36710 – 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 13: WED, OCT 14, 2020

LAST TIME: L^p SPACES

$(\Omega, \mathcal{F}, \mu)$ MEASURE SPACE. FOR $p \geq 1$, THE L^p SPACE IS THE SPACE OF

- EQUIVALENCE CLASSES OF FUNCTIONS f S.T. $\left(\int |f|^p d\mu\right)^{1/p} < \infty$

  THE MAPPING TAKING EQUIVALENCE CLASS $[f]$ TO THE ABOVE INTEGRAL, $\|f\|_p$, IS A NORM. WHEN $p = \infty$,

  $$\|f\|_\infty = \text{ess sup}(f).$$

- IF $f \in L^p$, THEN $a \cdot f \in L^p$ ALL $a \in \mathbb{R}$
- IF $f, g \in L^p$, THEN $f+g \in L^p$ $\left[|a+b|^p \leq (|a|+|b|)^p \leq \left(2\max\{|a|,|b|\}\right)^p \leq 2^p(|a|^p + |b|^p) \right]$

- HÖLDER INEQUALITY

Thm IF $f \in L^p$ AND $g \in L^q$, p, q CONJUGATE $\rightarrow \frac{1}{p} + \frac{1}{q} = 1$, THEN

$$\| fg \|_1 \le \| f \|_p \| g \|_q$$

IF $q = \infty$
$\| fg \|_1 \le \| f \|_1 \| g \|_\infty$

GENERALIZATION: $f_1, \dots f_k$ ARE S.T. $f_i \in L^{p_i}$ AND $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ THEN

$$\| \prod_i |f_i| \|_1 \le \prod_{i=1}^{k} \| f_i \|_{p_i}.$$

Corollary: (CAUCHY-SCHWARTZ INEQUALITY) $p = q = 2$

$$\int |fg| \, d\mu \le \sqrt{\int f^2 \, d\mu} \sqrt{\int g^2 \, d\mu}$$

IF X AND Y ARE R.V.'S THIS IMPLIES

$$\mathbb{E}[|XY|] \le \sqrt{\mathbb{E}[X^2] \, \mathbb{E}[Y^2]}$$

Pf OF HÖLDER / IF $a, b > 0$ $\lambda \in (0,1)$ THEN

$$\lambda a + (1-\lambda) b \ge a^\lambda b^{1-\lambda}$$

[ASIDE: YOUNG'S INEQUALITY: $|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$]

ASSUME p OR q ARE NOT ∞. LET

$u = |f|^p$ AND $v = |g|^q \Rightarrow u, v \in L_1$. SO

APPLY THE ABOVE INEQ. WITH $\lambda = \frac{1}{p}$ AND $1 - \lambda = \frac{1}{q}$ TO GET

$$\left(\frac{u}{\int u \, d\mu} \right)^{1/p} \left(\frac{v}{\int v \, d\mu} \right)^{1/q} \le \frac{1}{p} \frac{u}{\int u \, d\mu} + \frac{1}{q} \frac{v}{\int v \, d\mu}$$

HOLDS POINT-WISE (FOR EACH ω)

TAKE INTEGRAL ON BOTH SIDES TO OBTAIN

$$\int u^{1/p} v^{1/q} \, d\mu \le \left(\int u \, d\mu \right)^{1/p} \left(\int v \, d\mu \right)^{1/q} \quad \blacksquare$$

USING HÖLDER INEQ. WE CAN OBTAIN VARIOUS OTHER RESULTS ABOUT L^p SPACES:

- (MINKOWSKI INEQ.) IF $f, g \in L^p \Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$

- RELATIONSHIPS AMONG L^p SPACES.

IN GENERAL, IF $p<q$, WE CANNOT CONCLUDE THAT $L^p \subset L^q$ OR $L^q \subset L^p$. FOR EXAMPLE $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}, \mathcal{B}^2, \underset{\text{LEB. MEAS.}}{\lambda})$

LET

$$f(x) = \begin{cases} x^{-3/8} & 0 < x < 1 \\ x^{-1} & 1 \leq x < \infty \\ 0 & \text{OTHERWISE} \end{cases}$$

$\hookrightarrow \quad f \notin L^1, \quad f \in L^2, \quad f \notin L^3$

- IF $\mu(\Omega) < \infty$, THEN WE HAVE AN ORDERING:

$p < q \leq \infty \Rightarrow L^q \subset L^p$ (IF $f \in L^q$ THEN $f \in L^p$)

TO SEE THIS, CONSIDER FIRST OF $q = \infty$. THEN

$$\|f\|_p^p = \int |f|^p d\mu \leq \|f\|_\infty \underbrace{\int_\Omega d\mu}_{<\infty}$$

$$\left[\text{IN FACT} \quad \lim_{p\to\infty} \|f\|_p = \|f\|_\infty \right]$$

WHILE $q < \infty$, THEN USE HÖLDER:

$$\int |f|^p d\mu = \int |f|^p \cdot \underset{\hookrightarrow \text{ TRIVIAL FUNCTION } \omega \mapsto 1}{1} \, d\mu \qquad q/p > 1$$

$$\leq \underbrace{\left[\int \left(|f|^{\cancel{p}}\right)^{q/\cancel{p}} \right]^{p/q}}_{\|f\|_q^p} \underbrace{\left[\int 1^{\frac{q-p}{q}} d\mu \right]^{1-p/q}}_{\text{CONSTAN } C(p,q,\mu(\Omega))}$$

IN PARTICULAR, IF $\mu(\Omega) = 1$ (SO μ IS A PROB. MEASURE)

$$\|f\|_1 \leq \|f\|_p \leq \|f\|_q \leq \|f\|_\infty \qquad p < q$$

- IF WE ARE DEALING WITH ℓ^p SPACES, INTERESTINGLY, THE
  ↳ μ COUNTING MEASURE
  ABOVE RELASHIONSHIP IS REVERSED:

$$p < q \leq \infty \implies \ell^p \subset \ell^q$$

AND

$$\|f\|_\infty \leq \|f\|_q \leq \|f\|_p \leq \|f\|_1$$

■ SOME OTHER IMPORTANT PROPERTIES:

<u>MARKOV'S INEQUALITY</u>: IF $f \geq 0$ THEN $\mu(\{\omega : f(\omega) \geq c\}) \leq \frac{\int f d\mu}{c}$

ALL $c > 0$.

↳ IF μ IS A PROB. MEASURE P AND f A R.V. $X \geq 0 \implies P(X \geq c) \leq \frac{E[X]}{c}$

ALL $c > 0$

PROBABILITY OF AN EVENT

$$\mathbb{P}(X \geq c) = \text{Prob.}(X \geq c)$$

mathbb{P}

$$= P(\{\omega : X(\omega) \geq c\})$$

SOMETIMES WRITTEN AS $P(X \geq c)$.

THIS NOTATION IS OFTEN TIMES ABUSED, AS FOLLOWS. SAY P IS THE PROB. DISTR. OF X. THEN PEOPLE OFTEN WRITE $P(\underline{X \geq c})$, WHEN IT SHOULD BE WRITTEN AS $P(\{x \in \mathbb{R} : x \geq c\})$

NOT A SET IN $\mathcal{B}^1$ BUT IN $\mathcal{F}$.

<u>TCHEBYCHEV INEQ.</u> LET X BE A R.V. WITH FINITE VARIANCE AND MEAN μ. THEN

$$\mathbb{P}(|X - \mu| \geq c) \leq \frac{Var(X)}{c^2}, \qquad c > 0$$

↳ $\mathbb{E}[(X-\mu)^2]$

C_r - INEQUALITY : $\mathbb{E}\left[|X+Y|^r\right] \leq C_r\left(\mathbb{E}\left[|X|^r\right] + \mathbb{E}\left[|Y|^r\right]\right)$

WHERE
$$C_r = \begin{cases} 1 & \text{IF } r \in (0,1] \\ 2^{r-1} & \text{IF } r > 1. \end{cases}$$

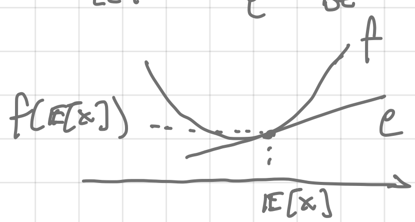
$$\left[\quad \mathbb{E}\left[(X+Y)^2\right] \leq 2\left(\mathbb{E}[X^2] + \mathbb{E}[Y^2]\right)\right]$$

JENSEN'S INEQUALITY : LET f BE A REAL VALUED FUNCTION THAT IS DEFINED OVER (a,b), $-\infty \leq a < b \leq \infty$, AND CONVEX.

IF X IS A R.V. ST. $\mathbb{P}(X \in (a,b)) = 1$, THEN

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$$

Pf/ LET ℓ BE SUPPORTING FUNCTION OF f AT $\mathbb{E}[X]$



BY CONSTRUCTION

$$\mathbb{E}[f(X)] \geq \mathbb{E}[\ell(X)] = \ell(\mathbb{E}[X]) = f(\mathbb{E}[X])$$

FIRST INEQ. STEMS FROM THE FACT THAT $f(x) \geq \ell(x)$ a.e. $\mathbb{P}$

ALSO $\mathbb{E}[f(X)] = \mathbb{E}[\ell(X)]$ IIF $f(x) = \ell(x)$

IIF f IS LINEAR OR f IS CONSTANT

$\blacksquare$

CONVERGENCE IN PROBABILITY AND WLLN

Def (CONVERGENCE IN PROB.) LET $\{X_n\}_{n=1}^{\infty}$ BE A SEQUENCE OF R.V.'S DEFINED ON THE SAME PROB. SPACE $(\Omega, \mathcal{F}, P)$. LET X BE ANOTHER RV ON THE SAME SPACE. THEN WE SAY THAT $\{X_n\}_n$ CONVERGES IN PROBABILITY TO X, $X_n \xrightarrow{P} X$, WHEN

$$\forall \varepsilon > 0, \quad P\left(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\right) \to 0 \quad \text{AS } n \to \infty$$

Remark IF X_n'S AND X TAKES VALUES ON SOME METRIC SPACE $(\mathcal{X}, \underbrace{d}_{\text{METRIC}})$, THEN THE ABOVE DEFINITION CAN BE EXTENDED TO

$$\forall \varepsilon > 0, \quad \lim_{n \to \infty} P\left(\{\omega : d(X_n(\omega), X(\omega)) > \varepsilon\}\right) = 0$$

- NOTE THAT THE DEF. DOES NOT REQUIRE $X_n(\omega) \to X(\omega)$ FOR ANY GIVEN ω.

- ANOTHER IMPORTANT POINT IS THAT CONVERGENCE IN PROB. IS A STATEMENT ABOUT THE JOINT DISTR. OF X_n AND X. EXAMPLE:

ASSUME THAT, FOR EACH n, $P(X_n = 1) = 1 - P(X_n = 0) = \frac{1}{2} \frac{n+1}{n}$ AND $X \sim \text{Bernoulli}(1/2)$. DOES IT HOLD THAT

$$X_n \xrightarrow{P} X \quad ?$$

ANSWER: WE DON'T KNOW! IF $X_n \perp\!\!\!\perp X$ ALL n, THEN NO!

SOME $\varepsilon > 0$,
$$P(|X_n - X| > \varepsilon) = P(X_n = 0 \text{ AND } X = 1) + P(X_n = 1 \text{ AND } X = 0)$$
$$= 1/2$$

ON THE OTHER HAND, IF $\mathbb{P}(X_n = 1 \mid X = 1) = 1$ AND

$\mathbb{P}(X_n = 1 \mid X = 0) = \frac{1}{n}$

THEN FOR $\varepsilon > 0$, $\mathbb{P}(|X_n - X| > \varepsilon) = \frac{1}{2n} \to 0$ AS $n \to \infty$

$\Downarrow$

SO YES! $X_n \xrightarrow{P} X$.