36710 – 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 15: WED, OCT 21, 2020

SOME RECOMMENDED REFERENCES:

1) NOTES FOR A GRADUATE-LEVEL COURSE IN ASYMPTOTIC STATISTICS BY D. HUNTER

http://personal.psu.edu/drh20/asymp/lectures/asymp.pdf

2) ELEMENTS OF LARGE SAMPLE THEORY, BY E. LEHMAN

3) ASYMPTOTIC STATISTICS, BY A. VAN DER VAART

MODES OF STOCHASTIC CONVERGENCE

$\{X_n\}$ SEQUENCE OF RV'S. QUESTION: IN WHAT SENSE DOES X_n "CONVERGE" TO X, ANOTHER RV?

SO FAR, WE HAVE SEEN 3 MODES TO DESCRIBE HOW A SEQUENCE $\{f_n\}$ OF MEAS. FUNCTIONS ON $(\Omega, \mathcal{F}, \mu)$ OR $(\Omega, \mathcal{F}, P)$ CONVERGES:

1) L^p CONVERGENCE: $f_n \xrightarrow{L^p} f$ MEANS $\|f_n - f\|_p \to 0$, $1 \leq p \leq \infty$

2) a.e. OR a.s. CONVERGENCE: $f_n \xrightarrow{a.s.} f$ MEANS THAT

$$\mu\left(\left\{\omega : \lim_{n\to\infty} f_n(\omega) \neq f(\omega)\right\}\right) = 0$$

3) CONVERGENCE IN MEASURE: $f_n \xrightarrow{\mu} f$ OR $f_n \xrightarrow{P} f$ MEANS

$$\forall \varepsilon > 0 \quad \lim \mu\left(\left\{\omega : |f_n(\omega) - f(\omega)| > \varepsilon\right\}\right) = 0$$

L^p CONVERGENCE $\not\Leftrightarrow$ a.s. CONVERGENCE

EXAMPLE: $\Omega = [0,1]$, $\mathcal{F}$ = BOREL σ-FIELD ON $(0,1)$, λ LEBESGUE

CONSIDER THE SEQUENCE OF FUNCTIONS $1, \ 1_{(0,1/2]}, \ 1_{(1/2,1)}$

$1_{(0,1/3]} \ 1_{(1/3,2/3]} \ 1_{(2/3,1)}$

$1_{(0,1/4]} \ 1_{(1/4,1/2]} \ \dots$

- THESE FUNCTIONS CONVERGE TO 0 IN L^p FOR ALL p
- HOWEVER $f_n \overset{a.s.}{\not\to} 0$ (IT DOES NOT CONVERGE AT ALL):

$\forall \omega \in \Omega \qquad f_n(\omega) = 1$ AND 0 i.o.

- $f_n \xrightarrow{P} 0$ BECAUSE $\forall \varepsilon > 0$ $\quad \lambda(\{\omega : |f_n(\omega)| > \varepsilon\}) = \frac{1}{k}$ IF

$$\frac{k(k+1)}{2} < n \le \frac{(k+1)(k+2)}{2}$$

EXAMPLE. SAME SETTING OF PREVIOUS EXAMPLE

$$f_n(\omega) = \begin{cases} n & 0 < \omega < 1/n \\ 0 & \text{OTHERWISE} \end{cases}$$

- THEN $f_n(\omega) \to 0$ ALL $\omega \in \Omega$ (SO $f_n \xrightarrow{a.s.} 0$)

BUT f_n DOES NOT CONVERGE TO ZERO IN L^p BECAUSE

$$\|f_n\|_p = n^{p-1} \to \infty$$

- OF COURSE $f_n \xrightarrow{P} 0$

Proposition LET $(\Omega, \mathcal{F}, P)$ BE A PROB. SPACE. IF $f_n \xrightarrow{L^\infty} f$ THEN $f_n \xrightarrow{a.e} f$

PF/ LET $A_n = \{\omega : |f_n(\omega)| \le \|f_n\|_\infty\}$. THEN $P(A_n) = 1$ ALL n

AND $P(\bigcap_n A_n) = 1$. SO $\forall \omega \in \bigcap_n A_n$

$$f_n(\omega) \le \|f_n\|_\infty \to 0$$

THIS PROVES THE RESULT FOR $f=0$. WHEN f IS NOT THE ZERO FUNCTION

USE THESE ARGUMENTS ON $f_n - f$. ■

PARTIAL CONVERSE (EGOROFF THEOREM): $(\Omega, \mathcal{F}, \mu)$, $\mu(\Omega) < \infty$.

IF $f_n \xrightarrow{a.s.} f$ THEN $\forall \varepsilon > 0$ $\exists A \in \mathcal{F}$ ST. $\mu(A) < \varepsilon$ AND

$$\sup_{\omega \in A^c} |f_n(\omega) - f(\omega)| \to 0.$$ ALMOST UNIFORM CONVERGENCE

L^p CONVERGENCE $\Longrightarrow$ CONVERGENCE IN PROBABILITY

BECAUSE IF $\|X_n - X\|_p \to 0$ THEN, BY MARKOV'S INEQ.

$$\mathbb{P}(|X_n - X| > \varepsilon) \leq \frac{\|X_n - X\|_p^p}{\varepsilon^p} \to 0 \qquad \forall \varepsilon > 0$$

CONVERGENCE WITH PROB 1 $\Longrightarrow$ CONVERGENCE IN PROBABILITY

(a.e. CONVERGENCE)

TO SEE THIS RECALL THAT $X_n \xrightarrow{a.e.} X$ WHEN

$$P(|X_n - X| > \varepsilon \text{ i.o.}) = 0 \qquad \forall \varepsilon > 0$$

[BECAUSE $\{\omega : \lim_{n \to \infty} X_n(\omega) = X(\omega)\}^c = \bigcup_{\varepsilon} \{\omega : |X_n(\omega) - X(\omega)| > \varepsilon \text{ i.o.}\}$]

ε ↳ RATIONAL

LET $\varepsilon > 0$ BE ARBITRARY. SET

$$A_{n,\varepsilon} = \{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}.$$ THEN, SINCE $X_n \xrightarrow{a.s.} X_n$,

$$P(\limsup A_{n,\varepsilon}) = 0$$

FROM THIS WE CONCLUDE THAT $\lim_{n \to} P(A_{n,\varepsilon}) = 0$ BECAUSE

$$\lim_{n \to \infty} P(A_{n,\varepsilon}) \leq \limsup_{n \to \infty} P(A_{n,\varepsilon}) \leq P(\limsup_n A_{n,\varepsilon}) = 0.$$

REMARK IF WE ARE DEALING $(\Omega, \mathcal{F}, \mu)$ AND $\mu(\Omega) = \infty$.

THEN a.e. CONVERGENCE DOES NOT IMPLY CONVERGE IN MEASURE.

HERE IS AN EXAMPLE: $\Omega = \mathbb{R}$, $\mu = \lambda$ (LEBESGUE MEASURE)

$\mathcal{F} = \mathcal{B}^1$

$$f_n(\omega) = 1_{[n,\infty)}(\omega)$$

THEN $f_n \xrightarrow{a.e.} 0$ BUT $\lambda(\{\omega : f_n(\omega) > \varepsilon\}) = \infty$ $\forall \varepsilon > 0$.

$$\begin{tikzcd}
\boxed{X_n \xrightarrow{a.s.} X} \arrow[dr] & & \boxed{X_n \xrightarrow{L^p} X} \arrow[dl] \\
& \boxed{X_n \xrightarrow{P} X} &
\end{tikzcd}$$

<u>SCHEFFE' LEMMA</u>: LET μ_n AND μ BE MEASURES ON $(\Omega, \mathcal{F})$

S.T. $\mu_n(A) = \int_A f_n \, d\nu$ AND $\mu(A) = \int_A f \, d\nu$ SOME

σ-FINITE MEASURE ν ON $(\Omega, \mathcal{F})$, $\forall A \in \mathcal{F}$.

IF $\mu_n(\Omega) = \mu(\Omega) < \infty$ ALL n AND $f_n \xrightarrow{a.e.} f$, $\rightarrow [\nu]$

THEN

$d_{TV}(\mu_n, \mu)$

$$\underline{\sup_{A \in \mathcal{F}} |\mu_n(A) - \mu(A)|} \le \int_\Omega |f_n - f| \, d\nu \to 0$$

PF/ LET $g_n = f_n - f \Rightarrow g_n^+ = \max\{g_n, 0\} \to 0$ a.e. $[\nu]$

AND $g_n^+ \le f$ ALL n

SO BY DCT $\int_\Omega g_n^+ \, d\nu \to 0$. NEXT $\int_\Omega g_n \, d\nu = 0$ ALL n.

SO

$$\int_\Omega |g_n| \, d\nu = \int_{\{g_n \ge 0\}} g_n \, d\nu - \int_{\{g < 0\}} g_n \, d\nu$$

$$= 2 \int_{\{g_n \ge 0\}} g_n \, d\nu = 2 \int_\Omega g_n^+ \, d\nu \to 0 \quad \blacksquare$$

◼ ALMOST SURE CONVERGENCE AND STRONG LAW OF LARGE NUMBERS (SLLN)

<u>Thm</u> (KOLMOGOROV 0-1 LAW). LET $\{X_n\}$ BE A SEQUENCE OF INDEP. RV'S. LET $\mathcal{T}_n = \sigma(\{X_i, i \geq n\})$ AND $\mathcal{T} = \bigcap_n \mathcal{T}_n$. $\mathcal{T}$ IS CALLED THE TAIL σ-FIELD. THE, $\forall A \in \mathcal{T}$, THE PROB. OF A IS EITHER 1 OR 0.

Pf/ $\mathcal{U}_n = \sigma(\{X_i, i \leq n\})$ AND $\mathcal{U} = \bigcup_n \mathcal{U}_n$. LET $B \in \mathcal{U}$ SO $B \in \mathcal{U}_n$ FOR SOME n. LET $A \in \mathcal{T}$. BECAUSE $A \in \mathcal{T}_{n+1}$, A AND B ARE INDEP. EVENTS.

SO $\mathcal{U}$ AND $\mathcal{T}$ ARE INDEPENDENT COLLECTION OF MEAS. SETS.

WE ALSO HAVE THAT $\sigma(\mathcal{U})$ AND $\mathcal{T}$ ARE INDEP.

BUT $\mathcal{T} \subseteq \sigma(\mathcal{U})$ SO $\mathcal{T}$ IS INDEP OF ITSELF AND

$\forall A \in \mathcal{T}$, $P(A) = P(A \cap A) = (P(A))^2 \Rightarrow P(A) = 1$ OR 0. ◼

<u>BOREL-CANTELLI LEMMAS</u>

1) <u>FIRST BOREL-CANTELLI LEMMA</u>: $(\Omega, \mathcal{F}, \mu)$. IF $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ THEN $\mu(\limsup A_n) = 0$ $[\mu(A_n \text{ i.o.}) = 0]$.

2) <u>SECOND BOREL-CANTELLI LEMMA</u>: $(\Omega, \mathcal{F}, P)$. IF $\sum_{n=1}^{\infty} P(A_n) = \infty$ AND $\{A_n\}$ ARE MUTUALLY INDEP EVENTS, THEN

$$P(\limsup A_n) = P(A_n \text{ i.o.}) = 1$$

Corollary IF $X_n \xrightarrow{P} X$ THEN $\exists$ SUBSEQUENCE $\{X_{n_k}\}_k$ S.T. $X_{n_k} \xrightarrow{a.s.} X$.

Pf/ LET $\{n_k\}_k$ AN INCREASING SEQUENCE S.T.

$$\mathbb{P}\left(|X_{n_k} - X| > 1/2^k \right) < 1/2^k$$

BECAUSE $\sum_{k=1}^{\infty} \mathbb{P}\left(|X_{n_k} - X| > 1/2^k \right) < \infty$ THEN, BY FIRST BC LEMMA

$$\mathbb{P}\left(|X_{n_k} - X| > 1/2^k \text{ i.o.} \right) = 0$$

$\hookrightarrow$ $X_{n_k} \to X$ WITH PROB 1. [TAKE $A = \{ |X_{n_k} - X| > 1/2^k \text{ i.o.} \}$ THEN $P(A^c) = 1$]

$\blacksquare$

THESE RESULTS ALLOW US TO DERIVE:

Thm (COMPLETENESS OF L^p SPACES). EACH L^p SPACE, $1 \le p \le \infty$, IS COMPLETE.

Def LET $(\mathcal{X}, d)$ BE A METRIC SPACE. A SEQUENCE $\{x_n\}_n$ IS $\mathcal{X}$ IS SAID TO BE A CAUCHY SEQUENCE IF $\forall \varepsilon > 0$, $\exists N = N(\varepsilon)$ S.T. $d(x_n, x_m) < \varepsilon$ $\forall n, m$