SDS 387, Fall 2024
Homework 1

Due September 17, by midnight on Canvas.

Warning: more problems will be added this and next week.

1. Let $\{x_n\}$ be a sequence of numbers. Describe the mathematical statements: $x_n = \Omega(1)$, $x_n = \omega(1)$ and $x_n = \Theta(1)$.

2. **Limit superior and limit inferior.**

    (a) Let $\{A_n\}$ be a sequence of events (an event is a collection of outcomes). Argue that an outcome belongs to $\limsup_n A_n$ if and only if it belongs to infinitely many events A_n's and that it belongs to $\liminf_n A_n$ if and only if there exists an integer N such that the outcome belongs to all the events A_n with $n \geq N$. Conclude that $\liminf_n A_n \subseteq \limsup_n A_n$.

    (b) Consider the same setting above. De Morgan's Laws state that $(\cup_n A)^c = \cap_n A_n^c$ and $(\cap_n A)^c = \cup_n A_n^c$, where A^c is the complement of the set A. Use De Morgan's law to show that $(\liminf_n A_n)^c = \limsup_n A_n^c$.

    (c) Let A_n be $(-1/n, 1]$ if n is odd and $(-1, 1/n]$ if n is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.

    (d) **Bonus Problem**. Let A_n the interior of the ball in $\mathbb{R}^2$ with unit radius and center $\left(\frac{(-1)^n}{n}, 0\right)$. Find $\limsup_n A_n$ and $\liminf_n A_n$.

3. Let $X_1, X_2, \ldots$ be a sequence of 0-1 Bernoulli random variables such $X_n \sim \text{Bernoulli}(1/n^2)$. Let $X = \sum_{n=1}^{\infty} X_n$. What is $\mathbb{P}(X < \infty)$?

4. Ferguson, problem 5, page 12.

5. Prova Markov's inequality: if X is a non-negative random variable, then for any $\epsilon > 0$

$$\mathbb{P}(X \geq \epsilon) \leq \frac{\mathbb{E}[X]}{\epsilon}.$$

Markov's inequality is almost always a loose upper bound, but there are rare cases when it is sharp. Find an example in which it holds exactly. *Hint: take X to be the indicator function of a set and select the right ϵ.*

Prove the PaleyZygmund inequality, a reverse Markov inequality of sort: if X is a non-negative random variable with two or more moments, then, for any $\alpha \in (0, 1)$,

$$\mathbb{P}(X \geq \alpha \mathbb{E}[X]) \geq (1 - \alpha)^2 \frac{\mathbb{E}[^2 X]}{\mathbb{E}[X^2]}.$$

6. Let $X_1, \ldots, X_n$ *i.i.d.* univariate random variables with common distribution function F_X. Given $\alpha \in (0, 1)$, use the DKW inequality given in class to construct a $1 - \alpha$ confidence band for F_X, a pair of random functions (random because dependent on $X_1, \ldots, X_n$), say $\hat{F}_\alpha^{\text{lower}}$ and $\hat{F}_\alpha^{\text{upper}}$, such that

$$\mathbb{P}\left(\hat{F}_\alpha^{\text{lower}}(x) \leq F_X(x) \leq \hat{F}_\alpha^{\text{upper}}(x), \forall x \in \mathbb{R}\right) \geq 1 - \alpha.$$