

Math-640: Bayesian Statistics
Assignment 1 – Due Wed. Feb. 3

1. BDA – Chapter 1, exercise 6.

Conditional probability: approximately $1/125$ of all births are fraternal twins and $1/300$ of births are identical twins. Elvis Presley had a twin brother (who died at birth). What is the probability that Elvis was an identical twin? (You may approximate the probability of a boy or girl birth as $1/2$).

2. BDA – Chapter 2, exercise 8.

Normal distribution with unknown mean: a random sample of n students is drawn from a large population, and their weights are measured. The average weight of the n sampled students is $\bar{y} = 150$ pounds. Assume the weights in the population are normally distributed with unknown mean θ and known standard deviation 20 pounds. Suppose your prior distribution for θ is normal with mean 180 and standard deviation 40.

- (a) Give your posterior distribution for θ . (Your answer will be a function of n .)
- (b) A new student is sampled at random from the same population and has a weight of \tilde{y} pounds. Give a posterior predictive distribution for \tilde{y} . (Your answer will still be a function of n .)
- (c) For $n = 10$, give a 95% posterior interval for θ and a 95% posterior predictive interval for \tilde{y} .
- (d) Do the same for $n = 100$.

3. Suppose y_1, \dots, y_n is a random sample from a negative binomial distribution with parameters (r, θ) , where r is known and θ is unknown. Here, r denotes the number of successes and θ denotes the success probability.

Suppose $\theta \sim \text{Beta}(\alpha, \beta)$.

- (a) Derive the posterior distribution of θ .
- (b) Show that the posterior mean is a weighted average of the MLE and the prior mean.
- (c) Suppose $r = 5$ and the following $n = 10$ observations are made:

7 10 5 8 6 12 6 9 7 5

For each of the the following priors for θ :

beta(1, 1), beta(0.5, 0.5), beta(5, 5), beta(1, 5), beta(5, 1),

- i. plot the prior and posterior distributions of θ on the same graph;
- ii. provide the posterior mean and a 95% credible interval for θ .

4. Let x_1, \dots, x_n be random samples from a Pareto(k, θ) distribution with probability density function

$$f(x|\theta) = \frac{\theta k^\theta}{x^{\theta+1}}, \quad x > k > 0, \quad \theta > 0$$

- (a) For the Pareto sampling model with known k , show that the conjugate prior for inference about θ is a gamma distribution.
- (b) Derive the posterior distribution of θ for the conjugate prior Gamma(α, β).