36710 – 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 2 : WED, SEP 2, 2020

- DO NOT SHARE THE ZOOM LINK OR THE LINKS TO THE CLASS RECORDINGS !!

- I WILL POST HW 1 LATER TODAY !!

- RECALL THE DEFINITION OF FIELD AND σ-FIELD FROM LAST TIME:

Ω UNIVERSE SET. A COLLECTION $\mathcal{F}$ OF SUBSETS OF Ω IS A FIELD WHEN :

$\mathbb{R}$

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \rightarrow A^c \in \mathcal{F}$ $\rightarrow \phi \in \mathcal{F}$

(iii) $A_1, A_2 \in \mathcal{F} \rightarrow A_1 \cup A_2 \in \mathcal{F}$ $\rightarrow$ BY (ii) $A_1 \cap A_2 \in \mathcal{F}$

$\Downarrow$

$\mathcal{F}$ IS CLOSED WRT FINITE UNIONS AND INTERSECTIONS

- A FIELD $\mathcal{F}$ IS A σ-FIELD IF PROPERTY (iii) IS REPLACED BY:

(iii)' FOR EVERY SEQUENCE $\{A_n\}$ OF SETS IN $\mathcal{F}$, $\bigcup_n A_n \in \mathcal{F}$

$\bigcap_n A_n \in \mathcal{F}$

$\Downarrow$

CLOSED WRT TO COUNTABLE UNIONS AND INTERSECTIONS

$\Omega = \mathbb{R}$

- EXAMPLE: LET $\mathcal{U}$ BE THE COLLECTIONS OF UNIONS OF FINITELY MANY DISJOINT SETS OF THE FORM $(a,b] = \{x \in \mathbb{R} : a < x \le b\}$

$(-\infty, b]$ $\quad -\infty < a \le b < \infty$

(a, ∞)

$(-\infty, \infty)$

$\downarrow$

THIS IS A FIELD

BUT IT IS NOT A σ-FIELD:

(a,b) IS NOT IN $\mathcal{U}$ BUT $(a,b) = \bigcup_n \left(a, b - \frac{1}{n}\right]$

$\{a\}$ IS NOT IN $\mathcal{U}$ BUT $\{a\} = \bigcap_n \left(a - \frac{1}{n}, a\right]$

$\downarrow$ SINGLETON

DEF. (MEASURABLE SPACE): $(\Omega, \mathcal{F})$

Ω: UNIVERSE SET

$\mathcal{F}$: σ-FIELD OVER Ω

$\hookrightarrow$ COLLECTION OF MEASURABLE SETS

- EXAMPLES: TRIVIAL σ-FIELD: $\mathcal{F} = \{\emptyset, \Omega\}$

POWER SET: $\mathcal{F} = 2^{\Omega}$

- DEF: (GENERATED σ-FIELDS). LET C BE A COLLECTION OF SUBSETS OF Ω

THE GENERATED σ-FIELD $\sigma(C)$: INTERSECTION OF ALL σ-FIELDS CONTAINING C

EXAMPLE: $C = \{A\}$, $A \subseteq \Omega$

$$\sigma(C) = \{\Omega, \emptyset, A, A^c\}$$

DEF (BOREL σ-FIELD): LET Ω BE A TOPOLOGICAL SPACE. LET C BE THE COLLECTION OF OPEN SETS. $\sigma(C)$ IS THE BOREL σ-FIELD. IF $\Omega = \mathbb{R}^1$, $\mathcal{B}^1$ BOREL σ-FIELD

$\downarrow$

$$C = \{(a,b), -\infty < a < b < \infty\}$$

- THERE ARE SUBSETS OF $\mathbb{R}$ THAT ARE <u>NOT</u> BOREL MEASURABLE (THEY DO NOT BELONG TO THE BOREL σ-FIELD)

MEASURES

<u>NOTATION</u>: $\overline{\mathbb{R}}$: EXTENDED REAL LINE $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$

<u>DEF</u>: LET $(\underbrace{\Omega}_{\text{UNIVERSE SET}}, \underbrace{\mathcal{F}}_{\sigma\text{-FIELD}})$ BE A MEASURABLE SPACE. A FUNCTION $\mu : \mathcal{F} \to \overline{\mathbb{R}}_+$

IS A <u>MEASURE</u> IF

(i) $\mu(\emptyset) = 0$

(ii) FOR EVERY SEQUENCE $\{A_n\}$ OF MUTUALLY DISJOINTS MEASURABLE SETS,

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$$

COUNTABLE ADDITIVITY

- A MEASURE SPACE $(\overbrace{\Omega}^{\text{UNIVERSE SET}}, \overbrace{\mathcal{F}}^{\sigma\text{-FIELD}}, \overbrace{\mu}^{\text{MEASURE}})$.
- A MEASURE ON A FIELD $\mathcal{F}'$, IS A FUNCTION $\mu : \mathcal{F}' \to \overline{\mathbb{R}}_+$ THAT SATISFIES (i) AND (ii) PROVIDED THAT $\bigcup_n A_n \in \mathcal{F}'$
- A MEASURE CAN BE FINITE $(\mu(\Omega) < \infty)$ OR INFINITE $(\mu(\Omega) = \infty)$.
- <u>DEF</u> A PROBABILITY MEASURE IS A MEASURE S.T. $\mu(\Omega) = 1$

IF P IS A PROB., $(\Omega, \mathcal{F}, P)$ IS A PROBABILITY SPACE.

THE PROBABILITY OF AN EVENT IS JUST $P(A)$ SOME $A \in \mathcal{F}$

EXAMPLES : 1) Ω COUNTABLE : $\Omega = \{\omega_1, \omega_2, \omega_3, \dots\}$

$\mathcal{F} = 2^{\Omega}$

LET $\{p_i\}_{i=1,2,\dots}$ BE S.T. $p_i \in [0,1]$ $\sum_i p_i = 1$

THEN THE FUNCTION

$P : 2^{\Omega} \to [0,1]$ GIVEN BY $P(A) = \sum_{i : \omega_i \in A} p_i$

IS A PROBABILITY MEASURE.

2) LET $\Omega = \mathbb{R}$ AND $\mathcal{F} = \mathcal{B}^1$. DEFINE :

$$P((-\infty, a]) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \qquad \text{ALL } (-\infty, a],\ a \in \mathbb{R}$$

DOES THIS DEFINE A PROB. MEASURE ON $\mathcal{B}^1$?

DEF (COUNTING MEASURE) Ω ANY SET , $\mathcal{F} = 2^{\Omega}$ AND FOR ANY $A \in \mathcal{F}$

LET $\mu(A) = |A|$

↳ CARDINALITY OF A .

→ REGULARITY CONDITION

DEF (σ - FINITE MEASURE) $(\Omega, \mathcal{F}, \mu)$ — UNIVERSE SET, σ-FIELD, MEASURE — μ IS SAID TO BE σ-FINITE

IF THERE EXISTS A COUNTABLE COLLECTION OF MEASURABLE SETS

$\{A_1, A_2, \dots\}$ S.T. $\mu(A_n) < \infty$ ALL n AND $\bigcup_n A_n = \Omega$

EXAMPLES : • LET Ω BE COUNTABLY INFINITE AND μ BE THE COUNTING MEASURE ON IT. μ IS σ-FINITE

• WHAT IF Ω IS UNCOUNTABLE ? μ (COUNTING MEASURE) IS NOT σ - FINITE

• IF μ IS FINITE (IN PARTICULAR, IF μ IS A PROBABILITY MEASURE) THEN IT IS σ-FINITE.

PROPERTIES OF MEASURES

ASSUME THROUGHOUT A MEASURE $(\Omega, \mathcal{F}, \mu)$

IF $A \subseteq B$ (BOTH MEASURABLE) $\rightarrow \mu(A) \leq \mu(B)$

PF/
$$\mu(B) = \mu\big(\underbrace{A \uplus}_{\text{DISJOINT UNION}} (\underbrace{B \cap A^c}_{B \setminus A})\big) \underset{\text{COUNTABLE ADDITIVITY}}{=} \mu(A) + \underbrace{\mu(B \cap A^c)}_{\geq 0} \geq \mu(A) \blacksquare$$

MORE GENERALLY, IF $\{A_n\}$ IS A SEQUENCE OF MEASURABLE SETS (NOT NECESSARILY DISJOINT)

$$\rightarrow \mu\Big(\bigcup_n A_n\Big) \leq \sum_n \mu(A_n)$$

PF/ DEFINE A NEW SEQUENCE $\{B_n\}$ OF MEASURABLE SETS AS FOLLOWS:

$B_1 = A_1$ AND FOR $n \geq 2$ $\quad B_n = A_n \setminus \bigcup_{i=1}^{n-1} B_i$

$$= A_n \cap A_{n-1}^c \cap \dots \cap A_1^c$$

$$\rightarrow \underbrace{\biguplus_n}_{\text{DISJOINT UNION}} B_n = \bigcup_n A_n$$

SO
$$\mu\Big(\bigcup_n A_n\Big) = \mu\Big(\biguplus_n B_n\Big) \underset{\text{COUNTABLE ADDITIVITY}}{=} \sum_n \mu(B_n) \leq \sum_n \mu(A_n)$$

BECAUSE $B_n \subseteq A_n$ $\blacksquare$

IF μ IS A PROB. MEASURE, THIS PROPERTY IS KNOWN AS

"THE UNION BOUND"

- 2 INTERESTING PROPERTIES OF PROBABILITY MEASURES:

(i) IF $\mu(A_n) = 0$ ALL n $\rightarrow \mu\Big(\bigcup_n A_n\Big) = 0$

(ii) IF $\mu(A_n) = 1$ ALL n $\rightarrow \mu\Big(\bigcap_n A_n\Big) = 1$

- <u>DEF</u> (ALMOST SURE / ALMOST EVERYWHERE) SUPPOSE THAT A CERTAIN PROPERTY HOLDS FOR ALL $\omega \in A^c$ WHERE $\mu(A) = 0$

THEN, WE SAY THAT THE PROPERTY HOLDS ALMOST EVERYWHERE, OR a.e. $[\mu]$. IF $\mu = P$ A PROB. MEASURE, WE SAY INSTEAD ALMOST SURELY, a.s. $[P]$.

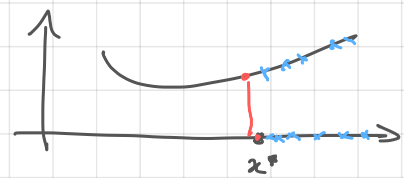
- <u>DEF</u> (SUPPORT). $(\underset{\mathbb{R}}{\Omega}, \underset{\mathcal{B}}{\mathcal{F}}, P)$. THE <u>SUPPORT</u> OF P IS THE SMALLEST CLOSED SET S IN $\mathbb{R}$ S.T. $P(S) = 1$

IF $A \subseteq S^c \Rightarrow P(A) = 0$

CONTINUITY OF MEASURES

RECALL THAT IF $f : \mathbb{R} \to \mathbb{R}$ IS A CONTINUOUS FUNCTION ON ITS DOMAIN THEN

$$f(x^*) = \lim_{x_n \to x^*} f(x_n)$$



THE SAME IS TRUE FOR MEASURES!