36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 16: MON, OCT 26, 2020

Thm (COMPLETENESS OF L^p SPACES). L^p IS COMPLETE, FOR EACH $1 \leq p \leq \infty$.

↳ USES 1^{st} BOREL-CANTELLI + FATOU LEMMA

Def LET $(\mathcal{X}, d)$ BE A METRIC SPACE. A SEQUENCE $\{x_n\}_n$ IS $\mathcal{X}$ IS SAID TO BE A CAUCHY SEQUENCE IF

$\forall \varepsilon > 0$, $\exists N = N(\varepsilon)$ S.T. $d(x_n, x_m) < \varepsilon$ $\forall n, m > N$.

IF A SEQUENCE IS CONVERGENT ($x_n \to x$, SOME $x \in \mathcal{X}$), THEN IT IS ALSO CAUCHY. $(\mathcal{X}, d)$ IS COMPLETE WHEN EVERY CAUCHY SEQUENCE IS CONVERGENT.

EXAMPLE: $\mathcal{X} = \mathbb{Q}$ (SET OF RATIONALS). $d(x,y) = |x-y|$

LET $x_n = \left(1 + \frac{1}{n}\right)^n \in \mathbb{Q}$ ALL n

$x_n \to e$ AS $n \to \infty$

BUT $e \notin \mathbb{Q}$

REMARK: IF $\{x_n\}$ IS A SEQUENCE IN A COMPLETE SPACE THAT IS CAUCHY, AND IF A SUBSEQUENCE CONVERGES TO $x \in \mathcal{X}$

THEN THE WHOLE SEQUENCE CONVERGES TO x.

Corollary IF $\{f_n\}$ IS A SEQUENCE IN L^p S.T. $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$

THEN $\sum_n f_n \in L^p$

$\left[\text{ LET } \quad g_k = \sum_{n=1}^{k} f_k \right]$

$$\lim_{k \to \infty} \| g_k - g_\infty \|_p = 0$$

$$\Downarrow$$

$$\sum_n f_n$$

Thm (BASIC L^2 THEOREM) LET $X_1, X_2, \ldots$ BE A SEQUENCE OF INDEP. RV'S ST. $\mathbb{E}[X_n] = 0$ AND $\mathrm{Var}[X_n] = \sigma_n^2$. LET $S_n = \sum_{n=1}^{n} X_n$. IF $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$ THEN $S_n \to S_\infty$ a.e. AND IN L_2. FOR SOME S_∞ S.T. $\mathbb{E}[S_\infty^2] = \sum_{n=1}^{\infty} \sigma_n^2$.

Pf/ FOR L^2 CONVERGENCE, THIS FOLLOWS FROM THE FACT THAT L^2 IS COMPLETE (SEE COROLLARY ABOVE).

TO SHOW a.s. CONVERGENCE, WE WILL NEED THIS RESULT:

(KOLMOGOROV'S MAXIMAL INEQ.): LET $X_1, \ldots X_n$ BE INDEP. RV'S WITH MEAN 0 AND FINITE VARIANCE. THEN

$$\mathbb{P}\left(\max_{1 \le k \le n} |S_k| \ge \varepsilon \right) \le \frac{\mathrm{Var}[S_n]}{\varepsilon^2}, \quad \varepsilon > 0.$$

Pf (OF KOLGOROV MAX INEQ.) / $n > 1$. LET, FOR SOME $\varepsilon > 0$,

$$A_k = \{ |S_k| \ge \varepsilon \text{ AND } |S_j| < \varepsilon \text{ ALL } j = 1, \ldots k \}$$

THEN $A_1, A_2, \ldots, A_n$ ARE DISJOINT AND

$$\{ \max_n |S_k| \ge \varepsilon \} = \biguplus_{k=1}^{n} A_k$$

SO $(\Omega, \mathcal{F}, P)$

$$\mathbb{E}\left[S_n^2\right] \geq \sum_{k=1}^{n} \int_{\{\omega \in \Omega,\ \omega \in A_k\}} S_n^2(\omega)\, dP(\omega)$$

$$= \sum_{k=1}^{n} \int_{A_k} S_n^2\, dP$$

$$= \sum_{k=1}^{n} \int_{A_k} \Big[S_k^2 + 2 S_k (S_n - S_k) + \underbrace{(S_n - S_k)^2}_{\geq 0}\Big]\, dP$$

$$\geq \sum_{k=1}^{n} \int_{A_k} S_k^2 + 2 S_k (S_n - S_k)\, dP$$

$$= \sum_{k=1}^{n} \int_{A_k} S_k^2\, dP$$

[BECAUSE $1_{A_k} S_k$ AND $(S_n - S_k)$ ARE INDEP. AND $\mathbb{E}[S_n - S_k] = 0$]

BY DEFINITION OF A_k $\leftarrow$ $$\geq \varepsilon^2 \sum_{k=1}^{n} P(A_k)$$

$$= \varepsilon^2\, \mathbb{P}\left(\max_k |S_k| \geq \varepsilon\right)$$

[A_k'S ARE DISJOINT]

$\blacksquare$

BACK TO THE PROOF THAT $S_n \xrightarrow{a.s.} S_\infty$.

LET $M_n = \sup_{p,q \geq n} |S_p - S_q|$. NOW, IF $\forall \varepsilon > 0$

$\mathbb{P}(M_n > \varepsilon) \to 0$ AS $n \to \infty$, THEN $M_n \downarrow 0$ a.s.

THIS CLAIM FOLLOWS FROM THIS RESULT: $X_n \xrightarrow{a.s.} 0$ IF AND ONLY IF $\sup_{k \geq n} |X_k| \xrightarrow{P} 0$ [IN YOUR HW]

LET $M_n^* = \sup_{p \geq n} |S_p - S_n|$. THEN, BY TRIANGLE INEQ.,

$$|S_p - S_q| \leq |S_p - S_n| + |S_q - S_n|$$

SO $M_n^* \leq M_n \leq 2 M_n^*$

SO, WE ONLY NEED TO SHOW THAT $M_n^* \xrightarrow{P} 0$. NEXT

$$\mathbb{P}\left(\sup_{p \geq n} |S_p - S_n| > \varepsilon \right) = \lim_{N \to \infty} \mathbb{P}\left(\max_{n \leq p \leq N} \underbrace{|S_p - S_n|}_{\sum_{p=1}^{N} X_{n+p}} > \varepsilon \right)$$

BY CONTINUITY

$$\leq \lim_{N \to \infty} \sum_{p=n+1}^{N} \frac{\sigma_p^2}{\varepsilon^2}$$

$$= \sum_{p=n+1}^{\infty} \frac{\sigma_p^2}{\varepsilon^2}$$

THE LAST TERM CONVERGES TO ZERO AS $n \to \infty$ BECAUSE

$\sum_{p=1}^{\infty} \sigma_p^2 < \infty$. ∎

PLEASE READ UP THE 3-SERIE THEOREM

SLLN (ALMOST SURE OF SAMPLE AVERAGES).

Lemma (KRONECKER'S LEMMA) LET $\{x_n\}$ BE A SEQUENCE OF NUMBERS AND $\{a_n\}$ ANOTHER SEQUENCE OF POSITIVE NUMBERS S.T. $a_n \uparrow \infty$ AND $\sum_{k=1}^{\infty} \frac{x_k}{a_k} < \infty$. THEN $\frac{\sum_{k=1}^{n} x_k}{a_n} \to 0$.

Corollary: LET $X_1, X_2, \ldots$ BE INDEPENDENT, MEAN ZERO RV'S AND LET $S_n = \sum_{k=1}^{n} X_k$. IF $\sum_{k=1}^{\infty} \frac{\mathbb{E}[X_k^2]}{a_k^2} < \infty$ THEN BY THE L^2/a.s. CONVERGENCE THEOREM,

$$\frac{S_n}{a_n} \xrightarrow{a.s.} 0$$

IF $\mathbb{E}[X_n^2] = \sigma^2$ TAKE $a_n = n$ AND CONCLUDE THAT $\frac{S_n}{n} \xrightarrow{a.s.} 0$

IN FACT WE CAN TAKE $a_n = n^{1/2+\delta}$, $\delta > 0$. THEN, SINCE

$$\sum_{k=1}^{\infty} \frac{\sigma^2}{k^{1+2\delta}} < \infty, \text{ THEN } \frac{S_n}{n^{1/2+\delta}} \xrightarrow{a.s.} 0$$

<u>KOLMOGOROV'S SLLN</u>: LET $X_1, X_2, \ldots$ BE iid S.T. $\mathbb{E}|X_1| < \infty$.

THEN $\frac{S_n}{n} \xrightarrow{a.s.} \mathbb{E}[X_1]$ AS $n \to \infty$.

PROOF USES TRUNCATION, DCT, L^2/a.s. CONVERGENCE THEOREM.
BOREL-CANTELLI.

LAW OF ITERATED LOGARITHMS

$X_1, X_2, \ldots$ iid MEAN ZERO, COMMON VARIANCE σ^2.

WE JUST SAW THAT $\frac{S_n}{n^{1/2+\delta}} \xrightarrow{a.s.} 0$ $\forall \delta > 0$.

<u>LAW OF ITERATED LOGARITHMS</u>:

$$\left.\begin{aligned} \limsup_{n\to\infty} \frac{S_n}{\sigma\sqrt{n \log(\log(n))}} &= \sqrt{2} \\ \liminf_{n\to\infty} \frac{S_n}{\sigma\sqrt{n \log(\log(n))}} &= -\sqrt{2} \end{aligned}\right\} \text{a.s.}$$

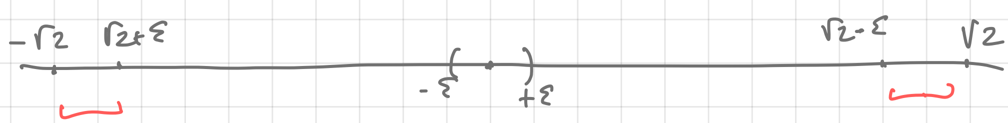
INTERPRET THIS PLOT AS A SINGLE REALIZATION OF $\to$ PATH

$$\omega \mapsto \left(S_1(\omega), S_2(\omega), \ldots \right)$$

IT IS POSSIBLE TO SHOW THAT $Z_n = \dfrac{S_n}{\sigma\sqrt{n \log(\log(n))}} \xrightarrow{P} 0$

SO $Z_n \xrightarrow{P} 0$ AND $\left.\begin{array}{l} \limsup Z_n = \sqrt{2} \\ \liminf Z_n = -\sqrt{2} \end{array}\right\}$ a.s.

THIS MEANS THAT $\forall \varepsilon > 0$ SMALL, FOR EACH ω OUTSIDE OF A SET OF PROBABILITY ZERO $Z_n(\omega) \in (\sqrt{2}-\varepsilon, \sqrt{2}) \cup (-\sqrt{2}, -\sqrt{2}+\varepsilon)$ I.O., AT THE SAME TIME FOR EACH n LARGE ENOUGH THE SET OF ω's S.T. $Z_n(\omega) \in (-\varepsilon, \varepsilon)$ HAS PROB. CLOSE TO 1.



- SIMILARLY: ASSUME X_i's ARE IID $N(\mu, 1)$. THEN

$$\frac{S_n}{n} \pm \frac{2}{\sqrt{n}} \text{ IS A } \sim 95\% \text{ CI FOR } \mu.$$

ASSUME $\mu = 0$. THEN THE INTERVAL WILL CONTAIN 0 IF

$$\left|\frac{S_n}{n}\right| \leq \frac{2}{\sqrt{n}} \iff |Z_n| \leq \frac{\sqrt{2}}{\sqrt{\log \log n}}$$

BUT FOR n LARGE ENOUG $\dfrac{Z_n}{\sqrt{\log \log n}} << (\sqrt{2}-\varepsilon, \sqrt{2})$.

$\hookrightarrow$ FOR EACH ω, WE WILL BE OUTSIDE OF CI INFINITELY OFTEN !!