AMB Design Spreadsheet Equations

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1 General Motor Equations

1.1 Motor System Equations

$\tilde{x}$ denotes the property of a system of equivalent motors connected 1:1 in a gearbox

$$\tilde{\omega_f} = \omega_f \cdot \left(\frac{V}{V_{spec}}\right) \tag{1}$$

$$\tilde{\tau_s} = \tau_s \cdot \eta n \left(\frac{V}{V_{spec}}\right) \tag{2}$$

$$\tilde{i_f} = i_f \cdot n \left(\frac{V}{V_{spec}}\right) \tag{3}$$

$$\tilde{i_s} = i_s \cdot n \left(\frac{V}{V_{spec}}\right) \tag{4}$$

$$P = \frac{2\pi\omega_f \cdot \tau_s}{4} \tag{5}$$

$$\tilde{P} = \frac{2\pi\tilde{\omega_f} \cdot \tilde{\tau_s}}{4} = P \cdot \eta n \left(\frac{V}{V_{spec}}\right)^2 \tag{6}$$

$$K_T = \frac{\tilde{\tau_s}}{\tilde{i_s} - \tilde{i_f}} \tag{7}$$

ω_f = Motor Free Speed
$\tilde{\omega_f}$ = Adjusted Free Speed
τ_s = Motor Stall Torque
$\tilde{\tau_s}$ = Adjusted Stall Torque
i_f = Motor Free Current
$\tilde{i_f}$ = Adjusted Free Current
i_s = Motor Stall Current
$\tilde{i_s}$ = Adjusted Stall Current
P = Motor Power
$\tilde{P}$ = Adjusted Motor Power
n = # of Motors
η = Gearbox Efficiency
V = Applied Voltage
V_{spec} = Specification Voltage (Almost Always 12)
K_T = Motor Torque Constant

1.2 Instantaneous Motor Equations

$$\omega = \tilde{\omega_f} \cdot (1 - \frac{\tau}{\tilde{\tau_s}}) \tag{8}$$

$$i = (\tilde{i_s} - \tilde{i_f})\frac{\tau}{\tilde{\tau_s}} + \tilde{i_f} \tag{9}$$

$$\eta_{motor} = \frac{W_{out}}{W_{in}} = \frac{\tau \cdot \omega(\tau)}{V \cdot i(\tau)} = \frac{\tau(\tilde{\tau_s} - \tau)\tilde{\omega_f}}{V(\tilde{i_s}\tau + \tilde{i_f}(\tilde{\tau_s} - \tau))} \tag{10}$$

ω = Instantaneous Motor Speed
τ = Instantaneous Motor Torque
i = Instantaneous Motor Current
η_{motor} = Instantaneous Motor Efficiency

2 Mechanism Gear Ratio Calculator

2.1 General Equations

$$\omega_{free} = \frac{\tilde{\omega_f}}{G} \tag{11}$$

$$F_s = \frac{\tilde{\tau_s} G}{r} \tag{12}$$

$$\omega_{load} = \omega_{free} \cdot (1 - \frac{F}{F_s}) \tag{13}$$

$$v_{free} = \omega_{free} \cdot 2\pi r \tag{14}$$

$$v_{load} = \omega_{load} \cdot 2\pi r \tag{15}$$

$$i = \frac{rF}{K_T G n} + \frac{\tilde{i_f}}{n} \tag{16}$$

$$V_s = V_{spec} \frac{rF}{\tilde{\tau_s} \eta n G} \tag{17}$$

G = Gear Ratio
η = Gearbox Efficiency
n = # of Motors
F = Load Applied
r = Load Radius
ω_{free} = Output Rotational Free Speed
ω_{load} = Output Rotational Loaded Speed
v_{free} = Output Linear Free Speed
v_{load} = Output Linear Loaded Speed
i = Current Per Motor
F_s = Stall Load
V_s = Stall Voltage

2.2 Gear Ratio Calculations

2.2.1 Maximum Power

$$F_s = 2F \tag{18}$$

Substituting into (12) and solving for G gives:

$$G = \frac{2rF}{\tilde{\tau_s}} \tag{19}$$

2.2.2 Maximum Efficiency

$$G = \frac{rF}{\tilde{\tau_s}} \left(1 + \sqrt{\frac{\tilde{i_s}}{\tilde{i_f}}}\right) \tag{20}$$

For derivation, see Appendix A

2.2.3 At Stall

$$F_s = F \tag{21}$$

Substituting into (12) and solving for G gives:

$$G = \frac{rF}{\tilde{\tau_s}} \tag{22}$$

2.2.4 By Rotational Speed (ω_{load})

We can substitute (8) and (9) into (10):

$$\omega_{load} = \frac{\tilde{\omega_f}}{G} \cdot (1 - \frac{F}{\frac{\tilde{\tau_s} G}{r}}) \tag{23}$$

Solving for G, we get:

$$G = \frac{\tilde{\omega_f}}{2\omega_{load}} \left(1 + \sqrt{1 - 4rF\frac{\omega_{load}}{\tilde{\tau_s}\tilde{\omega_f}}}\right) \tag{24}$$

2.2.5 By Linear Speed (v_{load})

We can solve (15) for ω_{load}:

$$\omega_{load} = \frac{v_{load}}{2\pi r} \tag{25}$$

Now knowing ω_{load} we can use (24) to calculate G

2.2.6 By Per-Motor Current (i)

Solving (16) for G, we get:

$$G = \frac{rF}{K_T(ni - \tilde{i_f})} \tag{26}$$

2.2.7 By Stall Load (F_s)

We can solve (12) for G:

$$G = \frac{rF_s}{\tilde{\tau_s}} \tag{27}$$

2.2.8 By Stall Voltage (V_s)

Solving (17) for G, we get:

$$G = 12\frac{rF}{\tau_s \eta n V_s} \tag{28}$$

3 Gear Options Calculator

This calculator uses a brute-force search algorithm to find combinations of gears that meet the set criteria. All gears in the list must have $dp = 20$. The formulas used to check the criteria are:

$$\text{OD}_x = \frac{n_x + 2}{20} \text{ [in]} \tag{29}$$

$$\text{C-C}_{xy} = \frac{n_x + n_y}{40} \text{ [in]} \tag{30}$$

$$\text{clearance}_{xyz} = \text{C-C}_{xy} - \tfrac{1}{2}\text{OD}_z \tag{31}$$

n_x = Number of teeth of gear $x \in [1, 2, 3, 4]$
G = Total gear ratio
OD_x = Outer diameter of gear x
C-C_{xy} = Center-to-center distance between gears x and y
clearance_{xyz} = Clearance distance of gear x shaft

4 Chain/Belt C-C Calculator

This method of solving for D is inspired by and based on the formulas used in Clem1640's Chain/Belt Calculator. The language in this section will only reference chain, sprockets, and chain links; but all equations also apply to belts, pulleys, and belt teeth, respectively.

First we start with basic equations for chain and sprockets:

$$2\pi r_i = n_i p \tag{32}$$

$$L = lp \tag{33}$$

$$C = D - (r_1 + r_2) \tag{34}$$

We can construct an equation to represent the path length of the chain relative to the center-to-center distance. A derivation of this equation can be found in Appendix B.

$$L = 2\sqrt{D^2 - (r_1 - r_2)^2} + 2(r_1 - r_2)\cos^{-1}\left(\frac{r_1 - r_2}{D}\right) + 2\pi r_2 \tag{35}$$

Unfortunately we cannot solve this equation analytically to get D, even with the help of a computer. So we solve the function numerically using the Newton-Raphson Method. For the forward calculation our initial guess for D is $\frac{L}{2}$, which would be the case if both sprockets had diameters of 0. For the reverse calculation, we use the estimated value of D as the initial guess. The Newton-Raphson Method then improves the accuracy of D to the correct solution until the error is less than 10^{-12}.

L = Chain Length
l = # of Links
D = Center to Center Distance
n_i = # of Teeth on Sprocket $i \in [1, 2]$
r_i = Radius of Sprocket $i \in [1, 2]$
p = Chain Pitch
C = Clearance

5 Drivetrain Calculator

5.1 Forward Calculation

Equations taken from JVN's Mechanical Design Calculator.

$$v_{free} = \frac{\tilde{\omega_f} \cdot \pi d}{G} \tag{36}$$

$$v_{adj} = v_{free} \cdot k_{SL} \tag{37}$$

$$i_{slip} = \frac{\tilde{W} \mu d}{2K_T \cdot nG} + \frac{\tilde{i_f}}{n} \tag{38}$$

5.2 Reverse Calculation

$$i_{tot} = \frac{V_{batt} - V_{min}}{R_{batt} + R_{main} + \frac{1}{n} R_{br}} \tag{39}$$

$$G_{slip} = \frac{\mu d \tilde{W}}{2K_T \left(i_{tot} - \tilde{i_f}\right)} \tag{40}$$

Formulas for v_{free} and v_{adj} are the same as for the Forward Calculation – (36) and (37) respectively

v_{free} = Free Speed
v_{adj} = Adjusted Speed
i_{slip} = Wheel Slip Current
i_{tot} = Maximum Total Current Draw
G_{slip} = Maximum Gear Ratio to Slip Wheels above V_{min}
K_T = Motor Torque Constant
ω_f = Motor Free Speed
i_f = Motor Free Current
d = Wheel Diameter
μ = Wheel Coefficient of Friction
$\tilde{W}$ = Adjusted Robot Weight (i.e. weight resting on driven wheels)
G = Total Gear Ratio
k_{SL} = Speed Loss Constant
V_{batt} = Battery Resting Voltage
V_{min} = Minimum Allowable System Voltage
R_{batt} = Battery Internal Resistance
R_{main} = Resistance of Main Power Wiring
R_{br} = Resistance of Each Branch Circuit

6 Projectile Trajectory Calculator

6.1 Forward Calculation

The Forward Calculation uses equations taken from this paper by the University of Illinois. Three forces can be identified as acting on the ball: gravity, F_g, air drag, F_D, and lift, F_L. The acceleration from these forces is applied at each timestep until the ball reaches the target or hits the ground.

$$F_g = mg\ (-\hat{y}) \tag{41}$$

$$F_D = \frac{1}{2}\rho|\bar{v}|^2 A_c C_D\ (-\hat{v}) = \frac{\pi}{2}\rho r^2 v^2 C_D\ (-\hat{v}) \tag{42}$$

$$F_L = \frac{1}{2}\rho|\bar{v}|^2 A_c C_L\ (\hat{\omega} \times \hat{v}) = \frac{\pi}{2}\rho r^2 v^2 C_L\ (\hat{\omega} \times \hat{v}) \tag{43}$$

Combining these gives

$$\begin{aligned} \sum F_x = ma_x &= -F_L \sin\theta_v - F_D \cos\theta_v \\ &= -\frac{\pi}{2}\rho r^2 v^2 (C_L \sin\theta_v + C_D \cos\theta_v) \\ &= -\frac{\pi}{2}\rho r^2 v (C_L v_y + C_D v_x) \end{aligned} \tag{44}$$

$$\begin{aligned} \sum F_y = ma_y &= F_L \cos\theta_v - F_D \sin\theta_v - F_g \\ &= \frac{\pi}{2}\rho r^2 v^2 (C_L \cos\theta_v - C_D \sin\theta_v) - mg \\ &= \frac{\pi}{2}\rho r^2 v (C_L v_x - C_D v_y) - mg \end{aligned} \tag{45}$$

To find the drag coefficient we use an empirically determined graph correlating the drag coefficient with the Reynolds number, $Re = \rho v D \mu^{-1}$. Since this is a strongly non-linear correlation, we will use an estimated average velocity to calculate an approximate drag coefficient to be used throughout the trajectory.

To find the lift coefficient we use the spin factor, $S = r\omega v^{-1}$. The Illinois paper finds a strong correlation between S and C_L, with $C_L = 1.6S$ for $S < 0.1$ and $C_L = 0.6S + 0.1$ for $S > 0.1$. At every timestep, S and therefore C_L are recalculated based on the updated velocity v.

C_D = Drag Coefficient
C_L = Lift Coefficient
ρ = Air Density
r = Ball Radius
v = Ball Velocity
ω = Ball Rotational Velocity
m = Ball Mass
g = Gravity Constant
θ_v = Velocity Angle (from $+\hat{x}$)

These equations are good approximations of drag and lift for spherical rotating balls. For any object in flight you can use the following traditional trajectory equations to calculate the object's path with negligible air drag and lift. These will give the same outputs as the simulation equations above when C_D and ω are set to 0.

$$h_f = h_i + d\tan\theta_i - \frac{g \cdot d^2}{2v_i^2\cos^2\theta_i} \tag{46}$$

$$\theta_f = \tan^{-1}\left(2\frac{h_f - h_i}{d} - \tan\theta_i\right) \tag{47}$$

6.2 Reverse Calculation

$$\theta_i = \tan^{-1}\left(2\frac{h_f - h_i}{d} - \tan\theta_f\right) \tag{48}$$

$$v_i = \sec\theta_i \cdot \sqrt{\frac{g \cdot d}{|\tan\theta_i - \tan\theta_f|}} \tag{49}$$

h_i = Release (Initial) Height
h_f = Target (Final) Height
θ_i = Release (Initial) Angle
θ_f = Target (Final) Angle
v_i = Release (Initial) Velocity
d = Horizontal Distance to Target
g = Acceleration Due to Gravity

For derivation see Appendix D.

7 Pneumatics Calculator

The "simulation" is run at timesteps of $dt = 1$ second for a duration of 150 seconds.

The pressure at the current step of the simulation can be calculated by:

$$P_{n+1} = P_n + \frac{W_{comp} - \sum W_{cyl}}{V} \tag{50}$$

$$W_{comp} = \dot{V}_{comp}(P) \cdot dt \cdot 1\text{atm} \tag{51}$$

$$W_{cyl} = \left(\frac{\pi D^2}{4} P_{push} + \frac{\pi (D^2 - d^2)}{4} P_{pull} \right) L \cdot m \tag{52}$$

$\dot{V}_{comp}(P)$ is the compressor flow-rate as a function of the output pressure, taken from a fourth-degree polynomial interpolation of the data provided online. The volume of air is measured at atmospheric pressure, not at system pressure. m is the number of actuations per second (can be 0 when not firing, 1 when firing, or $m > 1$ if firing more than once per second).

Derivations of these equations can be found in Appendix E.

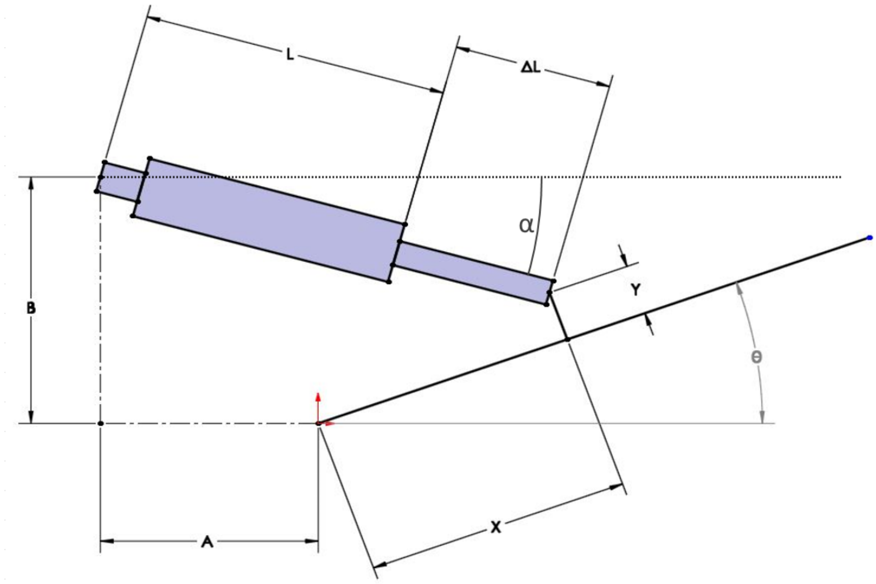
We can also calculate the pushing and pulling force of each cylinder using the formulas:

$$F_{push} = \frac{\pi D^2}{4} P_{push} \tag{53}$$

$$F_{pull} = \frac{\pi (D^2 - d^2)}{4} P_{pull} \tag{54}$$

P_n = System Pressure at Step n
V = System Volume
W_{comp} = Work Done by the Compressor
W_{cyl} = Work Done by Each Cylinder
D = Cylinder Diameter
d = Cylinder Rod Diameter
L = Cylinder Length
P_{push} = Pushing Pressure
P_{pull} = Pulling Pressure
F_{push} = Pushing Force
F_{pull} = Pulling Force

8 Pneumatic Linkage Calculator



Let θ_1 and α_1 represent the system when the cylinder is retracted, and θ_2 and α_2 represent it when extended.

In the horizontal direction we have:

$$A + X\cos\theta_1 - Y\sin\theta_1 = L\cos\alpha_1 \tag{55}$$

$$A + X\cos\theta_2 - Y\sin\theta_2 = (L + \Delta L)\cos\alpha_2 \tag{56}$$

And in the vertical direction we have:

$$X\sin\theta_1 + Y\cos\theta_1 = B - L\sin\alpha_1 \tag{57}$$

$$X\sin\theta_2 + Y\cos\theta_2 = B - (L + \Delta L)\sin\alpha_2 \tag{58}$$

We can combine (55) and (57) to remove α_1 and get:

$$L^2 = A^2 + B^2 + X^2 + Y^2 + 2(AX - BY)\cos\theta_1 - 2(BX + AY)\sin\theta_1 \tag{59}$$

And combine (56) and (58) to remove α_2 and get:

$$(L + \Delta L)^2 = A^2 + B^2 + X^2 + Y^2 + 2(AX - BY)\cos\theta_2 - 2(BX + AY)\sin\theta_2 \tag{60}$$

To derive the forward formulas, you can solve (59) and (60) separately to get expressions for θ_1 and θ_2. To derive the reverse formulas, you can solve (59) and (60) as a system of equations to get expressions for A and B. I will not include these expressions in this document, because they are extremely long.

9 Lead Screw Calculator

The basic transformation between rotational and linear motion is given by:

$$v = \omega \cdot np \tag{61}$$

Force & torque equations are taken from Shigley's Mechanical Engineering §8-2

$$d_m = d - \frac{p}{2} \tag{62}$$

$$T_R = \frac{Fd_m}{2}\left(\frac{\pi \mu d_p + np\cos(\alpha)}{\pi d_p \cos(\alpha) - \mu np}\right) \tag{63}$$

$$T_L = \frac{Fd_m}{2}\left(\frac{\pi \mu d_p - np\cos(\alpha)}{\pi d_p \cos(\alpha) + \mu np}\right) \tag{64}$$

Backdrive-able if $T_L < 0$.

$$\eta = \frac{T_R|_{\mu=0}}{T_R} = \frac{np}{\pi d_p} \cdot \frac{\pi d_p \cos(\alpha) - \mu np}{\pi \mu d_p + np\cos(\alpha)} \tag{65}$$

Equivalent radius/load formulas are designed to allow for lead screws as outputs in the Mechanism Gear Ratio Calculator. Derivations can be found in Appendix F.

$$r_{eq} = \frac{np}{2\pi} \tag{66}$$

$$L_{R,eq} = \frac{T_R}{r_{eq}} \tag{67}$$

$$L_{L,eq} = \frac{T_L}{r_{eq}} \tag{68}$$

d = Screw Diameter
p = Screw Pitch
n = # of Starts
α = Half Thread Angle (i.e. $\frac{\text{thread angle}}{2}$)
μ = Coefficient of Friction of Screw
F = Applied Force
v = Instantaneous Linear Speed
ω = Instantaneous Rotational Speed
d_m = Mean (Average) Diameter
T_R = Raise Torque
T_L = Lower Torque
η = Efficiency
r_{eq} = Equivalent Radius
$L_{R,eq}$ = Equivalent Raise Load
$L_{L,eq}$ = Equivalent Lower Load
v_{free} = Free Speed
v_{adj} = Adjusted Speed
i_{slip} = Wheel Slip Current
i_{tot} = Maximum Total Current Draw
G_{slip} = Maximum Gear Ratio to Slip Wheels above V_{min}

K_T = Motor Torque Constant
ω_f = Motor Free Speed
i_f = Motor Free Current
d = Wheel Diameter
μ = Wheel Coefficient of Friction
$\tilde{W}$ = Adjusted Robot Weight (i.e. weight resting on driven wheels)
G = Total Gear Ratio
k_{SL} = Speed Loss Constant

$$v_{shift} = \frac{v_1 v_2}{v_1 + v_2} \tag{69}$$

v_{shift} = Optimal Shifting Speed
v_2 = Gear 1 Free Speed
v_2 = Gear 2 Free Speed

For derivation see Appendix C.

9.1 Reverse Calculation

$$i_{tot} = \frac{V_{batt} - V_{min}}{R_{batt} + R_{main} + \frac{1}{n} R_{br}} \tag{70}$$

$$G_{slip} = \frac{\mu d \tilde{W}}{2K_T \left(i_{tot} - \tilde{i_f}\right)} \tag{71}$$

Formulas for v_{free} and v_{adj} are the same as for the Forward Calculation – (36) and (37) respectively

V_{batt} = Battery Resting Voltage
V_{min} = Minimum Allowable System Voltage
R_{batt} = Battery Internal Resistance
R_{main} = Resistance of Main Power Wiring
R_{br} = Resistance of Each Branch Circuit

10 Beam Bend Calculator

10.1 Geometrical Deformation Resistance

Formulas for deformation resistance constants of different cross-sectional geometries are taken from StructX.

I = Area (Second) Moment of Inertia
J = Torsion Constant

10.1.1 Hex

$$I = 0.0601a^4 \tag{72}$$

$$J = 0.1154a^4 \tag{73}$$

a = Distance Between Flats

10.1.2 Round

$$I = \frac{\pi}{64} d^4 \tag{74}$$

$$J = \frac{\pi}{32} d^4 \tag{75}$$

d = Diameter

10.1.3 Round Tube

$$I = \frac{\pi}{8} d^3 h \tag{76}$$

$$J = \frac{\pi}{4} d^3 h \tag{77}$$

d = Diameter
h = Wall Thickness

10.1.4 Square

$$I = \frac{a^4}{12} \tag{78}$$

$$J = \frac{9}{64} a^4 \tag{79}$$

a = Side Length

10.1.5 Rectangular Tube

$$I = \frac{1}{3} x^2 y h \tag{80}$$

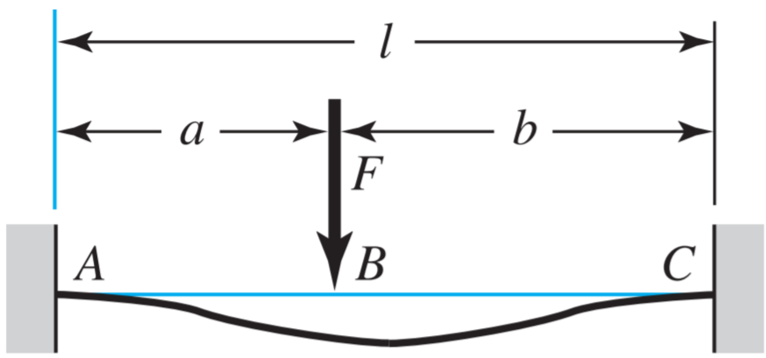
$$J = \frac{2h^2(x-h)^2(y-h)^2}{h(x+y-h)} \tag{81}$$

x = Length Parallel to Force
y = Length Perpendicular to Force
h = Wall Thickness

10.2 Displacement Equations

Displacement formulas are taken from Shigley's Mechanical Engineering Appendix A-9

10.2.1 Force Between Supports



$$
\begin{aligned}
y(x \le a) &= \frac{Fb^2x^2}{6EIl^3}\left(x(3a+b)-3al\right) \\
y(x \ge a) &= \frac{Fa^2(l-x)^2}{6EIl^3}\left((l-x)(3b+a)-3bl\right)
\end{aligned}
\tag{82}
$$

Substitute $b = l - a$, where $a \le \frac{l}{2} \le b$:

$$
\begin{aligned}
y(x \le a) &= \frac{Fx^2(l-a)^2}{6EIl^3}\left(x(l+2a)-3al\right) \\
y(x \ge a) &= \frac{Fa^2(l-x)^2}{6EIl^3}\left(a(l+2x)-3lx\right)
\end{aligned}
\tag{83}
$$

Take the derivative, set equal to zero, and solve for x:

$$
\begin{aligned}
x_{x\le a} &= 0,\ \frac{2al}{2a+l} \\
x_{x\ge a} &= \frac{l^2}{3l-2a},\ l
\end{aligned}
\tag{84}
$$

$x = 0$ and $x = l$ are the trivial solutions. We must test the non-trivial solutions to make sure they fulfil their domain requirement:

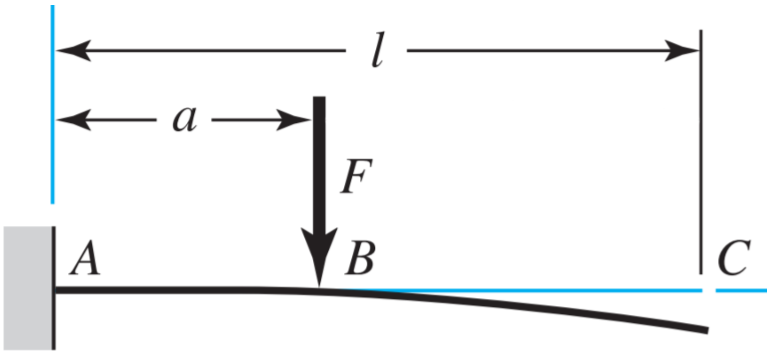
$$
\begin{aligned}
\frac{2al}{2a+l} \le a &\implies a \ge \frac{l}{2}\ , \\
\frac{l^2}{3l-2a} \ge a &\implies a \le \frac{l}{2} \cup l \le a < \frac{3}{2}l
\end{aligned}
\tag{85}
$$

The domain $a \ge \frac{l}{2}$ for the first solution contradicts our definition of $a < \frac{l}{2} < b$ therefore that solution is not valid. The maximum deflection occurs at $x = \frac{l^2}{3l-2a}$. Plugging this into (83) gives:

$$
y_{max} = \frac{2Fa^2(a-l)^3}{3EI(2a-3l)^2}
\tag{86}
$$

$y =$ Vertical Displacement
$F =$ Load Force
$a =$ Distance to Closer Support
$l =$ Distance Between Supports
$E =$ Modulus of Elasticity
$I =$ Area Moment of Inertia

10.2.2 Cantilevered Force



$$y_{max} = \frac{Fa^2}{6EI}(a - 3l) \tag{87}$$

y_{max} = Largest Displacement
F = Load Force
a = Distance to Support
l = Total Length
E = Modulus of Elasticity
I = Area Moment of Inertia

10.3 Buckling Force

This formula is taken from the Wikipedia page on Euler's Critical Load

$$F_{max} = \frac{\pi^2 EI}{(KL)^2} \tag{88}$$

Buckled shape of column shown by dashed line						
Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value K	0.65	0.80	1.2	1.0	2.10	2.0
End condition key	Rotation fixed and translation fixed Rotation free and translation fixed Rotation fixed and translation free Rotation free and translation free					

F_{max} = Maximum Force
E = Modulus of Elasticity
I = Area Moment of Inertia
K = End Condition Constant
L = Column Length

10.4 Twisting Torque

This formula is taken from Shigley's Mechanical Engineering §3-12

$$\theta_{max} = \frac{TL}{GJ} \tag{89}$$

θ_{max} = Largest Angular Displacement
T = Applied Torque
L = Distance Between Torque and Support
G = Shear Modulus
J = Torsion Constant

Derivations

A Maximum Efficiency Derivation

To find the maximum efficiency, we take the derivative of (10) and set it equal to 0:

$$0 = \frac{\partial \eta}{\partial \tau} = \frac{(\tilde{i_f}(\tau - \tilde{\tau_s})^2 - \tilde{i_s}\tau^2)\tilde{\omega_f}}{V(\tilde{i_s}\tau + \tilde{i_f}(\tilde{\tau_s} - \tau))^2} \tag{90}$$

Solving for the value of τ at which η is maximized:

$$\tau_{\eta_{max}} = \frac{\tilde{\tau_s}\sqrt{\tilde{i_f}}}{\sqrt{\tilde{i_f}} + \sqrt{\tilde{i_s}}} \tag{91}$$

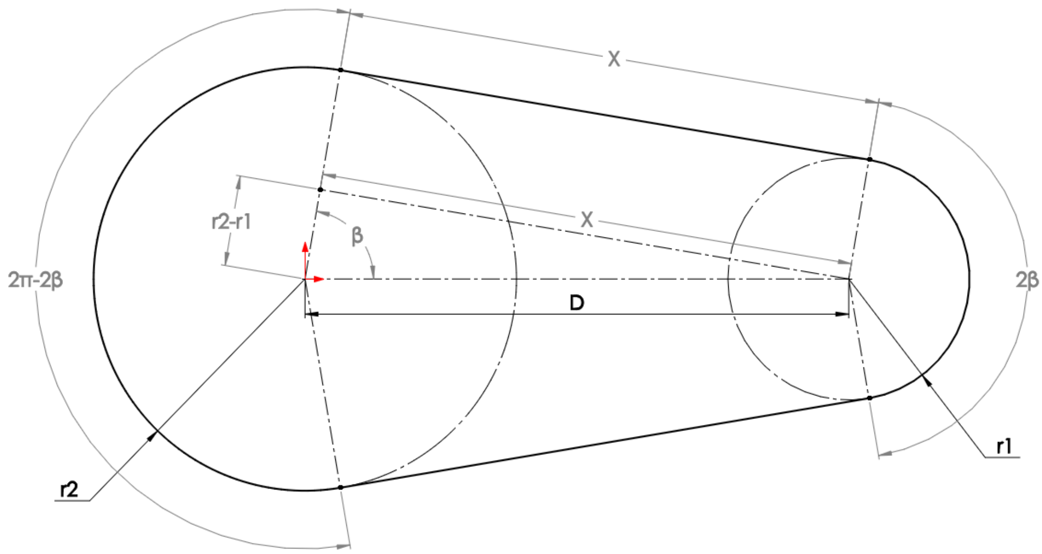
We know that $\tau_{manip} = G \cdot \tau_{motor}$, so we can substitute $\tau_{\eta_{max}}$ for τ_{motor}:

$$G = \frac{\tau_{manip}}{\tau_{motor}} = \frac{r \cdot F}{\tau_{\eta_{max}}} = \frac{r \cdot F}{\frac{\tilde{\tau_s}\sqrt{\tilde{i_f}}}{\sqrt{\tilde{i_f}} + \sqrt{\tilde{i_s}}}} \tag{92}$$

And simplifying gives:

$$G = \frac{rF}{\tilde{\tau_s}}\left(1 + \sqrt{\frac{\tilde{i_s}}{\tilde{i_f}}}\right) \tag{20}$$

B Chain Length Derivation



It is clear that:

$$L = 2X + r_1 \cdot 2\beta + r_2 \cdot (2\pi - 2\beta) \tag{93}$$

Looking at the right triangle in the center of the image, we can see that:

$$D^2 = X^2 + (r_2 - r_1)^2 \implies X = \sqrt{D^2 - (r_1 - r_2)^2} \tag{94}$$

$$\beta = \cos^{-1}\left(\frac{r_2 - r_1}{D}\right) = \cos^{-1}\left(\frac{r_1 - r_2}{D}\right) \tag{95}$$

Substituting (94) and (95) into (93) gives:

$$L = 2\sqrt{D^2 - (r_1 - r_2)^2} + r_1 \cdot 2\cos^{-1}\left(\frac{r_1 - r_2}{D}\right) + r_2 \cdot \left(2\pi - 2\cos^{-1}\left(\frac{r_1 - r_2}{D}\right)\right) \tag{96}$$

And rearranging gives:

$$L = 2\sqrt{D^2 - (r_1 - r_2)^2} + 2(r_1 - r_2)\cos^{-1}\left(\frac{r_1 - r_2}{D}\right) + 2\pi r_2 \tag{35}$$

C Optimal Shifting Speed Derivation

The optimal shifting speed is the speed where both high and low gears produce the same force at the wheel surface.

We will start with the definition of instantaneous motor speed from (8). This can be solved for the motor torque τ

$$\tau = \tau_s \left(1 - \frac{\omega}{\omega_f}\right) \tag{97}$$

Using the equations,

$$v_w = \frac{\omega r}{G} \tag{98}$$

$$F_w = \frac{\tau G}{r} \tag{99}$$

we can transform the rotational speed and torque at the motor into the linear speed and force at the wheel

$$F_w = \frac{\tau_s G}{r} \left(1 - \frac{v_w G}{r \omega_f}\right) \tag{100}$$

To find the optimal shifting speed, we will set the wheel forces at both gear ratios, G_1 and G_2, and both wheel radii, r_1 and r_2, equal at the shifting speed v_{shift}

$$\frac{\tau_s G_1}{r_1} \left(1 - \frac{v_{shift} \cdot G_1}{r_1 \omega_f}\right) = \frac{\tau_s G_2}{r_2} \left(1 - \frac{v_{shift} \cdot G_2}{r_2 \omega_f}\right) \tag{101}$$

This can then be simplified to

$$v_{shift} = \frac{r_1 r_2 \omega_f}{G_2 r_1 + G_1 r_2} \tag{102}$$

Plugging in $\omega = \omega_f$ into (98) gives the free speeds v_1 and v_2 for each gear ratio

$$\begin{aligned} v_1 &= \frac{\omega_f r_1}{G_1} \\ v_2 &= \frac{\omega_f r_2}{G_2} \end{aligned} \tag{103}$$

Then it is easy to see that

$$v_{shift} = \frac{1}{\frac{1}{v_1} + \frac{1}{v_2}} = \frac{v_1 v_2}{v_1 + v_2} = v_1 \parallel v_2 \tag{69}$$

D Projectile Equation Derivations

Starting with basic projectile motion equations:

$$\Delta x = v_i \cos\theta_i \cdot t \tag{104}$$

$$\Delta y = v_i \sin\theta_i \cdot t - \tfrac{1}{2} g t^2 \tag{105}$$

$$\theta_f = \tan^{-1}\left(\frac{v_{f,y}}{v_{f,x}}\right) = \tan^{-1}\left(\frac{v_i \sin\theta_i - gt}{v_i \cos\theta_i}\right) \tag{106}$$

Solving (104) for t and substituting into (105) gives:

$$\Delta y = \Delta x \tan\theta_i - \frac{g \cdot (\Delta x)^2}{2 v_i^2 \cos^2\theta_i} \tag{107}$$

We can then substitute in $\Delta x = d$ and $\Delta y = h_f - h_i$:

$$h_f = h_i + d \tan\theta_i - \frac{g \cdot d^2}{2 v_i^2 \cos^2\theta_i} \tag{46}$$

Solving (104) for t and substituting into (106) gives:

$$\theta_f = \tan^{-1}\left(\frac{v_i \sin\theta_i - g \cdot \frac{\Delta x}{v_i \cos\theta_i}}{v_i \cos\theta_i}\right) = \tan^{-1}\left(\tan\theta_i - \frac{g \cdot d}{v_i^2 \cos^2\theta_i}\right) \tag{108}$$

Further substituting in (46) into (108) gives:

$$\theta_f = \tan^{-1}\left(\tan\theta_i - \frac{2}{d}(d \tan\theta_i - (h_f - h_i))\right) \tag{109}$$

And by simplifying we get:

$$\theta_f = \tan^{-1}\left(2\frac{h_f - h_i}{d} - \tan\theta_i\right) \tag{47}$$

To calculate the reverse equations, we can solve (109) for θ_i:

$$\theta_i = \tan^{-1}\left(2\frac{h_f - h_i}{d} - \tan\theta_f\right) \tag{48}$$

Then substitute (46) into (48):

$$\tan\theta_i + \tan\theta_f = \frac{2}{d}\left(d \tan\theta_i - \frac{g \cdot d^2}{2 v_i^2 \cos^2\theta_i}\right) \tag{110}$$

And solve that for v_i:

$$v_i = \sec\theta_i \cdot \sqrt{\frac{g \cdot d}{|\tan\theta_i - \tan\theta_f|}} \tag{49}$$

E Pneumatics Simulation Derivation

We know that the energy of the system at any point can be represented by:

$$E = P \cdot V \tag{111}$$

We can find the energy of the system at any timestep by adding the total work done on the system in that time to the energy of the system at the previous step:

$$E_{n+1} = E_n + W_{tot} \tag{112}$$

Substituting (111) into (112) gives:

$$(PV)_{n+1} = (PV)_n + W_{comp} - \sum W_{cyl} \tag{113}$$

Since the volume of the system remains constant, we can divide by V to get:

$$P_{n+1} = P_n + \frac{W_{comp} - \sum W_{cyl}}{V} \tag{50}$$

Basic thermodynamics says that the work done by compression/expansion of gas at constant pressure can be represented by $W = P \cdot \Delta V$. In each actuation of a cylinder gas is compressed twice, once for extension and once for retraction. So for each timestep, the work done on the cylinder can be expressed as:

$$W_{cyl} = ((PAL)_{push} + (PAL)_{pull}) \cdot m \tag{114}$$

L is constant, so it can be factored out. We can also substitute A for the effective piston areas, giving:

$$W_{cyl} = \left(\frac{\pi D^2}{4} P_{push} + \frac{\pi(D^2 - d^2)}{4} P_{pull} \right) L \cdot m \tag{52}$$

Similarly, we can represent the work done by the compressor as:

$$W_{comp} = \dot{V}_{comp}(P) \cdot dt \cdot 1\text{atm} \tag{51}$$

Note: since $\dot{V}_{comp}$ is measured at atmospheric pressure, P must be 1 atm and not the system pressure

F Equivalent Radius/Load Derivation

To find the equivalent radius, we equate the rotational $\rightarrow$ linear motion transformation equations for both a wheel (15) and lead screw (61)

$$2\pi r \cdot \omega = v = \omega \cdot np \tag{115}$$

We can then solve for $r = r_{eq}$ to find the radius that will provide the same rotational $\rightarrow$ linear motion transformation as the given lead screw

$$r_{eq} = \frac{np}{2\pi} \tag{66}$$

We want to conserve the work done by a lead screw and equivalent wheel across one rotation (where x can be L or R).

$$W = 2\pi r_{eq} \cdot L_{x,eq} = 2\pi T_x \tag{116}$$

Solving for $L_{x,eq}$, we get:

$$L_{x,eq} = \frac{T_x}{r_{eq}} \tag{117}$$