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Math 611 Stochastic Simulation
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Homework 3

Exercise 1

Since $Area_{square} = 1$, $Area_{circle} = \frac{\pi}{2^2} = \frac{\pi}{4}$ as stated in the exercise. Therefore the number of points that are expected in the circle is $\frac{\pi/4}{1} = \pi/4$ Given that the orgin of the circle is at $(1/2, 1/2)$ with a radius of $r = 1/2$, the circle equation is $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$, so we should look for points with coordinates such that $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{1}{4}$. I assume that the circumference doesn't count as in the circle. Since the area of the circle is $\frac{\pi}{4}$, then the value of the percentage of points must be multipled by 4.

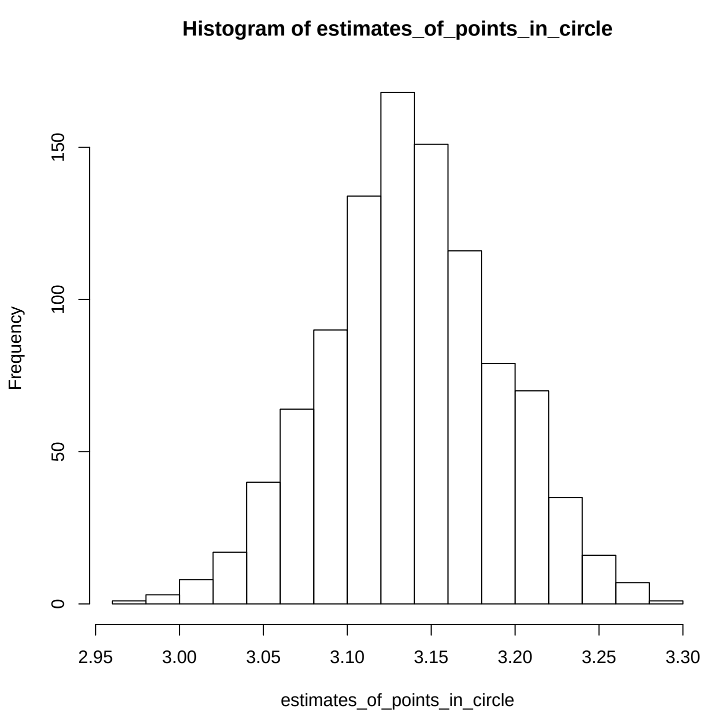
```
points_in_circle = function(z){
  points= data.frame(x = runif(1e3), y = runif(1e3))
  mean((points$x-1/2)^2+(points$y-1/2)^2<1/4)
}
estimates_of_points_in_circle = 4*sapply(1:1e3, points_in_circle)
mean(estimates_of_points_in_circle)

## [1] 3.140304

var(estimates_of_points_in_circle)

## [1] 0.002683615

hist(estimates_of_points_in_circle)
```



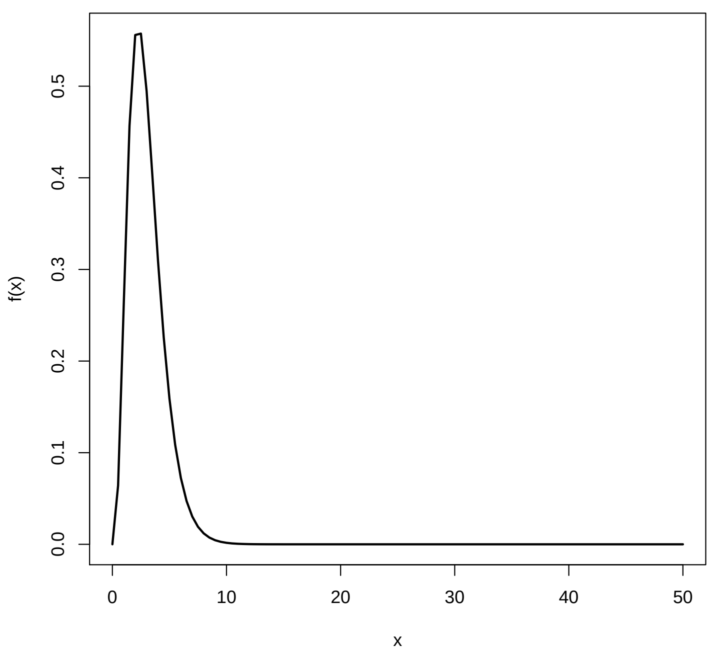
Based on the estimated mean and the variance, the fixed range is very close to π.

Exercise 2

Please note: I used the methods from "http://math.stackexchange.com/questions/1200443/evaluating-difficult-monte-carlo-integration-in-r" to do monte carlo integration in R

Part A

```
f = function(x){
  return(exp(-4*x/3)*x^3)
}
curve(f, lwd=2,to = 50)
```



Based on the graph, it is clear that $f(x)$ converges to 0 close to 10, so using the method described in class, we will do the following:

```
n = 1e4
a = 0
b = 10
x = runif(n, a, b)
y = f(x)
(b-a)/n*sum(y)

## [1] 1.917542
```

Part B

$$\begin{aligned}
\int_0^\infty e^{\frac{-4x}{3}} x^3 \delta x \implies \\
\int_0^\infty x^3 * e^{\frac{-4x}{3}} \delta x \implies \\
\int_0^\infty x^{4-1} * e^{-\frac{x}{3/4}} \delta x \implies \\
\Gamma(4) * (\frac{3}{4})^4 \int_0^\infty \frac{1}{\Gamma(4) * (\frac{3}{4})^4} * x^{4-1} * e^{-\frac{x}{3/4}} \delta x
\end{aligned} \tag{1}$$

please recognize that $\frac{1}{\Gamma(4)*(\frac{3}{4})^4} * x^{4-1} * e^{\frac{x}{3/4}} \sim gamma(4, 3/4)$

$\therefore \Gamma(4) * (\frac{3}{4})^4 \int_0^\infty \frac{1}{\Gamma(4)*(\frac{3}{4})^4} * x^{4-1} * e^{\frac{x}{3/4}} \delta x = \Gamma(4) * (\frac{3}{4})^4 = 1.898$

Exercise 3

I picked $(a, b) = (-10, 10)$ based on $X \sim N(0, 2^2)$

```
n = 1e4
a = -10
b = 10
x = runif(n, a, b)
y = dnorm(x,0,2)*exp(-x^2)
(b-a)/n*sum(y)

## [1] 0.3327544
```

$$\begin{aligned}
E_f(h(x)) = \int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2} * e^{-x^2} \delta x = \\
\int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2 - \frac{2\sigma^2}{2\sigma^2}x^2} \delta x = \\
\int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2(1+2\sigma^2)} \delta x = \\
\int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{\frac{2\sigma^2}{(1+2\sigma^2)}}x^2} \delta x = \\
1/\sqrt{(1+2\sigma^2)} \int_x \frac{1}{\sqrt{2\pi\sigma^2/(1+2\sigma^2)}} e^{-\frac{1}{\frac{2\sigma^2}{(1+2\sigma^2)}}x^2} \delta x = 1/\sqrt{(1+2\sigma^2)}
\end{aligned} \tag{2}$$

This is true because $\frac{1}{\sqrt{2\pi\sigma^2/(1+2\sigma^2)}} e^{-\frac{1}{\frac{2\sigma^2}{(1+2\sigma^2)}}x^2} \sim N(0, \sigma^2/(1+2\sigma^2))$

Exercise 4

The min probably of transiting to state 3 is $min(0.1, 0.2, 0.4) = 0.1 \therefore P_3(T_3 > n) \leq (0.9)^n$ That's because all other chains possible, given n, are $\leq (0.9)^n$ where $(0.9)^n \to 0$ as $n \to \infty$, so state 3 is recurrent.

For stage 1 and 2, the same logic is applied:

The min probably of transiting to state 2 is $min(0.2, 0.5, 0.4) = 0.2 \therefore P_2(T_2 > n) \leq (0.8)^n$ where $(0.8)^n \to 0$ as $n \to \infty$, so state 2 is recurrent.

The min probably of transiting to state 1 is $min 0.7, 0.3, 0.2 = 0.2 \therefore P_1(T_1 > n) \leq (0.8)^n$ where $(0.8)^n \to 0$ as $n \to \infty$, so state 1 is recurrent.

Exercise 5

The state space is 1...d

$(x, y) \sim hypergeometric(d, 2d, 2x)$ where $k = y \therefore$

$$p(x, y) = \frac{\left(\begin{array}{c} 2x \\ y \end{array}\right)\left(\begin{array}{c} 2d-2x \\ d-y \end{array}\right)}{\left(\begin{array}{c} 2d \\ d \end{array}\right)} \tag{3}$$

To move from 0 to 0, $p(0, 0) = \frac{\left(\begin{array}{c} 0 \\ 0 \end{array}\right)\left(\begin{array}{c} 2d-0 \\ d-0 \end{array}\right)}{\left(\begin{array}{c} 2d \\ d \end{array}\right)} = 1$ Thus, 0 is an aborbing state.

To move from d to d, $p(d, d) = \frac{\left(\begin{array}{c} 2d \\ d \end{array}\right)\left(\begin{array}{c} 2d-2d \\ d-d \end{array}\right)}{\left(\begin{array}{c} 2d \\ d \end{array}\right)} = 1$ Thus, d is an aborbing state.