Math-640: Bayesian Statistics
Assignment 2 – Due Wed. Feb. 17

1. BDA – Chapter 3, exercise 3.

   Estimation from two independent experiments: an experiment was performed on the effects of magnetic fields on the flow of calcium out of chicken brains. Two groups of chickens were involved: a control group of 32 chickens and an exposed group of 36 chickens. One measurement was taken on each chicken, and the purpose of the experiment was to measure the average flow μ_c in untreated (control) chickens and the average flow μ_t in treated chickens. The 32 measurements on the control group had a sample mean of 1.013 and a sample standard deviation of 0.24. The 36 measurements on the treatment group had a sample mean of 1.173 and a sample standard deviation of 0.20.

   (a) Assuming the control measurements were taken at random from a normal distribution with mean μ_c and variance σ_c^2, what is the posterior distribution of μ_c? Similarly, use the treatment group measurements to determine the marginal posterior distribution of μ_t. Assume a uniform prior distribution on $(\mu_c, \mu_t, \log \sigma_c, \log \sigma_t)$.

   (b) What is the posterior distribution for the difference, $\mu_t - \mu_c$? To get this, you may sample from the independent t distributions you obtained in part (a) above. Plot a histogram of your samples and give an approximate 95% posterior interval for $\mu_t - \mu_c$.

   The problem of estimating two normal means with unknown ratio of variances is called the Behrens-Fisher problem.

2. BDA – Chapter 3, exercise 13.

   Multivariate normal model: derive equations (3.13) by completing the square in vector-matrix notation.

   Equation (3.13) states that for a multivariate normal model with known variance such that

$$\begin{aligned} \boldsymbol{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma} &\sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \\ \boldsymbol{\mu} &\sim N(\boldsymbol{\mu}_0, \boldsymbol{\Delta}_0) \end{aligned}$$

   the posterior distribution for $\boldsymbol{\mu}$ is given by

$$p(\boldsymbol{\mu}|\boldsymbol{y}, \boldsymbol{\Sigma}) \sim N(\boldsymbol{\mu}_n, \boldsymbol{\Delta}_n)$$

   where

$$\begin{aligned} \boldsymbol{\mu}_n &= (\boldsymbol{\Delta}_0^{-1} + n\boldsymbol{\Sigma}^{-1})^{-1}(\boldsymbol{\Delta}_0^{-1}\boldsymbol{\mu}_0 + n\boldsymbol{\Sigma}^{-1}\bar{\boldsymbol{y}}) \\ \boldsymbol{\Delta}_n^{-1} &= \boldsymbol{\Delta}_0^{-1} + n\boldsymbol{\Sigma}^{-1}. \end{aligned}$$

3. Suppose $y_1, \ldots, y_n$ are the lifetimes of n light bulbs put on test under similar conditions. Suppose we model these lifetimes as iid exponential random variables with mean $1/\theta$.

   (a) Obtain Jeffreys prior for θ. Is this a proper prior?

   (b) Given that for $n = 10$ bulbs we observe a total lifetime of $\sum_{i=1}^{10} y_i = 1012$ hours, what is the posterior distribution of θ using the Jeffreys prior from (a)?

   (c) Obtain the 90% credible interval and the 90% HPD interval for θ. Plot the posterior density along with both intervals.

   (d) An 11-th bulb is put on test and is observed to fail after 95 hours. What is your new posterior distribution for $\theta, p(\theta|y_1, \ldots, y_{11})$? Show that the following two methods are equivalent for obtaining this posterior:

      i. Combine the information for the 11-th bulb with the other 10 bulbs, i.e., $p(y_1, \ldots, y_{11}|\theta)$, and compute the posterior using the Jeffreys prior for θ from (a).

      ii. Take the prior to be $\pi(\theta) = p(\theta|y_1, \ldots, y_{10})$ and combine with the likelihood function of the 11-th bulb.

   (e) Derive and plot the predictive posterior distribution of a future observation z based on the $n = 11$ observations.

4. The file `school.dat` contain data on the amount of time a random sample of students from a high school spent studying during an exam period. Consider the following model

$$\begin{aligned} y_i|\mu, \sigma^2 &\overset{iid}{\sim} N(\mu, \sigma^2), \qquad i = 1, \ldots, n \\ \mu|\sigma^2 &\sim N(\mu_0, \sigma^2/\kappa_0) \\ \sigma^2 &\sim \text{Inv-Gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0}{2}\sigma_0^2\right) \end{aligned}$$

   You can use the results provided in the textbook and the lecture notes (you don't have to derive the posterior distributions).

   (a) Obtain posterior samples (μ, σ^2) from the joint posterior distribution.

   (b) Provide posterior means and 95% credible intervals for μ and σ^2.

   (c) Using the marginal posterior distribution of μ, provide its posterior mean and 95% credible interval.

   (d) Based on the fact that the posterior predictive distribution for a future observation, $\tilde{y}$, can be written as

$$p(\tilde{y}|y_1, \ldots, y_n) = \int_{\sigma^2} \int_{\mu} f(\tilde{y}|\mu, \sigma^2) \cdot f(\mu, \sigma^2|y_1, \ldots, y_n)\ d\mu\ d\sigma^2$$

   use a two-stage sampling strategy to sample from the posterior predictive distribution.

   (e) Show using a quantile-quantile plot, that the sampling strategy in (d) is equivalent to sampling directly from the posterior predictive distribution, which follows a t-distribution.