3: Obtaining the solution.

$$\begin{pmatrix} m^{\tilde{X}}_{1,1} \\ m^{\tilde{X}}_{1,2} \\ m^{\tilde{X}}_{1,3} \\ \hline \alpha^{\tilde{X}}_{1,1} \\ \alpha^{\tilde{X}}_{1,2} \\ \alpha^{\tilde{X}}_{1,3} \\ \hline \beta^{\tilde{X}}_{1,1} \\ \beta^{\tilde{X}}_{1,2} \\ \beta^{\tilde{X}}_{1,3} \end{pmatrix} = \left(\begin{array}{ccc|ccc|ccc} -27 & -18 & -33 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -12 & -22 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -36 & -27 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -24 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -111 & -94 & -109 & 84 & 76 & 76 & 0 & 0 & 0 \\ -186 & -164 & -174 & 168 & 152 & 152 & 0 & 0 & 0 \\ -26 & -112 & -99 & 20 & 76 & 72 & 0 & 0 & 0 \\ -44 & -176 & -162 & 40 & 152 & 144 & 0 & 0 & 0 \\ \hline -57 & -58 & -43 & 84 & 76 & 76 & 0 & 0 & 0 \\ -108 & -102 & -92 & 126 & 114 & 114 & 0 & 0 & 0 \\ -14 & -40 & -45 & 20 & 76 & 72 & 0 & 0 & 0 \\ -26 & -90 & -90 & 30 & 114 & 108 & 0 & 0 & 0 \end{array}\right)^{\dagger} \begin{pmatrix} 420 \\ 280 \\ 327 \\ 218 \\ \hline 2376 \\ 4192 \\ 1787 \\ 3138 \\ \hline 1536 \\ 2654 \\ 1133 \\ 1972 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} m^{\tilde{X}}_{1,1} \\ m^{\tilde{X}}_{1,2} \\ m^{\tilde{X}}_{1,3} \end{pmatrix} \\ \begin{pmatrix} \alpha^{\tilde{X}}_{1,1} \\ \alpha^{\tilde{X}}_{1,2} \\ \alpha^{\tilde{X}}_{1,3} \end{pmatrix} \\ \begin{pmatrix} \beta^{\tilde{X}}_{1,1} \\ \beta^{\tilde{X}}_{1,2} \\ \beta^{\tilde{X}}_{1,3} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -5.41699 \\ -3.31542 \\ -6.48678 \end{pmatrix} \\ \begin{pmatrix} 1.68041 \\ 4.13539 \\ 3.95475 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$\begin{aligned} \tilde{X} &= \left( (m^{\tilde{X}}_{1,1}, \alpha^{\tilde{X}}_{1,1}, \beta^{\tilde{X}}_{1,1}) \quad (m^{\tilde{X}}_{1,2}, \alpha^{\tilde{X}}_{1,2}, \beta^{\tilde{X}}_{1,2}) \quad (m^{\tilde{X}}_{1,3}, \alpha^{\tilde{X}}_{1,3}, \beta^{\tilde{X}}_{1,3}) \right) \\ &= \left( (-5.41699, 1.68041, 0) \quad (-3.31542, 4.13539, 0) \quad (-6.48678, 3.95475, 0) \right) \end{aligned}$$

# Step 4: Verification of the solution.

$$\tilde{A}\tilde{X} = \begin{pmatrix} (-3,1,7) \\ (-2,4,10) \end{pmatrix} \otimes ((-5.41699, 1.68041, 0)\ (-3.31542, 4.13539, 0)\ (-6.48678, 3.95475, 0))$$
$$= \begin{pmatrix} (16.251, 44.6406, 12.1386) & (9.94626, 39.7495, 19.857) & (19.4603, 61.2265, 22.3058) \\ (10.834, 67.6132, 31.7504) & (6.63084, 66.2373, 38.074) & (12.9736, 96.5058, 49.6756) \end{pmatrix}.$$

$$\tilde{A}\tilde{X}\tilde{B} = \begin{pmatrix} (16.251, 44.6406, 12.1386) & (9.94626, 39.7495, 19.857) & (19.4603, 61.2265, 22.3058) \\ (10.834, 67.6132, 31.7504) & (6.63084, 66.2373, 38.074) & (12.9736, 96.5058, 49.6756) \end{pmatrix}$$
$$\otimes \begin{pmatrix} (9,2,12) & (2,1,3) \\ (6,3,13) & (12,2,7) \\ (11,4,8) & (9,4,9) \end{pmatrix}$$
$$= \begin{pmatrix} (420, 2376, 1536) & (327, 1787, 1133) \\ (280, 4192, 2654) & (218, 3138, 1972) \end{pmatrix}$$
$$= \tilde{C}.$$

Thus, the solution is verified.

# Conclusion

This study contributes to a simple and direct method for solving the arbitrary FFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$.

The contribution should be beneficial to researchers from diverse fields, such as linear algebra, fuzzy theory, as well as social sciences.

The contributions also should be applicable for real-life applications, particularly in the field of control system engineering.

Suggestion for future research-

Considering other type of linear and non-linear matrix equations, such as $AX + XA^T = C$,

$AXA^T - X = C$ and $AXB + CXD = E$, that are also crucial in the real control system applications such as in medical imaging acquisition system, image restoration, model reduction, signal processing and stochastic control.

# THANK YOU