36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 5 : WED, SEP 16, 2020

- LET $F : \mathbb{R}^k \to \mathbb{R}$ S.T.

  1) F IS RIGHT-CONTINUOUS: IF $x \downarrow y$ (MEANING $x_i \downarrow y_i$ FOR ALL $i = 1, \dots, k$)

MONOTONICITY $\rightarrow$ 2) $\Delta_A F \geq 0$ FOR ALL HYPER-RECTANGLES

$$A = (a_1, b_1] \times \dots \times (a_k, b_k] \qquad -\infty \leq a_i \leq b_i < \infty$$

THEN, THERE EXISTS A UNIQUE MEASURE μ ON $(\mathbb{R}^k, \mathcal{B}^k)$ S.T.

BOREL σ-FIELD

$$\mu(A) = \Delta_A F$$

FOR ALL HYPER-RECTANGLES A.

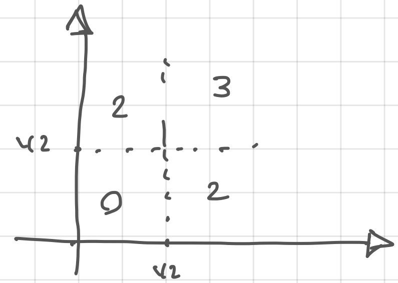
- IF $F(x) = \prod_{i=1}^{k} x_i$ THEN THE CORRESPONDING μ IS THE LEBESGUE MEASURE, USUALLY DENOTED WITH λ. IN PARTICULAR IF A IS A HYPER-RECTANGLE,

$$\lambda(A) = \prod_{i=1}^{k} (b_i - a_i) \qquad \text{VOLUME OF } A.$$

THIS IS TRUE FOR ALL $A \in \mathcal{B}^k$.

- <u>EXAMPLE OF WHY CONDITION 2 IS NEEDED</u>.

LET $F : \mathbb{R}^2 \to \mathbb{R}$ BE S.T.



$$F(x,y) = \begin{cases} 3 & \min\{x,y\} \geq 1/2 \\ 0 & \max\{x,y\} > 1/2 \\ 2 & \text{OTHERWISE} \end{cases}$$

$A = (1/4, 3/4] \times (1/4, 3/4]$ THEN

$$\Delta_A F = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$$
$$= 3 - 2 - 2 + 0$$
$$= -1$$

MEASURABLE FUNCTIONS

$(\Omega, \mathcal{F}, P)$ $\qquad$ $(S, \mathcal{A})$

PROBABILITY $\qquad$ MEASURABLE SPACE

- LET $f : \Omega \to S$. CAN f AND P BE USED TO CONSTRUCT A MEASURE (PROBABILITY) ON $(S, \mathcal{A})$?

$[a,b]$

<u>EXAMPLE</u>: $f : \Omega \to \mathbb{R}$ THEN FOR SOME "A" $\in \mathcal{A}$ WHAT IS THE PROBABILITY THAT THE IMAGE OF f IS IN A?

- ANOTHER REASON IS TO DEFINE THE NOTION OF INTEGRAL

Def (MEASURABLE FUNCTION): LET $(\Omega, \mathcal{F})$ AND $(S, \mathcal{A})$ BE TWO MEASURABLE SPACES. A FUNCTION $f: \Omega \to S$ IS $\mathcal{F}/\mathcal{A}$ MEASURABLE OR MEASURABLE WHEN, FOR EACH $A \in \mathcal{A}$, $f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in A\}$ IS IN $\mathcal{F}$. (IF PRE-IMAGE OF ANY MEASURABLE SET IS MEASURABLE).

EXAMPLES

1) IF $\mathcal{F} = 2^{\Omega}$, THEN ANY f IS MEASURABLE.
2) IF $\mathcal{A} = \{\emptyset, S\}$, THEN ANY FUNCTION f IS MEASURABLE.

REMARK: • f MEASURABLE DOES NOT IMPLY THAT, FOR ANY $B \in \mathcal{F}$,

$$f(B) = \{s \in S : s = f(\omega) \text{ some } \omega \in \Omega\}$$

IS MEASURABLE.

CONSIDER EXAMPLE 2): IF $B \in \mathcal{F}$ S.T. $f(B) \neq \emptyset$ AND $f(B) \neq S$ THEN f IS NOT MEASURABLE.

• ANOTHER REASON WHY MEASURABILITY IS DEFINED THROUGH f^{-1} IS THE FOLLOWING: IF $\{A_\alpha\}_{\alpha \in I}$ IS A COLLECTION OF MEASURABLE SUBSETS OF S, INDEXED BY AN ARBITRARY SET I,

i) $$f^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(A_\alpha)$$

ii) $$f^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right) = \bigcap_{\alpha \in I} f^{-1}(A_\alpha)$$

iii) $$f^{-1}(A^c) = \left(f^{-1}(A)\right)^c \quad \text{ALL } A \in \mathcal{A}$$

IN PARTICULAR, NOTICE THAT, IN GENERAL, $f(B^c) \neq (f(B))^c$

[EXAMPLE: TAKE $f(\omega) = c$ CONSTANT ALL $\omega \in \Omega$]

Def (σ-FIELD GENERATED BY f). LET f BE A MEAS. FUNCTION FROM $(\Omega, \mathcal{F})$ INTO $(S, \mathcal{A})$. THE COLLECTION OF SETS

$$\{ f^{-1}(A),\ A \in \mathcal{A} \}$$ → SHOW THIS IS A σ-FIELD

IS THE σ-FIELD GENERATED BY f, INDICATED WITH $\sigma(f)$.
↓ OVER Ω

- $\sigma(f)$ IS THE SMALLEST σ-FIELD S.T. f IS $\sigma(f)/\mathcal{A}$ MEAS.

Proposition LET $f: \underset{(\Omega, \mathcal{F})}{\Omega} \to \underset{(S, \mathcal{A})}{S}$. SUPPOSE THAT $\mathcal{A} = \sigma(\mathcal{C})$, SOME COLLECTION $\mathcal{C}$ OF SUBSETS OF S. THEN f IS $\mathcal{F}/\mathcal{A}$ MEAS. IFF $f^{-1}(\mathcal{C}) = \{ f^{-1}(C),\ C \in \mathcal{C} \} \subseteq \mathcal{F}$.

IF DIRECTION:

Pf/ THE COLLECTION $\mathcal{A}' = \{ A \in \mathcal{A} : f^{-1}(A) \in \mathcal{F} \}$ IS A σ-FIELD. $\mathcal{A}'$ BY DEFINITION CONTAINS $\mathcal{C}$, SO $\mathcal{A} = \sigma(\mathcal{C}) \subseteq \mathcal{A}'$. BUT $\mathcal{A}' \subseteq \mathcal{A} \Rightarrow \mathcal{A}' = \mathcal{A}$. ∎

Corollary IF f IS A CONTINUOUS FUNCTION FROM A TOPOLOGICAL SPACE TO ANOTHER, THEN f IS MEASURABLE, WITH RESPECT TO THE BOREL σ-FIELDS.

Pf/ TRUE BECAUSE, BY CONTINUITY, $f^{-1}(U)$ IS OPEN WHENEVER U IS OPEN ∎

- A GENERAL WAY TO CHECK MEASURABILITY OF FUNCTIONS TAKING VALUES IN $\mathbb{R}^k$. LET $f: (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$. TO PROVE THAT f IS MEASURABLE, WE ONLY NEED TO CHECK THAT THE PRE-IMAGE OF SETS OF THE FORM $(-\infty, a]$ IS MEAS.

WE COULD CONSIDER SETS OF THE FORM $(-\infty, a)$ OR $(a,b]$ OR (a,b) OR $[a, \infty)$ OR (a, ∞).

EXAMPLE IF $f: \mathbb{R} \to \mathbb{R}$ AND IS MONOTONE $\Rightarrow f$ IS MEAS.
ENDOWED BY BOREL σ-FIELDS.

- EXTENDED REAL VALUED FUNCTIONS: $f: \Omega \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

THEN $f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} \{\omega : f(\omega) > n\}$

- FOR FUNCTIONS OF THE FORM $f: (\Omega, \mathcal{F}) \to (\mathbb{R}^k, \mathcal{B}^k)$, WRITE

$$f(\omega) = \begin{bmatrix} f_1(\omega) \\ f_2(\omega) \\ \vdots \\ f_k(\omega) \end{bmatrix}.$$

THEN, f IS MEAS. IIF EACH f_i IS MEAS.

Pf/ ASSUME EACH f_i IS MEAS. THEN, WE ONLY NEED TO WORRY ABOUT THE PRE-IMAGE OF HYPER-RECTANGLES, BECAUSE

$$\mathcal{B}^k = \sigma\left(\left\{ (a_1, b_1] \times \dots \times (a_k, b_k] \;,\; \underset{\text{ALL } i}{a_i < b_i} \right\} \right)$$

↳ OTHER CHOICES ARE POSSIBLE. FOR EXAMPLE

$$\sigma\left(\left\{ (-\infty, a_1] \times \dots \times (-\infty, a_k] \;\; \underset{\text{ALL } i}{a_i \in \mathbb{R}} \right\} \right)$$

(*) $$f^{-1}\left((a_1, b_1] \times \dots \times (a_k, b_k] \right) = \{\omega : f_i(\omega) \in (a_i, b_i] \text{ ALL } i\}$$

$$= \bigcap_{i=1}^{k} f_i^{-1}\left((a_i, b_i] \right)$$

WHICH IS MEAS. BECAUSE EACH $f_i^{-1}((a_i, b])$ IS MEAS.

IF f IS MEASURABLE, THEN WE CLAIM THAT EACH SET OF THE FORM $f_i^{-1}((a_i, b_i])$ IS MEAS. $\Rightarrow$ EACH f_i IS MEAS.

TO SEE THIS, LOOK AT (*). FIX A COORDINATE, SAY i.

FOR ALL $j \neq 1$, LET $a_j \to -\infty$ AND $b_j \to \infty$, THEN

$$\bigcap_{e=1}^{n} f_e^{-1}\left(\left(a_e, b_e\right]\right) \uparrow f_1^{-1}\left(\left(a_1, b_1\right]\right)$$

SO $f^{-1}\left(\left(a_1, b_1\right]\right)$ IS MEAS. ∎

Properties of meas. real-valued functions:

i) IF f IS MEAS., THEN SO IS $a \cdot f$ ANY $a \in \mathbb{R}$.

APPLIES TO GENERAL FUNCTIONS ← ii) IF $f : \Omega \to S$ AND $g : S \to T$ ARE MEAS., SO IS

$g(f)$ OR $g \circ f : \Omega \to T$ [BECAUSE $[g(f)]^{-1}(B) = g^{-1}(f^{-1}(B))$]

iii) IF f AND g ARE REAL VALUED FUNCTIONS, SO IS

$f+g$, $\max\{f, g\}$, $\sqrt{|f-g|}$, ...

COMPOSITION OF $h(x,y) = x+y$ AND $\begin{bmatrix} f \\ g \end{bmatrix}$ TAKING VALUES IN $\mathbb{R}^2$.