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ADVANCED PROB. OVERVIEW

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LECTURE 1: MON, AUG 31, 2020

- WHAT IS THE PROBABILITY OF AN EVENT ?
- CONCEPT OF A MEASURE AND INTEGRATION
- NOTION OF CONVERGENCE

BASIC CONCEPTS IN SET THEORY

- Ω : UNIVERSE SET $\qquad \omega \in \Omega$
- $A, B \subseteq \Omega \qquad A \cup B, \quad A \cap B, \quad A^c \qquad \emptyset$ (EMPTY SET)
- POWER SET $\quad 2^{\Omega} = \{ A : A \subseteq \Omega \}$
- CARTESIAN PRODUCT $\quad A \times B = \{ (x, y) : x \in A, y \in B \}$

$$k \in \mathbb{N}_+ \qquad A^k = \underbrace{A \times A \times \dots \times A}_{k \text{ TIMES}}$$

$$A^B = \{ f : \quad f : A \to B \}$$

- SEQUENCES OF SETS : $A_1, A_2, A_3, \dots$ BE SUBSETS OF Ω

IF $\quad A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \qquad A = \bigcup_n A_n$

THEN $\{A_n\}$ IS AN INCREASING SEQUENCE $A_n \uparrow A$

IF $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ $A = \bigcap_n A_n$, THEN

$\{A_n\}$ IS A DECREASING SEQUENCE $A_n \downarrow A$

- IN BOTH CASES A IS THE <u>LIMIT</u> OF THE SEQUENCE (MONOTONE SEQUENCES)

- IF $\{A_n\}$ IS A SEQUENCE, WE DEFINE

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$\omega \in \limsup_n A_n$ IIF $\omega \in A_n$ FOR INFINITELY MANY n
(OR INFINITELY OFTEN)

$\forall n$, $\omega \in A_k$ FOR SOME $k \geq n$

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

$\omega \in \liminf_n A_n$ IIF $\omega \in A_k$ FOR ALL $k \geq n$, SOME n

$\omega \in A_n$ EVENTUALLY

- RECALL THAT IF $\{x_n\}$ IS A SEQUENCE OF NUMBERS, THEN

$$\limsup_n x_n = \inf_n \sup_{k\geq n} x_k \qquad \liminf_n x_n = \sup_n \inf_{k \geq n} x_k$$

IF, FOR A SET A, WE DEFINE

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$ INDICATOR FUNCTION

THEN

$$\mathbb{1}_{\limsup_n A_n} = \limsup_n \mathbb{1}_{A_n} \quad \text{AND} \quad \mathbb{1}_{\liminf_n A_n} = \liminf_n \mathbb{1}_{A_n}$$

$\{A_n\}$ HAS A LIMIT IF $\limsup_n A_n = \liminf_n A_n$

EXAMPLE IF $A_n = \begin{cases} (a,b) & n \text{ EVEN} \\ (c,d) & n \text{ ODD} \end{cases}$

WHERE $a, b, c, d \in \mathbb{R}$ $\quad (a,b) \cap (c,d) = \emptyset$

$$\limsup_n A_n = (a,b) \cup (c,d) \qquad \liminf_n A_n = \emptyset$$

- DE MORGAN LAWS: $\left(\bigcup_n A_n\right)^c = \bigcap_n A_n^c$

$$\left(\bigcap_n A_n\right)^c = \bigcup_n A_n^c$$

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

- $a, b \in \mathbb{R}$ $\quad a < b$

$$(a,b) = \{ x \in \mathbb{R} : a < x < b \} = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, \; b - \frac{1}{n}\right]$$

$$[a,b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, \; b + \frac{1}{n}\right)$$

- THE CARDINALITY OF A, DENOTE WITH $|A|$ OR $\# A$ OR card(A), IS THE NUMBER OF ELEMENTS IN A.

  A IS FINITE IF $|A| < \infty$ AND INFINITE OTHERWISE

- A SET A IS COUNTABLE IF $\exists f : A \to \mathbb{N}$ THAT IS INJECTIVE (IF $x \neq y$ THEN $f(x) \neq f(y)$)

  A COUNTABLE SET CAN BE FINITE OR INFINITE

- AN INFINITE SET THAT IS NOT COUNTABLE IS UNCOUNTABLE

<u>Claim</u> IF $A_1, A_2, \ldots$ ARE COUNTABLE SETS, THEN SO IS $\bigcup_n A_n$

(COUNTABLE UNION OF COUNTABLE SETS IS COUNTABLE)

Pf/ IT IS ENOUGH TO SHOW THAT $\mathbb{N}^2$ IS COUNTABLE

LET p_1 AND p_2 BE TWO PRIME NUMBERS. THEN

$f: \mathbb{N}^2 \to \mathbb{N}$ GIVEN BY $(n, m) \to p_1^n \, p_2^m$ IS

AN INJECTION. ∎

↳ THIS ALSO IMPLIES THAT $\mathbb{N}^k$ ANY INTEGER k IS COUNTABLE.

<u>Claim</u> IF $A_1, A_2, \ldots$ ARE COUNTABLE SETS, $A_1 \times A_2 \times A_3 \times \ldots$

IS NOT COUNTABLE (IT IS UNCOUNTABLE)!

Pf/ TAKE $A_n = \{0, 1\}$ FOR ALL n. THEN $A = \prod_{n=1}^{\infty} A_n$

IS THE SET OF ALL INFINITE BINARY SEQUENCES.

ASSUME THAT A IS COUNTABLE. THEN WE CAN WRITE

$$A = \{ s^{(1)}, s^{(2)}, s^{(3)}, \ldots \}$$

WHERE $s^{(i)}$ IS AN INFINITE BINARY SEQUENCE.

LET s BE AN INFINITE BINARY SEQUENCE S.T.

ith ELEMENT OF s ← $s_i = |1 - s_i^{(i)}|$ (FLIP $s_i^{(i)}$). THEN $s \neq s^{(i)}$ ALL i.

$\Downarrow$

$s \notin A$

SO A IS UNCOUNTABLE! ∎

SEE EXAMPLE 5 IN NOTES TO SEE HOW THIS RESULT CAN BE USED TO PROVE THAT $(0, 1]$ IS AN UNCOUNTABLE SET.

FIELDS AND σ-FIELDS

Ω UNIVERSE SET. A COLLECTION $\mathcal{F}$ OF SUBSETS OF Ω IS A FIELD WHEN:

- $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \rightarrow A^c \in \mathcal{F}$
- $A_1, A_2 \in \mathcal{F} \rightarrow A_1 \cup A_2 \in \mathcal{F}$

$\downarrow$

$\mathcal{F}$ IS CLOSED WRT FINITE UNIONS OR INTERSECTIONS

- A FIELD $\mathcal{F}$ IS A σ-FIELD IF, IN ADDITION, FOR EVERY SEQUENCE $\{A_n\}$ OF SETS IN $\mathcal{F}$, $\bigcup_n A_n \in \mathcal{F}$

$\downarrow$

CLOSED WRT TO COUNTABLE UNIONS AND INTERSECTIONS