

AMB Design Spreadsheet Equations

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Version 1.1

1 General Motor Equations

1.1 Motor System Equations

\tilde{x} denotes the property of a system of equivalent motors connected 1:1 in a gearbox

$$\tilde{\omega}_f = \omega_f \cdot \left(\frac{V}{V_{spec}} \right) \quad (1)$$

$$\tilde{\tau}_s = \tau_s \cdot \eta n \left(\frac{V}{V_{spec}} \right) \quad (2)$$

$$\tilde{i}_f = i_f \cdot n \left(\frac{V}{V_{spec}} \right) \quad (3)$$

$$\tilde{i}_s = i_s \cdot n \left(\frac{V}{V_{spec}} \right) \quad (4)$$

$$P = \frac{2\pi\omega_f \cdot \tau_s}{4} \quad (5)$$

$$\tilde{P} = \frac{2\pi\tilde{\omega}_f \cdot \tilde{\tau}_s}{4} = P \cdot \eta n \left(\frac{V}{V_{spec}} \right)^2 \quad (6)$$

$$K_T = \frac{\tilde{\tau}_s}{\tilde{i}_s - \tilde{i}_f} \quad (7)$$

ω_f = Motor Free Speed

$\tilde{\omega}_f$ = Adjusted Free Speed

τ_s = Motor Stall Torque

$\tilde{\tau}_s$ = Adjusted Stall Torque

i_f = Motor Free Current

\tilde{i}_f = Adjusted Free Current

i_s = Motor Stall Current

\tilde{i}_s = Adjusted Stall Current

P = Motor Power

\tilde{P} = Adjusted Motor Power

n = # of Motors

η = Gearbox Efficiency

V = Applied Voltage

V_{spec} = Specification Voltage (Almost Always 12)

K_T = Motor Torque Constant

1.2 Instantaneous Motor Equations

$$\omega = \tilde{\omega}_f \cdot \left(1 - \frac{\tau}{\tilde{\tau}_s} \right) \quad (8)$$

$$i = (\tilde{i}_s - \tilde{i}_f) \frac{\tau}{\tilde{\tau}_s} + \tilde{i}_f \quad (9)$$

$$\eta_{motor} = \frac{W_{out}}{W_{in}} = \frac{\tau \cdot \omega(\tau)}{V \cdot i(\tau)} = \frac{\tau(\tilde{\tau}_s - \tau)\tilde{\omega}_f}{V(\tilde{i}_s\tau + \tilde{i}_f(\tilde{\tau}_s - \tau))} \quad (10)$$

ω = Instantaneous Motor Speed

τ = Instantaneous Motor Torque

i = Instantaneous Motor Current

η_{motor} = Instantaneous Motor Efficiency

2 Mechanism Gear Ratio Calculator

2.1 General Equations

$$\omega_{free} = \frac{\tilde{\omega}_f}{G} \quad (11)$$

$$F_s = \frac{\tilde{\tau}_s G}{r} \quad (12)$$

$$\omega_{load} = \omega_{free} \cdot \left(1 - \frac{F}{F_s}\right) \quad (13)$$

$$v_{free} = \omega_{free} \cdot 2\pi r \quad (14)$$

$$v_{load} = \omega_{load} \cdot 2\pi r \quad (15)$$

$$i = \frac{rF}{K_T G n} + \frac{\tilde{i}_f}{n} \quad (16)$$

$$V_s = V_{spec} \frac{rF}{\tilde{\tau}_s \eta n G} \quad (17)$$

G = Gear Ratio

η = Gearbox Efficiency

n = # of Motors

F = Load Applied

r = Load Radius

ω_{free} = Output Rotational Free Speed

ω_{load} = Output Rotational Loaded Speed

v_{free} = Output Linear Free Speed

v_{load} = Output Linear Loaded Speed

i = Current Per Motor

F_s = Stall Load

V_s = Stall Voltage

2.2 Gear Ratio Calculations

2.2.1 Maximum Power

$$F_s = 2F \quad (18)$$

Substituting into (12) and solving for G gives:

$$G = \frac{2rF}{\tilde{\tau}_s} \quad (19)$$

2.2.2 Maximum Efficiency

$$G = \frac{rF}{\tilde{\tau}_s} \left(1 + \sqrt{\frac{\tilde{i}_s}{\tilde{i}_f}}\right) \quad (20)$$

For derivation, see Appendix [A](#)

2.2.3 At Stall

$$F_s = F \quad (21)$$

Substituting into (12) and solving for G gives:

$$G = \frac{rF}{\tilde{\tau}_s} \quad (22)$$

2.2.4 By Rotational Speed (ω_{load})

We can substitute (8) and (9) into (10):

$$\omega_{load} = \frac{\tilde{\omega}_f}{G} \cdot \left(1 - \frac{F}{\tilde{\tau}_s G}\right) \quad (23)$$

Solving for G , we get:

$$G = \frac{\tilde{\omega}_f}{2\omega_{load}} \left(1 + \sqrt{1 - 4rF \frac{\omega_{load}}{\tilde{\tau}_s \tilde{\omega}_f}}\right) \quad (24)$$

2.2.5 By Linear Speed (v_{load})

We can solve (15) for ω_{load} :

$$\omega_{load} = \frac{v_{load}}{2\pi r} \quad (25)$$

Now knowing ω_{load} we can use (24) to calculate G

2.2.6 By Per-Motor Current (i)

Solving (16) for G , we get:

$$G = \frac{rF}{K_T(ni - i_f)} \quad (26)$$

2.2.7 By Stall Load (F_s)

We can solve (12) for G :

$$G = \frac{rF_s}{\tilde{\tau}_s} \quad (27)$$

2.2.8 By Stall Voltage (V_s)

Solving (17) for G , we get:

$$G = 12 \frac{rF}{\tau_s \eta n V_s} \quad (28)$$

3 Lead Screw Calculator

The basic transformation between rotational and linear motion is given by:

$$v = \omega \cdot np \quad (29)$$

Force & torque equations are taken from [Shigley's Mechanical Engineering](#) §8-2

$$d_m = d - \frac{p}{2} \quad (30)$$

$$T_R = \frac{Fd_m}{2} \left(\frac{\pi \mu d_p + np \cos(\alpha)}{\pi d_p \cos(\alpha) - \mu np} \right) \quad (31)$$

$$T_L = \frac{Fd_m}{2} \left(\frac{\pi \mu d_p - np \cos(\alpha)}{\pi d_p \cos(\alpha) + \mu np} \right) \quad (32)$$

Backdrive-able if $T_L < 0$.

$$\eta = \frac{T_R|_{\mu=0}}{T_R} = \frac{np}{\pi d_p} \cdot \frac{\pi d_p \cos(\alpha) - \mu np}{\pi \mu d_p + np \cos(\alpha)} \quad (33)$$

Equivalent radius/load formulas are designed to allow for lead screws as outputs in the Mechanism Gear Ratio Calculator. Derivations can be found in [Appendix B](#).

$$r_{eq} = \frac{np}{2\pi} \quad (34)$$

$$L_{R,eq} = \frac{T_R}{r_{eq}} \quad (35)$$

$$L_{L,eq} = \frac{T_L}{r_{eq}} \quad (36)$$

d = Screw Diameter

p = Screw Pitch

n = # of Starts

α = Half Thread Angle (i.e. $\frac{\text{thread angle}}{2}$)

μ = Coefficient of Friction of Screw

F = Applied Force

v = Instantaneous Linear Speed

ω = Instantaneous Rotational Speed

d_m = Mean (Average) Diameter

T_R = Raise Torque

T_L = Lower Torque

η = Efficiency

r_{eq} = Equivalent Radius

$L_{R,eq}$ = Equivalent Raise Load

$L_{L,eq}$ = Equivalent Lower Load

4 Drivetrain Calculator

4.1 Forward Calculation

Equations taken from [JVN's Mechanical Design Calculator](#).

$$v_{free} = \frac{\tilde{\omega}_f \cdot \pi d}{G} \quad (37)$$

$$v_{adj} = v_{free} \cdot k_{SL} \quad (38)$$

$$i_{slip} = \frac{\tilde{W} \mu d}{2K_T \cdot nG} + \frac{\tilde{i}_f}{n} \quad (39)$$

v_{free} = Free Speed

v_{adj} = Adjusted Speed

i_{slip} = Wheel Slip Current

i_{tot} = Maximum Total Current Draw

G_{slip} = Maximum Gear Ratio to Slip Wheels above V_{min}

K_T = Motor Torque Constant

ω_f = Motor Free Speed

i_f = Motor Free Current

d = Wheel Diameter

μ = Wheel Coefficient of Friction

\tilde{W} = Adjusted Robot Weight (i.e. weight resting on driven wheels)

G = Total Gear Ratio

k_{SL} = Speed Loss Constant

$$v_{shift} = \frac{v_1 v_2}{v_1 + v_2} \quad (40)$$

v_{shift} = Optimal Shifting Speed

v_2 = Gear 1 Free Speed

v_2 = Gear 2 Free Speed

For derivation see [Appendix C](#).

4.2 Reverse Calculation

$$i_{tot} = \frac{V_{batt} - V_{min}}{R_{batt} + R_{main} + \frac{1}{n}R_{br}} \quad (41)$$

$$G_{slip} = \frac{\mu d \tilde{W}}{2K_T (i_{tot} - \tilde{i}_f)} \quad (42)$$

Formulas for v_{free} and v_{adj} are the same as for the Forward Calculation – (37) and (38) respectively

V_{batt} = Battery Resting Voltage

V_{min} = Minimum Allowable System Voltage

R_{batt} = Battery Internal Resistance

R_{main} = Resistance of Main Power Wiring

R_{br} = Resistance of Each Branch Circuit

5 Beam Bend Calculator

5.1 Geometrical Deformation Resistance

Formulas for deformation resistance constants of different cross-sectional geometries are taken from [StructX](#).

I = Area (Second) Moment of Inertia

J = Torsion Constant

5.1.1 Hex

$$I = 0.0601a^4 \quad (43)$$

$$J = 0.1154a^4 \quad (44)$$

a = Distance Between Flats

5.1.2 Round

$$I = \frac{\pi}{64}d^4 \quad (45)$$

$$J = \frac{\pi}{32}d^4 \quad (46)$$

d = Diameter

5.1.3 Round Tube

$$I = \frac{\pi}{8}d^3h \quad (47)$$

$$J = \frac{\pi}{4}d^3h \quad (48)$$

d = Diameter

h = Wall Thickness

5.1.4 Square

$$I = \frac{a^4}{12} \quad (49)$$

$$J = \frac{9}{64}a^4 \quad (50)$$

a = Side Length

5.1.5 Rectangular Tube

$$I = \frac{1}{3}x^2yh \quad (51)$$

$$J = \frac{2h^2(x-h)^2(y-h)^2}{h(x+y-h)} \quad (52)$$

x = Length Parallel to Force

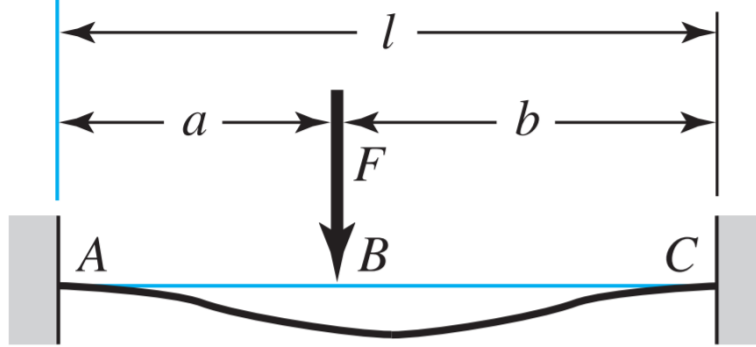
y = Length Perpendicular to Force

h = Wall Thickness

5.2 Displacement Equations

Displacement formulas are taken from [Shigley's Mechanical Engineering](#) Appendix A-9

5.2.1 Force Between Supports



$$y = \frac{Fx^2b^2}{6EI l^3} [3al - x(3a + b)] \quad (53)$$

Substitute $b = l - a$ to get:

$$y = \frac{Fx^2(a - l)^2}{6EI l^3} [3al - x(2a + l)] \quad (54)$$

Take the derivative, set equal to zero, and solve for x :

$$0 = \frac{\partial y}{\partial x} = \frac{Fx(l - a)^2}{2EI l^3} (2a(l - x) - lx) \quad (55)$$

$$x = \frac{2al}{2a + l} \quad (56)$$

Substitute (56) into (54) to get:

$$y_{max} = \frac{2Fa^3(l - a)^2}{3EI(2a + l)^2} \quad (57)$$

y = Vertical Displacement

F = Load Force

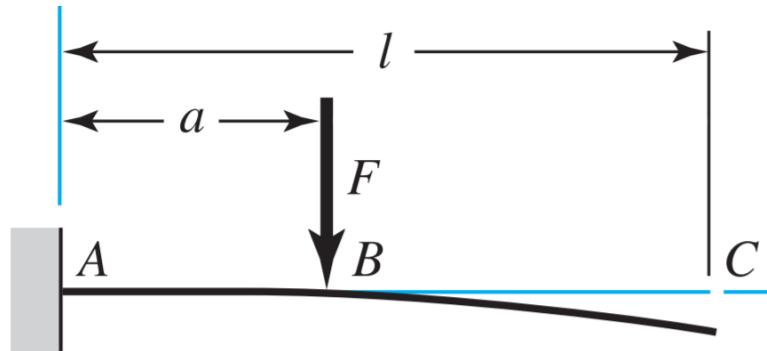
a = Distance to Closer Support

l = Distance Between Supports

E = Modulus of Elasticity

I = Area Moment of Inertia

5.2.2 Cantilevered Force



$$y_{max} = \frac{Fa^2}{6EI}(a - 3l) \quad (58)$$

y_{max} = Largest Displacement

F = Load Force

a = Distance to Support

l = Total Length

E = Modulus of Elasticity

I = Area Moment of Inertia

5.3 Buckling Force

This formula is taken from the Wikipedia page on [Euler's Critical Load](#)

$$F_{max} = \frac{\pi^2 EI}{(KL)^2} \quad (59)$$

Buckled shape of column shown by dashed line						
Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value K	0.65	0.80	1.2	1.0	2.10	2.0
End condition key	 Rotation fixed and translation fixed Rotation free and translation fixed Rotation fixed and translation free Rotation free and translation free					

F_{max} = Maximum Force

E = Modulus of Elasticity

I = Area Moment of Inertia

K = End Condition Constant

L = Column Length

5.4 Twisting Torque

This formula is taken from [Shigley's Mechanical Engineering](#) §3-12

$$\theta_{max} = \frac{TL}{GJ} \quad (60)$$

θ_{max} = Largest Angular Displacement

T = Applied Torque

L = Distance Between Torque and Support

G = Shear Modulus

J = Torsion Constant

6 Projectile Trajectory Calculator

6.1 Forward Calculation

$$h_f = h_i + d \tan \theta_i - \frac{g \cdot d^2}{2v_i^2 \cos^2 \theta_i} \quad (61)$$

$$\theta_f = \tan^{-1} \left(2 \frac{h_f - h_i}{d} - \tan \theta_i \right) \quad (62)$$

6.2 Reverse Calculation

$$\theta_i = \tan^{-1} \left(2 \frac{h_f - h_i}{d} - \tan \theta_f \right) \quad (63)$$

$$v_i = \sec \theta_i \cdot \sqrt{\frac{g \cdot d}{|\tan \theta_i - \tan \theta_f|}} \quad (64)$$

h_i = Release (Initial) Height

h_f = Target (Final) Height

θ_i = Release (Initial) Angle

θ_f = Target (Final) Angle

v_i = Release (Initial) Velocity

d = Horizontal Distance to Target

g = Acceleration Due to Gravity

For derivation see Appendix [D](#).

7 Chain/Belt C-C Calculator

This method of solving for D is inspired by and based on the formulas used in [Clem1640's Chain/Belt Calculator](#). The language in this section will only reference chain, sprockets, and chain links; but all equations also apply to belts, pulleys, and belt teeth, respectively.

First we start with basic equations for chain and sprockets:

$$2\pi r_i = n_i p \quad (65)$$

$$L = lp \quad (66)$$

$$C = D - (r_1 + r_2) \quad (67)$$

We can construct an equation to represent the path length of the chain relative to the center-to-center distance. A derivation of this equation can be found in Appendix [E](#).

$$L = 2\sqrt{D^2 - (r_1 - r_2)^2} + 2(r_1 + r_2) \cos^{-1} \left(\frac{r_1 - r_2}{D} \right) + 2\pi r_2 \quad (68)$$

Unfortunately, we cannot solve this equation empirically to get D , even with the help of a computer. So we solve the function numerically, using the Newton-Raphson Method. For the forward calculation our initial guess for D is $\frac{L}{2}$, which would be the case if both sprockets had diameters of 0. For the reverse calculation, we use the estimated value of

D as the initial guess. The Newton-Raphson Method then improves the accuracy of D to the correct solution until the error is less than 10^{-12} .

L = Chain Length

l = # of Links

D = Center to Center Distance

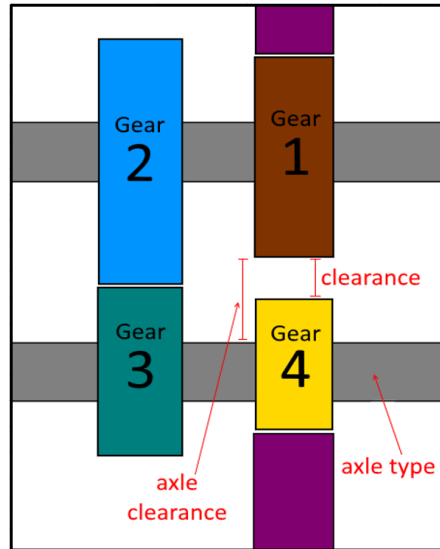
n_i = # of Teeth on Sprocket $i \in [1, 2]$

r_i = Radius of Sprocket $i \in [1, 2]$

p = Chain Pitch

C = Clearance

8 Gear Clearance Calculator



Starting with the basic equations:

$$r_p = \frac{n}{2p} \quad (69)$$

$$r_o = \frac{n+2}{2p} \quad (70)$$

$$C_a = r_{o2} + r_{o3} - r_{o1} - \frac{d_{axle}}{2} \quad (71)$$

$$C_g = r_{o2} + r_{o3} - r_{o1} - r_{o4} \quad (72)$$

Substituting (70) into (71) and (72), respectively, gives:

$$C_a = \frac{n_2 + n_3}{2p_{23}} - \frac{n_1}{2p_1} - \frac{d_{axle}}{2} \quad (73)$$

$$C_g = \frac{n_2 + n_3}{2p_{23}} - \frac{n_1}{2p_1} - \frac{n_4}{2p_4} \quad (74)$$

The axle or gear clears if $C_a > 0$ or $C_g > 0$, respectively.

C_a = Axle Clearance

C_g = Gear Clearance

r_p = Pitch Radius

r_o = Outside Radius

n = # of Teeth

p = Diametrical Pitch (a.k.a. dp)

d_{axle} = Axle Diameter

9 Ratio & Distance Calculator

We have the basic equations:

$$G = \frac{N}{n} \quad (75)$$

$$d = \frac{N + n}{2p} \quad (76)$$

Solving these equations for n and N gives:

$$n = \frac{2dp}{G + 1} \quad (77)$$

$$N = \frac{2dp}{G + 1}G \quad (78)$$

For each gear we can get the pitch diameter and outside diameter using (69) and (70), respectively.

N = # of Teeth on Large Gear

n = # of Teeth on Small Gear

G = Gear Ratio

d = Center-to-Center Distance

p = Diametrical Pitch (a.k.a. dp)

10 Pneumatics Calculator

The "simulation" is run at timesteps of $dt = 1$ second for a duration of 150 seconds.

The pressure at the current step of the simulation can be calculated by:

$$P_{n+1} = P_n + \frac{W_{comp} - \sum W_{cyl}}{V} \quad (79)$$

$$W_{comp} = \dot{V}_{comp}(P) \cdot dt \cdot 1\text{atm} \quad (80)$$

$$W_{cyl} = \left(\frac{\pi D^2}{4} P_{push} + \frac{\pi(D^2 - d^2)}{4} P_{pull} \right) L \cdot m \quad (81)$$

$\dot{V}_{comp}(P)$ is the compressor flow-rate as a function of the output pressure, taken from a fourth-degree polynomial interpolation of the data provided online. The volume of air is measured at atmospheric pressure, not at system pressure. m is the number of actuations per second (can be 0 when not firing, 1 when firing, or $m > 1$ if firing more than once per second).

Derivations of these equations can be found in Appendix [F](#).

We can also calculate the pushing and pulling force of each cylinder using the formulas:

$$F_{push} = \frac{\pi D^2}{4} P_{push} \quad (82)$$

$$F_{pull} = \frac{\pi(D^2 - d^2)}{4} P_{pull} \quad (83)$$

P_n = System Pressure at Step n

V = System Volume

W_{comp} = Work Done by the Compressor

W_{cyl} = Work Done by Each Cylinder

D = Cylinder Diameter

d = Cylinder Rod Diameter

L = Cylinder Length

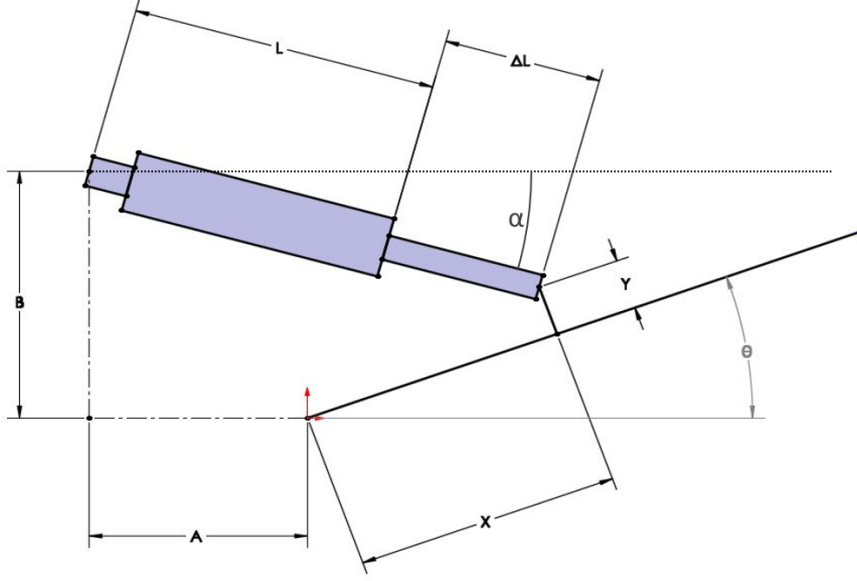
P_{push} = Pushing Pressure

P_{pull} = Pulling Pressure

F_{push} = Pushing Force

F_{pull} = Pulling Force

11 Pneumatic Linkage Calculator



Let θ_1 and α_1 represent the system when the cylinder is retracted, and θ_2 and α_2 represent it when extended.

In the horizontal direction we have:

$$A + X \cos \theta_1 - Y \sin \theta_1 = L \cos \alpha_1 \quad (84)$$

$$A + X \cos \theta_2 - Y \sin \theta_2 = (L + \Delta L) \cos \alpha_2 \quad (85)$$

And in the vertical direction we have:

$$X \sin \theta_1 + Y \cos \theta_1 = B - L \sin \alpha_1 \quad (86)$$

$$X \sin \theta_2 + Y \cos \theta_2 = B - (L + \Delta L) \sin \alpha_2 \quad (87)$$

We can combine (84) and (87) to remove α_1 and get:

$$L^2 = A^2 + B^2 + X^2 + Y^2 + 2AX \cos \theta_1 - 2BY \cos \theta_1 - 2BX \sin \theta_1 - 2AY \sin \theta_1 \quad (88)$$

And combine (85) and (87) to remove α_2 and get:

$$(L + \Delta L)^2 = A^2 + B^2 + X^2 + Y^2 + 2AX \cos \theta_2 - 2BY \cos \theta_2 - 2BX \sin \theta_2 - 2AY \sin \theta_2 \quad (89)$$

To derive the forward formulas, you can solve (88) and (89) separately to get expressions for θ_1 and θ_2 . To derive the reverse formulas, you can solve (88) and (89) as a system of equations to get expressions for A and B . I will not include these expressions in this document, because they are extremely long.

Derivations

A Maximum Efficiency Derivation

To find the maximum efficiency, we take the derivative of (10) and set it equal to 0:

$$0 = \frac{\partial \eta}{\partial \tau} = \frac{(\tilde{i}_f(\tau - \tilde{\tau}_s)^2 - \tilde{i}_s\tau^2)\tilde{\omega}_f}{V(\tilde{i}_s\tau + \tilde{i}_f(\tilde{\tau}_s - \tau))^2} \quad (90)$$

Solving for the value of τ at which η is maximized:

$$\tau_{\eta_{max}} = \frac{\tilde{\tau}_s\sqrt{\tilde{i}_f}}{\sqrt{\tilde{i}_f} + \sqrt{\tilde{i}_s}} \quad (91)$$

We know that $\tau_{manip} = G \cdot \tau_{motor}$, so we can substitute $\tau_{\eta_{max}}$ for τ_{motor} :

$$G = \frac{\tau_{manip}}{\tau_{motor}} = \frac{r \cdot F}{\tau_{\eta_{max}}} = \frac{r \cdot F}{\frac{\tilde{\tau}_s\sqrt{\tilde{i}_f}}{\sqrt{\tilde{i}_f} + \sqrt{\tilde{i}_s}}} \quad (92)$$

And simplifying gives:

$$G = \frac{rF}{\tilde{\tau}_s} \left(1 + \sqrt{\frac{\tilde{i}_s}{\tilde{i}_f}} \right) \quad (20)$$

B Equivalent Radius/Load Derivation

To find the equivalent radius, we equate the rotational \rightarrow linear motion transformation equations for both a wheel (15) and lead screw (29)

$$2\pi r \cdot \omega = v = \omega \cdot np \quad (93)$$

We can then solve for $r = r_{eq}$ to find the radius that will provide the same rotational \rightarrow linear motion transformation as the given lead screw

$$r_{eq} = \frac{np}{2\pi} \quad (34)$$

We want to conserve the work done by a lead screw and equivalent wheel across one rotation (where x can be L or R).

$$W = 2\pi r_{eq} \cdot L_{x,eq} = 2\pi T_x \quad (94)$$

Solving for $L_{x,eq}$, we get:

$$L_{x,eq} = \frac{T_x}{r_{eq}} \quad (95)$$

C Optimal Shifting Speed Derivation

The optimal shifting speed is the speed where both high and low gears produce the same force at the wheel surface.

We will start with the definition of instantaneous motor speed from (8). This can be solved for the motor torque τ

$$\tau = \tau_s \left(1 - \frac{\omega}{\omega_f} \right) \quad (96)$$

Using the equations,

$$v_w = \frac{\omega r}{G} \quad (97)$$

$$F_w = \frac{\tau G}{r} \quad (98)$$

we can transform the rotational speed and torque at the motor into the linear speed and force at the wheel

$$F_w = \frac{\tau_s G}{r} \left(1 - \frac{v_w G}{r \omega_f} \right) \quad (99)$$

To find the optimal shifting speed, we will set the wheel forces at both gear ratios, G_1 and G_2 , and both wheel radii, r_1 and r_2 , equal at the shifting speed v_{shift}

$$\frac{\tau_s G_1}{r_1} \left(1 - \frac{v_{shift} \cdot G_1}{r_1 \omega_f} \right) = \frac{\tau_s G_2}{r_2} \left(1 - \frac{v_{shift} \cdot G_2}{r_2 \omega_f} \right) \quad (100)$$

This can then be simplified to

$$v_{shift} = \frac{r_1 r_2 \omega_f}{G_2 r_1 + G_1 r_2} \quad (101)$$

Plugging in $\omega = \omega_f$ into (97) gives the free speeds v_1 and v_2 for each gear ratio

$$\begin{aligned} v_1 &= \frac{\omega_f r_1}{G_1} \\ v_2 &= \frac{\omega_f r_2}{G_2} \end{aligned} \quad (102)$$

Then it is easy to see that

$$v_{shift} = \frac{1}{\frac{1}{v_1} + \frac{1}{v_2}} = \frac{v_1 v_2}{v_1 + v_2} = v_1 \parallel v_2 \quad (40)$$

D Projectile Equation Derivations

Starting with basic projectile motion equations:

$$\Delta x = v_i \cos \theta_i \cdot t \quad (103)$$

$$\Delta y = v_i \sin \theta_i \cdot t - \frac{1}{2}gt^2 \quad (104)$$

$$\theta_f = \tan^{-1} \left(\frac{v_{f,y}}{v_{f,x}} \right) = \tan^{-1} \left(\frac{v_i \sin \theta_i - gt}{v_i \cos \theta_i} \right) \quad (105)$$

Solving (103) for t and substituting into (104) gives:

$$\Delta y = \Delta x \tan \theta_i - \frac{g \cdot (\Delta x)^2}{2v_i^2 \cos^2 \theta_i} \quad (106)$$

We can then substitute in $\Delta x = d$ and $\Delta y = h_f - h_i$:

$$h_f = h_i + d \tan \theta_i - \frac{g \cdot d^2}{2v_i^2 \cos^2 \theta_i} \quad (61)$$

Solving (103) for t and substituting into (105) gives:

$$\theta_f = \tan^{-1} \left(\frac{v_i \sin \theta_i - g \cdot \frac{\Delta x}{v_i \cos \theta_i}}{v_i \cos \theta_i} \right) = \tan^{-1} \left(\tan \theta_i - \frac{g \cdot d}{v_i^2 \cos^2 \theta_i} \right) \quad (107)$$

Further substituting in (61) into (107) gives:

$$\theta_f = \tan^{-1} \left(\tan \theta_i - \frac{2}{d} (d \tan \theta_i - (h_f - h_i)) \right) \quad (108)$$

And by simplifying we get:

$$\theta_f = \tan^{-1} \left(2 \frac{h_f - h_i}{d} - \tan \theta_i \right) \quad (62)$$

To calculate the reverse equations, we can solve (108) for θ_i :

$$\theta_i = \tan^{-1} \left(2 \frac{h_f - h_i}{d} - \tan \theta_f \right) \quad (63)$$

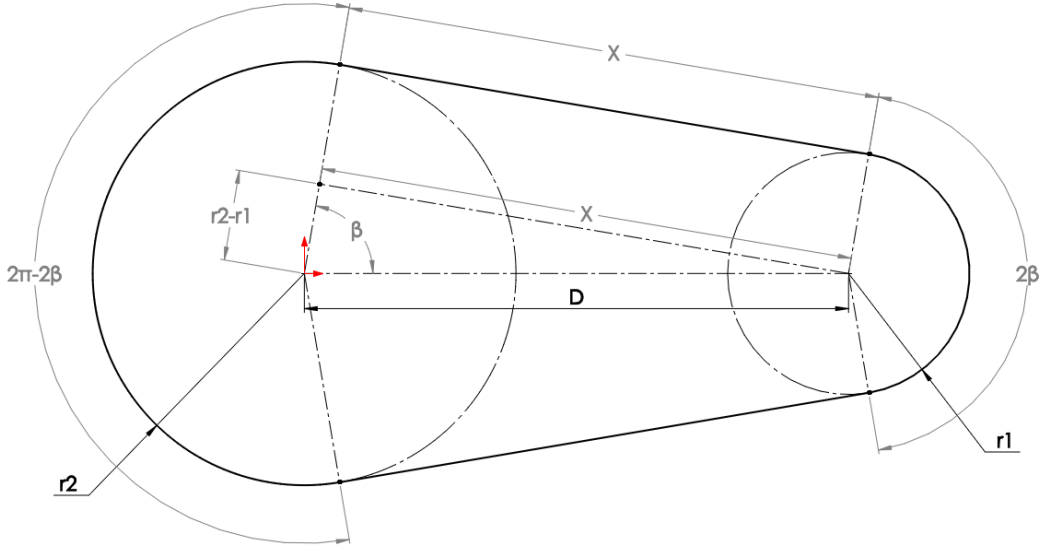
Then substitute (61) into (63):

$$\tan \theta_i + \tan \theta_f = \frac{2}{d} \left(d \tan \theta_i - \frac{g \cdot d^2}{2v_i^2 \cos^2 \theta_i} \right) \quad (109)$$

And solve that for v_i :

$$v_i = \sec \theta_i \cdot \sqrt{\frac{g \cdot d}{|\tan \theta_i - \tan \theta_f|}} \quad (64)$$

E Chain Length Derivation



It is clear that:

$$L = 2X + r_1 \cdot 2\beta + r_2 \cdot (2\pi - 2\beta) \quad (110)$$

Looking at the right triangle in the center of the image, we can see that:

$$D^2 = X^2 + (r_2 - r_1)^2 \implies X = \sqrt{D^2 - (r_1 - r_2)^2} \quad (111)$$

$$\beta = \cos^{-1} \left(\frac{r_2 - r_1}{D} \right) = \cos^{-1} \left(\frac{r_1 - r_2}{D} \right) \quad (112)$$

Substituting (111) and (112) into (110) gives:

$$L = 2\sqrt{D^2 - (r_1 - r_2)^2} + r_1 \cdot 2 \cos^{-1} \left(\frac{r_1 - r_2}{D} \right) + r_2 \cdot \left(2\pi - 2 \cos^{-1} \left(\frac{r_1 - r_2}{D} \right) \right) \quad (113)$$

And rearranging gives:

$$L = 2\sqrt{D^2 - (r_1 - r_2)^2} + 2(r_1 - r_2) \cos^{-1} \left(\frac{r_1 - r_2}{D} \right) + 2\pi r_2 \quad (68)$$

F Pneumatics Simulation Derivation

We know that the energy of the system at any point can be represented by:

$$E = P \cdot V \quad (114)$$

We can find the energy of the system at any timestep by adding the total work done on the system in that time to the energy of the system at the previous step:

$$E_{n+1} = E_n + W_{tot} \quad (115)$$

Substituting (114) into (115) gives:

$$(PV)_{n+1} = (PV)_n + W_{comp} - \sum W_{cyl} \quad (116)$$

Since the volume of the system remains constant, we can divide by V to get:

$$P_{n+1} = P_n + \frac{W_{comp} - \sum W_{cyl}}{V} \quad (79)$$

Basic thermodynamics says that the work done by compression/expansion of gas at constant pressure can be represented by $W = P \cdot \Delta V$. In each actuation of a cylinder gas is compressed twice, once for extension and once for retraction. So for each timestep, the work done on the cylinder can be expressed as:

$$W_{cyl} = ((PAL)_{push} + (PAL)_{pull}) \cdot m \quad (117)$$

L is constant, so it can be factored out. We can also substitute A for the effective piston areas, giving:

$$W_{cyl} = \left(\frac{\pi D^2}{4} P_{push} + \frac{\pi(D^2 - d^2)}{4} P_{pull} \right) L \cdot m \quad (81)$$

Similarly, we can represent the work done by the compressor as:

$$W_{comp} = \dot{V}_{comp}(P) \cdot dt \cdot 1\text{atm} \quad (80)$$

Note: since \dot{V}_{comp} is measured at atmospheric pressure, P must be 1 atm and not the system pressure