To the Quantum Future

Assignment 0

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Solution 1

Eventown Problem

Consider a vector space $V \subset \mathbb{F}_2^n$ over $\mathbb{F}_2$. For any vector $v_i \in V$, the j^{th} element is 1 if the j^{th} member is present in the i^{th} club and 0 otherwise. Consider the inner-product $\langle v_i, v_j \rangle$. This value represents the parity of the the number of people in the intersection of the i^{th} and j^{th} club. It is obvious that in $\mathbb{F}_2$ it is $0 \, \forall \, i, j$.So for any $v \in V, v \perp w \quad \forall w \in V$. In other words $\forall v \in V, v \in V^{\perp}$ or $V \subset V^{\perp}$. Also we know that $V \bigoplus V^{\perp} = \mathbb{F}_2^n$. Hence $2dimV \leq dimV + dimV^{\perp} = \dim \mathbb{F}_2^n = n$. So $dimV \leq n/2$. And since we defined V over $\mathbb{F}_2$, $m = |V| \leq 2^{n/2}$

Oddtown Problem

Define a matix $A \in \mathbb{F}_2^{n \times n}$ where $a_{ij} = 1$ if the i^{th} member belongs to the j^{th} club. The inner product of two columns of this matrix represent the parity of the number of members in the intersection of those two clubs which is 1 if both are same and 0 otherwise. Therefore it is easy to conclude that the matrix $A^t A$ is $I_{m \times m}$ or $dim(A^t A) = m$. We also know that $dim(AB) \leq min(dim(A), dim(B))$. Therefore $m = dim(A^t A) \leq min(dim(A), dim(A^t)) \leq n$

Solution 2

$$\begin{aligned} &\langle Ax, x \rangle = \langle x, Ax \rangle = \langle A^* x, x \rangle \\ \implies \quad &\langle (A - A^*)x, x \rangle = \langle Bx, x \rangle = 0 \quad \forall x \in V \end{aligned} \tag{1}$$

We need to prove that $B = \boldsymbol{O}$. We begin by replacing putting $(x+cy)$ where $x, y \in V$ instead of x in eqn(1).

$$\langle B(x + cy), (x + cy) \rangle = \langle (x + cy), B(x + cy) \rangle$$

Simplifying we get, $c\langle Bx, y \rangle = \bar{c}\langle Bx, y \rangle$. Put $c = 1, i$ to get the result that $\langle Bx, y \rangle = 0 \quad \forall x, y \in V$. Now put $y = Bx$ to get $||Bx|| = 0$. This implies $Bx = 0 \quad \forall x \in V$. Hence nullity of B is n or rank(B) = 0. Which means that $B = \boldsymbol{O}$.The converse is trivial.

$$\begin{aligned} &\langle (A - A^*)x, x \rangle = 0 \quad \forall x \in V \\ \implies \quad &\langle Ax, x \rangle = \langle A^* x, x \rangle = \langle x, Ax \rangle \end{aligned}$$

Note: This solution is inspired by one of the questions we discussed in MA106 course

Solution 3

Here we use the fact that : *Eigen values of Kronecker product of two matrices is the product of the eigen values of the indivisual matrices.*
Proof: Suppose v_a and v_b are the eigen values of A and B wrt eigen values λ and μ. Now, $(A \otimes B)(v_a \otimes v_b) = Av_a \otimes Bv_b = \lambda v_a \otimes \mu v_b = \lambda\mu(v_a \otimes v_b)$.

Moving on to the question it is easy to observe that the following matices are: (a):$A \otimes X$, (b) $X \otimes A$ and (c) $A \otimes A$, where $A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ and $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with eigen values $\frac{7 \pm \sqrt{57}}{2}$ and ± 1 respectively. Hence the eigen values for the question are $\pm(\frac{7 \pm \sqrt{57}}{2})$ for part (a) and (b) and $(53 \pm 7\sqrt{57})$ and -2 for part (c)

Solution 4

1)$\sqrt{\langle x, x \rangle}$ is a norm beacuse :

- $\sqrt{\langle x, x \rangle} = 0 \Rightarrow \langle x, x \rangle = 0 \Rightarrow \sum_i x_i^2 = 0 \Rightarrow x_i = 0 \quad \forall i$
- $\sqrt{\langle sx, sx \rangle} = \sqrt{s\bar{s}\langle x, x \rangle} = \sqrt{|s|^2 \langle x, x \rangle} = |s|\sqrt{\langle x, x \rangle}$
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$$\begin{aligned} |x+y|^2 &= \langle x+y, x+y \rangle \\ &= x^*x + y^*x + x^*y + y^*y \\ &\leq |x|^2 + 2|x||y| + |y|^2 \\ &= (|x|+|y|)^2 \end{aligned} \tag{2}$$

  eqn(2) is possible due to Cauchy-Schwartz theorem.

2) $|x-y|$ is mertic because:

- $|x-y| \geq 0$ and equality exists when $x-y=0$ or $x=y$ as proven above
- $|x-y| = \sqrt{\langle x-y, x-y \rangle} = \sqrt{\langle y-x, y-x \rangle} = |y-x|$
- $|x-z| = |(x-y)+(y-z)| \leq |x-y| + |y-z|$ as proven above in the third point.

$\langle f_n | f_m \rangle$ can be calculated directly using integration. WLOG, assume $n \geq m$:

$$\langle f_n | f_m \rangle = 1 + \frac{3n-m}{6n^2}$$

To prove it is cauchy, we can find $|f_n - f_m| = \sqrt{\langle f_n - f_m | f_n - f_m \rangle}$ which turns out to be $(1 - \frac{m}{n})\frac{1}{\sqrt{m}}$. It is obvious that for any n, we can set m as large as we want such that $|f_n - f_m| \leq \epsilon \quad \forall n, m \geq N$ for some N. Hence the sequence f_n is cauchy.

Solution 5

Here we use the fact that : $\langle \alpha v_1 \otimes w_1 | \beta v_2 \otimes w_2 \rangle = \alpha^* \beta \langle v_1 | v_2 \rangle \langle w_1 | w_2 \rangle$. Therefore:

$$\begin{aligned} \langle U | V \rangle_{HS} &= \langle TU | TV \rangle = \langle u | v \rangle \\ &= \langle \sum_{i,j} \alpha_{ij} v_i \otimes w_j | \sum_{i',j'} \beta_{i'j'} v_i' \otimes w_j' \rangle \\ &= \sum_{i,j,i',j'} \alpha_{ij}^* \beta_{i'j'} \langle v_i | v_i' \rangle \langle w_j | w_j' \rangle \end{aligned}$$

Now since $\{v_i\}$ and $\{w_j\}$ are orthonormal basis the above will be simplified to

$$\sum_{ij} \alpha_{ij}^* \beta_{ij}$$

Which is precisely equal to $\mathrm{tr}(U^*V)$
For the second part , note that the tensor product $v_i \otimes w_j$ can be considered as a Kronecker product which under the basis $\mathcal{B}$ reduces to

$$[a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{mn}]^T$$

where $a_{ij} = 1$ and 0 otherwise.

$$[0, 0, \ldots, 1, \ldots, 0]^T$$

Hence now it is easy to see that $(TU)_{\mathcal{B}} = (u)_{\mathcal{B}}$ reduces to the given form

$$[\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}, \alpha_{21}, \ldots, \alpha_{2n}, \ldots, \alpha_{mn}]^T$$