2/6/14

Suppose we have the subtraction game but $S = \{1, 3, 4\}$ is the number of chips you can take

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P	N	P	N	N	N	N	P	N	P	N	N	N	N	P

- State is in N if at least <u>one</u> move to P
- P if all moves lead to N

N - next player to move has a winning strategy

P - previous player to move has a winning strategy

x is P if and only if $x \bmod 7 = 0$ or 2

<u>Induction</u>

$x = 0 \bmod 7$

can move to:

	$x-1$,	$x-3$,	$x-4$
mod 7	6	4	3
	N	N	N

$\Rightarrow x \in P$

$x = 1 \bmod 7$

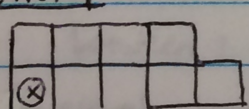
	$x-1$	$x-3$	$x-4$
mod 7	0	5	4
	P	N	N

$\Rightarrow x \in N$

$7k, 7k+1, \ldots, 7k+6 \xrightarrow{7k+7}$

P N P N N N N

<u>Chomp</u>

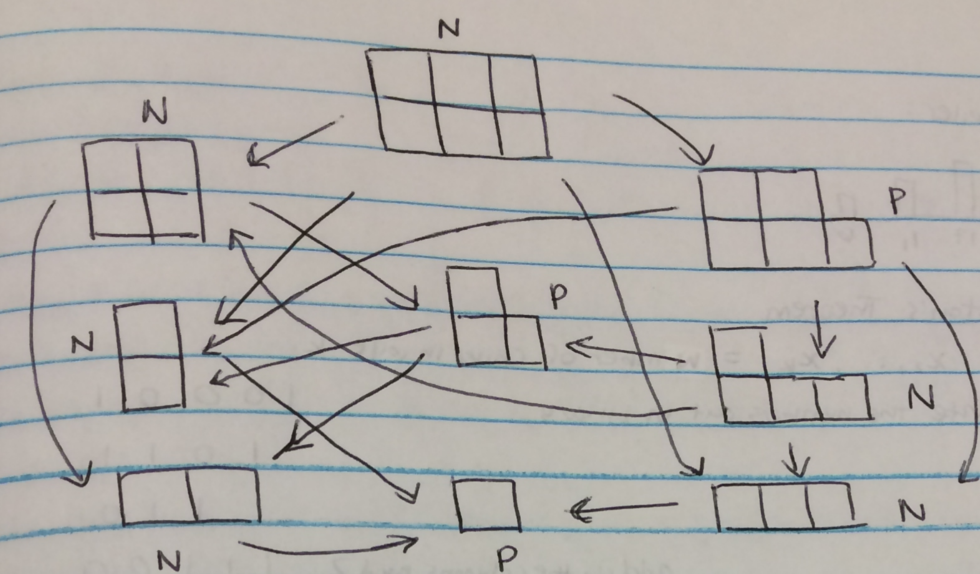


<u>goal</u>: force other player to remove final square

<u>Moves</u>

pick square, remove any above or to the right of that one.

impartial and progressively bounded



If state is rectangle R then $R \in N$.

R^- is the state you get to by removing the top-right square.

2 cases: $R^- \in P \Rightarrow R \in N$

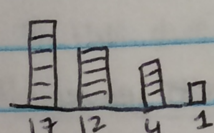
$R^- \in N$, $\exists$ a move from R^- to S and $S \in P$

- If go from R^- to S then also go R to S. $\Rightarrow R \in N$

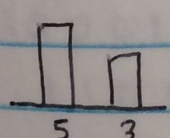
With every move, you remove the top-right block because it is above and to th
right of every other block.

We have shown that if you start with a rectangle, it is in N. There is
a winning strategy. You don't necessarily know what that strategy is.

<u>Nim</u>



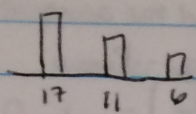
Can remove any number of chips from a single pile. Want to remove the la
chip.



N - piles are unequal

P - piles are equal

3 piles:



<u>Bouton's Theorem</u>

$x_1, x_2, \ldots, x_k$ = number of chips in pile k

Write the numbers out in binary

$$\begin{array}{r} 10001 \\ 1011 \\ 110 \\ \hline 11100 \end{array}$$

add up the columns mod 2 11100

= nim sum

<u>Theorem</u> State $x_1, x_2, \ldots, x_k$ is in P iff $x_1 \oplus x_2 \oplus \ldots \oplus x_k = 0$

For 2 piles, $x_1 \oplus x_2 = 0$ iff $x_1 = x_2$

2/10/14

<u>Misere Play</u>

$S = \{1, 3, 4\}$

0 1 2 3 4 5 6 7 8 9 10

N N P N P

$P = \begin{cases} n \equiv 1 \mod 7 \\ n \equiv 3 \mod 7 \end{cases}$

<u>Empty & Transfer</u>

$\boxed{m}$ $\boxed{n}$

te[illegible]al position (1,1) you lose if no legal moves (you can't leave [illegible] boxes empty)

[illegible]2) = N

[illegible],2) = N

(3,3) = P because end up with (2,1) no matter what

(5,3) = P

(4,1)

(3,1)

(2,1)

(1,1)

if both odd then P
if one is even then N
(6,3) = N (8,5) = N (k,4) = N
(3,3) (10,9) = N
P: any move → move → P position again
both odd → 1 even 1 odd → both odd
an even can always be written as sum of 2 odds
terminal is (1,1,1)
(odd, odd, odd) = P but unknown if those are all the P positions
(3,5,7) (2,3,7) (1,1,7)
dynamic subtraction game: # you can subtract depends on previous s[tate]
of game
first move: can remove as many as want but not whole pile
2nd move onwards: can remove at most the number removed last time
get to 0 wins
all odd positions are N positions.
to win: look at binary representation
powers of 2 only have one 1
delete last 1 in binary representation
2/11/14
Monday is President's Day
⇒ HW due in class Tuesday next week, no section either
<u>Bouton's Theorem</u>
If a position has piles $x_1, x_2, \ldots, x_k$
P-position iff $x_1 \oplus x_2 \oplus \ldots \oplus x_k = 0$ where $\oplus$ is nim sum
17, 13, 9

$$\begin{array}{r} 10001 \\ 1101 \\ \underline{1001} \\ 10101 \end{array}$$

Let $\hat{P}$ be set position with nim sum 0.
· terminal position 0 in all piles in P.

• From all $x \notin \hat{P}$ can move to $\hat{P}$

• all moves from $\hat{P}$ leave $\hat{P}$

$x_1 \oplus x_2 \oplus \ldots \oplus x_k = 0$

$(x_1, \ldots, x_k) \in \hat{P}$

The only way they add up to 0 is if the nim sum of $k-1$ add up to x_k

11, 13, 6

1011	$11 \oplus 13 = 6$	$11 \oplus 6 = 13$
1101	1011	$13 \oplus 6 = 11$
110	1101	
0000	110 = 6	

For any $x \in \hat{P}$, move to $x' \notin \hat{P}$

Now show that for every $x \notin \hat{P}$, can move to $\hat{P}$

```
10001 = 17
 1101 = 13
 1001 = 9
10101      ← look here for 1's and flip bits in 17 so nim sum = 0.
```

⇓

```
00100 = 4    only move in this case because can't
 1101 = 13   flip leading 0 bits to be 1's because
 1001 = 9    can't increase size of pile
00000
```

$x_1, x_2, \ldots, x_k$, nim sum $\neq 0$.

```
1011010
1000100
0001001
0010111
```

Let z be the nim sum and let k be the column of first 1 ($k=4$)

(largest power of 2 dividing nim sum)

- must be a pile with a 1 in position k, pile l
- reduce pile l from x_l to $x_l \oplus z$
- have to check $x_l \oplus z < x_l$ and new nim sum is 0.

replace x_ℓ with $x_\ell \oplus z$

$x_1 \oplus x_2 \oplus \dots \oplus (x_\ell \oplus z) \oplus \dots \oplus x_k$

$= z \oplus x_1 \oplus \dots \oplus x_k$

$= z \oplus z = 0$

$x_\ell \oplus z < x_\ell$

$x_\ell = 1\,0\,0\,1\,1\,0\,1\,0$

$z = 0\,0\,0\,1\,0\,1\,1\,0$

$1\,0\,0\,0 \dots$ smaller than x_ℓ

If more than 1 pile has a 1 in column k, you can pick any of those as x_ℓ.

$z = z_k z_{k-1} \dots z_0, \; z_k = 1$

$x_\ell = w_m w_{m-1} \dots w_{k+1} w_k w_{k-1} \dots w_0$

$x_\ell \oplus z = w_m w_{m-1}, \dots, w_{k+1}, 0, a_{k-1} \dots a_0 \qquad a_i = w_i + z_i \bmod 2.$

digits m to k+1 are unchanged and k^{th} digit goes from 1 to 0

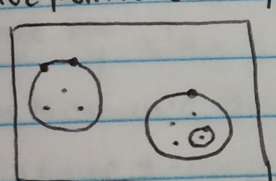
$\hat{P} = P$ yay.

Rim

have points on a plane

Move draw a line through at least one point not intersecting any others

- last point taken is the winner



can think of the curves as piles $(3,3,1) \rightarrow (1,1,3,1)$

- same moves as nim plus an extra move, reduce a pile + split it into 2

P positions same, piles nim sum is 0

want to check $x_1 \oplus x_2 \oplus \dots \oplus x_k = 0$. split x_k into u and v

$x_1 \oplus x_2 \oplus \dots \oplus x_{k-1} \oplus u \oplus v$

$= x_k \oplus u \oplus v \neq 0$ this sum only equals 0 if

$x_k = u \oplus v$

$x_k > u + v$

$> u \oplus v$ but by this, $x_k \neq u \oplus v$

$u \oplus v < u + v$

2/13/14

Two-person zero-sum games

- paper, scissors, rock

	P	R	S
P	0	1	-1
R	-1	0	1
S	1	-1	0

Set of strategies

- m player 1
- n player 2
- payoff matrix $m \times n$ matrix (A_{ij}) $1 \le i \le m$, $1 \le j \le n$

if player 1 chooses i

" 2 " j

then player 1 gets a_{ij} and player 2 gets $-a_{ij}$

Holder

$$\text{Chooser} \quad \begin{array}{c|cc} & L_2 & R_2 \\ \hline L_1 & 1 & 0 \\ R_1 & 0 & 2 \end{array}$$

$0 = \max_i \min_j a_{ij}$ best deterministic for player 1

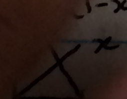
$1 = \min_j \max_i a_{ij}$ " " " player 2

Player 1 choose L with probability x.

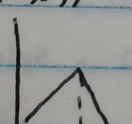
L_2 ... $(x \cdot 1 + (1-x) \cdot 0,\ x \cdot 0 + (1-x) \cdot 2)$

...$n(x, 2(1-x))$

...$(1-x)$ x



$\Rightarrow$



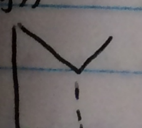
$x = \frac{2}{3}$

Bes... with random strategy $\frac{2}{3}$, with $x = \frac{2}{3}$

... goes left with probability y

... $2(1-y))$

$\Rightarrow$



$y = \frac{2}{3}$

player 1's strategy guaranteed to earn at least $\frac{2}{3}$

" 2's " payout at most $\frac{2}{3}$

value of the game $= \frac{2}{3}$

Pure strategy choosing a fixed strategy i.e. no randomness

mixed strategies, choose randomly according to some distribution

$\Delta_m = \{ x = (x_1 \dots x_m), x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \}$

$\Delta_n = \{ y = (y_1 \dots y_n), y_j \geq 0, \sum y_j = 1 \}$

payoff (expected) if player 1 use strategy x, player 2 use strategy y

X: strategy of player 1 $\mathbb{P}[X = i] = x_i$

Y: strategy of player 2 $\mathbb{P}[Y = j] = y_j$

$$E\, a_{XY} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \mathbb{P}[X = i, Y = j]$$
$$= \sum_i \sum_j x_i a_{ij} - y_j$$
$$= x^T A y$$

safety strategy for player 1

If $x^* \in \Delta_m$ so that $\min_{y \in \Delta_n} (x^*)^T A y = \max_x \min_y x^T A y$

$y^* \in \Delta_n$, $\max_x x^T A y^* = \min_y \max_x x^T A y$

$\max_x \min_y x^T A y \leq \min_y \max_x x^T A y$

Von Neuman's Theorem (min max)

in a finite 2-player zero-sum game, there is a V such that

$V = \max_x \min_y x^T A y = \min_y \max_x x^T A y$ = value of the game

safety strategies are optimal strategies

(x, y) is a Nash equilibrium if

- $\forall y' \in \Delta_n$: $x^T A y' \geq x^T A y$ Payout is always at least as big as strategy y
- $\forall x' \in \Delta_m$: $(x')^T A y \leq x^T A y$ player 1 can't change strategy to win more \$.

neither side has an incentive to change

$V = x^T A y$

$V \leq (x^*)^T A y \leq x^T A y$

$V \leq x^T A y$

$x^T A y \leq x^T A y^* \leq V$

$x^T A y \leq V \Rightarrow V = x^T A y$

$$\begin{array}{ccc} & \multicolumn{2}{c}{\text{goal}} \\ & L & R \end{array}$$

$$\text{kicker} \quad \begin{array}{c} L \\ R \end{array} \begin{pmatrix} 0.6 & 1 \\ 1 & 0.8 \end{pmatrix}$$

$x = (x_1, 1-x_1)$, $y = (y_1, 1-y_1)$

$\min_y x^T A y = \min_j (x^T A)_j$ (general fact)

$z = x^T A$, $z = (z_1, \ldots, z_n)$

$$\min_y x^T A y = \min_y z y = \min_y \sum_{j=1}^{n} z_j y_j$$

z_j is payoff if player 2 chooses state j $= \sum x_j a_{ij}$

2/18/14

Two player zero sum games

$A \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ payoff matrix

- each game value V
- player 1 strategy x for any strategy y $x^T A y \geq V$
- player 2 strategy y $\forall x$ $x^T A y \leq V$

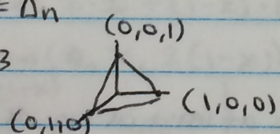
For any fixed x $\exists j$ such that the pure strategy $e_j = (0, 0, \ldots, \underset{\text{position } j}{1}, 0, 0)$ is best reply

i.e. $\min_y x^T A y = \min_j x^T A e_j = \min_j (x^T A)_j$

$\Delta_n = \{(y_1, \ldots, y_n) : y_i \geq 0, \sum_{i=1}^{n} y_i = 1\}$

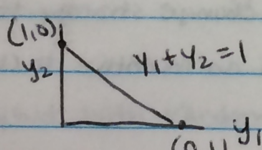
$\forall j$ $e_j \in \Delta_n$

$n = 3$



The vertices of the triangle are the pure strategies and the triangle is the set of all strategies

$n = 2$



a point (y_1, y_2, y_3) on the triangle means probability of playing strategy $1 = y_1$, $2 = y_2$, $3 = y_3$ - add up to 1.

$z \in$ convex hull of $w_1, \ldots, w_n$ if $\exists \alpha_1, \ldots, \alpha_n \geq 0$ where $\sum_{i=1}^{n} \alpha_i = 1$ $z = \sum_{i=1}^{n} \alpha_i w_i$

Δ_n convex combinations of e_j $y = (y_1, \ldots, y_n) = \sum_{i=1}^{n} y_i e_j$

Player 1 plays x

$x^T A y = x^T A (\sum_{j=1}^{n} y_i e_j)$

$= \sum_{j=1}^{n} y_j (x^T A e_j)$

$= \sum_{j=1}^{n} y_j (x^T A)_j$

Let j^* be j minimizing $(x^T A)_j = z_j$, $z_{j^*} \le z_j \ \forall j$

$x^T A y = \sum y_j z_j$

$\ge \sum y_i z_{j^*} = z_{j^*}$

Take $y = e_{j^*}$, then $x^T A e_{j^*} = 1 \cdot z_{j^*} = z_{j^*}$

Optimal strategies according to Bayes-Nash will be mixed strategies. But best response for other player is a pure strategy.

(x^*, y^*) optimal strategies

y^* minimizes $(x^*)^T A y \rightarrow \min_y (x^*)^T A y = (x^*)^T A y^*$

Both $(x^{*T} A)_1 = (x^{*T} A)_2 = \min_j ((x^*)^T A)_j$

$$x_T A = \begin{pmatrix} \sum x_i a_{i1} \\ \vdots \\ \sum x_i a_{in} \end{pmatrix} \leftarrow \text{entry } j \text{ expected payoff if player 2 plays } j$$

best response strategy will be a linear combination of strategies

Example

$$\begin{array}{ccc} & L & R \\ \begin{array}{c} L \\ R \end{array} & \multicolumn{2}{c}{\begin{pmatrix} 0.8 & 1 \\ 1 & 0.7 \end{pmatrix}} \end{array}$$

Player 1's strategy $(x_1, 1-x_1) = x$

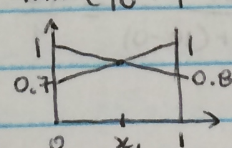
$\min_j (x^T A)_j = \min(0.8x_1 + (1-x_1), x_1 + 0.7(1-x_1))$

Pure strategy: $x_1 = 0$ or 1. mixed strategy: $x_1 \in (0,1)$

player 1 is trying to maximize the value, player 2 trying to minimize

$\max_x \min_j (x^T A)_j = \max_{x_1} \min(\frac{8}{10} x_1 + (1-x_1), x_1 + \frac{7}{10}(1-x_1)) = f(x_1)$

Want to minimize $f(x_1)$



$0.8x_1 + (1-x_1) = x_1 + \frac{7}{10}(1-x_1)$

$0.3 - 0.3x_1 = 0.2x_1 \Rightarrow x_1 = \frac{3}{5}$

$V = \min(0.8 \cdot \frac{3}{5} + 1 \cdot \frac{2}{5}, \frac{3}{5} + 0.7 \cdot \frac{2}{5})$

$= \min(0.88, 0.88) = 0.88$

$x^* = (\frac{3}{5}, \frac{2}{5})$ $V = 0.88$ Player 1 should go left with probability $\frac{3}{5}$

no matter what player 2's strategy is, V stays same. (expected payoff)

If x^*, y^* optimal $\forall j$ $y_j^* > 0$ then $(x^*)^T A e_j = V$

(Principle of Indifference)

Suppose payoff matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & 17 \end{pmatrix}$

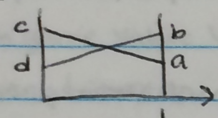
(e_1, e_1) is an optimal strategy, saddle point (i_*, j_*) is a saddle point if $\max_i a_{ij_*} = \min_j a_{i_*j} = a_{i_*j_*}$

17 can be any number.

Solving 2x2 games $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

1. Check for saddle point.
   - (e_{i*}, e_{j*}) equilibrium
   - $V = a_{i*j*}$
2. $x = (x_1, 1-x_1)$

$$\min_y x^T A y = \min(ax_1 + c(1-x_1),\ bx_1 + d(1-x_1))$$



Solve

$$ax_1 + c(1-x_1) = bx_1 + d(1-x_1)$$
$$(a-b)x_1 = (d-c)(1-x_1)$$
$$((a-b)+(d-c))x_1 = d-c$$
$$x_1^* = \frac{d-c}{(a-b)+(d-c)}$$

$$V = ax_1^* + c(1-x_1^*)$$

Example

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow x_1 = \frac{(2-0)}{(2-0)+(1-0)} = \frac{2}{3}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 10 \end{pmatrix} \Rightarrow x_1 = \frac{(10-0)}{(1-2)+(10-0)} = \frac{10}{9} > 1$$ there is a saddle point.

2/20/14

2 player zero sum games

Dominating strategies: If $\forall j$ $a_{ij} \geq a_{i'j}$ then i dominates i' for player 1.

example:

$$\begin{array}{c} i \\ i' \end{array} \begin{pmatrix} 7 & 12 & 3 & 15 \\ 0 & 11 & 1 & 2 \end{pmatrix}$$

In any optimal strategy $x_{i'} = 0$ (probability of playing this strategy is 0)

$x' = (x'_1, x'_2, \ldots, x'_m)$ $x'_{i'} > 0$

$x_i = (x'_1, x'_2, \ldots, x'_i + x'_{i'}, 0, \ldots, x'_m)$ $\forall y$ $xAy > x'Ay$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 6 \end{pmatrix}$$ row 1 < row 2

$$\Rightarrow \begin{pmatrix} 4 & 5 & 6 \\ 7 & 1 & 6 \end{pmatrix}$$

Player 2 : j dominates j' if $a_{ij} \le a_{ij'}$ $\forall i$ - never play j'

$5,1 \le 6,6$ so $\Rightarrow \begin{pmatrix} 4 & 5 \\ 7 & 1 \end{pmatrix}$

Go less

- each player picks number between $\{1, 2, \dots, n\}$.
- minimum win \$1, 0 tie

$$\begin{pmatrix} 0 & & 1 \\ & 0 & \\ -1 & & 0 \end{pmatrix}$$ $\forall i > 1, \forall j$ $a_{ij} \ge a_{ij}$ $(0\ 1\ 1\ 1\ 1 \dots 1)$

both play 1, value of the game is 0

Plus One

- pick from $\{1, \dots, n\}$
- get 1 if your number is the other player's +1
- lose 2 if your number is more than 1 larger

$$\begin{pmatrix} 0 & -1 & & 2 \\ 1 & 0 & -1 & \\ & \ddots & \ddots & \ddots \\ -2 & & 1 & 0 & -1 \end{pmatrix}$$ Row 1 dominates rows $4, \dots, n$

Column 1 dominates columns $4, \dots, n$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$ matrix is antisymmetric $\Rightarrow$ value is 0

$a_{ij} = -a_{ji}$ $\forall i, j$

$a_{ii} = -a_{ii} = 0$

V = expected payoff for player 1

$-V$ = expected payoff for player 2

$V = -V = 0$

$V = 0$

Player 1 $x = (x_1, x_2, x_3)$

$\min(x_2 - 2x_3, -x_1 + x_3, 2x_1 - x_2) = V$

$x_2 - 2x_3 = a_1 \ge 0$

$-x_1 + x_3 = a_2 \ge 0$

$2x_1 - x_2 = a_3 \ge 0$

$(x_2 - 2x_3) + 2(x_3 - x_1) + 2x_1 - x_2 = a_1 + 2a_2 + a_3$

$0 = a_1 + 2a_2 + a_3 \Rightarrow a_1 = a_2 = a_3 = 0$

$x_1 = x_3$

$x_2 = 2x_1 = 2x_3$

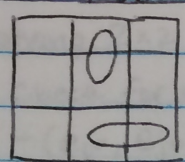
$x_1 + x_2 + x_3 = 1$

$x_1 + 2x_1 + x_1 = 1 \Rightarrow x_1 = x_3 = \frac{1}{4}, \; x_2 = \frac{1}{2}$

$x = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$

optimal strategy for player 2 is also $y = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$

<u>Bomber, Submarine</u>



Submarine takes 2 adjacent squares (player 2)

player 1 bombs one square

payoff player 1 is 1 if hit sub.

9 options (12 options)

<u>Player 1 options</u>	<u>player 2 options</u> corner	center
corner,	1/4	0
midside	1/4	1/4
center	0	1

top row is dominated by middle row

$\Rightarrow \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$ Value of $\frac{1}{4}$ and its a saddle point

<u>example</u> $\begin{pmatrix} 7 & 0 \\ 0 & 7 \\ 3 & 3 \end{pmatrix} \leftarrow$ dominated by the top 2 which average to $(\frac{7}{2}, \frac{7}{2})$

If $\alpha_1, \alpha_2, \dots, \alpha_{m-1} \geq 0$ $\sum_{i=1}^{m-1} \alpha_i = 1$ and $\forall j$ $a_{mj} \leq \sum_{i=1}^{m-1} a_{ij}\alpha_i$

then row m is dominated (pick row i with probability α_i, then $\sum_{i=1}^{m-1} a_{ij}\alpha_i$ is payoff against j and a_{mj} is payoff against j using strategy m).

$\Rightarrow \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$

$x^* = y^* = (\frac{1}{2}, \frac{1}{2})$ $V = \frac{7}{2}$

Dog, Cat, Bird
- payoff 1 same word $\Rightarrow x^* = y^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

2/24/14

<u>Two Person Zero Sum Games</u>

$$\begin{array}{c} p \\ 1-p \end{array} \begin{pmatrix} 3 & 2 & 4 & 0 \\ -2 & 1 & -4 & 5 \end{pmatrix}$$

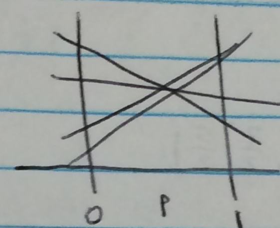
column 2 is dominated by columns 1 and 4

① $3p - 2(1-p) = 5p - 2$

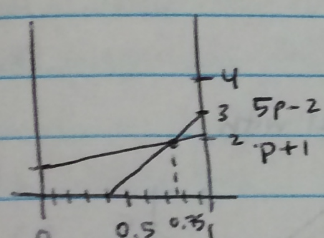
② $2p + 1 - p = p + 1$

③ $8p - 4$

④ $5 - 5p$



find the minimum and maximize over p, this will give your strategy for player 1.



Too lazy to draw the other lines but maximized at 0.7 (so optimal strategy for player 1)

$v = 1.5$

* graph shows that strategies 2 and 3 have no effect. So you can reduce the matrix to a 2x2 matrix and solve normally. to find player 2's strategy.

$$\begin{pmatrix} 0 & 8 & 5 \\ 8 & 4 & 6 \\ 12 & -4 & 3 \end{pmatrix}$$

3rd column is dominated by first two
$\frac{3}{8}$(column 1) + $\frac{5}{8}$(column 2)

$$\Downarrow$$

$$\begin{pmatrix} 0 & 8 \\ 8 & 4 \\ 12 & -4 \end{pmatrix}$$

① $8 - 8q$ $\quad q = \frac{1}{3}$

② $4q + 4$

③ $16q - 4$

Dog, Cat, Bird

- payoff 1 same word $\Rightarrow x^* = y^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

2/24/14

Two Person Zero Sum Games

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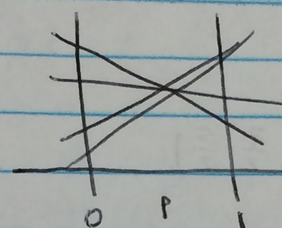

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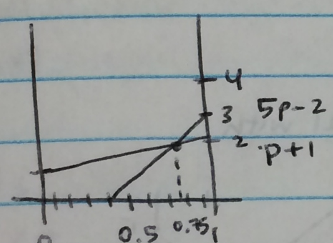

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$q = \frac{1}{3}$

$$\begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

If in each row, you have an element that is largest in its column and $\Leftrightarrow$, there is dominance

this is a magic square. entries are from ~~row~~ $\{1, \dots, n^2\}$ and sum of each column and row is equal

both players strategy is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

Prove that $\max_x \min_y x^T A y = (x^*)^T A y^*$

2/25/14

$$A = \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \\ 1 & 0 & 3 & 2 \end{pmatrix}$$

$\frac{1}{2}$ 1st row + $\frac{1}{2}$ 2nd row $= (3, 2, \frac{7}{2}, \frac{5}{2})$

$$A = \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$

In general, $2 \times m$ matrices are easier to deal with

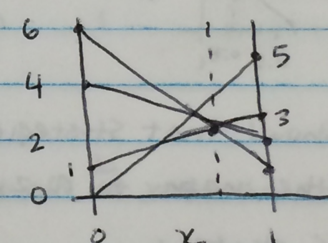
$x = (x_1, x_2) = (x_1, 1 - x_1)$

graph: ① $2x_1 + 4(1 - x_1)$

② $3x_1 + (1 - x_1)$

③ $x_1 + 6(1 - x_1)$

④ $5x_1$



$3x_1 + (1 - x_1) = x_1 + 6(1 - x_1)$

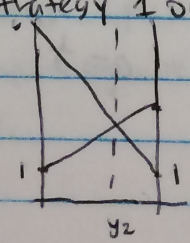
$7x_1 = 5x_1 \Rightarrow x_1 = \frac{5}{7}$

$V = 3 \cdot \frac{5}{7} + \frac{2}{7} = \frac{17}{7} = \max_x \min_j (x^T A)_j$

$x^* = (\frac{5}{7}, \frac{2}{7}, 0)$

Player 2 would not use strategy 1 or 4 against this strategy because doesn't minimize

$$\begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}$$



$\min_y \max_i (Ay)_i$

Player 1 plays $(\frac{5}{7}, \frac{2}{7}, 0)$

① $2 \cdot \frac{5}{7} + 4 \cdot \frac{2}{7} = \frac{18}{7}$

② $3 \cdot \frac{5}{7} + 1 \cdot \frac{2}{7} = \frac{17}{7}$

$\frac{17}{7}$

$\frac{25}{7}$

$y^* = (0, \frac{5}{7}, \frac{2}{7}, 0)$

Check x^*, y^* optimal?

- player 1 payoff using x^* against each j

$\min_j ((x^*)^T A)_j = \max_i (Ay^*)_i = 17/7$

player 1: ① $2 \cdot 0 + 3 \cdot \frac{5}{7} + 1 \cdot \frac{2}{7} + 5 \cdot 0 = \frac{17}{7}$

② $4 \cdot 0 + 1 \cdot \frac{5}{7} + 6 \cdot \frac{2}{7} + 0 \cdot 0 = \frac{17}{7}$

③ $1 \cdot 0 + 0 \cdot \frac{5}{7} + 3 \cdot \frac{2}{7} + 2 \cdot 0 = \frac{6}{7}$

$\max = \frac{17}{7}$

Principal of Indifference / Equilibrium Theorem

If (x, y) is optimal, $(Ay)_i = V$ whenever $x_i > 0$

$(x^T A)_j = V$ whenever $y_i > 0$

For all i, $(Ay)_i \le V$

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

Suppose $y_1, y_2 > 0$

$0 \cdot x_1 + (-1) x_2 = V$

$-x_1 + 2x_2 = V$

$x_1 + x_2 = 1$

$x_2 = -V, \ x_1 = -3V, \ -4V = 1, \ V = -\frac{1}{4}, \ x_1 = \frac{3}{4}, \ x_2 = \frac{1}{4}$

Solving non-singular games: $n = m$, A^{-1} exists

$\sum_{j=1}^{n} a_{1j} y_j = V$ $\qquad \mathbf{1} = (1, \dots, 1)$

$\sum_{j=1}^{n} a_{2j} y_j = V$ $\qquad Ay = V\mathbf{1}$

$\vdots$ $\qquad y = A^{-1}Ay$

$\sum_{j=1}^{n} a_{nj} y_j = V$ $\qquad = A^{-1}V\mathbf{1}$

$\sum_{j=1}^{n} y_j 1 = 1$ $\qquad \mathbf{1}^T y = 1$ $\qquad 1 = \mathbf{1}^T y = V \mathbf{1}^T A^{-1} \mathbf{1}$ $\qquad V = \frac{1}{\mathbf{1}^T A^{-1} \mathbf{1}}$

$y = \frac{A^{-1}\mathbf{1}}{\mathbf{1}^T A^{-1} \mathbf{1}}$ $\qquad x^T = \frac{\mathbf{1}^T A^{-1}}{\mathbf{1}^T A^{-1} \mathbf{1}}$

Examples $A = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$

$$V = \frac{1}{1^T A^{-1} 1} = \sum_{i,j} 1 \cdot a_{ij} \cdot 1 = -\frac{1}{4}$$

add up all the entries in the matrix

$$x = \frac{1^T A^{-1}}{-4} = \frac{(-3,-1)}{4} = \left(\frac{3}{4}, \frac{1}{4}\right)$$

Paper, Scissor, Rock

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad V = 0, \text{ singular matrix}$$

look at A+1 $B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, $V = 1$

$$B^{-1} = \frac{1}{9}\begin{pmatrix} 1 & -2 & 4 \\ 4 & 1 & -2 \\ -2 & 4 & 1 \end{pmatrix}$$

$$1^T B^{-1} 1 = \frac{1}{9}(9) = 1 \Rightarrow V = \frac{1}{1} = 1$$

$$x = \frac{1^T B^{-1}}{1} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Diagonal Games

$$\begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{pmatrix} = D, \quad d_1, d_2, \dots, d_n > 0$$

$$D^{-1} = \begin{pmatrix} 1/d_1 & & 0 \\ & 1/d_2 & \\ 0 & & \ddots \; 1/d_n \end{pmatrix}$$

$$1^T D^{-1} 1 = \sum_{i=1}^{n} \frac{1}{d_i} \Rightarrow V = 1 / \sum_{i=1}^{n} \frac{1}{d_i}$$

$$x = y = \frac{1^T D^{-1}}{1^T D^{-1} 1} = \left(d_1^{-1} \cdot V, d_2^{-1} \cdot V, \dots, V \cdot d_n^{-1}\right)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow V = \frac{1}{1 + \frac{1}{2} + \frac{1}{3}} \Rightarrow x = \left(\frac{1}{1} \cdot V, \frac{1}{2} \cdot V, \frac{1}{3} \cdot V\right)$$

$$= \left(\frac{6}{11}, \frac{3}{11}, \frac{2}{11}\right) = y$$

Triangular Games

everything below the diagonal $= 0$

$$\begin{pmatrix} 1 & 2 & \cdots & 2 \\ & \ddots & & \\ 0 & & & 5 \end{pmatrix}$$

example: strategies $\{1, 2, 3, 4\}$

player 1 gets 1 if $i = j$, loses $-j$ if $j > i$

Constraints 2-player zero-sum

$x_i \geq 0, \sum_{i=1}^{m} x_i = 1$

maximize $\min_j (x^T A)_j$

add in v, value of game and add constraint $(x^T A)_j \geq v$, maximize v

Crocodile - Zebra Game

1 2 3 4
4 3 2 1 symmetric around midpoint

- choose same croc wins with prob. 1
- adjacent points croc wins w/ p. $\frac{1}{2}$
- otherwise zebra escapes

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

$x + \frac{1}{2}x_2 = v$

$\frac{1}{2}x_1 + x_2 + \frac{1}{2}x_3 = v$

$\vdots$

Calculate A^{-1}

$v = \frac{1}{1^T A^{-1} 1}$

$x = 1^T A^{-1} / 1^T A^{-1} 1$

$$\begin{array}{ccc} & \text{side} & \text{middle} \\ \text{side} & \frac{1}{2} & \frac{1}{4} \\ \text{middle} & \frac{1}{4} & \frac{3}{4} \end{array} \Rightarrow \begin{pmatrix} 1/4 & 0 \\ 0 & 1/2 \end{pmatrix} \quad v = \frac{1}{6}$$

$v = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

$x = (\frac{2}{3}, \frac{1}{3})$

$x = (\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3})$

If $\exists$ permutation π, σ such that $\forall i,j$ $a_{ij} = a_{\pi(i)\sigma(j)}$ then $\exists$ optimal strategy with $\forall i$ $x_i^* = x^*_{\pi(i)}$, $\forall j$ $y^*_j = y^*_{\sigma(i)}$

$\pi(i) = \sigma(i) = 5 - i$

$(1,2,3,4) \longrightarrow (4,3,2,1)$

$a_{ij} = a_{\pi(i)\pi(j)}$

__Lemma__ $\forall x, y$, function $f(x,y)$

$$\max_x \min_y f(x,y) \le \min_y \max_x f(x,y)$$

Take (x^*, y^*)

$$\min_y f(x^*, y^*) \le f(x^*, y^*) \le \max_x f(x, y^*)$$

Fix $\forall y^*$, $\max_x \min_y$ ~~[illegible]~~ $f(x,y) \le \max_x f(x, y^*)$

$$\max_x \min_y f(x,y) \le \min_y \max_x f(x,y)$$

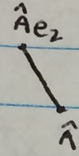
$$f(x,y) = x^T A y$$

Assume theorem is false. $\exists \lambda$ s.t. $\max_x \min_y x^T A y < \lambda < \min_y \max_x x^T A y$

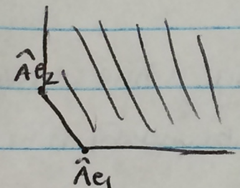
$\hat{A}$ where $\hat{a}_{ij} = a_{ij} - \lambda$

$$\max_x \min_y x^T \hat{A} y < 0 < \min_y \max_x x^T \hat{A} y$$

$\hat{A}y$



$\{\hat{A}y : y \in \Delta n\}$



$$K = \{\hat{A}y + v : \underset{v \ge 0}{y \in \Delta n,}\}$$

$\hat{A}y + v$, $\hat{A}y' + v'$

$\frac{y+y'}{2} \in \Delta_n$ $\quad \frac{v+v'}{2} \ge 0$

$$\frac{(\hat{A}y + v) + (\hat{A}y' + v')}{2} = \hat{A}\left(\frac{y+y'}{2}\right) + \left(\frac{v+v'}{2}\right) \in K$$ K is convex.

$\exists z$ such that $\forall w \in K$, $z^T w > c > 0$. $z \ne 0$, $z_i \ge 0$

Suppose $z_1 < 0$. any y, $v = (R, 0, 0, \dots, 0)$

$z^T(\hat{A}y + v) = z^T \hat{A} y + z_1 R < 0$ Contradiction.

$x = \frac{z}{\sum_{i=1} z_i}$, $x_i \ge 0$, $\sum_{i=1}^m x_i = 1$, $x \in \Delta m$

$\forall y$ $x^T \hat{A} y > 0$, $\max_x \min_y x^T \hat{A} y > 0$ contradiction.

__Linear Programming__

Set of variables $x_1, \dots, x_m$

Constraints linear $x_1 + 2x_2 \ge 7$

$x_3 \ge 0$, $x_1 - x_3 \ge 2$

Objective function to maximize: $x_1 + 3x_2 - x_3$

$$\begin{pmatrix} 1 & -2 & -3 & -4 \\ & 1 & -3 & -4 \\ 0 & & 1 & -4 \\ & & & 1 \end{pmatrix}$$

Strategies of player 2

① $x_1 = v$

② $-2x_1 + x_2 = v$

③ $-3x_1 - 3x_2 + x_3 = v$

④ $-4x_1 - 4x_2 - 4x_3 + x_4 = v$

Solve for x_1 first, plug in, solve for x_2, etc.

$x_1 = v$

$x_2 = v + 2v = 3v$

$x_3 = 3v + 3(3v) + v = 13v$

$x_4 = 69v$

$x_1 + x_2 + x_3 + x_4 = 1 \Rightarrow 86v = 1 \Rightarrow v = \frac{1}{86}$

$x^* = \left(\frac{1}{86}, \frac{3}{86}, \frac{13}{86}, \frac{69}{86}\right)$

2/27/14

Von Neuman's Minimax Theorem

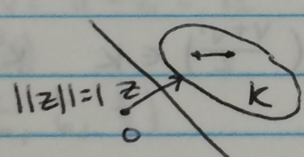
For all $n \times m$ matrices A, $\min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T A y = \max_x \min_y x^T A y = V$

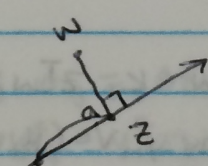
Theorem Hyperplane Separating Theorem

If $K \in \mathbb{R}^n$ closed convex, $0 \notin K$ (convex means a line between any two points is in the set - so not a donut, closed means boundary is in the set) you can draw a hyperplane separating K and 0.

There exist z such that

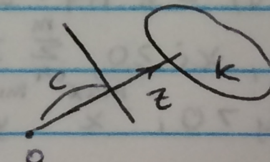
$\forall w \in K$, $z^T w > c > 0$



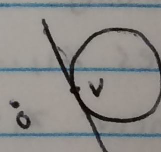


$z^T w$

$|a| = z^T w$



Let v = closest point in K to 0.



$z = \frac{v}{||v||}$

$z^T w = ||v||$