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Math 611 Stochastic Simulation
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Homework 3

Exercise 1

Since $Area_{square} = 1$, $Area_{circle} = \frac{\pi}{2^2} = \frac{\pi}{4}$ as stated in the exercise. Therefore the number of points that are expected in the circle is $\frac{\pi/4}{1} = \pi/4$ Given that the orgin of the circle is at $(1/2, 1/2)$ with a radius of $r = 1/2$, the circle equation is $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$, so we should look for points with coordinates such that $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}$. I assume that the circumference counts as in the circle. Since the area of the circle is $\frac{\pi}{4}$, then the fraction of points bounded by the circle must be multipled by 4.

```
points_in_circle = function(z, r=.5){
  points= data.frame(x = runif(1e3), y = runif(1e3))
  mean((points$x-r)^2+(points$y-r)^2<=r^2)
}
estimates_of_points_in_circle = 4*sapply(1:1e3, points_in_circle)
mean(estimates_of_points_in_circle)

## [1] 3.139868

var(estimates_of_points_in_circle)

## [1] 0.002590909
```

Based on the estimated mean and the variance, the fixed range is very close to the real value of π.

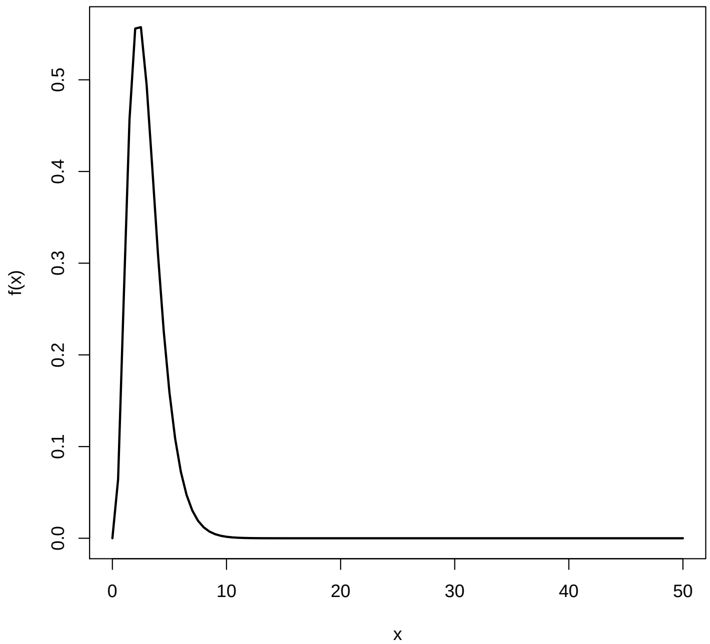
Exercise 2

Please note: I used the methods from "http://math.stackexchange.com/questions/1200443/evaluating-difficult-monte-carlo-integration-in-r" to do monte carlo integration in R

Part A

Solving via Simulation:

```
f = function(x){
  return(exp(-4*x/3)*x^3)
}
curve(f, lwd=2,to = 50)
```



Based on the graph, it is clear that $f(x)$ converges to 0 close to 10, so using the method described in class, we will do the following:

```
MCint = function(n =1e4, a, b, f){
  x = runif(n, a, b)
  y = f(x)
  return((b-a)*mean(y))
}
MCint(1e4,0,10, f)

## [1] 1.908676
```

Based on the solution in part B, it is clear that the approximated value from the simulation is close to the actual value.

Part B

$$\begin{aligned}
\int_0^\infty e^{\frac{-4x}{3}} x^3 \delta x &\implies \\
\int_0^\infty x^3 * e^{\frac{-4x}{3}} \delta x &\implies \\
\int_0^\infty x^{4-1} * e^{-\frac{x}{3/4}} \delta x &\implies \\
\Gamma(4) * (\frac{3}{4})^4 \int_0^\infty \frac{1}{\Gamma(4) * (\frac{3}{4})^4} * x^{4-1} * e^{-\frac{x}{3/4}} \delta x
\end{aligned} \tag{1}$$

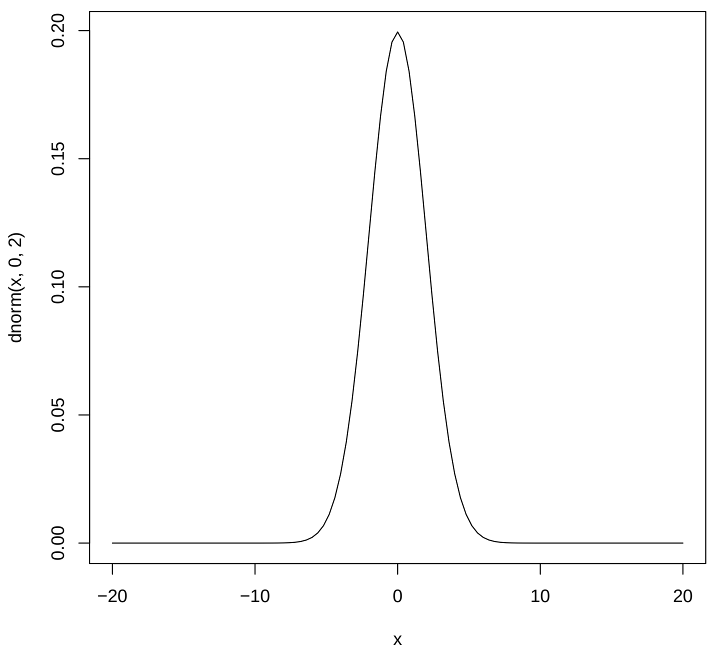
please recognize that $\frac{1}{\Gamma(4)*(\frac{3}{4})^4} * x^{4-1} * e^{\frac{x}{3/4}} \sim gamma(4, 3/4)$

$\therefore \Gamma(4) * (\frac{3}{4})^4 \int_0^\infty \frac{1}{\Gamma(4)*(\frac{3}{4})^4} * x^{4-1} * e^{\frac{x}{3/4}} \delta x = \Gamma(4) * (\frac{3}{4})^4 = 1.898$ which is close to the estimation from 2a.

Exercise 3

Using the same logic in code from 2a I picked $(a, b) = (-10, 10)$ based on the plot of the curve of $X \sim N(0, 2^2)$

```r
curve(dnorm(x,0,2), from = -20, 20)
```




```r
MCint(1e4, -10,10, f = function(x){dnorm(x,0,2)*exp(-x^2)})

## [1] 0.3320746
```

$$\begin{aligned} E_f(h(x)) = \int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2} * e^{-x^2} \delta x = \\ \int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2 - \frac{2\sigma^2}{2\sigma^2}x^2} \delta x = \\ \int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2(1+2\sigma^2)} \delta x = \\ \int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{\frac{2\sigma^2}{(1+2\sigma^2)}}x^2} \delta x = \\ 1/\sqrt{(1+2\sigma^2)} \int_x \frac{1}{\sqrt{2\pi\sigma^2/(1+2\sigma^2)}} e^{-\frac{1}{\frac{2\sigma^2}{(1+2\sigma^2)}}x^2} \delta x = 1/\sqrt{(1+2\sigma^2)} \end{aligned} \tag{2}$$

This is true because $\frac{1}{\sqrt{2\pi\sigma^2/(1+2\sigma^2)}} e^{-\frac{1}{\frac{2\sigma^2}{(1+2\sigma^2)}}x^2} \sim N(0, \sigma^2/(1+2\sigma^2))$

Based on $\sigma^2 = 4, E_f(h(x)) = \frac{1}{\sqrt{1+2*4}} = \frac{1}{3}$ This value is close to the estimation.

Exercise 4

Please recall the transtional matrix:

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} \tag{3}$$

The min probability of transiting from state i to state 3 is $min(0.1, 0.2, 0.4) = 0.1 \therefore P_3(T_3 > n) \leq (0.9)^n$ That's because all other chains possible, given n, are $\leq (0.9)^n$ Thus, $(0.9)^n \to 0$ as $n \to \infty$, so state 3 is recurrent.

For stage 1 and 2, the same logic is applied:

The min probability of transiting from state i to state 2 is $min(0.2, 0.5, 0.4) = 0.2 \therefore P_2(T_2 > n) \leq (0.8)^n$ where $(0.8)^n \to 0$ as $n \to \infty$, so state 2 is recurrent.

The min probability of transiting from state i to state 1 is $min0.7, 0.3, 0.2 = 0.2 \therefore P_1(T_1 > n) \leq (0.8)^n$ where $(0.8)^n \to 0$ as $n \to \infty$, so state 1 is recurrent.

Exercise 5

The state space is $1...d$ The daughter genes will inherit d subunit from the parent gene, which has $2d$ subunits. Since x_0 represents the number of mutant subunits the parent has before the duplication, the daughter will get m of those genes where $m < d$. There are $\binom{2x_0}{m}$ possiblities of that happening. The daughter also require $d - m$ normal genes from $2d - 2x_0$ normal subunits from the parent with $\binom{2d - 2x_0}{d - m}$ possibilities. There are a total of $\binom{2d}{d}$ different possibilities from the parent to daughter. Subbing x_0 for x and m for y:

$$p(x, y) = \frac{\binom{2x}{y}\binom{2d - 2x}{d - y}}{\binom{2d}{d}} \tag{4}$$

This is a hypergeometric distribution.

To move from d to d, $p(d, d) = \frac{\binom{2d}{d}\binom{2d - 2d}{d - d}}{\binom{2d}{d}} = 1$ Thus, d is an aborbing state.