CS215: Data Analysis and Interpretation
Assignment 3

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Contents

1 Question 1

(a)

Since no book has been picked yet, any book taken will be a new book. Hence $X_1 = 1$. When $(i-1)$ distinct books have picked, the next book to be picked should be among the $(n-i+1)$ books . Since there are n books to be picked from, the probability is $\frac{n-i+1}{n}$

(b)

The head probability of the Bernoulli trial corresponding to the geometric trial is the probability whether we select a distinct book other than the $(i-1)$ books selected. This probability as calculated above is $\frac{n-i+1}{n}$. Hence the parameter $p = \frac{n-i+1}{n}$

(c)

Suppose $p - 1 = q$

$$\begin{aligned}
\mathbb{E}[Z] &= \sum_{i=1}^{\infty} iP(Z=i) \\
&= p + 2pq + 3pq^2 + 4pq^3 + \ldots && (1) \\
q\mathbb{E}[Z] &= \quad pq + 2pq^2 + 3pq^3 + \ldots && (2) \\
\text{Subtracting (2) from (1), we get} & \\
(1-q)\mathbb{E}[Z] &= p + pq + pq^2 + pq^3 + \ldots \\
&= \frac{p}{1-q} \quad \text{Sum of GP and } q < 1 \\
\implies \mathbb{E}[Z] &= \frac{p}{(1-q)^2} = \frac{1}{p} && (3)
\end{aligned}$$

Now, for finding the variance,

$$\begin{aligned}
\mathbb{E}\left[Z^2\right] &= \sum_{i=1}^{\infty} i^2 P(Z=i) \\
&= p + 4pq + 9pq^2 + 16pq^3 + 25pq^4 \ldots && (4) \\
q\mathbb{E}\left[Z^2\right] &= \quad pq + 4pq^2 + 9pq^3 + 16pq^4 \ldots && (5) \\
\text{Subtracting (5) from (4), we get} & \\
(1-q)\mathbb{E}\left[Z^2\right] &= p + 3pq + 5pq^2 + 7pq^3 + \ldots && (6) \\
q(1-q)\mathbb{E}\left[Z^2\right] &= \quad pq + 3pq^2 + 5pq^3 + \ldots && (7) \\
\text{Subtracting (7) from (6), we get} & \\
(1-q)^2\mathbb{E}\left[Z^2\right] &= p + 2pq + 2pq^2 + 2pq^3 + \ldots \\
&= \frac{2p}{1-q} - p \quad \text{Sum of GP and } q < 1 \\
\implies \mathbb{E}\left[Z^2\right] &= \frac{2p}{(1-q)^3} - \frac{p}{(1-q)^2} = \frac{2}{p^2} - \frac{1}{p} \\
Var(Z) &= \mathbb{E}\left[Z^2\right] - \mathbb{E}[Z]^2 \\
&= \frac{1-p}{p^2} && (8)
\end{aligned}$$

(d)

From part (b) and 3 we can conclude ,

$$\mathbb{E}\left[X^{(n)}\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}\left[X_i\right] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n\sum_{i=1}^{n} \frac{1}{i}$$

(e)

For finding the variance we first need $Var(X_i)$. Again lets call $1 - p = q$

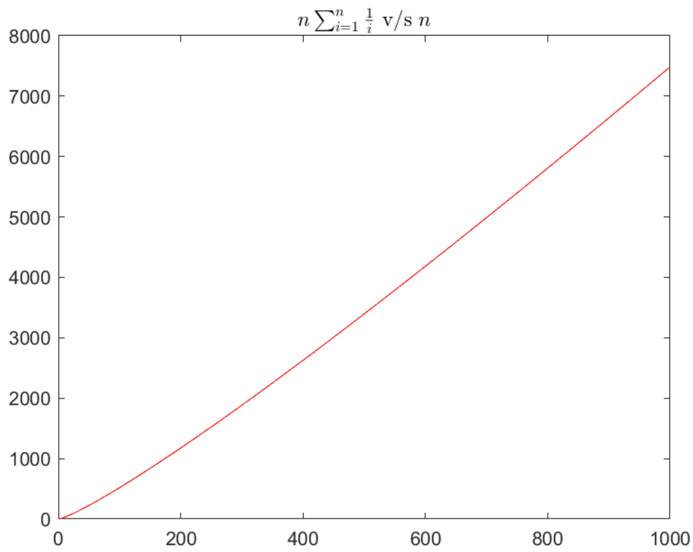
$$\begin{aligned}
Var(X^{(n)}) &= Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) \quad X_i' sare\text{ independent} \\
&= \sum_{i=1}^{n} \mathbb{E}\left[X_i^2\right] - \mathbb{E}\left[X_i\right]^2 \\
&= \sum_{i=1}^{n} \frac{1}{p_i^2} - \frac{1}{p_i} \quad \text{From 8} \\
&= \sum_{i=1}^{n} \left(\frac{n}{n-i+1}\right)^2 - \left(\frac{n}{n-i+1}\right) \\
&= \sum_{i=1}^{n} \left(\frac{n}{i}\right)^2 - \left(\frac{n}{i}\right) \\
&< \sum_{i=1}^{n} \left(\frac{n}{i}\right)^2 < \frac{n^2\pi^2}{6}
\end{aligned} \tag{9}$$

(f)

We know for a fact that , $\exists c > 0, N_0 \in \mathbb{N}$, such that

$$n\sum_{i=1}^{n} \frac{1}{i} < n(1 + log(n)) < c \cdot nlog(n) \quad \forall n > N_0$$

Hence we can say $\mathbb{E}\left[X^{(n)}\right] \in \Theta(nlog(n))$ or $f(n) = nlog(n)$



$\mathbb{E}\left[X^{(n)}\right]$ v/s n

2 Question 2

(a)

Suppose U is a uniform random variable.We know that for a uniform distribution, $P(U \leq x) = x$ Consider the random variable $V = F^{-1}(U)$. we are concerned with finding the CDF of this random variable V. (Since v_i are samples taken from V). This can be found by finding:

$$\begin{aligned} P(V \leq x) &= P(F^{-1}(U) \leq x) \\ &= P(U \leq F(x)) \quad \text{Because F is monotonically increasing, Apply F on both sided} \\ &= F(x) \end{aligned}$$

(b)

Claim: For the samples $Y_i \sim F$, for some distribution F, $F(Y_i) \sim U[0, 1]$, where $U[0, 1]$ represents a uniform distribution.

$$\begin{aligned} P(F(Y_i) \leq y) &= P(Y_i \leq F^{-1}(y)) \\ &= F(F^{-1}(y)) = y \qquad \text{From defintion of CDF of F} \end{aligned}$$

This is identical to the uniform distribution CDF which says $P(U_i \leq u) = u$. Hence, our claim is proven.

Now, for the question, Proving $P(D < d) = P(E < d)$ is equivalent, Now

$$P(D < d) = P(\max_x |F_e(x) - F(x)| < d) = P(\forall_x |F_e(x) - F(x)| < d)$$

. Now, let us substitute $y = F(x)$. As x varies over all possible values, y will vary from 0 to 1.

Also $x = F^{-1}(y)$.

$$\begin{aligned}
P(D < d) &= P(\forall_{y\in[0,1]} |F_e(F^{-1}(x)) - y| < d) \\
&= P(\forall_{y\in[0,1]} |\frac{1}{n}\sum \mathbf{1}(Y_i \leq F^{-1}(y)) - y| < d) \\
&= P(\forall_{y\in[0,1]} |\frac{1}{n}\sum \mathbf{1}(F(Y_i) \leq y) - y| < d) \\
&= P(\forall_{y\in[0,1]} |\frac{1}{n}\sum \mathbf{1}(U_i \leq y) - y| < d) \\
&= P(\max_{y\in[0,1]} E < d) = P(E < d)
\end{aligned}$$

Last step is due to the claim proven above.

This means that $P(D \geq d)$ is independent of the distribution F we are taking. Hence, this can be used to check whether a given set of samples Y_i belongs to a particular distribution F or not, as if it does belong to F, then the probability of D being higher than a particular value should be almost similar to probability that the max difference between the empirical CDF and the true uniform distribution (Uniform(0,1)) corresponding to the same number of samples is greater than that value. That is given a large set of values, the distribution of D tends to the distribution of E. Hence, we can use this property to check whether a distribution of samples belong to a particular distribution or not.

3 Question 3

(a)

First, observe that every coordinate z_i is a sample taken from the distribution $Z_i \sim \mathcal{N}(ax_i + by_i + c, \sigma^2)$, therefore, the likelihood is.

$$\begin{aligned}
f(z_1, z_2, z_3 \ldots z_n; \theta) &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{z_i - ax_i - by_i - c}{\sigma}\right)^2\right) \\
&= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n exp\left(\sum_{i=1}^{n} -\frac{1}{2}\left(\frac{z_i - ax_i - by_i - c}{\sigma}\right)^2\right)
\end{aligned}$$

Now ,

$$\mathcal{L} = log(f(z_1, z_2, z_3 \ldots z_n; \theta)) = -\sum_{i=1}^{n} \frac{1}{2}\left(\frac{z_i - ax_i - by_i - c}{\sigma}\right)^2 - nlog(\sigma\sqrt{2\pi})$$

We can get the required equations by setting :

$$\frac{\partial \mathcal{L}}{\partial a} = 0 \qquad \frac{\partial \mathcal{L}}{\partial b} = 0 \qquad \frac{\partial \mathcal{L}}{\partial c} = 0$$

We get the following 3 equations:

$$\sum x_i(ax_i + by_i + c) = \sum z_i x_i \tag{10}$$

$$\sum y_i(ax_i + by_i + c) = \sum z_i y_i \tag{11}$$

$$\sum (ax_i + by_i + c) = \sum z_i \tag{12}$$

In matrix form :

$$\begin{bmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i & \sum y_i & \sum 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum z_i x_i \\ \sum z_i y_i \\ \sum z_i \end{bmatrix}$$

(b)

Using similar logic as above we can get

$$\mathcal{L} = log(f(z_1, z_2, z_3 \dots z_n; \theta)) = -\sum_{i=1}^{n} \frac{1}{2} \left(\frac{z_i - a_1x_i^2 - a_2y_i^2 - a_3x_iy_i - a_4x_i - a_5y_i - a_6}{\sigma} \right)^2 - nlog(\sigma\sqrt{2\pi})$$

The required eqations can be obtained by setting :

$$\frac{\partial \mathcal{L}}{\partial a_i} = 0 \qquad i \in \{1, 2, 3, 4, 5, 6\}$$

We get the following equations:

$$\sum x_i^2(a_1x_i^2 + a_2y_i^2 + a_3x_iy_i + a_4x_i + a_5y_i + a_6) = \sum x_i^2 z_i \tag{13}$$
$$\sum y_i^2(a_1x_i^2 + a_2y_i^2 + a_3x_iy_i + a_4x_i + a_5y_i + a_6) = \sum y_i^2 z_i \tag{14}$$
$$\sum x_iy_i(a_1x_i^2 + a_2y_i^2 + a_3x_iy_i + a_4x_i + a_5y_i + a_6) = \sum x_iy_iz_i \tag{15}$$
$$\sum x_i(a_1x_i^2 + a_2y_i^2 + a_3x_iy_i + a_4x_i + a_5y_i + a_6) = \sum x_iz_i \tag{16}$$
$$\sum y_i(a_1x_i^2 + a_2y_i^2 + a_3x_iy_i + a_4x_i + a_5y_i + a_6) = \sum y_iz_i \tag{17}$$
$$\sum (a_1x_i^2 + a_2y_i^2 + a_3x_iy_i + a_4x_i + a_5y_i + a_6) = \sum z_i \tag{18}$$

In matrix form:

$$\begin{bmatrix} \sum x_i^4 & \sum x_i^2y_i^2 & \sum x_i^3y_i & \sum x_i^3 & \sum x_i^2y_i & \sum x_i^2 \\ \sum x_i^2y_i^2 & \sum y_i^4 & \sum x_iy_i^3 & \sum x_iy_i^2 & \sum y_i^3 & \sum y_i^2 \\ \sum x_i^3y_i & \sum x_iy_i^3 & \sum x_i^2y_i^2 & \sum x_i^2y_i & \sum x_iy_i^2 & \sum x_iy_i \\ \sum x_i^3 & \sum x_iy_i^2 & \sum x_i^2y_i & \sum x_i^2 & \sum x_iy_i & \sum x_i \\ \sum x_i^2y_i & \sum y_i^3 & \sum x_iy_i^2 & \sum x_iy_i & \sum y_i^2 & \sum y_i \\ \sum x_i^2 & \sum y_i^2 & \sum x_iy_i & \sum x_i & \sum y_i & \sum 1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} \sum x_i^2z_i \\ \sum y_i^2z_i \\ \sum x_iy_iz_i \\ \sum x_iz_i \\ \sum y_iz_i \\ \sum z_i \end{bmatrix}$$

(c)

On solving the equation on MATLAB, the predicted equation of the plane is :

$$z = 10.002208x + 19.998022y + 29.951579$$

To find the noise variance we first found out the predicted z values : $Z_{predicted}$, then subtracted the predicted values from the actual values to get the noise and then found its variance. **the noise variance we got was 23.068503**

4 Question 4

(a)

```
n = 1000;
data = normrnd(0,4,1,n);
indices = randperm(n, 750);
T = data(indices);
remaining_indices = setdiff(1:n, indices);
V = data(remaining_indices);
```

These lines of codes draw n = 1000 independent samples from $\mathcal{N}(0{,}16)$ and then distribute them such that 750 are in subset T and the rest 250 are in V and thus making sure they are disjoint.

(b)

The estimate for the PDF built from T with bandwidth parameter σ is:

$$\hat{p_n}(x;\sigma) = \sum_{j=1}^{750} \frac{e^{-(x-T_j)^2/2\sigma^2}}{750 \cdot \sigma\sqrt{2\pi}}$$

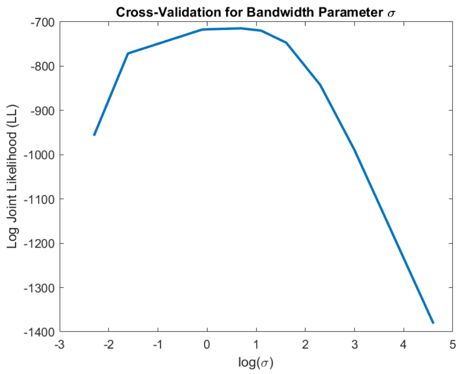
The expression for the joint likelihood of the samples in V, based on the estimate of the PDF built from T with bandwidth parameter σ is:

$$f(x_1, x_2, x_3 \ldots x_{250};\sigma) = \prod_{i=1}^{250}\sum_{j=1}^{750} \frac{e^{-(V_i-T_j)^2/2\sigma^2}}{750 \cdot \sigma\sqrt{2\pi}}$$

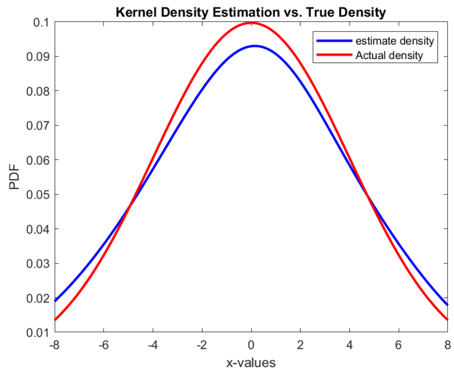
This follows as the samples in V are independent and identically distributed.

(c)

The plots for LL vs $log(\sigma)$ and the density estimation function vs the true pdf.



(a) Cross-Validation for σ

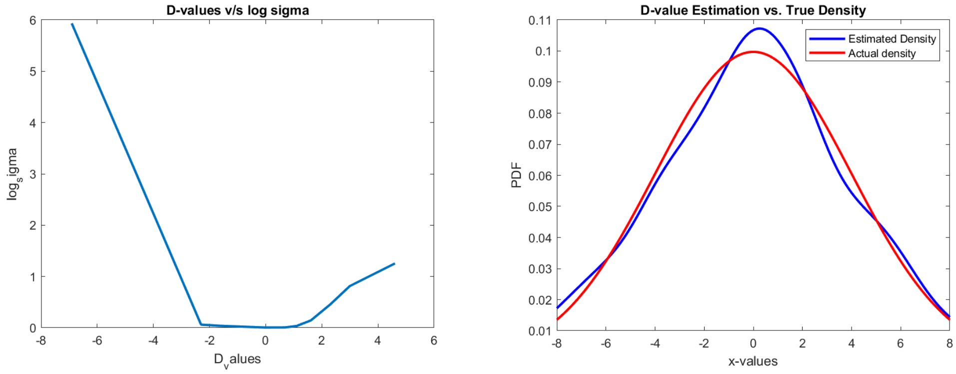


(b) Kernel Density Estimation vs. True Density

The best value of σ we got was **2.0** and this gave LL as **-714.574944**;

(d)

The plots for LL vs $log(\sigma)$ and the density estimation function vs the true pdf.



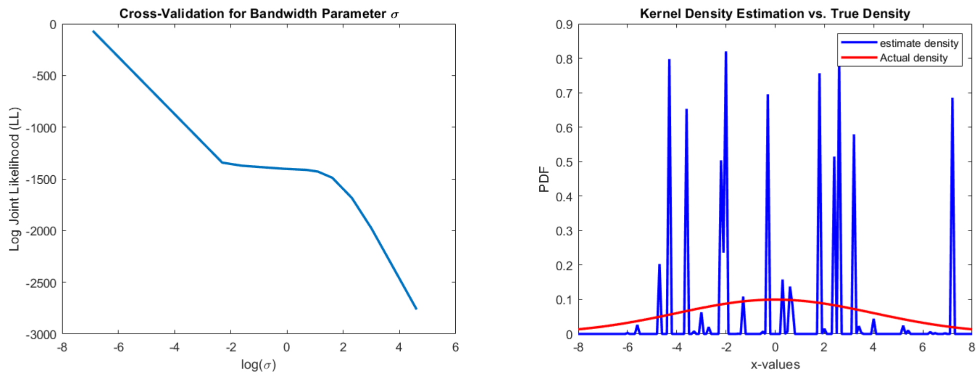
(a) D-values for σ

(b) Kernel Density Estimation vs. True Density

The best value of σ we got was **1.0** and this has the D-value as **0.005436**. The D-value we got for $\sigma = 2.0$ (the one for the best LL) is **0.007035**.

(e)

When we have T and V both equal (with both having 500 elements), we get the following graph for LL vs log σ.



(a) LL vs σ

(b) Kernel Density Estimation vs. True Density

This happens because in the term for the joint-likelihood of the samples in V, based on the estimate of the PDF built from T with bandwidth parameter σ, we will have one term as $\frac{e^0}{n\sigma\sqrt{2\pi}} = \frac{1}{\sigma\sqrt{2\pi}}$ (as T and V are equal, $T_j - V_i = 0$ for one term in the summation). Now when $\sigma \to 0$, $\frac{1}{\sigma\sqrt{2\pi}} \to \infty$ and $\frac{exp(-(T_i - V_j)^2/2\sigma^2)}{\sigma\sqrt{2\pi}} \to 0$. Thus the LL function gets huge for small σ and in the cross-validation step, we will always get the best σ as the smallest possible value we are taking, which might not give the best kernel estimation for the distribution, as shown in the (b) subfigure.

5 Question 5

$$P(S_n - E[S_n] > t) = P(e^{s(S_n - E[S_n])} > e^{st}) \text{for some} s > 0$$

Using Markov's Inequality we know that

$$P(e^{s(S_n - E[S_n])} > e^t) \leq \frac{E[e^{s(S_n - E[S_n])}]}{e^{st}}$$

Now consider the random variables $Y_i = e^{s(X_i - E[X_i])}$.
As the X_i are independent, we know that the Y_is are independent as well.

$$\begin{aligned} e^{s(S_n - E[S_n])} &= e^{s(\sum_{i=1}^n X_i - \sum_{i=1}^n E[X_i])} \\ &= e^{\sum_{i=1}^n s(X_i - E[X_i])} \\ &= \prod_{i=1}^n e^{s(X_i - E[X_i])} \\ &= \prod_{i=1}^n Y_i \end{aligned} \tag{19}$$

$$\begin{aligned} E[e^{s(S_n - E[S_n])}] = E[\prod_{i=1}^n Y_i] &= \prod_{i=1}^n E[Y_i] \\ &= \prod_{i=1}^n E[e^{s(X_i - E[X_i])}] \\ &\leq \prod_{i=1}^n e^{\frac{s^2(b_i^2 - a_i^2}{8}} \\ &= e^{\frac{s^2}{8}\sum_{i=1}^n (b_i - a_i)^2} \\ &= e^{\frac{s^2\lambda}{8}} \end{aligned}$$

where $\lambda = \sum_{i=1}^n (b_i - a_i)^2$

$$\begin{aligned} P(e^{s(S_n - E[S_n])} > e^t) &\leq \frac{E[e^{s(S_n - E[S_n])}]}{e^{st}} \\ &= \frac{e^{\frac{s^2\lambda}{8}}}{e^{st}} \\ &= e^{\frac{s^2\lambda}{8} - st} \end{aligned}$$

As we vary s, $\frac{s^2\lambda}{8} - st$ achieves a minima which can be regarded as the upper bound on $P(S_n - E[S_n] > t)$
Minima is achieved at

$$\frac{\partial}{\partial s}\left(\frac{s^2\lambda}{8} - st\right) = 0 \implies \frac{s\lambda}{4} = t \implies s = \frac{4t}{\lambda}$$

Therefore

$$P(S_n - E[S_n] > t) = P(e^{s(S_n - E[S_n])} > e^{st}) \leq e^{\left(\frac{4t}{\lambda}\right)^2 \frac{\lambda}{8} - \frac{4t}{\lambda}t} = e^{\frac{-2t^2}{\lambda}}$$

$$P(S_n - E[S_n] > t) \leq e^{\frac{-2t^2}{\sum_{i=1}^n (b_i - a_i)^2}}$$

if $t \geq 0$ If $t < 0$, then $s = \frac{4t}{\lambda}$ is not allowed because $s > 0$. In this case best bound is attained at s = 0 that $P(S_n - E[S_n] > t) \leq 1$

Now, for the proof of the intermediate result

1. Without loss of generality, we consider $E(X) = 0$, because X can be replaced by X - E(X) anyways. Hence, we consider $a \leq 0 \leq b$. The function e^{sx} is a convex function of x, and

hence a line segment joining two distinct points of the graph always lies above the graph of the function between the two points. Hence

$$e^{sx} \leq e^{sa} + \frac{e^{sb} - e^{sa}}{b-a}(x-a)$$

$$e^{sx} \leq \frac{(b-x)e^{sa}}{b-a} + \frac{(x-a)e^{sb}}{b-a}$$

2. Taking Expectation on both sides, and using $\mathbb{E}[X] = 0$

$$\begin{aligned}
\mathbb{E}\left[e^{sx}\right] &\leq \frac{e^{sa}}{b-a}\mathbb{E}[b-x] + \frac{e^{sb}}{b-a}\mathbb{E}[x-a] \\
\mathbb{E}\left[e^{sx}\right] &\leq \frac{be^{sa} - ae^{sb}}{b-a} = \frac{be^{sa} - ae^{sa} + ae^{sa} - ae^{sb}}{b-a} \\
&= e^{sa}\left(1 + \frac{a - ae^{s(b-a)}}{b-a}\right) \\
&= e^{\frac{s(b-a)a}{b-a}}\left(1 + \frac{a - ae^{s(b-a)}}{b-a}\right) \\
&= e^{L(s(b-a))}
\end{aligned}$$

where $L(h) = \frac{ha}{b-a} + \log\left(1 + \frac{a-ae^h}{b-a}\right)$

3.
$$L(h) = \frac{ha}{b-a} + \log\left(1 + \frac{a - ae^h}{b-a}\right)$$

$$\begin{aligned}
L'(h) &= \frac{a}{b-a} + \frac{1}{1 + \frac{a-ae^h}{b-a}}\frac{-ae^h}{b-a} \\
&= \frac{a}{b-a} - \frac{ae^h}{b-ae^h}
\end{aligned}$$

$$\begin{aligned}
L''(h) &= \frac{d}{dh}\left(\frac{-ae^h}{b-ae^h}\right) \\
&= \frac{-abe^h}{(b-ae^h)^2}
\end{aligned}$$

Now $\frac{(b-ae^h)^2}{4} \geq -abe^h$ if a and b are of different signs by AM-GM. If they are of the same sign, then its obviously true. Thus we have that

$$L''(h) \leq \frac{1}{4}$$

4. By Taylor's theorem we can write that

$$L(h) = L(0) + hL'(0) + \frac{h^2}{2}L''(c)$$

for all $h \geq 0$ and $0 \leq c \leq h$. Further as $L(0) = L'(0) = 0$

$$\implies L(h) \leq 0 + 0 + \frac{h^2}{2}\frac{1}{4}\ \forall h \geq 0$$

Thus we have that

$$\mathbb{E}\left[e^{sx}\right] \leq e^{L(s(b-a))} \leq e^{\frac{s^2(b-a)^2}{8}}$$