Math 640 Homework #1

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Exercise 1

Some things to note, identitical twins have the same gender; however fraternal twins can have different genders.

The sets below, are 1 if possible, 0 otherwise. Each case is assumed the have the same weigh in each set.

$(identical) : \{Boy\&Boy|identical, Boy\&Girl|identical, Girl\&Boy|identical, Girl\&Girl|identical\} = \{1, 0, 0, 1\}$

$P(Boy|identical) = \frac{1}{2}$

$(fraternity) : \{Boy\&Boy|fraternity, Boy\&Girl|fraternity, Girl\&Boy|fraternity, Girl\&Girl|fraternity\} = \{1, 1, 1, 1\}$

$P(Boy|fraternity) = \frac{1}{4}$

$P(identical|Boy) = \frac{P(identical\ twin \cap twin \cap Boy)}{P(Boy)} = \frac{P(Boy|identical)*P(identical)}{P(Boy|identical)*P(identical)+P(Boy|fraternity)} =$

$\frac{1/2*1/300}{1/2*1/300+1/4*1/125}$

Exercise 2

Part A

$\bar{y} = 150$ and $\sigma = 20$

$p(y_i|\theta) = \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{1}{2\sigma^2}(y_i - \theta)^2\right\} \quad y = \{y_1, y_2, \ldots y_n\}\, are\ iid \sim N(\theta, \sigma)$

$p(y|\theta) = \prod_{i=1}^{n} p(y_i|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{1}{2\sigma^2}(y_i - \theta)^2\right\} =$

$\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n exp\left\{\sum_{i=1}^{n} -\frac{1}{2\sigma^2}(y_i - \theta)^2\right\} =$

$\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \theta)^2\right\}$

$P(\theta) = \frac{1}{\sqrt{2\pi}r_0} exp\left\{-\frac{1}{2r_0^2}(\mu_0 - \theta)^2\right\}$ where $r_0 = 40$ and $\mu_0 = 180$

$$p(\theta|y) = \frac{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\theta)^2\right\} * \frac{1}{\sqrt{2\pi}r_0} exp\left\{-\frac{1}{2r_0^2}(\mu_0-\theta)^2\right\}}{\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\theta)^2\right\} * \frac{1}{\sqrt{2\pi}r_0} exp\left\{-\frac{1}{2r_0^2}(\mu_0-\theta)^2\right\}\delta\theta} =$$

$$\frac{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n * \frac{1}{\sqrt{2\pi}r_0} exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\theta)^2 - \frac{1}{2r_0^2}(\mu_0-\theta)^2\right\}}{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n * \frac{1}{\sqrt{2\pi}r_0}\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\theta)^2 - \frac{1}{2r_0^2}(\mu_0-\theta)^2\right\}\delta\theta} =$$

$$\frac{exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\left(\sum_{i=1}^n (y_i)^2 - \sum_{i=1}^n 2\theta y_i + n\theta^2\right) + \frac{1}{r_0^2}\left(\mu_0^2 - 2\theta\mu_0 + \theta^2\right)\right)\right\}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\left(\sum_{i=1}^n (y_i)^2 - \sum_{i=1}^n 2\theta y_i + n\theta^2\right) + \frac{1}{r_0^2}\left(\mu_0^2 - 2\theta\mu_0 + \theta^2\right)\right)\right\}\delta\theta} =$$

$$\frac{exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\left(-\sum_{i=1}^n 2\theta y_i + n\theta^2\right) + \frac{1}{r_0^2}\left(-2\theta\mu_0 + \theta^2\right)\right)\right\} * exp\left\{-\frac{1}{2}\left(\frac{\sum_{i=1}^n (y_i)^2}{\sigma^2} + \frac{\mu_0^2}{r_0^2}\right)\right\}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\left(-\sum_{i=1}^n 2\theta y_i + n\theta^2\right) + \frac{1}{r_0^2}\left(-2\theta\mu_0 + \theta^2\right)\right)\right\}\delta\theta * exp\left\{-\frac{1}{2}\left(\frac{\sum_{i=1}^n (y_i)^2}{\sigma^2} + \frac{\mu_0^2}{r_0^2}\right)\right\}} =$$

note: let $-\sum_{i=1}^n 2\theta y_i = 2\theta n\bar{y}$

$$\frac{exp\left\{-\frac{1}{2r_0^2\sigma^2}\left(r_0^2\left(-2\theta n\bar{y} + n\theta^2\right) + \sigma^2\left(-2\theta\mu_0 + \theta^2\right)\right)\right\}}{exp\left\{-\frac{1}{2r_0^2\sigma^2}\left(r_0^2(-2\theta n\bar{y} + n\theta^2) + \sigma^2(-2\theta\mu_0 + \theta^2)\right)\right\}} =$$

$$\frac{exp\left\{-\frac{1}{2r_0^2\sigma^2}\left(-2\theta n\bar{y} * r_0^2 + n\theta^2 * r_0^2 - 2\theta\mu_0 * \sigma^2 + \theta^2 * \sigma^2\right)\right\}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2r_0^2\sigma^2}\left(-2\theta n\bar{y} * r_0^2 + n\theta^2 * r_0^2 - 2\theta\mu_0 * \sigma^2 + \theta^2 * \sigma^2\right)\right\}\delta\theta} =$$

$$\frac{exp\left\{-\frac{1}{2r_0^2\sigma^2}\left(\theta^2 * \sigma^2 + n\theta^2 * r_0^2 - 2\theta n\bar{y} * r_0^2 - 2\theta\mu_0 * \sigma^2\right)\right\}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2r_0^2\sigma^2}\left(\theta^2 * \sigma^2 + n\theta^2 * r_0^2 - 2\theta n\bar{y} * r_0^2 - 2\theta\mu_0 * \sigma^2\right)\right\}\delta\theta} =$$

$$\frac{exp\left\{-\frac{1}{2r_0^2\sigma^2}\left(\theta^2 * \left(\sigma^2 + n * r_0^2\right) - 2\theta\left(n\bar{y} * r_0^2 + \mu_0 * \sigma^2\right)\right)\right\}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2r_0^2\sigma^2}\left(\theta^2 * \left(\sigma^2 + n * r_0^2\right) - 2\theta\left(n\bar{y} * r_0^2 + \mu_0 * \sigma^2\right)\right)\right\}\delta\theta} =$$

let's substitue $x = \left(\sigma^2 + n * r_0^2\right)$ and $y = \left(n\bar{y} * r_0^2 + \mu_0 * \sigma^2\right)$

$$\frac{exp\left\{-\frac{1}{2r_0^2\sigma^2}\left(\theta^2 * x - 2\theta y\right) * \frac{1/x}{1/x}\right\}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2r_0^2\sigma^2}(\theta^2 * x - 2\theta y) * \frac{1/x}{1/x}\right\}\delta\theta} = \frac{exp\left\{-\frac{1}{2r_0^2\sigma^2 * \frac{1}{x}}\left(\theta^2 - 2\theta y/x\right)\right\}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2r_0^2\sigma^2 * \frac{1}{x}}(\theta^2 - 2\theta y/x)\right\}\delta\theta} =$$

$$\frac{exp\left\{-\frac{1}{2r_0^2\sigma^2 * \frac{1}{x}}\left(\theta^2 - 2\theta y/x\right)\right\}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2r_0^2\sigma^2 * \frac{1}{x}}(\theta^2 - 2\theta y/x)\right\}\delta\theta} * \frac{exp\left\{-\frac{1}{2r_0^2\sigma^2 * \frac{1}{x}}\left((y/x)^2\right)\right\}}{exp\left\{-\frac{1}{2r_0^2\sigma^2 * \frac{1}{x}}\left((y/x)^2\right)\right\}} =$$

$$\frac{exp\left\{-\frac{1}{2r_0^2\sigma^2 * \frac{1}{x}}(\theta - y/x)^2\right\}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2r_0^2\sigma^2 * \frac{1}{x}}(\theta - y/x)^2\right\}\delta\theta} * \frac{\left(\sqrt{2\pi r_0^2\sigma^2 * \frac{1}{x}}\right)^{-1}}{\left(\sqrt{2\pi r_0^2\sigma^2 * \frac{1}{x}}\right)^{-1}}$$

$$\left(\sqrt{2\pi r_0^2\sigma^2 * \frac{1}{x}}\right)^{-1} * exp\left\{-\frac{1}{2r_0^2\sigma^2 * \frac{1}{x}}(\theta - y/x)^2\right\} \sim N(y/x, r_0^2\sigma^2/x)$$

$$\theta|y \sim N(y/x, r_0^2\sigma^2 x) \equiv N\left(\frac{\left(n\bar{y} * r_0^2 + \mu_0 * \sigma^2\right)}{\left(\sigma^2 + n * r_0^2\right)}, r_0^2\sigma^2/\left(\sigma^2 + n * r_0^2\right)\right)$$

After resubstituting everything back in

$\theta|y \sim N\left(\frac{n*150*40^2+180*20^2}{20^2+10*40^2}, \frac{40^2 20^2}{20^2+n*40^2}\right)$

Part B

$\theta|y \sim N\left(\frac{n*150*40^2+180*20^2}{20^2+10*40^2}, \frac{40^2 20^2}{20^2+n*40^2}\right)$

please note: To simplify $p(\theta|y)$, the book's notation will be used, thus:

$\theta|y \sim N\left(\mu_1, r_1^2\right)$

$p(\tilde{y}|\theta) = \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{1}{2\sigma^2}\left(\tilde{y}-\theta\right)^2\right\}$

$p(\tilde{y}|y) = \int_{-\infty}^{\infty} p(\tilde{y}|\theta) * p(\theta|y)\delta\theta =$

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{1}{2\sigma^2}\left(\tilde{y}-\theta\right)^2\right\} * \frac{1}{\sqrt{2\pi}r_1} * exp\left\{-\frac{1}{2r_1^2}\left(\theta-\mu_1\right)^2\right\}\delta\theta =$

$\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma r_1} exp\left\{-\frac{1}{2\sigma^2}\left(\tilde{y}-\theta\right)^2 - \frac{1}{2r_1^2}\left(\theta-\mu_1\right)^2\right\}\delta\theta =$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\left(\tilde{y}-\theta\right)^2 + \frac{1}{r_1^2}\left(\theta-\mu_1\right)^2\right)\right\}\delta\theta =$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\left(\tilde{y}^2-2\tilde{y}\theta+\theta^2\right) + \frac{1}{r_1^2}\left(\theta^2-2\theta\mu_1+\mu_1^2\right)\right)\right\}\delta\theta =$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\frac{\left(r_1^2\left(\tilde{y}^2-2\tilde{y}\theta+\theta^2\right)+\sigma^2\left(\theta^2-2\theta\mu_1+\mu_1^2\right)\right)}{\sigma^2 r_1^2}\right\}\delta\theta =$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\frac{\left(r_1^2\tilde{y}^2-r_1^2 2\tilde{y}\theta+r_1^2\theta^2+\sigma^2\theta^2-\sigma^2 2\theta\mu_1+\sigma^2\mu_1^2\right)}{\sigma^2 r_1^2}\right\}\delta\theta =$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\frac{r_1^2\theta^2+\sigma^2\theta^2-r_1^2 2\tilde{y}\theta-\sigma^2 2\theta\mu_1+r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2}\right\}\delta\theta =$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\frac{r_1^2\theta^2+\sigma^2\theta^2-r_1^2 2\tilde{y}\theta-\sigma^2 2\theta\mu_1}{\sigma^2 r_1^2}\right\} * exp\left\{-\frac{1}{2}\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2}\right\}\delta\theta =$

$\frac{1}{\sqrt{2\pi}} * exp\left\{-\frac{1}{2}\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\frac{\left(r_1^2+\sigma^2\right)\theta^2-2\theta\left(r_1^2\tilde{y}+\sigma^2\mu_1\right)}{\sigma^2 r_1^2}\right\}\delta\theta =$

As this stage, addition substitution is needed $\therefore x = r_1^2+\sigma^2 \; and \; y = r_1^2\tilde{y}+\sigma^2\mu_1$

$\frac{1}{\sqrt{2\pi}} * exp\left\{-\frac{1}{2}\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\frac{x\theta^2-2\theta y}{\sigma^2 r_1^2} * \frac{x^{-1}}{x^{-1}}\right\}\delta\theta =$

$\frac{1}{\sqrt{2\pi}} * exp\left\{-\frac{1}{2}\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2}\right\} * exp\left\{\frac{1}{2} * \frac{\left(yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\frac{\theta^2-2\theta yx^{-1}}{\sigma^2 r_1^2 x^{-1}}\right\} * exp\left\{-\frac{1}{2} * \frac{\left(yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\}\delta\theta =$

$$\frac{1}{\sqrt{2\pi}} * exp\left\{-\frac{1}{2}\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2} + \frac{1}{2} * \frac{\left(yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1} exp\left\{-\frac{1}{2}\frac{\theta^2-2\theta yx^{-1}+\left(yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\}\delta\theta =$$

$$x^{-1}\frac{1}{\sqrt{2\pi}} * exp\left\{-\frac{1}{2}\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2} + \frac{1}{2} * \frac{\left(yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1 * x^{-1}} exp\left\{-\frac{1}{2}\frac{\left(\theta-yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\}\delta\theta =$$

The integral $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma r_1 * x^{-1}} exp\left\{-\frac{1}{2}\frac{\left(\theta-yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\}\delta\theta$ can be recognized as a PDF$\sim N(yx^{-1}, \sigma^2 r_1^2\sqrt{x^{-1}})$ $\therefore$the value is 1

$$\frac{1}{\sqrt{2\pi x}} * exp\left\{-\frac{1}{2}\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2} + \frac{1}{2} * \frac{\left(yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\} * 1 =$$

$$\frac{1}{\sqrt{2\pi x}} * exp\left\{-\frac{1}{2}\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2} + \frac{1}{2} * \frac{\left(yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\} =$$

$$\frac{1}{\sqrt{2\pi x}} * exp\left\{-\frac{1}{2}\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2} + \frac{1}{2} * \frac{\left(yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right\} =$$

$$\frac{1}{\sqrt{2\pi x}} * exp\left\{-\frac{1}{2}\left[\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2} - \frac{\left(yx^{-1}\right)^2}{\sigma^2 r_1^2 x^{-1}}\right]\right\} =$$

$$\frac{1}{\sqrt{2\pi x}} * exp\left\{-\frac{1}{2}\left[\frac{r_1^2\tilde{y}^2+\sigma^2\mu_1^2}{\sigma^2 r_1^2} - \frac{x*\left(yx^{-1}\right)^2}{\sigma^2 r_1^2}\right]\right\} =$$

$$\frac{1}{\sqrt{2\pi x}} * exp\left\{-\frac{1}{2\sigma^2 r_1^2}\left[r_1^2\tilde{y}^2 + \sigma^2\mu_1^2 - x^{-1} * y^2\right]\right\} =$$

At this point, I'm replacing the variables x and y

$$\frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}} * exp\left\{-\frac{1}{2\sigma^2 r_1^2}\left[r_1^2\tilde{y}^2 + \sigma^2\mu_1^2 - \left(r_1^2+\sigma^2\right)^{-1} * \left(r_1^2\widetilde{y} + \sigma^2\mu_1\right)^2\right]\right\} =$$

$$\frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}}*exp\left\{-\frac{1}{2\sigma^2 r_1^2}\left[r_1^2\tilde{y}^2 + \sigma^2\mu_1^2 - \left(r_1^2+\sigma^2\right)^{-1} * \left(\left(r_1^2\widetilde{y}\right)^2 + 2 * r_1^2\widetilde{y} * \sigma^2\mu_1 + \left(\sigma^2\mu_1\right)^2\right)\right]\right\} = \frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}}*$$
$$exp\left\{-\frac{1}{2\sigma^2 r_1^2}\left[\left(r_1^2\tilde{y}^2 + \sigma^2\mu_1^2\right) * \frac{r_1^2+\sigma^2}{r_1^2+\sigma^2} - \left(r_1^2+\sigma^2\right)^{-1} * \left(\left(r_1^2\widetilde{y}\right)^2 + 2 * r_1^2\widetilde{y} * \sigma^2\mu_1 + \left(\sigma^2\mu_1\right)^2\right)\right]\right\} =$$

$$\frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}} * exp\left\{-\frac{1}{2\sigma^2 r_1^2 * \left(r_1^2+\sigma^2\right)}\left[\left(r_1^2\tilde{y}^2 + \sigma^2\mu_1^2\right) * \left(r_1^2+\sigma^2\right) - \left(\left(r_1^2\widetilde{y}\right)^2 + 2 * r_1^2\widetilde{y} * \sigma^2\mu_1 + \left(\sigma^2\mu_1\right)^2\right)\right]\right\} =$$

$$\frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}} * exp\left\{-\frac{1}{2\sigma^2 r_1^2 * \left(r_1^2+\sigma^2\right)}\left[\left(r_1^2\tilde{y}^2 + \sigma^2\mu_1^2\right) * \left(r_1^2+\sigma^2\right) - \left(\left(r_1^2\widetilde{y}\right)^2 + 2 * r_1^2\widetilde{y} * \sigma^2\mu_1 + \left(\sigma^2\mu_1\right)^2\right)\right]\right\} =$$

$$\frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}}*exp\left\{-\frac{1}{2\sigma^2 r_1^2 * \left(r_1^2+\sigma^2\right)}\left[\left(r_1^2\tilde{y}^2 * r_1^2 + \sigma^2\mu_1^2 * r_1^2 + r_1^2\tilde{y}^2 * \sigma^2 + \sigma^2\mu_1^2 * \sigma^2\right) - \left(\left(r_1^2\widetilde{y}\right)^2 + 2 * r_1^2\widetilde{y} * \sigma^2\mu_1 + \left(\sigma^2\mu_1\right)^2\right)\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}} * exp\left\{-\frac{1}{2\sigma^2 r_1^2 * \left(r_1^2+\sigma^2\right)}\left[\left(\sigma^2\mu_1^2 * r_1^2 + r_1^2\tilde{y}^2 * \sigma^2\right) - \left(2 * r_1^2\widetilde{y} * \sigma^2\mu_1\right)\right]\right\} =$$

$$\frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}} * exp\left\{-\frac{1}{2\sigma^2 r_1^2 * \left(r_1^2+\sigma^2\right)}\left[\sigma^2 * r_1^2\left(\mu_1^2 - 2 * \widetilde{y} * \mu_1 + \tilde{y}^2\right)\right]\right\} =$$

$\frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}} * exp\left\{-\frac{1}{2\left(r_1^2+\sigma^2\right)}\left[\left(\mu_1-\tilde{y}\right)^2\right]\right\} \equiv \frac{1}{\sqrt{2\pi\left(r_1^2+\sigma^2\right)}} * exp\left\{-\frac{1}{2\left(r_1^2+\sigma^2\right)}\left[\left(\tilde{y}-\mu_1\right)^2\right]\right\}$ (This is the format of a normal distribution's PDF) $\therefore \tilde{y}|y \sim N(\mu_1, r_1^2+\sigma^2) \equiv N\left(\frac{n*150*40^2+180*20^2}{20^2+10*40^2}, \frac{40^2 20^2}{20^2+n*40^2}+20^2\right)$

Part C

Please note, for 2C and 2D, I used the equations from the book to derive the mean and standard deviation; however, the numbers match with respect to my results from 2A and 2B.

```
n = 10
theta = c((1/(40^2)*180+n/(20^2)*150)/(1/(40^2)+n/(20^2)), 1/(1/40^2+n/20^2))
yhat = c((1/(40^2)*180+n/(20^2)*150)/(1/(40^2)+n/(20^2)), 1/(1/40^2+n/20^2)+20^2)

qnorm(c(0.025, 0.975), mean = theta[1], sd = sqrt(theta[2]))
qnorm(c(0.025, 0.975), mean = yhat[1], sd = sqrt(yhat[2]))
```

The 95% posterior interval for θ is $(138.4879, 162.9755)$ and the 95% posterior predictive interval for $\tilde{y}$ is $(109.6648, 191.7987)$

Part D

```
n = 100
theta = c((1/(40^2)*180+n/(20^2)*150)/(1/(40^2)+n/(20^2)), 1/(1/40^2+n/20^2))
yhat = c((1/(40^2)*180+n/(20^2)*150)/(1/(40^2)+n/(20^2)), 1/(1/40^2+n/20^2)+20^2)

qnorm(c(0.025, 0.975), mean = theta[1], sd = sqrt(theta[2]))
qnorm(c(0.025, 0.975), mean = yhat[1], sd = sqrt(yhat[2]))
```

The 95% posterior interval for θ is $(146.1598, 153.9899)$ and the 95% posterior predictive interval for $\tilde{y}$ is $(110.6805, 189.4691)$

Exercise 3

Part A

$p(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$

$p(y_i|\theta) = \begin{pmatrix} y_i+r-1 \\ y_i \end{pmatrix}\theta^r(1-\theta)^{y_i}$ Note: Since r is known, I don't consider it as a "given" parameter Since $y = \{y_i, y_2 \ldots, y_n\} \sim iid\ Negative\ binomial(r, \theta)$

$p(y|\theta) = \prod_{i=1}^n \begin{pmatrix} y_i+r-1 \\ y_i \end{pmatrix}\theta^r(1-\theta)^{y_i} = \theta^{rn} * (1-\theta)^{\sum_{i-1}^n y_i} * \prod_{i=1}^n \begin{pmatrix} y_i+r-1 \\ y_i \end{pmatrix}$

$$p(\theta|y) = \frac{p(y|\theta)*p(\theta)}{p(y)} = \frac{\theta^{rn}*(1-\theta)^{\sum_{i-1}^n y_i}*\prod_{i=1}^n \begin{pmatrix} y_i+r-1 \\ y_i \end{pmatrix}*\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\int_0^1 \theta^{rn}*(1-\theta)^{\sum_{i-1}^n y_i}*\prod_{i=1}^n \begin{pmatrix} y_i+r-1 \\ y_i \end{pmatrix}*\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\delta\theta} =$$

$$\frac{\theta^{rn+\alpha-1} * (1-\theta)^{\left(\sum_{i-1}^{n} y_i\right)+\beta-1} * \prod_{i=1}^{n} \begin{pmatrix} y_i + r - 1 \\ y_i \end{pmatrix} * \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}{\prod_{i=1}^{n} \begin{pmatrix} y_i + r - 1 \\ y_i \end{pmatrix} * \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{rn+\alpha-1} * (1-\theta)^{\sum_{i-1}^{n} y_i+\beta-1} \delta\theta} =$$

$$\frac{\theta^{rn+\alpha-1} * (1-\theta)^{\sum_{i-1}^{n} y_i+\beta-1}}{\int_0^1 \theta^{rn+\alpha-1} * (1-\theta)^{\sum_{i-1}^{n} y_i+\beta-1} \delta\theta} =$$

$$\frac{\theta^{rn+\alpha-1} * (1-\theta)^{\sum_{i-1}^{n} y_i+\beta-1}}{\frac{\Gamma(rn+\alpha)\Gamma(\sum_{i-1}^{n} y_i+\beta)}{\Gamma\left(rn+\alpha+\sum_{i-1}^{n} y_i+\beta\right)} \int_0^1 \frac{\Gamma\left(rn+\alpha+\sum_{i-1}^{n} y_i+\beta\right)}{\Gamma(rn+\alpha)\Gamma(\sum_{i-1}^{n} y_i+\beta)} * \theta^{rn+\alpha-1} * (1-\theta)^{\sum_{i-1}^{n} y_i+\beta-1} \delta\theta} =$$

Note:$\int_0^1 \frac{\Gamma\left(rn+\alpha+\sum_{i-1}^{n} y_i+\beta\right)}{\Gamma(rn+\alpha)\Gamma(\sum_{i-1}^{n} y_i+\beta)} * \theta^{rn+\alpha-1} * (1-\theta)^{\sum_{i-1}^{n} y_i+\beta-1} \delta\theta = 1$ because this is the PDF of $Beta(rn + \alpha, \sum_{i-1}^{n} y_i + \beta)$

$$\frac{\theta^{rn+\alpha-1} * (1-\theta)^{\sum_{i-1}^{n} y_i+\beta-1}}{\frac{\Gamma(rn+\alpha)\Gamma(\sum_{i-1}^{n} y_i+\beta)}{\Gamma\left(rn+\alpha+\sum_{i-1}^{n} y_i+\beta\right)} * 1} =$$

$$\frac{\Gamma\left(rn+\alpha+\sum_{i-1}^{n} y_i+\beta\right)}{\Gamma(rn+\alpha)\Gamma(\sum_{i-1}^{n} y_i+\beta)} * \theta^{rn+\alpha-1} * (1-\theta)^{\sum_{i-1}^{n} y_i+\beta-1}$$

$\theta|y \sim\sim Beta(rn + \alpha, \sum_{i-1}^{n} y_i + \beta)$

Part B

posterior mean

Since $\theta|y \sim Beta(rn + \alpha, \sum_{i-1}^{n} y_i + \beta) \equiv Beta(rn + \alpha, n\bar{y} + \beta)$

$E(\theta|y) = \frac{rn+\alpha}{rn+\alpha+\sum_{i-1}^{n} y_i+\beta} = \frac{rn+\alpha}{rn+\alpha+n\bar{y}+\beta}$

MLE

$$L(\theta) = \prod_{i=1}^{n} \begin{pmatrix} y_i + r - 1 \\ y_i \end{pmatrix} \theta^r (1-\theta)^{y_i}$$

$$log(L(\theta)) = log\left(\prod_{i=1}^{n} \begin{pmatrix} y_i + r - 1 \\ y_i \end{pmatrix} \theta^r (1-\theta)^{y_i}\right) =$$

$$\sum_{i=1}^{n} log\left(\begin{pmatrix} y_i + r - 1 \\ y_i \end{pmatrix} \theta^r (1-\theta)^{y_i}\right) = \sum_{i=1}^{n} log\left(\begin{pmatrix} y_i + r - 1 \\ y_i \end{pmatrix}\right) + \sum_{i=1}^{n} log\left(\theta^r\right) + \sum_{i=1}^{n} log\left((1-\theta)^{y_i}\right) =$$

$$\sum_{i=1}^{n} log\left(\begin{pmatrix} y_i + r - 1 \\ y_i \end{pmatrix}\right) + \sum_{i=1}^{n} r * log\left(\theta\right) + \sum_{i=1}^{n} y_i log\left((1-\theta)\right) =$$

note: $\sum_{i=1}^{n} y_i = n\bar{y}$ since $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$$\sum_{i=1}^{n} log\left(\begin{pmatrix} y_i + r - 1 \\ y_i \end{pmatrix}\right) + n * r * log\left(\theta\right) + n\bar{y} log\left((1-\theta)\right)$$

$\delta/d\theta(log(L(\theta|y))) = 0 + n * r * \frac{1}{\theta} + n\bar{y} * (-1) * \frac{1}{1-\theta} =$

$\frac{nr}{\theta} - \frac{n\bar{y}}{1-\theta} = \frac{nr}{\theta} + \frac{n\bar{y}}{\theta-1} = 0 \implies nr(\theta - 1) + n\bar{y} * \theta = (nr + n\bar{y})\,\theta - nr = 0 \implies (nr + n\bar{y})\,\theta = nr$

$\therefore \hat{\theta} = \frac{nr}{nr+n\bar{y}} = \frac{r}{r+\bar{y}}$

To confirm this is a maximum critical point, the second derivative of the log likelihood must be taken

$\delta^2/d\theta(log(L(\theta))) = \delta^2/d\theta\left(\frac{nr}{\theta} - \frac{n\bar{y}}{1-\theta}\right) =$

$\delta^2/d\theta\left(nr * \theta^{-1} - n\bar{y} * (1-\theta)^{-1}\right) =$

$-nr * \theta^{-2} - n\bar{y} * (-1) * (-1) * (1-\theta)^{-2} =$

$-nr * \theta^{-2} - n\bar{y} * (1-\theta)^{-2}$

By substituting $\hat{\theta}$ into the second derivative of the log likelihood, we get

$-nr * \left(\frac{r}{r+\bar{y}}\right)^{-2} - n\bar{y} * \left(1 - \left(\frac{r}{r+\bar{y}}\right)\right)^{-2}$

Thus the value is negative, which indicates that $\hat{\theta}$ is a maximum.

weighted average of the MLE and the prior mean

note the prior mean is $\frac{\alpha}{\alpha+\beta}$

$\frac{r}{r+\bar{y}} * \omega + (1-\omega) * \frac{\alpha}{\alpha+\beta} = \frac{rn+\alpha}{rn+\alpha+n\bar{y}+\beta}$

$\omega = \frac{n(r+\bar{y})}{rn+\alpha+n\bar{y}+\beta}$ Note: This was choosen in order to cancel out the MLE's denomintor and multiply the numerator by n

$1 - \omega = 1 - \frac{n(r+\bar{y})}{rn+\alpha+n\bar{y}+\beta} = \frac{rn+\alpha+n\bar{y}+\beta}{rn+\alpha+n\bar{y}+\beta} - \frac{n(r+\bar{y})}{rn+\alpha+n\bar{y}+\beta} =$

$\frac{rn+\alpha+n\bar{y}+\beta}{rn+\alpha+n\bar{y}+\beta} - \frac{(nr+n\bar{y})}{rn+\alpha+n\bar{y}+\beta} = \frac{\alpha+\beta}{rn+\alpha+n\bar{y}+\beta}$

$\therefore \frac{r}{r+\bar{y}} * \omega + (1-\omega) * \frac{\alpha}{\alpha+\beta} =$

$\frac{r}{r+\bar{y}} * \frac{n(r+\bar{y})}{rn+\alpha+n\bar{y}+\beta} + \frac{\alpha+\beta}{rn+\alpha+n\bar{y}+\beta} * \frac{\alpha}{\alpha+\beta} =$

$r * \frac{n}{rn+\alpha+n\bar{y}+\beta} + \frac{1}{rn+\alpha+n\bar{y}+\beta} * \alpha =$

$\frac{rn+\alpha}{rn+\alpha+n\bar{y}+\beta}$

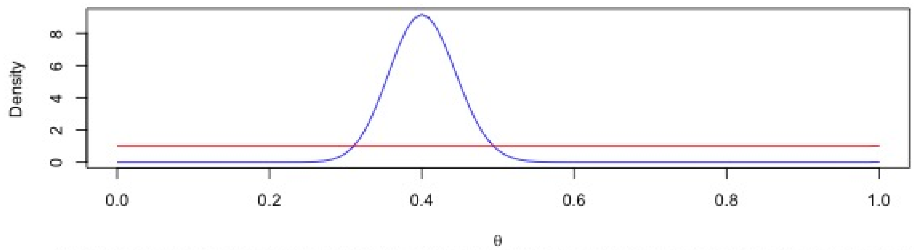
Part C

The following is the R code used.

```
r = 5
y = c(7,10,5,8,6,12,6,9,7,5)
alpha = c(1,.5,5,1,5)
beta = c(1,.5,5,5,1)
par(mfrow = c(3,1))
for(i in 1:length(alpha)){
        theta = seq(0,1,0.01)
        credible_interval = qbeta(c(0.025,0.975), r*length(Y)+alpha[i], sum(Y)+beta[i])
        posterior_mean = (r*length(Y)+alpha[i])/(r*length(Y)+alpha[i]+sum(Y)+beta[i])
        plot(theta,  dbeta(theta, r*length(Y)+alpha[i], sum(Y)+beta[i]),
                type = "l", col = "blue", xlab = expression(theta),
                ylab = "Density",
                main = paste("alpha = ", alpha[i],"and", "beta = ", beta[i],
                "- blue is posterior and red is prior"),
                sub = paste("The Credible Interval is", "(",credible_interval[1],",",
credible_interval[2], ")",
                "and the posterior mean is", posterior_mean))
  lines(theta, dbeta(theta, alpha[i], beta[i]), col = "red")
        }
```

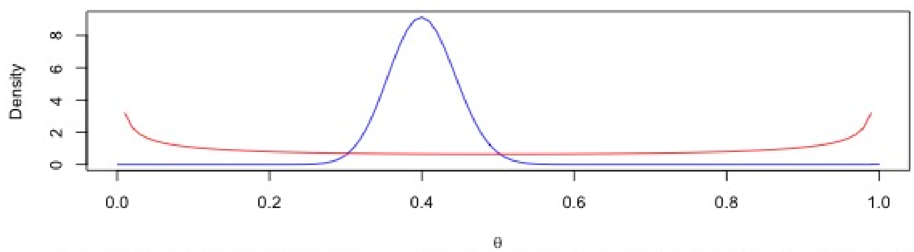
The red line in the following graphs represent the prior distributions of θ and the blue line represent the posterior distributions of θ. The subtitle contains the 95% credible interval as well as the posterior mean for each α and β pairing.

alpha = 1 and beta = 1 - blue is posterior and red is prior



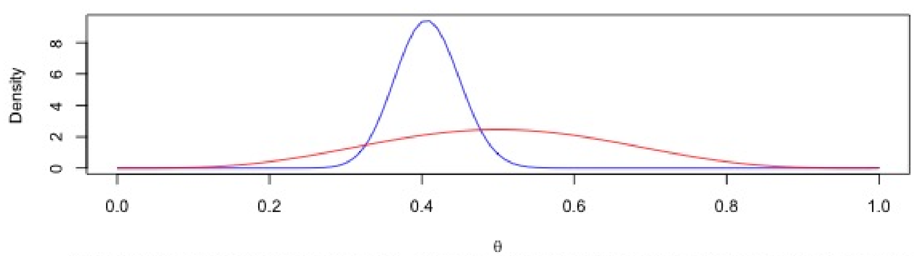
The Credible Interval is (0.318271686343006 , 0.487815027115156) and the posterior mean is 0.401574803149606

alpha = 0.5 and beta = 0.5 - blue is posterior and red is prior

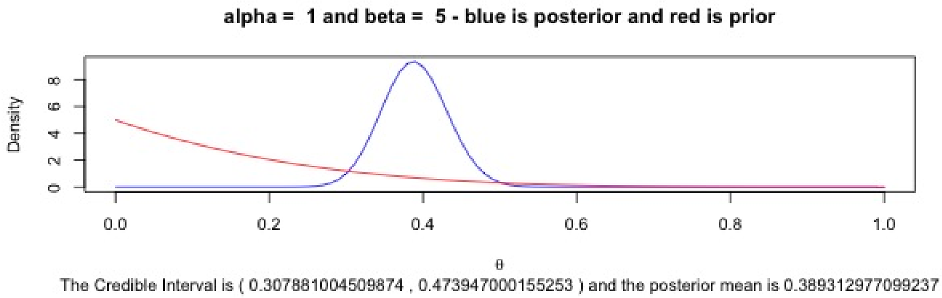


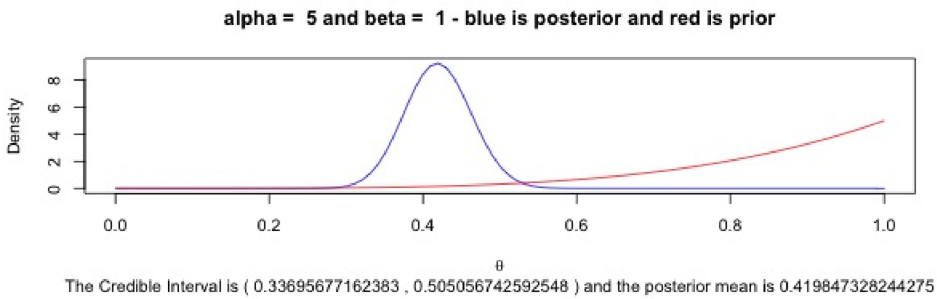
The Credible Interval is (0.317209575524597 , 0.487361640436785) and the posterior mean is 0.400793650793651

alpha = 5 and beta = 5 - blue is posterior and red is prior



The Credible Interval is (0.32626611085171 , 0.49114801037534) and the posterior mean is 0.407407407407407





Exercise 4

Part A

Setting Pareto distribution into exponential family form.

$$f(x|\theta) = \frac{\theta k^\theta}{x^{\theta+1}} = \frac{\theta k^\theta}{x^1 x^\theta} = x^{-1}\theta\frac{k^\theta}{x^\theta} = x^{-1}\theta * \left(\frac{k}{x}\right)^\theta = x^{-1}\theta * exp\left\{log\left(\left(\frac{k}{x}\right)^\theta\right)\right\} = x^{-1}\theta * exp\left\{\theta log\left(\left(\frac{k}{x}\right)\right)\right\}$$

$h(x) = x^{-1}$, $g(\theta) = \theta$, $\phi(\theta) = \theta$, and $t(x) = log\left(\frac{k}{x}\right)$

$$p(\theta) \propto (\theta)^{n_0} * exp\{\theta v\} = (\theta)^{n_0} e^{\theta v} = \theta^{(n_0+1)-1} e^{-(-\theta v)}$$

From the conjugate prior, it can be recognized as the kernel of the un-normalized density of the gamma distribution

$\therefore \theta \sim gamma(n_0 + 1, -v)$

Part B

From 4b, we are given that the prior distribution is $gamma(\alpha, \beta)$ $\quad p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}$

$c = log(k/x_i)$

$p(\theta|x) = \frac{p(x|\theta)*p(\theta)}{\int_0^1 p(x|\theta)*p(\theta)\delta\theta}$

Since $x_1 \ldots x_n$ are $iid \sim Pareto(k, \theta)$

$p(x|\theta) = \prod_{i=1}^n p(x_i|\theta) = \prod_{i=1}^n x_i^{-1}\theta * exp\left\{\theta log(k/x_i\right\} = \theta^n * exp\left\{\sum_{i=1}^n \theta log(k/x_i\right\} * \prod_{i=1}^n x_i^{-1}$

$p(\theta|x) = \frac{\theta^n * exp\left\{\sum_{i=1}^n \theta log(k/x)_i\right\} * \prod_{i=1}^n x_i^{-1} * \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}}{\int_0^1 \theta^n * exp\left\{\sum_{i=1}^n \theta log(k/x_i)\right\} * \prod_{i=1}^n x_i^{-1} * \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\delta\theta} = \frac{\theta^n * exp\left\{\sum_{i=1}^n \theta log(k/x_i)\right\} * \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}exp\{-\beta\theta\}}{\int_0^1 \theta^n * exp\left\{\sum_{i=1}^n \theta log(k/x_i)\right\} * \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}exp\{-\beta\theta\}\delta\theta} =$

$= \frac{\theta^n * exp\left\{\sum_{i=1}^n \theta * log(k/x_i)\right\} * \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}exp\{-\beta\theta\}}{\int_0^1 \theta^n * exp\left\{\sum_{i=1}^n \theta * log(k/x_i)\right\} * \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}exp\{-\beta\theta\}\delta\theta} = \frac{\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha+n-1}exp\left\{-\beta\theta + \sum_{i=1}^n \theta * log(k/x_i)\right\}}{\int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha+n-1}exp\left\{-\beta\theta + \sum_{i=1}^n \theta * log(k/x_i)\right\}\delta\theta} =$

$\frac{\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha+n-1}exp\left\{-\theta\left(\beta - \sum_{i=1}^n log(k/x_i)\right)\right\}}{\int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha+n-1}exp\left\{-\theta\left(\beta - \sum_{i=1}^n log(k/x_i)\right)\right\}\delta\theta} * \frac{\Gamma(\alpha)}{\Gamma(\alpha)} * \frac{\frac{1}{\Gamma(\alpha+n)}}{\frac{1}{\Gamma(\alpha+n)}} =$

$\frac{\frac{\beta^\alpha}{\Gamma(\alpha+n)}\theta^{\alpha+n-1}exp\left\{-\theta\left(\beta - \sum_{i=1}^n log(k/x_i)\right)\right\}}{\int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha+n)}\theta^{\alpha+n-1}exp\left\{-\theta\left(\beta - \sum_{i=1}^n log(k/x_i)\right)\right\}\delta\theta} * \frac{\frac{(\beta - \sum_{i=1}^n log(k/x_i))^{\alpha+n}}{\beta^\alpha}}{\frac{(\beta - \sum_{i=1}^n log(k/x_i))^{\alpha+n}}{\beta^\alpha}} =$

$\frac{\frac{(\beta - \sum_{i=1}^n log(k/x_i))^{\alpha+n}}{\Gamma(\alpha+n)}\theta^{\alpha+n-1}exp\left\{-\theta\left(\beta - \sum_{i=1}^n log(k/x_i)\right)\right\}}{\int_0^1 \frac{(\beta - \sum_{i=1}^n log(k/x_i))^{\alpha+n}}{\Gamma(\alpha+n)}\theta^{\alpha+n-1}exp\left\{-\theta\left(\beta - \sum_{i=1}^n log(k/x_i)\right)\right\}\delta\theta} =$ Please note that both the numerator and the formula within the integral of the denomintor are PDFs of gamma: $gamma(\alpha + n, \beta + \sum_{i=1}^n log(k/x_i))$

$\frac{(\beta - \sum_{i=1}^n log(k/x_i))^{\alpha+n}}{\Gamma(\alpha+n)}\theta^{\alpha+n-1}exp\left\{-\theta\left(\beta - \sum_{i=1}^n log(k/x_i)\right)\right\}$

$\theta|x \sim gamma(\alpha + n, \beta - \sum_{i=1}^n log(k/x_i))$