36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 11 : WED, OCT 7, 2020

- LAST TIME:
  - PRODUCT σ-FIELDS, PRODUCT MEASURES AND TONELLI / FUBINI THMS
  - INDEPENDENCE

- IF X AND Y ARE RANDOM VARIABLES ON A COMMON PROBABILITY SPACE $(\Omega, \mathcal{F}, P)$ THEY ARE INDEPENDENT WHEN $\sigma(X)$ AND $\sigma(Y)$ ARE INDEPENDENT [FOR EACH $A \in \sigma(X)$ AND $B \in \sigma(Y)$, $P(A \cap B) = P(A) \cdot P(B)$]

<u>Def</u> (OF MUTUAL INDEP. FOR A COLLECTION OF RV'S). LET $(\Omega, \mathcal{F}, P)$. LET $\{(S_t, \mathcal{A}_t), t \in T\}$ BE MEASURE SPACES.
ARBITRARY SET

LET $X_t : \Omega \to S_t$ BE $\mathcal{F}/\mathcal{A}_t$ - MEAS. FUNCTIONS. (THESE ARE RV'S!) FOR EACH $t \in T$. THEN $\{X_t, t \in T\}$ ARE MUTUALLY INDEP. WHEN THEIR σ-FIELDS $\{\sigma(X_t), t \in T\}$ ARE MUTUALLY INDEPENDENT.

<u>Thm</u> LET $(\Omega, \mathcal{F}, P)$ BE A PROB. SPACE AND $(S_1, \mathcal{A}_1)$ AND $(S_2, \mathcal{A}_2)$ BE MEASURABLE SPACES. LET $X_i : \Omega \to S_i$ $i = 1, 2$ BE RANDOM VARIABLES AND SET $X = (X_1, X_2)$ THE DISTRIBUTION OF $X : \Omega \to (S_1, S_2)$, μ_X, IS THE PRODUCT MEASURE $\mu_{X_1} \otimes \mu_{X_2}$ IFF X_1 AND X_2 ARE INDEPENDENT.

Pf/ IF DIRECTION. ASSUME X_1 AND X_2 ARE INDEP. THEN FOR ANY MEAS. RECT. $A_1 \times A_2$

$\{\omega : X_1(\omega) \in A_1\}$

$$\mu_X(A_1 \times A_2) = P(\{X_1 \in A_1\} \cap \{X_2 \in A_2\})$$
$$= P(\{X_1 \in A_1\})\, P(\{X_2 \in A_2\})$$
$$= \mu_{X_1}(A_1) \cdot \mu_{X_2}(A_2)$$

FOR THE OTHER DIRECTION, ASSUME μ_X IS THE PRODUCT MEASURE. THEN

$$\Pr(X_1 \in A_1, X_2 \in A_2) = P(\{X_1 \in A_1\} \cap \{X_2 \in A_2\})$$
$$= \mu_X(A_1 \times A_2)$$
$$= \mu_{X_1}(A_1) \cdot \mu_{X_2}(A_2)$$
$$= P(\{X_1 \in A_1\})\, P(\{X_2 \in A_2\}) \qquad \blacksquare$$

EXTENSION TO MUTUAL INDEPENDENCE IS DIRECT.

INFINITE PRODUCT SPACES AND STOCHASTIC PROCESSES

Def LET $(\Omega, \mathcal{F}, P)$ BE A PROBABILITY SPACE AND T AN ARBITRARY SET. SUPPOSE THAT FOR EACH $t \in T$ THERE EXISTS A MEASURABLE SPACE $(\mathcal{X}_t, \mathcal{F}_t)$ AND A RANDOM QUANTITY $X_t : \Omega \mapsto \mathcal{X}_t$. THE COLLECTION $\{X_t, t \in T\}$ IS CALLED A STOCHASTIC PROCESS AND T ITS INDEX SET.

REMARK $\omega \longmapsto \{X_t(\omega), t \in T\}$

TYPICALLY $\mathcal{X}_t = \mathbb{R}$ ALL t $(\mathcal{F}_t = \mathcal{B}^1)$, $T = \mathbb{N}$ OR $T = \mathbb{R}_{\geq 0}$

$T = \mathbb{N}$ $\downarrow$ DISCRETE TIME PROCESS

$T = \mathbb{R}_{\geq 0}$ $\downarrow$ CONTINUOUS TIME PROCESS

IF $T = \{1, \dots, k\}$ AND EACH X_n IS A R.V. (ON A COMMON PROB. SPACE!) THE THE RANDOM VECTOR $(X_1, \dots, X_k)$ IS A WAY TO REPRESENT $\{X_n, n \in T\}$.

EXAMPLE (RANDOM PROBABILITY MEASURE: MIXTURE).

LET $\Theta : \Omega \to \mathbb{R}^k$ BE A RANDOM VECTOR, WITH DISTRIBUTION μ_Θ

LET $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ BE MEAS S.T. $\int_{\mathbb{R}} f(x, \underline{\theta})\, dx = 1$ ALL $\underline{\theta} \in \mathbb{R}^k$.

[TAKE $k = 1$ LET μ_Θ BE $N(0,1)$ AND

$$f(x, \theta) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\theta)^2}{2\sigma^2} \right\} \quad \text{SOME } \sigma > 0. \,]$$

$\downarrow$ pdf OF $N(\theta, \sigma^2)$

LET $B \in T = \mathcal{B}^1$ AND DEFINE

$$X_B(\omega) = \int_B f(x, \Theta(\omega))\, dx$$

$\hookrightarrow$ LEBESGUE MEASURE.

ANOTHER EXAMPLE OF A RANDOM PROBABILITY MEASURE IS THE EMPIRICAL MEASURE: $X_1, X_2, \dots, X_n$ ARE INDEPENDENT RV'S WITH COMMON DISTRIBUTION. LET P_n BE THE RANDOM PROB. MEASURE:

$$B \in \mathcal{B}^1 \longmapsto P_n(B) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_i \in B\}$$

$\downarrow$

THEY ARE BOTH STOCHASTIC PROCESSES WITH $T = \mathcal{B}^1$ AND $\mathcal{X}_t = [0,1]$ ALL t.

Def (FUNCTIONAL REPRESENTATION OF PRODUCT SETS) LET $\mathcal{X} = \prod_{t \in T} \mathcal{X}_t$ (→ CARTESIAN PRODUCT) BE THE SET OF ALL FUNCTIONS $f: T \to \bigcup_{t \in T} \mathcal{X}_t$ SUCH THAT FOR EVERY t, $f(t) \in \mathcal{X}_t$. WHEN $\mathcal{X}_t = \mathcal{Y}$ ALL t, WE WRITE THIS AS $\mathcal{Y}^T$

<u>EXAMPLE</u> - THINK OF $\mathbb{R}^k$ AS $\mathbb{R}^{\{1,\dots,k\}}$ AND OF A VECTOR $(x_1, \dots, x_k) \in \mathbb{R}^k$ AS A FUNCTION S.T. $f(i) = x_i$ $i = 1, \dots, k$.

- FOR THE RANDOM PROB. EXAMPLE $\mathcal{X} = [0,1]^{\mathcal{B}^1}$ AND WE CAN THINK OF EACH PRB. MEASURE ON $(\mathbb{R}, \mathcal{B}^1)$ AS A FUNCTION FROM $\mathcal{B}^1$ INTO $[0,1]$. <u>WARNING</u> THE PRODUCT SET $\mathcal{X}$ CONTAINS ALSO FUNCTIONS THAT ARE NOT PROBABILITIES. FOR EXAMPLE THE FUNCTION $f(B) = 1$ ALL $B \in \mathcal{B}^1$.

<u>KEY INTUITION</u>: WE CAN NOW THINK OF A STOCHASTIC PROCESS AS A RANDOM FUNCTION!

Def (COORDINATE PROJECTION AND CYLINDER SETS). LET $\mathcal{X} = \prod_{t \in T} \mathcal{X}_t$. FOR EACH $t \in T$ LET $\pi_t : \mathcal{X} \to \mathcal{X}_t$ BE DEFINED AS

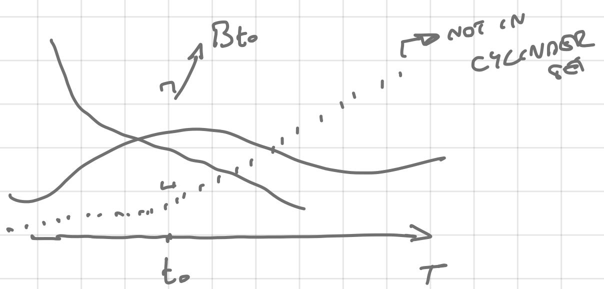
$$\pi_t(f) = f(t)$$

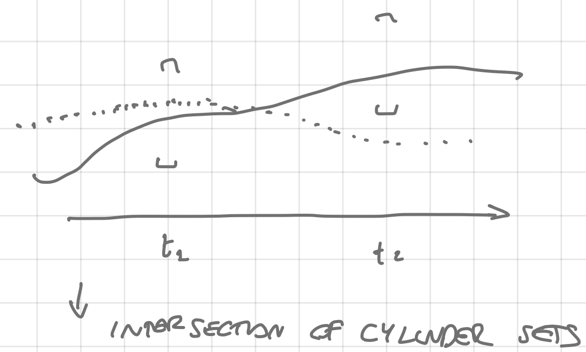
THIS IS THE t-COORDINATE PROJECTION FUNCTION.

A ONE-DIMENSIONAL CYLINDER SET IS A SET OF THE FORM

$$\prod_{t \in T} B_t \quad \text{s.t.} \quad B_{t_0} \in \mathcal{F}_{t_0} \quad \text{SOME } t_0 \in T$$
$$B_t = \mathcal{X}_t \quad \text{ALL } t \neq t_0.$$

DEFINE ON $\mathcal{X}$ THE PRODUCT σ-FIELD $\bigotimes_{t \in T} \mathcal{F}_t$ AS THE σ-FIELD GENERATED BY ALL CYLINDERS.





Lemma 34 THE PRODUCT σ-FIELD IS THE SMALLEST σ-FIELD S.T. ALL COORDINATE PROJECTIONS ARE MEAS. FUNCTIONS.

Thm 35 LET $(\Omega, \mathcal{F}, P)$ BE PROB. SPACE, AND T BE A SET INDEXING MEASURABLE SPACE $\{(\mathcal{X}_t, \mathcal{F}_t), t \in T\}$ S.T. $X_t : \Omega \to \mathcal{X}_t$ IS A FUNCTION FOR ALL t. DEFINE $X : \Omega \to \mathcal{X} = \prod_t \mathcal{X}_t$ BY SETTING $X(\omega)$ TO BE THE FUNCTION $f \in \mathcal{X}$ DEFINED BY $f_t = X_t(\omega)$, ALL t.

THEN X IS $\mathcal{F} / \bigotimes_t \mathcal{F}_t$ IIF EACH X_t IS $\mathcal{F}/\mathcal{F}_t$ MEAS

↳ X IS A RANDOM FUNCTION DESCRIBING THE STOCHASTIC PROCESS

· $X(\omega)$ IS A REALIZATION OR PATH OF THE PROCESS

HOW DO WE CONSTRUCT PROB. DISTRIBUTIONS OVER $\left(\mathcal{X}, \bigotimes_{t\in T} \mathcal{F}_t\right)$ TO HANDLE STOCHASTIC PROCESSES?

↓

■ KOLMOGOROV EXTENSION THEOREM.

<u>Def</u> (FINITE DIMENSIONAL PROJECTIONS). T. LET $v \subset T$ FINITE, SO $v = \{t_1, t_2, \dots, t_n\}$. LET $\mathcal{X}_v = \prod_{t\in v} \mathcal{X}_t$ AND, SIMILARLY,

↳ FINITE PRODUCT

LET $\mathcal{F}_v$ BE THE PRODUCT σ-FIELD OVER $\mathcal{X}_v$. FOR ANY $u \subset v$ OF THE FORM $u = \{t_{i_1}, \dots, t_{i_m}\}$ $m < n$ AND FOR ANY $x = (x(t_1), \dots, x(t_n)) \in \mathcal{X}_v$ LET $x_u = (x(t_{i_1}), \dots, x(t_{i_m}))$ BE THE CORRESPONDING SUB-VECTOR.

LET P_v IS A PROB. MEASURE ON $(\mathcal{X}_v, \mathcal{F}_v)$. THE PROJECTION OF P_v ON $(\mathcal{X}_u, \mathcal{F}_u)$ IS THE PROB. MEASURE $\pi_u(P_v)$ GIVEN BY

$$\pi_u(P_v)(B) = P_v\left(\{x \in \mathcal{X}_v : x_u \in B\}\right) \qquad B \in \mathcal{F}_u$$

SIMILARLY IF Q IS A PROB DISTR OVER $\left(\prod_{t\in T} \mathcal{X}_t, \bigotimes_{t\in T} \mathcal{F}_t\right)$ THE PROJECTION OF Q ON $(\mathcal{X}_v, \mathcal{F}_v)$ IS

$$\pi_v(Q)(B) = Q\left(\{x \in \prod_{t\in T} \mathcal{X}_t : x_v \in B\}\right) \qquad B \in \mathcal{F}_v$$