36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 10 : MON, OCT 5, 2020

PRODUCT SPACES

$\mathbb{R}^k = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{k\text{-TIMES}}$. WE ALREADY KNOW WHAT A RANDOM VECTOR IS: MEASURABLE FUNCTION FROM $(\Omega, \mathcal{F}, P)$ INTO $\mathbb{R}^k$, ENDOWED WITH BOREL σ-FIELD $\mathcal{B}^k$.

- SIGMA FIELD GENERATED BY OPEN SETS IN $\mathbb{R}^k$
- GENERATED BY HYPER-RECTANGLES $(a_1, b_1] \times \dots \times (a_k, b_k]$

- PRODUCT σ-FIELD IS CONSTRUCTED USING σ-FIELDS OF COMPONENT OF PRODUCT SPACE

<u>Def</u> LET $(\Omega_1, \mathcal{F}_1)$ AND $(\Omega_2, \mathcal{F}_2)$ BE MEASURE SPACES. THE PRODUCT σ-FIELD ON $\Omega_1 \times \Omega_2$ IS THE SMALLEST σ-FIELD CONTAINING THE CLASS $\{A_1 \times A_2, \ A_1 \in \mathcal{F}_1, \ A_2 \in \mathcal{F}_2\}$. $A_1 \times A_2$ IS CALLED A MEASURABLE RECTANGLE. THE PRODUCT

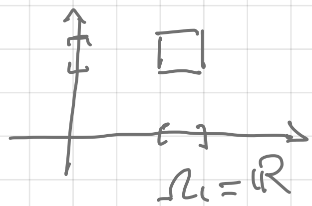
σ-FIELD IS DENOTED WIT $\mathcal{F}_1 \otimes \mathcal{F}_2$ (OR $\mathcal{F}_1 \times \mathcal{F}_2$)

<u>REMARK</u>: THE PRODUCT σ-FIELD IS <u>NOT</u> THE CARTESIAN PRODUCT OF $\mathcal{F}_1$ AND $\mathcal{F}_2$. IT IS LARGER THAN THAT.

$\downarrow$

NOT σ-FIELD BECAUSE IT IS NOT CLOSED UNDER COMPLEMENTS OR UNIONS

$\Omega_2 = \mathbb{R}$



<u>Lemma</u> LET $(\Omega_1, \mathcal{F}_1)$ AND $(\Omega_2, \mathcal{F}_2)$ BE MEASURABLE SPACES SUCH THAT THE CLASS $\mathcal{C}_i$ GENERATES $\mathcal{F}_i$, $i=1,2$ (I.E. $\mathcal{F}_i = \sigma(\mathcal{C}_i)$). LET $\mathcal{C} = \{ C_1 \times C_2,\ C_1 \in \mathcal{C}_1,\ C_2 \in \mathcal{C}_2 \}$

THEN $\sigma(\mathcal{C}) = \mathcal{F}_1 \otimes \mathcal{F}_2$.

$\hookrightarrow$ $\mathcal{B}^k = \underbrace{\mathcal{B} \otimes \dots \otimes \mathcal{B}}_{\text{PRODUCT } \sigma\text{-FIELD}}$

PF/ FIX $C_1 \in \mathcal{C}_1$. THE CLASS OF SET $\{ A \subseteq \Omega_2 : C_1 \times A \} \in \sigma(\mathcal{C})$ IS CLOSED UNDER COMPLEMENTS AND COUNTABLE UNIONS $\Rightarrow$ IS A σ-FIELD. SO $C_1 \times A_2$, $A_2 \in \mathcal{F}_2$, BELONGS TO $\sigma(\mathcal{C})$. SIMILARLY, FOR EACH FIXED $A_2 \in \mathcal{F}_2$, $\sigma(\mathcal{C})$ CONTAINS SETS OF THE FORM $A_1 \times A_2$, $A_1 \in \mathcal{F}_1$. $\Rightarrow$ $\sigma(\mathcal{C})$ CONTAINS MEASURABLE RECTANGLES SO $\sigma(\mathcal{C}) = \mathcal{F}_1 \otimes \mathcal{F}_2$. $\blacksquare$

REMARK IF $\mathcal{C}_1$ AND $\mathcal{C}_2$ ARE π-SYSTEMS, SO IS $\mathcal{C}_1 \times \mathcal{C}_2$

IF $C_1, D_1 \in \mathcal{C}_1$ AND $C_2, D_2 \in \mathcal{C}_1$ THEN

$$(C_1 \times C_2) \cap (D_1 \times D_2) = (C_1 \cap D_1) \times (C_2 \cap D_2).$$

Claim LET'S CONSIDER $\Omega_1 \times \Omega_2$. A COORDINATE PROJECTION, SAY f_1, IS THE FUNCTION $f_1 : S_1 \times S_2 \to S_1$ OF THE FORM $(\omega_1, \omega_2) \mapsto f(\omega_1, \omega_2) = \omega_1$.

THE PRODUCT σ-FIELD IS THE SMALLEST σ-FIELD FOR WHICH ALL COORDINATE PROJECTIONS ARE MEASURABLE.

LET $A_1 \in \mathcal{F}_1$. THEN $f_1^{-1}(A_1) = A_1 \times \Omega_2$

NOTICE THAT, FOR $A_2 \in \mathcal{F}_2$,

$$f_1^{-1}(A_1) \cap f_2^{-1}(A_2) = A_1 \times A_2$$

$\blacksquare$

Proposition 6: LET $(\Omega_1, \mathcal{F}_1)$ AND $(\Omega_2, \mathcal{F}_2)$ BE MEASURABLE SPACES

i) LET $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$, $\omega_1 \in \Omega_1$. THE ω_1-SECTION OF B IS THE SET $B_{\omega_1} = \{\omega_2 \in \Omega_2, (\omega_1, \omega_2) \in B\}$.

SEE ASH

THEN $B_{\omega_1} \in \mathcal{F}_2$.

ii) IF μ_2 IS A σ-FINITE MEASURE ON $(\Omega_2, \mathcal{F}_2)$. THEN THE FUNCTION $\omega_1 \in \Omega_1 \mapsto \mu_2(B_{\omega_1})$ IS MEASURABLE, FOR EACH $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$

iii) IF μ_2 IS A σ-FINITE MEASURE $(\Omega_2, \mathcal{F}_2)$. IF $f: \Omega_1 \times \Omega_2 \to \mathbb{R}$ IS NON-NEGATIVE AND MEASURABLE, THEN

$$\omega_1 \longmapsto \int_{\Omega_2} f(\omega_1, \omega_2)\, d\mu_2(\omega_2)$$

IS A MEASURABLE FUNCTION

SO WE ARE NOW READY TO DEFINE PRODUCT MEASURE

Theorem) LET $(\Omega_1, \mathcal{F}_1, \mu_1)$ AND $(\Omega_2, \mathcal{F}_2, \mu_2)$ BE MEASURE SPACES. THERE EXISTS A UNIQUE MEASURE μ ON $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ THAT SATISFIES

$$\mu(A_1 \times A_2) = \mu_1(A_1)\,\mu_2(A_2) \qquad \text{FOR ALL } A_1 \in \mathcal{F}_1,\ A_2 \in \mathcal{F}_2.$$

Pf/ (SKETCH). UNIQUENESS FOLLOWS FROM THE FACT THAT MEASURABLE RECTANGLES FORM A π SYSTEM. AS FOR EXISTENCE, CONSIDER THE MEAS. FUNCTION $\omega_1 \mapsto \mu_2(B_{\omega_1})$ $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$. DEFINE

$$\mu(B) = \int_{\Omega_1} \mu_2(B_{\omega_1})\, d\mu(\omega_1)$$

IF $B = A_1 \times A_2$ THEN $B_{\omega_1} = A_2$ IF $\omega_1 \in A_1$

AND

$$\mu(B) = \int_{\Omega_1} \mu_2(A_2)\, 1_{A_1}(\omega_1)\, d\mu_1(\omega_1)$$

$$= \mu_2(A_2) \underbrace{\int_{\Omega_1} 1_{A_1}(\omega_1)\, d\mu_1(\omega_1)}_{\mu_1(A_1)} = \mu_1(A_1)\mu_2(A_2)$$

THIS DEFINES A MEASURE ON FIELD OF FINITE DISJOINT UNIONS OF MEAS. RECTANGLES. APPLY EXTENSION THEOREM ∎

THE MEASURE μ IS CALLED THE <u>PRODUCT MEASURE</u>

THE LEBESGUE MEASURE ON $(\mathbb{R}^k, \mathcal{B}^k)$ IS A PRODUCT MEASURE.

<u>Thm (TONELLI THEOREM)</u> LET $(\Omega_1, \mathcal{F}_1, \mu_1)$ AND $(\Omega_2, \mathcal{F}_2, \mu_2)$ BE σ-FINITE MEASURE SPACES AND $f: \Omega_1 \times \Omega_2 \to \mathbb{R}_{\geq 0}$ A NON-NEGATIVE FUNCTION THAT IS $\mathcal{F}_1 \otimes \mathcal{F}_2 / \mathcal{B}^1$ MEAS.

THEN

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) \, d\underbrace{\mu_1 \otimes \mu_2}_{\text{PRODUCT MEASURE}}(\omega_1, \omega_2) = \int_{\Omega_2} \left[\int_{\Omega_1} f(\omega_1, \omega_2) \, d\mu_1(\omega_1) \right] d\mu(\omega_2)$$

$$= \int_{\Omega_1} \left[\int_{\Omega_2} f(\omega_1, \omega_2) \, d\mu_2(\omega_2) \right] d\mu_1(\omega_1)$$

THE INTEGRAL CAN BE INFINITE.

THE PROOF RELIES ON STANDARD MACHINERY.

<u>Thm</u> (FUBINI) UNDER THE SAME SETTINGS, IF $f: \Omega_1 \times \Omega_2 \to \mathbb{R}$ IS A MEAS. FUNCTION THAT IS INTEGRABLE wrt $\mu_1 \otimes \mu_2$ THEN THE ABOVE RESULT HOLDS

$$\Downarrow$$

$$\int_{\Omega_1 \times \Omega_2} |f(\omega_1, \omega_2)| \, d\mu_1 \otimes \mu_2(\omega_1, \omega_2) < \infty$$

$\downarrow$

SEE EXAMPLE 14 IN NOTES #4

INDEPENDENCE

Def (INDEPENDENCE BETWEEN SETS). LET $(\Omega, \mathcal{F}, P)$ BE A PROBABILITY SPACE. LET $\mathcal{C}_1$ AND $\mathcal{C}_2$ BE TWO CLASSES OF SETS IN $\mathcal{F}$. WE SAY THAT $\mathcal{C}_1$ AND $\mathcal{C}_2$ ARE INDEPENDENT WHEN

$$P(A_1 \cap A_2) = P(A_1) P(A_2) \qquad \text{FOR ALL } A_1 \in \mathcal{C}_1, \; A_2 \in \mathcal{C}_2$$

Def (INDEPENDENCE BETWEE RANDOM VARIABLES)

LET $(\Omega, \mathcal{F}, P)$ BE A PROB. SPACE AND $X_i : \Omega \to S_i$ $i = 1, 2$. BE RANDOM VARIABLES. THEN X_1 AND X_2 ARE INDEPENDENT IF $\sigma(X_1)$ AND $\sigma(X_2)$ ARE INDEPENDENT.

Def (INDEPENDENCE OF MULTIPLE COLLECTIONS OF SETS) $(\Omega, \mathcal{F}, P)$ AND A COLLECTION $\{\mathcal{C}_t, \underbrace{t \in T}_{\text{ARBITRARY}}\}$ OF CLASSES OF MEAS. SUBSETS. THIS COLLECTION IS SAID TO BE INDEP WHEN FOR ANY $n \leq |T|$ AND ANY DISTINCT $t_1, \dots, t_n$ IN T AND ARBITRARY CHOICES OF SETS A_{t_i} IN $\mathcal{C}_{t_i}$

$$P\left(\bigcap_{i=1}^{n} A_{t_i}\right) = \prod_{i=1}^{n} P(A_{t_i})$$

REMARK MUTUAL INDEP. IS NOT IMPLIED BY PAIRWISE INDEP. !!

EXAMPLE: TOSS A COIN TWICE AND EACH OUTCOME HAS PROB 1/4

$A_1 = \{$ IDENTICAL OUTCOMES $\}$

$A_2 = \{$ FIRST OUTCOME IS H $\}$

$A_3 = \{$ SECOND OUTCOME IS H $\}$

THEN $P(A_i) = 1/2$ ALL $i = 1, 2, 3$ AND

$$P(A_i \cap A_j) = \frac{1}{4} = P(A_i)\,P(A_j) \quad \text{ALL } i \neq j$$

↳ PAIRWISE INDEPENDENCE HOLDS

BUT $P(A_1 \cap A_2 \cap A_3) = P(\{HH\}) = \frac{1}{4} \neq P(A_1)\,P(A_2)\,P(A_3)$
$= 1/8$

↳ MUTUAL INDEPENDENCE DOES NOT HOLD!