36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 18: MON, NOV 2, 2020

- TOMORROW: VOTE!

- LAST TIME: CONVERGENCE IN DISTRIBUTION OR $\underset{\text{VAGUE}}{\text{WEAK}}$ CONVERGENCE

LET $(\mathcal{X}, d)$ BE A METRIC SPACE ENDOWED WITH BOREL σ-FIELD

LET $\{X_n\}_n$ AND X BE RANDOM VARIABLES TAKING VALUES IN $\mathcal{X}$. THEN $X_n \Rightarrow X$ WHEN

$$\mathbb{E}[f(X_n)] \longrightarrow \mathbb{E}[f(X)] \quad \text{AS } n \to \infty$$

FOR ALL $f \in C_b$ → SET OF CONTINUOUS BOUNDED FUNCTIONS ON $\mathcal{X}$

EQUIVALENTLY,

$$\mu_{X_n} \Rightarrow \mu_X$$

WHERE μ_{X_n} AND μ_X ARE THE PROB. DISTR. OF X_n AND X RESPECTIVELY.

<u>REMARK</u>: THE X_n'S AND X NEED NOT BE DEFINED ON THE SAME PROB. SPACE!

IF $\mathcal{X} = \mathbb{R}^d$ THEN $\mu_{X_n} \Rightarrow \mu_X$ IS EQUIVALENT TO

$$F_{X_n}(x) \to F_X(x) \quad \text{AS } n \to \infty$$

FOR ALL CONTINUITY POINTS x OF F_X, WHERE F_{X_n} AND F_X ARE THE c.d.f. OF X_n AND X, RESPECTIVELY.

<u>EXAMPLE</u>: LET μ_n BE A PROB. MEASURE THAT PUTS MASS $1/n$ OVER

$$\left\{ 0, \frac{1}{n}, \dots, \frac{n-1}{n} \right\} \subset [0,1]. \text{ THEN, THE cdf IS}$$

$$F_n(x) = \frac{\lfloor nx \rfloor + 1}{n} \to x = \underset{\substack{\downarrow \\ \text{cdf of Unifor}(0,1)}}{F(x)} \qquad x \in (0,1)$$

SO THE LIMITING DISTRIBUTION IN A WEAK CONVERGENCE SENSE IS Uniform $(0,1)$. LET $B = \mathbb{Q} \cap [0,1]$.

↳ SET OF RATIONALS

THEN

$$\mu_n(B) = 1 \quad \text{ALL } n.$$

BUT $\mu(B) = 0$, μ IS THE Uniform$(0,1)$ DISTRIBUTION

THIS DOES NOT VIOLATE THE PORTMANTEAU THEOREM:

$$\mu(\partial B) = \mu([0,1]) = 1$$

<u>EXAMPLE</u>: LET Φ BE THE cdf OF STANDARD NORMAL AND

$$F_n(x) = \begin{cases} 0 & x < -n \\ \dfrac{\Phi(x) - \Phi(-n)}{\Phi(n) - \Phi(-n)} & -n \le x \le n \\ 1 & x > n \end{cases} \qquad x \in \mathbb{R}$$

cdf OF A TRUNCATED $N(0,1)$

THEN $F_n(x) \to \Phi(x)$ ALL $x \in \mathbb{R}$

NOW CONSIDER F_n TO BE THE cdf OF Uniform$(-n,n)$.

BUT $F_n(x) \to 0$ ALL x AS $n \to \infty$

↳ $\{F_n\}_n$ DOES NOT CONVERGE TO ANY cdf!!

IN THE SECOND EXAMPLE, FAILURE OF CONVERGENCE IS DUE TO THE MASS ESCAPING TO INFINITY...

↳ EXTREME EXAMPLE: LET $X_n = c_n$ a.e. AND $c_n \to \infty$

THE, THE CORRESPONDING cdf F_n IS SUCH THAT

$F_n(x) \to 0$ AS $n \to \infty$ FOR ALL x

↓ DOES NOT CONVERGE TO A cdf.

TIGHTNESS

WE TAKE $\mathcal{X} = \mathbb{R}^d$.

Def A COLLECTION $\{P_n\}$ OF PROB. MEASURE ON $(\mathbb{R}^d, \mathcal{B}^d)$ IS TIGHT, OR BOUNDED IN PROBABILITY, IF FOR ANY $\varepsilon > 0$, $\exists K = K(\varepsilon)$ S.T. $P_n(K) \geq 1 - \varepsilon$ FOR ALL n.

↳ COMPACT

[IF $\{X_n\}$ IS A SEQUENCE OF RV'S IN $\mathbb{R}^d$, IT IS TIGHT IF

$\forall \varepsilon > 0 \quad \exists M = M(\varepsilon) \quad$ S.T. $\Pr(\|X_n\| > M) < \varepsilon$]

FOR ALL n

THE COLLECTION $\{P_n\}$ IS SAID TO BE RELATIVELY COMPACT IF EVERY SUB-SEQUENCE CONTAINS ANOTHER SUBSEQUENCE THAT CONVERGES.

Thm 23 (HELLY-BRAY SELECTION THM). LET $\{P_n\}$ BE A TIGHT SEQUENCE OF PROB. MEAS. ON $(\mathbb{R}^d, \mathcal{B}^d)$. THEN THERE EXISTS A SUBSEQUENCE CONVERGING IN DISTRIBUTION.

TO SUMMARIZE: LET $\{X_n\}$ BE A SEQUENCE OF RV'S IN $\mathbb{R}^d$

THEN

1) IF $X_n \Rightarrow X$ THEN $\{X_n\}$ IS TIGHT

2) IF $\{X_n\}$ IS TIGHT, THEN $\exists \{n_j\}$ ST.

$$X_{n_j} \Rightarrow X \quad \text{SOME } X.$$

PART 2) IS HELLY-BRAY SELECTION THM. AS FOR PART 1) THE PROOF IS THIS: LET $\varepsilon > 0$ BE FIXED AND LET $M = M(\varepsilon)$ ST. $\Pr(\|X\| \geq M) < \varepsilon$. NEXT, BY PORTMANTEAU THM,

$$\Pr(\|X_n\| \geq M) \leq \Pr(\|X\| \geq M) + \varepsilon$$
$$< 2\varepsilon \quad \text{FOR ALL } n > n_0(\varepsilon, M).$$

POSSIBLY INCREASE M SO THAT THE INEQUALITY HOLDS FOR ALL n.

CONTINUOUS MAPPING THEOREMS

THM LET $\{X_n\}$ BE A SEQUENCE OF RV'S ON SOME METRIC SPACE $(\mathcal{X}, d)$ AND LET X BE ANOTHER RV ON $\mathcal{X}$ S.T.

$$X_n \xrightarrow{D} X$$

LET $\mathcal{Y}$ BE A METRIC SPACE AND $g: \mathcal{X} \to \mathcal{Y}$.

LET $C_g = \{ x \in \mathcal{X} : g \text{ IS CONTINUOUS AT } x \}$

IF $\Pr(X \in C_g) = 1$ THEN $g(X_n) \xrightarrow{D} g(X)$.

PF/ LET Q_n BE THE DISTR. OF $g(X_n)$ AND Q THE DISTR. OF $g(X)$. LET P_n AND P BE THE DISTR. OF X_n AND X.

LET $B \subseteq \mathcal{Y}$ CLOSED. IF $x \in \overline{g^{-1}(B)}$ BUT $x \notin g^{-1}(B)$

$\hookrightarrow \{x : g(x) \in B\}$

THEN $x \notin C_g$ [g IS NOT CONTINUOUS AT x]

SO $$\overline{g^{-1}(B)} \subseteq g^{-1}(B) \cup C_g^c.$$

NEXT

$$\limsup_{n\to\infty} \underbrace{Q_n(B)}_{Pr(g(X_n)\in B)} = \limsup_{n\to\infty} P_n\left(g^{-1}(B)\right)$$

$$\leq \limsup_{n\to\infty} P_n\left(\overline{g^{-1}(B)}\right)$$

BY PORTMANTEAU

$$\leq P\left(\overline{g^{-1}(B)}\right)$$

BY UNION BOUND

$$\leq P\left(g^{-1}(B)\right) + \underbrace{P\left(C_g^c\right)}_{=0}$$

$$= \underbrace{Q(B)}_{Pr(X\in B)}$$

$\blacksquare$

<u>Thm</u> THE SAME RESULT HOLDS IF $\xrightarrow{D}$ IS REPLACED BY $\xrightarrow{P}$

SEE: PROOF IN ASYMPTOTIC STATISTICS BY A. VAN DER VAART

<u>Thm (15)</u> LET $\{X_n\}$ AND $\{Y_n\}$ BE SEQUENCE OF RV'S ON SAME METRIC SPACE $(\mathcal{X}, d)$. IF $X_n \xrightarrow{D} X$ AND $d(X_n, Y_n) \xrightarrow{P} 0$, THEN $Y_n \xrightarrow{D} X$.

EXAMPLE: $\dfrac{S_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{D} N(0,1)$ $\qquad S_n = \sum_{i=1}^{n} X_i$

X_i ARE IID (μ, σ^2)

THEN $\dfrac{(S_n - n\mu)^2}{n\sigma^2} \xrightarrow{D} \chi^2_1$

EXAMPLE: $X_1, X_2, \ldots,$ IID (μ, σ^2) UNKNOWN

WANT TO ESTIMATE σ^2: OF COURSE

$$\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2\right] = \sigma^2 \quad \text{ALL } n$$

$\frac{1}{n}\sum_{i=1}^{n} X_n$

SO ESTIMATE σ^2 WITH $\dfrac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 = \hat{\sigma}^2$

WE WOULD LIKE FOR $\hat{\sigma}^2 \xrightarrow{P} \sigma^2$.

WRITE $$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2 - (\bar{X}_n - \mu)^2\right]$$

BUT $\bar{X}_n \xrightarrow{P} \mu$ SO $(\bar{X}_n - \mu)^2 \xrightarrow{P} 0$ BY CMT

AND $\dfrac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2 \xrightarrow{P} \sigma^2$ BY WLLN

THEN $\hat{\sigma}^2 \xrightarrow{P} \sigma^2$ USING CMT AGAIN BECAUSE

ADDITION IS A CONTINUOUS FUNCTION

Thm 16: IF $X_n \xrightarrow{D} X$ AND $Y_n \xrightarrow{P} c$ $\quad$ → CONSTANT

THEN $$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} X \\ c \end{pmatrix}$$

Thm (SLUTSKY'S THEOREM) IF $X_n \xrightarrow{D} X$ AND $Y_n \xrightarrow{D} c$

THEN

1) $$X_n + Y_n \xrightarrow{D} X + c$$

2) $$X_n Y_n \xrightarrow{D} cX$$

3) $$\frac{X_n}{Y_n} \xrightarrow{D} \frac{X}{c} \quad \text{IF } c \neq 0$$

EXAMPLE:

$$U_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{D} N(0,1) = Z$$

$$\hat{\sigma}_n \xrightarrow{P} \sigma$$

THEN $$\begin{pmatrix} U_n \\ \hat{\sigma}_n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} Z \\ \sigma \end{pmatrix}$$

SO LET $g(z, s) = \frac{z\sigma}{s}$ THEN

$$g(U_n, \hat{\sigma}_n) = \frac{S_n - n\mu}{\sqrt{n}\,\hat{\sigma}_n} \xrightarrow{D} Z$$

↳ USE THIS FOR ASYMPTOTIC CI'S! FOR μ