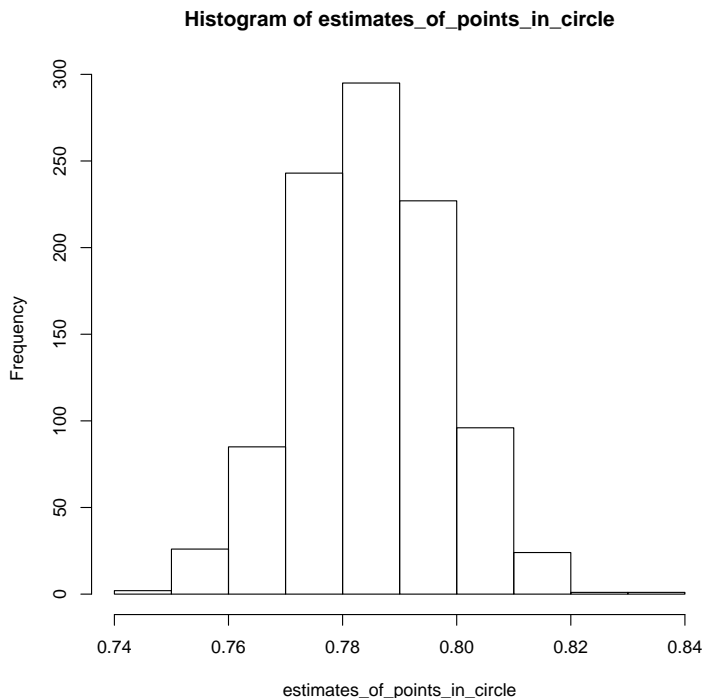


## Homework 3

### Exercise 1

Since  $Area_{square} = 1$ ,  $Area_{circle} = \frac{\pi}{2^2} = \frac{\pi}{4}$  as stated in the exercise. Therefore the number of points that are expected in the circle is  $\frac{\pi/4}{1} = \pi/4$ . Given that the origin of the circle is at  $(1/2, 1/2)$  with a radius of  $r = 1/2$ , the circle equation is  $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$ , so we should look for points with coordinates such that  $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{1}{4}$ . I assume that the circumference doesn't count as in the circle.

```
points_in_circle = function(z){  
  points= data.frame(x = runif(1e3), y = runif(1e3))  
  mean((points$x-1/2)^2+(points$y-1/2)^2<1/4)  
}  
estimates_of_points_in_circle = sapply(1:1e3, points_in_circle)  
mean(estimates_of_points_in_circle)  
  
## [1] 0.785465  
  
var(estimates_of_points_in_circle)  
  
## [1] 0.0001660909  
  
hist(estimates_of_points_in_circle)
```



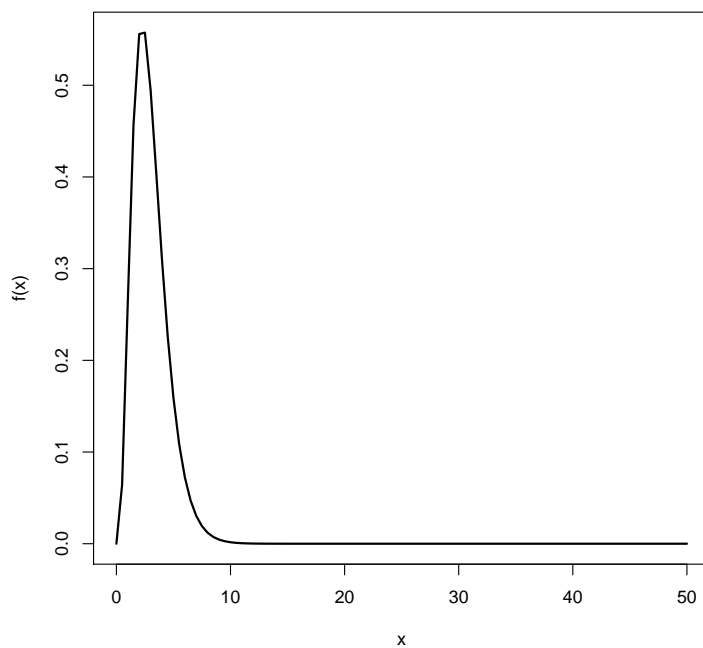
Based on the estimated mean and the variance, the fixed range is very close to the expected fraction.

## Exercise 2

Please note: I used the methods from "<http://math.stackexchange.com/questions/1200443/evaluating-difficult-monte-carlo-integration-in-r>" to do monte carlo integration in R

### Part A

```
f = function(x){  
  return(exp(-4*x/3)*x^3)  
}  
curve(f, lwd=2,to = 50)
```



Based on the graph, it is clear that  $f(x)$  converges to 0 close to 10, so using the method described in class, we will do the following:

```
n = 1e4  
a = 0  
b = 10  
x = runif(n, a, b)  
y = f(x)  
(b-a)/n*sum(y)  
## [1] 1.876069
```

## Part B

$$\begin{aligned}
 & \int_0^\infty e^{-\frac{4x}{3}} x^3 \delta x \implies \\
 & \int_0^\infty x^3 * e^{-\frac{4x}{3}} \delta x \implies \\
 & \int_0^\infty x^{4-1} * e^{-\frac{x}{3/4}} \delta x \implies \\
 & \Gamma(4) * \left(\frac{3}{4}\right)^4 \int_0^\infty \frac{1}{\Gamma(4) * \left(\frac{3}{4}\right)^4} * x^{4-1} * e^{-\frac{x}{3/4}} \delta x
 \end{aligned} \tag{1}$$

please recognize that  $\frac{1}{\Gamma(4) * \left(\frac{3}{4}\right)^4} * x^{4-1} * e^{-\frac{x}{3/4}} \sim \text{gamma}(4, 3/4)$

$$\therefore \Gamma(4) * \left(\frac{3}{4}\right)^4 \int_0^\infty \frac{1}{\Gamma(4) * \left(\frac{3}{4}\right)^4} * x^{4-1} * e^{-\frac{x}{3/4}} \delta x = \Gamma(4) * \left(\frac{3}{4}\right)^4 = 1.898$$

## Exercise 3

```

n = 1e4
a = -10
b = 10
x = runif(n, a, b)
y = exp(-(dnorm(x, 0, 2))^2)
(b-a)/n*sum(y)

## [1] 19.86586

```

$$\begin{aligned}
 E_f(h(x)) &= \int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2} * e^{-x^2} \delta x = \\
 & \int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2 - \frac{2\sigma^2}{2\sigma^2}x^2} \delta x = \\
 & \int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2(1+2\sigma^2)} \delta x = \\
 & \int_x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\frac{1}{2\sigma^2}x^2}{(1+2\sigma^2)}} \delta x = \\
 & 1/\sqrt{(1+2\sigma^2)} \int_x \frac{1}{\sqrt{2\pi\sigma^2/(1+2\sigma^2)}} e^{-\frac{\frac{1}{2\sigma^2}x^2}{(1+2\sigma^2)}} \delta x = 1/\sqrt{(1+2\sigma^2)}
 \end{aligned} \tag{2}$$

This is true because  $\frac{1}{\sqrt{2\pi\sigma^2/(1+2\sigma^2)}} e^{-\frac{\frac{1}{2\sigma^2}x^2}{(1+2\sigma^2)}} \sim N(0, \sigma^2/(1+2\sigma^2))$

## Exercise 4

## Exercise 5