# Empirical likelihood ratios applied to goodness-of-fit tests based on sample entropy

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# Introduction

## Goal

To implement empirical likelihood (EL) ratio tests for goodness-of-fit

## The outline of the EL approach

$X_1, \ldots, X_n$ – i.i.d.
The EL function:

$$L_p = \prod_{i=1}^{n} p_i,$$

$$s.t. \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i X_i = 0$$

and $p_i$, $i = 1, \ldots, n$ maximize the likelihood $L_p$.

# EL ratio test for normality

$X_1, \ldots, X_n$ – independent
$H_0 : X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2); \quad \mu, \sigma$ are unknown
The likelihood ratio:

$$\frac{\prod_{i=1}^{n} f_{H_1}(X_i)}{\prod_{i=1}^{n} f_{H_0}(X_i)} = \frac{\prod_{i=1}^{n} f_{H_1}(X_i)}{(2\pi e s^2)^{-\frac{n}{2}}}, \quad s^2 = \frac{1}{n}\sum_{j=1}^{n}\left(X_j - \frac{1}{n}\sum_{k=1}^{n} X_k\right)^2 \qquad (1)$$

## The maximum EL technique to nonparametric estimate

$$L_f = \prod_{i=1}^{n} f(X_i) = \prod_{i=1}^{n} f(X_{(i)}) = \prod_{i=1}^{n} f_i, \quad f_i = f(X_{(i)})$$

## Statement 1:

$$\Delta_m \leqslant 1, \quad \Delta_m = \frac{1}{2m} \sum_{j=1}^{n} \int_{X_{(j-m)}}^{X_{(j+m)}} f(x)dx$$

In the empirical form:

$$\tilde{\Delta}_m \leqslant 1, \quad \tilde{\Delta}_m = \frac{1}{2m} \sum_{j=1}^{n} \left(X_{(j+m)} - X_{(j+m)}\right) f_j$$

## The likelihood ratio test statistic

Thus, values of $f_1, \ldots, f_n$, which maximize $\log(L_f)$ and satisfy $\tilde{\Delta}_m \leqslant 1$:

$$f_i = \frac{2m}{n\left(X_{(i+m)} - X_{(i-m)}\right)} \Rightarrow T_{mn} = (2\pi e s^2)^{\frac{n}{2}} \prod_{i=1}^{n} \frac{2m}{n\left(X_{(i+m)} - X_{(i-m)}\right)}$$

# Reconsideration of the test statistic

## First reconsideration

The maximum EL:

$$\min_{1\leqslant m<\frac{n}{2}} \max_{f_1,\ldots,f_n:\tilde{\Delta}_m\leqslant 1} \prod_{i=1}^{n} f_i$$

The empirical modification of the test statistic (1):

$$V_n^1 = \min_{1\leqslant m<\frac{n}{2}} (2\pi e s^2)^{\frac{n}{2}} \prod_{i=1}^{n} \frac{2m}{n\left(X_{(i+m)} - X_{(i-m)}\right)} \tag{2}$$

## Second reconsideration

$$V_n^2 = \min_{1\leqslant m<n^{1-\delta}} (2\pi e s^2)^{\frac{n}{2}} \prod_{i=1}^{n} \frac{2m}{n\left(X_{(i+m)} - X_{(i-m)}\right)}, \quad 0<\delta<1 \tag{3}$$

# Testing

## Condition

We reject the null hypothesis iff

$$\log\left(V_n^j\right) > C, \tag{4}$$

where $C$ is a test-threshold, $j = 1, 2$, and $V_n^j$ – test statistics (2), (3).

## Significance level of the test

Since,

$$\sup_{\mu,\sigma} P_{H_0}\left\{\log\left(V_n^j\right) > C\right\} = P_{X_1,\dots,X_n \sim \mathcal{N}(0,1)}\left\{\log\left(V_n^j\right) > C\right\}, \quad j = 1, 2$$

the type I error of the tests (4) can be calculated **exactly**.

# Results of the experiment

Set up $\delta = 0.5$. Fig.1 and Fig.2 plot Monte-Carlo roots $C_\alpha$ of $P_{X_1,\dots,X_n\sim\mathcal{N}(0,1)}\left\{\log\left(V_n^j\right) > C_\alpha\right\} = \alpha$, for different $\alpha$ and $n$.

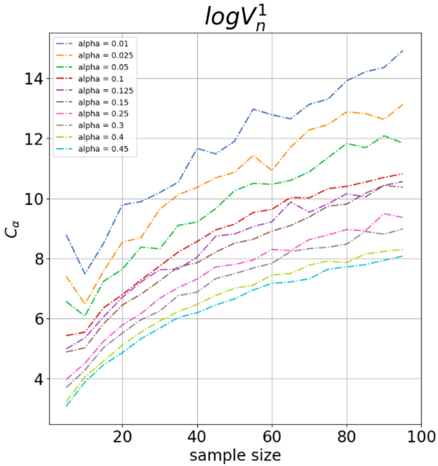


Figure 1: The values of thresholds $C_\alpha$ for the test (4) with $j = 1$.

# Results of the experiment

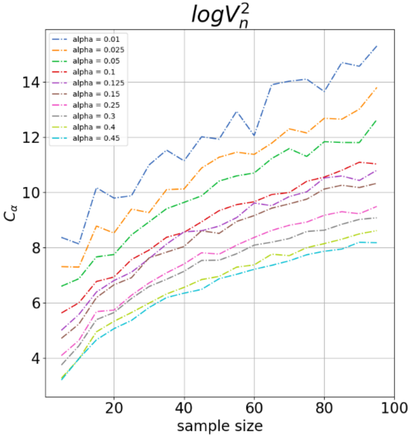


Figure 2: The values of thresholds $C_\alpha$ for the test (4) with $j = 2$.

# Results

| $n=20$ | $T_{1n}$ | $T_{2n}$ | $T_{3n}$ | $T_{4n}$ | $T_{5n}$ | $T_{6n}$ | $T_{7n}$ |
|---|---|---|---|---|---|---|---|
| Exp(1) | 0.68 | 0.779 | 0.835 | 0.813 | 0.83 | 0.821 | 0.786 |
| Gamma(2,1) | 0.278 | 0.398 | 0.466 | 0.429 | 0.463 | 0.447 | 0.391 |
| Unif(0,1) | 0.315 | 0.38 | 0.435 | 0.393 | 0.465 | 0.477 | 0.426 |
| Beta(2,1) | 0.296 | 0.387 | 0.447 | 0.415 | 0.474 | 0.47 | 0.423 |
| Cauchy(0,1) | 0.74 | 0.77 | 0.753 | 0.69 | 0.629 | 0.563 | 0.48 |

| $n=20$ | $T_{8n}$ | $T_{9n}$ | $T_{10n}$ | $T_{nn}$ | $V_n^1$ | $V_n^2$ | $W$ |
|---|---|---|---|---|---|---|---|
| Exp(1) | 0.75 | 0.742 | 0.654 | 0.122 | 0.835 | 0.84 | 0.839 |
| Gamma(2,1) | 0.347 | 0.342 | 0.28 | 0.091 | 0.462 | 0.471 | 0.509 |
| Unif(0,1) | 0.39 | 0.461 | 0.403 | 0.529 | 0.426 | 0.441 | 0.196 |
| Beta(2,1) | 0.406 | 0.437 | 0.365 | 0.318 | 0.443 | 0.449 | 0.293 |
| Cauchy(0,1) | 0.427 | 0.407 | 0.384 | 0.0 | 0.527 | 0.758 | 0.871 |

| $n=50$ | $V_n^1$ | $V_n^2$ | $W$ | $n=70$ | $V_n^1$ | $V_n^2$ | $W$ |
|---|---|---|---|---|---|---|---|
| Exp(1) | 1.000 | 1.000 | 1.000 | Exp(1) | 1.000 | 1.000 | 1.000 |
| Gamma(2,1) | 0.929 | 0.938 | 0.947 | Gamma(2,1) | 0.987 | 0.991 | 0.993 |
| Unif(0,1) | 0.959 | 0.960 | 0.755 | Unif(0,1) | 0.995 | 0.996 | 0.933 |
| Beta(2,1) | 0.958 | 0.959 | 0.832 | Beta(2,1) | 0.996 | 0.996 | 0.965 |
| Cauchy(0,1) | 0.443 | 0.992 | 0.998 | Cauchy(0,1) | 0.359 | 0.999 | 1.000 |

Figure 3: Monte Carlo power estimates of some tests for normality; $\alpha = 0.05$.

$V_n^j$, $j = 1, 2$ by (4), $\delta = 0.5$; $W$ corresponds to the Shapiro-Wilk $W$ test.

# Referenced paper

Vexler A., Gurevich G.
Empirical likelihood ratios applied to goodness-of-fit tests based on sample entropy
https://doi.org/10.1016/j.csda.2009.09.025

# Our Team

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