SDS 391-3, Fall 2025
Homework 1

Due ??, by midnight on Canvas.

1. **Limit superior and limit inferior.**

   (a) Let $\{A_n\}$ be a sequence of events (an event is a collection of outcomes). Argue that an outcome belongs to $\limsup_n A_n$ if and only if it belongs to infinitely many events A_n's and that it belongs to $\liminf_n A_n$ if and only if there exists an integer N such that the outcome belongs to all the events A_n with $n \geq N$ (so it belongs to the A_n's eventually). Conclude that $\liminf_n A_n \subseteq \limsup_n A_n$.
   Recalling the definition $\limsup_n A_n = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m$, if a point x belongs to $\limsup_n A_n$ then, for every n, it belongs to the set $\cup_{m=n}^{\infty} A_m$. Equivalently, for every n, there exists a $k \geq n$ such that $x \in A_k$. That is x belongs to infinitely many events A_n's. Similarly, since $\liminf_n A_n = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m$, if x belongs to $\liminf_n A_n$, there exists a N such that x belongs to each A_m with $m \geq N$.

   (b) Let A_n be $(-1/n, 1]$ if n is odd and $(-1, 1/n]$ if n is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.
   Note that for any $k \in \mathbb{N}$, $A_k \cup A_{k+1} = (-1, 1]$. Hence $\bigcup_{k=n}^{\infty} A_k = (-1, 1]$ for all $n \in \mathbb{N}$, and hence

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} (-1, 1] = (-1, 1].$$

   Also, note that for any $m \in \mathbb{N}$, $\bigcap_{k=m}^{\infty} A_{2k-1} = [0, 1]$ and $\bigcap_{k=m}^{\infty} A_{2k} = (-1, 0]$. Hence $\bigcap_{k=n}^{\infty} A_k = \{0\}$ for any $n \in \mathbb{N}$, and hence

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \{0\} = \{0\}.$$

   (c) **On the relationship between** $\liminf$ **and** $\limsup$ **of events and numbers.** Recall that for a sequence of numbers $\{x_n\}_{n=1,2\ldots}$,

$$\liminf_n x_n = \inf_{n \geq 1} \sup_{m \geq n} x_m \quad \text{and} \quad \limsup_n x_n = \sup_{n \geq 1} \inf_{m \geq n} x_m$$

   For an event A_n, denote with I_{A_n} the $0-1$ random varibale that is 1 if A_n takes place and 0 otherwise. Show that

$$I_{\limsup_n A_n} = \limsup_n I_{A_n} \quad \text{and} \quad I_{\liminf_n A_n} = \liminf_n I_{A_n}$$

   (d) **Bonus Problem**. Let A_n the interior of the ball in $\mathbb{R}^2$ with unit radius and center $\left(\frac{(-1)^n}{n}, 0\right)$. Find $\limsup_n A_n$ and $\liminf_n A_n$.

Let $D := \{x \in \mathbb{R}^2 : \|x\|_2 < 1\}$ and $B := \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_2 = 1,\, x_1 \neq 0\}$. We will show that $\liminf_n A_n = D$ and $\limsup_n A_n = D \cup B$.

For $\liminf_n A_n$, note that $x \in \liminf_n A_n$ if and only if $x \in A_n$ for all but finite n. Suppose $x \in D$. Then $\|x\|_2 < 1$, so choose N large enough so that $\frac{1}{N} < 1 - \|x\|_2$. Then for all $n \geq N$,

$$\begin{aligned}\left\| x - \left(\frac{(-1)^n}{n}, 0\right)\right\|_2 &\leq \|x\|_2 + \left\|\left(\frac{(-1)^n}{n}, 0\right)\right\|_2 \\ &= \|x\|_2 + \frac{1}{n} \leq \|x\|_2 + \frac{1}{N} < 1.\end{aligned}$$

Then $x \in A_n$ for all $n \geq N$, and hence $x \in \liminf_n A_n$, which implies $D \subset \liminf_n A_n$. Now, suppose $x \notin D$ and $x_1 \geq 0$. Then for all odd n,

$$\left\| x - \left(\frac{(-1)^n}{n},\, 0\right)\right\|_2 = \left\|\left(x_1 - \frac{1}{n},\, x_2\right)\right\|_2 > \|(x_1,\, x_2)\|_2 \geq 1,$$

Hence $x \notin A_n$ for all odd n, and hence $x \notin \liminf_n A_n$. Similarly, when $x \notin D$ and $x_1 \leq 0$, then $x \notin A_n$ for all even n, and hence $x \notin \liminf_n A_n$. These imply $\liminf_n A_n \subset D$, and hence

$$\liminf_n A_n = D.$$

For $\limsup_n A_n$, note that $x \in \limsup_n A_n$ if and only if $x \in A_n$ for infinitely many n. Suppose $x \in D \cup B$. We have already shown that $D = \liminf_n A_n \subset \limsup_n A_n$, and hence if $x \in D$ then $x \in \limsup_n A_n$. Now, suppose $x \in B$ and $x_1 > 0$. Then $\|x\|_1 = 1$. Choose N large enough so that $\frac{1}{N} < |x_1|$. Then for all even n with $n \geq N$, $\left|x_1 - \frac{1}{n}\right| \leq |x_1|$, and hence

$$\begin{aligned}\left\| x - \left(\frac{(-1)^n}{n}, 0\right)\right\|_2 &= \left\|\left(x_1 - \frac{1}{n},\, x_2\right)\right\|_2 \\ &< \left\|(x_1,\, x_2)\right\|_2 = 1.\end{aligned}$$

Hence $x \in A_n$ for all even n with $n \geq N$, and hence $x \in \limsup_n A_n$. Similarly, when $x \in B$ and $x_1 < 0$, $x \in A_n$ for all odd n with $n \geq N$, and hence $x \in \limsup_n A_n$. These imply that $D \cup B \subset \limsup_n A_n$. Now, suppose $x \notin D \cup B$. Then $\|x\|_2 > 1$ or $x = (0, \pm 1)$. When $\|x\|_2 > 1$, choose N large enough so that $\frac{1}{N} < 1 - \|x\|_2$. Then for all $n \geq N$,

$$\begin{aligned}\left\| x - \left(\frac{(-1)^n}{n}, 0\right)\right\|_2 &\geq \|x\|_2 - \left\|\left(\frac{(-1)^n}{n}, 0\right)\right\|_2 \\ &= \|x\|_2 - \frac{1}{n} \geq \|x\|_2 - \frac{1}{N} > 1.\end{aligned}$$

Then $x \notin A_n$ for all n with $n \geq N$, and hence $x \notin \limsup_n A_n$. Also, when $x = (0, \pm 1)$, then for all n,

$$\left\| x - \left(\frac{(-1)^n}{n}, 0 \right) \right\|_2 = \left\| \left(-\frac{(-1)^n}{n}, \pm 1 \right) \right\|_2 = \sqrt{1 + \frac{1}{n^2}} > 1,$$

Then $x \notin A_n$ for all n, and hence $x \notin \limsup_n A_n$. These show $\limsup_n A_n \subset D \cup B$, and hence

$$\limsup_n A_n = D \cup B.$$

2. Let $X_1, X_2, \ldots$ be a sequence of 0-1 Bernoulli random variables such $X_n \sim \text{Bernoulli}(1/n^2)$. Let $X = \sum_{n=1}^{\infty} X_n$. What is $\mathbb{P}(X < \infty)$?
Use Borel Cantelli's first Lemma. Define the events $A_n = \{X_n = 1\}$, $n = 1, 2, \ldots$. Then, $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6} < \infty$. So $\mathbb{P}(\limsup A_n) = 0$. Now, $\limsup A_n = \mathbb{P}(X = \infty)$.

3. Recall that Borel-Cantelli's Second Lemma says that if $\{A_n\}$ is a sequence of *independent*[1] events such that $\sum_n \mathbb{P}(A_n) = \infty$ then $\mathbb{P}(\limsup_n A_n) = 1$. You might wonder whether the requirement of independence is needed. The answer is yes. Find an example in which all the conditions of the lemma are met except for independence and the conclusion is false.

4. Ferguson, problem 5, page 12.
$X_n \xrightarrow{p} 0$ for all values of α. By Borel-Cantelli's Second Lemma, if $\alpha \geq 1$ then, for any $\epsilon > 0$ $|X_n| > \epsilon$ infinitely often with probability 1, by independence and because $\sum_n \frac{1}{n} \sim \log n \to \infty$. On the other hand, when $\alpha < 1$, Borel-Cantelli's First Lemma will imply that the probability that $|X_n| > \epsilon$ infinitely often is equal to 0 for any ϵ. Therefore, $X_n \overset{w.p.\,1}{\to} 0$ if and only if $\alpha < 1$. Finally, by direct calculation

$$\mathbb{E}[|X_n|^p] = \frac{n^{\alpha p}}{n} \to 0$$

if and only if $\alpha\, p < 1$.

5. Prova Markov's inequality: if X is a non-negative random variable, then for any $\epsilon > 0$

$$\mathbb{P}(X \geq \epsilon) \leq \frac{\mathbb{E}[X]}{\epsilon}.$$

Markov's inequality is almost always a loose upper bound, but there are rare cases when it is sharp. Find an example in which it holds exactly. *Hint: take X to be the*

[1] meaning that the probability of any finite intersection of events in the sequence is equal to the product of their respective probabilities.

indicator function of a set and select the right ϵ.
Prove the PaleyZygmund inequality, a reverse Markov inequality of sort: if X is a non-negative random variable with two or more moments, then, for any $\alpha \in (0,1)$,

$$\mathbb{P}(X \geq \alpha \mathbb{E}[X]) \geq (1-\alpha)^2 \frac{\mathbb{E}[^2 X]}{\mathbb{E}[X^2]}.$$

We can write

$$\begin{aligned} X &= X \mathbb{1}\{X < \theta \mathbb{E}[X]\} + X \mathbb{1}\{X \geq \theta \mathbb{E}[X]\} \\ &\overset{(i)}{\leq} \theta \mathbb{E}[X] + \sqrt{\mathbb{E}[X^2] \mathbb{P}(X \geq \theta E[X])}, \end{aligned}$$

where in *(i)* we have used Cauchy-Schwartz inequality to bound the second term. So,

$$\mathbb{E}[X](1-\theta) \leq \sqrt{\mathbb{E}[X^2] \mathbb{P}(X \geq \theta E[X])}.$$

The result follow from taking the square.

6. Let $X_1, \ldots, X_n$ *i.i.d.* univariate random variables with common distribution function F_X. Given $\alpha \in (0,1)$, use the DKW inequality given in class to construct a $1-\alpha$ confidence band for F_X, a pair of random functions (random because dependent on $X_1, \ldots, X_n$), say $\hat{F}_\alpha^{\text{lower}}$ and $\hat{F}_\alpha^{\text{upper}}$, such that

$$\mathbb{P}\left(\hat{F}_\alpha^{\text{lower}}(x) \leq F_X(x) \leq \hat{F}_\alpha^{\text{upper}}(x), \forall x \in \mathbb{R}\right) \geq 1-\alpha.$$

The DKW inequality states that

$$\mathbb{P}\left(\sup_x |F_X(x) - \hat{F}_n(x)| > \epsilon\right) \leq 2\exp\{-2n\epsilon^2\}, \quad \forall \epsilon > 0.$$

Set the right hand side of the above inequality to α and solve for ϵ to conclude that

$$\hat{F}_\alpha^{\text{lower}}(x) = \min\left\{0, \hat{F}_n(x) - \sqrt{\frac{\log 2/\alpha}{2n}}\right\}, \quad x \in \mathbb{R}$$

and

$$\hat{F}_\alpha^{\text{upper}}(x) = \max\left\{1, \hat{F}_n(x) + \sqrt{\frac{\log 2/\alpha}{2n}}\right\}, \quad x \in \mathbb{R}.$$

7. **Joint and marginal convergence.** Below, $\{X_n\}$ is a sequence of random vectors in $\mathbb{R}^d$ and X another random vector in $\mathbb{R}^d$.

(a) Show that $X_n \xrightarrow{p} X$ if and only if $X_n(j) \xrightarrow{p} X(j)$ for all $j = 1, \ldots, d$. *Note: the same is true about convergence with probability one.*

By definition, $X_n \xrightarrow{p} X$ if and only if, for each $\epsilon > 0$,

$$\mathbb{P}(\|X_n - X\| \geq \epsilon) \to 0,$$

or, equivalently,

$$\mathbb{P}(\|X_n - X\| < \epsilon) \to 1,$$

which implies, since $\max_j |X_n(j) - X(j)| \leq \|X_n - X\|$, that $\mathbb{P}(\max_j |X_n(j) - X(j)| < \epsilon) \to 1$. In turn, this implies that, for any j, $X_n(j) \xrightarrow{p} X(j)$. Conversely, if, for any $\epsilon > 0$, $\mathbb{P}(|X_n(j) - X(j)| \geq \epsilon) \to 0$ for all j, then $\mathbb{P}(\|X_n - X\| \geq d\epsilon) \leq \sum_{j=1}^{d} \mathbb{P}(|X_n(j) - X(j)| \geq \epsilon) \to 0$. Since ϵ is arbitrary, $X_n \xrightarrow{p} X$.

(b) Show that if $X_n \xrightarrow{d} X$, then $X_n(j) \xrightarrow{d} X(j)$ for all $j = 1, \ldots, d$.
There is more than one way to prove this. One could use the definition of convergence in distribution and a limiting argument. A shorter way is to use characteristic functions. Let ϕ_{X_n}, ϕ_X, $\phi_{X_n(j)}$, $\phi_{X(j)}$ be the characteristics functions of X_n, X, $X_n(j)$ and $X(j)$ respectively. Then, if $X_n \xrightarrow{d} X$, $\phi_{X_n}(t) \to \phi_X(t)$ for all $t \in \mathbb{R}^d$. In particular, this is true for the vector $t_x = (t_1, \ldots, t_d)$ such that $t_i = 0$ if $i \neq j$ and $t_j = x$ if $i = j$, where x is any real number. As a result, we get that, for any $x \in \mathbb{R}$,

$$\phi_{X_n(j)}(x) = \phi_{X_n}(t_x) \to \phi_X(t_x) = \phi_{X(j)}(x)$$

(c) In class, we looked at this example in $d = 2$. Set $U \sim \text{Uniform}(0, 1)$ and let $X_n = U$ for all n and

$$Y_n = \begin{cases} U & n \text{ odd}, \\ 1 - U & n \text{ even}. \end{cases}$$

Then, $X_n \xrightarrow{d} U$ and $X_n \xrightarrow{d} U$. In class, I claimed that

$$\begin{bmatrix} X_n \\ Y_n \end{bmatrix}$$

does not converge in distribution (in fact, in any meaningful sense). Prove the claim.
One way to prove the claim is to show that the random vector $\begin{bmatrix} X_n \\ Y_n \end{bmatrix}$ take values on the line segment on the plane joining $(0, 0)$ to $(1, 1)$ for all odd n and on the line segment joining $(1, 0)$ to $(0, 1)$ for all odd n. A simpler way is to use the Cramer-Wald device: the characteristic function of the vector evaluated at the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is equal to e^i when n is even and to the characteristic function of $2U$ at 1 when n is odd. As a result, it does not converge.

8. Show that the c.d.f. of a random variable can have at most countably many points of discontinuity.
For every point of discontinuity, say x, of the c.d.f. F, the interval $(F(x-), F(x))$ is not empty, by definition. Take any rational number in this interval. Thus, for every point of discontinuity of F, we can find a distinct rational. Since the set of rationals is countable, the set of discontinuity points of F can be put in a one-to-one correspondence with a subset of the set of rational numbers, which is countable.

9. For each n, let X_n a random variable uniformly distributed on $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Show that X_n converges on distribution to $U \sim \text{Uniform}(0,1)$. Let A be the set of all rational numbers in $[0,1]$. Then $\mathbb{P}(X_n \in A) = 1$ for all n but $\mathbb{P}(X \in A) = 0$. Show that this does not violate condition *(v)* of the Portmanteau theorem, as stated in the lecture notes.
X_n converges on distribution to $U \sim \text{Uniform}(0,1)$ because the c.d.f of X_n is

$$\mathbb{F}_{X_n}(x) = \begin{cases} 0 & x < 0 \\ \frac{\lfloor nx \rfloor}{n} & x \in [0,1] \\ 1 & x > 1 \end{cases} \to F_U(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases}$$

for all $x \in \mathbb{R}$. In this example, condition *(v)* of the Portmanteau theorem is not violated because A is dense in $[0,1]$, so $\partial A = [0,1] \setminus \mathbb{Q}$. Therefore, since $\mathbb{P}(U \in \mathbb{Q}) = 0$,

$$\mathbb{P}(U \in \partial A) = \mathbb{P}(U \in [0,1] \setminus \mathbb{Q}) = \mathbb{P}(U \in [0,1]) = 1 \neq 0.$$