36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 23: WED, NOV 18, 2020

http://www.stat.cmu.edu/~arinaldo/Teaching/36710-36752/Lecture_Notes/lec_notes_9.pdf

CONDITIONAL EXPECTATION

Def LET $(\Omega, \mathcal{F}, P)$ BE A PROBABILITY SPACE AND $\mathcal{C} \subseteq \mathcal{F}$ A SUB-σ-FIELD. LET X BE A RV THAT IS $\mathcal{F}/\mathcal{B}^1$ MEAS. S.T. $\mathbb{E}(|X|) < \infty$. LET $\mathbb{E}[X \mid \mathcal{C}]$ STANDS FOR ANY FUNCTION $h: \Omega \to \mathbb{R}$ THAT IS $\mathcal{C}/\mathcal{B}^1$ MEAS. S.T.

$$(*) \qquad \int_C h \, dP = \int_C X \, dP \quad \text{FOR ALL } C \in \mathcal{C}.$$

WE CALL SUCH A FUNCTION h A VERSION OF THE CONDITIONAL EXPECTATION OF X GIVEN $\mathcal{C}$.

$$\omega \longmapsto \mathbb{E}[X \mid \mathcal{C}](\omega)$$

$\downarrow$ UNIQUE a.e. P

$\hookrightarrow$ ITSELF A RANDOM VARIABLE MEAS. WRT $\mathcal{C}$

INTERPRETATION OF CONDITIONAL EXPECTATION USING CONDITIONAL PROBABILITIES

$$X = 1_A \quad , \quad A \in \mathcal{F}$$

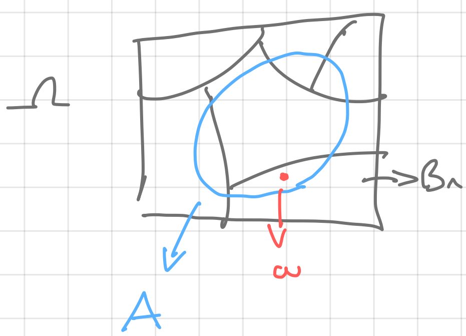
- THINK OF $(\Omega, \mathcal{F}, P)$ AS AN EXPERIMENT : OUTCOME ω

  WE ARE INTERESTED IN $\Pr(\omega \in A) = P(A)$ SOME FIXED $A \in \mathcal{F}$.

- WE ARE GIVEN ADDITIONAL INFORMATION ABOUT THE EXPERIMENT IN THE FORM OF A PARTITION OF $\Omega = \biguplus B_n$, EACH $B_n \in \mathcal{F}$.

  AND, FOR EACH ω, WE ARE TOLD THE B_n TO WHICH IT BELONGS

  $\hookrightarrow$ PARTIAL INFORMATION



LET $\mathcal{C}$ BE σ-FIELD GENERATED BY THE PARTITION

IN THIS SET UP, DEFINE THE FUNCTION

$$\omega \longmapsto h(\omega) = \begin{cases} \underbrace{\dfrac{P(A \cap B)}{P(B)}}_{P(A|B)} & \text{IF } \omega \in B \in \mathcal{C} \text{ AND } P(B) > 0 \\ f(\omega) & \text{OTHERWISE} \end{cases}$$

ARBITRARY INTEGRABLE $\mathcal{C}$-MEAS.

THIS FUNCTION $h(\omega)$ IS MEAS WRT $\mathcal{C}$ AND SATISFIES

$$P(A \cap B) = \int_B f(\omega)\, dP(\omega) \qquad \forall B \in \mathcal{C}$$

THIS IS THE (K) PROPERTY

MORE GENERALLY IF $\mathcal{C}$ IS A SUB-σ-FIELD OF $\mathcal{F}$ WE CAN REPEAT THE SAME ARGUMENTS: LET $A \in \mathcal{F}$ BE FIXED AND DEFINE A NEW MEASURE ON $\mathcal{C}$ GIVEN BY

$$\nu(B) = P(A \cap B) \qquad B \in \mathcal{C}$$

BY RN THEOREM $\exists\, h$ THAT IS $\mathcal{C}$-MEAS AND UNIQUE P-a.s SUCH THAT

$$\nu(B) = \int_B h \, dP$$

CALL THIS h $\quad P(A|\mathcal{C})$. THEN h HAS THESE PROPERTIES:

1) $h = P(A|\mathcal{C})$ IS $\mathcal{C}$-MEAS.

(*) ← 2) $\forall B \in \mathcal{C}, \quad \int_B h \, dP = \int_B P(A|\mathcal{C})(\omega) \, dP(\omega)$

$$= P(A \cap B) = \nu(B)$$

3) UNIQUE a.s.-P

IF $X = 1_A$ THESE ARE THE PROPERTIES USED IN DEFINING CONDITIONAL EXPECTATION!

- AGAIN, THIS ARGUMENT GENERALIZES TO R.V. X THAT ARE <u>NON-NEGATIVE</u>. DEFINE A NEW MEASURE ν ON $\mathcal{C}$ GIVEN BY $\nu(B) = \int_B X(\omega) \, dP(\omega), \quad B \in \mathcal{C}$.

ν IS FINITE. USE RN THEOREM TO

CONCLUDE THAT $\exists\, h$ $\mathcal{C}$-MEAS S.T.

$$\nu(B) = \underbrace{\int_B X \, dP}_{\mathcal{F}\text{-MEAS.}} = \underbrace{\int_B h \, dP}_{\mathcal{C}\text{-MEAS}} \qquad (*) \text{ PROPERTY}$$

CALL h A VERSION OF $\mathbb{E}[X \mid \mathcal{C}]$.

- GENERALIZE TO INTEGRABLE X.

<u>EXAMPLE</u> LET X AND Y BE TWO RV'S WITH JOINT pdf (DENSITY WRT LEBESGUE MEASURE) $f_{X,Y}$

FROM EARLIER PROB. CLASSES, THE CONDITIONAL DENSITY OF $X \mid Y$ IS

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{IF } f_Y(y) > 0$$

WHERE $f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y) \, dx$.

↳ LEBESGUE INTEGRATION

THEN, THE CONDITIONAL EXPECTATION OF X GIVEN $Y = y$ IS THE FUNCTION

$$g(y) = \int_{\mathbb{R}} x \, f_{X|Y}(x \mid y) \, dx$$

WRITTEN AS $\mathbb{E}[X \mid Y = y]$

LET $\mathcal{C} = \sigma(Y)$ SO THAT $g(Y)$ IS $\mathcal{C}$-MEAS

WE NEED TO SHOW THAT $\omega \to h(\omega) = g(Y(\omega))$ SATISFIES THE $(*)$ CONDITION AND THEREFORE IS A VERSION

OF $\mathbb{E}[X \mid \mathcal{C}]$:

(*) $$\int_C h\,dP = \int_C X\,dP \quad \text{FOR ALL } C \in \mathcal{C}.$$

PICK ANY $C \in \mathcal{C}$. THEN BECAUSE Y IS $\mathcal{C}$-MEAS, $\exists B \in \mathcal{B}^1$ S.T. $C = Y^{-1}(B)$. THEN

$$\boxed{1_C(\omega) = 1_B(Y(\omega))} \quad \text{FOR ALL } \omega. \text{ SO}$$

$$\int_C h\,dP = \int_\Omega 1_C(\omega)\,h(\omega)\,dP(\omega) = \int_\Omega 1_B(Y(\omega))\,h(\omega)\,dP(\omega)$$

$$= \int_{\mathbb{R}} 1_B(y)\,g(y)\,\underbrace{d\mu_Y(y)}_{f_Y(y)\,dy}$$

μ_Y PROB. DISTR. OF Y

$$= \int_{\mathbb{R}} 1_B(y)\left[\int_{\mathbb{R}} x\, f_{X|Y}(x|y)\,dx\right] f_Y(y)\,dy$$

$$= \int_{\mathbb{R}^2} 1_B(y)\,x\,f_{XY}(x,y)\,dx\,dy$$

$$= \mathbb{E}\left[1_B(Y)\,X\right]$$

$$= \int_\Omega 1_B(Y(\omega))\,X(\omega)\,dP(\omega)$$

$$= \int_\Omega 1_C(\omega)\,X(\omega)\,dP(\omega) = \int_C X\,dP$$

AND (*) IS SATISFIED

THIS CAN BE EXTENDED TO EVALUET

$\mathbb{E}[f(X) \mid Y]$ ARBITRARY (MEAS.) INTEGRABLE f.

SEE EXAMPLE 5 IN NOTES.

- THIS EXAMPLE SHOWS THAT THE CONDITIONAL EXPECTATION OF X GIVEN Y IS DEFINED AS THE CONDITIONAL EXPECTATION OF X GIVEN $\sigma(Y)$. IN THIS CASE WE WRITE $\mathbb{E}[X \mid Y]$ FOR $\mathbb{E}[X \mid \sigma(Y)]$. IT IS POSSIBLE TO SHOW THAT $\mathbb{E}[X \mid Y]$ IS A FUNCTION OF Y.

$\sigma(Y)/\mathcal{B}'$ MEAS

SOME REMARKS ABOUT $\mathbb{E}[X \mid \mathcal{C}]$:

1) IF X IS $\mathcal{C}$-MEAS THEN X IS ITSELF A VERSION OF $\mathbb{E}[X \mid \mathcal{C}]$ (THAT IS, WE CAN TAKE $\mathbb{E}[X \mid \mathcal{C}](\omega) = X(\omega)$)

2) IF $X = c$ a.s. THEN $\mathbb{E}[X \mid \mathcal{C}] = c$ a.s.

3) IF $\mathcal{C} = \{\emptyset, \Omega\}$, THEN $\mathbb{E}[X \mid \mathcal{C}] = \mathbb{E}[X]$

EXISTENCE OF COND. EXPECTATION USING L^2 PROJECTION

- IF $\mathbb{E}[X^2] < \infty$ YOU CAN SHOW THAT $\mathbb{E}[X]$ IS THE MINIMIZER OF

$$c \longmapsto \mathbb{E}\left[(X-c)^2\right]$$

- IF X AND Y ARE RV'S S.T. $\mathbb{E}[X^2] < \infty$. SUPPOSE WE WANT TO FIND A FUNCTION f S.T.

$$\mathbb{E}\left[\left(X - f(Y)\right)^2\right]$$

IS MINIMAL. THEN $f(Y) = \mathbb{E}[X|Y]$

$\downarrow$

= PROJECTION OF Y ONTO X "

- RECALL THAT $L^2(\Omega, \mathcal{F}, P)$ SPACES ARE HILBERT-SPACES WRT INNER PRODUCT $\mathbb{E}[X \cdot Y]$

<u>ORTHOGONAL PROJECTION FOR HILBERT SPACE</u>. IF V IS A HILBERT SPACE WITH INNER PRODUCT $\langle \cdot, \cdot \rangle$ AND NORM $\| \cdot \|$ [IN OUR CASE $\langle X, Y \rangle = \mathbb{E}[X \cdot Y]$ AND $\|X\| = \left(\mathbb{E}[X^2]\right)^{1/2}$]. LET V_0 BE A CLOSED SUB-SPACE OF V. FOR EACH $v \in V$ $\exists$ A UNIQUE $v_0 \in V_0$ CALLED THE ORTHOGONAL PROJECTION OF v ONTO V_0, SUCH THAT $v - v_0$ IS ORTHOGONAL TO ANY VECTOR IN V_0 [$\langle v - v_0, w \rangle = 0 \ \forall w \in V_0$] AND $\|v - v_0\| = \inf_{w \in V_0} \|v - w\|$.

<u>Thm 12</u> IF $\mathbb{E}[X]$ EXISTS THEN $\exists$ VERSION OF $\mathbb{E}[X \mid \mathcal{C}]$ FOR ANY σ-FIELD $\mathcal{C} \subseteq \mathcal{F}$.

Pf/ ASSUME $X \in L^2(\Omega, \mathcal{F}, P)$. THEN $L^2(\Omega, \mathcal{C}, P)$ IS A CLOSED LINEAR SUBSPACE. IF $X \in L^2(\Omega, \mathcal{F}, P)$ LET X_0 BE ITS UNIQUE PROJECTION ONTO $L^2(\Omega, \mathcal{C}, P)$.

$\to Y$ $\quad \to Y_0$

SO BY PROPERTIES OF ORTHOGONAL PROJECTIONS IN HILBERT SPACES:

$$\mathbb{E}[(X - X_0) \cdot Y] = 0 \quad \text{ALL } Y \in L^2(\Omega, \mathcal{C}, P)$$

TAKE $Y = 1_C$, $C \in \mathcal{C}$. THIS GIVES

$$\int_C X_0 \, dP = \int_C X \, dP \qquad (*) \text{ PROPERTY}$$

SO X_0 IS A VERSION OF $\mathbb{E}[X \mid \mathcal{C}]$.

IF X IS INTEGRABLE BUT NOT IN $L^2(\Omega, \mathcal{F}, P)$. ASSUME FIRST THAT $X \geq 0$. LET $X_n = \min\{X, n\}$.

SO $X_n \in L^2(\Omega, \mathcal{F}, P)$ ALL n. LET $X_{0,n}$ BE A VERSION OF $\mathbb{E}[X_n \mid \mathcal{C}]$ SO THAT

$$\square \quad \mathbb{E}[X_{0,n} 1_C] = \mathbb{E}[X_n 1_C] \quad \text{ALL } C \in \mathcal{C}$$

WE ALSO HAVE THAT $X_n \leq X_{n+1}$. THIS IMPLIES THAT

$$X_{0,n} \leq X_{0,n+1} \quad \text{a.e. } P$$

SO LET $X_0 = \lim_{n \to \infty} X_{n,0}$

THEN $\lim_{n\to\infty} \mathbb{E}\left[1_C X_{\sigma\wedge n}\right] = \mathbb{E}\left[1_C X_{\sigma}\right]$

|| BY $\triangle$ EQUATION

AND $\lim_{n\to\infty} \mathbb{E}\left[1_C X_n\right] = \mathbb{E}\left[1_C X\right]$

THEREFORE $\mathbb{E}\left[1_C X_{\sigma}\right] = \mathbb{E}\left[1_C X\right]$ (X) PROPERTY

- REPEAT FOR $X \leq 0$ AND FOR $X = X^+ - X^-$.

$\blacksquare$