Math-640: Bayesian Statistics
Assignment 1 – Due Wed. Feb. 3

1. BDA – Chapter 1, exercise 6.

   Conditional probability: approximately 1/125 of all births are fraternal twins and 1/300 of births are identical twins. Elvis Presley had a twin brother (who died at birth). What is the probability that Elvis was an identical twin? (You may approximate the probability of a boy or girl birth as 1/2).

2. BDA – Chapter 2, exercise 8.

   Normal distribution with unknown mean: a random sample of n students is drawn from a large population, and their weights are measured. The average weight of the n sampled students is $\bar{y} = 150$ pounds. Assume the weights in the population are normally distributed with unknown mean θ and known standard deviation 20 pounds. Suppose your prior distribution for θ is normal with mean 180 and standard deviation 40.

   (a) Give your posterior distribution for θ. (Your answer will be a function of n.)

   (b) A new student is sampled at random from the same population and has a weight of $\tilde{y}$ pounds. Give a posterior predictive distribution for $\tilde{y}$. (Your answer will still be a function of n.)

   (c) For $n = 10$, give a 95% posterior interval for θ and a 95% posterior predictive interval for $\tilde{y}$.

   (d) Do the same for $n = 100$.

3. Suppose $y_1, \ldots, y_n$ is a random sample from a negative binomial distribution with parameters (r, θ), where r is known and θ is unknown. Here, r denotes the number of successes and θ denotes the success probability.

   Suppose $\theta \sim \text{Beta}(\alpha, \beta)$.

   (a) Derive the posterior distribution of θ.

   (b) Show that the posterior mean is a weighted average of the MLE and the prior mean.

   (c) Suppose $r = 5$ and the following $n = 10$ observations are made:

   $$7 \quad 10 \quad 5 \quad 8 \quad 6 \quad 12 \quad 6 \quad 9 \quad 7 \quad 5$$

   For each of the the following priors for θ:
   beta$(1, 1)$, beta$(0.5, 0.5)$, beta$(5, 5)$, beta$(1, 5)$, beta$(5, 1)$,

   i. plot the prior and posterior distributions of θ on the same graph;

   ii. provide the posterior mean and a 95% credible interval for θ.

4. Let $x_1, \ldots, x_n$ be random samples from a Pareto(k, θ) distribution with probability density function

$$f(x|\theta) = \frac{\theta k^\theta}{x^{\theta+1}}, \qquad x > k > 0, \quad \theta > 0$$

(a) For the Pareto sampling model with known k, show that the conjugate prior for inference about θ is a gamma distribution.

(b) Derive the posterior distribution of θ for the conjugate prior Gamma(α, β).

5. Let $x_1, \ldots, x_n$ be random samples from a Gamma(α, θ) distribution

$$f(x|\alpha, \theta) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\theta x), \qquad x > 0, \alpha > 0, \theta > 0$$

where $\alpha = 2$ and θ is unknown. Suppose $n = 10$ and $\bar{x} = 4$.

(a) Derive Jeffreys prior for θ.

(b) Derive the posterior distribution of θ for the prior in (a). Does it lead to a proper posterior distribution?