Arif Ali
Math 611 Stochastic Simulation
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Homework 5

Exercise 1

$1, 2, 3$ are all aperiodical because $gcd(1) = gcd(2) = gcd(3) = 1$

Exercise 2

States $1, 2$ are periodic with $gcd(1) = gcd(2) = 2$ so the period for both stages is also 2.

Exercise 3

$$f(x|\theta) = \frac{1}{\theta + \frac{1}{2} - (\theta - \frac{1}{2})} = 1 \tag{1}$$

From $f(x|\theta)$, we know that $L(\theta|x) = \prod_{i=1}^{n} 1^n = 1$. This means that there is no unique value for θ

However $X_{(1)} > \theta - \frac{1}{2}$ and $X_{(n)} < \theta + \frac{1}{2}$

From Pitman 319, $f_{min}(x) = n(1 - F(x))^{n-1} f(x)$ and $f_{max}(x) = n(F(x))^{n-1} f(x)$

$$\begin{aligned} X_{(1)} = min(X_1 ... X_n) \sim n(1/2 + \theta - x)^{n-1} \\ X_{(n)} = max(X_1 ... X_n) \sim n(1/2 - \theta + x)^{n-1} \end{aligned} \tag{2}$$

The MLE of θ should occur when $X_{(1)} = X_{(n)}$

$$\begin{aligned} n(1/2 + \theta - x_{(1)})^{n-1} = n(1/2 - \theta + x_{(n)})^{n-1} \implies \\ 1/2 + \theta - x_{(1)} = 1/2 - \theta + x_{(n)} \implies \\ \theta - x_{(1)} = -\theta + x_{(n)} \implies \\ \theta = \frac{x_{(1)} + x_{(n)}}{2} \end{aligned} \tag{3}$$

$\hat{\theta} \sim \frac{X_{(1)} + X_{(n)}}{2}$

Exercise 4

Part A

Let the prior be $\theta \sim unif(0, 1)$ and $X|\theta \sim Bernoulli(\theta)$

$$\begin{aligned} p(\theta|X) \propto \prod_{i=1}^{n} p(X_i|\theta) * p(\theta) = \\ \prod_{i=1}^{n} \theta^{x_i} (1 - \theta)^{1 - x_i} = \\ \theta^{\sum_{i=1}^{n} x_i} (1 - \theta)^{\sum_{i=1}^{n} 1 - x_i} = \\ \theta^{\sum_{i=1}^{n} x_i} (1 - \theta)^{n - \sum_{i=1}^{n} x_i} \end{aligned} \tag{4}$$

Let $y = \sum_{i=1}^{n} x_i$ so,

$$\theta^y (1-\theta)^{n-y} = \theta^{(y+1)-1} * (1-\theta)^{(n-y+1)-1} \propto beta(y+1, n-y+1) \tag{5}$$

$\therefore$ the Bayes Estimate is $\frac{y+1}{y+1+n-y+1} = \frac{y+1}{n+2}$

Part B

$$\begin{aligned}
L(\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \implies \\
Log(L(\theta)) = \sum_{i=1}^{n} log(\theta^{x_i}(1-\theta)^{1-x_i}) = \\
\sum_{i=1}^{n} log(\theta^{x_i}) + \sum_{i=1}^{n} log((1-\theta)^{1-x_i}) = \\
\sum_{i=1}^{n} x_i log(\theta) + \sum_{i=1}^{n} 1 - x_i * log(1-\theta) = \\
n\bar{x} * log(\theta) + (n - n\bar{x}) * log(1-\theta)
\end{aligned} \tag{6}$$

$$\begin{aligned}
\delta/d\theta(log(L(\theta))) = \frac{n\bar{x}}{\theta} - \frac{n-n\bar{x}}{1-\theta} = 0 \implies \\
\frac{n\bar{x}}{\theta} = \frac{n-n\bar{x}}{1-\theta} \implies \\
n\bar{x} * (1-\theta) = (n-n\bar{x})(\theta) \implies \\
n\bar{x} = n\theta
\end{aligned} \tag{7}$$

$\therefore \hat{\theta} = \bar{x} = \frac{y}{n}$

Since the prior distribution is uniform, $E(\theta) = \frac{1}{2}$

$$\begin{aligned}
\frac{1}{2}(1-\omega) + \frac{y}{n}(\omega) = \frac{y+1}{n+2} \implies \\
\omega = \frac{n}{n+2} \implies \\
1 - \omega = \frac{2}{n+2}
\end{aligned} \tag{8}$$

$$\frac{1}{2}(\frac{2}{n+2}) + \frac{y}{n}(\frac{n}{n+2}) = \frac{y+1}{n+2} \tag{9}$$

Exercise 5

Let $f(x;\theta) = e^{-(x-\theta)}$, $Y_1 = min(X_i)$, and $T = T(X_1, ..., Xn)$

$$F(x) = \int_{\theta}^{x} e^{-(x-\theta)} dx = 1 - e^{\theta - x} \tag{10}$$

From Pitman 319, $f_{min}(x) = n(1 - F(x))^{n-1} f(x)$

$$\begin{aligned}
f_{min}(x) = n(1-(1-e^{\theta-x}))^{n-1} e^{-(x-\theta)} = \\
n(e^{\theta-x})^{n-1} e^{-(x-\theta)} = \\
n * (e^{\theta-x})^{n}
\end{aligned} \tag{11}$$

$$
\begin{aligned}
P(X_1, ..., X_n|T = t) = \frac{e^{n\theta - \sum x_i}}{n * (e^{\theta - t})^n} = \\
\frac{1}{n} e^{n\theta - \sum x_i - (\theta - t)n} = \\
\frac{1}{n} e^{-\sum x_i - (-t)n}
\end{aligned} \tag{12}
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Since the conditional density and range do not depend on θ, $Y_1 = min(X_i)$ is a sufficent Statistic