How many Binary Trees

$$c(n) = \sum_{k=0}^{n-1} c(k) \cdot c(n-k-1)$$

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \dots C_{n-2} C_1 + C_{n-1} C_0$$

$$(C_0 + C_1 x + \dots C_{n-1} x^{n-1})^2 = C_{n-1}^2 x^{2n-2} + \dots + (C_0 C_{n-1} + \dots + C_{n-1} C_0) x^{n-1} + \dots$$

Generating Functions

$\{a_n\} \Rightarrow$ let $F(x) = a_0 + a_1 x + \dots + a_n x^n + \dots$

(convergence - later)

$$F(x) = \sum_{k=0}^{\infty} C_k x^k \qquad (C_k = \#\text{ of B.T}_s\text{ of } k \text{ nodes})$$

$$F(x)^2 = \left(\sum_{k=0}^{\infty} C_k x^k\right) \cdot \left(\sum_{j=0}^{\infty} C_j x^j\right) = \sum_{n=1}^{\infty} \left(\sum_{j+k=n-1} C_j C_k\right) x^{n-1}$$

$$x F(x)^2 = \sum_{n=1}^{\infty} C_n x^n = F(x) - 1$$

$$x F(x)^2 - F(x) + 1 = 0$$

Solve for $F(x)$

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$F(0) = 1 = \frac{1 - \sqrt{1-4x}}{2x} \quad \text{as} \quad \lim_{x \to 0} \frac{1-\sqrt{1-4x}}{2x} \cdot \frac{1+\sqrt{1-4x}}{1+\sqrt{1-4x}} = \frac{4x}{2x(1+\sqrt{\ })} = 1$$

Need power series for $\sqrt{1-4x}$

$$(1+t)^p = \sum_{k=0}^{\infty} \binom{p}{k} t^k$$

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} = \frac{p(p-1)\dots(p-k+1)}{k!}$$

Define $\binom{z}{k} = \dfrac{z(z-1)\dots(z-k+1)}{k!}$

$$(1-4x)^{1/2} = \sum_{K=0}^{\infty} \binom{1/2}{K}(-4x)^K = \sum_{K=0}^{\infty} \binom{1/2}{K}(-4)^K x^K$$

$$= 1 + \sum_{K=1}^{\infty} \binom{1/2}{K}(-4)^K x^K$$

$$\sum_{K=0}^{\infty} C_K x^K = -\sum_{K=1}^{\infty} \binom{1/2}{K} \frac{(-4)^K}{2} x^{K-1} = -\sum_{K=0}^{\infty} -\binom{1/2}{K+1} \frac{(-4)^{K+1}}{2} x^K$$

$$\Rightarrow C_K = -\binom{\frac{1}{2}}{K+1} \frac{(-4)^{K+1}}{2} = -\binom{0.5}{K+1}\left(\frac{(-4)^{K+1}}{2}\right)$$