

# Homework 11

## Exercise 1

### Gibbs Sampler from Example 7.2

From Example 7.2 in the book, we are given the conditional distribution

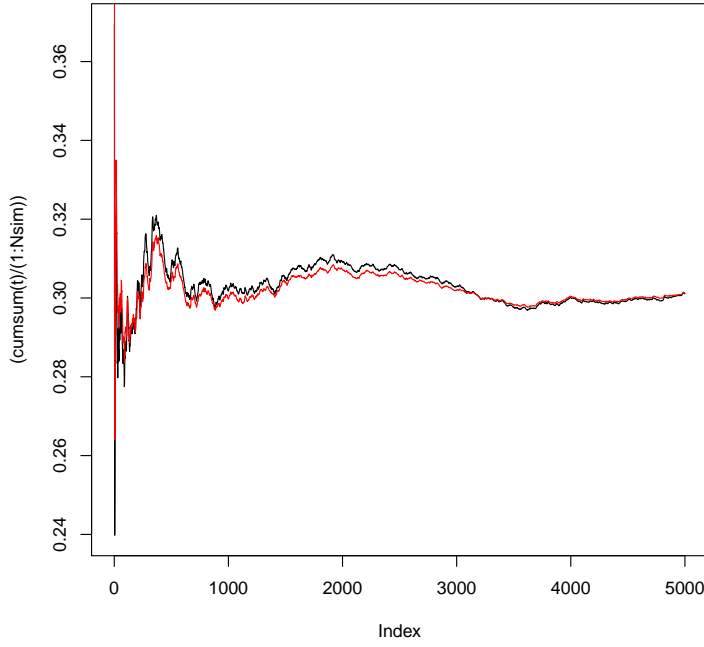
$$\theta|x \sim \text{Beta}(x+a, n-x+b) \quad (1)$$

By the properties of the beta distribution:

$$E[\theta|x] = \frac{x+a}{x+a+n-x+b} = \frac{x+a}{a+n+b} \quad (2)$$

```
## code from Example 7.2
## note: rb is the Rao-Blackwellization vectors I created to calculate
Nsim=5000
n=15
a=3
b=7
X=t=rb=array(0,dim=c(Nsim,1))
t[1]=rbeta(1,a,b)
X[1]=rbinom(1,n,t[1])
for (i in 2:Nsim){
  #rb[i-1] = 1/(i-1)*sum((X+a))/(a+n+b)
  X[i]=rbinom(1,n,t[i-1])
  t[i]=rbeta(1,a+X[i],n-X[i]+b)
}
rb = cumsum((X+a)/(a+n+b))/(1:Nsim)

plot((cumsum(t))/(1:Nsim),type="l")
lines(rb,col="red")
```



## Exercise 2

### Part A

please notes that:

$$f(y|p) = f(y|p, \theta) \quad (3)$$

By Bayes rule the following is true:

$$\begin{aligned} f(y|p, \theta) &\propto f(p, y, \theta) = f(\theta) * f(p|\theta) * f(y|p, \theta) = \\ &\frac{1}{a} * e^{-\frac{\theta}{a}} * \theta * p^{\theta-1} * \binom{n}{y} * p^y * (1-p)^{n-y} \propto \\ &p^{\theta-1} * p^y * (1-p)^{n-y} = \\ &p^{\theta+y-1} * (1-p)^{(n-y+1)-1} \propto \\ &\frac{1}{\beta(\theta+y, n-y+1)} * p^{\theta+y-1} * (1-p)^{(n-y+1)-1} \sim \text{beta}(\theta+y, n-y+1) \end{aligned} \quad (4)$$

### Part B

$$\begin{aligned} f(\theta|y, p) &\propto f(p, y, \theta) \propto e^{-\frac{\theta}{a}} * \theta * p^{\theta-1} \propto \\ &\theta^{2-1} * e^{-\frac{\theta}{a}} * p^{\theta} = \\ &\theta^{2-1} * e^{-\frac{\theta}{a}} * e^{\theta \log(p)} = \\ &\theta^{2-1} * e^{-\frac{\theta}{a}} * e^{(\theta) * \log(p)} = \\ &\theta^{2-1} * e^{-\frac{\theta}{a} + (\theta) * \log(p)} = \\ &\theta^{2-1} * e^{-\left(\frac{1}{a} - \log(p)\right) * \theta} = \\ &\theta^{2-1} * e^{-\frac{\theta}{\left(\frac{1}{a} - \log(p)\right)^{-1}}} \propto X \end{aligned} \quad (5)$$

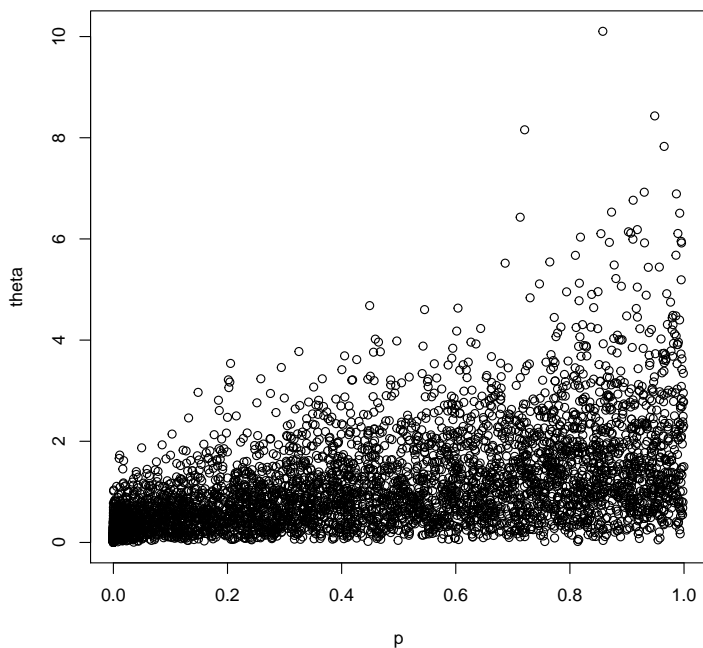
$$X \sim \text{Gamma}(2, (\frac{1}{a} + \log(p))^{-1}) \quad (6)$$

## Part C

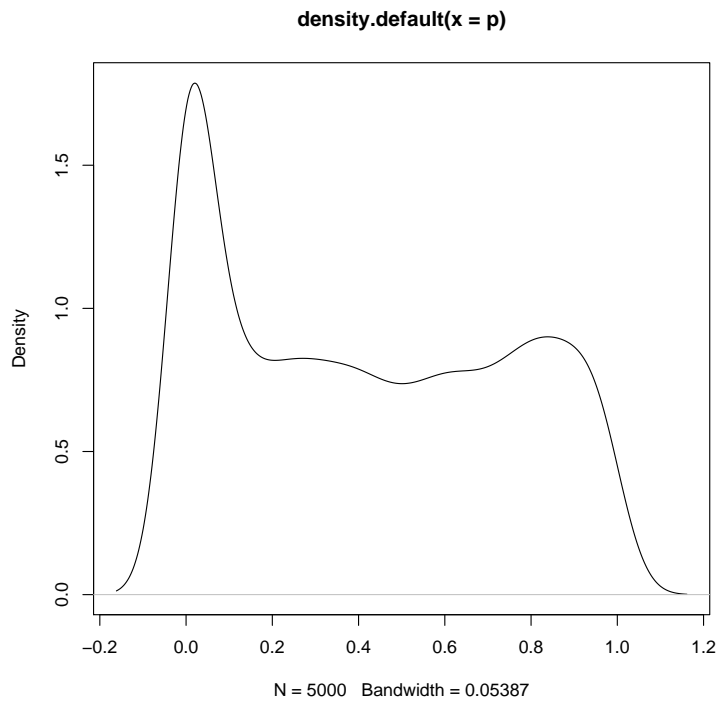
```

a = 1
p = 0.5
n = 10
theta = rgamma(1, a)
y = rbinom(1,n,p)
Nsim = 5e3
for (i in 2:Nsim){
  theta[i] = rgamma(1, 2, scale = (1/a-log(p[i-1]))^-1)
  y[i] = rbinom(1, n, p[i-1])
  p[i] = rbeta(1, y[i]+theta[i],n-y[i]+1)
}
plot(p, theta)

```



```
plot(density(p))
```



### Exercise 3

Note that  $\text{gamma}(1, \beta) = \text{exponential}(\beta)$

$$\begin{aligned}
 \int_0^y \beta * e^{-\beta * x} dx &= 0.5 = \\
 1 - e^{-\beta * y} &= 0.5 \implies \\
 -\exp(-\beta * y) &= -0.5 \implies \\
 \exp(-\beta * y) &= 0.5 \implies \\
 -\beta * y &= \ln(0.5) \implies \\
 -\beta * y &= -\ln(2) \implies \\
 y &= \frac{\ln(2)}{\beta}
 \end{aligned} \tag{7}$$

Using bcanon we obtain the following:

```

median_expo = function(beta){log(2)/beta}
library(bootstrap)
bootstrap_exponential = function(x, beta, N = 1e4, alpha_int = c(0.05, 0.95)){
  aa = bcanon(x, N, theta = median, alpha = alpha_int)
}
x = rexp(1e4, rate = 2)
bs_medians = bootstrap_exponential(x, 2)
bs_medians$confpoints[1,2]<median_expo(2)

## bca point
##      TRUE

```

```
bs_medians$confpoints[2,2]>median_expo(2)
```

```
## bca point  
## TRUE
```

The median of  $gamma(1, \beta)$  is 0.3465736, which is within the 90% BCa confidence Interval of (0.3339342, 0.350399). Instead of using a function from the the R package, we adapt the R function build in class.

```
boot.BCa <-  
function(x, th0, th, Stat, conf = .90) {  
  # BCa confidence interval  
  # th0: observed statistic  
  # th: vector of bootstrap distribution  
  # stat is the function to compute the statistic  
  
  x <- as.matrix(x)  
  n <- nrow(x) #observations in rows  
  N <- 1:n  
  alpha <- (1 + c(-conf, conf))/2  
  zalpha <- qnorm(alpha)  
  
  # the bias correction factor  
  z0 <- qnorm(sum(th < th0) / length(th))  
  
  # the acceleration factor (jackknife est.)  
  th.jack <- numeric(n)  
  for (i in 1:n) {  
    th.jack[i] <- Stat(x[-i, ]) #unlike the class example, we are calculating the median  
  }  
  L <- mean(th.jack) - th.jack  
  a <- sum(L^3)/(6 * sum(L^2)^1.5)  
  
  # BCa conf. limits  
  adj.alpha <- pnorm(z0 + (z0+zalpha)/(1-a*(z0+zalpha)))  
  limits <- quantile(th, adj.alpha, type=6)  
  list("est"=th0, "BCa"=limits)  
}  
x = rexp(1e4, rate = 2)  
boot.median = c()  
for(i in 1:1e4){  
  boot.median[i] = median(sample(x, size = 1e4, replace = T))  
}  
  
bca_int = boot.BCa(x = rexp(1e4, rate = 2), th0 = median(x),  
                   th = boot.median ,Stat = median)$BCa  
bca_int[1]<median_expo(2)  
  
## 5.057134%  
## TRUE  
  
bca_int[2]>median_expo(2)  
  
## 95.05662%  
## TRUE
```

The median of  $gamma(1, \beta)$  is 0.3465736, which is within the 90% BCa confidence Interval of (0.3372315, 0.3539623).  
If instead of BCa confidence intervals we did Percentile Confidence Intervals, that 90% Confidence interval would be (0.3370903, 0.3539155).