36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 25: MON, NOV 30, 2020

LAST TIME: REGULAR CONDITIONAL DISTRIBUTIONS

$(\Omega, \mathcal{F}, P)$, X A $\mathcal{F}$-MEAS R.V. AND $\mathcal{C} \subseteq \mathcal{F}$ SUB σ-FIELD

A REGULAR CONDITIONAL PROBABILITY OF X GIVEN $\mathcal{C}$ IS A COLLECTION

$$\mu_{X|\mathcal{C}}(\cdot)(\cdot) : \mathcal{B}' \times \Omega \longmapsto [0,1] \quad \text{S.T.}$$

1) FOR EACH $B \in \mathcal{B}'$, $\mu_{X|\mathcal{C}}(B)(\cdot)$ IS A VERSION OF

$$\mathbb{E}[1_{X \in B} | \mathcal{C}] = \Pr(X \in B | \mathcal{C})$$

2) FOR EACH FIXED ω

$\mu_{X|\mathcal{C}}(\cdot)(\omega)$ IS A PROB. MEAS ON $\mathcal{B}'$.

Def 27 (CONDITIONAL DISTRIBUTION OF X GIVEN Y)

$(\Omega, \mathcal{F}, P)$. LET X AND Y BE TWO RV'S ON Ω TAKING VALUES IN $(\mathcal{X}, \mathcal{B})$ AND $(\mathcal{Y}, \mathcal{D})$.

$\mathcal{B}$ ↳ σ-FIELD ON $\mathcal{X}$ $\mathcal{D}$ ↳ σ-FIELD ON $\mathcal{Y}$

ASSUME THAT $\mathcal{D}$ CONTAINS ALL SINGLETON SETS.

$\mu_{X|Y}$ WILL DENOTE THE CONDITIONAL DISTRIBUTION OF

X GIVEN Y IN THE FOLLOWING SENSE: FOR EACH $y \in \mathcal{Y}$ AND $\omega = Y^{-1}(y)$ AND $B \in \mathcal{B}$ WE LET

$$\mu_{X|Y}(B|y) = \underline{\mu_{X|\mathcal{C}}(B)(\omega)} \quad \text{WHERE } \mathcal{C} = \sigma(Y) \subseteq \mathcal{F}$$

REGULAR CONDITIONAL DISTRIBUTION

$$= \text{“}\Pr(X \in B \mid Y=y)\text{”}$$

- WE KNOW THAT $\mu_{X|\mathcal{C}}(B)(\omega)$ CAN BE EXPRESSED AS $h(Y(\omega))$ FOR SOME h. WE DEFINE

$$\mu_{X|Y}(B|y) = h(y).$$

- SO NOW IF $g: \mathcal{X} \to \mathbb{R}$ (MEAS.!!) IS S.T. $\mathbb{E}[g(X)]$ EXISTS, THEN

$$\int g(x)\, d\mu_{X|\mathcal{C}}(x) \text{ IS A VERSION OF } \mathbb{E}[g(X) \mid \mathcal{C}]$$

$\downarrow$

CAN COMPUTE USING E.G. CONDITIONAL pdf OR pmf.

<u>Thm 30</u> LET $(\Omega, \mathcal{F}, P)$ BE A PROB. SPACE AND $\mathcal{C} \subseteq \mathcal{F}$ A SUB σ-FIELD. LET X BE A RV. IF X TAKES VALUES IN A <u>NICE</u> SPACE $\mathcal{X}$ ($\exists$ A ONE-TO-ONE MAPPING FROM $\mathcal{X}$ INTO $\mathbb{R}$, ϕ, S.T. ϕ AND ϕ^{-1} ARE MEASURABLE), THE A RCD OF X GIVEN $\mathcal{C}$ EXISTS!

REMARKS:

1) LET $(\Omega, \mathcal{F}, P)$ BE $\Omega = [0,1]$ $P =$ RESTRICTION OF LEB. MEASURE ON $[0,1]$

$\mathcal{F}$ = BOREL σ-FIELD ON $[0,1]$

LET $\mathcal{C}$ BE σ-FIELD GENERATED BY SUBSETS OF $[0,1]$ THAT ARE COUNTABLE OR CO-COUNTABLE (THE COMPLEMENT IS COUNTABLE)

WE ARE INTERESTED IN $\Pr(A \mid \mathcal{C})$ SOME A BOREL MEAS. $A \subseteq [0,1]$

CLAIM THAT

$$\Pr(A \mid \mathcal{C})(\omega) = P(A) \quad \text{WITH PROB } 1.$$

THIS IS BECAUSE, FOR ANY $B \in \mathcal{C}$, $P(B) = 0$ OR 1

SO

$$\int_B \Pr(A \mid \mathcal{C})(\omega)\, dP(\omega) = P(A \cap B)$$

SO, IT SEEMS LIKE $\mathcal{C}$ IS COMPLETELY UN-INFORMATIVE.

HOWEVER $\mathcal{C}$ CONTAINS ALL SINGLETONS (ONE-POINT SETS)

SO, IF ω IS THE OUTCOME, THEN KNOWING WHICH SETS OF $\mathcal{C}$ CONTAINS ω IS THE SAME AS KNOWING ω ITSELF!

SO, IT WOULD SEEM THAT $\mathcal{C}$ IS FULLY INFORMATIVE.

BUT THIS IS NOT THE CASE BECAUSE IT IS NOT TRUE THAT

$$\Pr(A \mid \mathcal{C})(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

↳ SO INTUITION THAT $\mathcal{C}$ PROVIDES US WITH ADDITIONAL INFORMATION FAILS IN THIS EXAMPLE.

2) SUFFICIENCY : SUFFICIENT σ-FIELD

SEE p. 450 OF BILLINGSLEY'S BOOK

"PROBABILITY AND MEASURE"

MARTINGALES

SEE LECTURE NOTES 10:

http://www.stat.cmu.edu/~arinaldo/Teaching/36710-36752/Lecture_Notes/lec_notes_10.pdf

Def $(\Omega, \mathcal{F}, P)$. LET $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \ldots$ BE AN INCREASING SEQUENCE OF SUB-σ-FIELDS (SO $\mathcal{F}_n \subseteq \mathcal{F}$ FOR ALL n). THIS IS CALLED A FILTRATION. FOR EACH n, LET X_n BE A RV THAT IS $\mathcal{F}_n$ MEAS. THE SEQUENCE $\{X_n\}$ IS SAID TO BE ADAPTED TO THE FILTRATION $\{\mathcal{F}_n\}$. THE PAIRS $(\{X_n\}, \{\mathcal{F}_n\})$ DEFINES A MARTINGALE WHEN

1) $\{X_n\}$ IS ADAPTED TO $\{\mathcal{F}_n\}$
2) $\mathbb{E}[|X_n|] < \infty$ ALL n.
3) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ ALL n.

IT IS SAID TO BE A SUB-MARTINGALE (SUPER-MARTINGALE) IF 3) HOLDS WITH $\geq$ (RESP $\leq$).

REMARK: $\mathbb{E}[X_{n+k} \mid \mathcal{F}_n] = X_n$ BY TOWER PROPERTY OF COND. EXPECTATION.

FOR ALL $k \geq 1$

$\hookrightarrow$ SO $\mathbb{E}[X_n]$ IS CONSTANT IN n!

EXAMPLES (SUMS OF INDEP RV'S). LET $\{Y_n\}$ BE INDEP. RV WITH $\mathbb{E}[Y_n] = 0$ ALL n. LET $X_n = \sum_{i=1}^{n} Y_i$ AND $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$. THEN $(\{X_n\}, \{\mathcal{F}_n\})$ IS A MARTINGALE.

$$
\begin{aligned}
\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[\sum_{i=1}^{n+1} Y_i \mid \mathcal{F}_n\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} Y_i + Y_{n+1} \mid \mathcal{F}_n\right] \\
&= \mathbb{E}\Big[\underbrace{\sum_{i=1}^{n} Y_i}_{X_n} \mid \mathcal{F}_n\Big] + \underbrace{\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n]}_{= \mathbb{E}[Y_{n+1}] = 0} \\
&= X_n
\end{aligned}
$$

BECAUSE X_n IS $\mathcal{F}_n$-MEAS.

IT IS A SUB-MARTINGALE IF $\mathbb{E}[Y_n] \geq 0$

SUPER-MARTINGALE IF $\mathbb{E}[Y_n] \leq 0$.

EXAMPLE (GABLING). ASSUME THE SAME SETTINGS. THINK OF Y_n AS THE AMOUNT THAT A GAMBLER WINS PER UNIT OF CURRENCY ON THE n^{th} PLAY OF A SEQUENCE OF INDEPENDENT GAMES. LET Y_0 BE THE INITIAL FORTUNE OF THE GAMBLER WHICH WE CAN THINK OF A DETERMINISTIC AMOUNT

(TECHNICALLY, WE NEED TO INCREASE OUR FILTRATION WITH $\mathcal{F}_0$, THE TRIVIAL σ-FIELD. THEN $\mathbb{E}[Y_0] = Y_0$).

SUPPOSE THAT, AT EACH TIME n, THE GAMBLER CAN DECIDE HOW MUCH MONEY W_n TO BET ON THE n^{th} GAME BASED ON THE OUTCOMES OBSERVED SO FAR. FORMALLY W_n IS MEASURABLE WRT TO $\mathcal{F}_{n-1}$. [W_n IS SAID TO BE <u>PREVISIBLE</u>]. AT TIME n THE GAMBLER'S FORTUNE IS

$$Z_n = Y_0 + \sum_{k=1}^{n} Y_k W_k .$$

NOW

$$\mathbb{E}\left[Y_{n+1} \cdot W_{n+1} \mid \mathcal{F}_n\right] = W_{n+1}\, \mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right]$$
$$= W_{n+1}\, \mathbb{E}\left[Y_{n+1}\right]$$

THIS IS $=0$, ≥ 0 OR ≤ 0 DEPENDING ON WHETHER $\mathbb{E}[Y_n]$ IS $=0$, ≥ 0 OR ≤ 0.

SO $\{Z_n\}$ IS A MARTINGALE, SUB-MARTINGALE OR SUPER-MART.

IIF $\{X_n\}$ IS A " " "

<u>EXAMPLE</u> (RN-DERIVATES). $(\Omega, \mathcal{F}, P)$ $\{\mathcal{F}_n\}$ BE A FILTRATION.

LET ν BE A PROB. MEASURE ON $(\Omega, \mathcal{F}, P)$ SUCH THAT FOR EACH n, X_n IS THE RN OF ν RESTRICTED TO $\mathcal{F}_n$ WRT P (ALSO RESTRICTED TO $\mathcal{F}_n$).

$X_n = \dfrac{d\nu}{dP}$ ON $\mathcal{F}_n$. THEN $\{X_n\}$ IS ADAPTED TO $\{\mathcal{F}_n\}$.

AND INTEGRABLE!

IN ADDITION $\{X_n\}$ IS A MARTINGALE:

FOR ANY $A \in \mathcal{F}_n$

$$\nu(A) = \int_A X_n(\omega)\, dP(\omega) = \int_A X_{n+1}(\omega)\, dP(\omega)$$

SO $\mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] = X_n$