36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 17: WED, OCT 28, 2020

■ CONVERGENCE IN DISTRIBUTION, WEAK CONVERGENCE OF PROB. MEASURES

CONVERGENCE IN LAW

LET $(\mathcal{X}, \mathcal{B})$ BE A TOPOLOGICAL SPACE (IN FACT, THINK OF METRIC SPACES) ENDOWED ITS BOREL σ-FIELD. LET $\{X_n\}$ AND X BE RV'S TAKING VALUES IN $\mathcal{X}$ FROM SOME PROB. SPACE.

Def $\{X_n\}$ CONVERGES IN DISTRIBUTION TO X WHEN

$$\lim_{n \to \infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$$

FOR ALL BOUNDED CONTINUOUS FUNCTIONS $f: \mathcal{X} \to \mathbb{R}$.

WRITE THIS $X_n \xrightarrow{D} X$.

REMARK LET $Z_1, Z_2, \ldots \overset{iid}{\sim} N(0,1)$. LET $X_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i$

THEN, TRIVIALLY, $X_n \sim N(0,1)$ ALL n.

SO $X_n \xrightarrow{D} Z_1$. BUT $X_n \overset{P}{\not\to} Z_1$.

REMARK INSTEAD OF THE CLASS OF BOUNDED CONTINUOUS FUNCTIONS, WE COULD ALSO CONSIDER, IN $\mathbb{R}^d$, THE CLASS OF CONTINUOUS FUNCTIONS AND VANISH AT INFINITY (VANISHING OUTSIDE OF A COMPACT SET) OR OF BOUNDED LIPSCHITZ FUNCTIONS.

THEOREM (PORTMANTEAU THEOREM). ASSUME $\mathcal{X}$ TO BE A METRIC SPACE

HAS A DISTANCE FUNCTION

THEN $X_n \xrightarrow{D} X$ IS EQUIVALENT TO

i) FOR EACH CLOSED SET $C \subseteq \mathcal{X}$

$$\limsup_{n \to \infty} \mu_{X_n}(C) \leq \mu_X(C)$$

WHERE μ_{X_n} AND μ_X ARE THE PROB. DISTR. OF X_n AND X.

ii) FOR EACH OPEN SET $A \subseteq \mathcal{X}$

$$\liminf_{n \to \infty} \mu_{X_n}(A) \geq \mu_X(A)$$

iii) FOR EACH $B \in \mathcal{B}$ S.T. $\mu_X(\partial B) = 0$

$$\lim_{n \to \infty} \mu_{X_n}(B) = \mu_X(B).$$

$\partial B = \bar{B} \setminus B^\circ$

INTERIOR UNION OF ALL OPEN SETS IN B

CLOSURE INTERSECTION OF ALL CLOSED SETS CONTAINING IT

REMARK: CONVERGENCE IN DISTRIBUTION IS A PROPERTY OF THE SEQUENCE $\{\mu_{X_n}\}$ AND OF μ_X $\Rightarrow$ $\{X_n\}$ AND X NEED NOT BE DEFINED OVER THE SAME PROB. SPACE!

REMARK: IN DEF. OF CONVERGENCE IN DISTR. CONTINUITY IS NECESSARY:

LET $\{X_n\}$ S.T. $\Pr(X_n = 1/n) = 1 \Rightarrow X_n \xrightarrow{D} 0$

LET $f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 1 \end{cases}$

THEN $f(X_n) = 1$ ALL n BUT $f(X) = 0$

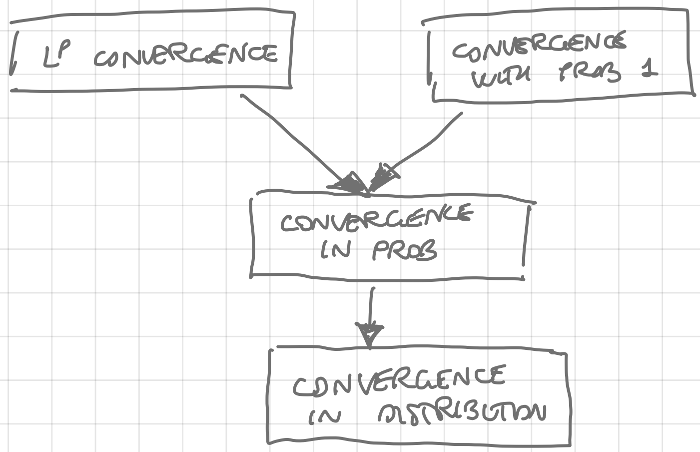
Prop. 5: LET $\{X_n\}$ AND X BE DEFINED ON $(\mathcal{X}, \mathcal{B})$
METRIC SPACE

THEN

1) IF $X_n \xrightarrow{P} X$, THEN $X_n \xrightarrow{D} X$

2) IF X IS DEGENERATE $\mu_X(c) = 1$ SOME $c \in \mathcal{X}$ THEN
$X_n \xrightarrow{D} X$ IMPLIES $X_n \xrightarrow{P} X$.

REMARK:



Pf/ LET C BE A CLOSED SET IN $\mathcal{X}$. FOR EACH INTEGER m

LET $C_m = \{x \in \mathcal{X} : d(x, C) \le 1/m\}$ [C_m IS ALSO CLOSED]

$$d(x,C) = \inf_{y \in C} d(x, y)$$

NEXT, NOTICE THAT

$$\{X \in C_m^c\} \cap \{d(X_n, X) \le 1/m\} \subseteq \{X_n \in C^c\}$$

$$\hookrightarrow \{X_n \in C\} \subseteq \{X \in C_m\} \cup \{d(X_n, X) > 1/m\}$$

SO $\mu_{X_n}(C) \le \mu_X(C_m) + P_n(d(X_n, X) > 1/m)$

OK, BECAUSE X_n AND X ARE DEFINED ON SAME PROB SPACE!

TAKE $\limsup_{n\to\infty}$ ON BOTH SIDES

$$\limsup_{n\to\infty} \mu_{X_n}(C) \le \mu_X(C_m) \qquad \left[\text{BECAUSE } \Pr(d(X_n, X) > 1/m) \to 0\right]$$

TAKE LIMIT AS $m \to \infty$ $\quad \mu_X(C_m) \downarrow \mu_X(C)$

$$\hookrightarrow \limsup_{n\to\infty} \mu_{X_n}(C) \le \mu_X(C) + \varepsilon \quad \forall \varepsilon > 0.$$

THE FIRST RESULT FOLLOWS.

FOR SECOND RESULT, PICK $\varepsilon > 0$ (SMALL), LET $B(c,\varepsilon)$ BE THE OPEN BALL OF RADIUS ε AND CENTER c $\left[B(c,\varepsilon) = \{x \in \mathcal{X} : d(x,c) < \varepsilon\}\right]$

THEN

$$\Pr(d(X_n, c) \ge \varepsilon) = \Pr(X_n \in \underbrace{B^c(c,\varepsilon)}_{\text{CLOSED SET}})$$

$$\le \limsup_{n\to\infty} \Pr(X_n \in B^c(c,\varepsilon))$$

BY PART i) OF PORTMANTEAU THEOREM

$$\le \Pr(X \in B^c(c,\varepsilon))$$

$$= 0$$

■

■ THE CASE $\mathcal{X} = \mathbb{R}$, $\mathcal{B} = \mathcal{B}^1$

BY PART iii) OF PORTMANTEAU THEOREM, TAKE $B = (-\infty, x]$ SO THAT

$$\mu_{X_n}((-\infty, x]) = \Pr(X_n \le x) \to \Pr(X \le x) = \mu_X((-\infty, x])$$

PROVIDED THAT $\mu_X(\underbrace{\{x\}}_{\partial B}) = 0$, IS NECESSARY FOR

$$X_n \xrightarrow{D} X.$$

THIS NECESSARY CONDITION IS EQUIVALENT TO:

$$\underbrace{F_{X_n}(x)}_{\text{cdf of } X_n} \to \underbrace{F_X(x)}_{\text{cdf } X} \quad \text{AT EACH POINT } x \text{ WHERE } F_X \text{ IS CONTINUOUS}$$

↳ SAYS THIS CONDITION IS ALSO SUFFICIENT!

Lemma 8 TAKE $(\mathcal{X}, \mathcal{B}) = (\mathbb{R}, \mathcal{B}^1)$. LET F_{X_n} BE THE cdf OF X_n AND F_X BE THE cdf OF X. THEN

$X_n \xrightarrow{D} X$ IIF $F_{X_n}(x) \to F_X(x)$ FOR ALL x AT WHICH F_X IS CONTINUOUS.

THE PROOF OF THIS RESULT RELIES ON

Thm (SKOROKHOD) ASSUME THAT $F_{X_n}(x) \to F_X(x)$ FOR ALL x AT WHICH F_X IS CONTINUOUS. THEN THERE EXISTS A PROB. SPACE $(\Omega, \mathcal{F}, P)$ $\left[([0,1], \mathcal{B}^1, \lambda) \right]$ AND $\{Y_n\}$ AND Y ON THIS SPACE S.T. $Y_n \xrightarrow{a.s} Y$ AND Y_n AND Y HAVE cdf F_{X_n} AND F_X

Pf SKETCH/ RECALL THAT IF G IS A cdf, THEN

$p \in (0,1)$, $G^{-1}(p) = \{ \inf x : G(x) \geq p \}$

IN PARTICULAR

$$G(G^{-1}(p)) \geq p \qquad p \in (0,1).$$

TAKE $([0,1], \mathcal{B}^1, \lambda)$ AND DEFINE $Y_n(\omega) = F_{X_n}^{-1}(\omega)$

λ ↓ LEBESGUE MEASURE

$$Y(\omega) = F_X^{-1}(\omega)$$

THEN

$$\begin{aligned} \Pr(Y \leq y) &= \Pr(F_X^{-1}(\omega) \leq y) \\ &= \Pr(F_X F_X^{-1}(\omega) \leq F_X(y)) \\ &\leq \Pr(\omega \leq F_X(y)) = F_X(y). \end{aligned}$$

TO FINISH, SHOW THAT

$$Y_n \xrightarrow{a.s.} Y.$$

THIS RESULT EXTENDS IMMEDIATELY TO THE CASE $\mathcal{X} = \mathbb{R}^d$, $\mathcal{B} = \mathcal{B}^d$.

IN PARTICULAR $X_n \xrightarrow{D} X$ IFF $F_{X_n}(x) \to F_X(x)$ AT ALL POINTS OF CONTINUITY x OF F_X. [$x \in \mathbb{R}^d$,

$$F_{X_n}(x) = P_n \begin{pmatrix} X_n(1) \le x(1) \\ \vdots \\ X_n(d) \le x(d) \end{pmatrix}]$$

<u>REMARK</u>: RECALL THAT $X_n \xrightarrow{P} X$ AND $Y_n \xrightarrow{P} Y$

IS EQUIVALENT TO $\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{P} \begin{pmatrix} X \\ Y \end{pmatrix}$ HW!

THIS IS NO LONGER THE CASE FOR CONVERGENCE IN DISTR.!

SUPPOSE $X_n \xrightarrow{D} X$ AND $Y_n \xrightarrow{D} Y$. THEN, IN GENERAL

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \not\xrightarrow{D} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

<u>EXAMPLE</u>: LET $X_n = \begin{cases} X & n \text{ EVEN} \\ 1 - X & n \text{ ODD} \end{cases}$

WHERE $X \sim \text{Uniform}(0,1)$

LET $Y_n = X$ ALL n.

THEN $Y_n \xrightarrow{D} X$ AND $X_n \xrightarrow{D} X$ BECAUSE

$X \sim \text{Uniform}(0,1)$ IFF $1-X \sim \text{Uniform}(0,1)$

BUT $\begin{pmatrix} X_n \\ Y_n \end{pmatrix}$ DOES NOT CONVERGE TO ANYTHING

HOWEVER IF $X_n \perp\!\!\!\perp Y_n$ ALL n THEN

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} X \\ Y \end{pmatrix}$$