36710 – 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 26: WED, DEC 2, 2020

- LAST TIME: MARTINGALES

<u>Def</u> $(\Omega, \mathcal{F}, P)$. LET $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \dots$ BE AN INCREASING SEQUENCE OF SUB-σ-FIELDS (SO $\mathcal{F}_n \subseteq \mathcal{F}$ FOR ALL n). THIS IS CALLED A <u>FILTRATION</u>. FOR EACH n, LET X_n BE A RV THAT IS $\mathcal{F}_n$ MEAS. THE SEQUENCE $\{X_n\}$ IS SAID TO BE <u>ADAPTED</u> TO THE FILTRATION $\{\mathcal{F}_n\}$. THE PAIRS $(\{X_n\}, \{\mathcal{F}_n\})$ DEFINES A MARTINGALE WHEN

1) $\{X_n\}$ IS ADAPTED TO $\{\mathcal{F}_n\}$
2) $\mathbb{E}[|X_n|] < \infty$ ALL n.
3) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ ALL n.

IT IS SAID TO BE A SUB-MARTINGALE (SUPER-MARTINGALE) IF 3) HOLDS WITH $\geq$ (RESP $\leq$).

WE SAW SOME EXAMPLES GAMBLING

EXAMPLE (RN-DERIVATES). $(\Omega, \mathcal{F}, P)$ $\{\mathcal{F}_n\}$ BE A FILTRATION.

LET ν BE A PROB. MEASURE ON $(\Omega, \mathcal{F}, P)$ SUCH THAT FOR EACH n, X_n IS THE RN OF ν RESTRICTED TO $\mathcal{F}_n$ WRT P (ALSO RESTRICTED TO $\mathcal{F}_n$).

$X_n = \frac{d\nu}{dP}$ ON $\mathcal{F}_n$. THEN $\{X_n\}$ IS ADAPTED TO $\{\mathcal{F}_n\}$.

AND INTEGRABLE!

IN ADDITION $\{X_n\}$ IS A MARTINGALE:

FOR ANY $A \in \mathcal{F}_n$

SINCE $A \in \mathcal{F}_{n+1}$

$$\nu(A) = \int_A X_n(\omega)\, dP(\omega) = \int_A X_{n+1}(\omega)\, dP(\omega)$$

SO $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$

APPLICATION OF THIS EXAMPLE: LIKELIHOOD RATIOS. $(\Omega, \mathcal{F}, P)$

LET $\{X_n\}$ BE A SEQUENCE OF RV'S AND LET $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

FOR EACH n, THE PROB. DISTRIBUTION OF $X_1, \dots, X_n$ HAS A STRICTLY POSITIVE DENSITY WITH RESPECT TO λ_n ON $(\mathbb{R}^n, \mathcal{B}^n)$.

DENOTED WITH p_n. LET Q BE ANOTHER PROBABILITY $(\Omega, \mathcal{F})$ SUCH THAT THE PROB. DISTR. ON $(\mathbb{R}^n, \mathcal{B}^n)$ GIVEN BY

$$B \in \mathcal{B}^n \longmapsto Q\left((X_1, \dots, X_n)^{-1}(B) \right)$$

HAS A DENSITY q_n WRT λ_n.

DEFINE

$$Y_n = \frac{q_n(X_1, \dots, X_n)}{p_n(X_1, \dots, X_n)}$$

THIS IS THE LIKELIHOOD RATIO. THIS IS $\frac{dQ}{dP}$ RESTRICTED TO $\mathcal{F}_n$

↳ SO $\{Y_n\}, \{\mathcal{F}_n\}$ IS A MARTINGALE!

TO SEE THIS, NOTICE THAT FOR ANY $B \in \mathcal{B}^n$

$$\int_{\{\omega : (X_1(\omega), \dots, X_n(\omega)) \in B\}} Y_n(\omega)\, dP(\omega) = \int_B \frac{q_n(x_1, \dots, x_n)}{\cancel{p_n(x_1 \dots x_n)}} \cancel{p_n(x_1 \dots, x_n)}\, \underbrace{dx_1 \dots dx_n}_{d\lambda^n(x_1 \dots x_n)}$$

$$= \int_B q_n(x_1 \dots x_n)\, dx_1 \dots dx_n$$

$$= \int_{\{\omega : (X_1(\omega), \dots, X_n(\omega)) \in B\}} dQ(\omega)$$

ALSO EQUAL $(X_1, \dots, X_n)^{-1}(B)$

SINCE $\mathcal{F}_n$ IS GENERATED BY $(X_1, .., X_n)$, EACH SET A OF $\mathcal{F}_n$ IS OF THE FORM $(X_1, \dots X_n)^{-1}(B)$ FOR SOME $B \in \mathcal{B}^n$. SO $Y_n = \frac{dQ}{dP}$, AS CLAIMED.

STOPPING TIMES

<u>Def</u> (STOPPING TIME) $(\Omega, \mathcal{F}, P)$ AND A FILTRATION $\{\mathcal{F}_n\}$ ON IT.

A POSITIVE, INTEGER VALUE R.V. IS A CALLED A STOPPING TIME WRT $\{\mathcal{F}_n\}$ IF $\{\tau = n\} \in \mathcal{F}_n$ FOR ALL n.

(SO $\{\tau \leq n\} \in \mathcal{F}_n$ AND $\{\tau > n\} \in \mathcal{F}_n$)

ASSOCIATED TO τ IS A SPECIAL σ-FIELD

$$\boxed{\mathcal{F}_\tau = \{ A \in \mathcal{F} : A \cap \{\tau \leq k\} \in \mathcal{F}_k \text{ FOR ALL } k \}}$$

↳ σ-FIELD "GENERATED BY τ"

<u>REMARK</u> τ CAN BE INFINITY

IF $\{X_n\}$ IS ADAPTED TO $\{\mathcal{F}_n\}$ AND $\tau < \infty$ a.s. THEN X_τ IS THE RV DEFINED AS $\omega \longmapsto \underline{X_{\tau(\omega)}(\omega)}$

IF $\tau = \infty$ THE DEFINE X_∞ AS SOME ARBITRARY RV.

<u>EXAMPLE</u> LET $\tau = n_0$ τ IS A STOPPING TIME

$\{\tau = n\}$ IS EITHER ϕ OR Ω, SO $\{\tau = n\} \in \mathcal{F}_n$ ALL n

ALSO, FOR ANY $A \in \mathcal{F}$

$$A \cap \underline{\{\tau \leq n\}} = \begin{cases} A & \text{IF } n_0 \leq n \\ \phi & \text{OTHERWISE} \end{cases}$$

SO $A \cap \{\tau \leq n\} \in \mathcal{F}_n$ IIF $A \in \mathcal{F}_{n_0} \iff \mathcal{F}_\tau = \mathcal{F}_{n_0}$

<u>EXAMPLE</u> (FIRST PASSAGE) $\{X_n\}$ ADAPTED TO FILTRATION $\{\mathcal{F}_n\}$

FOR A GIVEN $B \in \mathcal{B}^1$, LET

$$\tau = \inf\{n : X_n \in B\} \qquad (\inf \phi = \infty)$$

THIS IS A STOPPING TIME:

$$\{\tau = n\} = \underbrace{\{X_n \in B\}}_{\in \mathcal{F}_n} \bigcap_{i<n} \underbrace{\{X_i \in B^c\}}_{\in \mathcal{F}_i \subseteq \mathcal{F}_n} \in \mathcal{F}_n$$

NEXT, WE WANT TO SHOW THAT τ AND X_τ ARE MEAS. WRT $\mathcal{F}_\tau$. WE NEED TO SHOW THAT FOR ALL $B \in \mathcal{B}^1$

$X_\tau^{-1}(B) \in \mathcal{F}$ AND $\{\tau \le k\} \cap X_\tau^{-1}(B) \in \mathcal{F}_k$ ALL k.

BUT

$$X_\tau^{-1}(B) = \left(\bigcup_{k=1}^{\infty} \{\tau = k\} \cap \{X_k^{-1}(B)\}\right) \cup$$

$$\{\tau = \infty\} \cap \{X_\infty^{-1}(B)\} \in \mathcal{F}$$

NEXT, FOR ~~EACH~~ k

$$\{\tau \le k\} \cap \{X_\tau^{-1}(B)\} = \bigcup_{i=1}^{k} \Big(\underbrace{\{\tau = i\} \cap \{X_i^{-1}(B)\}}_{\in \mathcal{F}_i \subseteq \mathcal{F}_k}\Big)$$

$$\in \mathcal{F}_k$$

ALSO EASY TO SEE THAT IF $\tau_1 \le \tau_2$ (BOTH STOPPING TIMES WRT $\{\mathcal{F}_n\}$)

THEN

$$\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$$

REMARK ASSUME $\{X_n\}$ IS A MARTINGALE ASSOCIATED TO A RANDOM WALK: $Y_1, Y_2, \ldots$ IID MEAN 0 AND

$X_n = \sum_{i=1}^{n} Y_i$. LET $\mathbb{P}(Y_i = 1) = 1/2 = \mathbb{P}(Y_i = -1)$

LET $\tau = \inf\{n : X_n = 5\}$. (THEN $\tau < \infty$ a.e.)

SO $\mathbb{E}[X_\tau] = 5$ BUT $\mathbb{E}[X_n] = 0$ ALL n.

<u>EXAMPLE</u> (GAMBLER'S RUIN) RECALL THE GAMBLING SETTING FROM LAST TIME. Z_n: WEALTH ACCUMULATED THROUGH A SEQUENCE OF n FAIR GAMES. LET $\tau = \inf\{n : Z_n \geq x + Z_0\}$ (ORIGINAL WEALTH) SOME ARBITRARY $x > 0$. 2 ISSUES: DEPENDING ON THE GAME IT MAY BE THE CASE THAT $\tau = \infty$ (TRUE WHEN GAME IS NOT FAIR). BUT EVEN IF $\tau < \infty$ WE MAY NEED AN UNLIMITED AMOUNT OF RESOURCES TO BE ABLE TO SURVIVE UNTIL τ. FOR EXAMPLE IF THE GAME IS FAIR AND THE OUTCOME OF EACH ROUND IS $+1$ OR -1, THEN WE GET GAMBLER'S RUIN PROBLEM. THE PROB. THAT $Z_n = x$ BEFORE $Z_n = -k$ IS $\frac{k}{k+x}$

$\to 1$ WHEN $k \to \infty$.

ONCE WE HAVE A MARTINGALE AND A STOPPING TIME, WE CAN "COMPOSE" THEM. FOR EXAMPLE, IF $\{X_n\}$ IS A MARTINGALE AND τ IS STOPPING TIME WRT SOME FILTRATION

THEN $X_n^* = X_{\min\{\tau, n\}} = \begin{cases} X_n & \text{IF } n \leq \tau \\ X_\tau & \text{OTHERWISE} \end{cases}$

THEN $\{X_n^*\}$ IS ALSO A MARTINGALE!!

OPTIONAL STOPPING

LET $(\{X_n\}, \{\mathcal{F}_n\})$ BE A MARTINGALE. LET $\{\tau_k\}$ BE A SEQUENCE OF STOPPING TIMES S.T. $\tau_k \leq \tau_{k+1}$ ALL k.

THEN $$\left(\{X_{\tau_k}\}, \{\mathcal{F}_{\tau_k}\}\right)$$ IS A MARTINGALE

<u>Thm</u> (OPTIONAL STOPPING THEOREM) LET $(\{X_n\}, \{\mathcal{F}_n\})$ BE A MARTINGALE AND SUPPOSE THAT $\{\tau_k\}$ IS AN INCREASING SEQUENCE OF STOPPING TIMES ST. $\tau_k \leq M_k$ a.s. FOR SOME SEQUENCE OF NUMBERS $\{M_k\}$. THEN $(\{X_{\tau_k}\}, \{\mathcal{F}_{\tau_k}\})$ IS A MARTINGALE.

<u>REMARKS</u> 1) THE ASSUMPTION $\tau_k \leq M_k$ a.s. ALL k CAN BE REPLACED BY OTHER CONDITIONS:

i) $\tau_k < \infty$ a.s.

ii) $\mathbb{E}\left[|X_{\tau_k}|\right] < \infty$

iii) $\liminf\limits_{m \to \infty} \mathbb{E}\left[|X_m| \, 1\{\tau_k > m\}\right] \to 0$ ALL k.

↳ "UNIFORM INTEGRABILITY"

2) SAME RESULT HOLDS FOR SUPER AND SUB-MARTINGALES