Mathematics 640 Homework 3

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Exercise 1

$\mathbf{y}|\boldsymbol{\mu},\boldsymbol{\Sigma} \sim N(\boldsymbol{\mu},\boldsymbol{\Sigma})$

$\boldsymbol{\mu} \sim N(\boldsymbol{\mu_0}, \triangle_\mathbf{0})$

Note: we should focus solely on the the exponent of the multi-normal distribution.

Since the numeratpr and the formula within the intergral of the denominator are identical please assume the steps approached are done for both.

$$exp\left\{-\frac{1}{2} * \sum_{i=1}^{n} (y_i - \boldsymbol{\mu})^T * \boldsymbol{\Sigma}^{-1} * (y_i - \boldsymbol{\mu})\right\} * exp\left\{-\frac{1}{2}(-\boldsymbol{\mu}_0)^T \triangle_0^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\right\} =$$

$$exp\left\{-\frac{1}{2} * \sum_{i=1}^{n} y_i^T \boldsymbol{\Sigma}^{-1} y_i - \mu^T \boldsymbol{\Sigma}^{-1} y_i - y_i^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right\} * exp\left\{-\frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)^T \triangle_0^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\right\} =$$

$$exp\left\{-\frac{1}{2} * \left[\sum_{i=1}^{n} y_i^T \boldsymbol{\Sigma}^{-1} y_i - \sum_{i=1}^{n} \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} y_i - \sum_{i=1}^{n} y_i^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \sum_{i=1}^{n} \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + (\boldsymbol{\mu} - \boldsymbol{\mu}_0)^T \triangle_0^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\right]\right\} =$$

$$exp\left\{-\frac{1}{2} * \left[\sum_{i=1}^{n} y_i^T \boldsymbol{\Sigma}^{-1} y_i - \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} n\bar{y} - n\bar{y}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + n\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + (\boldsymbol{\mu} - \boldsymbol{\mu}_0)^T \triangle_0^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\right]\right\}$$

Since $\sum_{i=1}^{n} y_i^T \boldsymbol{\Sigma}^{-1} y_i$ does not contain μ, it can be factored out

$$exp\left\{-\frac{1}{2} * \left(-\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} n\bar{y} - n\bar{y}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + n\mu^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^T \triangle_0^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}_0^T \triangle_0^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}^T \triangle_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{\mu}_0^T \triangle_0^{-1}\boldsymbol{\mu}_0\right)\right\} =$$

$$exp\left\{-\frac{1}{2} * \left(-\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} n\bar{y} - n\bar{y}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + n\mu^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^T \triangle_0^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}_0^T \triangle_0^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}^T \triangle_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{\mu}_0^T \triangle_0^{-1}\boldsymbol{\mu}_0\right)\right\}$$

Since $\boldsymbol{\mu}_0^T \triangle_0^{-1}\boldsymbol{\mu}_0$ does not contain μ, it can be factored out

$$exp\left\{-\frac{1}{2} * \left(-\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} n\bar{y} - n\bar{y}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + n\mu^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^T \triangle_0^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}_0^T \triangle_0^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}^T \triangle_0^{-1}\boldsymbol{\mu}_0\right)\right\} =$$

$$exp\left\{-\frac{1}{2} * \left(\left(-n\bar{y}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}_0^T \triangle_0^{-1}\boldsymbol{\mu}\right) + \left(\boldsymbol{\mu}^T \triangle_0^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^T n\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right) + \left(-\boldsymbol{\mu}^T \triangle_0^{-1}\boldsymbol{\mu}_0 - \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} n\bar{y}\right)\right)\right\} =$$

$$exp\left\{-\frac{1}{2} * \left(-\left(\bar{y}^T n\boldsymbol{\Sigma}^{-1} + \boldsymbol{\mu}_0^T \triangle_0^{-1}\right)\boldsymbol{\mu} + \boldsymbol{\mu}^T \left(\triangle_0^{-1} + n\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu} - \boldsymbol{\mu}^T \left(\triangle_0^{-1}\boldsymbol{\mu}_0 + n\boldsymbol{\Sigma}^{-1}\bar{y}\right)\right)\right\}$$

Since both $\triangle_0^{-1}$ and $\boldsymbol{\Sigma}^{-1}$ are symmetric $\left(\triangle_0^{-1}\right)^T = \triangle_0^{-1}$ and $\left(\boldsymbol{\Sigma}^{-1}\right)^T = \boldsymbol{\Sigma}^{-1}$

$$exp\left\{-\frac{1}{2} * \left(-\left(\triangle_0^{-1}\boldsymbol{\mu}_0 + n\boldsymbol{\Sigma}^{-1}\bar{y}\right)^T \boldsymbol{\mu} + \boldsymbol{\mu}^T \left(\triangle_0^{-1} + n\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu} - \boldsymbol{\mu}^T \left(\triangle_0^{-1}\boldsymbol{\mu}_0 + n\boldsymbol{\Sigma}^{-1}\bar{y}\right)\right)\right\}$$

Let $X = \triangle_0^{-1}\boldsymbol{\mu}_0 + n\boldsymbol{\Sigma}^{-1}\bar{y}$ and $Y = \triangle_0^{-1} + n\boldsymbol{\Sigma}^{-1}$

note: Y is basically $\triangle_n^{-1}$ and $u_n = Y^{-1}X$ so the goal is to manipulate in the mutlivariate normal.

$exp\left\{-\frac{1}{2} * \left(-(X)^T\boldsymbol{\mu} + \boldsymbol{\mu}^T(Y)\boldsymbol{\mu} - \boldsymbol{\mu}^T(X)\right)\right\}$

multiply the numerator and denominator by $exp\left\{-\frac{1}{2} * X^TY^{-1}X\right\}$

$exp\left\{-\frac{1}{2}\left(-\boldsymbol{\mu}(X)^T - \boldsymbol{\mu}^T(X) + \boldsymbol{\mu}^T(Y)\boldsymbol{\mu}\right)\right\} * exp\left\{-\frac{1}{2} * X^TY^{-1}X\right\} =$

$exp\left\{-\frac{1}{2}\left(-\boldsymbol{\mu}(X)^T - \boldsymbol{\mu}^T(X) + \boldsymbol{\mu}^T(Y)\boldsymbol{\mu} + X^TY^{-1}X\right)\right\} =$

$exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}^T(Y)\boldsymbol{\mu} - \boldsymbol{\mu}(X)^T - \boldsymbol{\mu}^T(X) + X^TY^{-1}X\right)\right\}$

$I = Y^{-1}Y = YY^{-1}$ because Y is symmetric and invertible

$exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}^T(Y)\boldsymbol{\mu} - \boldsymbol{\mu}Y\left(Y^{-1}X\right)^T - \boldsymbol{\mu}^TY\left(Y^{-1}X\right) + \left(Y^{-1}X\right)^TY\left(Y^{-1}X\right)\right)\right\}$

Let $\triangle_n^{-1} = Y = \boldsymbol{\Sigma}^{-1}n + \triangle_0^{-1} = \triangle_0^{-1} + n\boldsymbol{\Sigma}^{-1}$ and $\mu_n = Y^{-1}X = \left(\triangle_0^{-1} + n\boldsymbol{\Sigma}^{-1}\right) * \left(\boldsymbol{\Sigma}^{-1}ny + \triangle_0^{-1}\boldsymbol{\mu}_0\right)$

$exp\left\{-\frac{1}{2}\left(\boldsymbol{\mu}^T\triangle_n^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}\triangle_n^{-1}\mu_n^T - \boldsymbol{\mu}^T\triangle_n^{-1}\mu_n + \mu_n^T\triangle_n^{-1}\mu_n\right)\right\} =$

$exp\left\{-\frac{1}{2}\left(\left(\boldsymbol{\mu}^T - \mu_n^T\right)\triangle_n^{-1}\left(\boldsymbol{\mu} - \mu_n\right)\right)\right\} =$

$exp\left\{-\frac{1}{2}\left(\left(\boldsymbol{\mu} - \mu_n\right)^T\triangle_n^{-1}\left(\boldsymbol{\mu} - \mu_n\right)\right)\right\}$ Based on this kernel, it can be concluded that

$\mu|Y \sim N\left(\mu_n, \triangle_n\right)$ where

$\triangle_n^{-1} = \boldsymbol{\Sigma}^{-1}n + \triangle_0^{-1} = \triangle_0^{-1} + n\boldsymbol{\Sigma}^{-1}$ and $\mu_n = \left(\triangle_0^{-1} + n\boldsymbol{\Sigma}^{-1}\right) * \left(\boldsymbol{\Sigma}^{-1}ny + \triangle_0^{-1}\boldsymbol{\mu}_0\right)$

Exercise 2

Part A

Under H_0: the following are true

$\pi_1 = \pi_2$

$X_j|\pi_1 \sim Binomial(n_j, \pi_1)$ where $j = 1, 2$

$\pi|H_0 \sim Beta(a, b)$

under the assumption that x_1 and x_2 are independent:

$p(x_1, x_2|\pi_1, \pi_2) = p(x_1|\pi_1) * p(x_2|\pi_2)\left[\begin{pmatrix} n_1 \\ x_1 \end{pmatrix}\pi_1^{x_1} * (1-\pi_1)^{n_1-x_1}\right] * \left[\begin{pmatrix} n_2 \\ x_2 \end{pmatrix}\pi_2^{x_2} * (1-\pi_2)^{n_2-x_2}\right]$

since $\pi_1 = \pi_2$ under H_0

$p(x_1, x_2|\pi) = p(x_1|\pi) * p(x_2|\pi) \left[\binom{n_1}{x_1} \pi^{x_1} * (1-\pi)^{n_1 - x_1} \right] * \left[\binom{n_2}{x_2} \pi^{x_2} * (1-\pi)^{n_2 - x_2} \right] =$

$\binom{n_1}{x_1} * \binom{n_2}{x_2} \pi^{x_1+x_2} * (1-\pi)^{n_1-x_1+n_2-x_2}$ Since $\pi|H_0 \sim Beta(a,b)$ then

$p(x_1, x_2|H_0) = \int_{\pi} p(x_1, x_2|\pi) * p(\pi|H_0) d\pi =$

$\int \binom{n_1}{x_1} \binom{n_2}{x_2} \pi^{x_1+x_2} (1-\pi)^{n_1-x_1+n_2-x_2} \left[\frac{1}{B(a,b)} \pi^{a-1} (1-\pi)^{b-1} \right] d\pi =$

$\binom{n_1}{x_1} \binom{n_2}{x_2} \frac{1}{B(a,b)} \int \pi^{x_1+x_2+a-1} (1-\pi)^{n_1-x_1+n_2-x_2+b-1} d\pi =$

$\binom{n_1}{x_1} \binom{n_2}{x_2} \frac{B(x_1+x_2+a, n_1-x_1+n_2-x_2+b)}{B(a,b)} \int \frac{1}{B(x_1+x_2+a, n_1-x_1+n_2-x_2+b)} \pi^{x_1+x_2+a-1} (1-\pi)^{n_1-x_1+n_2-x_2+b-1} d\pi =$

$\binom{n_1}{x_1} \binom{n_2}{x_2} \frac{B(x_1+x_2+a, n_1-x_1+n_2-x_2+b)}{B(a,b)}$

Part B

Under H_1: the following are true

$\pi_1 \neq \pi_2$

$X_j|\pi_1 \sim Binomial(n_j, \pi_1)$ where $j = 1, 2$

$\pi|H_1 \sim Beta(a,b)$ where $j = 1, 2$

under the assumption that x_1 and x_2 are independent:

$p(x_1, x_2|\pi_1, \pi_2) = p(x_1|\pi_1) * p(x_2|\pi_2) = \left[\binom{n_1}{x_1} \pi_1^{x_1} * (1-\pi_1)^{n_1-x_1} \right] * \left[\binom{n_2}{x_2} \pi_2^{x_2} * (1-\pi_2)^{n_2-x_2} \right]$

$p(x_1, x_2|H_1) = \int_{\pi_1} \int_{\pi_2} p(x_1, x_2|\pi_1, \pi_2) * p(\pi_1|H_1) * p(\pi_2|H_1) d\pi_2 d\pi_1 =$

$\int_{\pi_1} \int_{\pi_2} \left[\binom{n_1}{x_1} \pi_1^{x_1} * (1-\pi_1)^{n_1-x_1} \right] * \left[\binom{n_2}{x_2} \pi_2^{x_2} * (1-\pi_2)^{n_2-x_2} \right] *$

$\left(\frac{1}{B(a,b)} \pi_1^{a-1} (1-\pi_1)^{b-1} \right) * \left(\frac{1}{B(a,b)} \pi_2^{a-1} (1-\pi_2)^{b-1} \right) d\pi_2 d\pi_1 =$

$\binom{n_1}{x_1} \binom{n_2}{x_2} \left(\frac{1}{B(a,b)} \right)^2 \int_{\pi_1} \pi_1^{x_1} * (1-\pi_1)^{n_1-x_1} * \left(\pi_1^{a-1} (1-\pi_1)^{b-1} \right) *$

$\int_{\pi_2} \left[\pi_2^{x_2} * (1-\pi_2)^{n_2-x_2} \right] * \left(\pi_2^{a-1} (1-\pi_2)^{b-1} \right) d\pi_2 d\pi_1 =$

$$\begin{pmatrix} n_1 \\ x_1 \end{pmatrix}\begin{pmatrix} n_2 \\ x_2 \end{pmatrix}\left(\frac{1}{B(a,b)}\right)^2 \int_{\pi_1} \pi_1^{x_1+a-1} * (1-\pi_1)^{n_1-x_1+b-1} \int_{\pi_2} \pi_2^{x_2+a-1} * (1-\pi_2)^{n_2-x_2+b-1} d\pi_2 d\pi_1 =$$

$$\begin{pmatrix} n_1 \\ x_1 \end{pmatrix}\begin{pmatrix} n_2 \\ x_2 \end{pmatrix}\left(\frac{1}{B(a,b)}\right)^2 B(x_2+a, n_2-x_2+b) * B(x_1+a, n_1-x_1+b) \int_{\pi_1} \frac{1}{B(x_1+a,n_1-x_1+b)} \pi_1^{x_1+a-1} *$$

$$(1-\pi_1)^{n_1-x_1+b-1} \int_{\pi_2} \frac{1}{B(x_2+a,n_2-x_2+b)} \pi_2^{x_2+a-1} * (1-\pi_2)^{n_2-x_2+b-1} d\pi_2 d\pi_1 =$$

$$\begin{pmatrix} n_1 \\ x_1 \end{pmatrix}\begin{pmatrix} n_2 \\ x_2 \end{pmatrix}\left(\frac{1}{B(a,b)}\right)^2 B(x_2+a, n_2-x_2+b) * B(x_1+a, n_1-x_1+b)$$

Part C

$$p(H_0|x_1,x_2) = \frac{p(H_0)*p(x_1,x_2|H_0)}{p(x_1,x_2)} = \frac{p(H_0)*p(x_1,x_2|H_0)}{p(x_1,x_2|H_0)*p(H_0)+p(x_1,x_2|H_1)*p(H_1)} = \frac{\omega*p(x_1,x_2|H_0)}{p(x_1,x_2|H_0)*\omega+p(x_1,x_2|H_1)*(1-\omega)}$$

Part D

$$BF = \frac{p(H_1|x_1,x_2)/p(H_0|x_1,x_2)}{p(H_1)/p(H_2)} = \frac{\frac{(1\text{-}\omega)*p(x_1,x_2|H_1)}{p(x_1,x_2|H_0)*\omega+p(x_1,x_2|H_1)*(1-\omega)} / \frac{\omega*p(x_1,x_2|H_0)}{p(x_1,x_2|H_0)*\omega+p(x_1,x_2|H_1)*(1-\omega)}}{(1\text{-}\omega)/\omega} =$$
$$\frac{(1\text{-}\omega)*p(x_1,x_2|H_1)/\omega*p(x_1,x_2|H_0)}{(1\text{-}\omega)/\omega} = \frac{p(x_1,x_2|H_1)}{p(x_1,x_2|H_0)} =$$
$$\frac{\begin{pmatrix} n_1 \\ x_1 \end{pmatrix}\begin{pmatrix} n_2 \\ x_2 \end{pmatrix}\left(\frac{1}{B(a,b)}\right)^2 B(x_2+a,n_2-x_2+b)*B(x_1+a,n_1-x_1+b)}{\begin{pmatrix} n_1 \\ x_1 \end{pmatrix}\begin{pmatrix} n_2 \\ x_2 \end{pmatrix}\frac{B(x_1+x_2+a,n_1-x_1+n_2-x_2+b)}{B(a,b)}} =$$
$$\frac{B(x_2+a,n_2-x_2+b)*B(x_1+a,n_1-x_1+b)}{B(x_1+x_2+a,n_1-x_1+n_2-x_2+b)} * \left(\frac{1}{B(a,b)}\right)$$

Part E

$$log\left(\frac{B(x_2+a,n_2-x_2+b)*B(x_1+a,n_1-x_1+b)}{B(x_1+x_2+a,n_1-x_1+n_2-x_2+b)} * \left(\frac{1}{B(a,b)}\right)\right) = log\left(\frac{B(x_2+a,n_2-x_2+b)*B(x_1+a,n_1-x_1+b)}{B(x_1+x_2+a,n_1-x_1+n_2-x_2+b)}\right) - log(B(a,b)) =$$

```
logB = function(a,b){
        lgamma(a)+lgamma(b)-lgamma(a+b)
        }
x1=46
n1=143
x2=30
n2 = 151
omega = 0.05
a=2
b=8
exp(logB(x2+a,n2-x2+b)+logB(x1+a,n1-x1+b)-logB(x1+x2+a,n1-x1+n2-x2+b)-logB(a,b))
 4.534823
```

Under the Kass & Raftery – scale of evidence in favor of H_1, the interpretation is postive since $3 < 4.534823 \leq 20$

Part F

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prop.test(x=c(x1,x2),n=c(n1,n2))

        2-sample test for equality of proportions with
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        continuity correction
data:  c(x1, x2) out of c(n1, n2)
X-squared = 5.1733, df = 1, p-value = 0.02294
alternative hypothesis: two.sided
95 percent confidence interval:
0.01663715 0.22936850 sample estimates:
prop 1      prop 2   0.3216783 0.1986755
```

The p-value is significant, which leads us to reject H_0 in favor of H_1