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Math 611 Stochastic Simulation
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Homework 8

Exercise 1

Since n is odd, the median value of $i \in 1...n$ must be $\frac{n+1}{2}$. Using order statistics

$$\begin{aligned} f_{(\frac{n+1}{2})}(x) = n * f(x) * \begin{pmatrix} n-1 \\ \frac{n+1}{2}-1 \end{pmatrix} (F(x))^{\frac{n+1}{2}-1}(1-F(x))^{n-\frac{n+1}{2}} \propto \\ f(x) * (F(x))^{\frac{n+1}{2}-1}(1-F(x))^{n-\frac{n+1}{2}} = \\ f(x) * (F(x))^{\frac{n-1}{2}}(1-F(x))^{\frac{n-1}{2}} \end{aligned} \tag{1}$$

Assuming that y is exponential distributed with $\lambda = 1$

$$\begin{aligned} f(y) = e^{-y} * (1-e^{-y})^{\frac{n-1}{2}}(1-(1-e^{-y}))^{\frac{n-1}{2}} = \\ e^{-y} * (1-e^{-y})^{\frac{n-1}{2}}(e^{-y}))^{\frac{n-1}{2}} = \\ e^{-y} * (1-e^{-y})^{\frac{n-1}{2}} * e^{-y\frac{n-1}{2}} \end{aligned} \tag{2}$$

Exercise 2

```
log_f = function(y,n=101){
  -y+(n-1)/2*log(1-exp(-y))+(-y*(n-1)/2)
}
```

Exercise 3

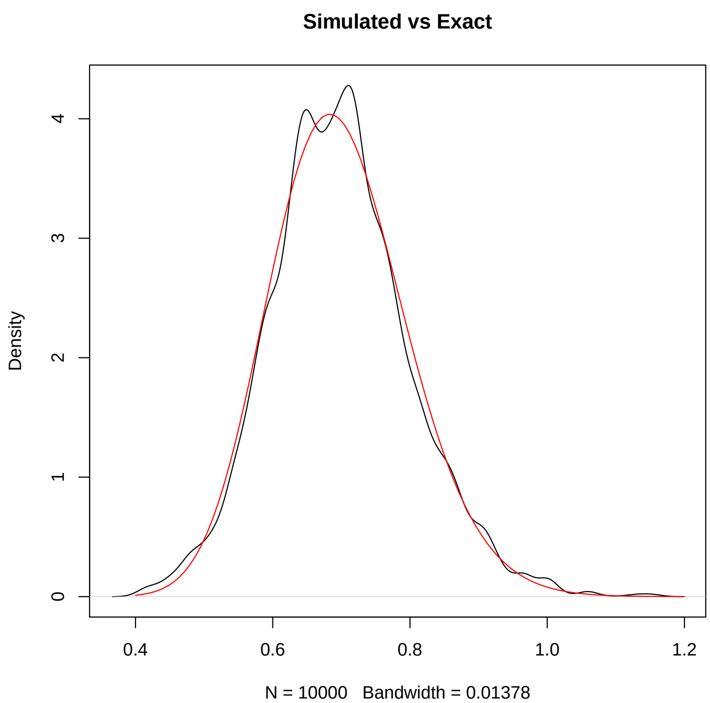
```
x = 1
accept = 0
for(t in 2:1e4){
  y = rexp(1, rate = x[t-1])
  rho = exp(log_f(y))*dexp(x[t-1], rate = y)/
    (exp(log_f(x[t-1]))*dexp(y, rate = x[t-1]))
    if(runif(1)<rho){
      x[t] = y
      accept[t] = 1
    }
    else{
      x[t] =x[t-1]
      accept[t] = 0
  }
}
summary(x)

##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.4077  0.6309  0.6942  0.6985  0.7604  1.1560
```

The acceptance rate is 0.1349.

Please note that in order to compare the simulated density with the exacty, we have to calculate the normalizing constant

$$n \begin{pmatrix} n-1 \\ \frac{n+1}{2} - 1 \end{pmatrix} = n * \frac{(n-1)!}{(n-1-((n+1)/2-1)) * \frac{n+1}{2} - 1} = \frac{n!}{(\frac{n-1}{2})^2} \tag{3}$$

```
plot(density(x),
     main="Simulated vs Exact")
#Exact Samples
norm = factorial(101)/((factorial((101-1)/2))^2)
curve(norm*exp(log_f(x)),0.4,1.2,add=T,col="red")
```



Exercise 4

Let $R \sim exp(1)$, $M \sim exp(\frac{1}{2})$, $S \sim exp(\frac{1}{3})$. Also note that 25 minutes is $\frac{5}{12}$ hours.

$$p(min(R, M, S) > 25 * 1/60) = 1 - exp(-(1^{-1} + (1/2)^{-1} + (1/3)^{-1}) * 5/12) = 1 - 0.082085 \tag{4}$$

Exercise 5

Let $p = \frac{1}{50}$, $n = 20$ $binom(20, 1/50)$,so $\frac{\lambda}{n} = \frac{\lambda}{20} = 1/50 \implies \lambda = 2/5$ and $\therefore binom(20, 1/50) \approx Poisson(2/5)$

$$p(k \geq 1) = 1 - p(k = 0) = 0.32968 \tag{5}$$

Exercise 6

T_1 be the expected value of the amount of time that the first navigation lasts. There are three possible senarios when that can happen, {1 or 2 dies before t and 3 dies at t}, {1 or 3 dies before t and 2 dies at t}, {2 or 3 dies at t and 1 dies at t}

$$\begin{aligned}
f(time) = (p(T_1 > t)p(T_2 \leq t) + p(T_1 \leq t)p(T_2 > t)) * p(T_3 = t) + \\
(p(T_1 > t)p(T_3 \leq t) + p(T_1 \leq t)p(T_3 > t)) * p(T_2 = t) + \\
(p(T_2 > t)p(T_3 \leq t) + p(T_2 \leq t)p(T_3 > t)) * p(T_1 = t) = \\
(exp(-t) * (1 - exp(-2/3t)) + (1 - exp(-t)) * (exp(-2/3t))) * 1/3 * exp(-1/3t) + \\
(exp(-t) * (1 - exp(-1/3t)) + (1 - exp(-t)) * (exp(-1/3t))) * 2/3 * exp(-2/3t) + \\
(exp(-2/3t) * (1 - exp(-1/3t)) + (1 - exp(-2/3t)) * (exp(-1/3t))) * 1 * exp(-t) = \\
4/3exp(-4/3t) - 4/3exp(-2t) + exp(-t) - exp(-2t) + 5/3exp(-5/3t) - 5/3exp(-2t) = \\
4/3exp(-4/3t) + exp(-t) + 5/3exp(-5/3t) - 4exp(-2t)
\end{aligned} \tag{6}$$

$$\begin{aligned}
E(t) = \int_0^\infty t * (4/3exp(-4/3t) + exp(-t) + 5/3exp(-5/3t) - 4exp(-2t))dt = \\
4/3 \int_0^\infty t * (exp(-4/3t))dt + \int_0^\infty t * (exp(-t))dt + \\
5/3 \int_0^\infty t * (exp(-5/3t))dt - 4 \int_0^\infty t * (exp(-2t))dt = \\
3/4 + 1 + 3/5 - 1 = 1.35
\end{aligned} \tag{7}$$