36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 21 : WED, NOV 11, 2020

- INVERSION FORMULA AND UNIQUENESS OF chf's

<u>Thm</u> LET ϕ_X BE THE chf OF THE RANDOM VECTOR $X \in \mathbb{R}^d$.
LET $A = \{ (x_1, \ldots, x_d) : a_j < x_j \leq b_j \text{ ALL } j \}$
WHERE $a_j < b_j$ ALL j. LET μ_X BE THE DISTRIBUTION OF X
AND ASSUME THAT $\mu_X(\partial A) = 0$. LET, FOR $T > 0$,

$$B_T = \{ (y_1, \ldots y_d) : |y_j| \leq T \text{ ALL } j \}$$

THEN

$P(X \in A)$

$$\mu_X(A) = \lim_{T \to \infty} \frac{1}{(2\pi)^d} \int_{B_T} \prod_{j=1}^{d} \left[\frac{\exp(-i y_j a_j) - \exp(-i y_j b_j)}{i y_j} \right] \phi_X(y) \, dy$$

$dy_1 \ldots dy_d$

DISTINCT PROB. MEASURES HAVE DISTINCT chf's.

<u>Corollary</u> (CRAMÉR - WOLD) LET X AND Y BE TWO RANDOM VECTORS IN $\mathbb{R}^d$. THEN $X \stackrel{d}{=} Y$ IFF $t^T X \stackrel{d}{=} t^T Y$ FOR ALL $t \in \mathbb{R}^d$.

$\hookrightarrow \mathbb{E}\left[\exp i(t^T x)\right]$

Pf/ $X \overset{d}{=} Y$ $[\mu_X = \mu_Y]$ IIF $\phi_X(t) = \phi_Y(t)$ ALL $t \in \mathbb{R}^d$

IIF $\phi_X(s\alpha) = \phi_Y(s\alpha)$ FOR ALL $s \in \mathbb{R}$ AND $\alpha \in \mathbb{R}^d$

BUT $\phi_X(s\alpha)$ IS THE chf OF $\alpha^T X$ AT s.

SO THE ABOVE STATEMENTS ARE EQUIVALENT TO $\alpha^T X = \alpha^T Y$

FOR ALL $\alpha \in \mathbb{R}^d$ ∎

ANOTHER PROPERTY OF chf IS CONTINUITY:

__Prop 13__ IF ϕ IS chf OF A RV IN $(\mathbb{R}^1, \mathcal{B}^1)$ WITH DISTRIBUTION μ_X AND ϕ IS INTEGRABLE THEN μ_X HAS A LEBESGUE DENSITY f GIVEN BY

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e(-i t x)\, \phi(t)\, dt$$

AND IS CONTINUOUS.

∎ CONTINUITY OF chf's

__Thm__ LET $\{P_n\}_n$ BE A SEQUENCE OF PROB. DISTRIBUTIONS ON $(\mathbb{R}^d, \mathcal{B}^d)$ WITH chf's $\{\phi_n\}_n$ AND LET P ANOTHER PROB. DISTR. ON $(\mathbb{R}^d, \mathcal{B}^d)$ WITH chf ϕ. THEN

$$P_n \Rightarrow P \quad \text{IIF} \quad \phi_n(t) \to \phi(t) \quad \text{FOR ALL } t \in \mathbb{R}^d.$$

__EXAMPLE__ $Y_1, Y_2, \ldots$ iid Uniform $[-1, 1]$ AND LET

$$X_n = \sqrt{\frac{3}{n}} \sum_{i=1}^{n} Y_i, \quad \text{FOR } n = 1, 2, \ldots$$

THEN, THE chf OF X_n IS

$$\phi_n(t) = \left(\frac{\sin(t\sqrt{3/n})}{t\sqrt{3/n}} \right)^n$$

THEN, SINCE

$$\sin(t) = t - t^3/6 + o(t^3) \quad \text{SO THAT}$$

$$\frac{\sin(t\sqrt{3/n})}{t\sqrt{3/n}} = 1 - \frac{t^2}{2n} + o(1/n) \quad \text{FOR EACH } t$$

THEN $\phi_n(t) \to \exp\{-t^2/2\}$ AS $n \to \infty$ FOR EACH t.

THE chf OF $N(0,1)$

<u>Corollary</u> IF $\lim_{n\to\infty} \phi_n(t)$ EXISTS FOR ALL t AND IS CONTINUOUS AT ZERO, THEN THE POINT-WISE LIMITS GIVE A chf AND THE DISTRIBUTIONS CONVERGE WEAKLY TO A DISTRIBUTION WITH THAT chf.

<u>REMARK</u> THE CONDITION OF CONTINUITY AT ZERO OF THE LIMITING FUNCTION IS NECESSARY. AS AN EXAMPLE LET $X_n \sim N(0,n)$

THEN $\phi_n(t) = \exp\{-n\frac{t^2}{2}\} \to 0$ AS $n \to \infty$ FOR ALL $t \neq 0$

BUT $\phi_n(0) = 1$ ALL n

SO $\lim_{n\to\infty} \phi_n(t)$ IS NOT CONTINUOUS AT ZERO!

<u>Corollary</u> (CRAMÉR-WALD DEVICE) IF $\{X_n\}$ IS A SEQUENCE OF

RANDOM VECTORS IN $\mathbb{R}^d$ AND X A RV IN $\mathbb{R}^d$

THEN

$$X_n \xrightarrow{D} X \quad \text{IIF} \quad a^T X_n \xrightarrow{D} a^T X \quad \text{FOR ALL } a \in \mathbb{R}^d$$

CENTRAL LIMIT THEOREM

<u>Theorem</u>: (LINDEBERG-FELLER CLT). LET $\{r_n\}_n$ BE AN INCREASING SEQUENCE OF INTEGERS. FOR EACH n LET $X_{n,1}, X_{n,2}, \ldots, X_{n,r_n}$ BE INDEPENDENT CENTERED RV'S SUCH THAT $X_{n,k}$ HAS VARIANCE $\sigma^2_{n,k}$.

$$\begin{array}{lllll} X_{1,1} & \ldots & X_{1,r_1} & & \\ X_{2,1} & \ldots & X_{2,r_1}, & \ldots & X_{2,r_2} \\ \vdots & & & & \end{array}$$

$\rightarrow$ INDEPENDENT ACROSS ROWS

DEFINE $\sigma_n^2 = \sum_{k=1}^{r_n} \sigma^2_{n,k}$ AND $S_n = \sum_{k=1}^{r_n} X_{n,k}$

ASSUME THAT, FOR EACH $\varepsilon > 0$,

LF CONDITION $\rightarrow$

$$\lim_{n \to \infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{r_n} \mathbb{E}\left[X_{n,k}^2 \, \mathbb{1}\{|X_{n,k}| \geq \varepsilon \sigma_n\}\right] = 0$$

THEN

$$\frac{S_n}{\sigma_n} \xrightarrow{D} N(0,1).$$

<u>Remark</u>: THE RESULT HOLDS UNIFORMLY OVER TRIANGULAR ARRAYS

ONLY REQUIRES SECOND MOMENT CONTROL!

PROOF USES chf's AND CONTINUITY THEOREM

<u>EXAMPLE</u>: $X_1, X_2, \ldots$ iid $(0, \sigma^2)$. THEN $r_n = n$ $\quad X_{n,k} = X_k$ ALL n.

$\sigma_n^2 = n\sigma^2$ AND LF CONDITION IS

$$\frac{1}{\sigma^2} \mathbb{E}\Big[\underbrace{X^2 \mathbb{1}\{|X| \geq \varepsilon\sqrt{n}\,\sigma\}}\Big] \to 0$$

BY DCT

$\downarrow$

SO IF $X_1, X_2, \ldots \overset{iid}{\sim} (\mu, \sigma^2)$

$$\frac{\sqrt{n}}{\sigma}\left(\bar{X}_n - \mu\right) \xrightarrow{D} N(0,1)$$

<u>Remark</u> THE LF CONDITION IS ESSENTIALLY NECESSARY: IF

$$\max_k \Pr\left(\frac{|X_{n,k}|}{\sigma_n} \geq \varepsilon\right) \to 0 \quad \text{AS } n \to \infty \text{ AND } \forall \varepsilon > 0$$

AND $\dfrac{S_n}{\sigma_n} \xrightarrow{D} N(0,1)$

THEN THE LF CONDITION HOLDS!

<u>EXAMPLE</u> (LYAPUNOV CONDITION FOR CLT) ASSUME $\exists\, \delta > 0$ S.T. $\mathbb{E}\left[|X_{n,k}|^{2+\delta}\right] < \infty$ ALL n AND k AND

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{\sigma_n^{2+\delta}} \left[\mathbb{E}\,|X_{n,k}|^{2+\delta}\right] = 0$$

THEN LF-CONDITION HOLDS.

<u>Example</u>: $Y_1, Y_2, \ldots$ INDEPENDENT S.T. $Y_k \sim \text{Poisson}(1/k)$

AND LET $X_{n,k} = Y_k - \frac{1}{k}$ ALL n AND $k \leq n$

THEN $\sigma_n^2 = \sum_{k=1}^{n} \frac{1}{k} = L_n$. USE LYAPUNOV'S CONDITION WITH $\delta = 1$. THEN $\mathbb{E}\left[X_{n,k}^3\right] = \frac{1}{k}$ AND

$$\mathbb{E}\left[|X_{n,k}|^3\right] \leq \frac{5}{k}$$

SO THE SUM IN LYAPUNOV CONDITION IS BOUNDED BY

$$\frac{5}{\sqrt{L_n}} \to 0 \quad \text{AS } n \to \infty$$

$$\hookrightarrow \quad \frac{\sum_{k=1}^{n} Y_k - L_n}{\sqrt{L_n}} \xrightarrow{D} N(0,1)$$

ALSO

$$\frac{\sum_{k=1}^{n} Y_k - \log n}{\sqrt{\log n}} \xrightarrow{D} N(0,1)$$

SEE ALSO THE BERNOULLI EXAMPLES IN NOTES (EXAMPLE 23)

CLT FOR $X_{n,k} \sim \text{Bernoulli}(1/k)$ HOLDS WITH $r_n = n$

BERRY-ESSEEN CLT

ASSUME A TRIANGULAR ARRAY SETTING $X_{n,k} \sim (\mu_{n,k}, \sigma_{n,k}^2)$ INDEPENDENT WITHIN EACH ROW OF THE ARRAY.

LET

$$Y_k = \frac{X_{n,k} - \mu_{n,k}}{\left(\sum_{k=1}^{r_n} \sigma_{n,k}^2\right)^{1/2}} \quad \text{AND} \quad W_{r_n} = \sum_{k=1}^{r_n} Y_k \sim (0,1)$$

THEN

SMALL UNIVERSAL CONSTANT $< 1/2$

$$\sup_{x \in \mathbb{R}} \left| \Pr(W_{r_n} \leq x) - \Pr(\underset{N(0,1)}{Z} \leq x) \right| \leq C \frac{\sum_{k=1}^{r_n} \mathbb{E}\left[|X_{n,k} - \mu_{n,k}|^3\right]}{\left(\sum_{k=1}^{r_n} \sigma_{n,k}^2\right)^{3/2}}$$

IF $X_{n,k} = X_k \overset{iid}{\sim} (0, \sigma^2)$ S.T. $\mathbb{E}[|X|^3] = \mu_3$

THE BERRY-ESSEEN BOUND IS

$$C \frac{n \mu_3}{(n\sigma^2)^{3/2}} = C \frac{\mu_3}{\sqrt{n}\,\sigma^3} \to 0$$

AS $n \to \infty$ LIKE $\frac{1}{\sqrt{n}}$