SDS 391-3, Fall 2025
Homework 1

Due Sept 24, by midnight on Canvas.

1. **Limit superior and limit inferior.**

   (a) Let $\{A_n\}$ be a sequence of events (an event is a collection of outcomes). Argue that an outcome belongs to $\limsup_n A_n$ if and only if it belongs to infinitely many events A_n's and that it belongs to $\liminf_n A_n$ if and only if there exists an integer N such that the outcome belongs to all the events A_n with $n \geq N$ (so it belongs to the A_n's eventually). Conclude that $\liminf_n A_n \subseteq \limsup_n A_n$.

   (b) Let A_n be $(-1/n, 1]$ if n is odd and $(-1, 1/n]$ if n is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.

   (c) **On the relationship between** $\liminf$ **and** $\limsup$ **of events and numbers.** Recall that for a sequence of numbers $\{x_n\}_{n=1,2\ldots}$,

   $$\liminf_n x_n = \inf_{n \geq 1} \sup_{m \geq n} x_m \quad \text{and} \quad \limsup_n x_n = \sup_{n \geq 1} \inf_{m \geq n} x_m$$

   For an event A_n, denote with I_{A_n} the $0-1$ random varibale that is 1 if A_n takes place and 0 otherwise. Show that

   $$I_{\limsup_n A_n} = \limsup_n I_{A_n} \quad \text{and} \quad I_{\liminf_n A_n} = \liminf_n I_{A_n}$$

   (d) **Bonus Problem**. Let A_n the interior of the ball in $\mathbb{R}^2$ with unit radius and center $\left(\frac{(-1)^n}{n}, 0\right)$. Find $\limsup_n A_n$ and $\liminf_n A_n$.

2. **On the WLLN for dependent variables**.

   (a) Suppose that $X_1, X_2, \ldots, X_n$ is a finite sequence of centered[1] random variables such that $\mathrm{Var}[X_n] \leq \sigma^2$ for all n and $\mathrm{Cov}[X_i, X_j] \geq c > 0$ for some $c > 0$ and all i and j. Show that,

   $$\lim_n \mathbb{P}(|\overline{X}_n| \geq \epsilon) \neq 0,$$

   for all sufficiently small $\epsilon > 0$. Therefore, the WLLN does not hold.

   (b) Now suppose instead that the variables are m-dependent: X_i and X_j are independent provided that $|i - j| > m$ (they may or may not be independent otherwise).

[1] the arguments can be modified to allow for non-zero means, but let's not do that.

Also assume that $\text{Cov}[X_i, X_j] \leq c$ for some c and all i and j with $|i - j| \leq m$. Show that if m is fixed, then

$$\lim_n \mathbb{P}(|\overline{X}_n| \geq \epsilon) = 0, \tag{1}$$

for all $\epsilon > 0$.

(c) Now let's allow m to grow with n (that is, for each n, $X_1, X_2, \ldots, X_n$ is m-dependent, where m is a function of n). Show that, as long as $m = o(n)$, (1) still holds true.

3. Recall that Borel-Cantelli's Second Lemma says that if $\{A_n\}$ is a sequence of *independent*[2] events such that $\sum_n \mathbb{P}(A_n) = \infty$ then $\mathbb{P}(\limsup_n A_n) = 1$. You might wonder whether the requirement of independence is needed. The answer is yes. Find an example in which all the conditions of the lemma are met except for independence and the conclusion is false.

4. Ferguson, problem 5, page 12.

5. Prova Markov's inequality: if X is a non-negative random variable, then for any $\epsilon > 0$

$$\mathbb{P}(X \geq \epsilon) \leq \frac{\mathbb{E}[X]}{\epsilon}.$$

Markov's inequality is almost always a loose upper bound, but there are rare cases when it is sharp. Find an example in which it holds exactly. *Hint: take X to be the indicator function of a set and select the right ϵ.*
Prove the PaleyZygmund inequality, a reverse Markov inequality of sort: if X is a non-negative random variable with two or more moments, then, for any $\alpha \in (0, 1)$,

$$\mathbb{P}(X \geq \alpha \mathbb{E}[X]) \geq (1 - \alpha)^2 \frac{\mathbb{E}[^2 X]}{\mathbb{E}[X^2]}.$$

6. Let $X_1, \ldots, X_n$ *i.i.d.* univariate random variables with common distribution function F_X. Given $\alpha \in (0, 1)$, use the DKW inequality given in class to construct a $1 - \alpha$ confidence band for F_X, a pair of random functions (random because dependent on $X_1, \ldots, X_n$), say $\hat{F}_\alpha^{\text{lower}}$ and $\hat{F}_\alpha^{\text{upper}}$, such that

$$\mathbb{P}\left(\hat{F}_\alpha^{\text{lower}}(x) \leq F_X(x) \leq \hat{F}_\alpha^{\text{upper}}(x), \forall x \in \mathbb{R}\right) \geq 1 - \alpha.$$

[2] meaning that the probability of any finite intersection of events in the sequence is equal to the product of their respective probabilities.

7. **Joint and marginal convergence.** Below, $\{X_n\}$ is a sequence of random vectors in $\mathbb{R}^d$ and X another random vector in $\mathbb{R}^d$.

   (a) Show that $X_n \xrightarrow{p} X$ if and only if $X_n(j) \xrightarrow{p} X(j)$ for all $j = 1, \ldots, d$. *Note: the same is true about convergence with probability one.*

   (b) Show that if $X_n \xrightarrow{d} X$, then $X_n(j) \xrightarrow{d} X(j)$ for all $j = 1, \ldots, d$.

   (c) In class, we looked at this example in $d = 2$. Set $U \sim \text{Uniform}(0,1)$ and let $X_n = U$ for all n and

   $$Y_n = \begin{cases} U & n \text{ odd}, \\ 1 - U & n \text{ even}. \end{cases}$$

   Then, $X_n \xrightarrow{d} U$ and $X_n \xrightarrow{d} U$. In class, I claimed that

   $$\begin{bmatrix} X_n \\ Y_n \end{bmatrix}$$

   does not converge in distribution (in fact, in any meaningful sense). Prove the claim.

8. Show that the c.d.f. of a random variable can have at most countably many points of discontinuity.

9. For each n, let X_n a random variable uniformly distributed on $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Show that X_n converges on distribution to $U \sim \text{Uniform}(0,1)$. Let A be the set of all rational numbers in $[0,1]$. Then $\mathbb{P}(X_n \in A) = 1$ for all n but $\mathbb{P}(X \in A) = 0$. Show that this does not violate condition *(v)* of the Portmanteau theorem, as stated in the lecture notes.