Arif Ali
Math 611 Stochastic Simulation
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Homework 6

Exercise 1

```
x = c(169.14353, 135.73850, 102.46566,  80.91151, 148.45425,
      144.68948, 106.56257, 104.83559,94.81216, 109.47048,
      95.94150, 123.84673,  87.18401, 104.73420, 111.94364,
      119.69467, 151.77627,  81.80692, 116.58660,  98.28933)

mu_1 <- c(100)
mu_2 <- c(120)
## as well as the latent variable parameters
#tau_1 <- c(1-0.7) # 1-W
W <- c(0.7) # W
for( i in 1:12) {
  ## Given the observed data, as well as the distribution parameters,
  ## what are the latent variables?
  T_1 <- (1-W[i]) * dnorm(x, mu_1[i], sd = sqrt(20))
  T_2 <- W[i] * dnorm(x, mu_2[i], sd = sqrt(25))
  P_1 <- T_1 / (T_1 + T_2)
  P_2 <- T_2 / (T_1 + T_2) ## note: P_2 = 1 - P_1
  #tau_1[i+1] <- mean(P_1)
  W[i+1] <- mean(P_1)
  ## Given the observed data, as well as the latent variables,
  ## what are the population parameters
  mu_1[i+1] <- sum( P_1 * x) / sum(P_1)
  mu_2[i+1] <- sum( P_2 * x) / sum(P_2)
}
data.frame(mu_1, mu_2, W)[-1,]

##         mu_1     mu_2         W
## 2   96.04482 133.4018 0.5074671
## 3   98.20225 138.5979 0.5979239
## 4   99.30204 140.9658 0.6365587
## 5   99.85074 142.2978 0.6561922
## 6  100.33812 143.4546 0.6728337
## 7  100.80388 144.7254 0.6894349
## 8  101.02890 145.4078 0.6977058
## 9  101.07789 145.5545 0.6994714
## 10 101.08642 145.5785 0.6997678
## 11 101.08782 145.5824 0.6998159
## 12 101.08804 145.5830 0.6998237
## 13 101.08808 145.5831 0.6998249
```

Exercise 2

Part A

Since $X \sim N(0,1) \implies X^2 \sim Chi-squared(1)$ then $Z_1^2 \sim Chi-squared(1)$.

By modifying the derivation from http://www.math.uah.edu/stat/special/ChiSquare.html: please note $f(x)$ and $F(x)$ refer to the PDF and CDF of $N(0,1)$

$$\begin{aligned} p(X = x | X \sim \theta_1 Z_1^2) \implies \\ p(-\sqrt{x/\theta_1} < Z_1 < \sqrt{x/\theta_1}) = 2 * F(\sqrt{x/\theta_1}) - 1 \end{aligned} \tag{1}$$

$$\begin{aligned} d/\delta x(2 * F(\sqrt{x/\theta_1}) - 1) = \\ 2 * 1/2 * 1/\sqrt{(\theta_1)} * x^{1/2-1} * f(\sqrt{x/\theta_1}) = \\ 1/\sqrt{\theta_1} * x^{1/2-1} * \frac{1}{\sqrt{2\pi}} e^{(\sqrt{x/\theta_1})^2/2} = \\ \frac{1}{\sqrt{\pi}} * \frac{1}{\sqrt{2\theta_1}} * x^{1/2-1} * e^{\frac{1}{2\theta_1}x} = \\ \frac{1}{\sqrt{\pi}} * (\frac{1}{2\theta_1})^{1/2} * x^{1/2-1} * e^{\frac{1}{2\theta_1}x} \sim Gamma(\frac{1}{2}, \frac{1}{2\theta_1}) \end{aligned} \tag{2}$$

Part B

From Part A, we know that $\theta_1 * Z_1^2 \sim Gamma(\frac{1}{2}, \frac{1}{2\theta_1})$. Using this same logic, $\theta_2 * Z_2^2 \sim Gamma(\frac{1}{2}, \frac{1}{2\theta_2})$, so by using Method of mommemts on one of the theta parameters, it should mirror the results of the other theta parameter.

$$\begin{aligned} E(\theta_1 * Z_1^{21}) = E(\theta_1 * Z_1^2) = \frac{\frac{1}{2}}{\frac{1}{2\theta_1}} = \\ \theta_1 E(\theta_1 * Z_1^2) = \\ Var(\theta_1 * Z_1^2) + (E(\theta_1 * Z_1^2))^2 = \\ 2(\theta_1)^2 + \theta_1^2 = 3\theta_1^2 \end{aligned} \tag{3}$$

Part C

```
theta_1 = 0.3
theta_2 = 1-theta_1
W = c()
for(i in 1:1e3){
  if(runif(1)< theta_1){
    W[i] = theta_1*rchisq(1,1)
  } else {
    W[i] = theta_2*rchisq(1,1)
  }
}

for( i in 1:12) {
  ## Given the observed data, as well as the distribution parameters,
  ## what are the latent variables?
  T_1 <- dgamma(W,0.5, 1/(2*theta_1))
  T_2 <- dgamma(W,0.5, 1/(2*theta_2))
  P_1 <- T_1 / (T_1 + T_2)
```

```
  P_2 <- T_2 / (T_1 + T_2)

  theta_1[i+1] <-  sum( P_1 * W) / sum(P_1)
  theta_2[i+1] <- sum( P_2 * W) / sum(P_2)

}
data.frame(theta_1, theta_2)[-1,]
```

```
##      theta_1   theta_2
## 2  0.3300647 0.8869361
## 3  0.3317461 0.8940444
## 4  0.3327202 0.8963451
## 5  0.3332192 0.8975735
## 6  0.3335825 0.8982493
## 7  0.3339054 0.8986130
## 8  0.3340001 0.8990231
## 9  0.3343020 0.8991573
## 10 0.3342301 0.8994933
## 11 0.3344372 0.8995990
## 12 0.3347586 0.8994143
## 13 0.3346795 0.8996898
```

Exercise 3

Part A

$$\begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.05 & 0.5 & 0.45 & 0 \\ 0 & 0.1 & 0.5 & 0.4 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix} \tag{4}$$

Part B

Let $\pi(0) = c$

$$\begin{aligned} \pi(1) = \pi(0) * \frac{p_0}{q_1} = \pi(0) * \frac{0.5}{0.05} &= 10c \\ \pi(2) = \pi(1) * \frac{.45}{0.1} &= 45c \\ \pi(3) = \pi(2) * \frac{.4}{0.3} &= 60c \end{aligned} \tag{5}$$

$$\begin{aligned} \pi(0) + \pi(1) + \pi(2) + \pi(3) = 1 = \\ c + 10c + 45c + 60c = 116c \implies \\ c = \frac{1}{116} \end{aligned} \tag{6}$$

So:

$$\begin{aligned} \pi(0) + \pi(1) + \pi(2) + \pi(3) = 1 = \\ c + 10c + 45c + 60c = 116c \implies \\ c = \frac{1}{116} \end{aligned} \tag{7}$$

$$\begin{aligned}
\pi(0) &= \frac{1}{116} \\
\pi(1) &= \frac{10}{116} \\
\pi(2) &= \frac{45}{116} \\
\pi(3) &= \frac{60}{116}
\end{aligned} \tag{8}$$

Exercise 4

From class, we know that:

$$\begin{pmatrix} 0 & 3/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 3/3 & 0 \end{pmatrix} \tag{9}$$

And that the stationary distribution follows a binomial distribution:

$$\pi(i) = \begin{pmatrix} n \\ i \end{pmatrix} = (\frac{1}{2})^i * (\frac{1}{2})^{n-i} = \pi(i) = \begin{pmatrix} n \\ i \end{pmatrix} (\frac{1}{2})^n \tag{10}$$

$$\begin{aligned}
p(i) * p(i, i+1) = \begin{pmatrix} n \\ i \end{pmatrix} (\frac{1}{2})^n * \frac{n-i}{n} = \\
\frac{n!}{i! * (n-i)!} (\frac{1}{2})^n * \frac{n-i}{n} = \\
\frac{n * (n-1)!}{i! * (n-i)(n-i-1)!} (\frac{1}{2})^n * \frac{n-i}{n} = \\
\frac{(n-1)!}{i! * (n-i-1)!} (\frac{1}{2})^n = \\
\frac{n!}{(i+1)! * (n-i-1)!} (\frac{1}{2})^n * \frac{i+1}{n} = \\
\pi(i+1)p(i+1, i)
\end{aligned} \tag{11}$$