Arif Ali
Math 611 Stochastic Simulation
Sept 29, 2016

# Homework 4

## Exercise 1

```
h = function(x){
  exp(-x)/(1+x^2)
}
n = 1e4

H = integrate(h, 0, 1)
```

The analytical evaluation of $\int_0^1 \frac{e^{-x}}{(1+x^2)}\delta x$ is 0.5247971.

### Part A

$$F_X(x) = c\int_0^x 1\delta t = c\big|_0^x t = cx \tag{1}$$

$$1 = F_X(1) = c * 1 \implies c = 1 \therefore F_X(x) = x \tag{2}$$

$$\hat{I} = \int_0^1 \frac{exp(-x)}{1+x^2} * 1/1dx \approx \frac{1}{n}\sum_{i=1}^{n} \frac{e^{-x}}{1+x^2} \tag{3}$$

```
u = runif(n,0,1)
x = u # F^-1(x) = x
I_hat = h(x)
summary(I_hat)

##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.1840  0.3057  0.4894  0.5281  0.7355  1.0000

var(I_hat)

## [1] 0.06001488
```

### Part B

$$\begin{aligned} G_X(x) = c * \int_0^x \frac{e^{-x}}{1-e^{-1}}\delta x = \\ c * \frac{1}{1-e^{-1}}\int_0^x e^{-x}\delta x = \\ c * \frac{1}{1-e^{-1}} * (1-e^{-x}) \end{aligned} \tag{4}$$

$$1 = G_X(1) = c * \frac{1}{1-e^{-1}} * (1-e^{-1}) = c \implies c = 1 \therefore G(x) = \frac{1}{1-e^{-1}} * (1-e^{-x}) \tag{5}$$

To find $G^{-1}(x)$, we do the following:

$$\begin{aligned} y = \frac{1}{1-e^{-1}} * (1-e^{-x}) &\implies \\ y * (1-e^{-1}) = 1-e^{-x} &\implies \\ 1 - y * (1-e^{-1}) = e^{-x} &\implies \\ ln(1 - y * (1-e^{-1})) = -x &\implies \\ x = -ln(1 - y * (1-e^{-1})) \\ \therefore G^{-1}(x) = -ln(1 - x * (1-e^{-1})) \end{aligned} \tag{6}$$

$$\begin{aligned} \hat{I} = \int_0^1 \frac{exp(-x)}{1+x^2} * \frac{1}{(\frac{e^{-x}}{1-e^{-1}})} dx \approx \\ \frac{1}{n}\sum_{i=1}^{n} \frac{\frac{e^{-x}}{1+x^2}}{\frac{e^{-x}}{1-e^{-1}}} = \\ \frac{1-e^{-1}}{n}\sum_{i=1}^{n} \frac{1}{1+x^2} \end{aligned} \tag{7}$$

```
u = runif(n,0,1)
x = -log(1-u*(1-exp(-1))) #G^-1(U)

I_hat = (1-exp(-1))*(1/(1+x^2))
summary(I_hat)

##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.3161  0.4455  0.5513  0.5236  0.6134  0.6321

var(I_hat)

## [1] 0.009490023
```

# Exercise 2

If $\int_0^1 exp(x)\delta x$ is evaluated analytically, the result is 1.7182818.

For the antithetic variable appoarch let $h(x) = e^x$ and $f(x) = 1$

$$F(x) = \int_0^x 1\delta x = x \implies F^{-1}(x) = x \tag{8}$$

```
MCint = function(n =1e4, a, b, f){
  x = runif(n, a, b)
  y = f(x)
  return((b-a)*(y))
}
n = 1e3
U = runif(n, 0, 1)
X = U
```

```
Y = 1 - U
anti = 1/(2)*(exp(U)+exp(1-U))
MC = MCint(n,0,1, function(x){exp(x)})

var(anti) < var(MC)

## [1] TRUE
```

The percent reduction in variance is approximately 98.3752672%.

When 1000 estimates using the antithetic variable appoarch and MC integration

```
anti = c()
MC = c()
for(i in 1:1000){
  n = 1e3
  U = runif(n, 0, 1)
  X = U
  Y = 1 - U
  anti[i] = mean(1/(2)*(exp(U)+exp(1-U)))
  MC[i] = mean(MCint(n,0,1, function(x){exp(x)}))
 }
var(anti) < var(MC)

## [1] TRUE

(var(MC) - var(anti))/var(MC)*100

## [1] 98.61364
```

The percent reduction in variance is approximately 98.6136392%.

# Exercise 3

States 1 and 5 are transient because there are one or more ways in which their states can never return.

States 2, 3, and 4 are recurrent.

$\{2,4\}$ is a closed set that is irreducible.

# Exercise 4

$$\begin{aligned}\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix}\begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.4 & 0 & 0.6 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix} \implies \\ \begin{pmatrix} 0.7p_1 + 0.6p_2 \\ 0.4p_3 + 0.4p_4 \\ 0.3p_1 + 0.4p_2 \\ 0.5p_3 + 0.6p_4 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}\end{aligned} \tag{9}$$

From the matrix we can see $p_1 = 2p_2$ and $0.6p_1 + 0.4p_2 = p_3 \implies p_2 = p_3$ and $0.5p_3 + 0.6p_4 = p_4 \implies p_4 = 1.25p_3$

Since $p_1 + p_2 + p_3 + p_4 = 1 = 2p_3 + p_3 + p_3 + 1.25p_3 = 1 \implies p_3 = \frac{1}{5.25}$

$$\begin{aligned} p_1 &= 2/5.25 \\ p_2 &= 1/5.25 \\ p_4 &= 1.25/5.25 \end{aligned} \tag{10}$$

The stationary distribution is $\begin{pmatrix} \frac{2}{5.25} & \frac{1}{5.25} & \frac{1}{5.25} & \frac{1.25}{5.25} \end{pmatrix}$.

## Exercise 5

We know that $X_n \sim poisson(t)$ and $X_{n+1} = \psi_{n+1} + R(X_n)$ where $\psi_{n+1} \sim poisson(\lambda)$ and $R(X_n) \sim poisson(pt)$. The addition of two poisson variables combines the lambda values so $X_{n+1} \sim poisson(\lambda + pt)$. In order for the stationary distribution to work $X_n = X_{n+1}$

$$\begin{aligned} t = \lambda + pt \implies \\ t - pt = \lambda \implies \\ t(1-p) = \lambda \implies \\ t = \frac{\lambda}{1-p} \end{aligned} \tag{11}$$

$\therefore$ the stationary distribution is $poisson(\frac{\lambda}{1-p})$