36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 27: MON, DEC 7, 2020

- LAST TIME:

Thm (OPTIONAL STOPPING THEOREM) LET $(\{X_n\}, \{\mathcal{F}_n\})$ BE A MARTINGALE AND SUPPOSE THAT $\{\tau_k\}$ IS AN INCREASING SEQUENCE OF STOPPING TIMES ST. $\tau_k \leq M_k$ a.s. FOR SOME SEQUENCE OF NUMBERS $\{M_k\}$. THEN $(\{X_{\tau_k}\}, \{\mathcal{F}_{\tau_k}\})$ IS A MARTINGALE.

Pf/ wlog ASSUME $M_n \leq M_{n+1}$ ALL n.

FIRST WE SHOW THAT $\mathbb{E}[|X_{\tau_n}|] < \infty$. INDEED, SINCE $\tau_n \leq M_n$ a.s.,

$$\mathbb{E}[|X_{\tau_n}|] = \sum_{k=1}^{M_n} \underbrace{\mathbb{E}[|X_{\tau_n}|; \tau_n = k]}_{\int_{\{\tau_n = k\}} |X_{\tau_n}|(\omega)\, dP(\omega)} = \sum_{k=1}^{M_n} \mathbb{E}[|X_k|; \tau_n = k]$$

$$\leq \sum_{k=1}^{M_n} \mathbb{E}[|X_k|] < \infty$$

BECAUSE $\mathbb{E}[|X_n|] < \infty$ ALL n AND $M_n < \infty$

NEXT WE NEED TO SHOW THAT

$$\mathbb{E}[X_{\tau_{n+1}} \mid \mathcal{F}_{\tau_n}] = X_{\tau_n}$$

WE KNOW THAT X_{τ_n} IS $\mathcal{F}_{\tau_n}$ - MEAS. LET $A \in \mathcal{F}_{\tau_n}$ BE ARBITRARY.

WE NEED TO SHOW, USING THE DEFINITION OF CONDITIONAL EXPECTATION, THAT

$$\int_A X_{\tau_{n+1}} \, dP = \int_A X_{\tau_n} \, dP$$

WRITE

$$\int_A (X_{\tau_{n+1}} - X_{\tau_n}) \, dP = \int_{A \cap \{\tau_n < \tau_{n+1}\}} [X_{\tau_{n+1}} - X_{\tau_n}] \, dP$$

NEXT,

$$X_{\tau_{n+1}(\omega)} - X_{\tau_n(\omega)} = \sum_{\substack{k : \\ \tau_n(\omega) < k \leq \tau_{n+1}(\omega)}} X_k(\omega) - X_{k-1}(\omega)$$

SO

$$\int_A [X_{\tau_{n+1}} - X_{\tau_n}] \, dP = \int_A \sum_{k=2}^{M_{n+1}} \mathbb{1}_{\{\tau_n < k \leq \tau_{n+1}\}} (X_k - X_{k-1}) \, dP$$

NOTICE THAT $\{\tau_n < k \leq \tau_{n+1}\} = \underbrace{\{\tau_n \leq k-1\}}_{\in \mathcal{F}_{k-1}} \cap \underbrace{\{\tau_{n+1} \leq k-1\}^c}_{\in \mathcal{F}_{k-1}} \in \mathcal{F}_{k-1}$

SO

$$B_k = A \cap \{\tau_n < k \leq \tau_{n+1}\} \in \mathcal{F}_{k-1} \quad \text{FOR EACH } k$$

BECAUSE $A \in \mathcal{F}_{\tau_n}$ SO $A \cap \{\tau_n \leq k-1\} \in \mathcal{F}_{k-1}$

SO

$$\int_A \left[X_{\tau_{n+1}} - X_{\tau_n}\right] dP = \sum_{k=2}^{M_{n+1}} \int_{B_k} (X_k - X_{k-1})\, dP$$

↳ INTEGRANDS NO LONGER DEPEND ON STOPPING-TIMES

$$(\square) \qquad = \sum_{k=2}^{M_{n+1}} \int_{B_k} \left(X_k - \mathbb{E}\left[X_k \mid \mathcal{F}_{k-1}\right]\right) dP$$

$$= 0$$

BECAUSE, FOR EACH k, $\displaystyle\int_{B_k} X_k \, dP = \int_{B_k} \mathbb{E}\left[X_k \mid \mathcal{F}_{k-1}\right] dP$

$\blacksquare$

REMARK: THE SAME PROOF CAN BE USED FOR SUB- AND SUPER- MARTINGALES

ALL WE NEED TO DO IS REPLACE "=" IN $(\square)$ WITH A "$\geq$" OR "$\leq$" RESPECTIVELY.

AS A COROLLARY OF THIS RESULT, WE OBTAIN

Corollary (WALD'S IDENTITY) IF $(\{X_n\}, \{\mathcal{F}_n\})$ IS A MARTINGALE AND τ IS A BOUNDED STOPPING TIME (IN FACT ALL IS NEEDED IS THAT $\mathbb{E}[\tau] < \infty$), THEN

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_1]$$

IN PARTICULAR, IF $Y_1, Y_2, \ldots$ IID WITH MEAN μ THEN

$$\mathbb{E}[S_\tau] = \mu\, \mathbb{E}[\tau] \quad \text{WHERE} \quad S_n = \sum_{i=1}^{n} Y_i$$

PF/ $S_n - n\mu$ IS A ZERO MEAN MARTINGALE SO

$\mathbb{E}[S_\tau - \tau\mu] = 0$ BY WALD'S IDENTITY

$\blacksquare$

EXAMPLE $Y_1, Y_2, \ldots$ iid RADEMACHER RV'S. $\left[Y_n = \begin{cases} 1 & \text{WITH PROB } 1/2 \\ -1 & \text{"} \end{cases} \right]$

$a, b \in \mathbb{Z}$ FOR $a < 0 < b$ LET $\tau = \inf\{n : S_n \notin (a, b)\}$

$S_n = \sum_{i=1}^{n} Y_i$

THEN $\mathbb{E}[S_\tau] = 0$ $\left[\mathbb{E}[\tau] < \infty \right]$

SO

$$b \, \mathbb{P}(S_\tau = b) + a \, \mathbb{P}(S_\tau = a) = 0$$

SO

$$\mathbb{P}(S_\tau = b) = \frac{b}{b-a} \qquad \mathbb{P}(S_\tau = a) = -\frac{a}{b-a}$$

WE CAN ALSO GET SPECIAL RESULT FOR SUPER-MARTINGALES

__Lemma__: IF $(\{X_n\}, \{\mathcal{F}_n\})$ IS A NON-NEGATIVE SUPER-MARTINGALE AND $\tau \leq \infty$ A STOPPING TIME. THEN

$$\mathbb{E}[X_\tau] \leq \mathbb{E}[X_1]$$

PF/ FOR A FIXED $m \in \mathbb{N}$. THEN $\mathbb{E}[X_{\min\{\tau, m\}}] \leq \mathbb{E}[X_1]$ BY OPTIONAL STOPPING THEOREM.

NOW,

$$\mathbb{E}[X_\tau ; \tau < \infty] = \lim_{m \to \infty} \mathbb{E}[X_{\min\{\tau, m\}} ; \tau < m]$$

BY MCT

ALSO

$$\mathbb{E}[X_\tau ; \tau = \infty] \leq \liminf_{m \to \infty} \mathbb{E}[X_{\min\{\tau, m\}} ; \tau > m]$$

BY FATOU'S LEMMA

SO, COMBINING, WE GET THAT

$$\mathbb{E}[X_\tau] \leq \liminf_{m \to \infty} \underbrace{\mathbb{E}[X_{\min\{\tau, m\}}]}_{\leq \mathbb{E}[X_1]} \leq \mathbb{E}[X_1]$$

∎

ANOTHER APPLICATION OF THE OPTIONAL STOPPING THEOREM IS

Thm (DOOB'S MAXIMAL INEQUALITY) IF $(\{X_n\}, \{\mathcal{F}_n\})$ IS A SUB-MARTINGALE, THEN, FOR ANY $\alpha > 0$,

$$\mathbb{P}\left(\max_{i=1,\dots,n} X_i \geq \alpha\right) \leq \frac{1}{\alpha}\, \mathbb{E}\left[|X_n|\right]$$

REMARK: USING CHEBYSHEV'S INEQ. WE GET THE WORSE BOUND

$$\frac{1}{\alpha}\, \mathbb{E}\left[|\max_i X_i|\right]$$

Pf/ LET $\tau_2 = n$ AND τ_1 BE SMALLEST k S.T. $X_k \geq \alpha$ IF THERE IS ONE AND n OTHERWISE. SO $\tau_1 \leq \tau_2$.

FOR $1 \leq k \leq n$, SET $M_k = \max_{i \leq k} X_i$. THEN

$$\{M_n \geq \alpha\} \cap \{\tau_1 \leq k\} = \{M_k \geq \alpha\} \in \mathcal{F}_k$$

SO $\{M_n \geq \alpha\} \in \mathcal{F}_{\tau_1}$ $\quad \left[= \{A \in \mathcal{F} : A \cap \{\tau_1 \leq k\} \in \mathcal{F}_k \text{ ALL } k\}\right]$

NEXT

$$\alpha\, P(\{M_n \geq \alpha\}) \leq \int_{\{M_n \geq \alpha\}} X_{\tau_1}\, dP \leq \int_{\{M_n \geq \alpha\}} X_{\tau_2}\, dP$$

BY THE OPTIONAL STOPPING THM FOR SUB-MARTINGALES

$$= \int_{\{M_n \geq \alpha\}} X_n\, dP \leq \int_{\{M_n \geq \alpha\}} X_n^+\, dP$$

$$\leq \mathbb{E}\left[X_n^+\right] \leq \mathbb{E}\left[|X_n|\right]$$

∎

THIS RESULT RECOVERS AS A SPECIAL CASE KOLMOGOROV'S MAXIMAL INEQUALITY:

IF $Y_1, Y_2, \ldots$ INDEP SEQUENCE OF CENTERED RV'S WITH FINITE VARIANCE

THEN

$$\mathbb{P}\left(\max_{1 \le n} |S_n| \ge \alpha \right) \le \frac{V[S_n]}{\alpha}, \qquad \alpha > 0.$$

THIS IS BECAUSE $S_1^2, S_2^2, \ldots$ IS A SUB-MARTINGALE

FINALLY, WE GIVE ONE LAST RESULT FOR NON-NEGATIVE SUPER-MARTINGALES:

<u>Thm</u> (VILLE'S INEQUALITY) LET $(\{X_n\}, \{\mathcal{F}_n\})$ BE A NON-NEGATIVE SUPER-MARTINGALE. THEN, FOR ANY $\alpha > 0$,

$$\mathbb{P}(\{\exists n : X_n \ge \alpha\}) \le \frac{\mathbb{E}[X_1]}{\alpha}$$

PF/ LET $\tau = \inf\{n : X_n \ge \alpha\}$. THEN, FOR EACH FIXED $m \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}(\tau \le m) &= \mathbb{P}(X_{\min\{\tau, m\}} \ge \alpha) \\ &\le \frac{\mathbb{E}[X_{\min\{\tau, m\}}]}{\alpha} \\ &\le \frac{\mathbb{E}[X_1]}{\alpha} \quad \text{BY OPTIONAL STOPPING THEOREM}\end{aligned}$$

NOW LET $m \to \infty$ AND USE DOMINATED CONVERGENCE THEOREM TO GET THAT

$$\mathbb{P}(\tau < \infty) \le \frac{\mathbb{E}[X_1]}{\alpha} \qquad \blacksquare$$

FOR MORE SEE THE BOOK "STOPPED RANDOM WALKS" BY ALLAN GUT

MARTINGALE CONVERGENCE

MARTINGALE PROCESSES HAVE NICE CONVERGENCE PROPERTIES

Thm LET $(\{X_n\}, \{\mathcal{F}_n\})$ BE A SUB-MARTINGALE S.T. $\sup_n \mathbb{E}|X_n| < \infty$. THEN $\lim_{n\to\infty} X_n = X_\infty$ EXISTS a.s. AND $\mathbb{E}[|X_\infty|] < \infty$

Corollary IF $(\{X_n\}, \{\mathcal{F}_n\})$ IS A NON-NEGATIVE SUPERMARTINGALE THE SAME RESULT HOLDS.

CLT FOR MARTINGALES

SEE HALL & HYDE'S BOOK ON THIS SUBJECT.

Thm LET $(\{X_n\}, \{\mathcal{F}_n\})$ BE A MARTINGALE WITH $\mathbb{E}[X_n]=0$ AND LET $Y_n = X_n - X_{n-1}$, SO $X_n = \sum_{i=1}^{n} Y_i$

$\downarrow$

MARTINGALE DIFFERENCE

TAKE $X_0=0$.

LET $\sigma_i^2 = \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}]$

$$V_n = \sum_{i=1}^{n} \sigma_i^2 \quad \text{AND} \quad s_n^2 = \mathbb{E}[V_n^2] = \mathbb{E}[X_n^2]$$

IF $\frac{V_n^2}{s_n^2} \xrightarrow{P} 1$ AND $\frac{1}{s_n^2} \sum_{i=1}^{N} \mathbb{E}[Y_i^2 \mathbb{1}\{|Y_i| \geq \varepsilon s_n\}] \to 0$ AS $N \to \infty$ AND $\forall \varepsilon > 0$

THEN $$\frac{X_n}{s_n} = \frac{\sum_{i=1}^{n} Y_i}{s_n} \xrightarrow{D} N(0,1)$$

THE SAME IS TRUE FOR $\frac{\sum_{i=1}^{n} Y_i}{V_n}$