THE FIXED POINTS OF *b*-BISTOCHASTIC-VOLTERRA QUADRATIC STOCHASTIC OPERATORS ON $S^1 \times S^1$

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CONTENTS

Introduction

- The simplest non-linear Markov Operator is a Quadratic Stochastic Operators (QSO) and yet it is not fully studied.
- QSO was first initiated by Bernstein which arises from some problems of population genetics.

Let $I = \{1,2,\dots,n\}$ be the n type of species in a population.

Given $\mathbf{x}^{(0)} = \left(x_1^{(0)},\dots,x_n^{(0)}\right)$, then $\mathbf{x}^{(1)} = \left(x_1^{(1)},\dots,x_n^{(1)}\right)$ can be obtain by applying the QSO

$$V(\mathbf{x})_k = x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}$$

Introduction

- Since the study of QSO is tricky in general setting, there are many researchers introduced various classes of QSO such as strictly non-Volterra QSO, Centered QSO, Bistochastic QSO, *b*-bistochastic QSO, F-QSO, separable QSOs and etc.
- Yet, all these classes do not cover the set of whole QSOs.
- This paper introduces a new class of QSO named *b*-bistochastic-Volterra QSO (bV-QSO) defined on one-dimensional simplex $S^1 \times S^1 \to S^1 \times S^1$

Objectives

- To develop the **form and conditions of bV-QSO** defined on one dimensional simplex.

- To list all of the **fixed points** of bV-QSOs and their stability defined on one dimensional simplex.

- To study the trajectories of bV-QSO defined on one dimensional simplex

Literature Review

Type of QSO	Author (year)	Findings
b-Bistochastic QSO	Ghanikhodzaev, R. N. (1993)	Bistochastic QSO is defined in terms of classical majorization where $V(\mathbf{x}) \prec \mathbf{x},\ \forall \mathbf{x} \in S^{(n-1)}$
	Parker, D. and Ram, P. (1996)	A new order called majorization was introduced to generalized the classical majorization under linear system. This majorization can also be defined as partial order on sequence.
	Mukhamedov, F. and Embong, A.F. (2015)	A new class named b-bistochastic QSO was introduced in terms of new majorization or simply, b-order.
Volterra QSO	Ghanikhodzaev, R. N., et al. (1993)	All necessary definitions and a brief descriptions of the results in discrete-time dynamical systems generated by Volterra QSO was presented in this paper.
	Rozikov, U. and Zhamilov, U. (2011)	Described all the fixed points of Volterra QSO of a two-sex population and constructed Lyapunov functions in order to investigate the dynamics.

Literature Review

Definition (b-bistochastic QSO)

Let V be a QSO defined on S^{n-1}, then V is called a b-bistochastic if

$$V(\mathbf{x}) \leq^{b} \mathbf{x}, \quad \forall \mathbf{x} \in S^{n-1},$$

where S is simplex and n is natural number.

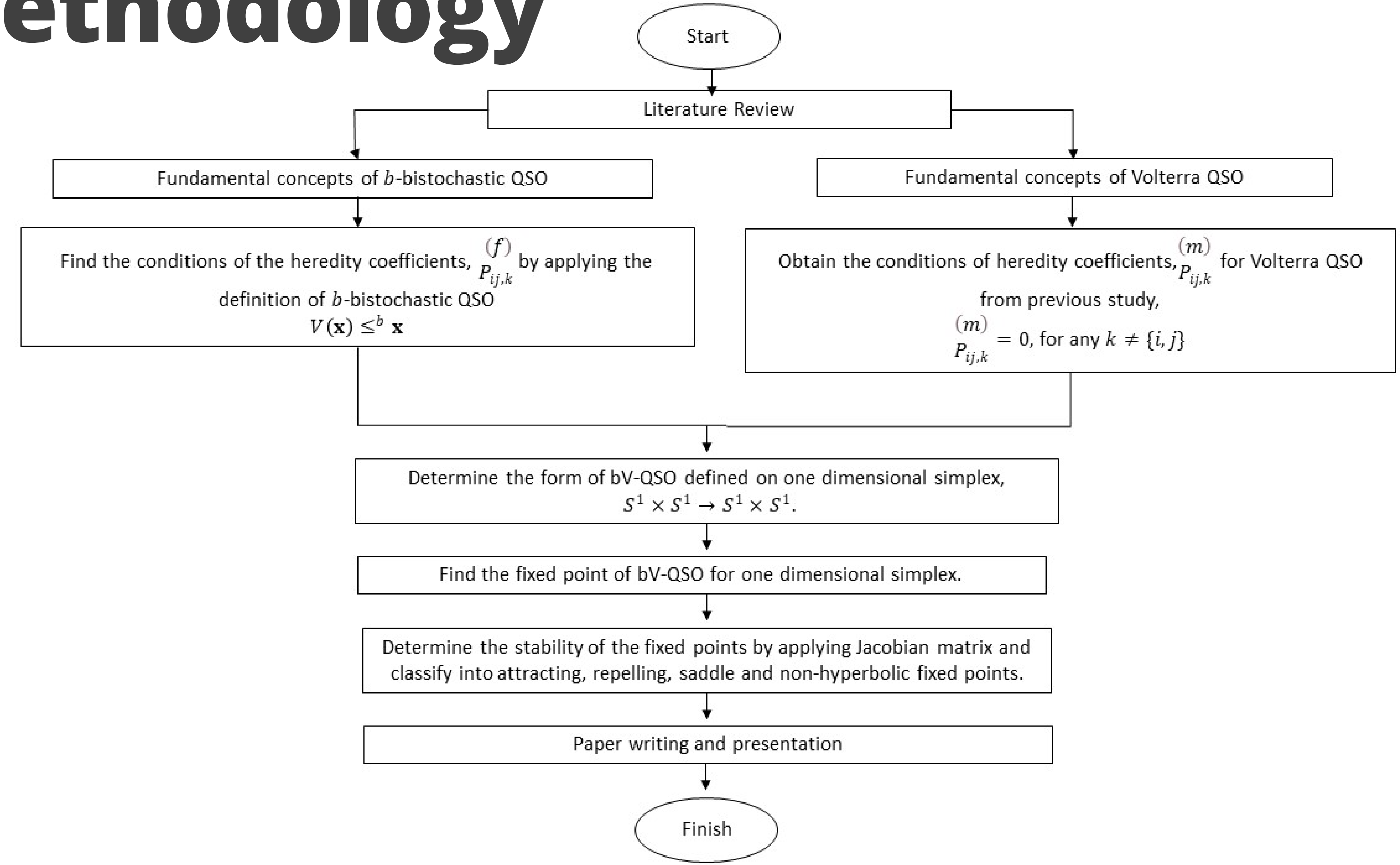
Properties of Volterra QSO

Given V defined on $S^{(n-1)} \times S^{(v-1)} \to S^{(n-1)} \times S^{(v-1)}$ is QSO.
Then V is called Volterra QSO if

$$P^{(f)}_{ij,k} = 0 \quad \text{for} \quad k \notin \{i,j\}, \tag{2}$$

$$P^{(m)}_{ij,l} = 0 \quad \text{for} \quad l \notin \{i,j\}. \tag{3}$$

Methodology



Methodology

Definition (bV-QSO)

A QSO V is called bV-QSO if

$$V_{\mathbf{x}} \leq^{b} \mathbf{x},$$

and the heredity coefficients for $V_{\mathbf{y}}$ satisfy $P_{ij,k}^{(m)} = 0$ for any $k \notin \{i, j\}$.

Results and Discussions

The simplex that is applied is as follows

$$S^1 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0, x_1 + x_2 = 1\}.$$

Theorem (Description of bV-QSO on 1D simplex)

V is a bV-QSO defined on $S^1 \times S^1$, where $\mathbf{x}, \mathbf{y} \in S^1$ if

$$V(\mathbf{x}, \mathbf{y}) = \begin{cases} \mathbf{x}' = axy \\ \mathbf{y}' = xy + b(x - 2xy + y), \end{cases}$$

where $a = P^{(f)}_{11,1}$ and $b = P^{(m)}_{12,1}$.

Results and Discussions

Proof

$$V(\mathbf{x},\mathbf{y}) = \begin{cases} x_1' = P^{(f)}_{11,1}x_1y_1 + P^{(f)}_{12,1}x_1y_2 + P^{(f)}_{21,1}x_2y_1 + P^{(f)}_{22,1}x_2y_2 \\ x_2' = P^{(f)}_{11,2}x_1y_1 + P^{(f)}_{12,2}x_1y_2 + P^{(f)}_{21,2}x_2y_1 + P^{(f)}_{22,2}x_2y_2 \\ y_1' = P^{(m)}_{11,1}x_1y_1 + P^{(m)}_{12,1}x_1y_2 + P^{(m)}_{21,1}x_2y_1 + P^{(m)}_{22,1}x_2y_2 \\ y_2' = P^{(m)}_{11,2}x_1y_1 + P^{(m)}_{12,2}x_1y_2 + P^{(m)}_{21,2}x_2y_1 + P^{(m)}_{22,2}x_2y_2. \end{cases}$$

$$V(\mathbf{x},\mathbf{y}) = \begin{cases} x' = P^{(f)}_{11,1}xy + P^{(f)}_{12,1}(x-2xy+y) + P^{(f)}_{22,1}(1-y-x+xy) \\ y' = P^{(m)}_{11,1}xy + P^{(m)}_{12,1}(x-2xy+y) + P^{(m)}_{22,1}(1-y-x+xy). \end{cases}$$

Results and Discussions

From the definition of bV-QSO:

$$xy(P^{(f)}_{11,1}-2P^{(f)}_{12,1}+P^{(f)}_{22,1})+x(P^{(f)}_{12,1}-P^{(f)}_{22,1}-1)+y(P^{(f)}_{12,1}-P^{(f)}_{22,1})+P^{(f)}_{22,1}\le 0.$$

x	y	Equation	$P^{(f)}_{ij,k}$
$x=0$	$y=0$	$P^{(f)}_{22,1}\le 0$	$P^{(f)}_{22,1}=0.$
$x=0$	$y=1$	$P^{(f)}_{12,1}-P^{(f)}_{22,1}+P^{(f)}_{22,1}\le 0$	$P^{(f)}_{12,1}=0.$
$x=1$	$y=1$	$P^{(f)}_{11,1}-1\le 0$	$P^{(f)}_{11,1}\in[0,1].$

$$P^{(m)}_{11,2}=P^{(m)}_{22,1}=0.$$

Results and Discussions

Theorem (Fixed Point)

Let V be bV-QSO defined on $S^1 \times S^1$. Then,

1. $(0,0)$ *is always the fixed point.*
2. *If* $a < 1$ *and* $b = 1$, $(0, y)$ *is the fixed point for any* $y \in (0, 1]$.
3. *If* $b < 1$ *and* $a = 1$, *then* $(1, 1)$ *is the fixed points.*
4. *If* $a = 1$ *and* $b = 1$, *then* $(0, y)$ *and* $(x, 1)$ *are the fixed points.*

$(0,0)$

1. For $b < 1$, $\lambda_1 < 1, \lambda_2 < 1$ which implies attracting fixed point.
2. For $b = 1$, $\lambda_1 < 1, \lambda_2 = 1$ which implies non-hyperbolic.

Results and Discussions

$(0, y), b = 1$, for $y \in [0, 1]$

1. For $ay < 1$, $\lambda_1 < 1, \lambda_2 = 1$ which implies non-hyperbolic.
2. For $ay = 1$, $\lambda_1 = 1, \lambda_2 = 1$ which implies non-hyperbolic.

$(1, 1), a = 1$

1. For $b < 1$, $\lambda_1 < 1, \lambda_2 > 1$ which implies saddle fixed point.
2. For $b = 1$, $\lambda_1 < 1, \lambda_2 = 1$ which implies non-hyperbolic.

$(x, 1), a = 1, b = 1$

1. $\lambda_1 < 1, \lambda_2 = 1$ which implies non-hyperbolic.

Results and Discussions

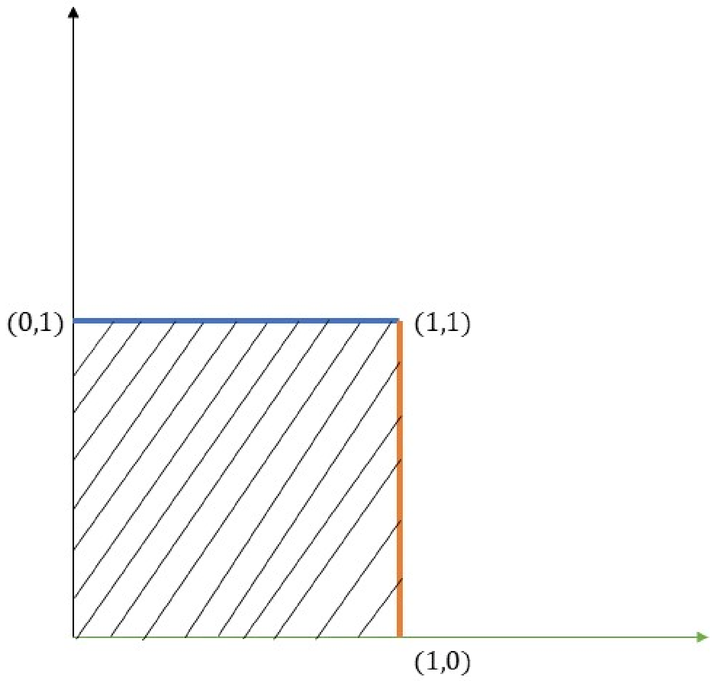
To study the trajectories, we consider the following figure which includes the internal region and boundaries which include 4 sides:

Side 1: $x_1 = 0$

Side 2: $y_1 = 0$

Side 3: $x_2 = 1$

Side 4: $y_2 = 1$



Results and Discussions

Theorem

Let V be a bV-QSO. Define a functional $\varphi : S^1 \times S^1 \to \mathbb{R}$ as follows:

$$\varphi(x, y) = x + y.$$

Then φ is a Lyapunov function for $V(x, y)$ if $a + b \leq 1$.

Theorem

If $a + b < 1$, then

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \lim_{n\to\infty} \mathbf{y}^{(n)} = 0.$$

Results and Discussions

Theorem

Let V be a bV-QSO on $S^1 \times S^1$. Any initial point (x, y) taken from the sides, the limit of $\lim_{n\to\infty} V^{(n)}(x, y)$ is converged. Moreover, the trajectories are described as follows:

- *Side 1.* $\lim_{n\to\infty} V^{(n)}(0, y) = \begin{cases} (0,0), & \text{if } b < 1, \\ (0,y), & \text{if } b = 1. \end{cases}$
- *Side 2.* $\lim_{n\to\infty} V^{(n)}(x, 0) = \begin{cases} (0,0), & \text{if } b < 1, \\ (0,x), & \text{if } b = 1. \end{cases}$
- *Side 3.* $\lim_{n\to\infty} V^{(n)}(1, y) = \begin{cases} (0,0), & \text{if } a \le 1,\ b < 1, \\ (0,1), & \text{if } a < 1,\ b = 1, \\ (y,1), & \text{if } a = 1,\ b = 1. \end{cases}$
- *Side 4.* $\lim_{n\to\infty} V^{(n)}(x, 1) = \begin{cases} (0,0), & \text{if } a \le 1,\ b < 1, \\ (0,1), & \text{if } a < 1,\ b = 1, \\ (x,1), & \text{if } a = 1,\ b = 1. \end{cases}$

Conclusions

- This paper introduces a **new class of QSO** namely *b*-bistochastic-Volterra QSO (**bV-QSO**) defined on one-dimensional simplex.
- All **fixed points** and their **stability** are presented in this paper.
- Since the operator converge for any initial point, therefore we concluded that bV-QSO defined on one dimensional simplex is a regular operator.
- The study of bV-QSO on two-dimensional simplex has been done which will be published in another work.
- The study of bV-QSO on higher dimensional simplex remain as open problem

THANK YOU