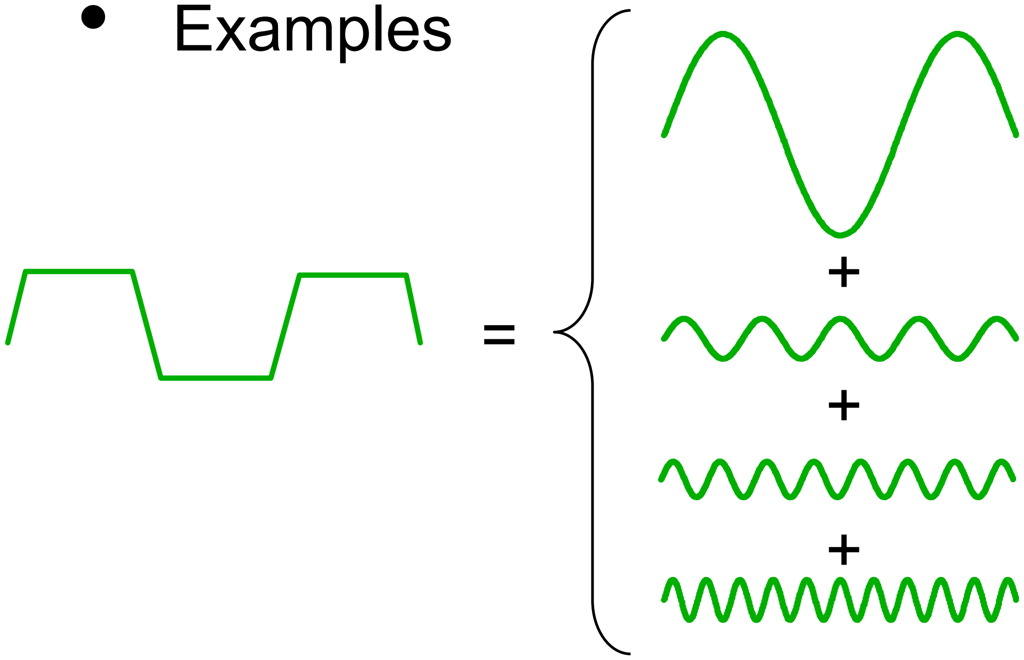
Fourier Transform 2D

- Discrete Fourier Transform - 2D
- Continues Fourier Transform - 2D
- Fourier Properties
- Convolution Theorem
- Examples



The 2D Discrete Fourier Transform

- For an image $f(x,y)$ $x=0..N-1$, $y=0..M-1$, there are two-indices basis functions $B_{u.v}(x,y)$:

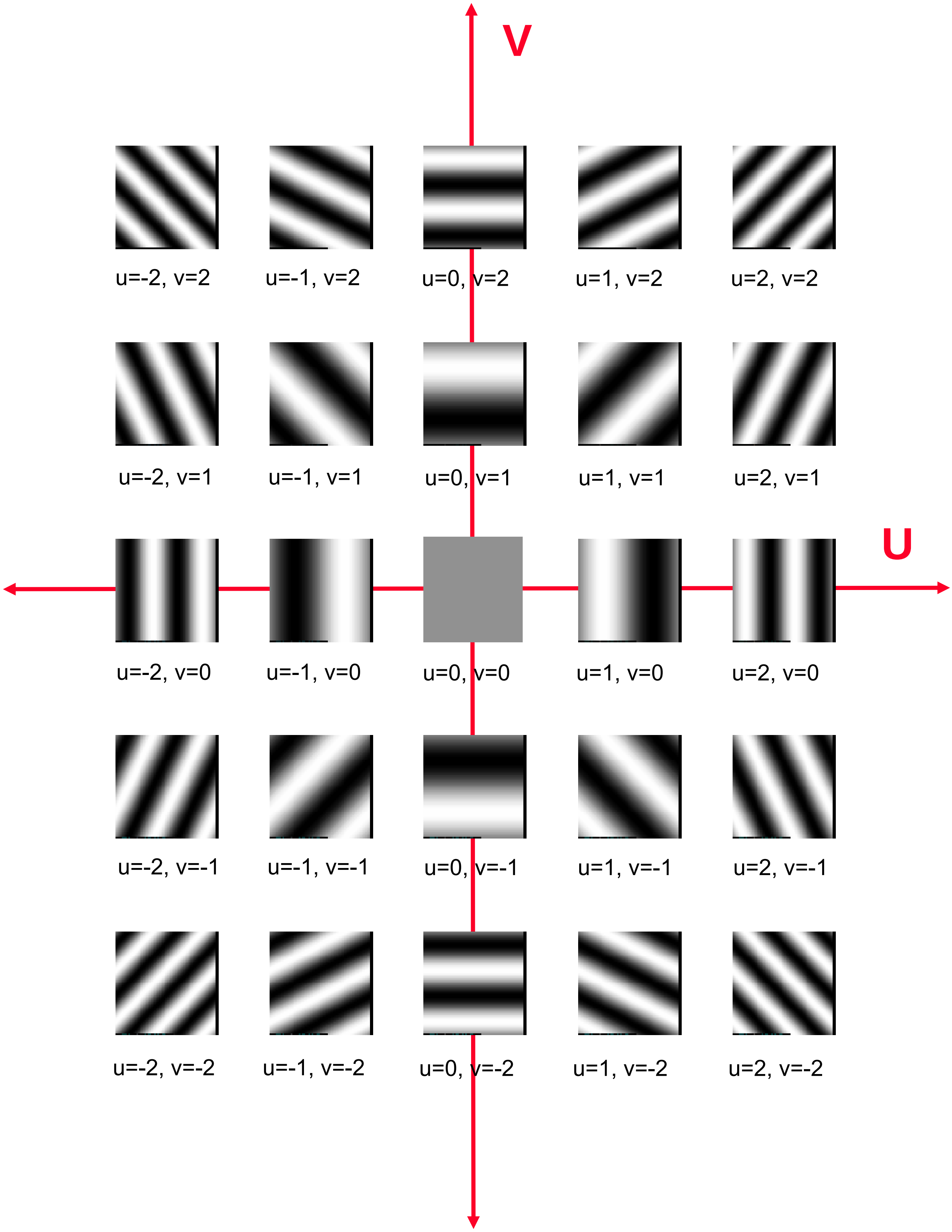
$$B_{u,v}(x,y) = \frac{1}{\sqrt{MN}} e^{2\pi i\left(\frac{ux}{N}+\frac{vy}{M}\right)}$$

$u=0..N-1$, $M=0..M-1$

- The inner product of 2 functions (in 2D) is defined similarly to the 1D case :

$$F(u,v) = \langle f(x,y), B_{u,v}(x,y)\rangle =$$

$$= \sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x,y) B^{*}_{u,v}(x,y)$$


V
U
u=-2, v=2
u=-1, v=2
u=0, v=2
u=1, v=2
u=2, v=2
u=-2, v=1
u=-1, v=1
u=0, v=1
u=1, v=1
u=2, v=1
u=-2, v=0
u=-1, v=0
u=0, v=0
u=1, v=0
u=2, v=0
u=-2, v=-1
u=-1, v=-1
u=0, v=-1
u=1, v=-1
u=2, v=-1
u=-2, v=-2
u=-1, v=-2
u=0, v=-2
u=1, v=-2
u=2, v=-2

The 2D Discrete Fourier Transform

- Image f(x,y) x = 0,1,...,N-1 y=0,1,...,M-1
- The 2D Discrete Fourier Transform (DFT) is defined as:

$$F(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x,y) e^{-2\pi i (ux/N + vy/M)}$$

u = 0, 1, 2, ..., N-1
v = 0, 1, 2, ..., M-1

Matlab: F=fft2(f);

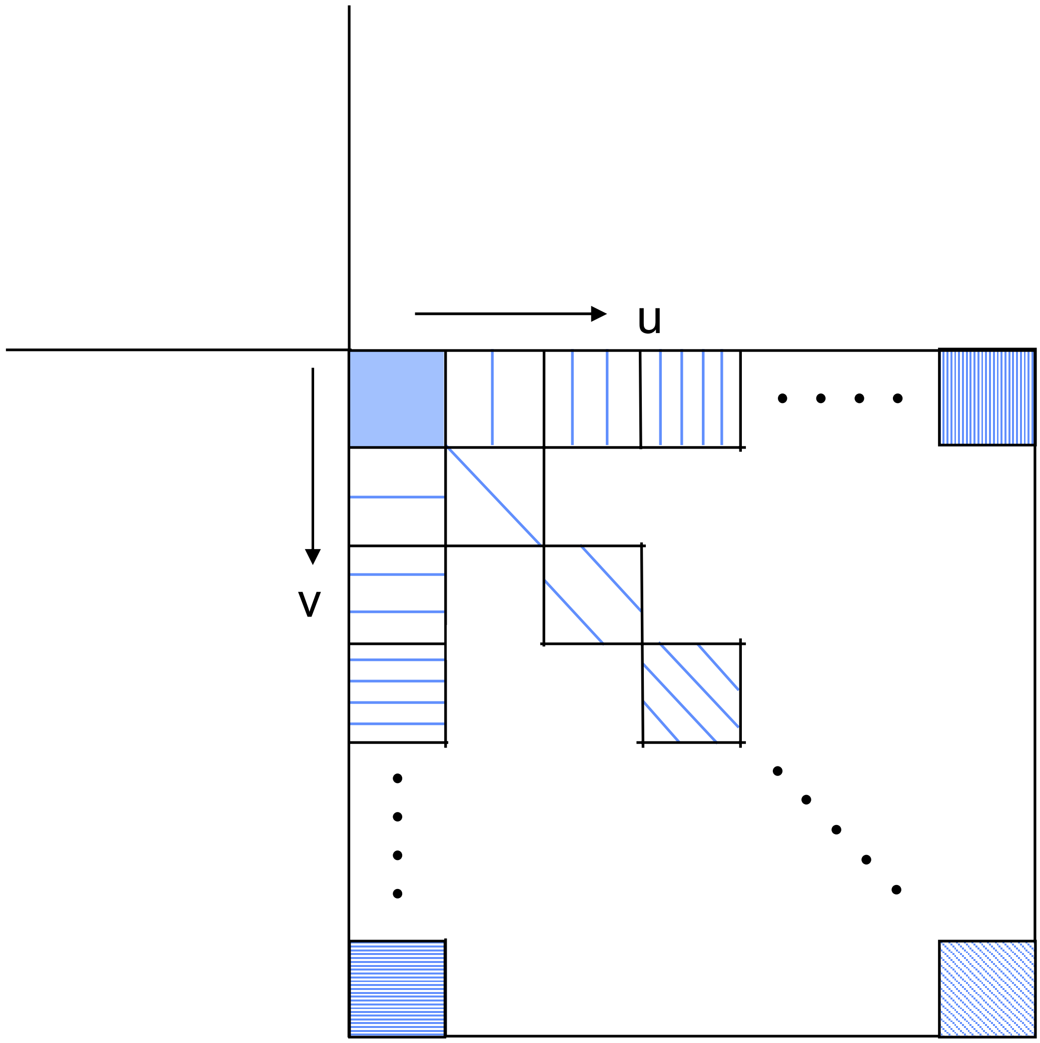
- The Inverse Discrete Fourier Transform (IDFT) is defined as:

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u,v) e^{2\pi i (ux/N + vy/M)}$$

y = 0, 1, 2, ..., N-1
x = 0, 1, 2, ..., M-1

Matlab: f=ifft2(F);

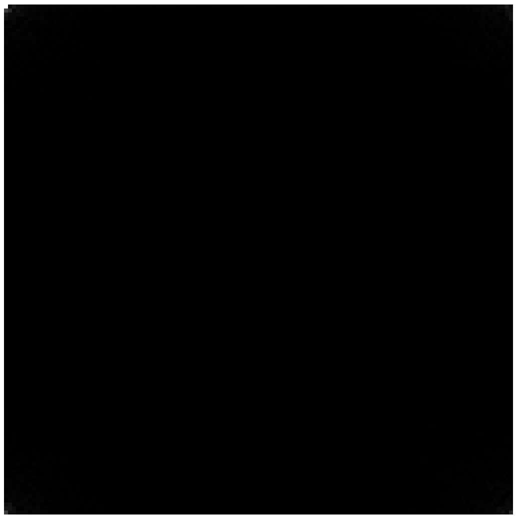
Fourier Transform - Image

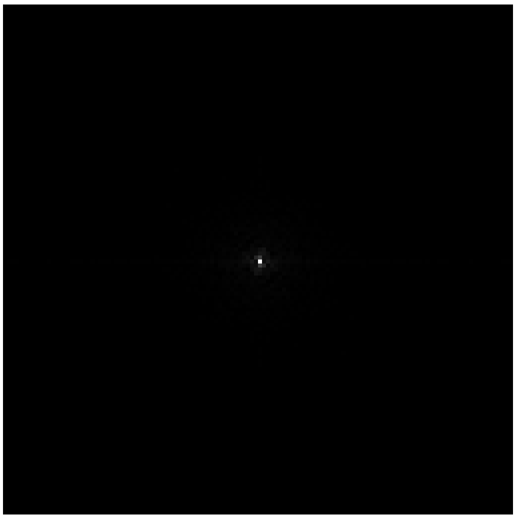


Fourier Image - Example

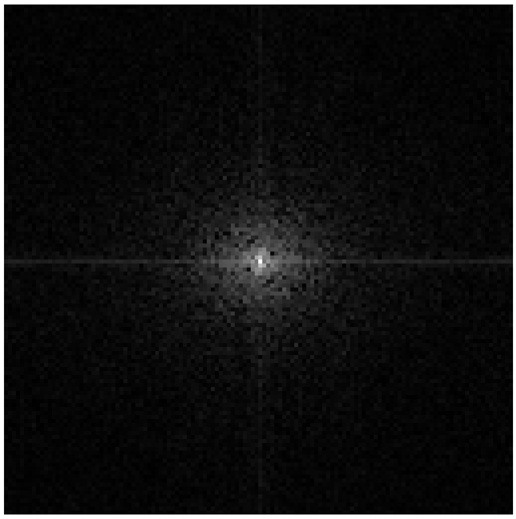
Original

Fourier Image = |F(u,v)|





Shifted Fourier Image



Shifted Log Fourier Image = log(1+ |F(u,v)|)

Visualizing the Fourier Transform Image using Matlab Routines

- F(u,v) is a Fourier transform of f(x,y) and it has complex entries.

F = fft2(f);

- In order to display the Fourier Spectrum |F(u,v)|
  - Reduce dynamic range of |F(u,v)| by displaying the log:

D = log(1+abs(F));

  - Cyclically rotate the image so that F(0,0) is in the center:

D = fftshift(D);

Example:

|F(u)| = 100 4 2 1 0 0 1 2 4

Display in Range([0..100]):

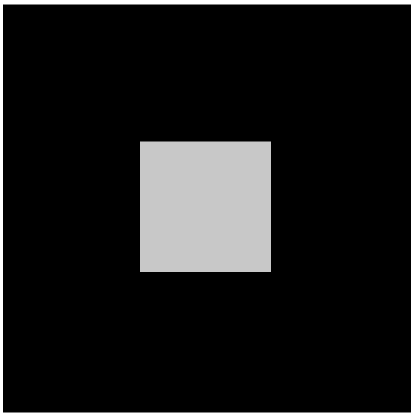
log(1+|F(u)|) = 4.62 1.61 1.01 0.69 0 0 0.69 1.01 1.61

log(1+|F(u)|)/0.0462 = 100 40 20 10 0 0 10 20 40

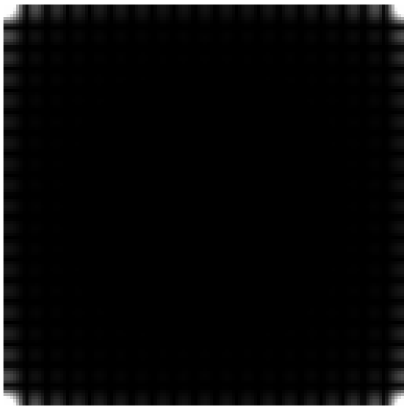
fftshift(log(1+|F(u)|) = 0 10 20 40 100 40 20 10 0

Visualizing the Fourier Image - Example

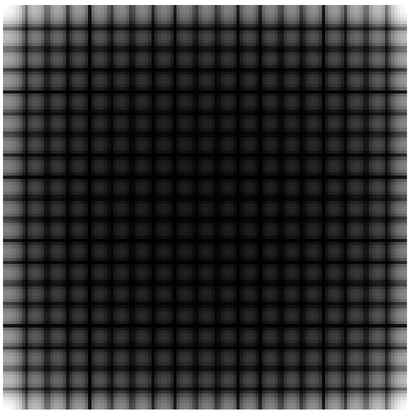
Original



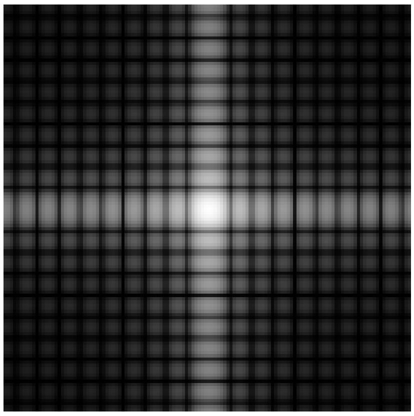
|F(u,v)|



log(1 + |F(u,v)|)



fftshift(log(1 + |F(u,v)|))



Properties of The Fourier Transform

- Linearity:

$$\tilde{F}[\alpha f] = \alpha \tilde{F}[f]$$

- Distributive (additivity):

$$\tilde{F}[f_1 + f_2] = \tilde{F}[f_1] + \tilde{F}[f_2]$$

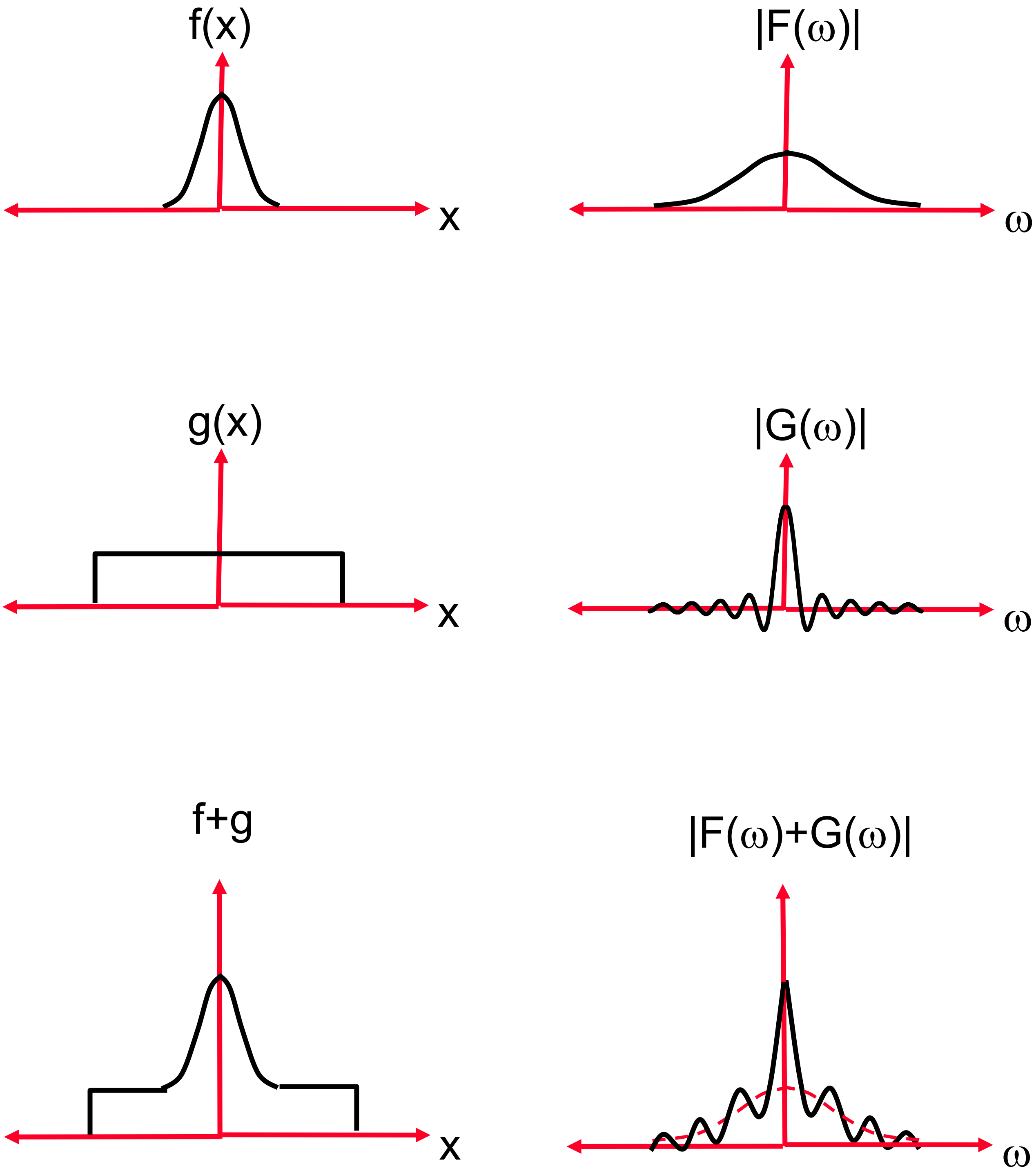
- DC (average):

$$F(0,0) = \sum_x \sum_y f(x,y) e^0$$

- Parseval

$$\sum_x \sum_y \| f(x,y) \|^2 = \sum_u \sum_v \| F(u,v) \|^2$$

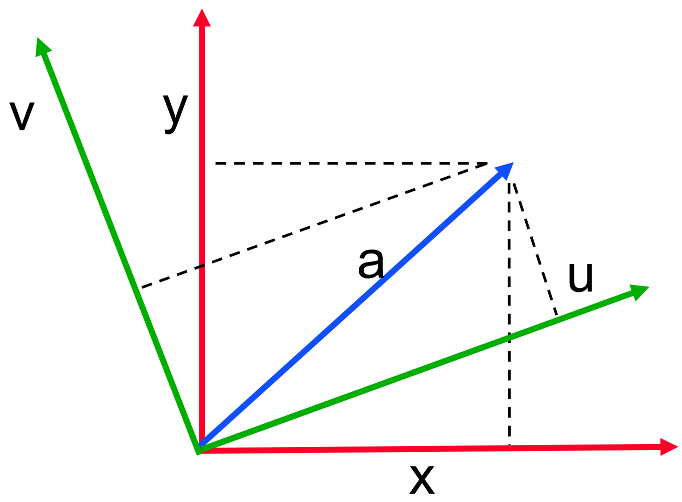
Distributive: $\tilde{F}\{f+g\} = \tilde{F}\{f\} + \tilde{F}\{g\}$



Parseval's Theorem

One more characteristic:

$$\boxed{\sum_x \sum_y |f(x,y)|^2 = \sum_u \sum_v |F(u,v)|^2}$$



Properties of The Fourier Transform

- Symmetric:

  If f(x,y) is real then,

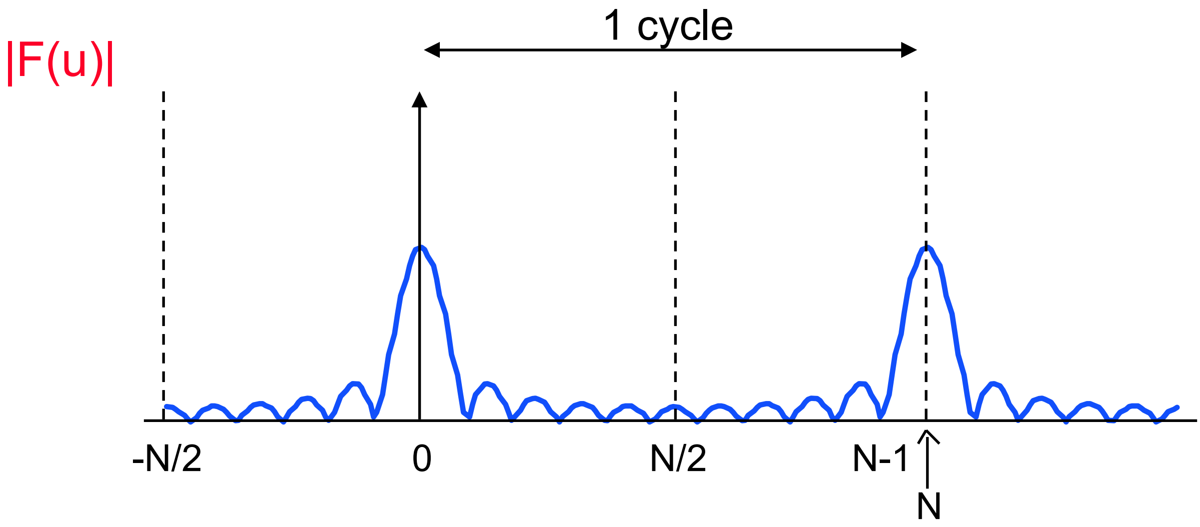
$$F(u,v) = F^{*}(-u,-v) \quad \text{thus} \quad |F(u,v)| = |F(-u,-v)|$$

- Cyclic:

  if f(x,y) is discrete

$$F(u,v) = F(u+N,v) = F(u,v+M) = F(u+N,v+M)$$

Cyclic and Symmetry of the Fourier Transform - 1D Example



Properties: Cont.

Separability

$$F(u,v)=\sum_x\sum_y f(x,y)e^{-2\pi i\left(\frac{ux}{N}+\frac{vy}{M}\right)}=$$

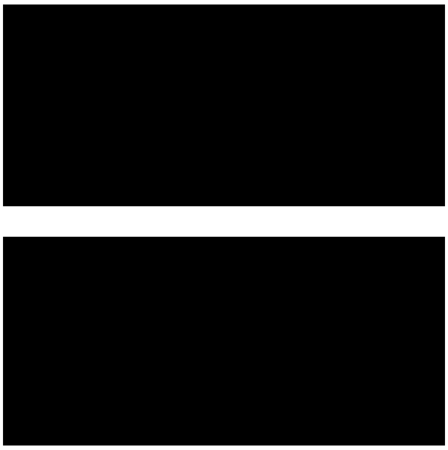
$$=\sum_x\left(\sum_y f(x,y)e^{-2\pi i\frac{vy}{N}}\right)e^{-2\pi i\frac{ux}{N}}=\sum_x F(x,v)e^{-2\pi i\frac{ux}{N}}$$

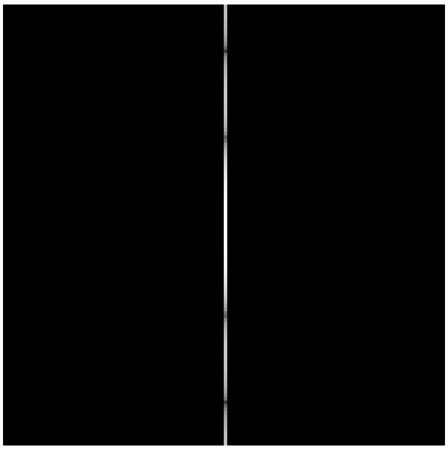
- Thus, performing a 2D Fourier Transform is equivalent to performing 2 1D transforms:
  1. Perform 1D transform on EACH column of image f(x,y), obtaining F(x,v).
  2. Perform 1D transform on EACH row of F(x,v), obtaining F(u,v).

- Higher Dimensions: Fourier in any dimension can be performed by applying 1D transform on each dimension.

Example - Separability

2D Image





Fourier Spectrum

Image Transformations

- **Translation**:

$$\tilde{F}\left[f\left(x-x_0,y-y_0\right)\right]=F\left(u,v\right)e^{-2\pi i\left(\frac{ux_0}{N}+\frac{vy_0}{M}\right)}$$

The Fourier Spectrum remains unchanged under translation:

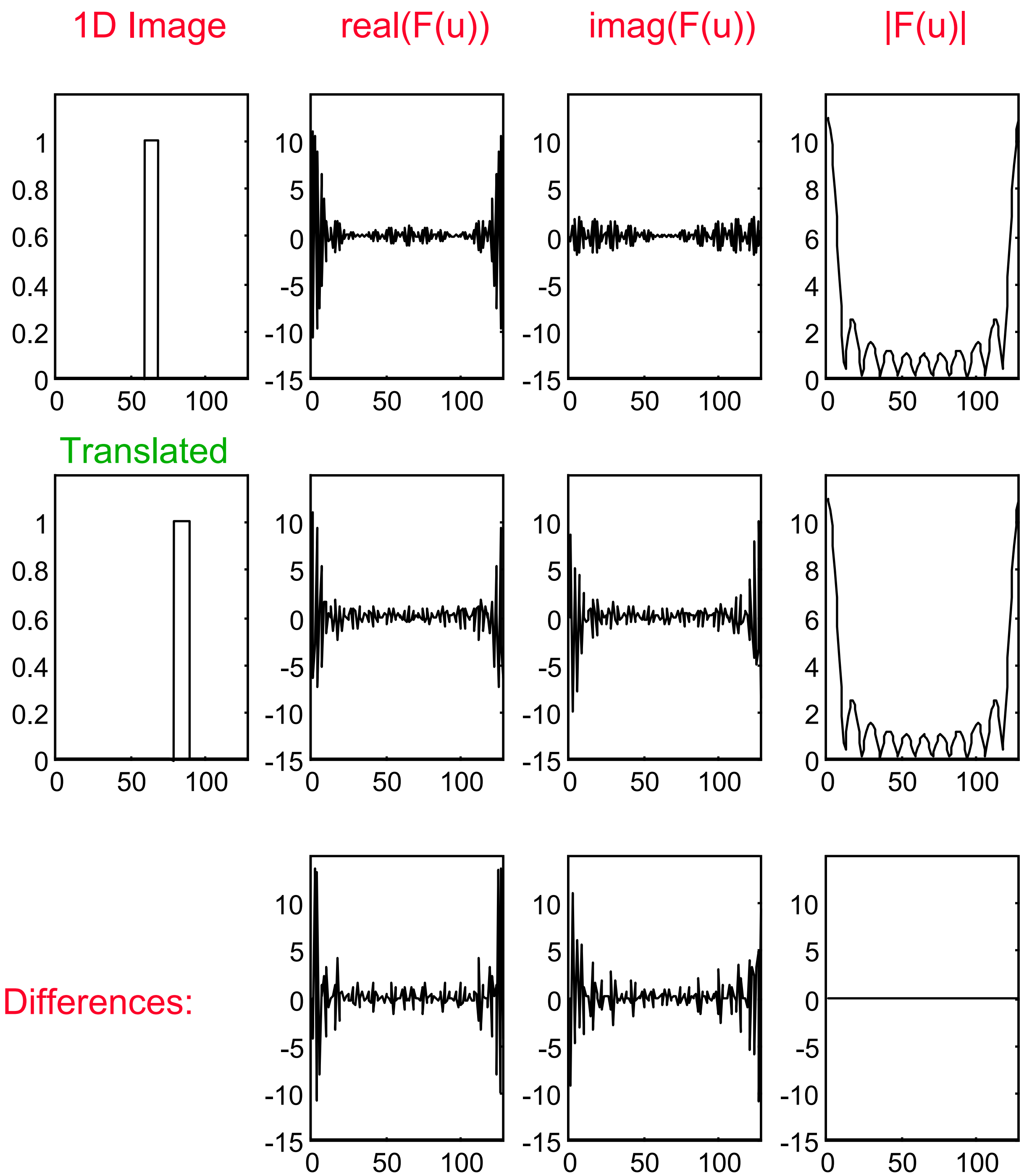
$$\left|F\left(u,v\right)\right|=\left|F\left(u,v\right)e^{-2\pi i\left(\frac{ux_0}{N}+\frac{vy_0}{M}\right)}\right|$$

- **Rotation**: Rotation of f(x,y) by θ $\rightarrow$ rotation of F(u,v) by θ

- **Scaling**:

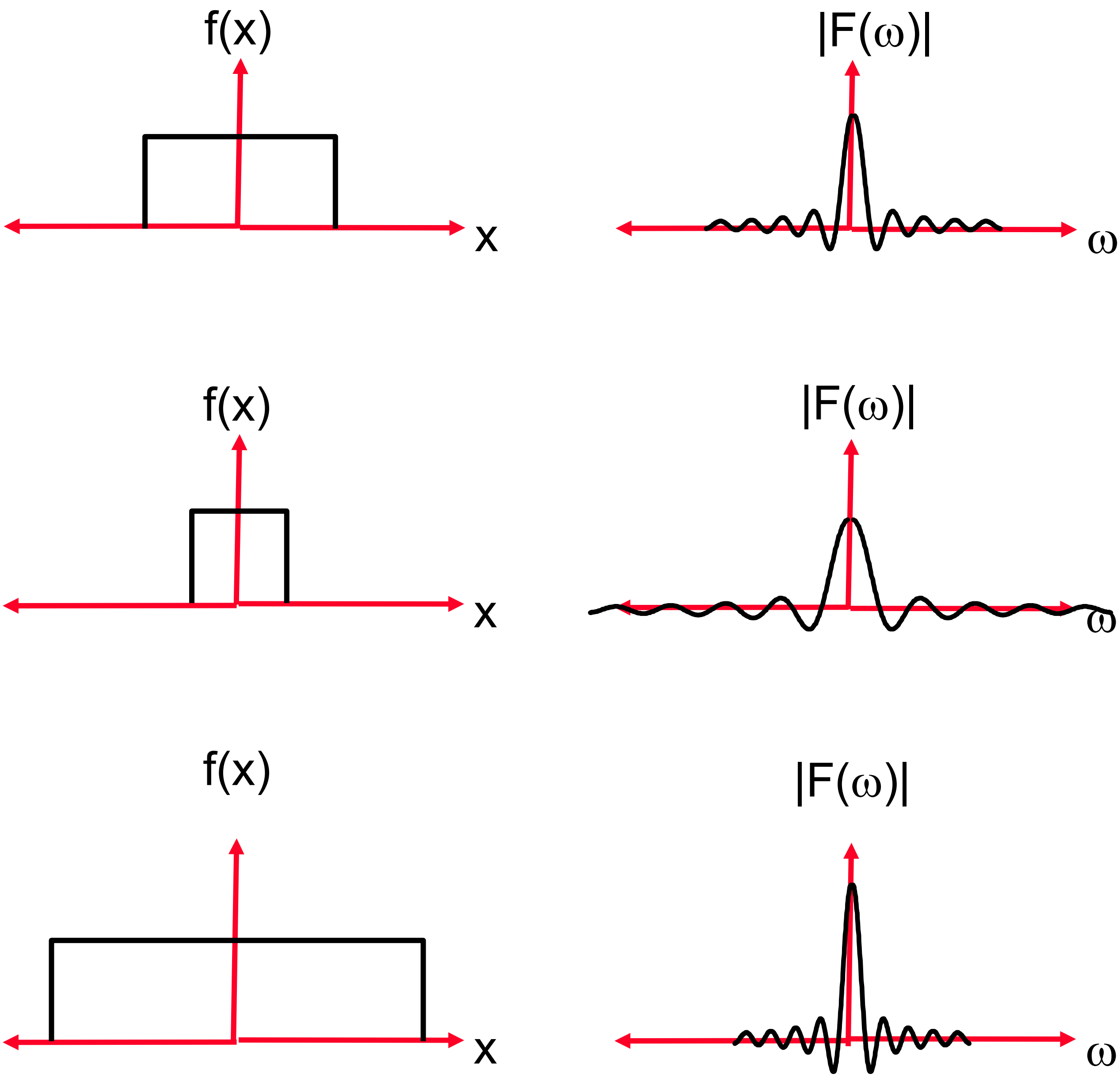
$$\tilde{F}\left[f\left(a\,x,b\,y\right)\right]=\frac{1}{\left|ab\right|}F\left(\frac{u}{a},\frac{v}{b}\right)$$

Example - Translation

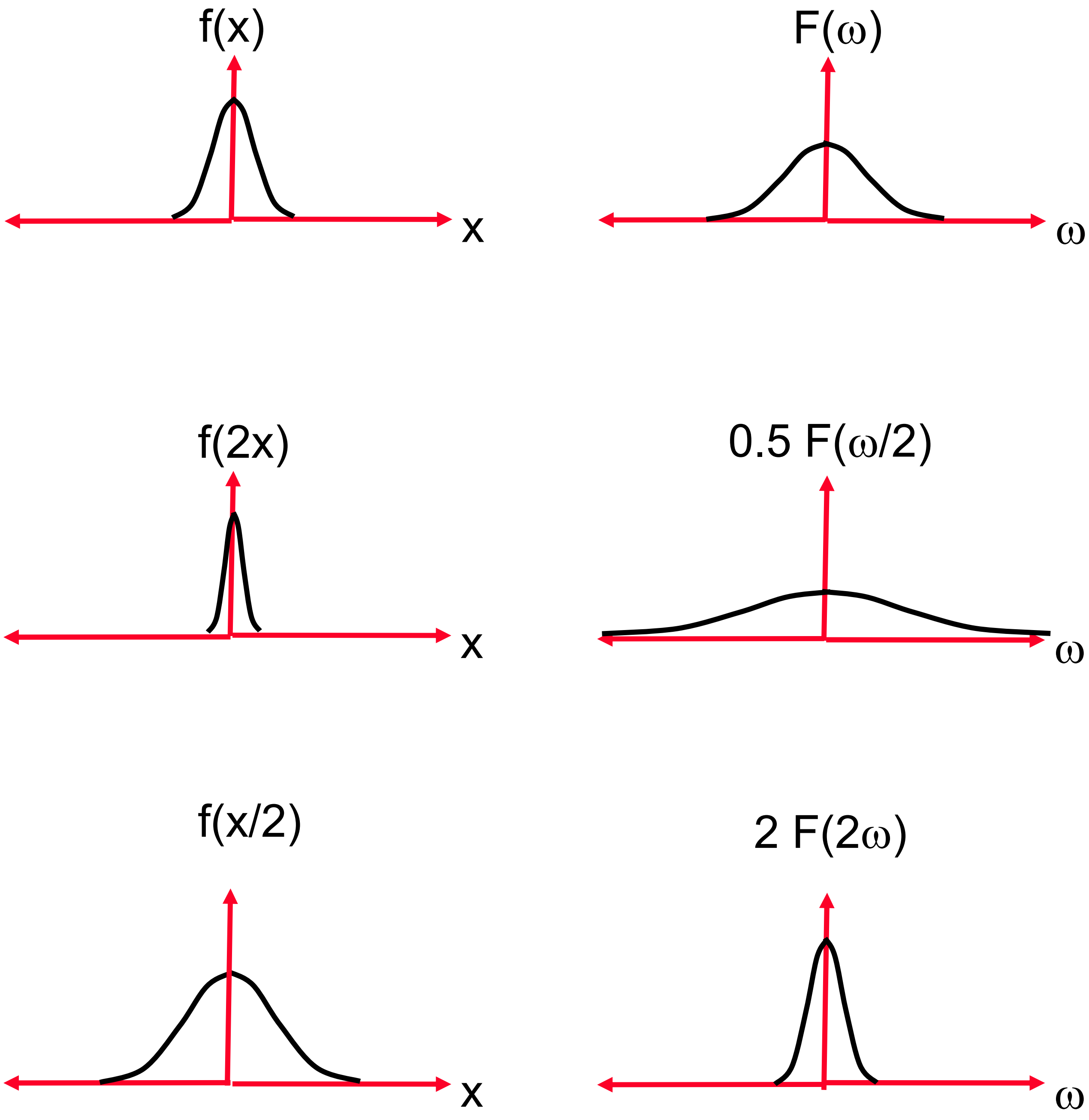


Change of Scale- 1D:

$$\text{if } \tilde{F}\{f(x)\}=F(\omega) \text{ then } \tilde{F}\{f(ax)\}=\frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

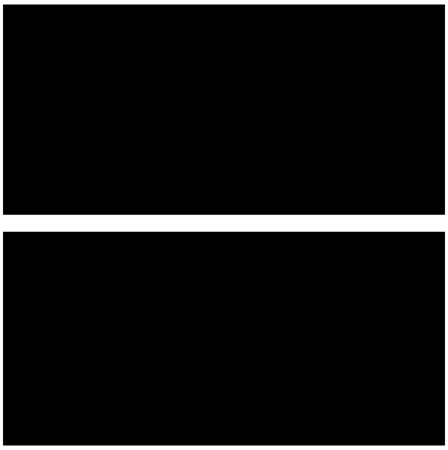


Change of Scale

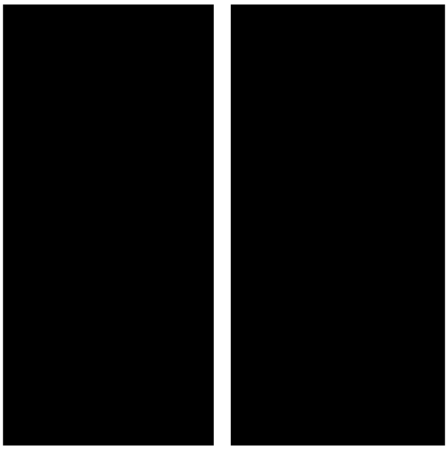


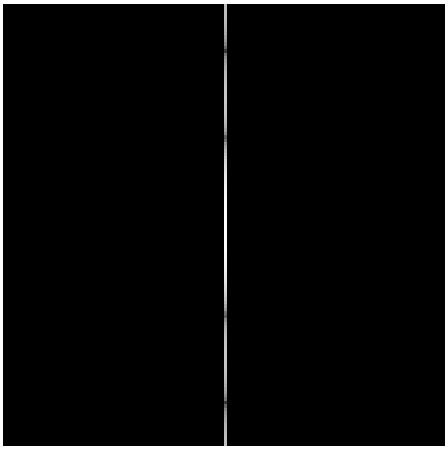
Example - Rotation

2D Image

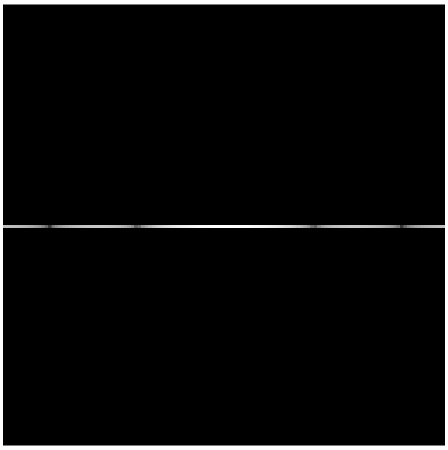


2D Image - Rotated





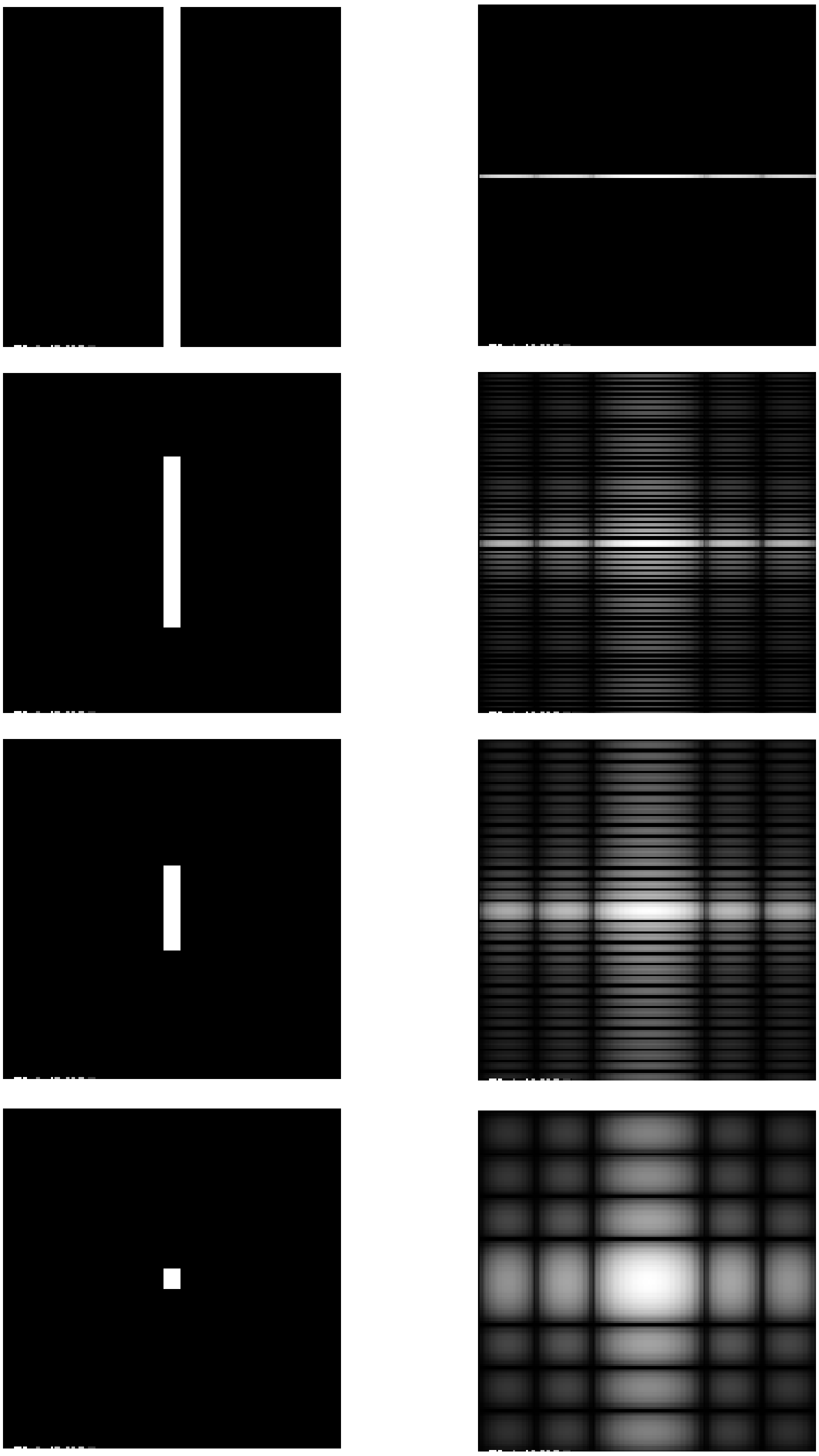
Fourier Spectrum



Fourier Spectrum

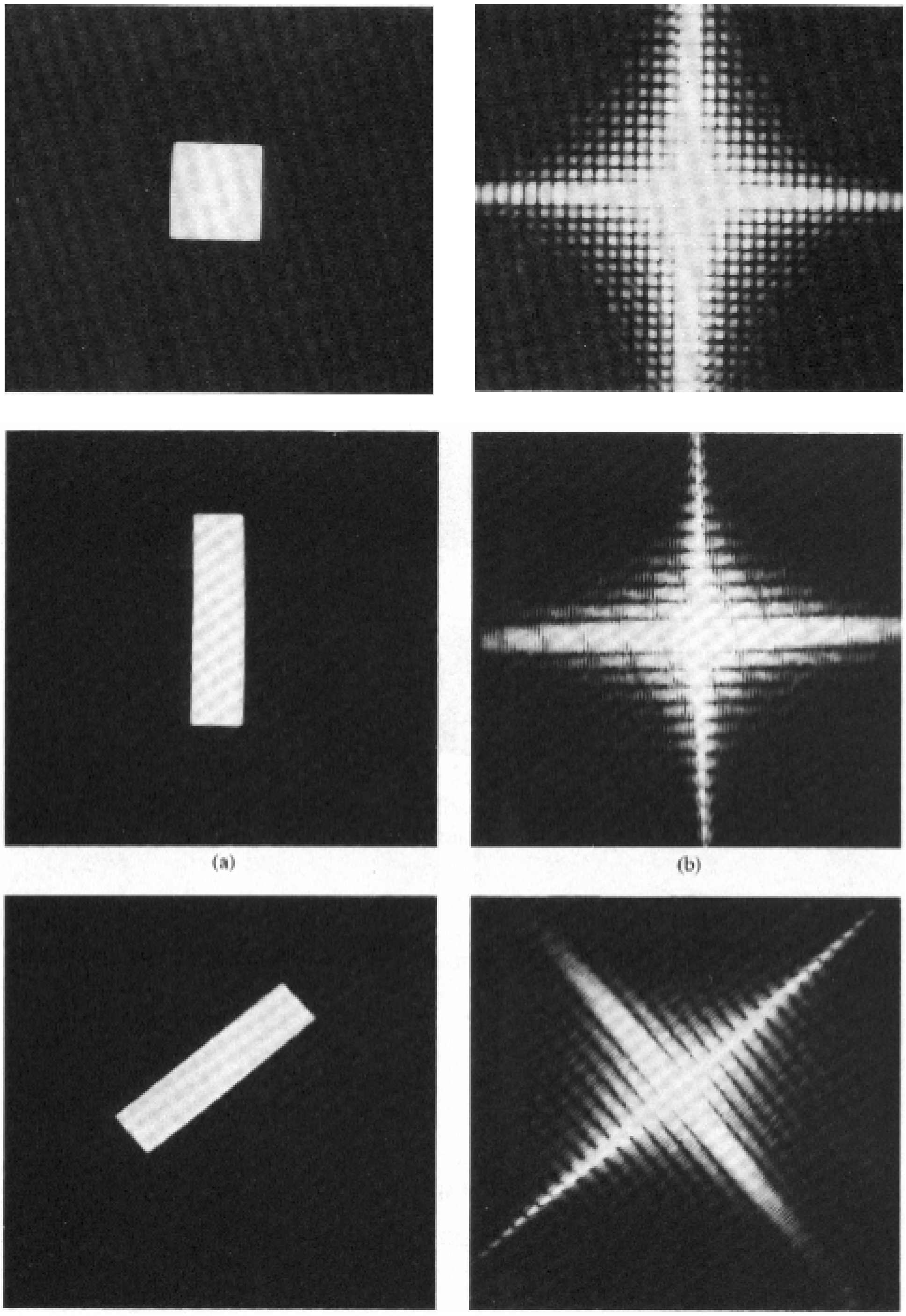
Fourier Transform Examples

Image Domain Frequency Domain



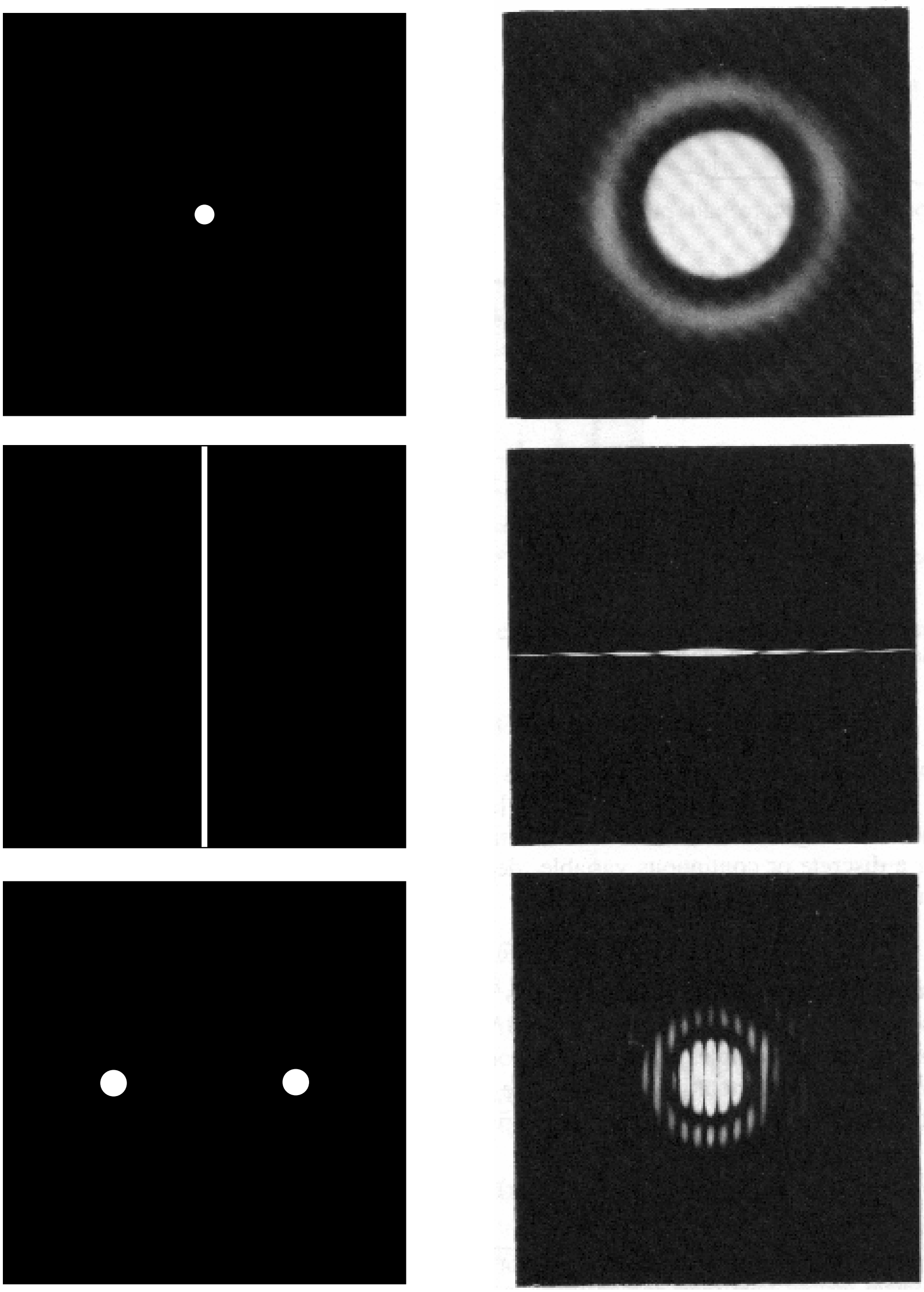
Fourier Transform Examples

Image Domain Frequency Domain



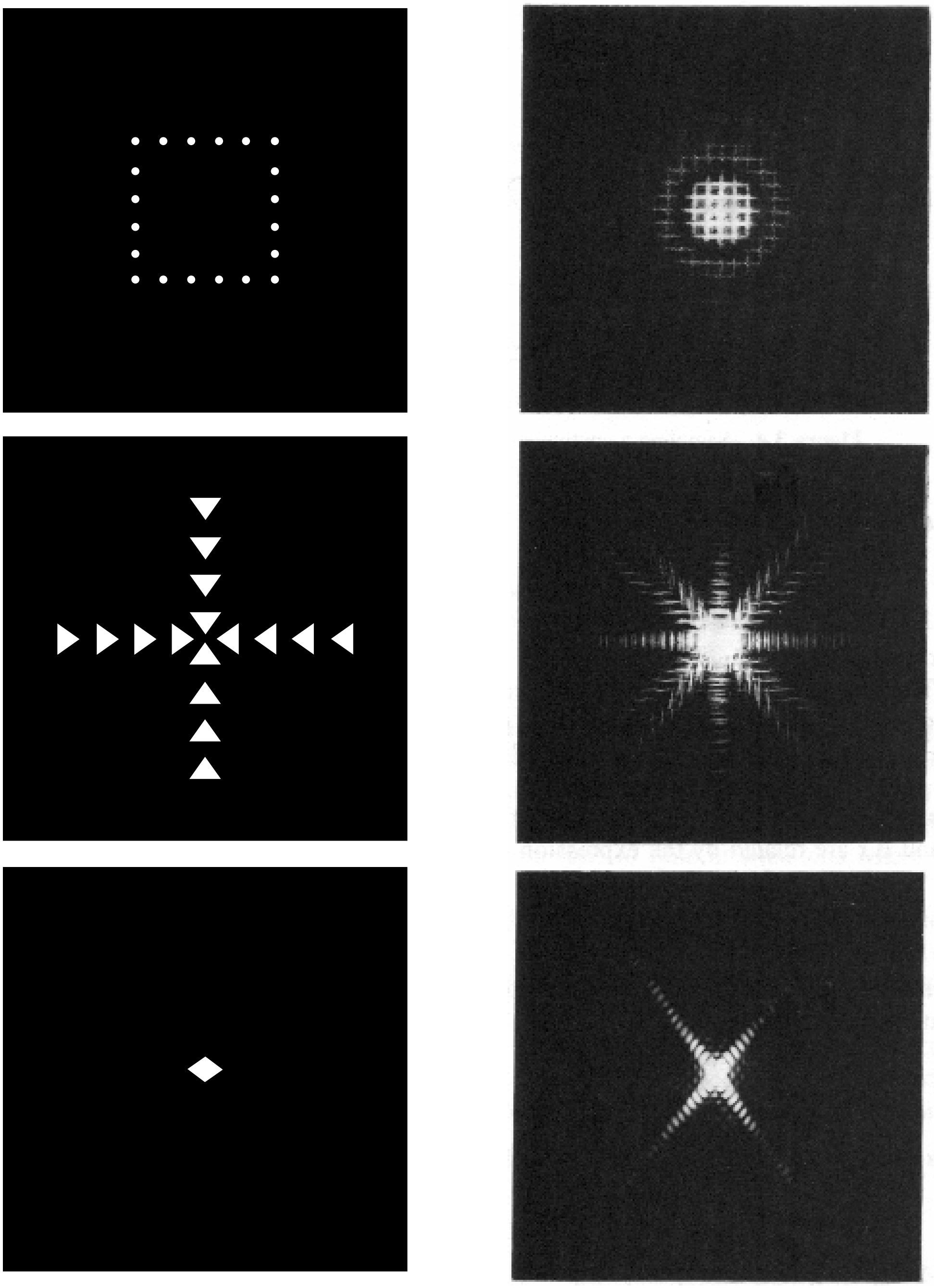
Fourier Transform Examples

Image Domain Frequency Domain

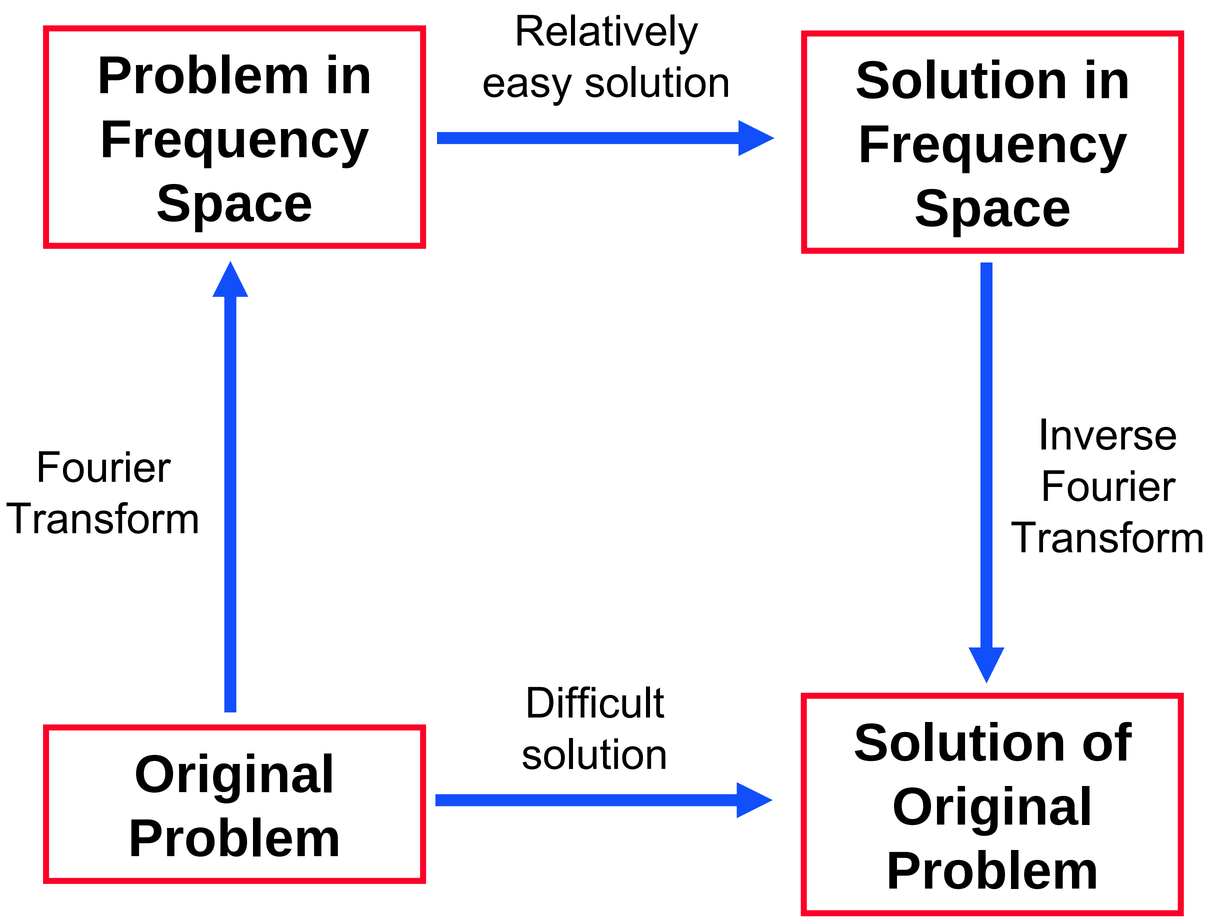


Fourier Transform Examples

Image Domain Frequency Domain



Why do we need representation in the frequency domain?



The Convolution Theorem

$g = f * h$	$g = f\,h$
implies	implies
$G = F\,H$	$G = F * H$

Convolution in one domain is multiplication in the other and vice versa

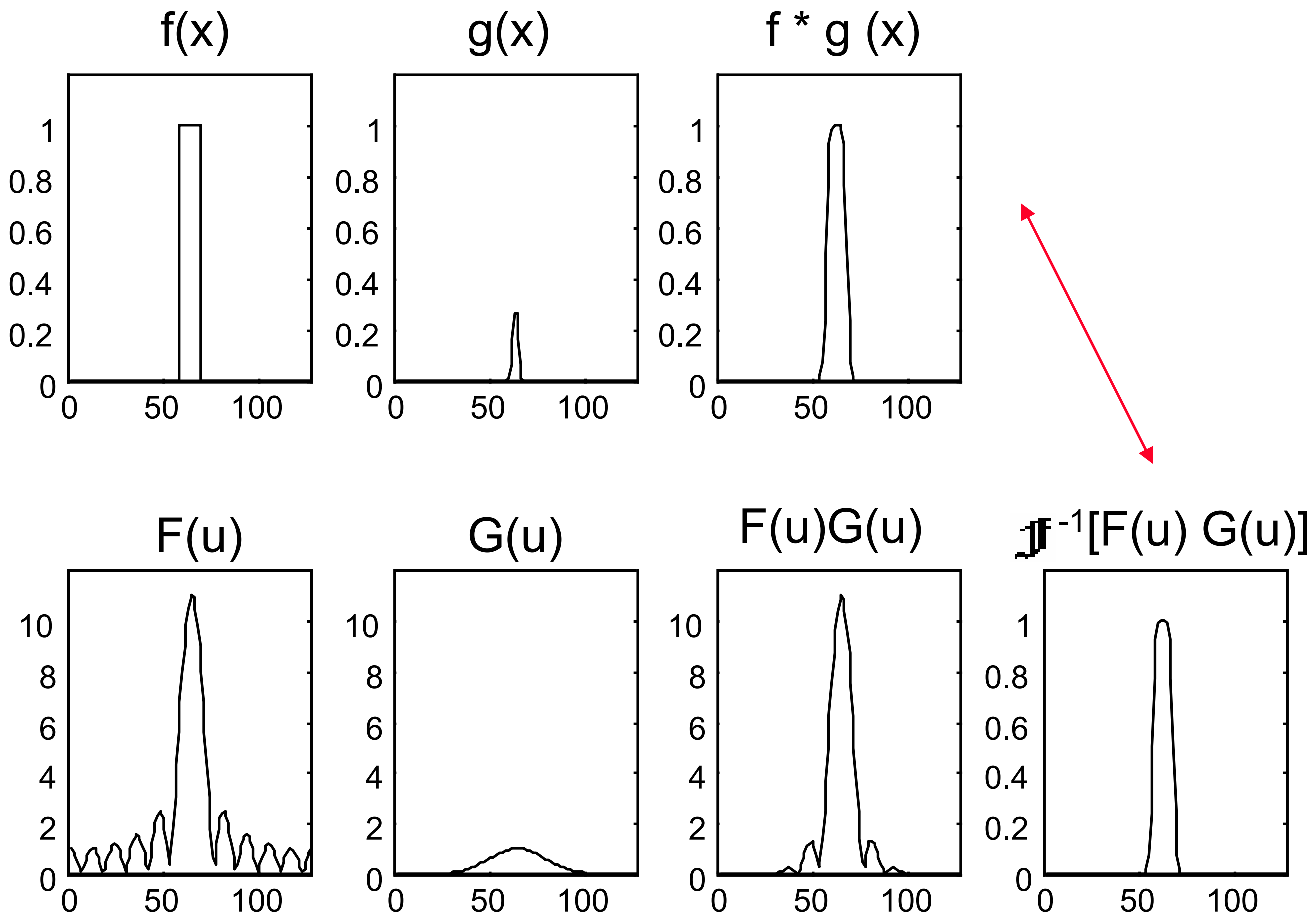
The Convolution Theorem

$$\boxed{\tilde{F}\{f(x) * g(x)\} = \tilde{F}\{f(x)\}\tilde{F}\{g(x)\}}$$

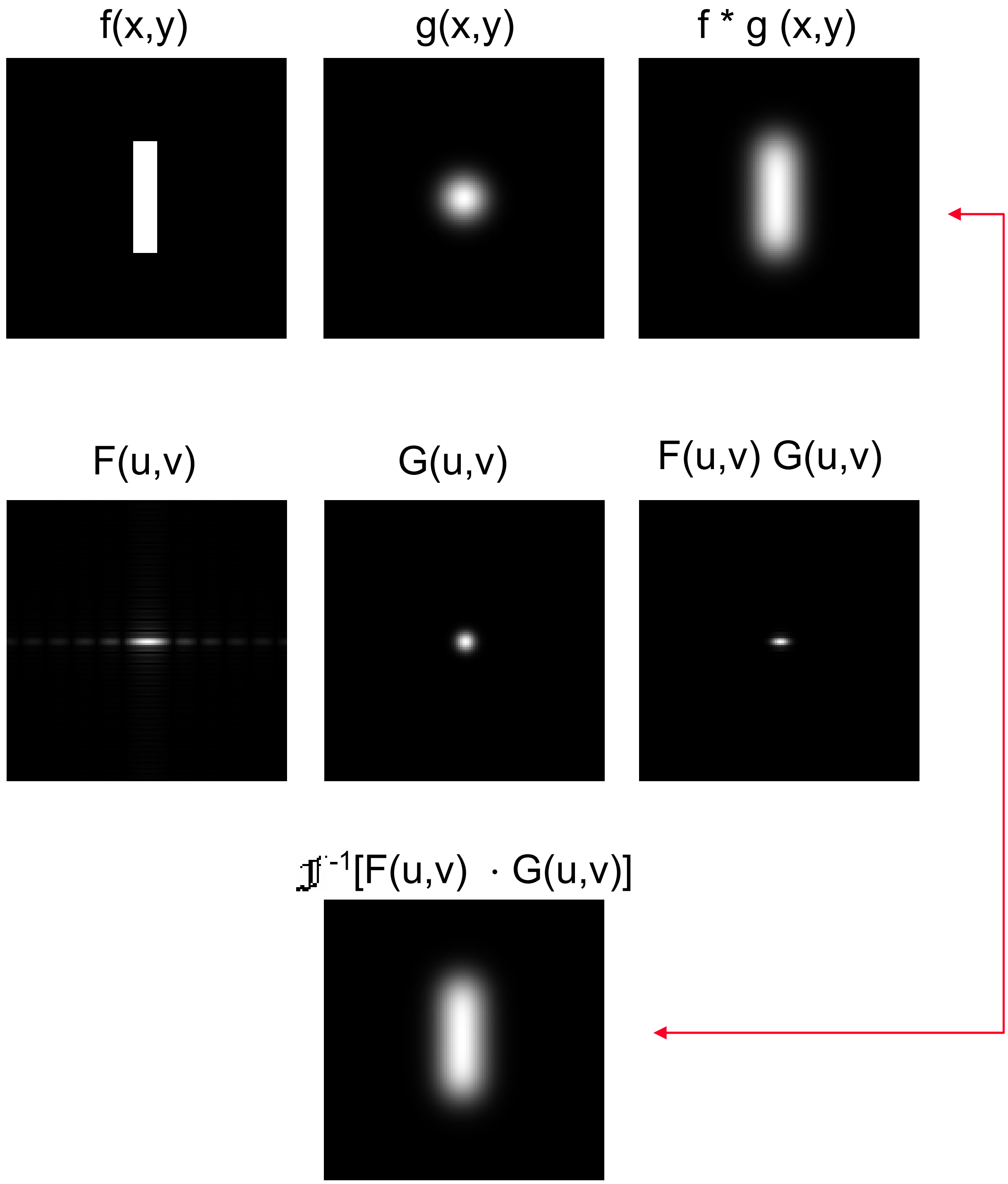
and likewise

$$\boxed{\tilde{F}\{f(x)g(x)\} = \tilde{F}\{f(x)\} * \tilde{F}\{g(x)\}}$$

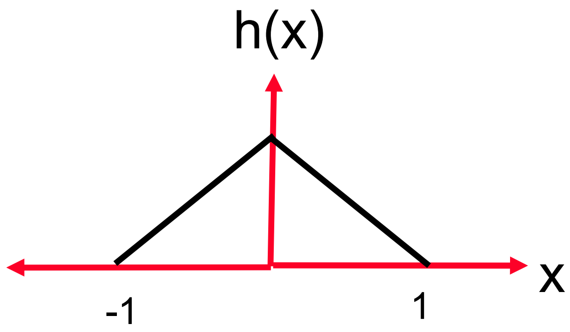
Convolution Theorem - Example



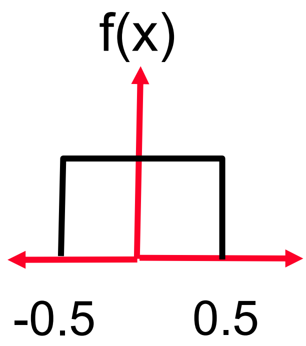
Convolution Theorem - 2D Example



Example: What is the Fourier Transform of:



h(x) **=**



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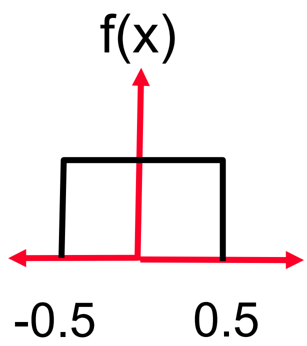
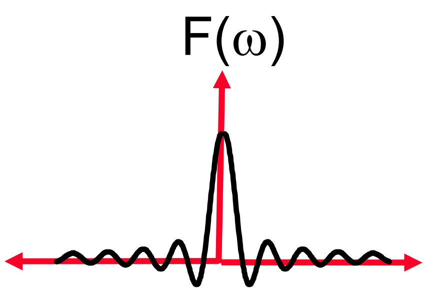


H(ω) **=**



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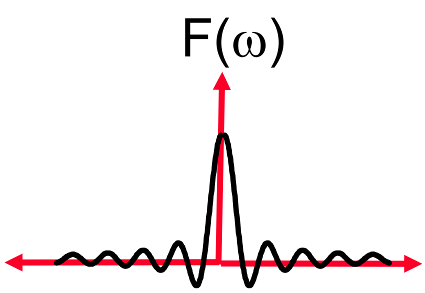

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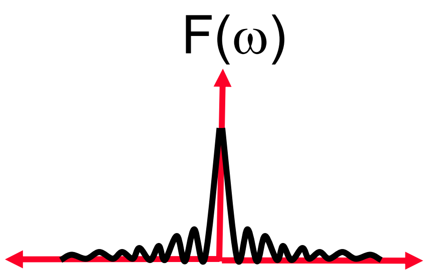


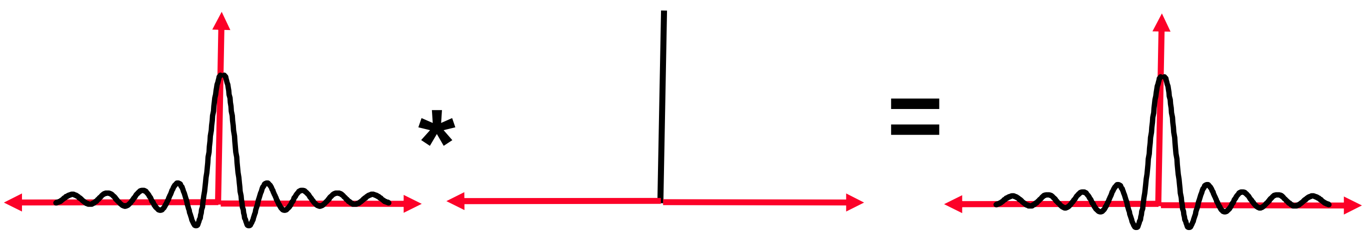
Example: What is the Fourier Transform of the Dirac Function?

$$\delta(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & otherwise \end{cases}$$

Proof : Consider any function **f(x)**

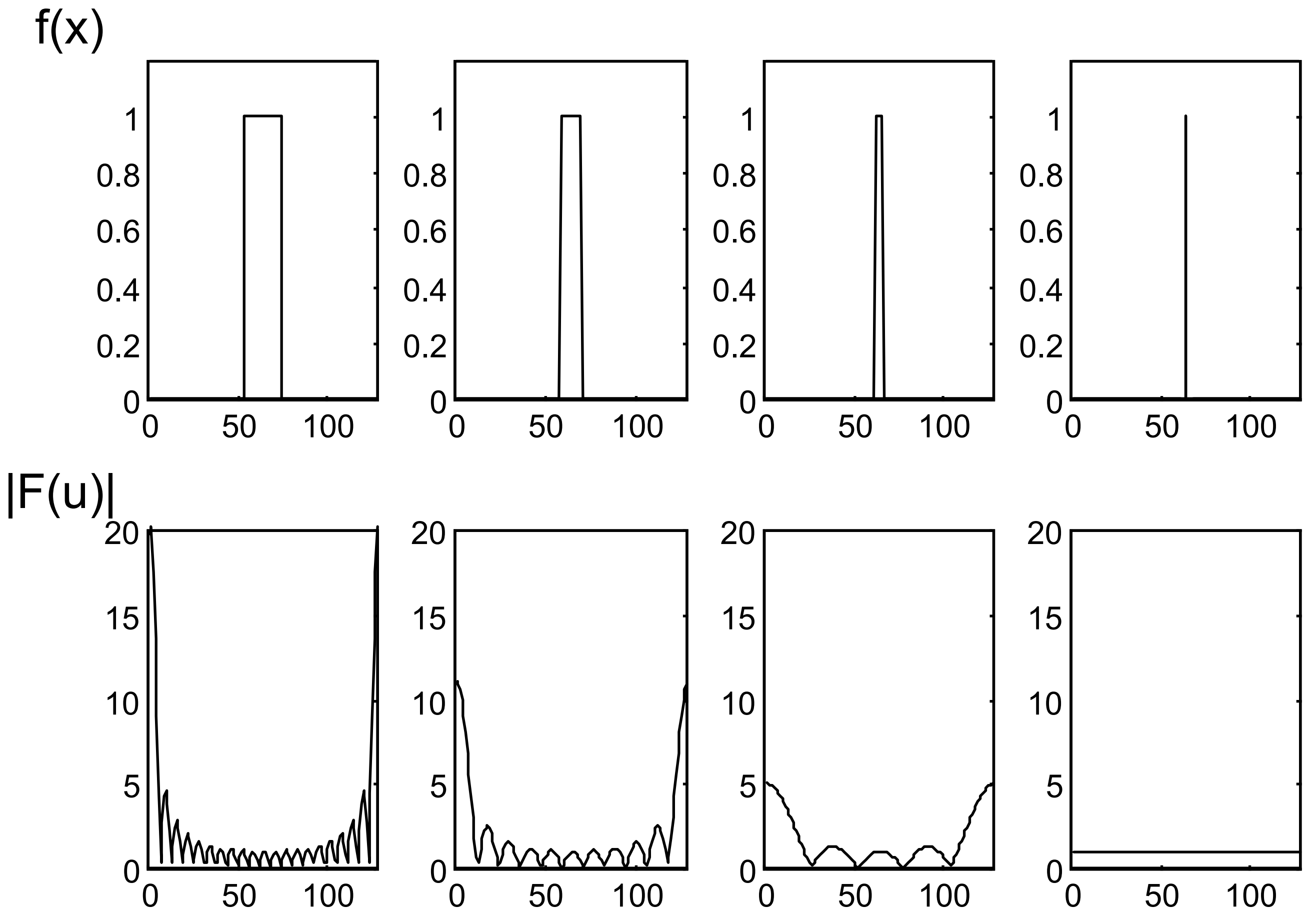
$$f(x) * \delta(x) = f(x)$$

$$F(u) \cdot \mathrm{F}[\delta(x)] = F(u)$$



$$\mathrm{F}[\delta(x)] = 1$$

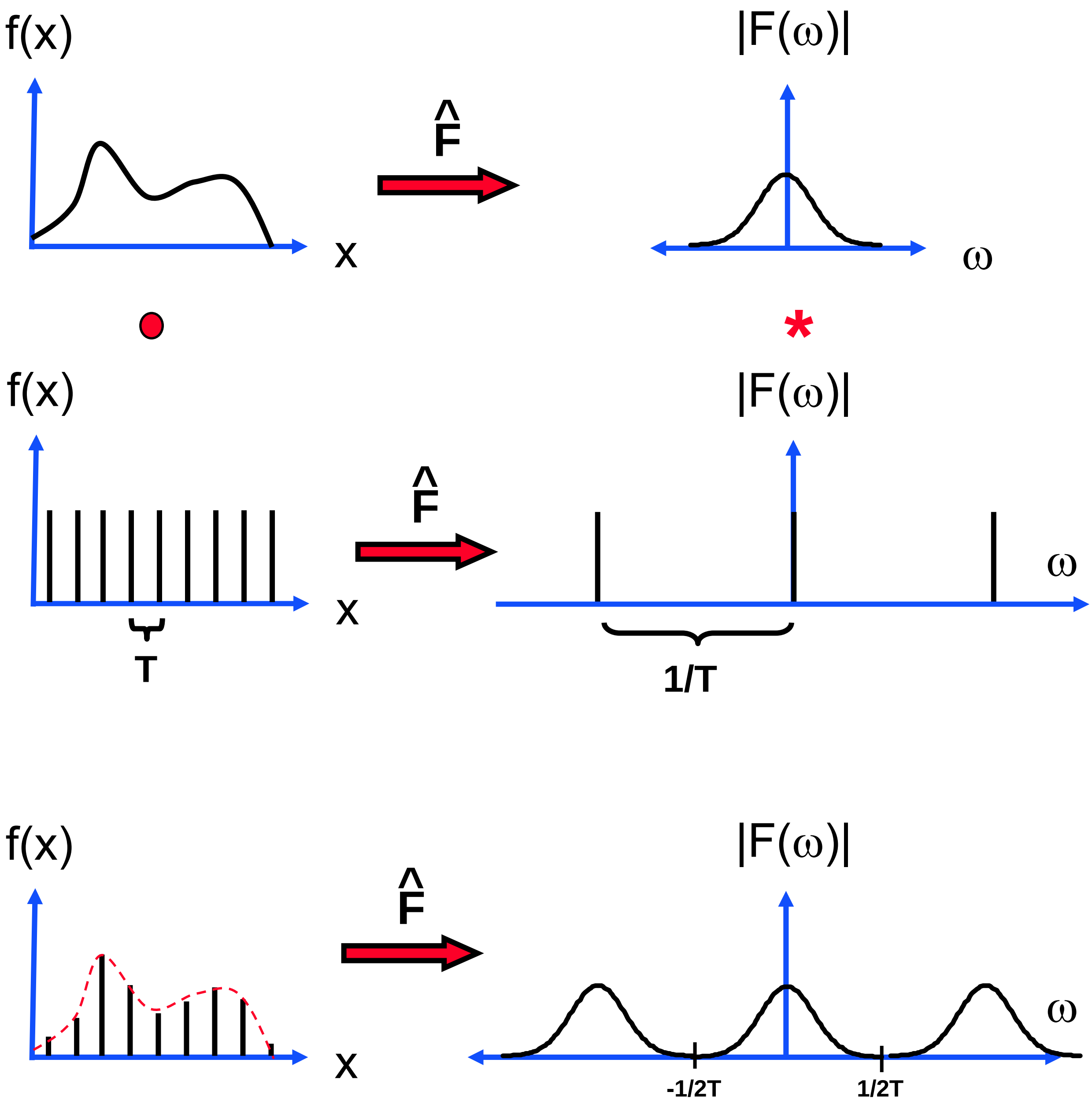
What is the Fourier Transform of the Dirac Function? Answer II:



What is the Fourier Transform of an image with constant gray value?

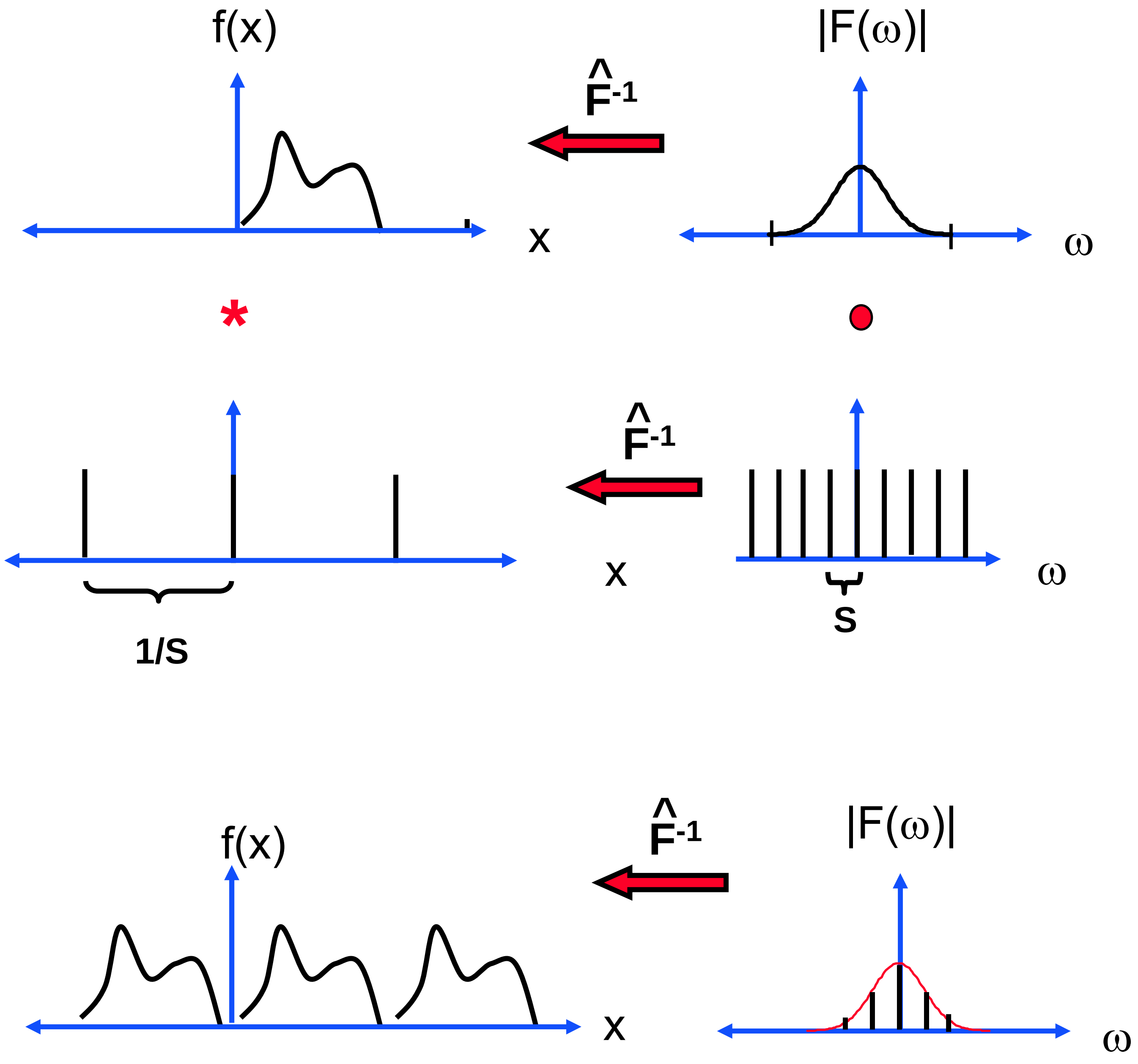
Sampling The Image

- Sampling a function f(x) with impulse train of cycle T produces replicas in the frequency domain with cycle 1/T:



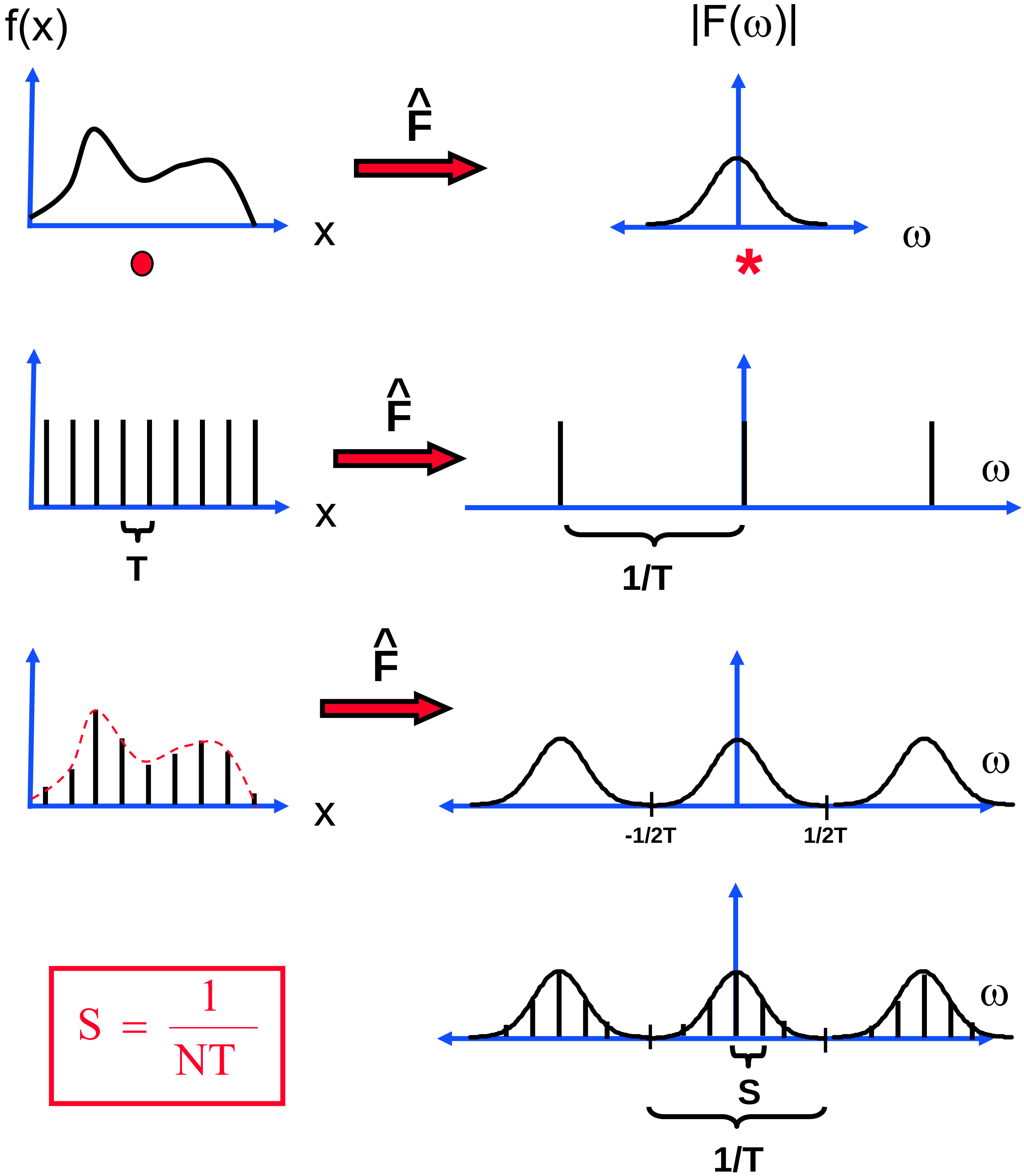
Sampling the Transform

- Sampling a function F(ω) with impulse train of cycle S produces replicas in the image domain with cycle 1/S:



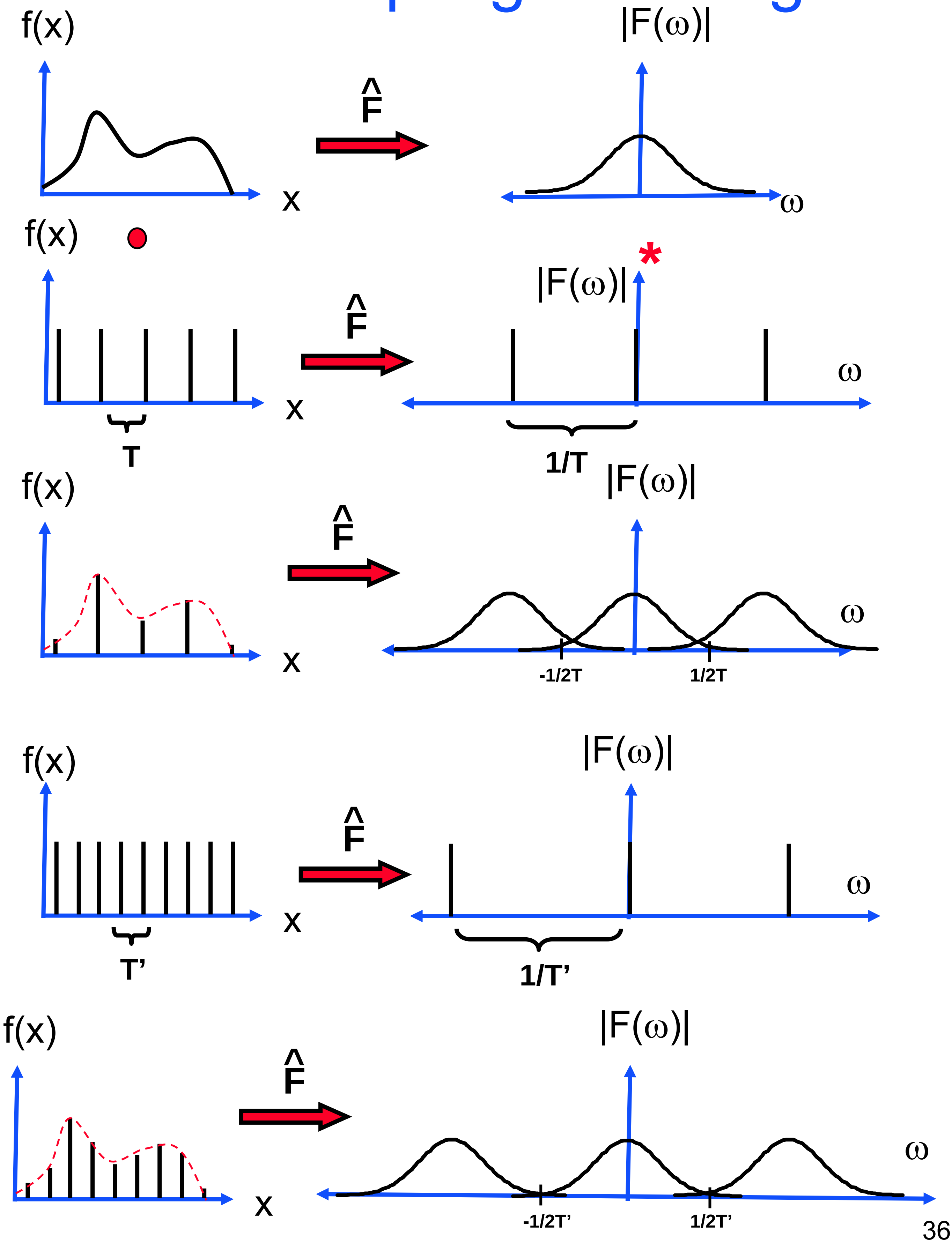
Sampling Image & Transform

- Sampling both f(x) with impulse train of cycle T and F(ω) with impulse train of cycle S:



$$S = \frac{1}{NT}$$

Undersampling the Image



Critical Sampling

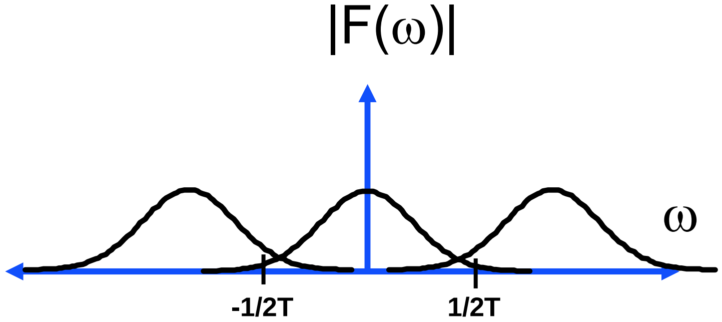
- If the maximal frequency of f(x) is ω_{max} , it is clear from the above replicas that ω_{max} should be smaller that 1/2T.

- Alternatively:

$$\frac{1}{T} > 2\omega_{\max}$$

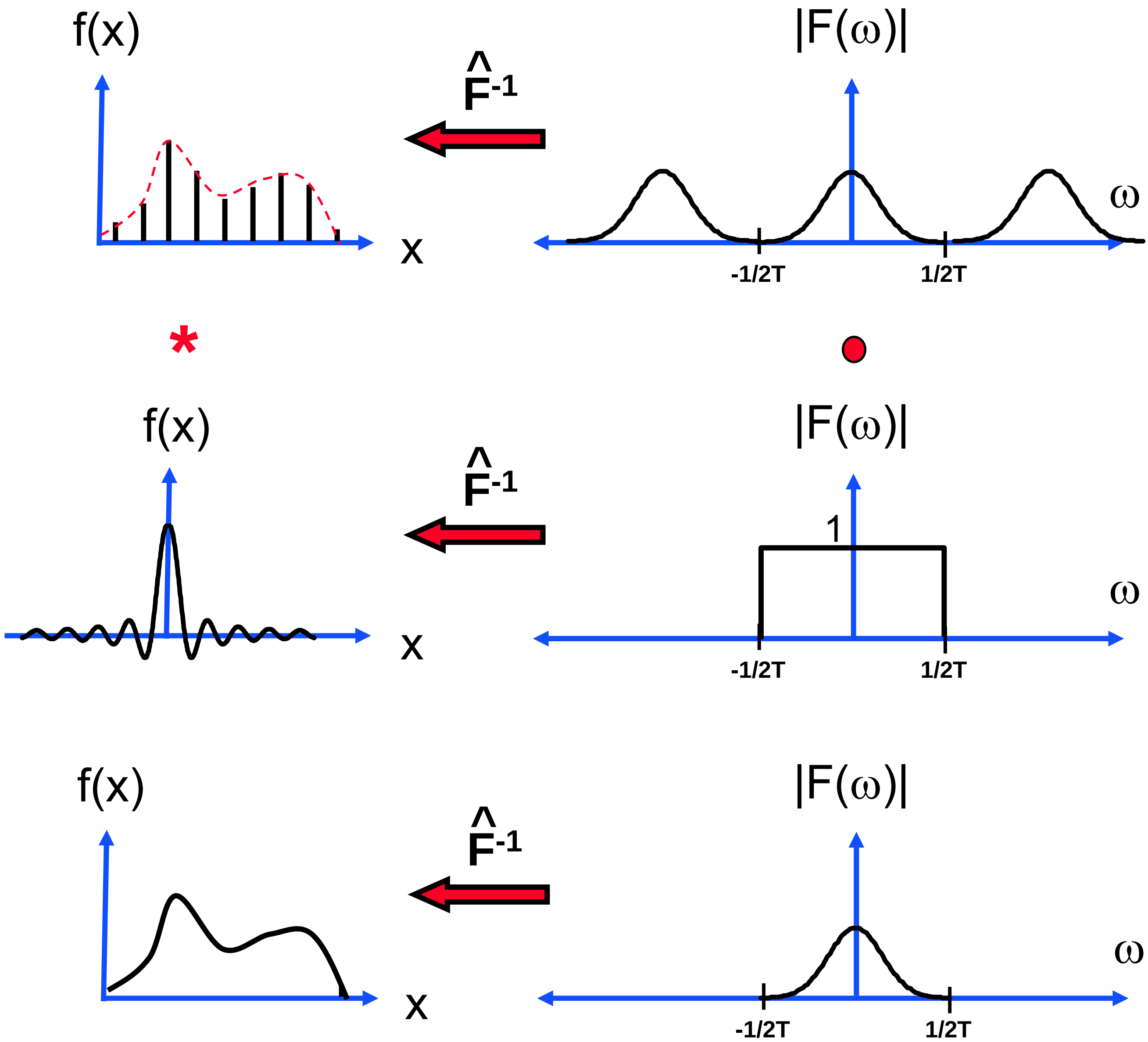
- **Nyquist Theorem**: If the maximal frequency of f(x) is ω_{max} the sampling rate should be larger than $2\omega_{max}$ in order to fully reconstruct f(x) from its samples.

- If the sampling rate is smaller than $2\omega_{max}$ overlapping replicas produce **aliasing**.

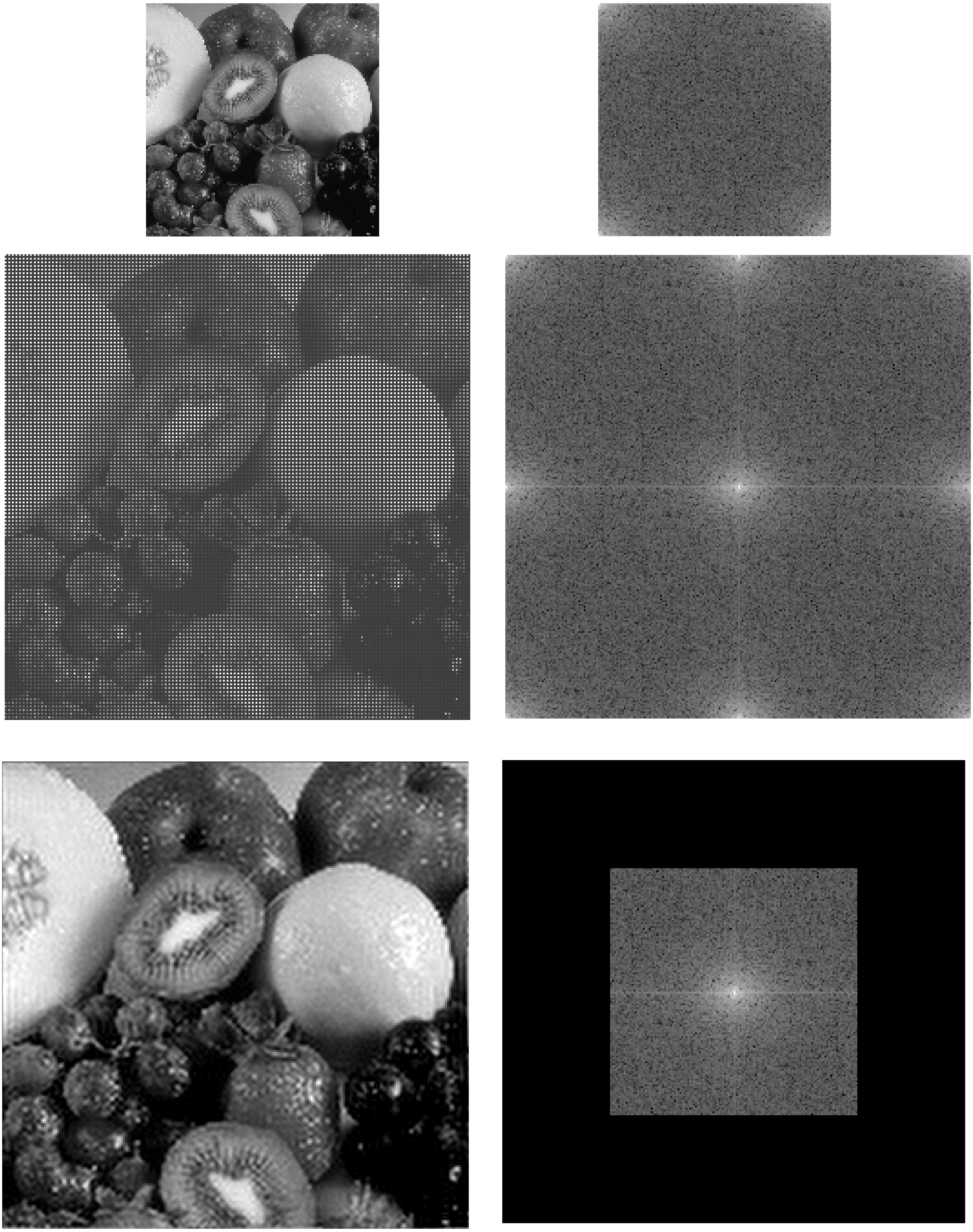


Optimal Interpolation

- It is possible to fully reconstruct f(x) from its samples:



Optimal Interpolation- Example



Optimal Interpolation- Example

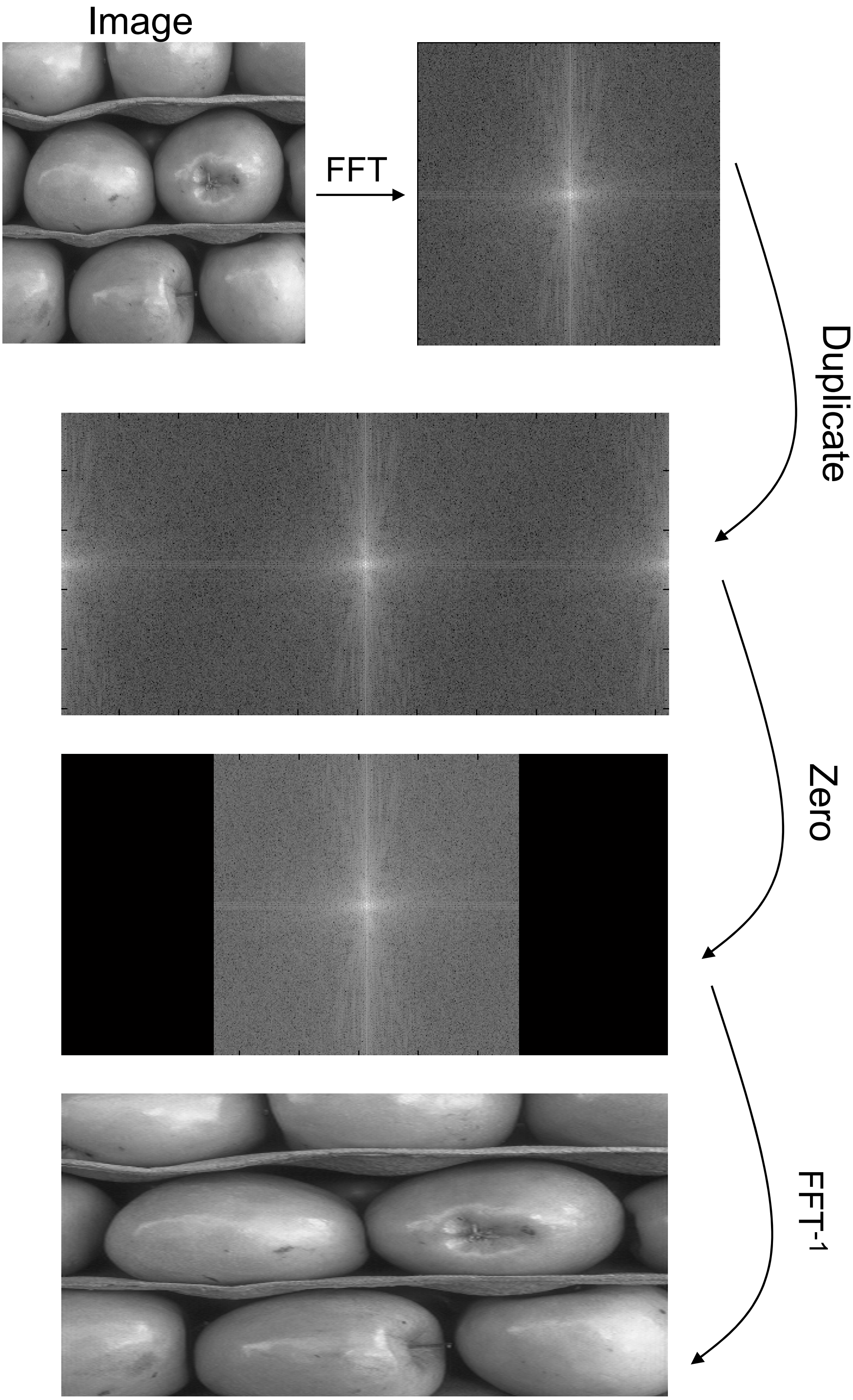
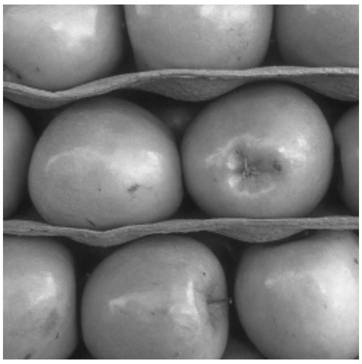
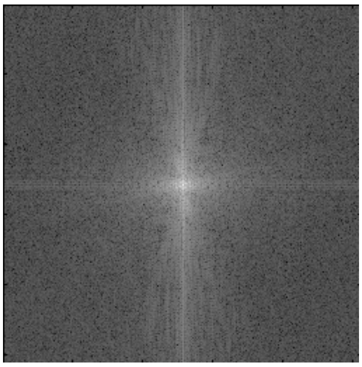
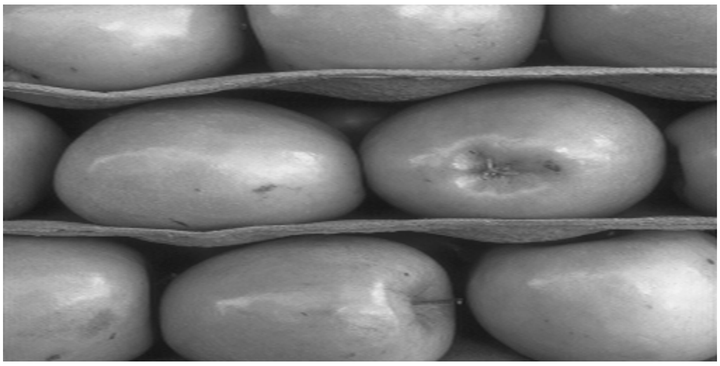
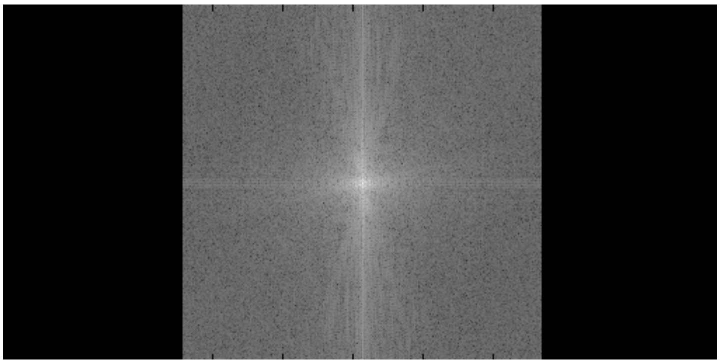


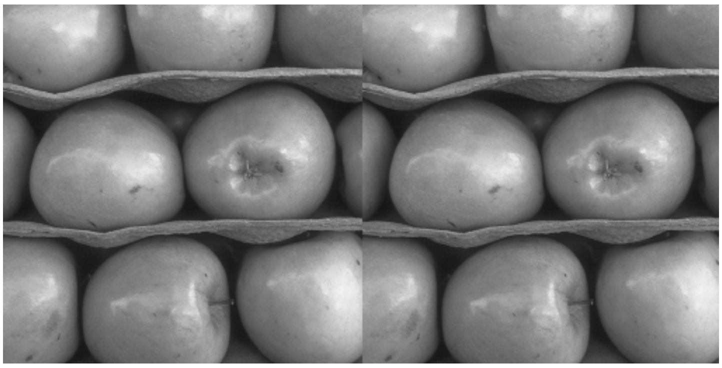
Image Domain | Frequency Domain

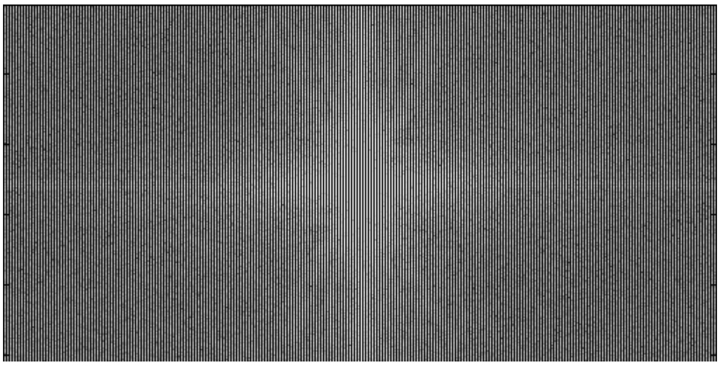












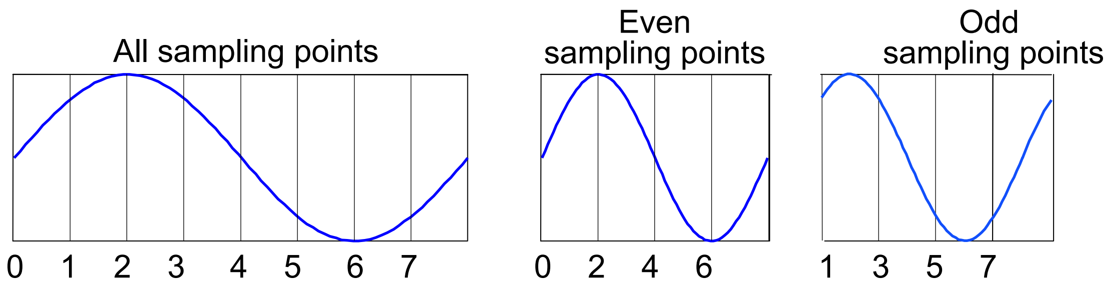
Fast Fourier Transform - FFT

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{\frac{-2\pi i u x}{N}} \qquad u = 0, 1, 2, ..., N-1$$

$O(n^2)$ operations

$$F(u) = \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x) e^{\frac{-2\pi i u 2x}{N}}}_{\text{even } x} + \underbrace{\frac{1}{N} \sum_{x=0}^{N/2-1} f(2x+1) e^{\frac{-2\pi i u (2x+1)}{N}}}_{\text{odd } x}$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x) e^{\frac{-2\pi i u x}{N/2}}}_{\substack{\text{Fourier Transform of} \\ \text{of N/2 even points}}} + e^{\frac{-2\pi i u}{N}} \underbrace{\frac{1}{N/2} \sum_{x=0}^{N/2-1} f(2x+1) e^{\frac{-2\pi i u x}{N/2}}}_{\substack{\text{Fourier Transform of} \\ \text{of N/2 odd points}}} \right]$$



The Fourier transform of N inputs, can be performed as 2 Fourier Transforms of N/2 inputs each + one complex multiplication and addition for each value i.e. O(N).

Note, that only N/2 different transform values are obtained for the N/2 point transforms.

$$F_N(u) = \frac{1}{2}\left[\frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x)e^{\frac{-2\pi iux}{N/2}} + \; e^{\frac{-2\pi iu}{N}} \frac{1}{N/2}\sum_{x=0}^{N/2-1} f(2x+1)e^{\frac{-2\pi iux}{N/2}}\right]$$

$$F_N(u) = \frac{1}{2}\left[F^e_{N/2}(u) + e^{\frac{-2\pi iu}{N}} F^o_{N/2}(u)\right]$$

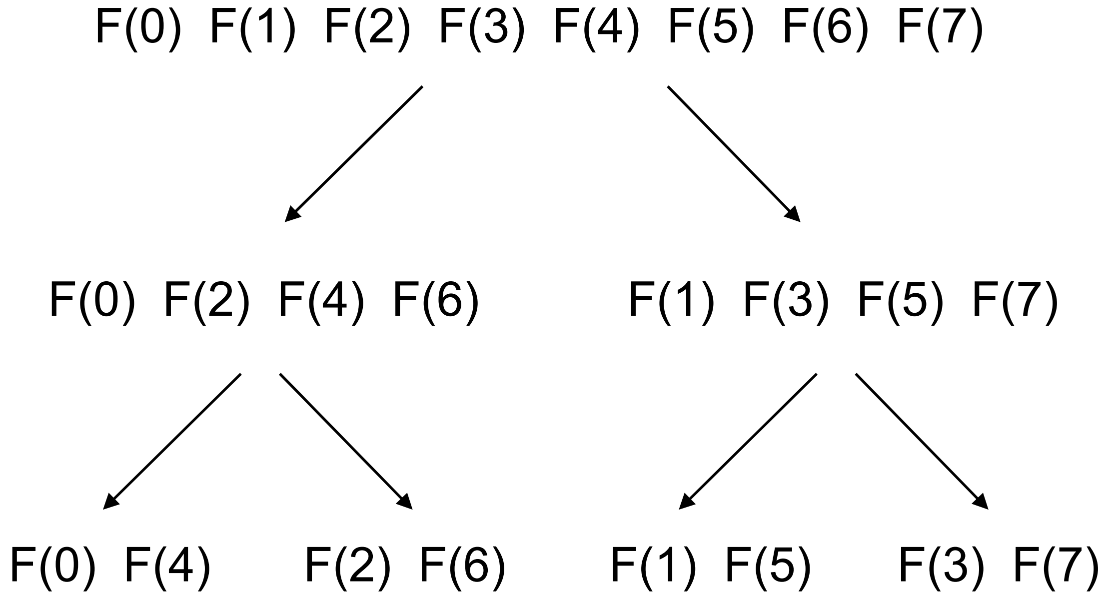
For u' = u+N/2 : $e^{\frac{-2\pi iu'}{N}} = e^{\frac{-2\pi i(u+N/2)}{N}} = e^{\frac{-2\pi iu}{N}} e^{-\pi i} = -e^{\frac{-2\pi iu}{N}}$

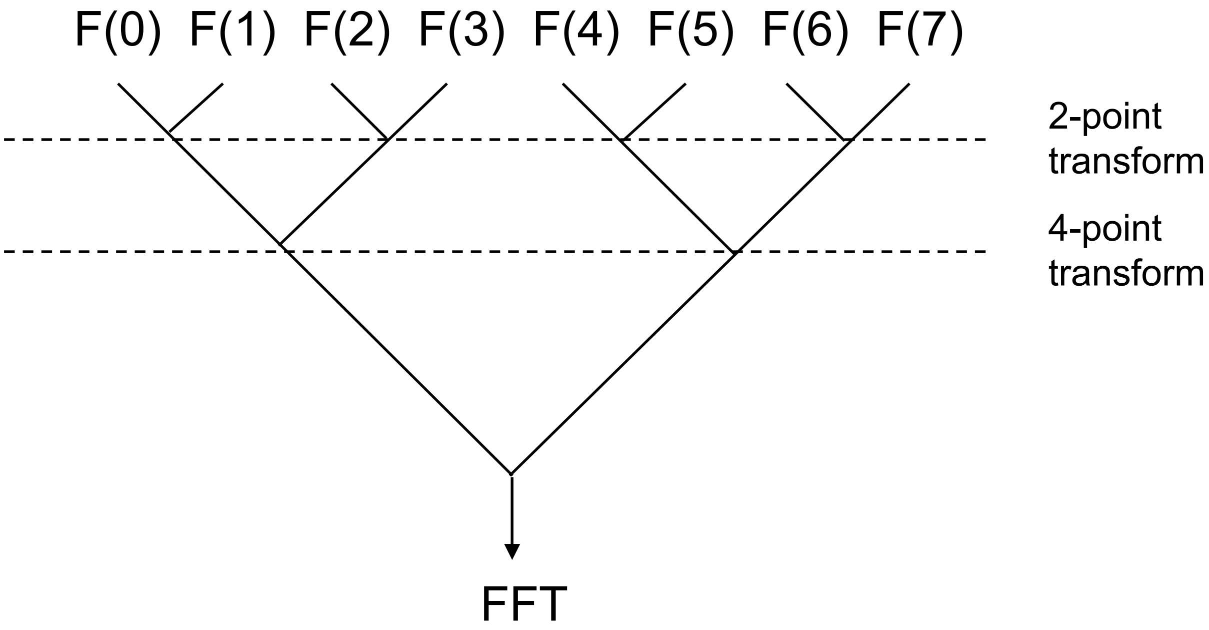
obtain :

$$\left.\begin{aligned} F_N(u) &= \frac{1}{2}\left[F^e_{N/2}(u) + e^{\frac{-2\pi iu}{N}} F^o_{N/2}(u)\right] \\ F_N(u+\tfrac{N}{2}) &= \frac{1}{2}\left[F^e_{N/2}(u) - e^{\frac{-2\pi iu}{N}} F^o_{N/2}(u)\right] \end{aligned}\right\} \text{For } u = 0, 1, 2, \ldots, N/2-1$$

Thus: only one complex multiplication is needed for two terms.

Calculating $F^e_{N/2}(u)$ and $F^o_{N/2}(u)$ is done recursively by calculating $F^e_{N/4}(u)$ and $F^o_{N/4}(u)$.

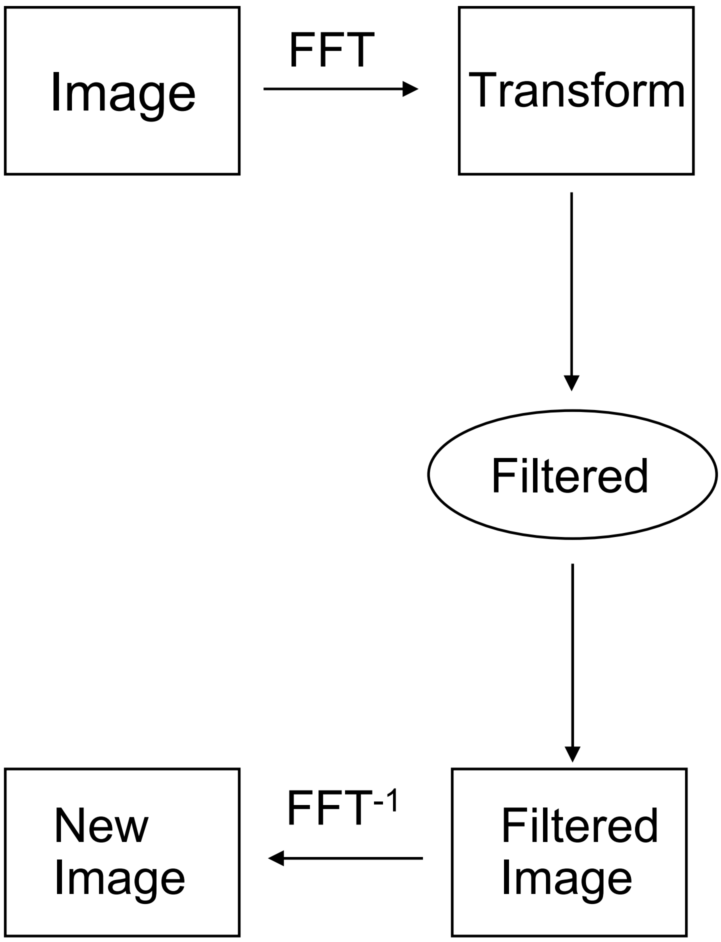




$\boxed{\text{FFT : } \quad O(n\log(n)) \text{ operations}}$

FFT of NxN Image: $O(n^2\log(n))$ operations

Frequency Enhancement



Demo