Mathematics 640 Homework 4

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Exercise 1

Part A

$U \sim Unif(0,1)$

$X = g(U) = log\left(\frac{U}{1-U}\right) \implies e^X = \frac{U}{1-U} \implies e^X = U(1+e^X)$ so $g^{-1}(U) = \frac{e^X}{1+e^X}$

Please note that $f_U(u) = \frac{1}{1-0} = 1$

$f_X(x) = f_U(g^{-1}(x)) * \left|\frac{d}{dx}g^{-1}(x)\right| = 1 * \left|\frac{d}{dx}\frac{e^x}{1+e^x}\right| = \frac{e^x}{(1+e^x)^2}$

Part B

$F_X(x) = \int_{-\infty}^{x} f_X(x)dx = \frac{e^x}{1+e^x}$

$F_X(F_X^{-1}(x)) = \frac{exp\left\{F_X^{-1}(x)\right\}}{1+exp\left\{F_X^{-1}(x)\right\}} \implies$

$U = \frac{exp\left\{F_X^{-1}(x)\right\}}{1+exp\left\{F_X^{-1}(x)\right\}} \implies$

$U\left(1+exp\left\{F_X^{-1}(x)\right\}\right) = exp\left\{F_X^{-1}(x)\right\} \implies$

$U * 1 + U * exp\left\{F_X^{-1}(x)\right\} = exp\left\{F_X^{-1}(x)\right\} \implies$

$U = exp\left\{F_X^{-1}(x)\right\}(1-U) \implies$

$exp\left\{F_X^{-1}(x)\right\} = \frac{U}{1-U}$

```
U = runif(1e6,0,1)
X = log(U/(1-U))
```

Part C

$F_X(-2) = e^{-2}/(1+e^{-2}) = 0.1192029$

```
mean(x<=-2)
# [1] 0.119257
```

Exercise 2

Part A

i

1. Generate $x \sim g$ where $g(x)$ is the evelope function and $U \sim Unif(0,1)$
2. The x value is accepted if x if $U < \frac{f(x|y)}{M*g(x)}$ else it is rejected

```
f = function(x) {
        result = 30*x^2*(1-x)^2
        if(result >= 0 & result <= 1){
                return(result)    }
        else{
        return(0)}
        }
g = function(x) return( dnorm(x,0.5,0.25) )
M = 1.2

K=1e6
x.rs = accpt = rep(NA, K)
for(i in 1:K) {
        x = rnorm(1,0.5,0.25); u = runif(1)
        if(u < f(x)/(M*g(x))) {
                x.rs[i]=x
                accpt[i]=1 }
        else accpt[i]=0
}
mean(x.rs, na.rm = T)
```

ii

```
g = function(x) return( dbeta(x,2,2) )
M = 1.25
x = seq(0, 1, length=1e6)

K=1e6
x.rs = accpt = rep(NA, K)
for(i in 1:K) {
        x = rbeta(1,2,2); u = runif(1)
        if(u < f(x)/(M*g(x))) {
                x.rs[i]=x
                accpt[i]=1 }
        else accpt[i]=0
```

```
}
mean(x.rs, na.rm = T)
```

Part B

From the code in Part A The Monte Carlo Estimates are:

$E(X_i) = 0.5004494$

$E(X_{ii}) = 0.5000574$

Part C

1. Obtain the importance weight $w(x) = \frac{f(x|y)}{M*g(x)}$
2. Estimate $E(h(x)|y) \approx \frac{\sum_{k=1}^{K} w(x^{(k)})*(x^{(k)})}{\sum_{k=1}^{K} w(x^{(k)})}$

```
samp.size=10000
x <- matrix(NA, samp.size, 2)
wts <- matrix(NA, samp.size, 2)

M = 1.2
x[,1] = rnorm(samp.size,0.5,0.25)
wts[,1] = f(x[,1])/(M*dnorm(x[,1],0.5,0.25))

M=1.25
x[,2] = rbeta(samp.size,2,2)
wts[,2] = f(x[,2])/(M*dbeta(x[,2],2,2))

mu.is = rep(NA, 0)

for(i in 1:2){
        mu.is[i] = sum((x[,i])*wts[,i])/sum(wts[,i])
}

mu.is
# [1] 0.4915551 0.5076251
```

Part D

$E(X) = \int_0^1 x * f(x)dx =$

$\int_0^1 x * 30x^2 * (1-x)^2 dx =$

$\int_0^1 30x^3 * \left(1 - 2x + x^2\right) dx =$

$\int_0^1 30x^3 - 60x^4 + 30x^5 dx =$

$\Big|_0^1 \frac{30}{4}x^4 - 12x^5 + 5x^6 dx = \frac{15-24+10}{2} = \frac{1}{2}$

In both b and c, the Monte Carlo estimation of the mean are very close to the Analytically evaluated $E(x)$

Exercise 3

Part A

$p(\boldsymbol{y}|\mu,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} * exp\left\{-\frac{1}{2\sigma^2}(y_i-\mu)^2\right\} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\}$

$$p(\mu|\sigma^2,\boldsymbol{y}) = \frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \pi(\mu) * \pi(\sigma^2)}{\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \pi(\mu) * \pi(\sigma^2) d\mu} =$$

$$\frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \left(\frac{1}{\sqrt{2\pi r_0^2}}\right) * exp\left\{-\frac{1}{2r_0^2}(\mu_0-\mu)^2\right\} * \pi(\sigma^2)}{\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \left(\frac{1}{\sqrt{2\pi r_0^2}}\right) * exp\left\{-\frac{1}{2r_0^2}(\mu_0-\mu)^2\right\} * \pi(\sigma^2) d\mu} =$$

$$\frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \left(\frac{1}{\sqrt{2\pi r_0^2}}\right) * exp\left\{-\frac{1}{2r_0^2}(\mu_0-\mu)^2\right\}}{\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \left(\frac{1}{\sqrt{2\pi r_0^2}}\right) * exp\left\{-\frac{1}{2r_0^2}(\mu_0-\mu)^2\right\} d\mu} =$$

$$\frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \left(\frac{1}{\sqrt{2\pi r_0^2}}\right) * exp\left\{-\frac{1}{2r_0^2}(\mu_0-\mu)^2\right\}}{\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \left(\frac{1}{\sqrt{2\pi r_0^2}}\right) * exp\left\{-\frac{1}{2r_0^2}(\mu_0-\mu)^2\right\} d\mu} =$$

Let $I = exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * exp\left\{-\frac{1}{2r_0^2}(\mu_0-\mu)^2\right\} =$

$exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * exp\left\{-\frac{1}{2r_0^2}(\mu_0-\mu)^2\right\} =$

$exp\left\{-\frac{1}{2}\left[\mu^2\left(\frac{n}{\sigma^2}+\frac{1}{r_0^2}\right) - 2\mu\left(\mu - \frac{\sum y_i}{\sigma^2} + \frac{\mu_0}{\sigma^2}\right) + \frac{\sum y_i^2}{\sigma^2} + \frac{\mu_0^2}{r_0^2}\right]\right\} =$

$$exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{r_0^2}\right)\left(\mu - \frac{\frac{\sum y_i}{\sigma^2}+\frac{\mu_0}{r_0^2}}{\frac{n}{\sigma^2}+\frac{1}{r_0^2}}\right)^2\right\} * exp\left\{-\frac{1}{2}\left[\frac{\sum y_i^2}{\sigma^2} + \frac{\mu_0^2}{r_0^2} - \frac{\left(\frac{\sum y_i}{\sigma^2}+\frac{\mu_0}{r_0^2}\right)^2}{\frac{n}{\sigma^2}+\frac{1}{r_0^2}}\right]\right\}$$

Thus,

$$exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{r_0^2}\right)\left(\mu - \frac{\frac{\sum y_i}{\sigma^2}+\frac{\mu_0}{r_0^2}}{\frac{n}{\sigma^2}+\frac{1}{r_0^2}}\right)^2\right\} \text{ is the kernel for } N\left(\frac{\frac{\sum y_i}{\sigma^2}+\frac{\mu_0}{r_0^2}}{\frac{n}{\sigma^2}+\frac{1}{r_0^2}}, \left[\frac{n}{\sigma^2}+\frac{1}{r_0^2}\right]^{-1}\right)$$

$$\therefore \mu|\sigma^2,\boldsymbol{y} \sim N\left(\frac{\frac{\sum y_i}{\sigma^2}+\frac{\mu_0}{r_0^2}}{\frac{n}{\sigma^2}+\frac{1}{r_0^2}}, \left[\frac{n}{\sigma^2}+\frac{1}{r_0^2}\right]^{-1}\right)$$

$$p(\mu|\sigma^2,\boldsymbol{y}) = \frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \pi(\mu) * \pi(\sigma^2)}{\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \pi(\mu) * \pi(\sigma^2) d\sigma^2} =$$

$$\frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \pi(\sigma^2)}{\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \pi(\sigma^2) d\sigma^2} =$$

$$\frac{\left(\frac{1}{\sqrt{\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * \left(\sigma^2\right)^{-(\alpha+1)} e^{-2\beta/2\sigma^2}}{\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{\sigma^2}}\right)^n * exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\right\} * (\sigma^2)^{-(\alpha+1)} e^{-2\beta/2\sigma^2} d\sigma^2} =$$

$$\frac{\left(\sigma^2\right)^{-n/2} * exp\left\{-\frac{1}{2\sigma^2}\left[\beta+\sum_{i=1}^n (y_i-\mu)^2\right]\right\} * \left(\sigma^2\right)^{-(\alpha+1)}}{\int_{-\infty}^{\infty}(\sigma^2)^{-n/2} * exp\left\{-\frac{1}{2\sigma^2}\left[\beta+\sum_{i=1}^n (y_i-\mu)^2\right]\right\} * \left(\sigma^2\right)^{-(\alpha+1)} d\sigma^2} =$$

$$\frac{exp\left\{-\frac{1}{2\sigma^2}\left[\beta+\sum_{i=1}^n (y_i-\mu)^2\right]\right\} * \left(\sigma^2\right)^{-(\alpha+1+n/2)}}{\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2\sigma^2}\left[\beta+\sum_{i=1}^n (y_i-\mu)^2\right]\right\} * (\sigma^2)^{-(\alpha+1+n/2)} d\sigma^2}$$

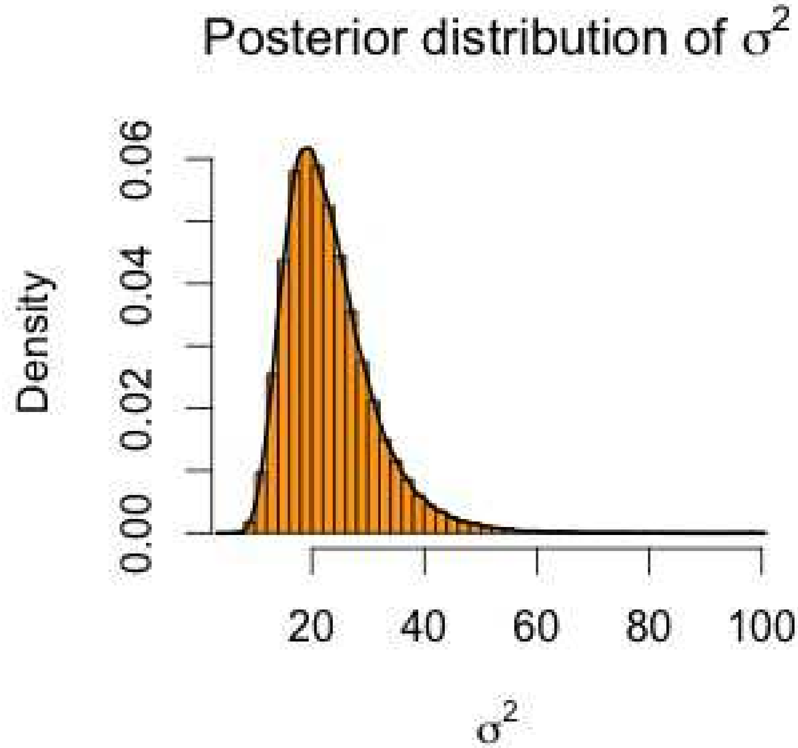
We recognize $exp\left\{-\frac{1}{2}\left[\beta + \sum_{i=1}^n (y_i-\mu)^2\right]/\sigma^2\right\} * \left(\sigma^2\right)^{-(\alpha+1+n/2)}$ as the kernel of the inverse gamma.
$\therefore \sigma^2|\mu, \boldsymbol{y} \sim innv - gamma\left(\alpha + n/2, \frac{1}{2}\left[\beta + \sum_{i=1}^n (y_i-\mu)^2\right]\right)$

Part B

```
y <-
read.table("~/Documents/Georgetown/Bayesian-Statistics-Mathematics-640/HW2/school.dat",
quote="\"", comment.char="")[,1]
n = length(y)
T = 1e5
mu0 = 5
r20 = 0.5
alpha = 1
beta = 4
theta = matrix(NA, T, 2)
theta[1,] = c(30, 5)
for (i in 2:T) {
        theta[i,1] = rnorm(1,
                        (sum(y)/theta[i-1,2]+mu0/r20)/(n/theta[i-1,2]+1/r20),
                                (n/theta[i-1,2]+1/r20)^-0.5)
        phi = rgamma(1, alpha+n/2,
                                rate = 1/2*(beta+sum((y-theta[i,1])^2) ))
        theta[i,2] = 1/phi
}
```

Part C

```
nburn = 1e3
hist(theta[nburn:T,2], breaks=50,xlab=expression(sigma^2), prob=T,
        col="orange",
        main=expression(paste("Posterior distribution of ",sigma^2)))
lines(density(theta[,2]), lwd=2, col="black")
```



Exercise 4

Part A

$$r = \frac{p(\theta^*|y)}{p(\theta^{(t-1)}|y)} = \frac{exp\left\{-\frac{(\theta^*)^2}{2}\right\}}{exp\left\{-\frac{\left(\theta^{(t-1)}\right)^2}{2}\right\}} = exp\left\{-\frac{(\theta^*)^2-\left(\theta^{(t-1)}\right)^2}{2}\right\}$$

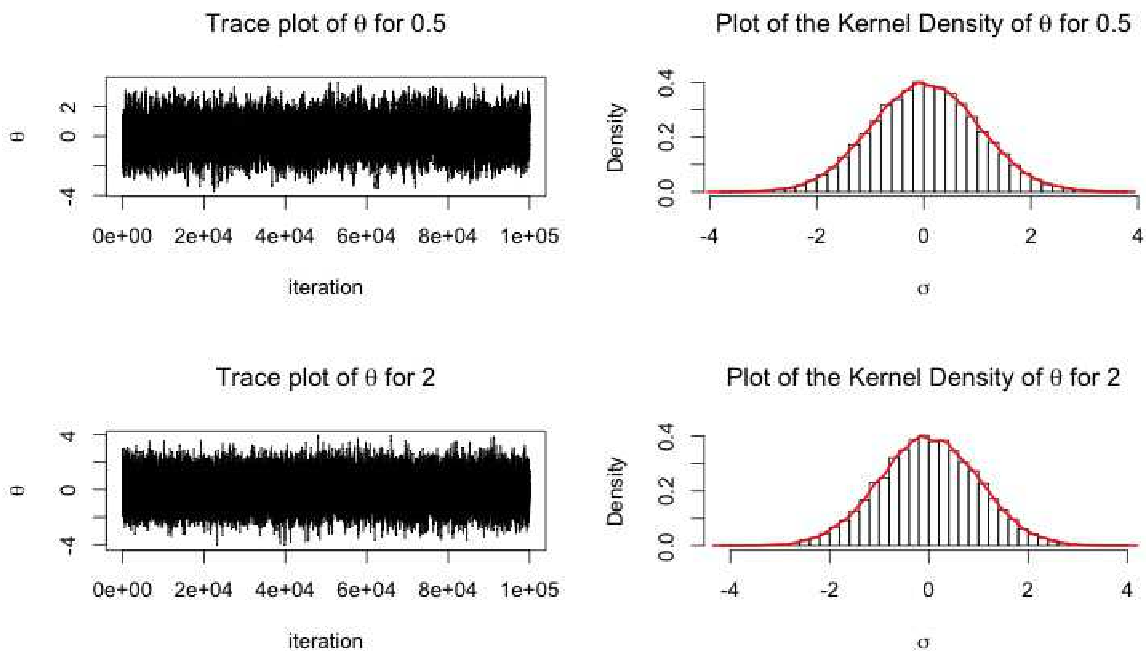
```
metro = function(sigma, theta0 = 0, T=1e5){
        theta = c(theta0)
        accept = c(0)
        print(sigma)
        for(t in 2:T){
                theta_star = rnorm(1, theta[t-1], sigma)
                r = exp(-(theta_star^2-theta[t-1]^2)/2)
                u = runif(1)
                if(u<r){
                        theta[t] = theta_star
                        accept[t] = 1
                }else{
                        theta[t] = theta[t-1]
                        accept[t] = 0
                                }
                        }
        return(data.frame(theta, accept))
        }
```

```
s1 = metro(0.5)
s2 = metro(2)
```

Part B

```
plot(s1[,1], type="l", xlab="iteration", ylab=expression(theta),
        main=expression(paste("Trace plot of ",theta," for " ,sigma=0.5)))
hist(s1[,1], breaks=50,xlab=expression(sigma), prob=T,
        main=expression(paste("Plot of the Kernel Density of ",theta, " for ", sigma=0.5))

plot(s2[,1], type="l", xlab="iteration", ylab=expression(theta),
        main=expression(paste("Trace plot of ",theta," for " ,sigma=0.5)))
hist(s2[,1], breaks=50,xlab=expression(sigma), prob=T,
        main=expression(paste("Plot of the Kernel Density of ",theta, " for ", sigma=0.5))
```



Part C

Acceptance rate

```
mean(s1[-(1:nburn),2]==1)
#        [1] 0.8435152
mean(s2[-(1:nburn),2]==1)
#        [1] 0.499798
```

At $\sigma = 0.5$, the acceptance rate is 84.35% and at $\sigma = 2$, the acceptance rate is 49.98%.

Autocorrelation

```
library(coda)
s1.mcmc = as.mcmc(s1[-(1:nburn),1])
s2.mcmc = as.mcmc(s2[-(1:nburn),1])
```

```
autocorr(s1.mcmc)
#               [,1]
# Lag 0   1.00000000
# Lag 1   0.91379243
# Lag 5   0.64000173
# Lag 10  0.41388263
# Lag 50  0.01565715

autocorr(s2.mcmc)
#                [,1]
# Lag 0   1.000000000
# Lag 1   0.639223395
# Lag 5   0.111507977
# Lag 10  0.006725854
# Lag 50  0.001244550
```

Effective Sample Size

```
effectiveSize(s1.mcmc)
# var1
# 4404.591
effectiveSize(s2.mcmc)
# var1
# 21589.28
```

The acceptance rate for $\sigma = 0.5$ is 84.35% and at $\sigma = 2$, the acceptance rate is 49.98%. The higher acceptance rate isn't good because the MCMC chain is not moving around and exploring the state space. Given the high autocorrelation for $\sigma = 0.5$, there is a higher number of repeating pattern within the MCMC as opposed to $\sigma = 2$. Therefore, using $\sigma = 2$ is a better option for the proposed function.