Chapter 2
Finite Two-Person Zero-Sum Games

This chapter deals with two-player games in which each player chooses from finitely many pure strategies or randomizes among these strategies, and the sum of the players' payoffs or expected payoffs is always equal to zero. Games like the 'Battle of the Bismarck Sea' and 'Matching Pennies', discussed in Sect. 1.3.1 belong to this class.

In Sect. 2.1 the basic definitions and theory are discussed. Section 2.2 shows how to solve $2 \times n$ and $m \times 2$ games, and larger games by elimination of strictly dominated strategies.

2.1 Basic Definitions and Theory

Since all data of a finite two-person zero-sum game can be summarized in one matrix, such a game is usually called a 'matrix game'.

Definition 2.1 (Matrix game). A *matrix game* is an $m \times n$ matrix A of real numbers, where the number of rows m and the number of columns n are integers greater than or equal to 1. A (*mixed*) *strategy* of player 1 is a probability distribution $\mathbf{p}$ over the rows of A, i.e., an element of the set

$$\Delta^m := \{\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} p_i = 1,\ p_i \geq 0 \text{ for all } i = 1, \ldots, m\}.$$

Similarly, a (*mixed*) *strategy* of player 2 is a probability distribution $\mathbf{q}$ over the columns of A, i.e., an element of the set

$$\Delta^n := \{\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} q_j = 1,\ q_j \geq 0 \text{ for all } j = 1, \ldots, n\}.$$

H. Peters, *Game Theory – A Multi-Leveled Approach*.

A strategy $\mathbf{p}$ of player 1 is called *pure* if there is a row i with $p_i = 1$. This strategy is also denoted by $\mathbf{e}^i$. Similarly, a strategy $\mathbf{q}$ of player 2 is called *pure* if there is a column j with $q_j = 1$. This strategy is also denoted by $\mathbf{e}^j$.

The interpretation of such a matrix game A is as follows. If player 1 plays row i (i.e., pure strategy $\mathbf{e}^i$) and player 2 plays column j (i.e., pure strategy $\mathbf{e}^j$), then player 1 receives payoff a_{ij} and player 2 pays a_{ij} (and, thus, receives $-a_{ij}$), where a_{ij} is the number in row i and column j of matrix A. If player 1 plays strategy[1] $\mathbf{p}$ and player 2 plays strategy $\mathbf{q}$, then player 1 receives the expected payoff[2]

$$\mathbf{p}A\mathbf{q} = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j a_{ij},$$

and player 2 receives $-\mathbf{p}A\mathbf{q}$.

For 'solving' matrix games, i.e., establishing what clever players would or should do, the concepts of maximin and minimax strategies are important, as will be explained below. First we give the formal definitions.

Definition 2.2 (Maximin and minimax strategies). A strategy $\mathbf{p}$ is a *maximin strategy* of player 1 in matrix game A if

$$\min\{\mathbf{p}A\mathbf{q} \mid \mathbf{q} \in \Delta^n\} \geq \min\{\mathbf{p}'A\mathbf{q} \mid \mathbf{q} \in \Delta^n\} \text{ for all } \mathbf{p}' \in \Delta^m.$$

A strategy $\mathbf{q}$ is a *minimax strategy* of player 2 in matrix game A if

$$\max\{\mathbf{p}A\mathbf{q} \mid \mathbf{p} \in \Delta^m\} \leq \max\{\mathbf{p}A\mathbf{q}' \mid \mathbf{p} \in \Delta^m\} \text{ for all } \mathbf{q}' \in \Delta^n.$$

In words: a maximin strategy of player 1 maximizes the minimal (with respect to player 2's strategies) payoff of player 1, and a minimax strategy of player 2 minimizes the maximum (with respect to player 1's strategies) that player 2 has to pay to player 1. Of course, the asymmetry in these definitions is caused by the fact that, by convention, a matrix game represents the amounts that player 2 has to pay to player 1.[3]

In order to check if a strategy $\mathbf{p}$ of player 1 is a maximin strategy it is sufficient to check that the first inequality in Definition 2.2 holds with $\mathbf{e}^j$ for every $j = 1, \ldots, n$ instead of every $\mathbf{q} \in \Delta^n$. This is not difficult to see but the reader is referred to Chap. 12 for a more formal treatment. A similar observation holds for minimax strategies. In other words, to check if a strategy is maximin (minimax) it is sufficient to consider its performance against every pure strategy, i.e., column (row).

Why would we be interested in such strategies? At first glance, such strategies seem to express a very conservative or pessimistic, worst-case scenario attitude. The reason for considering maximin/minimax strategies is provided by von

[1] Observe that here, by a 'strategy' we mean a mixed strategy: we add the adjective 'pure' if we wish to refer to a pure strategy.

[2] Since no confusion is likely to arise, we do not use transpose notations like $\mathbf{p}^T A\mathbf{q}$ or $\mathbf{p}A\mathbf{q}^T$.

[3] It can be proved by basic mathematical analysis that maximin and minimax strategies always exist.

Neumann [140]. Von Neumann shows[4] that for every matrix game A there is a real number $v = v(A)$ with the following properties:

1. A strategy $\mathbf{p}$ of player 1 guarantees a payoff of at least v to player 1 (i.e., $\mathbf{p}A\mathbf{q} \geq v$ for all strategies $\mathbf{q}$ of player 2) if and only if $\mathbf{p}$ is a maximin strategy.
2. A strategy $\mathbf{q}$ of player 2 guarantees a payment of at most v by player 2 to player 1 (i.e., $\mathbf{p}A\mathbf{q} \leq v$ for all strategies $\mathbf{p}$ of player 1) if and only if $\mathbf{q}$ is a minimax strategy.

Hence, player 1 can obtain a payoff of at least v by playing a maximin strategy, and player 2 can guarantee to pay not more than v – hence secure a payoff of at least $-v$ – by playing a minimax strategy. For these reasons, the number $v = v(A)$ is also called the *value* of the game A – it represents the worth to player 1 of playing the game A – and maximin and minimax strategies are called *optimal strategies* for players 1 and 2, respectively.

Therefore, 'solving' the game A means, naturally, determining the optimal strategies and the value of the game. In the 'Battle of the Bismarck Sea' in Sect. 1.3.1, the pure strategies N of both players guarantee the same amount 2. Therefore, this is the value of the game and N is optimal for both players. The analysis of that game is easy since it has a 'saddlepoint', namely position $(1,1)$ with $a_{11} = 2$. The formal definition of a saddlepoint is as follows.

Definition 2.3 (Saddlepoint). A position (i, j) in a matrix game A is a *saddlepoint* if

$$a_{ij} \geq a_{kj} \text{ for all } k = 1, \ldots, m \text{ and } a_{ij} \leq a_{ik} \text{ for all } k = 1, \ldots, n,$$

i.e., if a_{ij} is maximal in its column j and minimal in its row i.

Clearly, if (i, j) is a saddlepoint, then player 1 can guarantee a payoff of at least a_{ij} by playing the pure strategy row i, since a_{ij} is minimal in row i. Similarly, player 2 can guarantee a payoff of at least $-a_{ij}$ by playing the pure strategy column j, since a_{ij} is maximal in column j. Hence, a_{ij} must be the value of the game A: $v(A) = a_{ij}$, $\mathbf{e}^i$ is an optimal (maximin) strategy of player 1, and $\mathbf{e}^j$ is an optimal (minimax) strategy of player 2.

2.2 Solving $2 \times n$ Games and $m \times 2$ Games

In this section we show how to solve matrix games where at least one of the players has two pure strategies. We also show how the idea of strict domination can be of help in solving matrix games.

[4] See Chap. 12 for a more rigorous treatment of zero-sum games and a proof of von Neumann's result.

2.2.1 $2 \times n$ *Games*

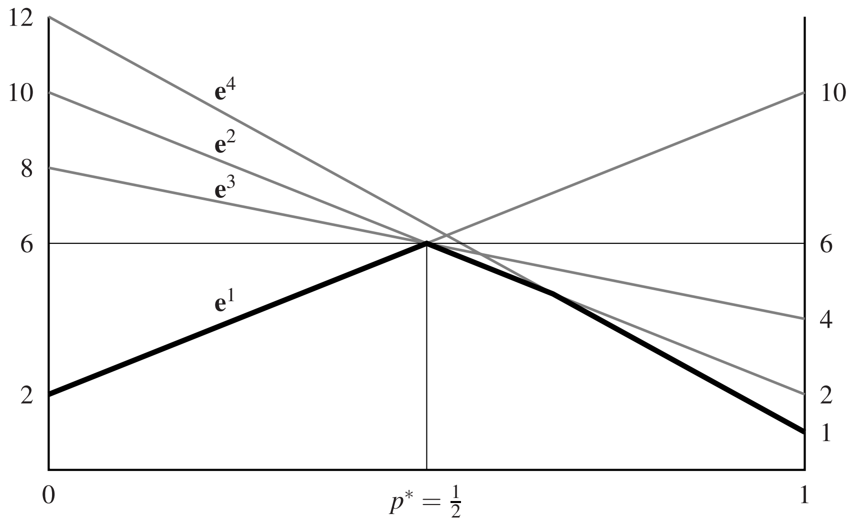
We demonstrate how to solve a matrix game with 2 rows and n columns graphically, by considering the following 2×4 example:

$$\begin{array}{cc} & \begin{array}{cccc} \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 & \mathbf{e}^4 \end{array} \\ A = & \begin{pmatrix} 10 & 2 & 4 & 1 \\ 2 & 10 & 8 & 12 \end{pmatrix}. \end{array}$$

We have labelled the columns of A, i.e., the pure strategies of player 2 for reference below. Let $\mathbf{p} = (p, 1-p)$ be an arbitrary strategy of player 1. The expected payoffs to player 1 if player 2 plays a pure strategy are equal to:

$$\begin{aligned}
\mathbf{p}A\mathbf{e}^1 &= 10p + 2(1-p) = 8p + 2 \\
\mathbf{p}A\mathbf{e}^2 &= 2p + 10(1-p) = 10 - 8p \\
\mathbf{p}A\mathbf{e}^3 &= 4p + 8(1-p) = 8 - 4p \\
\mathbf{p}A\mathbf{e}^4 &= p + 12(1-p) = 12 - 11p.
\end{aligned}$$

We plot these four linear functions of p in one diagram:



In this diagram the values of p are plotted on the horizontal axis, and the four straight gray lines plot the payoffs to player 1 if player 2 plays one of his four pure strategies, respectively. Observe that for every $0 \le p \le 1$ the minimum payoff that player 1 may obtain is given by the lower envelope of these curves, the thick black curve in the diagram: for any p, any combination (q_1, q_2, q_3, q_4) of the points on $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, and $\mathbf{e}^4$ with first coordinate p would lie on or above this lower envelope. Clearly, the lower envelope is maximal for $p = p^* = \frac{1}{2}$, and the maximal value is 6. Hence, we have established that player 1 has a unique optimal (maximin) strategy, namely $\mathbf{p}^* = (\frac{1}{2}, \frac{1}{2})$, and that the value of the game, $v(A)$, is equal to 6.

What are the optimal or minimax strategies of player 2? From the theory of the previous section we know that a minimax strategy $\mathbf{q} = (q_1, q_2, q_3, q_4)$ of player 2 should guarantee to player 2 to have to pay at most the value of the game. From the diagram it is clear that q_4 should be equal to zero since otherwise the payoff to player 1 would be larger than 6 if player 1 plays $(\frac{1}{2}, \frac{1}{2})$, and thus $\mathbf{q}$ would not be a minimax strategy. So a minimax strategy has the form $\mathbf{q} = (q_1, q_2, q_3, 0)$. Any such strategy, plotted in the diagram, gives a straight line that is a combination of the lines associated with $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ and which passes through the point $(\frac{1}{2}, 6)$ since all three lines pas through this point. Moreover, for no value of p should this straight line exceed the value 6, otherwise $\mathbf{q}$ would not guarantee a payment of at most 6 by player 2. Consequently, this straight line has to be horizontal. Summarizing this argument, we look for numbers $q_1, q_2, q_3 \geq 0$ such that

$$\begin{aligned} 2q_1 + 10q_2 + 8q_3 &= 6 \quad \text{(left endpoint should be } (0,6)) \\ 10q_1 + 2q_2 + 4q_3 &= 6 \quad \text{(right endpoint should be } (1,6)) \\ q_1 + q_2 + q_3 &= 1 \quad (\mathbf{q} \text{ is a probability vector).} \end{aligned}$$

This system of equations is easily reduced[5] to the two equations

$$\begin{aligned} 3q_1 - q_2 &= 1 \\ q_1 + q_2 + q_3 &= 1. \end{aligned}$$

The first equation implies that if $q_1 = \frac{1}{3}$ then $q_2 = 0$ and if $q_1 = \frac{1}{2}$ then $q_2 = \frac{1}{2}$. Clearly, q_1 and q_2 cannot be larger since then their sum exceeds 1. Hence the set of optimal strategies of player 2 is

$$\{\mathbf{q} = (q_1, q_2, q_3, q_4) \in \Delta^4 \mid \frac{1}{3} \leq q_1 \leq \frac{1}{2},\ q_2 = 3q_1 - 1,\ q_4 = 0\}.$$

2.2.2 $m \times 2$ Games

The solution method to solve $m \times 2$ games is analogous. Consider the following example:

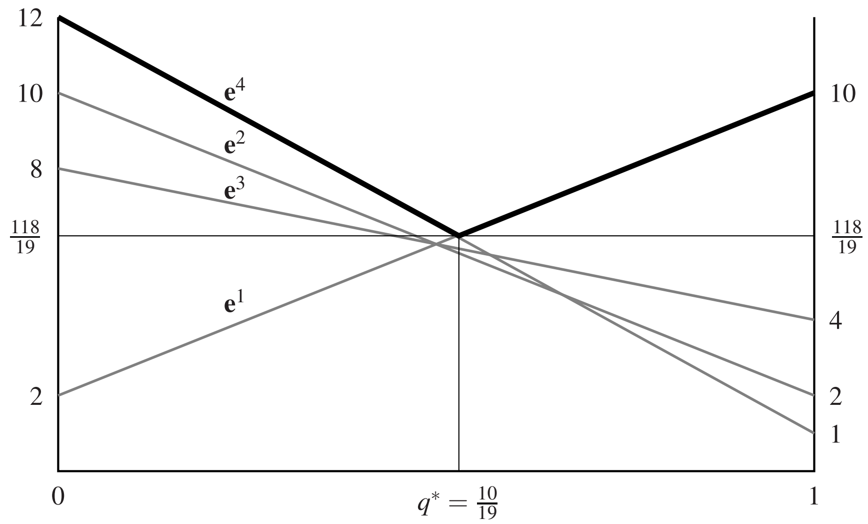
$$A = \begin{array}{c} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \\ \mathbf{e}^4 \end{array} \begin{pmatrix} 10 & 2 \\ 2 & 10 \\ 4 & 8 \\ 1 & 12 \end{pmatrix}.$$

[5] For instance, by substitution. In fact, one of the two first equations could be omitted to begin with, since we already know that any combination of the three lines passes through $(\frac{1}{2}, 6)$, and two points are sufficient to determine a straight line.

Let $\mathbf{q} = (q, 1-q)$ be an arbitrary strategy of player 2. Again, we make a diagram in which now the values of q are put on the horizontal axis, and the straight lines indicated by $\mathbf{e}^i$ for $i = 1,2,3,4$ are the payoffs to player 1 associated with his four pure strategies (rows) as functions of q. The equations of these lines are given by:

$$\begin{aligned}
\mathbf{e}^1 A\mathbf{q} &= 10q + 2(1-q) = 8q + 2 \\
\mathbf{e}^2 A\mathbf{q} &= 2q + 10(1-q) = 10 - 8q \\
\mathbf{e}^3 A\mathbf{q} &= 4q + 8(1-q) = 8 - 4q \\
\mathbf{e}^4 A\mathbf{q} &= q + 12(1-q) = 12 - 11q.
\end{aligned}$$

The resulting diagram is as follows.



Observe that the maximum payments that player 2 has to make are now located on the upper envelope, represented by the thick black curve. The minimum is reached at the point of intersection of $\mathbf{e}^1$ and $\mathbf{e}^4$ in the diagram, which has coordinates $(\frac{10}{19}, \frac{118}{19})$. Hence, the value of the game is $\frac{118}{19}$, and the unique optimal (minimax) strategy of player 2 is $\mathbf{q}^* = (\frac{10}{19}, \frac{9}{19})$.

To find the optimal strategy or strategies $\mathbf{p} = (p_1, p_2, p_3, p_4)$ of player 1, it follows from the diagram that $p_2 = p_3 = 0$, otherwise for $q = \frac{10}{19}$ the value $\frac{118}{19}$ of the game is not reached, so that $\mathbf{p}$ is not a maximin strategy. So we look for a combination of $\mathbf{e}^1$ and $\mathbf{e}^4$ that gives at least $\frac{118}{19}$ for every q, hence it has to be equal to $\frac{118}{19}$ for every q. This gives rise to the equations $2p_1 + 12p_4 = 10p_1 + p_4 = \frac{118}{19}$ and $p_1 + p_4 = 1$, with unique solution $p_1 = \frac{11}{19}$ and $p_4 = \frac{8}{19}$. So the unique optimal strategy of player 1 is $(\frac{11}{19}, 0, 0, \frac{8}{19})$.

2.2.3 Strict Domination

The idea of strict domination can be used to eliminate pure strategies before the graphical analysis of a matrix game. Consider the game

$$\begin{array}{cc} & \begin{array}{cccc} \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 & \mathbf{e}^4 \end{array} \\ A = & \begin{pmatrix} 10 & 2 & 5 & 1 \\ 2 & 10 & 8 & 12 \end{pmatrix}, \end{array}$$

which is almost identical to the game in Sect. 2.2.1, except that a_{13} is now 5 instead of 4. Consider a strategy $(\alpha, 1-\alpha, 0, 0)$ of player 2. The expected payments from this strategy from player 2 to player 1 are $8\alpha + 2$ if player 1 plays the first row and $10 - 8\alpha$ if player 1 plays the second row. For any value $\frac{1}{4} < \alpha < \frac{3}{8}$, the first number is smaller than 5 and the second number is smaller than 8. Hence, this is strictly better for player 2 than playing his pure strategy $\mathbf{e}^3$, no matter what player 1 does. But then, for any strategy $\mathbf{q} = (q_1, q_2, q_3, q_4)$ of player 2 with $q_3 > 0$, the expected payoff to player 2 would become strictly larger (his payment to player 1 strictly smaller) by transferring the probability q_3 to the first two pure strategies in some right proportion α, i.e., by playing $(q_1 + \alpha q_3, q_2 + (1-\alpha) q_3, 0, q_4)$ for some $\frac{1}{4} < \alpha < \frac{3}{8}$, instead of $\mathbf{q}$. Hence, in an optimal (minimax) strategy we must have $q_3 = 0$. This implies that, in order to solve the above game, we can start by deleting the third column of the matrix. In the diagram in Sect. 2.2.1, we do not have to draw the line corresponding to $\mathbf{e}^3$. The value of the game is still 6, player 1 still has a unique optimal strategy $\mathbf{p}^* = (\frac{1}{2}, \frac{1}{2})$, and player 2 now also has a unique optimal strategy, namely the one where $q_3 = 0$, which is the strategy $(\frac{1}{2}, \frac{1}{2}, 0, 0)$.

In general, strictly dominated pure strategies in a matrix game are not played with positive probability in any optimal strategy and can therefore be deleted before solving the game. Sometimes, this idea can also be used to solve matrix games in which each player has more than two pure strategies ($m, n > 2$). Moreover, the idea can be applied iteratively, that is, after deletion of a strictly dominated pure strategy, in the smaller game perhaps another strictly dominated pure strategy can be deleted, etc., until no more pure strategies are strictly dominated.[6]

We first provide a formal definition of strict domination for completeness' sake, and then discuss another example where iterated elimination of strictly dominated strategies is applied.

Definition 2.4 (Strict domination). Let A be an $m \times n$ matrix game and i a row. The pure strategy $\mathbf{e}^i$ is *strictly dominated* if there is a strategy $\mathbf{p} = (p_1, \ldots, p_m) \in \Delta^m$ with $p_i = 0$ such that $\mathbf{p}A\mathbf{e}^j > \mathbf{e}^i A \mathbf{e}^j$ for every $j = 1, \ldots, n$. Similarly, let j be a column. The pure strategy $\mathbf{e}^j$ is *strictly dominated* if there is a strategy $\mathbf{q} = (q_1, \ldots, q_n) \in \Delta^n$ with $q_j = 0$ such that $\mathbf{e}^i A \mathbf{q} < \mathbf{e}^i A \mathbf{e}^j$ for every $i = 1, \ldots, m$.[7]

[6] Of course, all this needs to be proved formally, but in this chapter it is just assumed. See Chap. 13 for a more formal treatment.

[7] An equivalent definition is obtained if the conditions $p_i = 0$ and $q_j = 0$ are omitted.

Example 2.5. Consider the following 3×3 matrix game:

$$A = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}.$$

For player 1, the third strategy $\mathbf{e}^3$ is strictly dominated by the strategy $\mathbf{p} = (\frac{7}{12}, \frac{5}{12}, 0)$, since $\mathbf{p}A = (3\frac{1}{2}, 2\frac{1}{12}, 2\frac{5}{6})$ has every coordinate strictly larger than $\mathbf{e}^3 A = (3, 2, 1)$. Hence, in any optimal strategy player 1 puts zero probability on the third row. Elimination of this row results in the matrix

$$B = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 5 & 4 \end{pmatrix}.$$

Now, player 2's third strategy $\mathbf{e}^3$ is strictly dominated by the strategy $\mathbf{q} = (\frac{1}{4}, \frac{3}{4}, 0)$, since $B\mathbf{q} = (\frac{3}{2}, 3\frac{3}{4})$, which has every coordinate strictly smaller than $B\mathbf{e}^3 = (2, 4)$. Hence, in any optimal strategy player 2 puts zero probability on the third column. Elimination of this column results in the matrix

$$C = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}.$$

This is a 2×2 matrix game, which can be solved by the method in Sect. 2.2.1 or Sect. 2.2.2. See Problem 2.1(a).

Problems

2.1. *Solving Matrix Games*

Solve the following matrix games, i.e., determine the optimal strategies and the value of the game. Each time, start by checking if the game has a saddlepoint.

(a)

$$\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

What are the optimal strategies in the original matrix game A in Example 2.5?

(b)

$$\begin{pmatrix} 2 & -1 & 0 & 2 \\ 2 & 0 & 0 & 3 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 16 & 12 & 2 \\ 2 & 6 & 16 \\ 8 & 8 & 6 \\ 0 & 7 & 8 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 1 \\ 3 & 1 & 3 \end{pmatrix}$$

2.2. *Saddlepoints*

(a) Let A be an arbitrary $m \times n$ matrix game. Show that any two saddlepoints must have the same value. In other words, if (i, j) and (k,l) are two saddlepoints, show that $a_{ij} = a_{kl}$.

(b) Let A be a 4×4 matrix game in which $(1,1)$ and $(4,4)$ are saddlepoints. Show that A has at least two other saddlepoints.

(c) Give an example of a 4×4 matrix game with exactly three saddlepoints.

2.3. *Rock–Paper–Scissors*

In the famous Rock–Paper–Scissors two-player game each player has three pure strategies: Rock, Paper, and Scissors. Here, Scissors beats Paper, Paper beats Rock, Rock beats Scissors. Assign a 1 to winning, 0 to a draw, and -1 to losing. Model this game as a matrix game, try to guess its optimal strategies, and then show that these are the unique optimal strategies. What is the value of this game?

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