On the Maximal Perimeter of a Convex Set in $\mathbb{R}^n$ with Respect to a Gaussian Measure

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Introduction

Let A be an $n \times n$ positive definite symmetric matrix and let

$$d\gamma_A(y) = \varphi_A(x)\,dx = (2\pi)^{-\frac{n}{2}}\sqrt{\det A}\,e^{-\frac{\langle Ax,x\rangle}{2}}\,dx$$

be the corresponding Gaussian measure. Let

$$\Gamma(A) = \sup\left\{\frac{\gamma_A(Q_h \setminus Q)}{h} \;:\; Q \subset \mathbb{R}^n \text{ is convex, } h > 0\right\}$$

where Q_h denotes the set of all points in $\mathbb{R}^n$ whose distance from Q does not exceed h.

Since, for convex Q, one has $Q_{h'+h''} \setminus Q = [(Q_{h'})_{h''} \setminus Q_{h'}] \cup [Q_{h'} \setminus Q]$, the definition of $\Gamma(A)$ can be rewritten as

$$\begin{aligned}\Gamma(A) &= \sup\left\{\limsup_{h\to 0+}\frac{\gamma_A(Q_h \setminus Q)}{h} \;:\; Q \subset \mathbb{R}^n \text{ is convex}\right\}\\ &= \sup\left\{\int_{\partial Q}\varphi_A(y)\,d\sigma(y) \;:\; Q \subset \mathbb{R}^n \text{ is convex}\right\}\end{aligned}$$

where $d\sigma(y)$ is the standard surface measure in $\mathbb{R}^n$.

Making the change of variable $x \to Bx$ where B is the (positive definite) square root of A, the last expression can be rewritten as

$$\sup\left\{\int_{\partial Q}\varphi(y)|B\nu_y|\,d\sigma(y) \;:\; Q \subset \mathbb{R}^n \text{ is convex}\right\}$$

where $\varphi(y) = (2\pi)^{-\frac{n}{2}}e^{-\frac{|y|^2}{2}}$ is the density of the standard Gaussian measure $d\gamma$ in $\mathbb{R}^n$ and ν_y is the unit normal vector to the boundary ∂Q of the body Q at the point $y \in \partial Q$.

Recall that the Hilbert-Schmidt norm $\|A\|_{\text{H-S}}$ of a positive definite symmetric matrix A is defined as the square root of the sum of squares of all entries of A or, which is the same, as the square root of the sum of squares of the eigenvalues of A. The aim of this paper is to prove the following

Theorem. *There exist absolute constants* $0 < c < C < +\infty$ *such that*

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$$c\sqrt{\|A\|_{\text{H-S}}} \leqslant \Gamma(A) \leqslant C\sqrt{\|A\|_{\text{H-S}}}.$$

A few words should, probably, be said about the history of the question. To the best of my knowledge, it was S. Kwapien who first pointed out that it would be desirable to have good estimates for $\Gamma(I_n)$ (i.e., for the maximal perimeter of a convex body with respect to the standard Gaussian measure). The only progress that has been made was due to K. Ball who in 1993 proved the inequality $\Gamma(I_n) \leqslant 4n^{\frac{1}{4}}$ for all $n \geqslant 1$ and observed that a cube in $\mathbb{R}^n$ may have its Gaussian perimeter as large as $\sqrt{\log n}$ (see [B]). Many people seemed to believe that the logarithmic order of growth must be the correct one and that it is the upper bound that needs to be improved. If it were the case, it would open a road to essentially improving some constants in various "convex probability" theorems (see [Be1],[Be2] for a nice example). Alas, as it turned out, K. Ball's estimate is sharp.

As to the proof of the theorem, I cannot shake the feeling that there should exist some simple and elegant way leading to the result. Unfortunately, what I can present is a pretty boring and technical computation. So I encourage the reader to stop reading the paper here and to (try to) prove the theorem by himself.

The Case $A = I_n$

We shall be primarily interested in the behavior of $\Gamma(I_n)$ for large n. Our first goal will be to prove the asymptotic upper bound

$$\limsup_{n\to\infty} \frac{\Gamma(I_n)}{n^{\frac{1}{4}}} \leqslant \pi^{-\frac{1}{4}} < 0.76\,,$$

which, with some extra twist, can be improved to

$$\limsup_{n\to\infty} \frac{\Gamma(I_n)}{n^{\frac{1}{4}}} \leqslant (2\pi)^{-\frac{1}{4}} < 0.64\,.$$

While this result is essentially equivalent to that of K. Ball, our proof will use different ideas and yield more information about the possible shapes of convex bodies with large Gaussian perimeter.

As to the estimates from below, we shall show that

$$\liminf_{n\to\infty} \frac{\Gamma(I_n)}{n^{\frac{1}{4}}} \geqslant e^{-\frac{5}{4}} > 0.28\,.$$

First of all, note that in the definition of $\Gamma(I_n)$ we may restrict ourselves to convex bodies Q containing the origin. One of the most natural ways to estimate the integral $\int_{\partial Q} \varphi(y)\,d\sigma(y)$ is to introduce some "polar coordinate system" $x = X(y,t)$ in $\mathbb{R}^n$ with $y \in \partial Q$, $t \geqslant 0$. Then we can write

$$1=\int_{\mathbb{R}^n}\varphi(x)\,dx=\int_{\partial Q}\left[\int_0^\infty \varphi(X(y,t))D(y,t)\,dt\right]d\sigma(y)=\int_{\partial Q}\varphi(y)\xi(y)\,d\sigma(y) \tag{$*$}$$

where $D(y,t)$ stands for the determinant of the differential $\frac{\partial X(y,t)}{\partial y\,\partial t}$ of the mapping $\partial Q\times(0,+\infty)\ni(y,t)\to X(y,t)\in\mathbb{R}^n$ and

$$\xi(y)=\varphi(y)^{-1}\int_0^\infty \varphi\big(X(y,t)\big)D(y,t)\,dt.$$

This yields the estimate

$$\int_{\partial Q}\varphi(y)\,d\sigma(y)\leqslant\frac{1}{\min\limits_{\partial Q}\xi}.$$

There are two natural polar coordinate systems associated with a convex body Q containing the origin. The first one is given by the mapping $X_1(y,t)=ty$. Then

$$D_1(y,t)=t^{n-1}|y|\alpha(y)$$

where $\alpha(y)$ is the cosine of the angle between the "radial vector" y and the unit outer normal vector ν_y to the surface ∂Q at the point y. So, in this case, we have

$$\begin{aligned}\xi_1(y)&=e^{\frac{|y|^2}{2}}\left[\int_0^\infty |y|t^{n-1}e^{-\frac{t^2|y|^2}{2}}\,dt\right]\alpha(y)\\&=|y|^{-(n-1)}e^{\frac{|y|^2}{2}}\left[\int_0^\infty t^{n-1}e^{-\frac{t^2}{2}}\,dt\right]\alpha(y)\,.\end{aligned}$$

It is not hard to see that the function $f(t):=t^{n-1}e^{-\frac{t^2}{2}}$ is nice enough for the application of the Laplace asymptotic formula. Since it attains its maximum at $t_0=\sqrt{n-1}$ and since $\frac{d^2}{dt^2}\log f(t_0)=-2$, we get

$$\int_0^\infty f(t)dt=\big[\sqrt{\pi}+o(1)\big]f(t_0).$$

Observing that $\frac{d^2}{dt^2}\log f(t_0)\leqslant -1$ for all $t>0$, we get

$$f(t)\leqslant f(t_0)e^{-\frac{(t-t_0)^2}{2}}\qquad\text{for all } t>0.$$

Bringing these estimates together, we conclude that

$$\xi_1(y)\geqslant e^{\frac{(|y|-\sqrt{n-1})^2}{2}}\big[\sqrt{\pi}+o(1)\big]\alpha(y).$$

Unfortunately, as one can easily see, $\alpha(y)$ can be very close to 0 at some points, so we cannot get an estimate for the Gaussian perimeter of an *arbitrary* convex body Q using ξ_1 alone. Nevertheless, let us mention here that if we know in advance that Q contains a ball of radius $R > 0$ centered at the origin, we may use the elementary inequality $\alpha(y) \geqslant \frac{R}{|y|}$ and conclude (after some not very hard computations) that

$$\min_{\partial Q} \xi_1 \geqslant \left[1 + o(1)\right] \frac{\sqrt{\pi}\, R}{\sqrt{n}}.$$

Thus, if R is much greater than $n^{\frac{1}{4}}$, the Gaussian perimeter of Q is much less than $n^{\frac{1}{4}}$. It is interesting to compare this observation with the construction of the convex body Q with large perimeter below: the body we shall construct will have the ball of radius $n^{\frac{1}{4}}$ as its inscribed ball!

Let us now consider the second natural "polar coordinate system" associated with Q, which is given by the mapping $X_2(y,t) = y + t\,\nu_y$. The reader may object that it is a coordinate system in $\mathbb{R}^n \setminus Q$, not in $\mathbb{R}^n$, but this makes things only better because now we can write $1 - \gamma(Q)$ instead of 1 on the left hand side of the inequality $(*)$ (it is this improvement that, exploited carefully, yields the extra factor of $2^{-\frac{1}{4}}$). It is not hard to check that $X_2(y,t)$ is an expanding map in the sense that $|X_2(y',t') - X_2(y'',t'')|^2 \geqslant |y' - y''|^2 + (t' - t'')^2$ and, therefore, $D_2(y,t) \geqslant 1$ for all $y \in \partial Q$, $t > 0$. This results in the inequality

$$\xi_2(y) \geqslant \int_0^\infty e^{-t|y|\alpha(y)} e^{-\frac{t^2}{2}}\, dt \geqslant \frac{1}{|y|\alpha(y) + 1}.$$

This expression can also be small, but only if $\alpha(y)$ is *large*. Thus, it seems to be a good idea to bring these two estimates together and to write

$$\int_{\partial Q} \varphi(y)\, \varXi(y)\, d\sigma(y) \leqslant 2$$

where

$$\varXi(y) = \xi_1(y) + \xi_2(y) \geqslant [1 + o(1)] \cdot \left\{ e^{\frac{(|y| - \sqrt{n-1})^2}{2}} \sqrt{\pi}\, \alpha(y) + \frac{1}{|y|\alpha(y) + 1} \right\}.$$

It is a simple exercise in elementary analysis now to show that the minimum of the right hand side over all possible values of $|y|$ and $\alpha(y)$ is $[2 + o(1)]\pi^{\frac{1}{4}} n^{-\frac{1}{4}}$ attained at $|y| \approx \sqrt{n-1}$, $\alpha(y) \approx (\pi n)^{-\frac{1}{4}}$.

Note that if $|y|$ or $\alpha(y)$ deviate much from these values ($|y|$ on the additive and $\alpha(y)$ on the multiplicative scale), the corresponding value of $\varXi(y)$ is much greater than $n^{-\frac{1}{4}}$. Thus, if a convex body Q with the Gaussian perimeter comparable to $n^{\frac{1}{4}}$ exists at all, a noticeable part of its boundary (in the sense of angular measure) should lie in the constant size neighborhood of the sphere S of radius $\sqrt{n}$ centered of the origin, $\alpha(y)$ being comparable to $n^{-\frac{1}{4}}$ on that part of the boundary. At first glance, this seems unfeasible because what it means is that the boundary of Q should simultaneously be very close to the sphere S and very transversal (almost orthogonal!) to it. Actually, it leaves one with essentially one possible choice of the body Q for which it is impossible to "do something" to essentially improve the upper bound: the regular polyhedron with inscribed radius of $n^{\frac{1}{4}}$ and circumscribed radius of $\sqrt{n}$ (there is no such deterministic thing, to be exact, but there is a good random substitute).

The fastest way to get the estimate $\Gamma(I_n) \geqslant \text{const}\, n^{\frac{1}{4}}$ seems to be the following. Observe, first of all, that the polar coordinate system $X_1(y,t)$ can be used to obtain the inequality

$$\int_{\partial Q} \varphi(y)\, d\sigma(y) \geqslant \int_{(\partial Q)'} \varphi(y)\, d\sigma(y) \geqslant \text{const}\; n^{\frac{1}{4}} \gamma(\mathcal{K}_Q)$$

where

$$(\partial Q)' = \left\{ y \in \partial Q \;:\; \big|\, |y| - \sqrt{n-1} \big| \leqslant 1,\ \tfrac{1}{2} n^{-\frac{1}{4}} \leqslant \alpha(y) \leqslant 2n^{-\frac{1}{4}} \right\}$$

and $\mathcal{K}_Q = \{ty \;:\; y \in (\partial Q)', t \geqslant 0\}$ is the cone generated by $(\partial Q)'$. Let now H be a hyperplane tangent to the ball of radius $n^{\frac{1}{4}}$ centered at the origin. Let S be the (smaller) spherical cap cut off from the sphere $\mathbb{S}$ of radius $\sqrt{n}$ centered at the origin by the hyperplane H, let $\widetilde{H} = \{y \in H \;:\; \sqrt{n} - 1 \leqslant |y| \leqslant \sqrt{n}\}$, and let $\widetilde{S}$ be the radial projection of $\widetilde{H}$ to the sphere $\mathbb{S}$. Now, instead of one hyperplane H, take N independent random hyperplanes H_j and consider the convex body Q that is the intersection of the corresponding half-spaces. A point $y \in \widetilde{H}_j$ belongs to $(\partial Q)'$ unless it is cut off by one of the other hyperplanes H_k. Note that if a point $y \in \widetilde{H}_j$ is cut off by a hyperplane H_k, then its radial projection to the sphere $\mathbb{S}$ belongs to S_k. Thus,

$$\gamma(\mathcal{K}_Q) \geqslant \sum_{j=1}^{N} \lambda\Big(\widetilde{S}_j \setminus \bigcup_{k:k\neq j} S_j \Big)$$

where $d\lambda$ is the normalized (by the condition $\lambda(\mathbb{S}) = 1$) angular measure on $\mathbb{S}$. Since the random hyperplanes H_j are chosen independently, the expectation of the right hand side equals $N(1-\lambda(S))^{N-1}\lambda(\widetilde{S})$. Observing that, for large n, $\lambda(S)$ is small compared to 1 and choosing $N \approx \lambda(S)^{-1}$, we get the estimate

$$\gamma(\mathcal{K}) \geqslant \text{const}\, \frac{\lambda(\widetilde{S})}{\lambda(S)}.$$

A routine computation shows that the ratio $\frac{\lambda(\widetilde{S})}{\lambda(S)}$ stays bounded away from 0 as $n \to \infty$, finishing the proof. While this (sketch of a) proof is missing a few technical details, I included it in the hope that it might give the reader a clearer picture of how the example was constructed than the completely formal reasoning below aimed at obtaining the largest possible coefficient in front of $n^{\frac{1}{4}}$ rather than at making the geometry transparent.

The formal construction runs as follows. Consider N (a large integer to be chosen later) independent random vectors x_j equidistributed over the unit sphere in $\mathbb{R}^{n+1}$ (this 1 is added just to avoid indexing φ) and define the (random) polyhedron

$$Q := \big\{x \in \mathbb{R}^{n+1} \,:\, \langle x, x_j\rangle \leqslant \rho\big\}.$$

In other words, Q is the intersection of N random half-spaces bounded by hyperplanes H_j whose distance from the origin is ρ. The expectation of the Gaussian perimeter of Q equals

$$N\frac{1}{\sqrt{2\pi}}e^{-\frac{\rho^2}{2}}\int_{\mathbb{R}^n}\varphi(y)\big(1-p(|y|)\big)^{N-1}\,dy$$

where $p(r)$ is the probability that a fixed point whose distance from the origin equals $\sqrt{r^2+\rho^2}$ is separated from the origin by one random hyperplane H_j. It is easy to compute $p(r)$ explicitly: it equals

$$\left[\int_{-\sqrt{r^2+\rho^2}}^{\sqrt{r^2+\rho^2}}\left(1-\frac{t^2}{r^2+\rho^2}\right)^{\frac{n-1}{2}}dt\right]^{-1}\int_{\rho}^{\sqrt{r^2+\rho^2}}\left(1-\frac{t^2}{r^2+\rho^2}\right)^{\frac{n-1}{2}}dt.$$

This is quite a cumbersome expression so let us try to find a good asymptotics for it when $\rho = e^{O(1)}n^{\frac{1}{4}}$ and $r = \sqrt{n-1}+w$, $|w| < O(1)$. The first integral then becomes a typical exercise example for the Laplace asymptotic formula and we get it equal to $\sqrt{2\pi}+o(1)$. Using the inequality $(1-a) \leqslant e^{-\frac{a^2}{2}}e^{-a}$ $(a>0)$, we can estimate the second integral by

$$\begin{aligned}\int_\rho^\infty &\exp\left\{-\frac{n-1}{4(r^2+\rho^2)^2}t^4\right\}\exp\left\{-\frac{n-1}{r^2+\rho^2}\frac{t^2}{2}\right\}dt\\ &\leqslant \exp\left\{-\frac{n-1}{4(r^2+\rho^2)^2}\rho^4\right\}\int_\rho^\infty \exp\left\{-\frac{n-1}{r^2+\rho^2}\frac{t^2}{2}\right\}dt\,.\end{aligned}$$

The first factor is asymptotically equivalent to $e^{-\frac{\rho^4}{4n}}$ in the ranges of ρ and r we are interested in. To estimate the second factor, let us observe that, for every $a>0$,

$$\int_\rho^\infty e^{-a\frac{t^2}{2}}\,dt \leqslant \int_\rho^\infty e^{-a\frac{\rho^2}{2}}e^{-a\rho(t-\rho)}\,dt = \frac{1}{a\rho}e^{-a\frac{\rho^2}{2}}.$$

Observe also that under our restrictions for r and ρ, we have

$$\frac{n-1}{r^2+\rho^2} = 1-\frac{2w}{\sqrt{n}}-\frac{\rho^2}{n}+o(n^{-\frac{1}{2}}).$$

Bringing all the estimates together, we arrive at the inequality

$$p(r) \leqslant \big[1+o(1)\big]\frac{1}{\sqrt{2\pi}}\frac{1}{\rho}\exp\left\{\frac{\rho^4}{4n}\right\}\exp\left\{\frac{w\rho^2}{\sqrt{n}}\right\}e^{-\frac{\rho^2}{2}} =: L(n,\rho)\exp\left\{\frac{w\rho^2}{\sqrt{n}}\right\}$$

(the reader should treat $o(1)$ in the definition of $L(n,\rho)$ as some hard to compute but definite quantity that depends on n only and tends to 0 as $n\to\infty$). Since we expect the main part of the integral $\int_{\mathbb{R}^n}\varphi(y)(1-p(|y|))^{N-1}dy$ to come from the points y with $|y|\approx\sqrt{n-1}$, which correspond to the values of w close to 0, let us look at what happens if we replace the last factor in our estimate for $p(r)$ by its value at $w=0$, which is just 1. Then $p(r)$ would not depend on r at all and, taking into account that $\int_{\mathbb{R}^n}\varphi(y)\,dy=1$, we would get the quantity

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{\rho^2}{2}}N\big[1-L(n,\rho)\big]^{N-1}$$

to maximize. Optimizing first with respect to N (note that $L(n,\rho)\to 0$ as $n\to\infty$), we see that we should take N satisfying the inequality $N\leqslant L(n,\rho)^{-1}\leqslant N+1$, which results in the value of the maximum being

$$\big[1+o(1)\big]e^{-1}\rho\exp\left\{-\frac{\rho^4}{4n}\right\}.$$

Optimizing with respect to ρ, we see that the best choice would be $\rho=n^{\frac14}$ which would yield the desired asymptotic lower bound $e^{-\frac54}n^{\frac14}$ for $\Gamma(I_n)$. Now let us use these values of N and ρ and make an accurate estimate of the integral

$$\int_{\mathbb{R}^n}\varphi(y)\big(1-p(|y|)\big)^{N-1}dy \geqslant c\int_{-W}^{W} f(\sqrt{n-1}+w)\Big(1-L(n,\rho)\exp\Big\{\frac{w\rho^2}{\sqrt{n}}\Big\}\Big)^{N-1}$$

where $f(t)=t^{n-1}e^{-\frac{t^2}{2}}$ as before, $c=\big(\int_0^\infty f(t)\,dt\big)^{-1}$, and W is some big positive number. Note again that the product $c\,f(\sqrt{n-1}+w)=[1+o(1)]\frac{1}{\sqrt{\pi}}e^{-w^2}$ for fixed w and $n\to\infty$. Also, for fixed w and $n\to\infty$, the second factor in the integral is asymptotically equivalent to $\exp\{-e^w\}$ (recall that $\rho=n^{\frac14}$ and, therefore, $\frac{\rho^2}{\sqrt{n}}=1$). Thus, we obtain the estimate

$$\Gamma_n \geqslant \big[1+o(1)\big]e^{-\frac14}n^{\frac14}\frac{1}{\sqrt{\pi}}\int_{-W}^{W}\exp\{-e^w\}e^{-w^2}\,dw.$$

The integral on the right looks scary, but, since everything except the factor $\exp\{-e^w\}$ is symmetric, we can replace it by

$$\int_{-W}^{W}\frac{\exp\{-e^w\}+\exp\{-e^{-w}\}}{2}\,e^{-w^2}\,dw.$$

Using the elementary inequality

$$\frac{\exp\{-a\} + \exp\{-\frac{1}{a}\}}{2} \geqslant \frac{1}{e} \quad \text{for all } a > 0,$$

we conclude that

$$\Gamma_n \geqslant \big[1 + o(1)\big] e^{-\frac{5}{4}} n^{\frac{1}{4}} \frac{1}{\sqrt{\pi}} \int_{-W}^{W} e^{-w^2}\, dw.$$

It remains to note that $\frac{1}{\sqrt{\pi}} \int_{-W}^{W} e^{-w^2}\, dw$ can be made arbitrarily close to 1 by choosing W large enough.

The General Case

Let us start with two simple reductions. First of all, observe that the estimate we want to prove is homogeneous with respect to A, so, without loss of generality, we may assume that $\operatorname{Tr} A = 1$.

Since the problem is rotation invariant, we may assume that both A and B are diagonal matrices. We shall primarily deal with B, so let us denote the diagonal entries of B by $b_1, \dots, b_n$ (our normalization condition $\operatorname{Tr} A = 1$ means that $\sum_j b_j^2 = 1$). Denote

$$\mathcal{D} := \sqrt[4]{\sum_j b_j^4}\,.$$

Note that $0 < \mathcal{D} \leqslant 1$, so $\mathcal{D}^2 \leqslant \mathcal{D}$ and so forth.

Proof of the Estimate $\boldsymbol{\Gamma(A) \leqslant C\sqrt{\|A\|_{\mathrm{H\text{-}S}}}}$

We shall follow the idea of K. Ball and use the Cauchy integral formula. Let us recall how it works. Suppose you have two functions $F, G : \mathbb{R}^n \to [0, +\infty)$, a nonnegative homogeneous of degree 1 function Ψ in $\mathbb{R}^n$, and a random unit vector $z_\omega \in \mathbb{R}^n$. Suppose that you can show that for every point $y \in \mathbb{R}^n$ and for every vector $\nu \in \mathbb{R}^n$,

$$\mathcal{E}_\omega \left[|\langle \nu, z_\omega \rangle| \int_{\mathbb{R}} G(y - t z_\omega) dt \right] \geqslant \kappa\, F(y) \Psi(\nu)$$

with some constant $\kappa > 0$, where $\mathcal{E}_\omega$ denotes the expectation with respect to z_ω. Then for any convex body $Q \subset \mathbb{R}^n$,

$$\int_{\partial Q} F(y) \Psi(\nu_y)\, d\sigma(y) \leqslant 2\kappa^{-1} \int_{\mathbb{R}^n} G(x)\, dx.$$

To make this general formula applicable to our special case, we have to choose $F(y) = \varphi(y)$ and $\Psi(\nu) = |B\nu|$. Unfortunately, there is no clearly forced choice of z_ω and G. To choose z_ω, let us observe that our task is to make the "typical value" of $|\langle \nu, z_\omega\rangle|$ approximately equal to $|B\nu|$. The standard way to achieve this is to take $z_\omega = BZ_\omega$ with $Z_\omega = \sum_j \varepsilon_j(\omega)e_j$ where e_j is the orthonormal basis in $\mathbb{R}^n$ in which B is diagonal and $\varepsilon_j(\omega)$ $(\omega \in \Omega)$ are independent random variables taking values ± 1 with probability $\frac{1}{2}$ each. Note that our normalization condition $\sum_j b_j^2 = 1$ guarantees that z_ω is always a unit vector in $\mathbb{R}^n$. The hardest part is finding an appropriate function G. We shall search for $G(x)$ in the form

$$G(x) = \varphi(x)\Xi(x)$$

where $\Xi(x)$ is some relatively tame function: after all, the integral of a function over a random line containing a fixed point is equal to the value at the point times something "not-so-important" (at least, I do not know a better way to evaluate it with *no a priori information*). If $\Xi(x)$ changes slower than $\varphi(x)$, then we may expect the main part of the integral $\int_{\mathbb{R}} G(y - tz_\omega)\,dt$ to come from the points t that lie in a small neighbourhood of $t_0 = \langle y, z_\omega\rangle$, which is the point where the function $t \to \varphi(y - tz_\omega)$ attains its maximum. To make this statement precise, let us observe that

$$\begin{aligned}\varphi\big(y - (t_0+\tau)z_\omega\big) &= \exp\left\{-\frac{\tau^2}{2}\right\}\exp\left\{\frac{\langle y, z_\omega\rangle^2}{2}\right\}\varphi(y)\\ &\geqslant \frac{1}{\sqrt{e}}\exp\left\{\frac{\langle y, z_\omega\rangle^2}{2}\right\}\varphi(y)\end{aligned}$$

when $|\tau| \leqslant 1$. If our function $\Xi(x)$ satisfies the condition

$$\max\big\{\Xi(x-\tau z_\omega), \Xi(x+\tau z_\omega)\big\} \geqslant \frac{1}{2}\Xi(x) \quad \text{for all } x \in \mathbb{R}^n,\ \omega \in \Omega,\ |\tau| \leqslant 1$$

(which we shall call "weak convexity" condition), then we may estimate the integral from below by $\frac{1}{2\sqrt{e}}\,\varphi(y)\,\Xi(y - \langle y, z_\omega\rangle z_\omega)\exp\{\langle y, z_\omega\rangle^2/2\}$. Then our "only" task will be to prove the inequality

$$\mathcal{E}_\omega\left[|\langle \nu, z_\omega\rangle| \cdot \Xi\big(y - \langle y, z_\omega\rangle z_\omega\big) \cdot \exp\left\{\frac{\langle y, z_\omega\rangle^2}{2}\right\}\right] \geqslant \kappa|B\nu|. \qquad (**)$$

Let us make a second "natural leap of faith" and assume that Ξ changes so slowly that $\Xi(y - \langle y, z_\omega\rangle z_\omega) \approx \Xi(y)$. Then we can just *compute* the expectation of the product of other two factors and *define* $\Xi(y)$ to be the factor that makes the desired inequality almost an identity (we should pray that after that the loop will close and we shall not have to make a second iteration). Thus, our first task will be to compute the quantity

$$\mathcal{E}_\omega\left[|\langle \nu, z_\omega\rangle| \cdot \exp\left\{\frac{\langle y, z_\omega\rangle^2}{2}\right\}\right] = \mathcal{E}_\omega\left[|\langle B\nu, Z_\omega\rangle| \cdot \exp\left\{\frac{\langle By, Z_\omega\rangle^2}{2}\right\}\right].$$

Since $B\nu$ and By are just two arbitrary vectors in $\mathbb{R}^n$, let us introduce some one-letter notation for them. Let, say, $B\nu = v$ and $By = u$. As usual, we shall write v_j and u_j for the coordinates of v and u in the basis e_j. The unpleasant thing we shall have to face on this way is that even $\mathcal{E}_\omega[\exp\{\langle u, Z_\omega\rangle^2/2\}]$ (forget about the factor $|\langle v, Z_\omega\rangle|$!) is not easy to compute when $|u| > 1$. Fortunately, if we assume in addition that $\varXi(\beta x) \geqslant \varXi(x)$ for all $x \in \mathbb{R}^n$, $\beta \geqslant 1$, then the left hand side of $(**)$ will satisfy a similar inequality with respect to y and, thereby, $(**)$ will hold for all points y with $|By| > 1$ as soon as it holds for all y with $|By| = 1$.

We shall use the formula

$$\exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \exp\{t\langle u, Z_\omega\rangle\}\, dt$$

and write

$$\begin{aligned}\mathcal{E}_\omega\left[\exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\mathcal{E}_\omega\left[\exp\{t\langle u, Z_\omega\rangle\}\right]\, dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\prod_j\left(\frac{e^{-tu_j}+e^{tu_j}}{2}\right) dt\,.\end{aligned}$$

Using the elementary inequality $\frac{e^{-s}+e^s}{2} \leqslant e^{\frac{s^2}{2}}\left[1+\frac{s^4}{26}\right]^{-1}$, we can estimate the last integral from above by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{(1-|u|^2)t^2}{2}\right\}\left[1+\frac{(t\|u\|_4)^4}{26}\right]^{-1} dt$$

where $\|u\|_4 := \sqrt[4]{\sum_j u_j^4}$. If $|u| \leqslant 1$, we can say that, from the L^1 point of view, the integrand is hardly distinguishable from the characteristic function of the interval $|t| \leqslant \varDelta(u)$ where $\varDelta(u) = 1/\max\{\sqrt{1-|u|^2}, \|u\|_4\}$, so the last integral should be, roughly speaking, $\varDelta(u)$. A reasonably accurate computation yields the upper bound $\min\{1/\sqrt{1-|u|^2}, 3/\|u\|_4\} \leqslant 3\varDelta(u)$. To estimate $\mathcal{E}_\omega[\exp\{\langle u, Z_\omega\rangle^2/2\}]$ from below, we shall use another elementary inequality $\frac{e^{-s}+e^s}{2} \geqslant e^{s^2/2}e^{-s^4/8}$. It yields the lower bound

$$\begin{aligned}\mathcal{E}_\omega\left[\exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] &\geqslant \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{(1-|u|^2)t^2}{2}\right\}\exp\left\{-\frac{(t\|u\|_4)^4}{8}\right\} dt \\ &\geqslant \varDelta(u)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}e^{-\frac{t^4}{8}}\, dt \geqslant \frac{1}{2}\varDelta(u).\end{aligned}$$

Let us now turn to the estimates for the expectation $\mathcal{E}_\omega[|\langle v, Z_\omega\rangle| \exp\{\langle u, Z_\omega\rangle^2/2\}]$. To this end, we shall first estimate $\mathcal{E}_\omega[\exp\{is\langle v, Z_\omega\rangle\} \exp\{\langle u, Z_\omega\rangle^2/2\}]$ where, as usual, $i = \sqrt{-1}$. Again, write

$$\begin{aligned}
&\mathcal{E}_\omega\left[\exp\{is\langle v, Z_\omega\rangle\}\exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\,\mathcal{E}_\omega\left[\exp\{is\langle v, Z_\omega\rangle + t\langle u, Z_\omega\rangle\}\right]\,dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\prod_j\left(\frac{e^{-(tu_j+isv_j)}+e^{(tu_j+isv_j)}}{2}\right)\,dt.
\end{aligned}$$

Now note that for all $\alpha, \beta \in \mathbb{R}$, one has

$$\left|\frac{e^{-(\alpha+i\beta)}+e^{(\alpha+i\beta)}}{2}\right| \leqslant \frac{e^{-\alpha}+e^{\alpha}}{2}$$

and this trivial estimate (the triangle inequality) can be improved to

$$\left|\frac{e^{-(\alpha+i\beta)}+e^{(\alpha+i\beta)}}{2}\right| \leqslant e^{-\delta\,\beta^2}\left(\frac{e^{-\alpha}+e^{\alpha}}{2}\right)$$

with some absolute $\delta \in (0,1)$ for $|\alpha|, |\beta| \leqslant 1$. Therefore,

$$\prod_j\left|\frac{e^{-(tu_j+isv_j)}+e^{(tu_j+isv_j)}}{2}\right| \leqslant \prod_j\left(\frac{e^{-tu_j}+e^{tu_j}}{2}\right)$$

for all $t, s \in \mathbb{R}$ and

$$\prod_j\left|\frac{e^{-(tu_j+isv_j)}+e^{(tu_j+isv_j)}}{2}\right| \leqslant e^{-\delta s^2|v|^2}\prod_j\left(\frac{e^{-tu_j}+e^{tu_j}}{2}\right)$$

if $|t| \leqslant \|u\|_\infty^{-1}$ and $|s| \leqslant \|v\|_\infty^{-1}$. Since $\|u\|_\infty^{-1} \geqslant \|u\|_4^{-1} \geqslant \Delta(u)$, we can write

$$\begin{aligned}
&\mathcal{E}_\omega\left[\exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] - \left|\mathcal{E}_\omega\left[\exp\{is\langle v, Z_\omega\rangle\}\exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right]\right| \\
&\geqslant \left(1-e^{-\delta s^2|v|^2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\Delta(u)}^{\Delta(u)} e^{-\frac{t^2}{2}}\prod_j\left(\frac{e^{-tu_j}+e^{tu_j}}{2}\right)\,dt \\
&\geqslant \left(1-e^{-\delta s^2|v|^2}\right)\Delta(u)\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{t^2}{2}}e^{-\frac{t^4}{8}}\,dt \geqslant \frac{1}{2}\left(1-e^{-\delta s^2|v|^2}\right)\Delta(u)\,.
\end{aligned}$$

On the other hand, the trivial inequality $|\alpha - \alpha e^{i\beta}| \leqslant 2\alpha\min\{|\beta|, 1\}$ ($\alpha > 0$, $\beta \in \mathbb{R}$) yields

$$\begin{aligned}
&\left|\mathcal{E}_\omega\left[\exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] - \mathcal{E}_\omega\left[\exp\{is\langle v, Z_\omega\rangle\}\exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right]\right| \\
&\qquad\qquad \leqslant 2\mathcal{E}_\omega\left[\min\{|s\langle v, Z_\omega\rangle|, 1\}\exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right].
\end{aligned}$$

Bringing these estimates together and taking $s = |v|^{-1}$, we obtain

$$\mathcal{E}_\omega \left[\min\{|\langle v, Z_\omega\rangle|, |v|\} \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \geqslant \frac{1}{4}(1-e^{-\delta})\varDelta(u)|v| = 2\eta\varDelta(u)|v|$$

where $\eta := \frac{1}{8}(1-e^{-\delta}) > 0$ is an absolute constant. Obviously, the expectation $\mathcal{E}_\omega[|\langle v, Z_\omega\rangle| \exp\{\langle u, Z_\omega\rangle^2/2\}]$ can be only greater.

This brings us to the idea to take $\varXi(y) = \varDelta(By)^{-1} = \max\{\|By\|_4, \sqrt{1-|By|^2}\}$. This formula makes little sense for $|By| > 1$, so, to be formally correct, we shall distinguish two cases: $|By|^2 \geqslant 1 - \|By\|_4^2$ and $|By|^2 < 1 - \|By\|_4^2$. We shall separate them completely and even construct two different functions $\varXi_1$ and $\varXi_2$ serving the first and the second case correspondingly. Note that all points y for which $|By| \geqslant 1$ are covered by the first case, so we need the condition that $\varXi$ be non-decreasing along each ray starting at the origin only for $\varXi_1$. Let us start with

Case 1: $1 - \|By\|_4^2 \leqslant |By|^2 \leqslant 1$.
In this case the natural candidate for $\varXi_1$ is $\varXi_1(y) = \|By\|_4$. We have no problem with the "weak convexity" condition because $\varXi_1$ is even strongly convex. Also, it obviously satisfies $\varXi_1(\beta y) \geqslant \varXi_1(y)$ for every $\beta \geqslant 1$. The only thing we should take care about is the assumption $\varXi_1(y - \langle y, z_\omega\rangle z_\omega) \approx \varXi_1(y)$. What we would formally need here is $\varXi_1(y - \langle y, z_\omega\rangle z_\omega) \geqslant \zeta\, \varXi_1(y)$ with some absolute $0 < \zeta \leqslant 1$. Unfortunately, it is futile to hope for such an estimate for all $y \in \mathbb{R}^n$ and $\omega \in \varOmega$ because it can easily happen that y is collinear with some z_ω and then we shall get $\varXi_1(y - \langle y, z_\omega\rangle z_\omega) = 0$. To exclude this trivial problem, let us bound $\varXi$ from below by some constant. Since our aim is to control the integral of $\varXi_1$ with respect to the Gaussian measure in $\mathbb{R}^n$, we may just take the maximum of $\varXi_1$ and its average value with respect to the Gaussian measure $d\gamma(x) = \varphi(x)\, dx$, which is almost the same as $\|\varXi_1\|_{L^4(\mathbb{R}^n, d\gamma)} = \sqrt[4]{3}\sqrt[4]{\sum_j b_j^4} = \sqrt[4]{3}\,\mathcal{D}$. This leads to the revised definition

$$\varXi_1(y) = \max\{\|By\|_4, \mathcal{D}\}$$

(note that this revised function $\varXi_1$ is still convex and non-decreasing along each ray starting at the origin). The condition $\varXi_1(y - \langle y, z_\omega\rangle z_\omega) \geqslant \zeta\, \varXi_1(y)$ is then trivially satisfied with $\zeta = 1$ if $\|By\|_4 \leqslant \mathcal{D}$. Assume that $\|By\|_4 \geqslant \mathcal{D}$. Then

$$\begin{aligned}\varXi_1(y - \langle y, z_\omega\rangle z_\omega) &\geqslant \|By - \langle y, z_\omega\rangle Bz_\omega\|_4 \geqslant \|By\|_4 - |\langle By, Z_\omega\rangle| \cdot \|B^2 Z_\omega\|_4 \\ &= \varXi_1(y) - |\langle By, Z_\omega\rangle| \sqrt[4]{\sum_j b_j^8} \geqslant \varXi_1(y) - \mathcal{D}^2 |\langle By, Z_\omega\rangle|.\end{aligned}$$

Uniting this estimate with the trivial lower bound $\varXi_1(y - \langle y, z_\omega\rangle z_\omega) \geqslant \mathcal{D}$, we can write

$$\varXi_1(y - \langle y, z_\omega\rangle z_\omega) \geqslant \frac{1}{1 + \mathcal{D}|\langle By, Z_\omega\rangle|} \varXi_1(y)$$

(we used here the elementary estimate $\max\{\alpha - \beta\mathcal{D}, \mathcal{D}\} \geqslant \frac{\alpha}{1+\beta}$). Therefore we shall be able to prove (∗∗) if we demonstrate that under the conditions $\sqrt{1-|u|^2} \leqslant \|u\|_4$ and $\|u\|_4 \geqslant \mathcal{D}$ (where, as before, $u = By$), the main part of the expectation $\mathcal{E}_\omega[\min\{|\langle v, Z_\omega\rangle|, |v|\}\exp\{\langle u, Z_\omega\rangle^2/2\}]$ comes from those $\omega \in \Omega$ for which $|\langle v, Z_\omega\rangle|$ is not much greater than $\mathcal{D}^{-1}$. To this end, we shall have to prove some "tail estimate" for $\mathcal{E}_\omega[\exp\{\langle u, Z_\omega\rangle^2/2\}]$. Using the inequality

$$\langle u, Z_\omega\rangle^2 \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\} \leqslant \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2}} \exp\bigl\{t\langle u, Z_\omega\rangle\bigr\}\, dt,$$

we get

$$\begin{aligned}
\mathcal{E}_\omega\left[\langle u, Z_\omega\rangle^2 \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] &\leqslant \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2}} \mathcal{E}_\omega\left[\exp\bigl\{t\langle u, Z_\omega\rangle\bigr\}\right]\, dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2}} \prod_j \left(\frac{e^{-tu_j} + e^{tu_j}}{2}\right)\, dt \\
&\leqslant \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2 \left[1 + \frac{(t\|u\|_4)^4}{26}\right]^{-1}\, dt \\
&= \frac{2}{3}\pi^{\frac{1}{2}} 26^{\frac{3}{4}} \|u\|_4^{-3} \leqslant 16\|u\|_4^{-3}.
\end{aligned}$$

This results in the tail estimate

$$\begin{aligned}
&\mathcal{E}_\omega\left[\chi_{\{|\langle u, Z_\omega\rangle| > \beta\|u\|_4^{-1}\}} \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \\
&\qquad \leqslant \beta^{-2}\|u\|_4^2 \mathcal{E}_\omega\left[\langle u, Z_\omega\rangle^2 \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \leqslant 16\beta^{-2}\|u\|_4^{-1}.
\end{aligned}$$

Choosing $\beta := \frac{4}{\sqrt{\eta}}$ and recalling that $\|u\|_4 \geqslant \mathcal{D}$, we finally get ($u = By$, $v = B\nu$)

$$\begin{aligned}
&\mathcal{E}_\omega\left[|\langle v, Z_\omega\rangle| \cdot \Xi_1(y - \langle y, z_\omega\rangle z_\omega) \cdot \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \\
&\geqslant \mathcal{E}_\omega\left[\min\bigl\{|\langle v, Z_\omega\rangle|, |v|\bigr\} \cdot \Xi_1(y - \langle y, z_\omega\rangle z_\omega) \cdot \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \\
&\geqslant \frac{1}{1+\beta}\Xi_1(y) \cdot \mathcal{E}_\omega\left[\chi_{\{|\langle u, Z_\omega\rangle| \leqslant \beta\mathcal{D}^{-1}\}} \cdot \min\bigl\{|\langle v, Z_\omega\rangle|, |v|\bigr\} \cdot \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \\
&\geqslant \frac{1}{1+\beta}\Xi_1(y) \cdot \mathcal{E}_\omega\left[\chi_{\{|\langle u, Z_\omega\rangle| \leqslant \beta\|u\|_4^{-1}\}} \cdot \min\bigl\{|\langle v, Z_\omega\rangle|, |v|\bigr\} \cdot \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \\
&\geqslant \frac{\Xi_1(y)}{1+\beta}\Bigl(\mathcal{E}_\omega\Bigl[\min\left\{|\langle v, Z_\omega\rangle|, |v|\right\} \cdot \exp\Bigl\{\tfrac{\langle u, Z_\omega\rangle^2}{2}\Bigr\}\Bigr]
\end{aligned}$$

$$-|v| \cdot \mathcal{E}_\omega \Big[\chi_{\{|\langle u,Z_\omega\rangle|>\beta\|u\|_4^{-1}\}} \exp\Big\{\tfrac{\langle u,Z_\omega\rangle^2}{2}\Big\}\Big]\Big)$$

$$\geqslant \frac{\Xi_1(y)}{1+\beta}\left(2\eta\|u\|_4^{-1}|v| - 16\beta^{-2}\|u\|_4^{-1}|v|\right) \geqslant \frac{\eta}{1+\beta}\Xi_1(y)\|u\|_4^{-1}|v| = \frac{\eta}{1+\beta}|v|.$$

This finishes Case 1.

Case 2: $|By|^2 \leqslant 1 - \|By\|_4^2$.
To make the long story short, the function Ξ_2 that we shall use for this case is

$$\Xi_2(y) = \max\Big\{\sqrt{(1-|By|^2)_+}, \mathcal{D}\Big\}.$$

It is easy to see that

$$\|\Xi_2\|^4_{L^4(\mathbb{R}^n,d\gamma)} \leqslant \mathcal{D}^4 + \int_{\mathbb{R}^n}(1-|Bx|^2)^2\,d\gamma(x) = 3\mathcal{D}^4.$$

To prove “weak convexity”, let us observe that

$$1-|B(x+\tau z_\omega)|^2 = 1-|Bx|^2 - 2\tau\langle Bx, Bz_\omega\rangle - \tau^2|Bz_\omega|^2 \geqslant 1-|Bx|^2-\mathcal{D}^4$$

if $|\tau|\leqslant 1$ and the sign of τ is opposite to that of $\langle Bx, Bz_\omega\rangle$. Therefore,

$$\sqrt{(1-|Bx+\tau z_\omega|^2)_+} \geqslant \sqrt{(1-|Bx|^2)_+} - \mathcal{D}^2 \geqslant \sqrt{(1-|Bx|^2)_+} - \mathcal{D}$$

for such τ, which is enough to establish the weak convexity property for Ξ_2. Now let us turn to the inequality $\Xi_2(y-\langle y,z_\omega\rangle z_\omega) \geqslant \zeta\,\Xi_2(y)$. Again, it is trivial if $\sqrt{1-|By|^2}\leqslant\mathcal{D}$. For other y, write

$$1-|B(y-\langle y,z_\omega\rangle z_\omega)|^2 = 1-|By|^2 + 2\langle y,z_\omega\rangle\langle By,Bz_\omega\rangle - \langle y,z_\omega\rangle^2|Bz_\omega|^2.$$

It will suffice to show that the main contribution to the mathematical expectation $\mathcal{E}_\omega[\min\{|\langle v,Z_\omega\rangle|,|v|\}\exp\{\langle u,Z_\omega\rangle^2/2\}]$ is made by those $\omega\in\Omega$ for which

$$2\langle y,z_\omega\rangle\langle By,Bz_\omega\rangle - \langle y,z_\omega\rangle^2|Bz_\omega|^2 \geqslant -K^2\mathcal{D}^2$$

where K is some absolute constant (for such ω, one has $\Xi_2(y-\langle y,z_\omega\rangle z_\omega)\geqslant \frac{\Xi_2(y)}{1+K}$). Since $|Bz_\omega|^2=\mathcal{D}^4$, we can use the tail estimate

$$\mathcal{E}_\omega\left[\chi_{\left\{|\langle u,Z_\omega\rangle|>\beta\frac{1}{\sqrt{1-|u|^2}}\right\}}\exp\left\{\frac{\langle u,Z_\omega\rangle^2}{2}\right\}\right]$$
$$\leqslant \beta^{-2}(1-|u|^2)\mathcal{E}_\omega\left[\langle u,Z_\omega\rangle^2\exp\left\{\frac{\langle u,Z_\omega\rangle^2}{2}\right\}\right] \leqslant \beta^{-2}\frac{1}{\sqrt{1-|u|^2}}$$

(which is proved in exactly the same way as the tail estimate in Case 1) to restrict ourselves to $\omega\in\Omega$ satisfying $|\langle y,z_\omega\rangle|\leqslant\beta\mathcal{D}^{-1}$. This allows to bound the subtrahend in the difference $2\langle y,z_\omega\rangle\langle By,Bz_\omega\rangle - \langle y,z_\omega\rangle^2|Bz_\omega|^2$ by $\beta^2\mathcal{D}^2$. To bound the minuend from below, we shall use the following

Correlation Inequality

Let $u, w \in \mathbb{R}^n$ satisfy $|u| < 1$, $\langle u, w\rangle \geqslant 0$. Then

$$\mathcal{E}_\omega \left[\chi_{\left\{\langle u, Z_\omega\rangle\langle w, Z_\omega\rangle < -\beta \frac{|w|}{\sqrt{1-|u|^2}}\right\}} \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \leqslant \frac{2}{\sqrt{3}} e^{-\beta} \frac{1}{\sqrt{1-|u|^2}}.$$

Let us first show that this correlation inequality implies the desired bound for the minuend $2\langle y, z_\omega\rangle\langle By, Bz_\omega\rangle$. Indeed, write

$$\langle y, z_\omega\rangle\langle By, Bz_\omega\rangle = \langle By, Z_\omega\rangle\langle B^3 y, Z_\omega\rangle = \langle u, Z_\omega\rangle\langle B^2 u, Z_\omega\rangle$$

where, as always, $u = By$. Observe that $\langle u, B^2 u\rangle = |Bu|^2 \geqslant 0$. Therefore, according to the correlation inequality, we may restrict ourselves to $\omega \in \Omega$ satisfying $\langle u, Z_\omega\rangle\langle B^2 u, Z_\omega\rangle \geqslant -\beta\frac{1}{\sqrt{1-|u|^2}}|B^2 u|$ where $\beta > 0$ is chosen so large that $\beta^{-2} + \frac{2}{\sqrt{3}}e^{-\beta} \leqslant \eta$. Now observe that

$$|B^2 u| = \sqrt{\sum_j b_j^4 u_j^2} \leqslant \sqrt{\sqrt{\sum_j b_j^8}\,\sqrt{\sum_j u_j^4}} \leqslant \mathcal{D}^2\|u\|_4.$$

Thus

$$2\langle u, Z_\omega\rangle\langle B^2 u, Z_\omega\rangle \geqslant -2\beta\frac{\|u\|_4}{\sqrt{1-|u|^2}}\mathcal{D}^2 \geqslant -2\beta\mathcal{D}^2$$

due to our assumption $|u|^2 < 1 - \|u\|_4^2$. To prove the correlation inequality, just take

$$\widetilde{u} := u - \frac{\sqrt{1-|u|^2}}{2|w|} w$$

and observe that $|\widetilde{u}|^2 \leqslant |u|^2 + \frac{1}{4}(1-|u|^2)$, so $\sqrt{1-|\widetilde{u}|^2} \geqslant \frac{\sqrt{3}}{2}\sqrt{1-|u|^2}$. Now we have

$$\mathcal{E}_\omega \left[\chi_{\left\{\langle u, Z_\omega\rangle\langle w, Z_\omega\rangle < -\beta \frac{|w|}{\sqrt{1-|u|^2}}\right\}} \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right] \leqslant e^{-\beta}\mathcal{E}_\omega\left[\exp\left\{\frac{\langle \widetilde{u}, Z_\omega\rangle^2}{2}\right\}\right]$$

because

$$\langle \widetilde{u}, Z_\omega\rangle^2 \geqslant \langle u, Z_\omega\rangle^2 - 2\frac{\sqrt{1-|u|^2}}{2|w|}\langle u, Z_\omega\rangle\langle w, Z_\omega\rangle \geqslant \langle u, Z_\omega\rangle^2 + \beta$$

under the condition $\langle u, Z_\omega\rangle\langle w, Z_\omega\rangle < -\beta\frac{|w|}{\sqrt{1-|u|^2}}$. It remains to recall that

$$\mathcal{E}_\omega\left[\exp\left\{\frac{\langle \widetilde{u}, Z_\omega\rangle^2}{2}\right\}\right] \leqslant \frac{1}{\sqrt{1-|\widetilde{u}|^2}} \leqslant \frac{2}{\sqrt{3}}\frac{1}{\sqrt{1-|u|^2}}.$$

The upper bound for $\Gamma(A)$ is now completely proved.

Proof of the Estimate $\boldsymbol{\Gamma(A) \geqslant c\sqrt{\|A\|_{\text{H-S}}}}$

Let $\varrho > 0$. Consider the family of random polyhedrons

$$Q(\varrho, N; \omega) := \left\{ x \in \mathbb{R}^n \,:\, \left|\left\langle x, x^{[k]} \right\rangle\right| \leqslant \varrho \quad \text{for all } k = 1, \dots, N \right\}$$

where $x^{[k]} = BZ_\omega^{[k]}$ and $Z_\omega^{[k]}$ ($k \geqslant 1$) is a sequence of independent random vectors equidistributed with $Z_\omega = \sum_j \varepsilon_j(\omega) e_j$. Let us observe that $|B\nu_y|$ identically equals $\sqrt{\sum_j b_j^4} = \mathcal{D}^2$ on $\partial Q(\varrho; \omega)$. Thus, the inequality $\Gamma(A) \geqslant c\sqrt{\|A\|_{\text{H-S}}} = c\mathcal{D}$ will be proved if we show that at least one polyhedron $Q(\varrho, N; \omega)$ has the Gaussian perimeter of $c\mathcal{D}^{-1}$ or greater.

I tried to use as few non-trivial statements about Bernoulli random variables in this note as possible but I still had to employ the following

Pinnelis Tail Lemma. *Let $u \in \mathbb{R}^n$, $\beta \geqslant 0$. Then*

$$\mathcal{P}_\omega\left\{ \langle u, Z_\omega \rangle \geqslant \beta |u| \right\} \leqslant K \frac{1}{\sqrt{2\pi}} \int_\beta^\infty e^{-\frac{t^2}{2}}\, dt \leqslant K \frac{1}{1+\beta} e^{-\frac{\beta^2}{2}}$$

where K is some universal constant. Informally speaking, this means that Bernoulli tails do not exceed Gaussian tails.

The simplest and most elegant proof of the Pinnelis Tail Lemma belongs to Sergei Bobkov, who observed that the function

$$\Phi(\beta) = \frac{1}{\sqrt{2\pi}} \int_\beta^\infty e^{-\frac{t^2}{2}}\, dt$$

satisfies the inequality

$$\Phi\left(\frac{\beta - a}{\sqrt{1 - a^2}} \right) + \Phi\left(\frac{\beta + a}{\sqrt{1 - a^2}} \right) \leqslant 2\Phi(\beta)$$

for all $\beta \geqslant \sqrt{3}$, $0 \leqslant a < 1$ (to prove it, just differentiate the left hand side with respect to a and check that the derivative is never positive), which allows to prove the lemma by induction with $K = \frac{1}{2\Phi(\sqrt{3}\,)} < 13$.

We shall show that the "average perimeter" of $Q(\varrho, N; \omega)$ is large. To formalize this, choose some nice continuous non-negative decreasing L^1-function $p : [0, +\infty) \to \mathbb{R}$ (which will serve as the weight with which we shall average with respect to ϱ) and some small $h > 0$.

Note that for each $\varrho > 0$, $N \geqslant 1$, and $\omega \in \Omega$,

$$\gamma\big(Q(\varrho + h, N; \omega)\big) - \gamma\big(Q(\varrho, N; \omega)\big) \leqslant h\Upsilon$$

where Υ is the supremum of all perimeters of our polyhedra with respect to the standard Gaussian measure. Therefore

$$\mathcal{E}_\omega\big[\gamma\big(Q(\varrho+h,N;\omega)\big)\big]-\mathcal{E}_\omega\big[\gamma\big(Q(\varrho,N;\omega)\big)\big]\leqslant h\Upsilon.$$

On the other hand,

$$\mathcal{E}_\omega\big[\gamma\big(Q(\varrho;\omega)\big)\big]=\int_{\mathbb{R}^n}\Big(1-\mathcal{P}_\omega\big\{|\langle Bx,Z_\omega\rangle|>\varrho\big\}\Big)^{N(\varrho)}d\gamma(x)\,.$$

Now take $\varrho_\ell=\ell h$ ($\ell=1,2,\dots$), choose some integer-valued positive increasing function $N(\varrho)$, and consider the sumtegral

$$\sum_{\ell=1}^{\infty}p(\varrho_\ell)\int_{\mathbb{R}^n}\Big[(1-\mathcal{P}_\omega\left\{|\langle Bx,Z_\omega\rangle|>\varrho_{\ell+1}\right\})^{N(\varrho_\ell)} \\ -(1-\mathcal{P}_\omega\left\{|\langle Bx,Z_\omega\rangle|>\varrho_\ell\right\})^{N(\varrho_\ell)}\Big]\,d\gamma(x)\,.$$

On one hand, this sumtegral does not exceed $\sum_{\ell=1}^{\infty}p(\varrho_\ell)h\Upsilon\leqslant\Upsilon\int_0^\infty p(\varrho)\,d\varrho$. On the other hand, since

$$(1-\alpha)^M-(1-\beta)^M\geqslant e^{-1}M(\beta-\alpha)\qquad\text{whenever }\alpha\leqslant\beta\leqslant\frac{1}{M}\,,$$

we can change the order of summation and integration (the sumtegrand is nonnegative) and estimate our sumtegral from below by

$$e^{-1}\!\!\int_S\bigg\{\sum_{\ell=1}^{\infty}p(\varrho_\ell)N(\varrho_\ell)\Big[\mathcal{P}_\omega\left\{|\langle Bx,Z_\omega\rangle|>\varrho_\ell\right\} \\ -\mathcal{P}_\omega\left\{|\langle Bx,Z_\omega\rangle|>\varrho_{\ell+1}\right\}\Big]\bigg\}\,d\gamma(x)$$

where $S\subset\mathbb{R}^n$ is the set of all points x for which $\mathcal{P}_\omega\left\{|\langle Bx,Z_\omega\rangle|>\varrho\right\}\leqslant N(\varrho)^{-1}$ for all $\varrho>0$. For each fixed $x\in S$, the integrand converges to the mathematical expectation $\mathcal{E}_\omega\big[p\big(|\langle Bx,Z_\omega\rangle|\big)\,N\big(|\langle Bx,Z_\omega\rangle|\big)\big]$ as $h\to0^+$. Therefore, the lower limit of the sumtegral is at least

$$e^{-1}\gamma(S)\mathcal{E}_\omega\Big[p\big(|\langle Bx,Z_\omega\rangle|\big)\,N\big(|\langle Bx,Z_\omega\rangle|\big)\Big]$$

as $h\to0^+$. Comparing the upper and the lower bound, we get the inequality

$$\Upsilon\int_0^\infty p(\varrho)\,d\varrho\geqslant e^{-1}\gamma(S)\mathcal{E}_\omega\Big[p\big(|\langle Bx,Z_\omega\rangle|\big)\,N\big(|\langle Bx,Z_\omega\rangle|\big)\Big]\,.$$

Our aim will be to choose the function $N(\rho)$ sufficiently small to make the set S large on one hand and sufficiently large to make the right hand side much larger than $\int_0^\infty p(\varrho)\,d\varrho$ on the other hand. Note that the demand that $N(\varrho)$ assume only integer values can be dropped because, given any non-negative function $N(\varrho)$, we can always replace it by the function $\widetilde{N}(\varrho)$ that takes value

1 if $0 \leqslant N(\varrho) \leqslant 2$ and value k if $k < N(\varrho) \leqslant k+1$, $k = 2, 3, \dots$. This will not reduce the set S and will reduce the mathematical expectation on the right not more than twice.

Now recall that

$$\int_{\mathbb{R}^n} \|Bx\|_4^4 \, d\gamma(x) = 3\mathcal{D}^4 \qquad \text{and} \qquad \int_{\mathbb{R}^n} \left(1 - |Bx|^2\right)^2 \, d\gamma(x) = 2\mathcal{D}^4.$$

Therefore, for at least one quarter (with respect to $d\gamma$) of the points $x \in \mathbb{R}^n$, one has both

$$\|Bx\|_4 \leqslant 2\mathcal{D} \qquad \text{and} \qquad \left|1 - |Bx|^2\right| \leqslant 2\mathcal{D}^2$$

(the measures of the exceptional sets do not exceed $\frac{3}{16}$ and $\frac{1}{2}$ correspondingly).

Now we can use Pinnelis Tail Lemma and observe that for such points x,

$$\mathcal{P}_\omega\big\{|\langle Bx, Z_\omega\rangle| > \varrho\big\} \leqslant 2K \left(1 + \frac{\varrho}{\sqrt{1+2\mathcal{D}^2}}\right)^{-1} \exp\left\{-\frac{1}{1+2\mathcal{D}^2}\,\frac{\varrho^2}{2}\right\}.$$

This leads to the choice

$$N(\varrho) := \frac{1}{2K}\left(1 + \frac{\varrho}{\sqrt{1+2\mathcal{D}^2}}\right) \exp\left\{\frac{1}{1+2\mathcal{D}^2}\,\frac{\varrho^2}{2}\right\}.$$

Let us now choose the weight p. Since the only mathematical expectations we can easily compute are those of slight perturbations of exponential functions, it seems reasonable to try

$$p(\varrho) := \exp\left\{-\frac{\mathcal{D}^2}{3}\,\frac{\varrho^2}{2}\right\}.$$

With such a choice, we have

$$\mathcal{E}_\omega\Big[p\big(|\langle Bx, Z_\omega\rangle|\big)\, N\big(|\langle Bx, Z_\omega\rangle|\big)\Big] \geqslant \frac{1}{2K}\mathcal{E}_\omega\left[(1 + |\langle u, Z_\omega\rangle|) \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\}\right]$$

where

$$u := \sqrt{\frac{1}{1+2\mathcal{D}^2} - \frac{\mathcal{D}^2}{3}}\; Bx\,.$$

Note that

$$\|u\|_4 \leqslant \|Bx\|_4 \leqslant 2\mathcal{D} \quad \text{and} \quad |u|^2 \geqslant \left(\frac{1}{1+2\mathcal{D}^2} - \frac{\mathcal{D}^2}{3}\right)(1 - 2\mathcal{D}^2) \geqslant 1 - 5\mathcal{D}^2\,.$$

Using the inequality

$$(1 + |\langle u, Z_\omega\rangle|) \exp\left\{\frac{\langle u, Z_\omega\rangle^2}{2}\right\} \geqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |t| e^{-\frac{t^2}{2}} \exp\big\{t\langle u, Z_\omega\rangle\big\}\, dt,$$

we conclude that

$$\mathcal{E}_\omega \left[(1 + |\langle u, Z_\omega \rangle|) \exp \left\{ \frac{\langle u, Z_\omega \rangle^2}{2} \right\} \right] \geqslant \mathcal{D}^{-2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |t| e^{-\frac{5t^2}{2}} e^{-2t^4} \, dt.$$

Since $\int_0^\infty p(\varrho)\, d\varrho = \sqrt{\frac{3\pi}{2}}\, \mathcal{D}^{-1}$, the desired bound $\Upsilon \geqslant c\, \mathcal{D}^{-1}$ follows.

The theorem is thus completely proved.

References

[B] Ball, K. (1993): The reverse isoperimetric problem for Gaussian measure. Discrete Comput. Geom., **10(4)**, 411–420

[Be1] Bentkus, V. (1986): Dependence of the Berry–Esseen bound on the dimension. Lithuanian Maqth. J., **26(2)**, 205–210

[Be2] Bentkus, V. (2001): On the dependence of the Berry–Esseen bound on dimension. Max Planck Instutute Preprint series 2001 (53), Bonn. Journal Statistical Planning Inference, to appear