Let X be a variable random represent number of defect item in each sample.

Let X_i is the observation of number of defect item in sample i-th size p ($i=1,2,\ldots,n$ represent the i-th sample). Let n is number of sample size p.

Suppose that X_i are auto-correlated and follow the identical normal distribution with known in control mean μ_o and standard deviation σ_0. The process can be modelled as follow by AR(1) model:

$$X_i-\mu_0=\rho\left(X_{i-1}-\mu_0\right)+\varepsilon_{i,}\, i=1,2,\ldots,n$$

where $\rho=(-1,1)$ is parameter of AR(1).

Suppose that the mean is shifting from μ_o to $\mu_1=\mu_o+\delta\sigma_0$ where δ is magnitude of mean shift in term of σ_0.

Note that the number of defects in each sample size n only concern that whether the sample is conforming (whether the number of defects $X\in[0,Lu]$ which Lu is the upper warning limit and Lu can be expressed:

$Lu=\mu_o+k\sigma_0$ which k is warning limit coefficient

Let $p_i^j(i=1,2,3,\ldots,n;\, j=0,1)$ is probability that sample i-th is non-conforming sample, i.e., sample X_i has number of defect item larger than Lu in state j. we use in this study $j=0,1$ indicates the process in control and out of control respectively.

Let $\phi(.)$ is CDF of standard normal distribution, and

$f(.)$ is PDF of standard normal distribution.

Define $p_i^j=P\{X_i>Lu\}$ is probability that number of nonconforming item in sample i-th larger than Lu in state-j.

When i = 1

$$p_1^j=P\{X_1>Lu\}=P\left\{\frac{X_1-\mu_j}{\sigma_0}\geq\frac{Lu-\mu_j}{\sigma_0}\right\}p_1^j=1-\phi\left(\frac{Lu-\mu_j}{\sigma_0}\right)$$

When i = 2

$$p_2^j = P\{X_2 > Lu\} \text{¿} P\{X_2 > Lu \vee X_1 > Lu\} P\{X_1 > Lu\} + P\{X_2 > Lu \vee X_1 < Lu\} P\{X_1 < Lu\}$$

$$\text{Note}: AR(1): X_2 - \mu_j = \rho(X_{2-1} - \mu_j) + \varepsilon_1 X_2 = \rho(X_1 - \mu_j) + \varepsilon_1 + \mu_j$$

$$p_2^j = p_1^j \int_{Lu}^{+\infty} P\{\rho(X_1 - \mu_j) + \varepsilon_1 + \mu_j > Lu\} f(X_1)\, dX_1 + (1 - p_1^j) \int_{-\infty}^{Lu} P\{\rho(X_1 - \mu_j) + \varepsilon_1 + \mu_j > Lu\} f(X_1)\, dX_1$$

$$\text{Note}: \varepsilon_i (i = 1, 2, \ldots, p) \text{ are } i.i.d. \text{ normal random variables and } \varepsilon_i \quad N(0, (1-\rho^2)\sigma_0^2)$$

$$p_2^j = p_1^j \int_{Lu}^{+\infty} \left\{ \frac{\varepsilon_1 - 0}{\sqrt{(1-\rho^2)\sigma_0^2}} > \frac{Lu - \rho(X_1 - \mu_j) - \mu_j}{\sqrt{(1-\rho^2)\sigma_0^2}} \right\} f(X_1)\, dX_1 + (1 - p_1^j) \int_{-\infty}^{Lu} \left\{ \frac{\varepsilon_1 - 0}{\sqrt{(1-\rho^2)\sigma_0^2}} > \frac{Lu - \rho(X_1 - \mu_j) - \mu_j}{\sqrt{(1-\rho^2)\sigma_0^2}} \right\} f(X_1)\, dX_1$$

$$p_2^j = p_1^j \int_{Lu}^{+\infty} \left\{ 1 - \phi\left(\frac{Lu - \rho(X_1 - \mu_j) - \mu_j}{\sqrt{(1-\rho^2)\sigma_0^2}} \right) \right\} f(X_1)\, dX_1 + (1 - p_1^j) \int_{-\infty}^{Lu} \left\{ 1 - \phi\left(\frac{Lu - \rho(X_1 - \mu_j) - \mu_j}{\sqrt{(1-\rho^2)\sigma_0^2}} \right) \right\} f(X_1)\, dX_1$$

In general:

$$p_i^j = p_{i-1}^j \int_{Lu}^{+\infty} \left\{ 1 - \phi\left(\frac{Lu - \rho(X_{i-1} - \mu_j) - \mu_j}{\sqrt{(1-\rho^2)\sigma_0^2}} \right) \right\} f(X_{i-1})\, dX_{i-1} + (1 - p_{i-1}^j) \int_{-\infty}^{Lu} \left\{ 1 - \phi\left(\frac{Lu - \rho(X_{i-1} - \mu_j) - \mu_j}{\sqrt{(1-\rho^2)\sigma_0^2}} \right) \right\} f(X_{i-1})\, dX_{i-1}$$

Thus, we can write that:

1. When i =1

$$p_1^j = 1 - \phi\left(\frac{Lu - \mu_j}{\sigma_0} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

2. When $2 \le i \le p$

$$p_i^j = p_{i-1}^j M_{i-1}^j + (1 - p_{i-1}^j) N_{i-1}^j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

which:

$$M_{i-1}^{j}=\int_{Lu}^{+\infty}\left\{1-\phi\left(\frac{Lu-\rho\left(X_{i-1}-\mu_{j}\right)-\mu_{j}}{\sqrt{\left(1-\rho^{2}\right)\sigma_{0}^{2}}}\right)\right\}f\left(X_{i-1}\right)d\,X_{i-1}$$

$$N_{i-1}^{j}=\int_{-\infty}^{Lu}\left\{1-\phi\left(\frac{Lu-\rho\left(X_{i-1}-\mu_{j}\right)-\mu_{j}}{\sqrt{\left(1-\rho^{2}\right)\sigma_{0}^{2}}}\right)\right\}f\left(X_{i-1}\right)d\,X_{i-1}$$

We found that the probability p_i^j depends on p_{i-1}^j as result of the autocorrelation property shown in equation (2).

Since the classification of conforming and non-conforming samples is a Bernoulli trial, we define $Y_i^j\left(i=1,2,\ldots,n\right)$ as a sequence of Bernoulli random variable for a fixed $j=0,1$ such that:

$$Y_i^j=\begin{cases}0\,;0\leq Xi\leq Lu\\1\,;Xi>Lu\end{cases}$$

Let Z_j denote the statistic sample of a $\mathrm{CCC_G}$ chart, i.e. the cumulative number of samples size-p inspected until the first non-conforming sample is encountered. Then, Z_j is generally considered to be a geometric random variable with parameter p_i^j.

Let $P_j=P\{Z_j<L\}$ be the probability that the total number of conforming samples smaller than *LCL* in state-*j*, i.e., the probability that control chart produces out-of-control signal when the process is in state-*j*.

$P_j=P\{Z_j<L\}=1-\left(1-p_i^j\right)^L$ with $p_i^j=\max_{1\leq i\leq n} p_i^j\rightarrow 0$

The probability of type I error α is $\alpha=P_0$ and the probability of type II error β is $\beta=1-P_1$

Therefore,

$$ARL_0 = \frac{1}{P_0}$$

$$ARL_0 = \frac{1}{1-\beta} = \frac{1}{P_1}$$

As it is difficult to derive the closed-form solution for ARL, we present a step-by-step description of procedure for calculating ARL in the following table:

Set $\mu_0, \sigma_0, \delta, \rho, k, n, LCL$	
Compute p_1^0 and p_1^1	Through Eq. 1
Set $\lambda_0 = p_1^0$ and $\lambda_1 = p_1^1$	
[cycle i] For i = 2 to n step 1	
Compute p_i^0 and p_i^1	Trough Eq. 2
Compute $\lambda_0 = \lambda_0 + p_i^0$ and $\lambda_1 = \lambda_1 + p_i^1$	
Next [cycle i]	
Compute P_0 and P_1	
Compute $ARL_0 = \frac{1}{P_0}$ and $ARL_0 = \frac{1}{P_1}$	
Stop	

Genetic Algorithm:

$$\underset{k,LCL}{\text{minimize}}\ ARL_1$$

$$s.t\ ARL_0 \geq 0 k > 0 LCL > 0$$