Curvature Comparison of Bézier Curve, Ball Curve and Trigonometric Curve in Preserving the Positivity of Real Data

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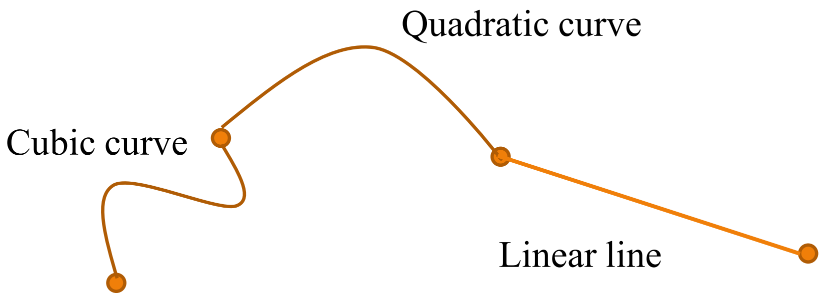
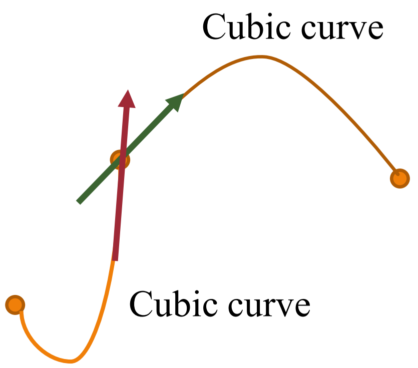
CONTENTS

INTRODUCTION

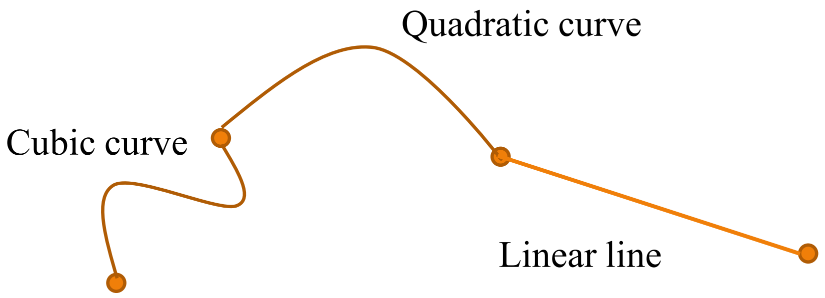
- It is important for us to be able to interpolate set of data points and preserve the properties of the data.
- Conditions were imposed to preserve the positivity of the data. However, it is not easy to decide which one is better than the other. Hence, in this paper, we proposed the use of curvature as a tool to analyse the fairness of curve.
- In this paper, the positive data will be interpolated using three different types of curves. The interpolated curves later will be compared using the curvature plot.
- Hence, curve interpolation schemes using three curves which are rational cubic Bézier, rational cubic Ball and non-rational cubic trigonometric splines will be proposed. Then, the curvature results for all three curves will be presented and discussed.

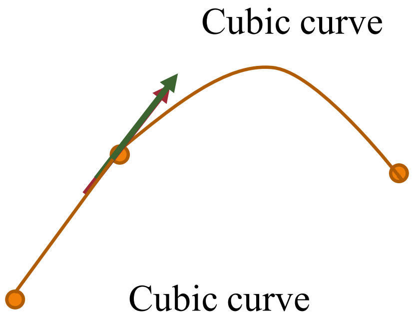
INTRODUCTION





INTRODUCTION





OBJECTIVES

- To present the sufficient conditions imposed to preserve positivity of the data using all three curves.
- To compare the curvature for all three splines to determine which curve is the best curve.

LITERATURE REVIEW

- Recently, there are lots of research that use the curvature to determine the smoothness of curves and surfaces such as (Othman et al., 2021) and (Safaruddin and Misro, 2021).
- Han (2015), Saaban et al. (2016), Zhu (2018) and Karim (2018) proposed a rational Bézier spline to interpolate positive data.
- Tahat et al. (2015), Karim (2016), (Karim and Saaban, 2017) and Karim and Nguyen (2021) proposed rational Ball spline to preserve positivity of data.
- Dube and Rana (2014), Sarfraz et al. (2015), Liu et al. (2015), and Munir et al. (2019) used trigonometric spline to preserve positivity of the data.

METHODOLOGY

Rational Cubic Bézier Curve

For $x \in [x_i, x_{i+1}]$ where $i = 1, 2, \ldots, n-1$, a local variable is introduced where,

$$\theta = \frac{(x - x_i)}{h_i}, \quad \text{i.e. } 0 \le \theta \le 1 .$$

Let $\{(x_i, f_i), i = 1, \ldots, n\}$ be a given set of data points where $x_1 < x_2 < \ldots < x_n$ and $f_1, f_2, \ldots, f_n$ are real numbers. Let

$$h_i = x_{i+1} - x_i, \quad \Delta_i = \frac{(f_{i+1} - f_i)}{h_i}, \quad i = 1, \ldots, n-1 .$$

A rational cubic/cubic (referring to a cubic numerator/cubic denominator) Bézier curve $Be(x)$ on the interval of $[x_i\ ,\ x_{i+1}]$ where $i = 1, 2, \ldots, n-1$ is defined as

$$Be(x) = Be(x_i + h_i\theta) \equiv Be_i(\theta) = \frac{Pe_i(\theta)}{Qe_i(\theta)}, \tag{1}$$

where

$$Pe_i(\theta) = \alpha_i L_i (1-\theta)^3 + c_i M_i 3\theta(1-\theta)^2 + g_i N_i 3\theta^2(1-\theta) + \beta_i R_i \theta^3,$$
$$Qe_i(\theta) = \alpha_i (1-\theta)^3 + c_i 3\theta(1-\theta)^2 + g_i 3\theta^2(1-\theta) + \beta_i \theta^3.$$

α_i, β_i, c_i and g_i are the weights. L_i, M_i, N_i and R_i are the control points.

METHODOLOGY

To make the function achieve C^1 continuity (segments share the same first derivative at joint), the following interpolating properties are used:

$$\begin{aligned} Be(x_i) &= f_i, \quad Be(x_{i+1}) = f_{i+1}\,, \\ Be'(x_i) &= d_i, \quad Be'(x_{i+1}) = d_{i+1}\,, \end{aligned} \tag{2}$$

$Be'(x_i)$ is the derivative with respect to x and d_i are the derivative values estimated on given knots. The formulae for the estimated values of the derivative parameters are:

$$\begin{aligned} d_1 &= \max\left\{\frac{(2h_1+h_2)\Delta_1 - h_1\Delta_2}{h_1+h_2}, 0\right\}, \\ d_i &= \frac{h_i\Delta_{i-1} + h_{i-1}\Delta_i}{h_i + h_{i-1}}, \quad i = 2, \ldots, n-1\,, \\ d_n &= \max\left\{\frac{(2h_{n-1}+h_{n-2})\Delta_{n-1} - h_{n-1}\Delta_{n-2}}{h_{n-1}+h_{n-2}}, 0\right\}. \end{aligned} \tag{3}$$

For a positive data where $Be_i(x) > 0$ for $i = 1, 2, \ldots, n$ in the form of (1), the interpolant produced using Bézier function will preserve positivity if the values for the shape parameters are:

$$\alpha_i, \beta_i > 0, \qquad c_i > max\left\{0, -\frac{\alpha_i h_i d_i}{3f_i}\right\}, \quad g_i > max\left\{0, \frac{\beta_i h_i d_{i+1}}{3f_{i+1}}\right\}. \tag{4}$$

METHODOLOGY

Rational Cubic Ball Curve

In this paper, we are looking at a piecewise rational cubic Ball function $B(x)$ in cubic/cubic form which is defined as

$$B(x) = B(x_i + h_i\theta) \equiv B_i(\theta) = \frac{p_i(\theta)}{q_i(\theta)}, \tag{5}$$

where,

$$\begin{aligned} p_i(\theta) &= u_iU_i(1-\theta)^2 + 2v_iV_i\theta(1-\theta)^2 + 2w_iW_i\theta^2(1-\theta) + z_iZ_i\theta^2, \\ q_i(\theta) &= u_i(1-\theta)^2 + 2a_i\theta(1-\theta)^2 + 2b_i\theta^2(1-\theta) + z_i\theta^2. \end{aligned}$$

U_i, V_i, W_i and Z_i are the control points and a_i, b_i, u_i, v_i, w_i and z_i are the weights or the shape parameters. The weights are taken to be non-negative since negative weight may cause singularities

METHODOLOGY

Rational Cubic Ball Curve

To make the function achieve C^1 continuity (segments share the same first derivative at joint), the following interpolating properties are used:

$$\begin{aligned} B(x_i) &= f_i, \quad B(x_{i+1}) = f_{i+1}\,, \\ B'(x_i) &= d_i, \quad B'(x_{i+1}) = d_{i+1}\,, \end{aligned} \tag{6}$$

$B'(x_i)$ is the derivative with respect to x and d_i are the derivative values estimated on given knots. The formulae for the estimated values of the derivative parameters are as in (3).

For a positive data where $B_i(x) > 0$ for $i = 1, 2, ..., n$, the interpolant produced using Ball function will preserve positivity if the values for the shape parameters a_i and b_i are:

$$u_i, z_i > 0, \qquad a_i > max\left\{0, -\frac{u_i}{2} - \frac{u_i h_i d_i}{2f_i}\right\}, \quad b_i > max\left\{0, -\frac{z_i}{2} + \frac{z_i h_i d_{i+1}}{2f_{i+1}}\right\}. \tag{7}$$

METHODOLOGY

Cubic Trigonometric Bézier Curve

Abbas et al. (2011) considered the data $\{(x_i, f_i) : i = 0, 1, 2, ..., n\}$ over the interval $[a, b]$ with the knots $a = x_0 < x_i < x_2 < ... < x_n = b$. The cubic trigonometric Bézier function over each subinterval $I_i = [x_i, x_{i+1}]$, $i = 0, 1, 2, ..., n-1$ is defined as:

$$S_i(x) = \sum_{i=0}^{3} \omega_i P_i, \tag{8}$$

where

$$\begin{aligned}
\omega_0 &= \left[1 - sin\left(\frac{\pi\theta}{2}\right)\right]^2 \left[1 - \gamma sin\left(\frac{\pi\theta}{2}\right)\right] \\
\omega_1 &= sin\left(\frac{\pi\theta}{2}\right)\left[1 - sin\left(\frac{\pi\theta}{2}\right)\right]\left[2 + \gamma - \gamma sin\left(\frac{\pi\theta}{2}\right)\right] \\
\omega_2 &= cos\left(\frac{\pi\theta}{2}\right)\left[1 - cos\left(\frac{\pi\theta}{2}\right)\right]\left[2 + \sigma - \sigma cos\left(\frac{\pi\theta}{2}\right)\right] \\
\omega_3 &= \left[1 - cos\left(\frac{\pi\theta}{2}\right)\right]^2 \left[1 - \sigma cos\left(\frac{\pi\theta}{2}\right)\right]
\end{aligned} \tag{9}$$

$$\begin{aligned}
P_0 &= f_i \\
P_1 &= \frac{\pi f_i(2 + \gamma_i) + 2h_i d_i}{\pi(2 + \gamma_i)} \\
P_2 &= \frac{\pi f_{i+1}(2 + \sigma_i) - 2h_i d_{i+1}}{\pi(2 + \sigma_i)} \\
P_3 &= f_{i+1} \\
\theta &= \frac{x - x_i}{h_i}, h_i = x_{i+1} - x_i, i = 0, 1, 2, ..., n-1
\end{aligned} \tag{10}$$

$\omega_0, \omega_1, \omega_2$ and ω_3 are the trigonometric cubic basis functions, γ and σ are the shape parameters and P_0, P_1, P_2 and P_3 are the control points.

METHODOLOGY

Cubic Trigonometric Bézier Curve

To make the function achieve C^1 continuity (segments share the same first derivative at joint), the following interpolating properties are used:

$$\begin{aligned} S(x_i) &= f_i, \quad S(x_{i+1}) = f_{i+1}\,, \\ S'(x_i) &= d_i, \quad S'(x_{i+1}) = d_{i+1}\,, \end{aligned} \tag{11}$$

where $S'(x_i)$ is the derivative with respect to x, and d_i are the derivatives.

Based on Abbas et al. (2011), for a positive data set $\{(t_i, f_i) : i = 0, 1, 2, \ldots, n\}$, the author presented a piecewise cubic trigonometric spline that preserves the positivity of the interpolant that satisfied the following conditions:

$$\begin{aligned} \gamma_i &> \frac{-2h_i d_i}{\pi f_i} & \sigma_i &> \frac{2h_i d_{i+1}}{\pi f_{i+1}} \\ \gamma_i &> max\left\{0, \frac{-2h_i d_i}{\pi f_i}\right\} & \sigma_i &> max\left\{0, \frac{2h_i d_{i+1}}{\pi f_{i+1}}\right\} \end{aligned} \tag{12}$$

METHODOLOGY

Curvature

Let $z(x) = (z_x(t), z_f(t))$ be two-dimensional parametric curve. $z'(x)$ and $z''(x)$ is the first and second order derivative of $z(x)$ respectively. The curvature equation of a curve is defined as follows:

$$\kappa(x) = \frac{|z'(x) \times z''(x)|}{|z'(x)|^3}. \tag{6}$$

Curvature plot is one of the ways to help determine the smooth shape (Miller, 2009). Based on the curvature plot, the inflection point can be found by detecting the value of curvature starts to change its sign.

RESULTS AND DISCUSSIONS

Table 1 shows a positive W-shaped data from Sarfraz (2007); consists of values of conductance which can never be negative.

Table 1: A positive data set of conductance values from Sarfraz (2007).

i	**1**	**2**	**3**	**4**	**5**	**6**	**7**
x_i	2	3	7	8	9	13	14
f_i	10	2	3	7	2	3	10

RESULTS AND DISCUSSIONS

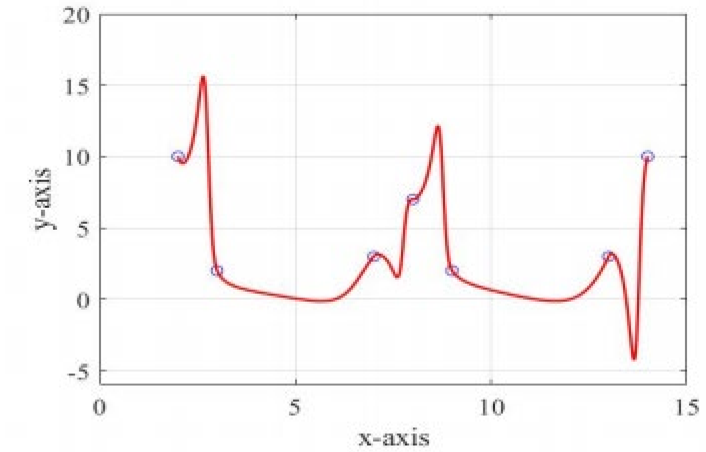
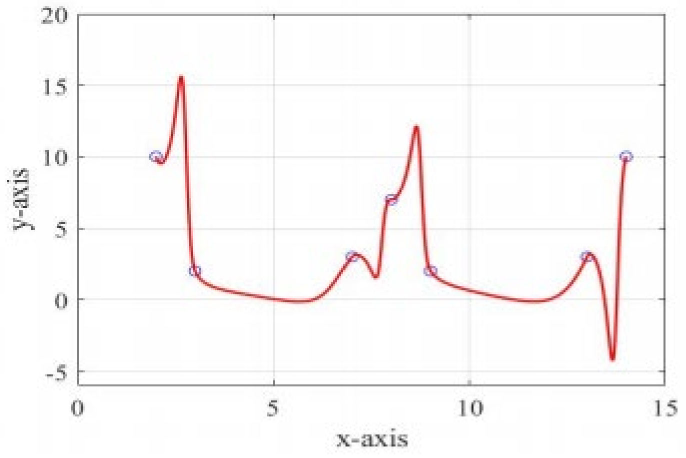


Figure 1: Cubic Bézier Curve with $\alpha_i = \beta_i = -1$

RESULTS AND DISCUSSIONS



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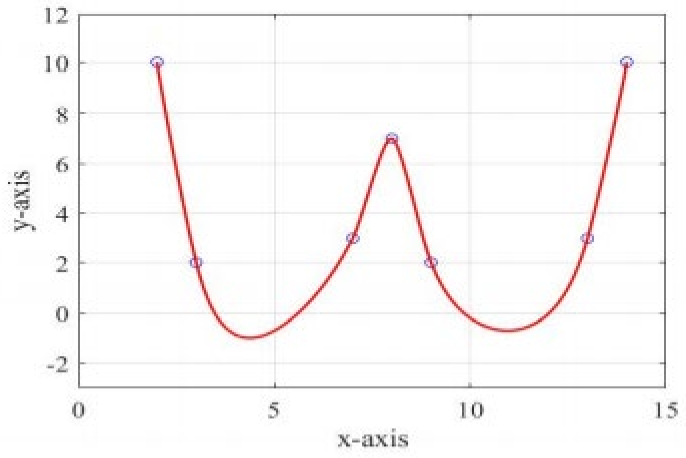
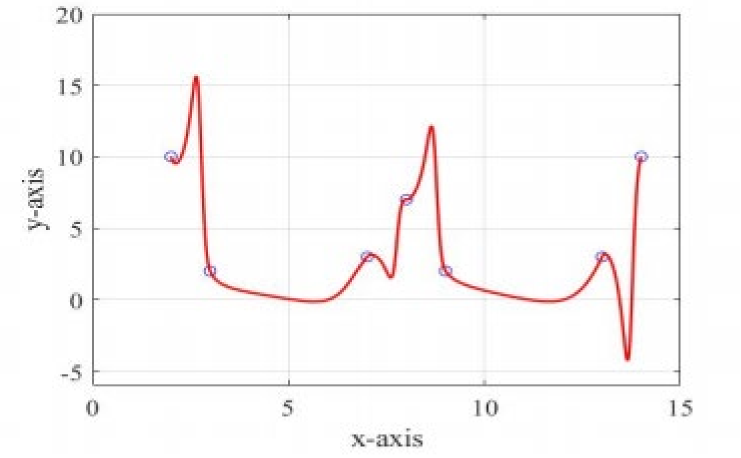


Figure 2: Cubic Bézier Curve with $\alpha_i = \beta_i = 0.05$

RESULTS AND DISCUSSIONS



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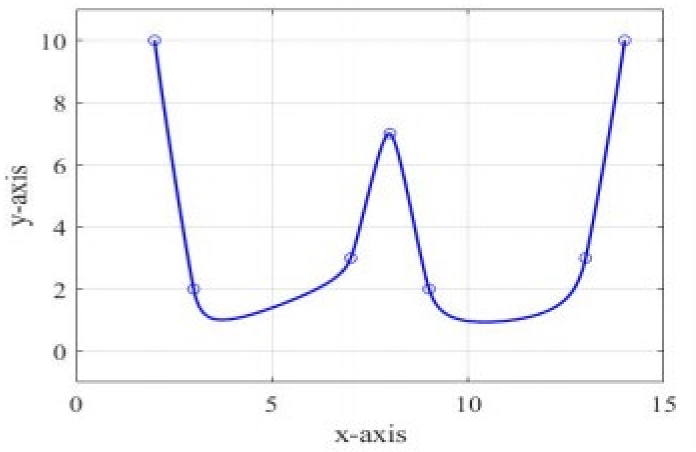
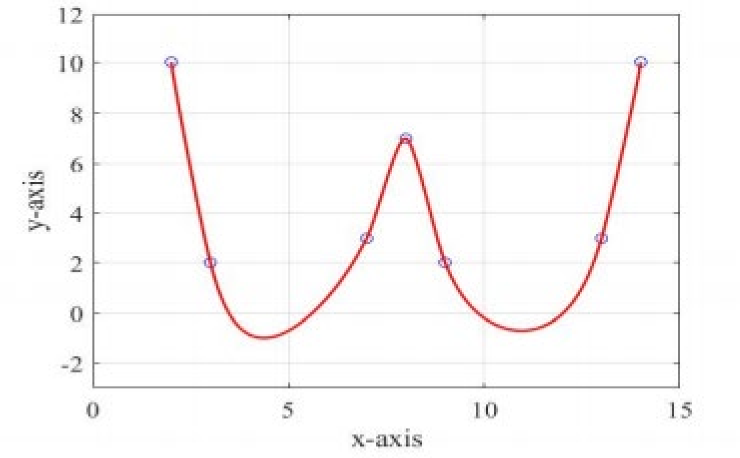


Figure 3: Cubic Bézier Curve with $\alpha_i = \beta_i = 10$



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RESULTS AND DISCUSSIONS

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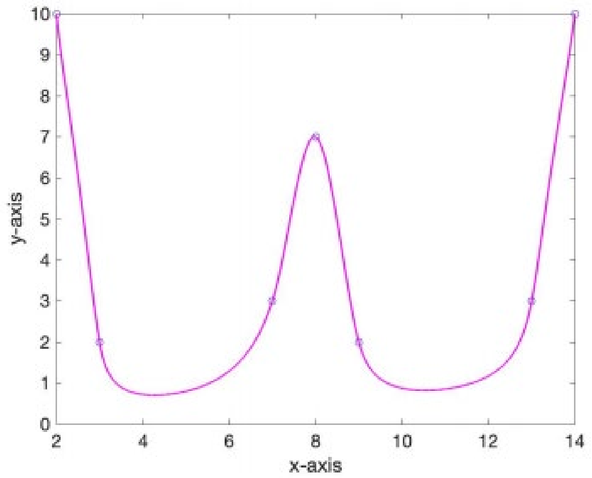
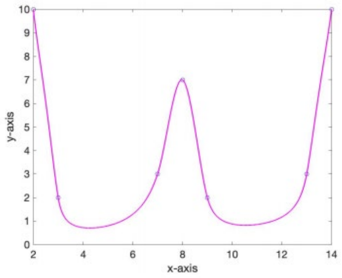


Figure 4: Cubic Bézier Curve with shape parameter equals to 0.1

RESULTS AND DISCUSSIONS



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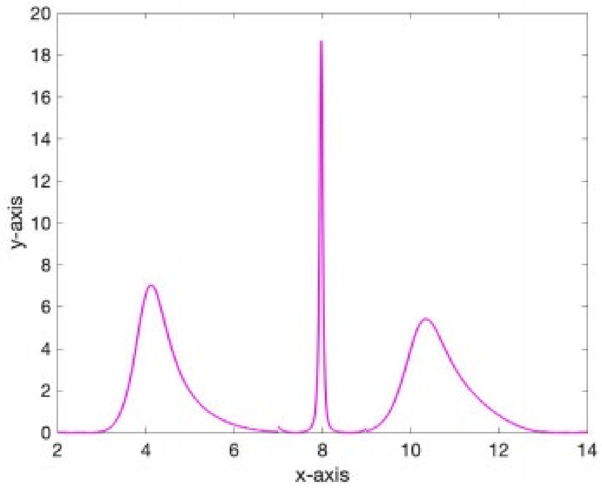
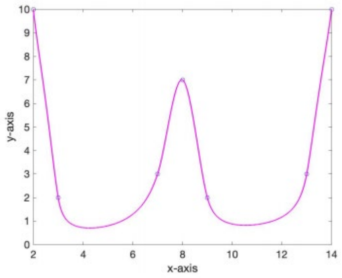
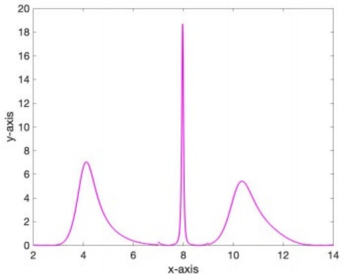


Figure 5: The curvature plot for Cubic Bézier Curve with shape parameter equals to 0.1

RESULTS AND DISCUSSIONS





Cubic Bézier Curve (left)
Curvature profile (right)

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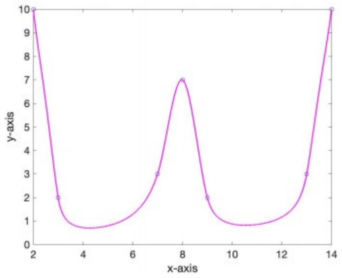

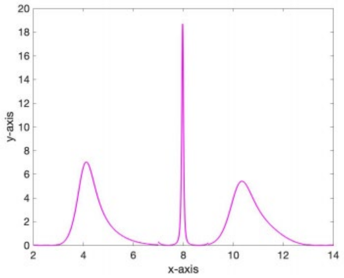

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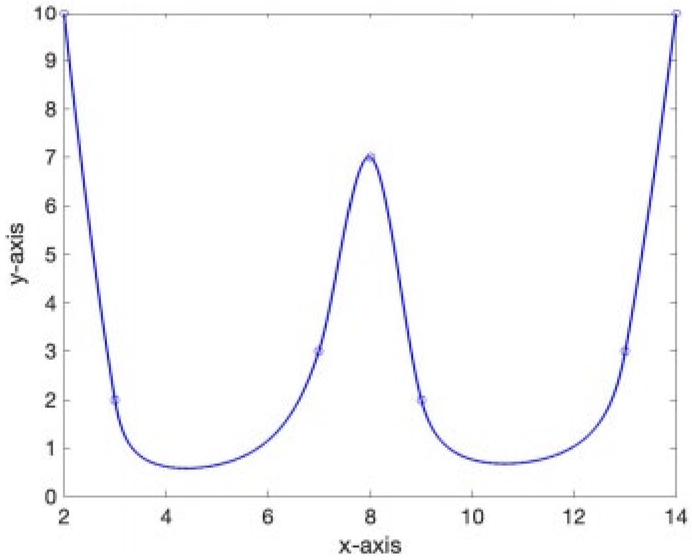
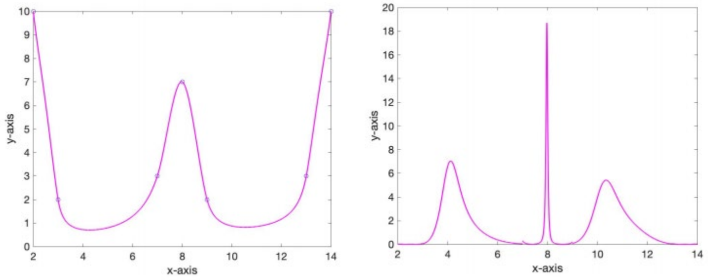
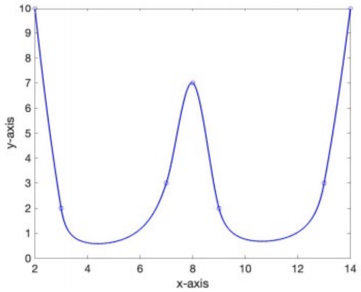


Figure 6: Cubic Ball Curve with shape parameter equals to 0.1

RESULTS AND DISCUSSIONS



Cubic Bézier Curve (left)
Curvature profile (right)



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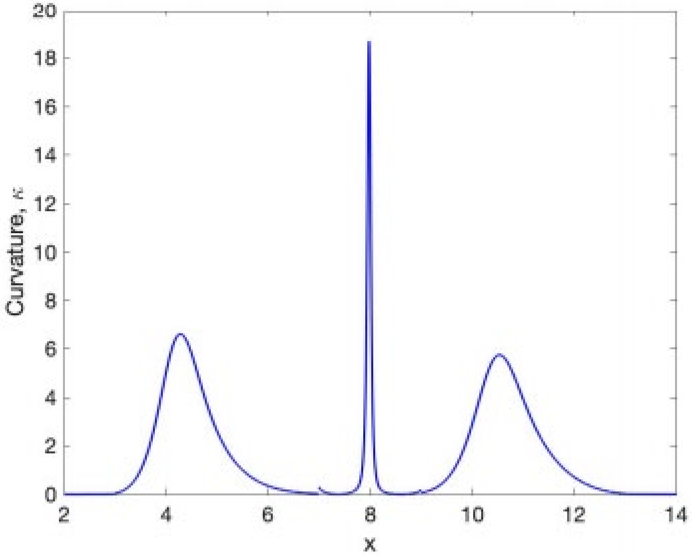
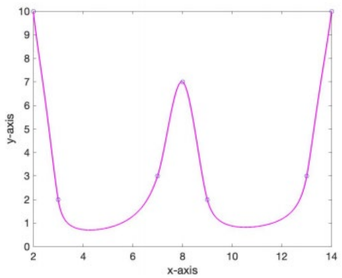
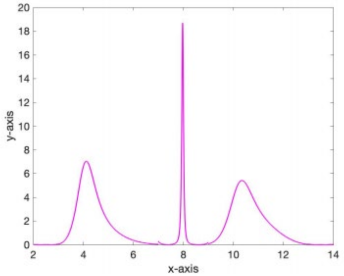


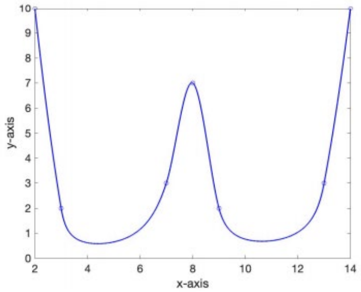
Figure 7: The curvature plot for Cubic Ball Curve with shape parameter equals to 0.1

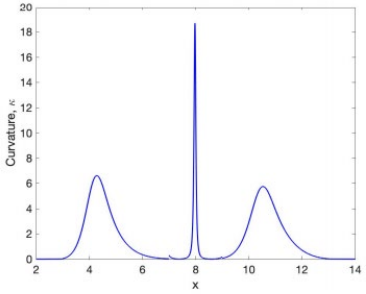
RESULTS AND DISCUSSIONS





Cubic Bézier Curve (left)
Curvature profile (right)





Cubic Ball Curve (left)
Curvature profile (right)

Table 1: A positive data set of conductance values from Sarfraz (2007)

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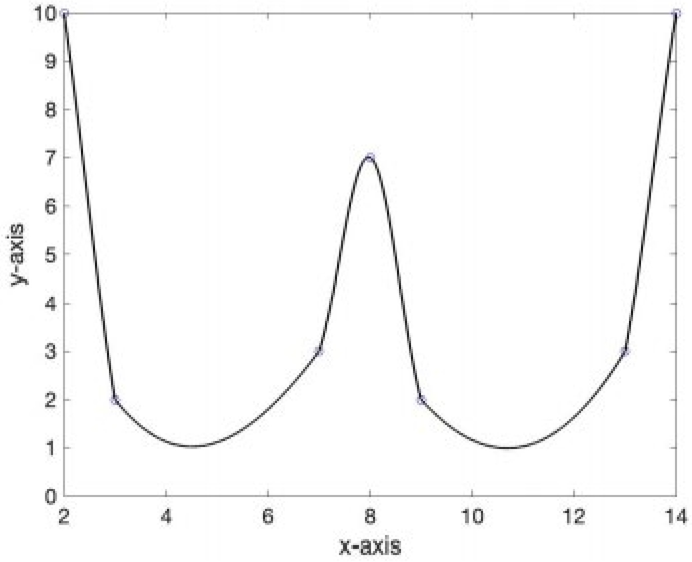
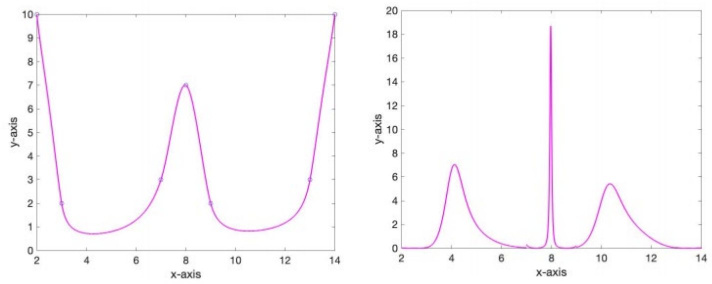
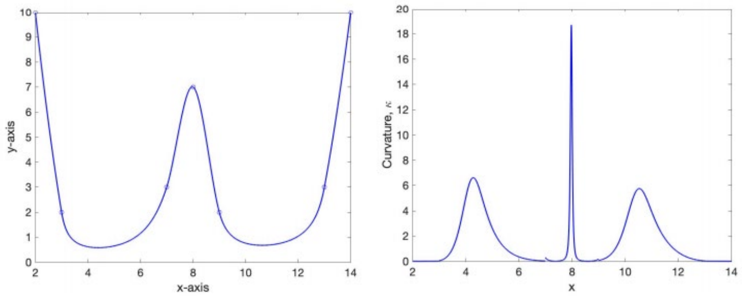


Figure 8: Trigonometric Bézier Curve with shape parameter equals to 0.1

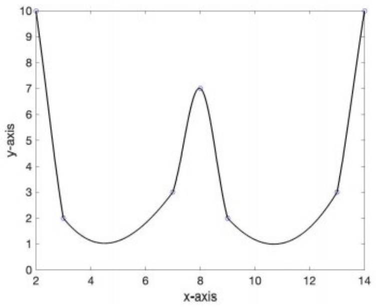
RESULTS AND DISCUSSIONS



Cubic Bézier Curve (left)
Curvature profile (right)



Cubic Ball Curve (left)
Curvature profile (right)



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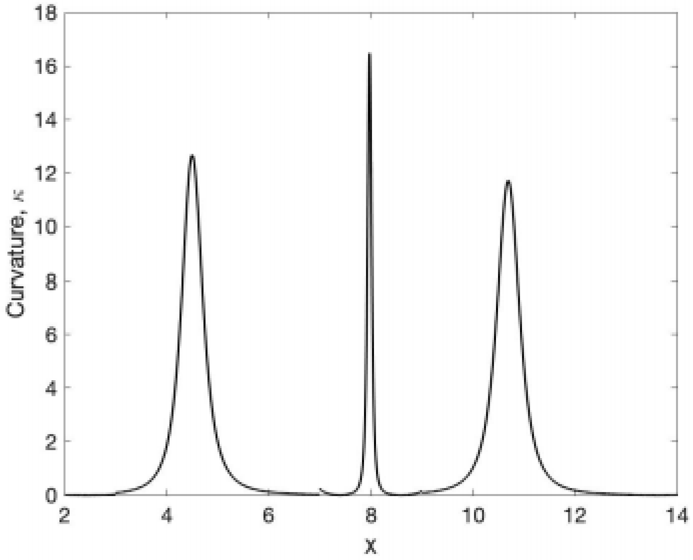
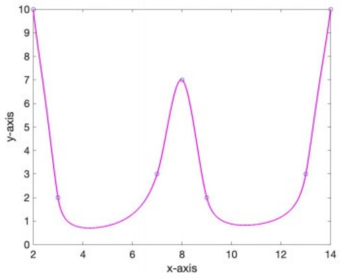
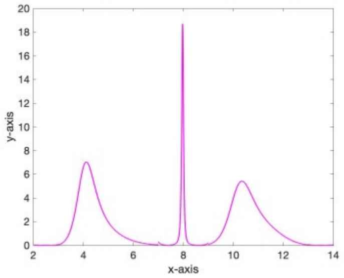


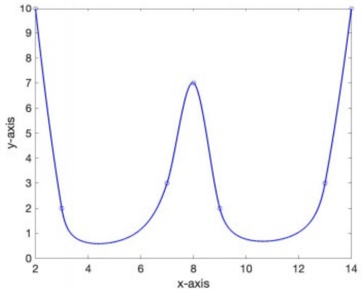
Figure 9: The curvature plot for Trigonometric Bézier Curve with shape parameter equals to 0.1

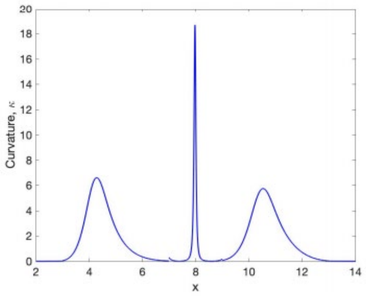
RESULTS AND DISCUSSIONS



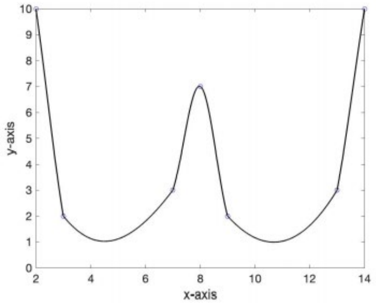


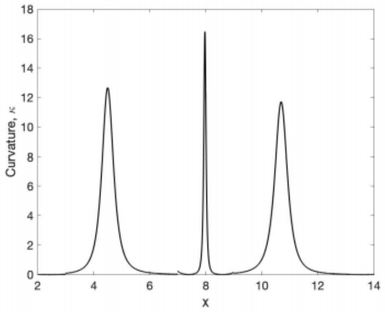
Cubic Bézier Curve (left)
Curvature profile (right)





Cubic Ball Curve (left)
Curvature profile (right)





Trigonometric Bézier Curve (left)
Curvature profile (right)

Table 1: A positive data set of conductance values from Sarfraz (2007)

i	**1**	**2**	**3**	**4**	**5**	**6**	**7**
x_i	2	3	7	8	9	13	14
f_i	10	2	3	7	2	3	10

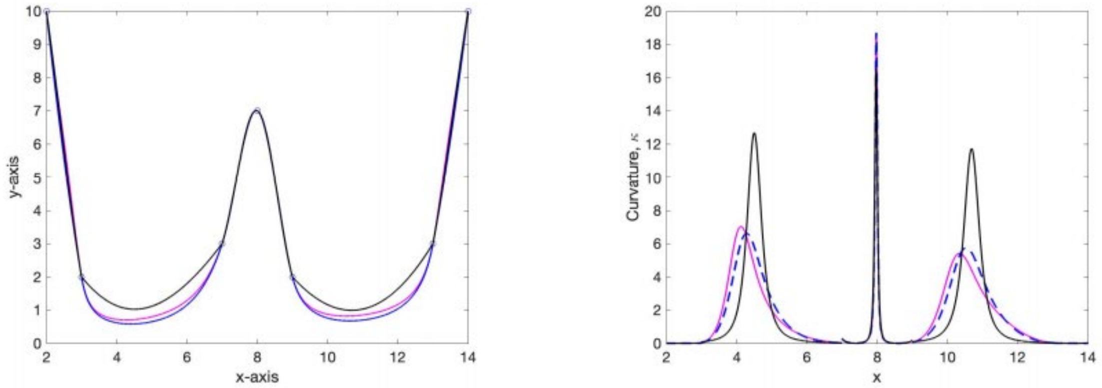


Figure 10: Combinations of all curves. (Pink: Bézier), (BLUE: Ball) and Black (Trigonometric)

RESULTS AND DISCUSSIONS

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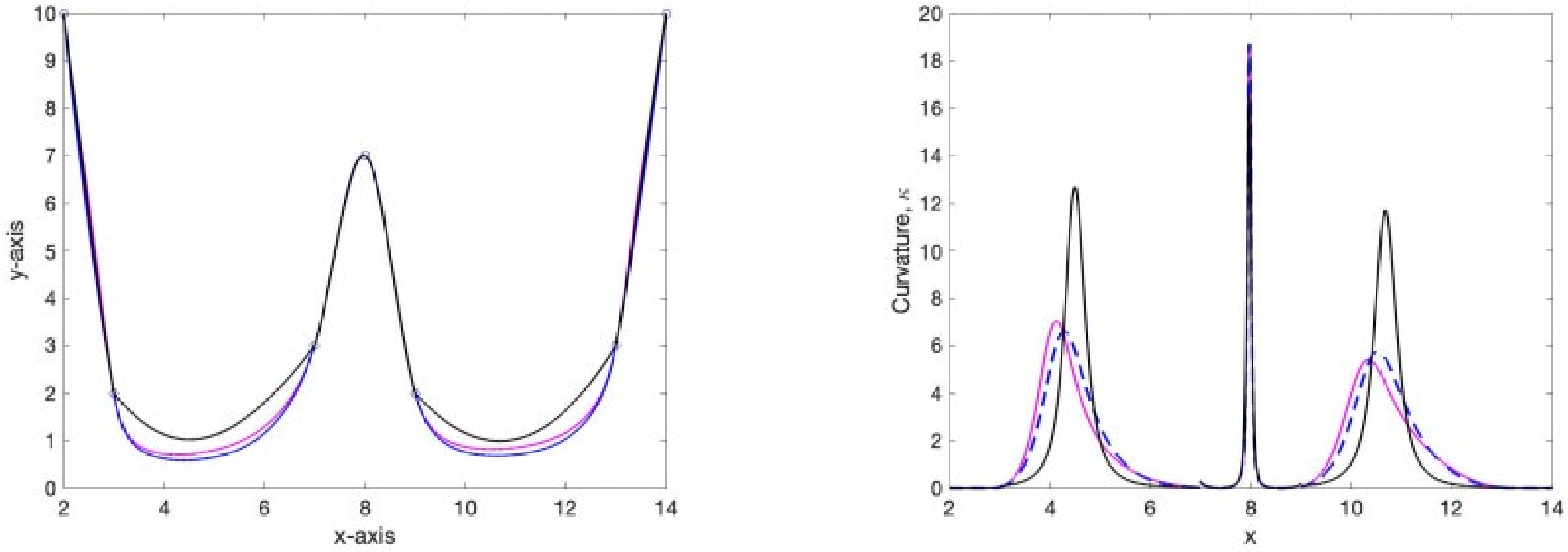

Figure 10: Combinations of all curves. (Pink: Bézier), (BLUE: Ball) and Black (Trigonometric)

CONCLUSIONS

- All three curves managed to preserved the positivity of the data and resulted in smooth interpolation and achieved C^1 continuity.
- The interpolated curve by cubic Bézier and Ball curve is the best based on the lowest amplitude value of curvature
- However, the cubic trigonometric Bézier curve shows higher curvature amplitude than the other two.
- It was found that the curvatures of all three functions are discontinuous. This is expected since curvature is a function of first and second derivatives.

FUTURE WORKS

- The result is based on one value of shape parameter which is 0.1 for all types of curves and it might give different indicator when different sets value of shape parameters for each curve is used in the future.
- C^2 continuity or curvature continuity conditions can be applied to meet curvature continuity and different value of shape parameters can be used to find the variation of the best fit curve.

THANK YOU