

## Homework 8

### Exercise 1

Since  $n$  is odd, the median value of  $i \in 1 \dots n$  must be  $\frac{n+1}{2}$ . Using order statistics

$$\begin{aligned} f_{(\frac{n+1}{2})}(x) &= n * f(x) * \left( \frac{n-1}{\frac{n+1}{2}-1} \right) (F(x))^{\frac{n+1}{2}-1} (1-F(x))^{n-\frac{n+1}{2}} \propto \\ &= f(x) * (F(x))^{\frac{n+1}{2}-1} (1-F(x))^{n-\frac{n+1}{2}} = \\ &= f(x) * (F(x))^{\frac{n-1}{2}} (1-F(x))^{\frac{n-1}{2}} \end{aligned} \quad (1)$$

Assuming that  $y$  is exponential distributed with  $\lambda = 1$

$$\begin{aligned} f(y) &= e^{-y} * (1 - e^{-y})^{\frac{n-1}{2}} (1 - (1 - e^{-y}))^{\frac{n-1}{2}} = \\ &= e^{-y} * (1 - e^{-y})^{\frac{n-1}{2}} (e^{-y})^{\frac{n-1}{2}} = \\ &= e^{-y} * (1 - e^{-y})^{\frac{n-1}{2}} * e^{-y \frac{n-1}{2}} \end{aligned} \quad (2)$$

### Exercise 2

```
log_f = function(y,n=101){
  -y+(n-1)/2*log(1-exp(-y))+(-y*(n-1)/2)
}
```

### Exercise 3

```
x = 1
accept = 0
for(t in 2:1e4){
  y = rexp(1, rate = x[t-1])
  rho = exp(log_f(y))*dexp(x[t-1], rate = y)/
    (exp(log_f(x[t-1]))*dexp(y, rate = x[t-1]))
  if(runif(1)<rho){
    x[t] = y
    accept[t] = 1
  }
  else{
    x[t] =x[t-1]
    accept[t] = 0
  }
}
summary(x)

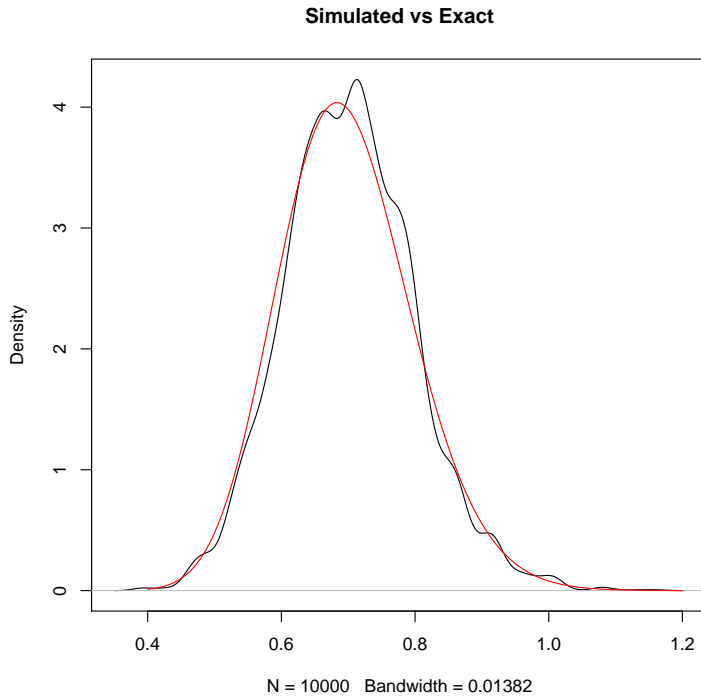
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
## 0.3916  0.6346  0.6989  0.7020  0.7667  1.1620
```

The acceptance rate is 0.1306.

Please note that in order to compare the simulated density with the exact, we have to calculate the normalizing constant

$$n \binom{\frac{n-1}{2}}{\frac{n+1}{2}-1} = n * \frac{(n-1)!}{(n-1 - ((n+1)/2 - 1)) * \frac{n+1}{2} - 1} = \frac{n!}{(\frac{n-1}{2})^2} \quad (3)$$

```
plot(density(x),
     main="Simulated vs Exact")
#Exact Samples
norm = factorial(101)/((factorial((101-1)/2))^2)
curve(norm*exp(log_f(x)),0.4,1.2,add=T,col="red")
```



## Exercise 4

Let  $R \sim \exp(1)$ ,  $M \sim \exp(\frac{1}{2})$ ,  $S \sim \exp(\frac{1}{3})$ . Also note that 25 minutes is  $\frac{5}{12}$  hours.

$$p(\min(R, M, S) > 25 * 1/60) = 1 - \exp(-(1^{-1} + (1/2)^{-1} + (1/3)^{-1}) * 5/12) = 1 - 0.082085 \quad (4)$$

## Exercise 5

Let  $p = \frac{1}{50}$ ,  $n = 20$   $\text{binom}(20, 1/50)$ , so  $\frac{\lambda}{n} = \frac{\lambda}{20} = 1/50 \implies \lambda = 2/5$  and  $\therefore \text{binom}(20, 1/50) \approx \text{Poisson}(2/5)$

$$p(k \geq 1) = 1 - p(k = 0) = 0.32968 \quad (5)$$

## Exercise 6

$T_1$  be the expected value of the amount of time that the first navigation lasts. There are three possible scenarios when that can happen, {1 or 2 dies before  $t$  and 3 dies at  $t$ }, {1 or 3 dies before  $t$  and 2 dies at  $t$ }, {2 or 3 dies at  $t$  and 1 dies at  $t$ }

$$\begin{aligned}
 f(time) &= (p(T_1 > t)p(T_2 \leq t) + p(T_1 \leq t)p(T_2 > t)) * p(T_3 = t) + \\
 &\quad (p(T_1 > t)p(T_3 \leq t) + p(T_1 \leq t)p(T_3 > t)) * p(T_2 = t) + \\
 &\quad (p(T_2 > t)p(T_3 \leq t) + p(T_2 \leq t)p(T_3 > t)) * p(T_1 = t) = \\
 &\quad (exp(-t) * (1 - exp(-2/3t)) + (1 - exp(-t)) * (exp(-2/3t))) * 1/3 * exp(-1/3t) + \\
 &\quad (exp(-t) * (1 - exp(-1/3t)) + (1 - exp(-t)) * (exp(-1/3t))) * 2/3 * exp(-2/3t) + \\
 &\quad (exp(-2/3t) * (1 - exp(-1/3t)) + (1 - exp(-2/3t)) * (exp(-1/3t))) * 1 * exp(-t) = \\
 &\quad 4/3exp(-4/3t) - 4/3exp(-2t) + exp(-t) - exp(-2t) + 5/3exp(-5/3t) - 5/3exp(-2t) = \\
 &\quad 4/3exp(-4/3t) + exp(-t) + 5/3exp(-5/3t) - 4exp(-2t)
 \end{aligned} \tag{6}$$

$$\begin{aligned}
 E(t) &= \int_0^\infty t * (4/3exp(-4/3t) + exp(-t) + 5/3exp(-5/3t) - 4exp(-2t))dt = \\
 &\quad 4/3 \int_0^\infty t * (exp(-4/3t))dt + \int_0^\infty t * (exp(-t))dt + \\
 &\quad 5/3 \int_0^\infty t * (exp(-5/3t))dt - 4 \int_0^\infty t * (exp(-2t))dt = \\
 &\quad 3/4 + 1 + 3/5 - 1 = 1.35
 \end{aligned} \tag{7}$$