SDS 387, Fall 2024
Homework 1

Due September 17, by midnight on Canvas.

1. Let $\{x_n\}$ be a sequence of numbers. Describe the mathematical statements: $x_n = \Omega(1)$, $x_n = \omega(1)$ and $x_n = \Theta(1)$.

2. **Limit superior and limit inferior.**

   (a) Let $\{A_n\}$ be a sequence of events (an event is a collection of outcomes). Argue that an outcome belongs to $\limsup_n A_n$ if and only if it belongs to infinitely many events A_n's and that it belongs to $\liminf_n A_n$ if and only if there exists an integer N such that the outcome belongs to all the events A_n with $n \geq N$. Conclude that $\liminf_n A_n \subseteq \limsup_n A_n$.

   (b) Consider the same setting above. De Morgan's Laws state that $(\cup_n A)^c = \cap_n A_n^c$ and $(\cap_n A)^c = \cup_n A_n^c$, where A^c is the complement of the set A. Use De Morgan's law to show that $(\liminf_n A_n)^c = \limsup_n A_n^c$.

   (c) Let A_n be $(-1/n, 1]$ if n is odd and $(-1, 1/n]$ if n is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.

   (d) **Bonus Problem**. Let A_n the interior of the ball in $\mathbb{R}^2$ with unit radius and center $\left(\frac{(-1)^n}{n}, 0\right)$. Find $\limsup_n A_n$ and $\liminf_n A_n$.

3. Let $X_1, X_2, \ldots$ be a sequence of 0-1 Bernoulli random variables such $X_n \sim \text{Bernoulli}(1/n^2)$. Let $X = \sum_{n=1}^{\infty} X_n$. What is $\mathbb{P}(X < \infty)$?

4. Ferguson, problem 5, page 12.

5. Prova Markov's inequality: if X is a non-negative random variable, then for any $\epsilon > 0$

$$\mathbb{P}(X \geq \epsilon) \leq \frac{\mathbb{E}[X]}{\epsilon}.$$

   Markov's inequality is almost always a loose upper bound, but there are rare cases when it is sharp. Find an example in which it holds exactly. *Hint: take X to be the indicator function of a set and select the right ϵ.*
   Prove the PaleyZygmund inequality, a reverse Markov inequality of sort: if X is a non-negative random variable with two or more moments, then, for any $\alpha \in (0, 1)$,

$$\mathbb{P}(X \geq \alpha \mathbb{E}[X]) \geq (1-\alpha)^2 \frac{\mathbb{E}[^2 X]}{\mathbb{E}[X^2]}.$$

6. Let $X_1, \ldots, X_n$ *i.i.d.* univariate random variables with common distribution function F_X. Given $\alpha \in (0, 1)$, use the DKW inequality given in class to construct a $1 - \alpha$

confidence band for F_X, a pair of random functions (random because dependent on $X_1, \ldots, X_n$), say $\hat{F}_\alpha^{\text{lower}}$ and $\hat{F}_\alpha^{\text{upper}}$, such that

$$\mathbb{P}\left(\hat{F}_\alpha^{\text{lower}}(x) \leq F_X(x) \leq \hat{F}_\alpha^{\text{upper}}(x), \forall x \in \mathbb{R}\right) \geq 1 - \alpha.$$

7. **Joint and marginal convergence.** Below, $\{X_n\}$ is a sequence of random vectors in $\mathbb{R}^d$ and X another random vector in $\mathbb{R}^d$.

   (a) Show that $X_n \xrightarrow{p} X$ if and only if $X_n(j) \xrightarrow{p} X(j)$ for all $j = 1, \ldots, d$. *Note: the same is true about convergence with probability one.*

   (b) Show that if $X_n \xrightarrow{d} X$, then $X_n(j) \xrightarrow{d} X(j)$ for all $j = 1, \ldots, d$.

   (c) In class, we looked at this example in $d = 2$. Set $U \sim \text{Uniform}(0, 1)$ and let $X_n = U$ for all n and

   $$Y_n = \begin{cases} U & n \text{ odd}, \\ 1 - U & n \text{ even}. \end{cases}$$

   Then, $X_n \xrightarrow{d} U$ and $X_n \xrightarrow{d} U$. In class, I claimed that

   $$\begin{bmatrix} X_n \\ Y_n \end{bmatrix}$$

   does not converge in distribution (in fact, in any meaningful sense). Prove the claim.

8. Show that the c.d.f. of a random variable can have at most countably many points of discontinuity.

9. For each n, let X_n a random variable uniformly distributed on $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Show that X_n converges on distribution to $U \sim \text{Uniform}(0, 1)$. Let A be the set of all rational numbers in $[0, 1]$. Then $\mathbb{P}(X_n \in A) = 1$ for all n but $\mathbb{P}(X \in A) = 0$. Show that this does not violate condition *(v)* of the Portmanteau theorem, as stated in the lecture notes.