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Math 640 Bayesian Statistics
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Homework #5

Exercise 1

Part A

The posterior means can be calculated the following way.

```
set.seed(4231423)
library(MASS)
data(UScrime)
library(LearnBayes)

y = UScrime$y; n=length(y)
X = as.matrix(cbind(rep(1,n), UScrime[,-ncol(UScrime)]))

T = 1e4; k=dim(X)[2]; beta = matrix(NA, T, k)
g = n;  nu0 = 2;  s20 = 1

XtX.inv = solve(t(X)%*%X)
Sg = t(y)%*%(diag(1,n) - g/(g+1)*X%*%XtX.inv%*%t(X))%*%y
v = g/(g+1)*XtX.inv
m = v%*%t(X)%*%y

sigma2 = 1/rgamma(T, (nu0+n)/2, (nu0*s20+Sg)/2 )
for(t in 1:T) {
  beta[t,] = mvrnorm(1, m, sigma2[t]*v)
}

names(beta) = c("(Intercept)",names(UScrime)[-ncol(UScrime)])
Beta_means = apply(beta, 2, mean)
names(Beta_means) = c("(Intercept)",names(UScrime)[-ncol(UScrime)])
Beta_means
##   (Intercept)             M            So            Ed           Po1
## -5854.0213914     8.6223843    -0.8114880    18.4902911    18.9415360
##           Po2            LF           M.F           Pop            NW
##   -10.8042356    -0.6578557     1.6918465    -0.7234731     0.4089468
##            U1            U2           GDP          Ineq          Prob
##    -5.6857842    16.3936888     0.9534558     6.9253094 -4780.8605899
##          Time
##    -3.4803122
```

The confidence intervals are

```
beta.ci = apply(beta, 2, quantile, c(0.025, 0.975))
colnames(beta.ci) = c("(Intercept)",names(UScrime)[-ncol(UScrime)])
t(beta.ci)
##                      2.5%        97.5%
## (Intercept) -9166.3045979 -2540.489511
```

```
## M               0.1356346    17.117501
## So           -304.4383476   303.383762
## Ed              5.7318849    31.062020
## Po1            -2.3172868    40.776414
## Po2           -34.9421674    12.459706
## LF             -3.5771120     2.346274
## M.F            -2.5033091     5.859471
## Pop            -3.3802632     1.876897
## NW             -0.9255687     1.725497
## U1            -14.4219511     3.126659
## U2             -1.1069140    33.359286
## GDP            -1.2410612     3.036100
## Ineq            2.2515701    11.566756
## Prob        -9514.3377455  -145.111528
## Time          -18.1444128    11.245169
```

The least square values for β can be calculated using the lm function.

```
Crime.fit = lm(y~.,data= UScrime)
```

With respect the the differences between the Bayesian Method and the least square methods, we can determine the difference with the following.

```
Beta_means - Crime.fit$coefficients
##   (Intercept)             M            So            Ed           Po1
## 130.266213115  -0.160633050   2.991962298  -0.342140358  -0.338897869
##           Po2            LF           M.F           Pop            NW
##   0.137956920   0.005970441  -0.048839080   0.009535018  -0.011499322
##            U1            U2           GDP          Ineq          Prob
##   0.141318531  -0.386278374  -0.008206634  -0.141900501  74.405225544
##          Time
##  -0.001294313
```

It's interesting to see that the difference between the posteior means and the values of β calculated by the least squares method are very close.

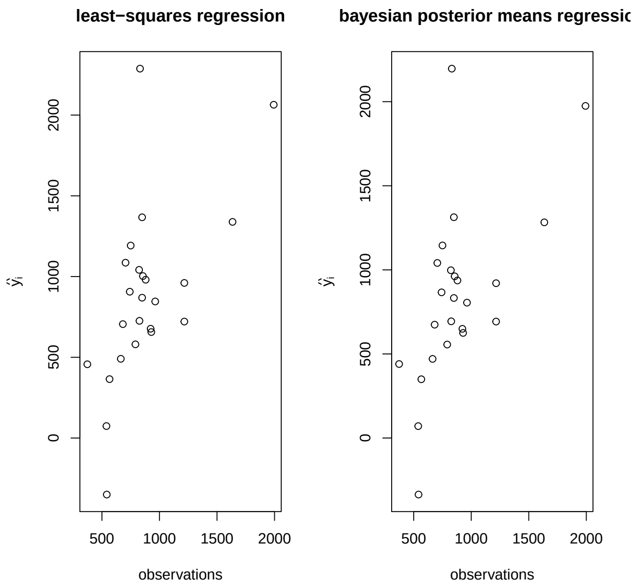
```
t(beta.ci) - confint(Crime.fit)
##                     2.5%        97.5%
## (Intercept) 138.960223850 122.82087598
## M            -0.139783539  -0.17311593
## So            2.753210816   3.79910375
## Ed           -0.437538765  -0.43341933
## Po1           0.043490411  -0.14523066
## Po2          -0.040273520  -0.55780301
## LF            0.084245856   0.01256842
## M.F          -0.092801085  -0.03240783
## Pop          -0.017189447  -0.02015995
## NW           -0.024228103  -0.01673541
## U1           -0.007907262   0.36682025
## U2           -1.094352421  -0.21320991
## GDP          -0.088440616  -0.03984540
## Ineq         -0.182574823  -0.13351934
## Prob        -24.533334845  75.61569183
## Time         -0.051719915   0.11051163
```

The difference between the Confidence Interval following the same pattern as with the means. That is attributed to the intercept and Prob; however, based on the values obtained, the differences aren't as significant.

Part B

```
Problem1b = function(seed = F, graph = F){
  if(seed){
    set.seed(seed)
    }
  train = sample(nrow(UScrime), nrow(UScrime)*0.5)
  UScrime.train = UScrime[train,]
  UScrime.test = UScrime[-train,]
  Crimeb.fit = lm(y~.,data= UScrime.train)
  Crimeb.fit$coefficients
  Crimeb.fittedvalues = predict(Crimeb.fit, UScrime.test)

  y = UScrime.train$y; n=length(y)
  X = as.matrix(cbind(rep(1,n), UScrime.train[,-ncol(UScrime.train)]))

  T = 1e4; k=dim(X)[2]; beta = matrix(NA, T, k)
  g = n;  nu0 = 2;  s20 = 1

  XtX.inv = solve(t(X)%*%X)
  Sg = t(y)%*%(diag(1,n) - g/(g+1)*X%*%XtX.inv%*%t(X))%*%y
  v = g/(g+1)*XtX.inv
  m = v%*%t(X)%*%y

  sigma2 = 1/rgamma(T, (nu0+n)/2, (nu0*s20+Sg)/2 )
  for(t in 1:T) {
    beta[t,] = mvrnorm(1, m, sigma2[t]*v)
  }

  Beta_means = apply(beta, 2, mean)
  names(Beta_means) = c("(Intercept)",names(UScrime)[-ncol(UScrime)])
  Beta_means
  bayesian_predictions = (as.matrix(UScrime.test[,-ncol(UScrime.test)])%*%(Beta_means[-1])+Beta_means[1

  mean((bayesian_predictions - UScrime.test$y)^2)

  if(graph){
    par(mfrow=c(1,2))
    plot(UScrime.test$y, Crimeb.fittedvalues,
         main = "least-squares regression",
         xlab = "observations", ylab = expression(hat(y[i])))
    plot(UScrime.test$y, bayesian_predictions,
         main = "bayesian posterior means regression",
         xlab = "observations", ylab = expression(hat(y[i])))

  }

  results = list(Crimeb.fit$coefficients, mean((Crimeb.fittedvalues-UScrime.test$y)^2),
                 Beta_means,mean((bayesian_predictions - UScrime.test$y)^2))
```

```
  names(results) = c("least-squares regression coefficients", "MSE",
                     " posterior mean", "MSE posterior mean")
  return(results)
}

Problem1b(1e8, T)
```




```
## $`least-squares regression coefficients`
##   (Intercept)             M            So            Ed           Po1
## -8.259814e+03  1.490893e+01  6.268855e+02  1.128908e+00 -1.292324e+01
##           Po2            LF           M.F           Pop            NW
##  2.694348e+01  8.477062e+00  1.030005e+00 -8.069709e-01  2.276716e-01
##            U1            U2           GDP          Ineq          Prob
##  1.929445e+00  1.392674e+01 -6.914113e-01 -2.167347e-01 -1.005828e+04
##          Time
##  5.982888e+00
##
## $MSE
## [1] 188881.3
##
## $` posterior mean`
##   (Intercept)             M            So            Ed           Po1
## -7948.3836292    14.3931292   598.8011211     1.0311238   -12.8756182
##           Po2            LF           M.F           Pop            NW
##    26.3771702     8.1412927     1.0008753    -0.7463585     0.2091552
##            U1            U2           GDP          Ineq          Prob
##     1.8268808    13.4431374    -0.6748428    -0.1946998 -9639.4103593
##          Time
##     5.7397735
##
## $`MSE posterior mean`
## [1] 176657.6
```

From the result, the method using the posterior mean performs better than one using the least squares estimates.

Part C

```
Problem1c = matrix(data = NA, nrow = 100, ncol = 2)
names(Problem1c) = c("least-squares regression MSE", "MSE  posterior mean")
for(i in 1:100){
  Problem1ci = Problem1b()
  Problem1c[i,]= c(Problem1ci[[2]], Problem1ci[[4]])
}
apply(Problem1c,2,mean, na.rm = T)
## [1] 173703.2 164701.3
```

Exercise 2

Part A

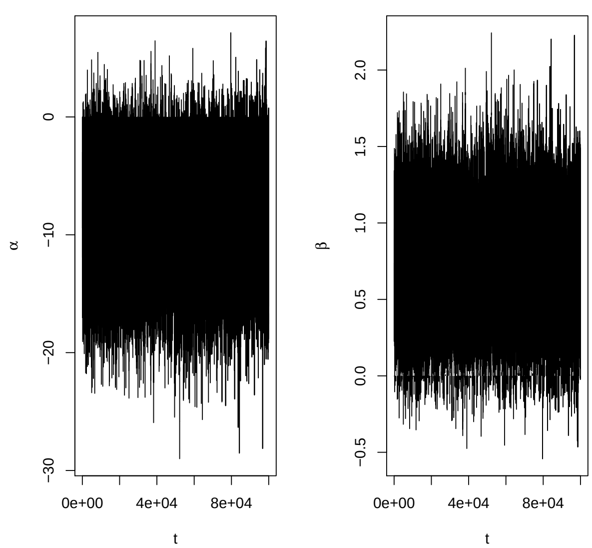
$logit(P(Y_i = 1|\alpha, \beta, x_i)) = log(P(Y_i = 1|\alpha, \beta, x_i)) - log(1 - P(Y_i = 1|\alpha, \beta, x_i)) = \alpha + \beta * x_i \implies$
$\frac{(P(Y_i=1|\alpha,\beta,x_i))}{1-(P(Y_i=1|\alpha,\beta,x_i))} = exp(\alpha + \beta * x_i) \implies (P(Y_i = 1|\alpha, \beta, x_i)) = \frac{exp(\alpha+\beta*x_i)}{1+exp(\alpha+\beta*x_i)}$
$\therefore (P(Y_i = 0|\alpha, \beta, x_i)) = \frac{1}{1+exp(\alpha+\beta*x_i)}$
$f(y_i|\alpha, \beta, x_i) = \left(\frac{exp(\alpha+\beta*x_i)}{1+exp(\alpha+\beta*x_i)}\right)^{y_i} * \left(\frac{1}{1+exp(\alpha+\beta*x_i)}\right)^{1-y_i} =$
$\left(\frac{exp(\alpha+\beta*x_i)}{1+exp(\alpha+\beta*x_i)}\right)^{y_i} * (1 + exp(\alpha + \beta * x_i))^{y_i} \left(\frac{1}{1+exp(\alpha+\beta*x_i)}\right) = (exp(\alpha + \beta * x_i))^{y_i} * \left(\frac{1}{1+exp(\alpha+\beta*x_i)}\right)$
$f(\boldsymbol{y}|\alpha, \beta, \boldsymbol{x}) = \prod_{i=1}^{n} (exp(\alpha + \beta * x_i))^{y_i} * \left(\frac{1}{1+exp(\alpha+\beta*x_i)}\right) =$
$exp\left\{log\left(\prod_{i=1}^{n} (exp(\alpha + \beta * x_i))^{y_i} * \left(\frac{1}{1+exp(\alpha+\beta*x_i)}\right)\right)\right\} =$
$exp\left\{\sum_{i=1}^{n} [log\,(exp(\alpha + \beta * x_i))^{y_i} - log\,(1 + exp(\alpha + \beta * x_i))]\right\} =$
$exp\left\{\sum_{i=1}^{n} [y_i * log\,(exp(\alpha + \beta * x_i)) - log\,(1 + exp(\alpha + \beta * x_i))]\right\}$

Part B

$r = \frac{p(\theta^*|\boldsymbol{y},\boldsymbol{x})}{p(\theta^{t-1}|\boldsymbol{y},\boldsymbol{x})} =$
$= \frac{p(\boldsymbol{y}|\theta^*,\boldsymbol{x})*p(\theta^*)}{p(\boldsymbol{y}|\theta^{t-1},\boldsymbol{x})*p(\theta^{t-1})} =$
$\frac{exp\left\{\sum_{i=1}^{n}[y_i*log(exp(\alpha^*+\beta^**x_i))-log(1+exp(\alpha^*+\beta^**x_i))]\right\}}{exp\left\{\sum_{i=1}^{n}[y_i*log(exp(\alpha^{t-1}+\beta^{t-1}*x_i))-log(1+exp(\alpha^{t-1}+\beta^{t-1}*x_i))]\right\}} * \frac{p(\alpha^*)*p(\beta^*)}{p(\alpha^{t-1})*p(\beta^{t-1})}$

```
library(MASS)
msparrownest <- read.table("~/Documents/Georgetown/Bayesian-Statistics-Mathematics-640/HW5/msparrownest
names(msparrownest) = c("nests", "wingspan")
y = msparrownest$nests; n=length(y)
x = msparrownest$wingspan
X = as.matrix(cbind(rep(1,n), msparrownest[,"wingspan"]))
proposal_var = n*mean(y)*(1-mean(y))*solve(t(X)%*%X)

ratio = function(thetastar, theta_pre){
  return(exp(sum(y*log(exp(thetastar[1]+thetastar[2]*x))-
           log(1+exp(thetastar[1]+thetastar[2]*x))))/
    (exp(sum(y*log(exp(theta_pre[1]+theta_pre[2]*x))-
             log(1+exp(theta_pre[1]+theta_pre[2]*x)))))*
      dnorm(thetastar[1], 0,10)*dnorm(thetastar[2],0,10)/
```

```
      (dnorm(theta_pre[1], 0,10)*dnorm(theta_pre[2],0,10)))
}


T = 1e5
theta = matrix(nrow=T, ncol=2)
theta[1,] = c(0,0)
accept = rep(0,T)

for(t in 2:T){
  theta_star = mvrnorm(1,theta[t-1,], Sigma = proposal_var)
  r = ratio(theta_star, theta[t-1,])
  u = runif(1)
  if(u<r){
    theta[t,] = theta_star
    accept[t] = 1
  } else{
    theta[t,] = theta[t-1,]
  }
}
par(mfrow = c(1,2))
plot(theta[,1], type = 'l', xlab = "t", ylab = expression(alpha))
plot(theta[,2], type = 'l', xlab = "t",ylab = expression(beta))
```



Based on trace the trace plots, there does not seem to be a failure of convergence to the stationary distribution for either α or β.

```
nburn = 1e3
mean(accept[-(1:nburn)])
## [1] 0.3978182
```

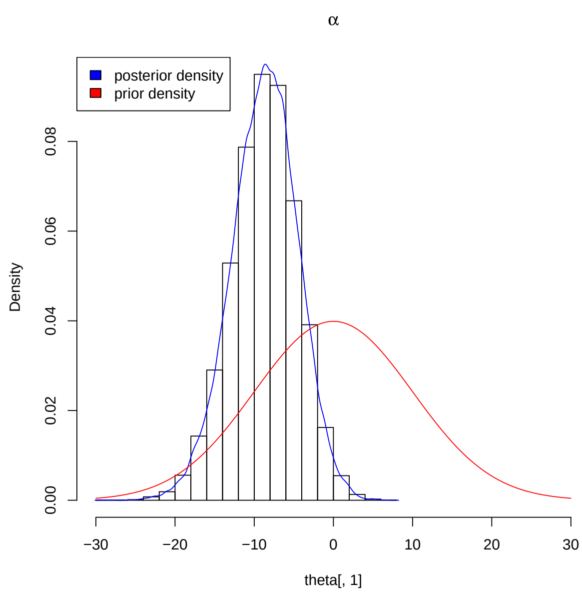
The acceptance rate is approximately 39.6%.

```
library(coda)
theta.mcmc = as.mcmc(theta[-(1:nburn),])
effectiveSize(theta.mcmc)
##     var1     var2
## 12584.82 12448.95
```

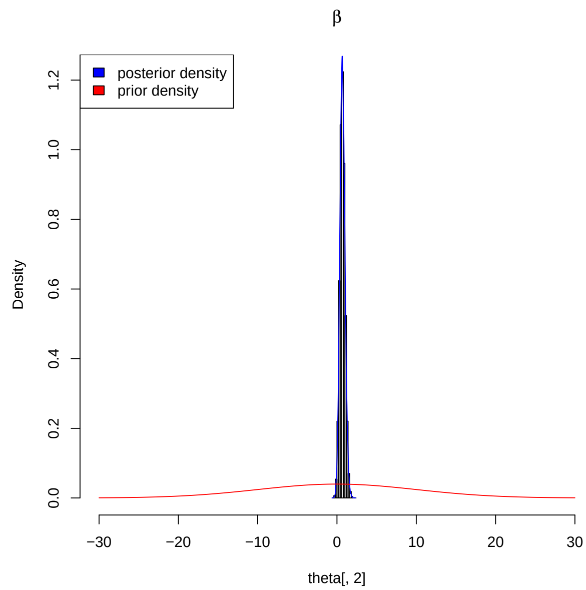
Based on the effectiveSize, > 1000 for both α and β have been obtained.

Part C

```
# The posterior estimates for alpha and beta.
apply(theta, 2, mean)
## [1] -8.7142122  0.6939065
hist(theta[,1], probability = TRUE,
     main = expression(alpha), xlim = c(-30,30))
lines(density(theta[,1]), col = "blue")
curve(dnorm(x, 0,10), add = TRUE, col = "red")
legend("topleft", legend = c("posterior density", "prior density"),
       fill = c("blue", "red"))
```



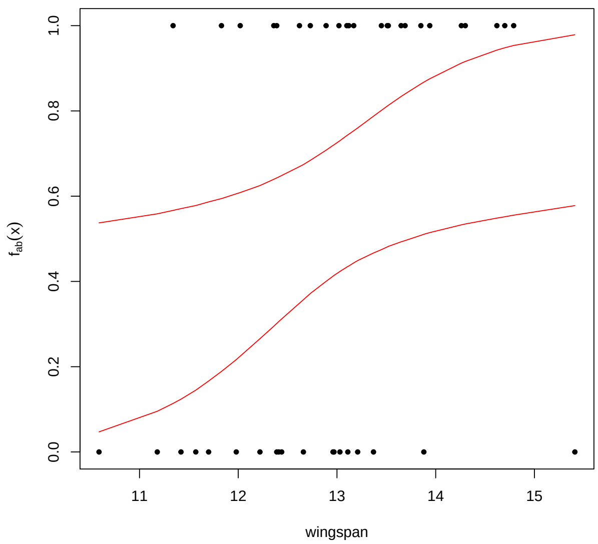

```
hist(theta[,2], probability = TRUE,
     main = expression(beta), xlim = c(-30,30))
lines(density(theta[,2]), col = "blue")
curve(dnorm(x, 0,10), add = TRUE, col = "red")
legend("topleft", legend = c("posterior density", "prior density"),
       fill = c("blue", "red"))
```



The prior estimates for both α and β are 0 because the densities for are defined as $N(0, 10^2)$. The difference between the posterior and prior estimate for α is approximately 8.63 where as the difference between those of beta are approximately 0.69. For the different in the posterior and prior densities for alpha, there is a shift to the left (hence the negative mean) and a lower standard deviation. For beta, the change in SD is much more significant.

Part D

```
fab = function(x,theta){
  return(exp(theta[1]+theta[2]*x)/(1+exp(theta[1]+theta[2]*x)))
}
confidence_band = matrix(nrow = length(x), ncol = 3)

confidence_band[,1] = x
for(i in 1:length(x)){
  fabresults = rep(0, nrow(theta))
  for(t in 1:nrow(theta)){
  fabresults[t] = fab(x[i], theta[t,])
  }
  confidence_band[i, 2:3] = quantile(fabresults, c(0.025,0.975))
  }
confidence_band = as.data.frame(confidence_band)
names(confidence_band) = c("x_i", "lower", "upper")
confidence_band = confidence_band[order(confidence_band$x_i),]
plot(nests~wingspan, data = msparrownest, pch =20,
     ylab = expression(f[ab](x)))
lines(upper~x_i, data = confidence_band, col = "red")
lines(lower~x_i, data = confidence_band, col = "red")
```



0.0
0.2
0.4
0.6
0.8
1.0
$f_{ab}(x)$
11
12
13
14
15
wingspan