36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 24 : MON, NOV 23, 2020

Def LET $(\Omega, \mathcal{F}, P)$ BE A PROBABILITY SPACE AND $\mathcal{C} \subseteq \mathcal{F}$ A SUB-σ-FIELD. LET X BE A RV THAT IS $\mathcal{F}/\mathcal{B}^1$ MEAS. S.T. $\mathbb{E}|X| < \infty$. LET $\mathbb{E}[X \mid \mathcal{C}]$ STANDS FOR ANY FUNCTION $h : \Omega \to \mathbb{R}$ THAT IS $\mathcal{C}/\mathcal{B}^1$ MEAS. S.T.

$$(*) \qquad \int_C h \, dP = \int_C X \, dP \qquad \text{FOR ALL } C \in \mathcal{C}.$$

WE CALL SUCH A FUNCTION h A VERSION OF THE CONDITIONAL EXPECTATION OF X GIVEN $\mathcal{C}$.

$$\omega \longmapsto \mathbb{E}[X \mid \mathcal{C}](\omega)$$

$\downarrow$ UNIQUE a.e. P

$\hookrightarrow$ ITSELF A RANDOM VARIABLE MEAS. WRT $\mathcal{C}$

LAST TIME : CONDITIONAL EXPECTATION DEFINED AS A PROJECTION OF X ONTO THE SPACE $L^2(\Omega, \mathcal{C}, P)$

<u>Corollary 13</u>: LET $X \in L^2(\Omega, \mathcal{F}, P)$ AND $\mathcal{C} \subseteq \mathcal{F}$ A SUB σ-FIELD. LET $Z \in L^2(\Omega, \mathcal{C}, P)$. THEN THE FOLLOWING ARE EQUIVALENT:

i) $Z = \mathbb{E}[X \mid \mathcal{C}]$ $\left[Z \text{ IS A VERSION OF } \mathbb{E}[X \mid \mathcal{C}]\right]$

ii) $\mathbb{E}[Z \cdot W] = \mathbb{E}[W \cdot X]$ FOR ALL $W \in L^2(\Omega, \mathcal{C}, P)$

$$\mathbb{E}[\,W \cdot (X - Z)\,] = 0$$

$\downarrow$

ORTHOGONAL PROJECTION OF X ONTO $L^2(\Omega, \mathcal{C}, P)$

SO $X - Z$ IS A "RESIDUAL" OF THIS PROJECTION

iii) Z IS THE ORTHOGONAL PROJECTION OF X ONTO $L^2(\Omega, \mathcal{C}, P)$

EXPLAINS WHY $\mathbb{E}[X \mid Y]$ MINIMIZES

$$\mathbb{E}\left[(X - g(Y))^2\right]$$

OVER ALL MEAS. g.

<u>EXAMPLES OF COND. EXPECTATIONS</u>: $(\Omega, \mathcal{F}, P)$ X S.T. $\mathbb{E}[|X|] < \infty$

AND Y ANOTHER RV.

<u>CASE i)</u> ASSUME Y IS DISCRETE : $\operatorname{supp}(Y) = \underbrace{\{y : Y(\omega) = y \text{ SOME } \omega\}}_{\text{RANGE OF } Y}$ IS COUNTABLE

$\mathbb{E}[X \mid Y] = ?$

$\mathbb{E}[X \mid \sigma(Y)]$

THEN $\mathbb{E}[X \mid Y] = g(Y)$, WHERE g TAKES VALUES IN $\operatorname{supp}(Y)$

AND IS GIVEN BY

$$y \longmapsto g(y) = \frac{1}{P(\underbrace{Y^{-1}(y)}_{\{\omega \,:\, Y(\omega) = y\}})} \int_{Y^{-1}(y)} X(\omega)\, dP(\omega)$$

IF $Y = 1_A$, $A \in \mathcal{F}$, THIS GIVES THE CONDITIONAL EXPECTATION OF X GIVEN EVENT A.

CASE 2): Y IS A CONTINUOUS R.V. LET X BE A R.V. S.T.

$$\mathbb{P}(X=1)=p \quad \text{AND} \quad \mathbb{P}(X=-1)=1-p, \quad p\in(0,1)$$

LET $Y = X + N$, $N \sim N(0,1)$ $N \perp\!\!\!\perp X$

$$\mathbb{E}[X \mid Y] = ?$$

WE KNOW THAT $\mathbb{E}[X \mid Y] = g(Y)$ SO WE NEED TO "GUESS" g AND VERIFY THE (*) PROPERTY FOR IT. I CLAIM THAT $g(\cdot)$ IS OF THE FORM:

$$y \in \mathbb{R} \longmapsto g(y) = \frac{p\, f_{Y|1}(y|1) - (1-p) f_{Y|-1}(y|-1)}{f_Y(y)}$$

WHERE $f_{Y|x}(y|x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y-x)^2}{2}\right\}$ $x \in \{1,-1\}$

AND

$$f_Y(y) = p\, f_{Y|1}(y|1) + (1-p)\, f_{Y|-1}(y|-1)$$

SO, WE NEED TO VERIFY THAT, FOR EACH $C \in \sigma(Y)$,

$$\mathbb{E}[X \mathbb{1}_C] = \mathbb{E}[X \mathbb{1}_{\{Y\in A\}}] = \mathbb{E}[\mathbb{1}_{\{Y\in A\}} g(Y)]$$

$A \in \mathcal{B}^1$ S.T. $Y^{-1}(A) = C$

SO

$$\mathbb{E}[\mathbb{1}_{\{Y\in A\}} g(Y)] = \int_{\mathbb{R}} \mathbb{1}_{\{y\in A\}} g(y)\, d\mu_Y(y)$$

$$= \int_{\mathbb{R}} \mathbb{1}_{\{y\in A\}} g(y)\, f_Y(y)\, dy$$

REPLACE $g(y)$ WITH ITS EXPRESSION AND USE FUBINI

$$= \int_{\mathbb{R}\times\{-1,1\}} \mathbb{1}_{\{y\in A\}}\, x \; d\mu_{X,Y}(x,y)$$

$$= \mathbb{E}[X \mathbb{1}_{\{Y\in A\}}]$$

$f_{Y|x}(y|x) \cdot P(X=x)$ IS THE RN DERIVATIVE OF THE DISTR. OF (X,Y) W.R.T λ + COUNTING MEASURE ON $\{-1,1\}$

PROPERTIES OF CONDITIONAL EXPECTATION:

"CONDITIONAL EXPECTATION BEHAVES LIKE EXPECTATION"

1) $$\mathbb{E}\left[\mathbb{E}[X \mid \mathcal{C}]\right] = \mathbb{E}[X]$$

"LAW OF ITERATED EXPECTATION": $\mathbb{E}\left[\mathbb{E}[X \mid Y]\right] = \mathbb{E}[X]$

MORE GENERALLY, WE HAVE THAT:

IF $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{F}$, $\mathcal{C}_1$ AND $\mathcal{C}_2$ σ-FIELDS, AND IF $\mathbb{E}[|X|] < \infty$, THEN $\mathbb{E}[X \mid \mathcal{C}_1]$ IS A VERSION OF

$$\mathbb{E}\left[\mathbb{E}[X \mid \mathcal{C}_2] \mid \mathcal{C}_1\right]$$

TOWER PROPERTY OF CONDITIONAL EXPECTATION

EXAMPLE: (X, Y, Z) THEN $\mathbb{E}[X \mid Y]$ IS A VERSION

$$\mathbb{E}\left[\mathbb{E}[X \mid \underbrace{Y, Z}_{\sigma(Y,Z)}] \mid Y\right]$$

2) LINEARITY: $\mathbb{E}[X + Y \mid \mathcal{C}] = \mathbb{E}[X \mid \mathcal{C}] + \mathbb{E}[Y \mid \mathcal{C}]$

IF X, Y AND $X+Y$ HAVE EXPECTATION

3) ASSUME THAT Y AND $X \cdot Y$ ARE INTEGRABLE AND Y IS $\mathcal{C}/\mathcal{B}'$-MEASURABLE. THEN

$Y \cdot \mathbb{E}[X \mid \mathcal{C}]$ IS A VERSION OF $\mathbb{E}[X \cdot Y \mid \mathcal{C}]$

4) CONDITIONAL EXPECTATION SATISFIES THE MCT, DCT AND JENSEN'S INEQ. HOLDS FOR CONDITIONAL EXPECTATION

IF $0 \le X_n \le X$ a.s. AND $X_n \to X$ a.s.

MCT

$$\mathbb{E}[X_n \mid \mathcal{C}] \xrightarrow{a.s.} \mathbb{E}[X \mid \mathcal{C}]$$

IF $X_n \xrightarrow{a.s.} X$ AND $|X_n| \le Y$ a.s., $\mathbb{E}[|Y|] < \infty$, THEN

$$\mathbb{E}[X_n | C] \xrightarrow{a.s} \mathbb{E}[X | C]$$ DCT

IF $\mathbb{E}[X]$ IS FINITE AND ϕ IS A CONVEX FUNCTION

S.T. $\phi(X)$ IS INTEGRABLE, THEN JENSEN

$$\mathbb{E}[\phi(X) | C] \ge \phi(\mathbb{E}[X | C]) \quad a.s$$

5) ASSUME $\mathbb{E}[X]$ AND $\sigma(X)$ IS INDEP OF C. THEN $\mathbb{E}[X]$ IS A VERSION OF $\mathbb{E}[X | C]$.

CONDITIONAL DISTRIBUTION

- FOR A $A \in \mathcal{F}$ WE DEFINE $\mathbb{P}(A | C) = \mathbb{E}[1_A | C]$.

  THIS IS WELL DEFINED FOR A FIXED A. WE WOULD LIKE THESE CONDITIONAL PROBABILITIES TO BEHAVE LIKE PROBABILITIES: THIS IS NOT AN OBVIOUS CONCLUSION BECAUSE WE ARE DEALING WITH $\omega \mapsto \mathbb{P}(A | C)(\omega)$, DEFINED OVER ALL $A \in \mathcal{F}$ AND $\omega \in \Omega$.

- IT IS EASY TO PROVE THAT a.s. $0 \le P(A | C) \le 1$ AND $P(\Omega | C) = 1$.

  BY LINEARITY IT IS ALSO POSSIBLE TO SHOW THAT, FOR ANY COUNTABLE SEQUENCE OF DISJOINT EVENTS $A_1, A_2, \dots$

$$\mathbb{P}(\biguplus_n A_n | C) = \sum_n \mathbb{P}(A_n | C) \quad a.s.$$

COUNTABLE ADDITIVITY

THE ABOVE STATEMENT HOLDS FOR ANY ω OUTSIDE OF A NULL SET $N(A_1, A_2, \dots)$ OF P- MEASURE ZERO.

SO THE SET OF ω'S FOR WHICH COUNTABLE ADDITIVITY FAILS IS

$$\bigcup \left\{ N(A_1, A_2, \dots) \ , \ \{A_1, A_2, \dots\} \ \begin{array}{l}\text{SEQUENCE OF} \\ \text{DISJOINT SETS} \\ \text{IN } \mathcal{F}\end{array} \right\}$$

<u>UNCOUNTABLE</u> UNION OF MEASURABLE SETS OF P- PROBABILITY ZERO

POSSIBLY NON- MEASURABLE OR OF POSITIVE MEASURE!

Def (CONDITIONAL DISTRIBUTION) $(\Omega, \mathcal{F}, P)$, X R.V. $\mathcal{C} \subseteq \mathcal{F}$

FOR EACH $B \in \mathcal{B}^1$, DEFINE

$$\omega \longmapsto \mu_{X|\mathcal{C}}(B)(\omega) = \mathbb{P}(X^{-1}(B) | \mathcal{C})(\omega)$$

A COLLECTION OF VERSIONS OF $\{\mu_{X|\mathcal{C}}(B)(\cdot), B \in \mathcal{B}^1\}$ IS CALLED A CONDITIONAL DISTRIBUTION OF X GIVEN $\mathcal{C}$.

IF IN ADDITION FOR EACH ω, $\mu_{X|\mathcal{C}}(\cdot)(\omega)$ IS A PROB. MEASURE ON $(\mathbb{R}, \mathcal{B}^1)$, THEN THIS COLLECTION IS CALLED A <u>REGULAR CONDITIONAL PROBABILITY</u>.

<u>REMARK</u> : A REG. COND. PROBABILITY IS A COLLECTION

$$\mu_{X|\mathcal{C}}(\cdot)(\cdot) : \mathcal{B}^1 \times \Omega \longmapsto [0,1] \quad \text{s.t.}$$

1) FOR EACH $B \in \mathcal{B}^1$ $\mu_{X|\mathcal{C}}(B)(\cdot)$ IS A VERSION OF

$$\mathbb{E}[\mathbb{1}_{X \in B} | \mathcal{C}]$$

2) FOR EACH FIXED ω

$$\mu_{X|\mathcal{C}}(\cdot)(\omega) \text{ IS A PROB. MEAS ON } \mathcal{B}^1.$$