1/21/14

Sly's OH: Tuesday 1:10-2:00 333 Evans

Thursday 11:30-12:30

HW will be due on Mondays

Class Topics:

- Auctions
- Combinatorial Games
- Zero Sum Games
- General Sum Games
- Signalling + Evolutionary Games
- Stable Matchings
- Elections

Auctions

- compare
- cost to produce = \$c , price > c
- X is a random value customer places on gadget

CDF $F(x) = P[X \le x]$

density $f(x)$

price is v

Profit from a random customer = $\begin{cases} v-c & \text{if } X > v \text{, buy} \\ 0 & \text{if } X < v \text{, no sale} \end{cases}$

$E(\text{profit}) = E(v-c)I(X > v) = (v-c)(1-F(v))$

$E\,I(X>v) = P[X > v] = 1 - P(X \le v) = 1 - F(v)$

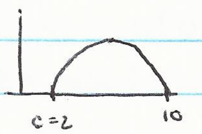
$X \sim U[0,10]$

$F(x) = \frac{x}{10} \quad 0 < x < 10$

$f(x) = \frac{1}{10} = F'(x)$

$c = 2$

$E(\text{profit}) = (v-2)(1 - \frac{v}{10}) = g(v)$



Parabola with max at $\frac{10+2}{2} = 6$

$g'(v) = (1 - \frac{v}{10}) - \frac{1}{10}(v-2)$

$= -\frac{2}{10}v + \frac{12}{10}$

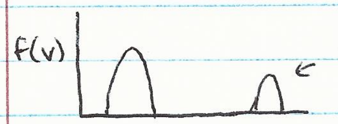
Profit = $(6-2)(1-\frac{6}{10}) = 4 \cdot 0.4 = 1.6$

General case

$g(v) = (v-c)(1-F(v))$

$g'(v) = 1-F(v) + (v-c)(-f(v)) = 0$, solve for v

$v = \frac{1-F(v) + cf(v)}{f(v)}$



the wealthy 1%

this distribution has more than one solution to $g'(v) = 0$

- you can either sell a lot and make less per sale, or sell less and make more per sale
- both of these prices might give you equal profit
- best strategy is to have two different prices, pick the one closer to what they are willing to pay
- internet marketplaces can tailor price to your demographics

Comparing → rich zip codes X_1 / business

→ poor zip codes X_2 / tourists

1/23/14

GSI OH: Th 3:30 – 5:30

F 2–4 Evans 446

starting next week

Quiz

① Let $X_1, X_2, \dots, X_n$ iid $U[0,1]$ $Y = \max_{1\le i\le n} X_i$, Find $E(Y)$

$\frac{n}{n+1}$. $P[Y \le y] = y^n = P[Y_1 \le y, \dots, Y_n \le y] = \prod_{i=1}^{n} P[Y_i \le y] = y^n$

$EY = \int_0^\infty P[Y > y] = \int_0^1 1 - y^n \, dy$

density Y $f_Y(y) = \frac{d}{dy} y^n = ny^{n-1}$

$E(Y) = \int_0^1 y \cdot f_Y(y)\, dy = \int_0^1 n y^n dy = \frac{ny^{n+1}}{n+1}\Big|_0^1 = \frac{n}{n+1}$

② Let $X, Y = \begin{cases} 1 & \text{with prob } 1/2 \\ -1 & \text{with prob } 1/2 \end{cases}$

$Z \sim U[0,1]$, X, Y, Z independent

Are X and XY independent?

Are X and XZ independent?

If you have Gaussian RVs, you can calculate covariance to know if they're independent. For regular RVs, you can't do that. $\mathrm{cov}(X,Y)=0 \not\Rightarrow X, Y$ indep.

Auctions, continued

① English Auction

- Successive bidding, highest bid wins

② Dutch Auction

③ First price sealed Auction

- highest bidder wins, pays highest bid

④ Second Price (Vickery) Auction

- Sealed bid, highest bidder wins but pays 2nd highest price

Value v (how much it's worth to you)

utility $= v-p$ where p is the price you paid $= E(V-p)\, I(\text{win auction})$

① If current level less than v, increase bid

3 bidders, values 50, 55, 60. Say increment by 1 cent. The bidder that values it at 60 will get it for \$55.01

④ Same end result as English auction, but less exciting

Claim Best strategy in 2nd price auction is to bid your value.

- Truthful mechanism. (people are incentivized to tell the truth)
- Agent 1,2 with values V_1, V_2; agent 2's bid is B_2
    - If $V_1 > B_2$, profit $= V_1 - B_2 > 0$ provided $B_1 > B_2$
      if $B_1 < B_2$, profit $=0$
    - If $V_1 < B_2$: win $(B_1 > B_2)$ then profit $= V_1 - B_2 < 0$
      lose $(B_1 < B_2)$ then profit $=0$
  - Set $B_1 = V_1$ is best, profit ≥ 0.
  - Dominant Strategy: best response to any strategy
    - this strategy in a first price auction means utility $=0$, this is not the best strategy for 3

$V_1, V_2 \sim U[0,1]$ independent

$R = \min(V_1, V_2)$, $E(R) =$

1/28/14

First HW due Monday in section

Correction: Q1 Find mean + variance of X. $N, X = Bin(N, \frac{1}{2})$

2nd price auction: best strategy is to be truthful

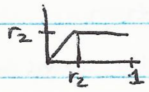
values $V_1, V_2 \sim u[0,1]$

$R = \min(V_1, V_2)$

$E(R) = ?\ \int_0^1 r \cdot f(r)\,dr = \int_0^1 r \cdot 2(1-r)\,dr = \frac{1}{3}$

$P[R \leq r] = 1 - P[R > r] = 1 - P(V_1 > r, V_2 > r) = 1 - (1-r)^2$

$f(r) = F'(r) = 2(1-r)$



$E(R) = \int_0^1\int_0^1 \min(r_1, r_2)\,dr_1\,dr_2$

$= \int_0^1 r_2 - \frac{1}{2}r_2^2\,dr_2$

$= \int_0^1 \left(\int_0^{r_2} r_1\,dr_1 + \int_{r_2}^1 r_2\,dr_1\right) dr_2$

1st price auction

utility = (value − bid) I(win)

values V_1, V_2 IID $u[0,1]$

Bidding strategy $\beta : [0,1] \to [0, \infty)$

value v, bid $\beta(v)$

$\beta(v) < v$

"Best" strategy: $\beta(v) = \frac{v}{2}$

Agents $1, 2, \ldots, n$ have values $V_1, \ldots, V_n$ IID distribution V

Bidding strategy profile $(\beta_1, \beta_2, \ldots, \beta_n)$ - bidding strategies

$a_i[b] = P[i$ wins, bidding b, given agents $j \neq i$ bid $\beta_j(v_j)]$

$\beta_1(v) = \beta_2(v) = \frac{v}{2}$

Agent 1 $a_1(b) = P[$Agent 1 wins bidding b when agent 2 bids $\beta(v_2) = \frac{v_2}{2}$

Agent 1 bids $\beta_1(v_1)$, agent 2 bids $\beta_2(v_2)$

$P_i[b]$ = expected payment of player i if bids b, others bid $\beta_j(v_j)$

$= b \cdot a_i[b]$ (in first price auction)

expected utility given value v_i, bid b

$u_i[b|v_i] = v_i a_i[b] - p_i[b]$

$(\beta_1, \beta_2, \beta_3)$ is Bayes-Nash equilibrium if for all i, v_i,

$u_i[\beta_i(v_i)|v_i] = \max u_i[b|v_i]$

no player can change strategy and improve their expected utility

Check $\beta_1(v) = \beta_2(v) = \frac{v}{2}$ is a B.N. equilibrium 2 person, 1st price auction $U[0,1]$

$a_1[b] = P[\text{agent 1 wins bidding } b] = P[b > \beta(v_2)] = P[b > \frac{v_2}{2}] = \begin{cases} 2b & b < \frac{1}{2} \\ 1 & b > \frac{1}{2} \end{cases}$

$P_i[b] = b \cdot a_i[b] = 2b^2$, $b < \frac{1}{2}$ never makes sense to bid more than $\frac{1}{2}$ because agent 2 will never bid more than $\frac{1}{2}$

$$u_1(b|v_1) = (v_1 - b_1)\, a_1[b] = (v_1 - b) \cdot 2b$$
$$= v \cdot 2b - 2b^2$$

how to maximize this equation? take the derivative.

$$\frac{d}{db} u_1(b|v_1) = 2v - 4b = 0 \Rightarrow b = \frac{v}{2}$$

if you know agent 2 will bid half their value, you should bid half your value.

First price $v_1, v_2 \sim U[0,1]$

$\beta_1 = \beta_2 = \beta$, increasing, differentiable

- Agent 2 bid $\beta(v_2)$
- $u_1(b|v) = (v-b)\, P[b > \beta(v_2)]$

$b = \beta(w)$ maximize

$$w = \beta^{-1}(b) \quad u_1(\beta(w)|v) = (v - \beta(w))\, P[\beta(w) > \beta(v_2)]$$
$$= (v - \beta(w))\, P[w > v_2]$$
$$= (v - \beta(w))\, w$$

maximized at $w = v$

$$\left.\frac{du(\beta(w)(v)}{dw}\right|_{w=v} = v - \beta(w) - w\beta'(w)$$

$$v = \beta(v) + v\beta'(v)$$
$$= (v\beta(v))'$$
$$\frac{v^2}{2} = v\,\beta(v)$$
$$\beta(v) = \frac{v}{2}$$

R = revenue, $E(R) = ? = E(\max(\frac{v_1}{2}, \frac{v_2}{2}))$

$$= \frac{1}{2} E(\max(v_1, v_2))$$
$$= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$ Same as expected value for revenue from 2nd price auction

1/30/14

All Pays Auction

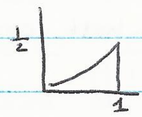
- highest bidder gets the item
- everyone pays their bid

Agents	1	2
values	0.8	0.6
bids	0.3	0.1
Payments	0.3	0.1
Profit	0.5	-0.1

$P_1[b] = b$ (expected payment of Agent 1)

$V_1, V_2 \sim U[0,1]$

$\beta(v) = \frac{v^2}{2}$ = strategy



$$a_1[b] = P[\beta(v_2) < b]$$
$$= P\left[\frac{v_2^2}{2} < b\right]$$
$$= P[v_2^2 < 2b]$$
$$= P[v_2 < \sqrt{2b}]$$
$$= \begin{cases} \sqrt{2b} & 0 < b \le \frac{1}{2} \\ 1 & b > \frac{1}{2} \end{cases}$$

$u_1[b|v] = v \cdot a[b] - b$

- To verify Bayes-Nash equilibrium check $u_1[\beta(v)|v] = \max_b u_1[b|v]$

$u[b|v] = v\sqrt{2b} - b$

$\frac{d}{db} u[b|v] = v \cdot \sqrt{2} \cdot \frac{1}{2\sqrt{b}} \cdot -1 = 0$

$1 = v \cdot \frac{1}{\sqrt{2b}}$

$\sqrt{2b} = v$

$b = \frac{v^2}{2}$

Any auction : <u>Payment Equivalence Theorem</u>

- n agents with IID values $V_1, \ldots, V_n$ - continuous distribution
- if highest bidder always wins
- β - symmetric Bayes Nash equilibrium (and is differentiable and increasing)

then $P_i[\beta(v)]$ is the same as in the 1st/2nd price auction.

$$P_i[\beta(v)] = F_V(v)^{n-1} E\left[\max_{1 \le i \le n-1} V_i \,\middle|\, \max_{1 \le i \le n-1} V_i \le v\right]$$

If your value is v, your expected payment does not depend on the auction at all.

$a_n[\beta(v)] = P[\beta(v) > \max \beta(V_1, \ldots, V_{n-1})]$

$= P[v > \max(V_1, \ldots, V_{n-1})]$ (here we use the fact that β is increasing)

$= \prod_{i=1}^{n-1} P[V_i < v]$

$= F(v)^{n-1}$

2nd price auction

expected payment win with bid $v = E[\max_{1\le i\le n-1} V_i \mid \max_{1\le i \le n-1} V_i \le v]$

$P_i[\beta(v)] = P[\text{win}]\,E[\text{payment} \mid \text{win}]$

$P[v] = F(v) \cdot \underset{\underset{U[0,v]}{\wr}}{E[V_1 \mid V_1 \le v]} = v \cdot \frac{v}{2} = \frac{v^2}{2}$

But in an all-pays auction $P[\beta(v)] = \beta(v)$ so $\beta(v) = \frac{v^2}{2}$

$u[\beta(w) \mid v] = a[\beta(w)] \cdot v - P[\beta(w)]$

$= v \cdot (F_v(w)^{n-1}) - P[\beta(w)]$

$\frac{d}{dw} u[\beta(w) \mid v]\big|_{w=v} = 0 = v \cdot \frac{d}{dv} a[\beta(v)] - \frac{d}{dv} P[\beta(v)]$

$P[\beta(v)] = \int_0^v w \frac{d}{dw} a[\beta(w)]\, dw = \int_0^v w \cdot \frac{d}{dw}(F_v(w))^{n-1}\, dw$ (independent of the auction)

<u>Revenue Equivalence</u>

If R revenue of the auction $E(R)$ does not depend on the auction.

$E(R) = E(\sum_{i=1}^{n} P_i[\beta(v_i)]) = \frac{1}{3}$ with 2 agents

New Zealand had an auction of its electromagnetic spectrum - had simultaneous 2nd price auctions which is a bad idea because if you just want one chunk of waves you have to decide if you want to bid small in many auctions or high in just one auction.

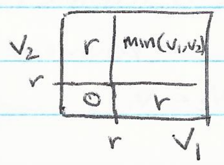
- result: top bid = \$100,000 and 2nd bid = \$6

Dutch Auction: flowers are destroyed with a guillotine

you can have a reserve price R

2 player $U[0,1] \sim V_1, V_2$; 2nd price auction, reserve r (Vickrey)

$R = \begin{cases} \max(r, \min(V_1, V_2)), & \max(V_1, V_2) > r \end{cases}$



$E(R) = r(1-r)r \cdot 2 + E(\min(V_1, V_2)\mathbb{1}(V_1 > r, V_2 > r))$

$= \frac{E[\min V_1, V_2 \mid V_1 > r, V_2 > r]}{P(V_1 > r, V_2 > r)}$

$= \int_r^1 \int_r^1 \min(V_1, V_2)\, dV_1\, dV_2$

$= \frac{1}{3} + r^2 - \frac{4}{3} r^3$ maximize this, $2r - 4r^2 = 0$, $r = \frac{1}{2}$

$\Rightarrow E(R) = \frac{5}{12}$

2/3/14

Bayes-Nash Equilibrium

Values: $V_1, V_2, \ldots, V_n$ $\quad V_i \sim F$

Strategy: $\beta_1, \ldots, \beta_n$

bid: $\beta_1(V_1), \beta_2(V_2), \ldots, \beta_n(V_n)$

$n=2$, $V_i \overset{iid}{\sim} Exp(1)$, First price auction

$\beta_1 = \beta_2 = \beta$

Player 1 value V_1, bid b

$E(\text{utility}) = (V_1 - b)\, P[\text{player 1 wins}]$

$= (V_1 - b)\, P[\beta(V_2) < b]$

Assume that $b = \beta(w)$

$= (V_1 - b)\, P[V_2 < w]$

$= (V_1 - \beta(w))(1 - e^{-w})$

This will be maximized when $w = V_1$ (by definition)

$\frac{\partial}{\partial w}[(V_1 - \beta(w))(1-e^{-w})]$

$= -\beta'(w)(1-e^{-w}) + (V_1 - \beta(w))e^{-w}$

B.N. eq $\Rightarrow -\beta'(w)(1-e^{-w}) + (w - \beta(w))e^{-w} = 0$

$\Rightarrow we^{-w} = ((1-e^{-w})\beta(w))'$

$\beta(w) = -\frac{ve^{-v}}{1-e^{-v}} + 1$

Revenue Equivalence Theorem: depends on fact that allocation is always in same manner

optimal strategy is one that is best for you, no matter what others are doing - always an equilibrium strategy

First Price Auction	Second Price Auction	
V_1, V_2	V_1, V_2	- values
$\beta(V_1), \beta(V_2)$	V_1, V_2	- bids
$\beta(V_1)\, P(1 \text{ wins})$ $= \beta(V_1)\, P(V_1 > V_2)$	$P[V_1 < V_2]\, E[V_2 \mid V_2 \leq V_1]$	- payment

From the Payment Equivalence Theorem, $\beta(V_1) = E[V_2 \mid V_2 \leq V_1]$

War of Attrition Auction

$V_1, V_2, V_3 \overset{iid}{\sim} U(0,1)$

2nd price w/ 3 agents: $E[\max(V_2,V_3) \mid \max(V_2,V_3) \leq V_1]\ P[\max(V_2,V_3) \leq V_1]$

WPA expected payment $= E[\max(\beta(V_2), \beta(V_3)) | \max(V_2, V_3) \le V_1] P[\max(V_2, V_3) \le V_1]$

$+ \beta(V_1) P[\max(V_2, V_3) > V_1]$

$= [\int_0^{V_1} \beta(w)\, 2w\, dw] + \beta(V_1)(1 - V_1^2)$

CDF: $P(\max(V_2, V_3) < w |$ $\quad = \frac{W^2}{V_1^2}$

Vickery Auction with reserve price

$V_1, V_2 \overset{iid}{\sim} Exp(1)$

reserve r

Find r to maximize expected revenue

$E[\text{revenue}] = 0 \cdot (1-e^{-r})^2 + r \cdot 2(1-e^{-r})(e^{-r}) + e^{-2r}(r + \frac{1}{2})$

memoryless property of exponential distribution

$\min(V_1, V_2) \sim exp(2)$ $\quad f(x) = \lambda e^{-\lambda x}$; $E(x) = \frac{1}{\lambda}$