36710 – 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 6 : MON, SEP 21, 2020

LAST TIME: MEASURABLE FUNCTIONS

Properties of meas. real-valued functions:

i) IF f IS MEAS., THEN SO IS $a \cdot f$ ANY $a \in \mathbb{R}$. → CHECK THAT $\{\omega : f(\omega) < \frac{c}{a}\}$ IS MEAS. $c \in \mathbb{R}$

ii) IF $f : \Omega \to S$ AND $g : S \to T$ ARE MEAS., SO IS $g(f)$ OR $g \circ f : \Omega \to T$ [BECAUSE $[g(f)]^{-1}(B) = g^{-1}(f^{-1}(B))$] ← APPLIES TO GENERAL FUNCTIONS

iii) IF f AND g ARE REAL VALUED FUNCTIONS, SO IS

$$f+g, \quad \max\{f, g\}, \quad \sqrt{|f-g|}, \quad \ldots$$

COMPOSITION OF $h(x,y) = x+y$ AND $\begin{bmatrix} f \\ g \end{bmatrix}$ TAKING VALUES IN $\mathbb{R}^2$.

ASIDE: IF $f : \Omega \to \overline{\mathbb{R}}$ (EXTENDED REAL LINE) WE CAN CHECK MEASURABILITY IN THE SAME WAY. IN PARTICULAR

$$f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} \{\omega : f(\omega) > n\}$$

OTHER EXAMPLES OF MEAS. FUNCTIONS:

Thm LET $f_n : \Omega \to \mathbb{R}$ BE MEAS. FOR ALL n. THEN, THE FOLLOWING ARE ALSO MEAS.:

i) $\limsup_n f_n$ AND $\liminf_n f_n$

EXERCISE

ii) $\lim_n f_n$ IF THE LIMIT EXISTS [THE SET $\{\omega : \lim_n f_n(\omega) \text{ EXISTS}\}$ IS MEAS]

Pf/ i) FOR $\omega \longmapsto \limsup_n f_n(\omega) = \inf_n \sup_{k \geq n} f_k(\omega)$, IT IS ENOUGH TO SHOW THAT

$\sup_n f_n$ IS MEAS. . BUT THIS IS TRUE BECAUSE, FOR ANY $c \in \mathbb{R}$,

$$\{\omega : \sup_n f_n(\omega) \leq c\} = \bigcap_n \{\omega : \underbrace{f_n(\omega) \leq c}_{\text{MEAS}}\}$$

THE PROOF FOR $\omega \longmapsto \liminf_n f_n(\omega) = \sup_n \inf_{k \geq n} f_k(\omega)$

ii) ASSUME $f_n(\omega) \to f(\omega)$ ALL $\omega \Rightarrow f$ IS MEAS. IT IS ENOUGH TO SHOW

THAT THE FOLLOWING SET IS MEAS. (FOR ALL $c \in \mathbb{R}$):

$$\{\omega : f(\omega) > c\} = \{\omega : \lim_n f_n(\omega) > c\} = \bigcup_{r=1}^{\infty} \{\omega : f_n(\omega) > c + \tfrac{1}{r} \text{ EVENTUALLY}\}$$

$$= \bigcup_{r=1}^{\infty} \liminf_n \{\omega : f_n(\omega) > c + \tfrac{1}{r}\}$$

$$= \bigcup_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{\omega : \underbrace{f_k(\omega) > c + \tfrac{1}{r}}_{\text{MEAS}}\}$$

■

■ RANDOM VARIABLES

LET $(\Omega, \mathcal{F}, P)$ BE A PROBABILITY SPACE. A RANDOM VARIABLE X IS

A MEASURABLE FUNCTION FROM Ω INTO $(\mathbb{R}^k, \underbrace{\mathcal{B}^k}_{\text{BOREL } \sigma\text{-FIELD}})$.

↳ IT IS POSSIBLE TO ALLOW X TO BE AN EXTENDED REAL-VALUED FUNCTION.

EXAMPLE: $\Omega = [0,1]$, $\mathcal{F}$ BOREL σ-FIELD ON IT AND P IS THE LEBESGUE MEASURE ON $[0,1]$

$$X(\omega) = \lfloor 2\omega \rfloor$$

$$Z(\omega) = \omega$$

Def (INDUCED MEASURE) LET $(\Omega, \mathcal{F}, \overset{\text{MEASURE}}{\mu})$ BE A MEASURE SPACE AND LET f BE A MEAS. FUNCTION FROM THIS SPACE INTO THE MEASURABLE SPACE $(S, \mathcal{A})$.

THEN, f INDUCES A MEASURE ON $(S, \mathcal{A})$ DEFINED BY:

$$\begin{aligned} \nu(A) &= \mu\left(f^{-1}(A)\right), \qquad \forall A \in \mathcal{A} \\ &= \mu\left(\{\omega : f(\omega) \in A\}\right) \end{aligned}$$

ν IS CALLED THE <u>INDUCED MEASURE</u>

Def (PROBABILITY DISTRIBUTION) LET X BE A RANDOM VARIABLE, DEFINED ON $(\Omega, \mathcal{F}, P)$ AND TAKING VALUE IN $(S, \mathcal{A})$. THE $\overset{\text{PROB.}}{\text{DISTRIBUTION}}$ OF X IS THE PROBABILITY MEASURE ON $(S, \mathcal{A})$ INDUCED BY X.

REMARK: WE WANT TO WRITE SOMETHING THAT X BELONGS TO A SET A.

$\Pr(X \in A)$, $A \in \mathcal{B}$. FOR THIS TO MAKE SENSE, WE NEED TO (i) HAVE AN ABSTRACT PROB. SPACE $(\Omega, \mathcal{F}, P)$. (ii) MAKE SURE X IS MEAS. THEN,

$$\Pr(X \in A) = P\left(\{\omega : X(\omega) \in A\}\right) = \underbrace{P\left(X^{-1}(A)\right)}_{\text{THIS IS THE DISTRIBUTION OF } X!}$$

LET μ_X BE THE DISTR. OF X.

$$\underline{\mu_X}(A) = \Pr(X \in A) = P\left(\{\omega : X(\omega) \in A\}\right)$$

SO BACK TO THE EXAMPLE: $X \sim \text{Bernoulli}(1/2)$ BECAUSE $P(\{\omega : X(\omega) = 1\}) = 1/2$

$Z \sim \text{Uniform}[0,1]$

BECAUSE, FOR ALL $c \in [0,1]$

$$P(\{\omega : Z(\omega) \le c\}) = c$$

- IN FACT, IN MOST CASES, WE CAN TAKE $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}$ ON $[0,1]$, $\mu =$ LEBESGUE MEASURE ON $[0,1]$

SUPPOSE WE WANT TO "GENERATE" A R.V. X THAT HAS A cdf F_X

TAKE $X(\omega) = F_X^{-1}(\omega) = \inf\{x : F_X(x) \ge \omega\}$. THEN, FOR ANY $c \in \mathbb{R}$,

$$\begin{aligned} Pr(X \le c) &= \lambda(\{\omega : X(\omega) \le c\}) = \lambda(\{\omega \in [0,1],\ F_X^{-1}(\omega) \le c\}) \\ &= \lambda(\{\omega : \omega \le F_X(c)\}) = \lambda([0, F_X(c)]) \\ &= F_X(c). \end{aligned}$$

INTEGRATION.

<u>Def</u> (SIMPLE FUNCTION) A SIMPLE FUNCTION IS A MEAS. FUNCTION TAKING ON FINITELY MANY VALUES. WE CAN REPRESENT A SIMPLE FUNCTION CANONICALLY AS:

$$f(\omega) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(\omega)$$

$a_1, \dots, a_n$ ARE REALS

$A_1, \dots, A_n$ ARE DISJOINT ELEMENTS OF $\mathcal{F}$

$A_i = \{\omega : f(\omega) = a_i\}$

<u>Lemma</u> LET f BE A MEAS. (POSSIBLY EXTENDED) REAL-VALUED FUNCTION, THAT IS <u>NON-NEGATIVE</u>. THEN, THERE EXISTS A SEQUENCE $\{f_n\}_n$

OF NON-NEGATIVE SIMPLE FUNCTIONS S.T. $f_n(\omega) \leq f(\omega)$ ALL n AND ω

AND $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ ALL ω $[f_n \uparrow f]$

Pf/ LET $$f_n(\omega) = \begin{cases} \frac{k-1}{2^n} & \text{IF } \frac{k-1}{2^n} \leq f(\omega) < \frac{k}{2^n} \quad 1 \leq k \leq n2^n \\ n & \text{OTHERWISE} \end{cases}$$

$$f_n = \frac{1}{2^n} \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{A_k}(\omega) + n\, 1_{A_\infty}(\omega)$$

$$A_k = f^{-1}\left(\left[\frac{k-1}{2^n}, \frac{n}{2^n}\right)\right) \qquad A_\infty = f^{-1}([n, \infty)).$$

$\blacksquare$

- IF f IS A MEAS. REAL VALUED FUNCTION, THEN WE EXPRESS IT AS:

$$f(\omega) = f^+(\omega) - f^-(\omega)$$

WHERE

$$f^+(\omega) = \max\{f(\omega), 0\}, \quad f^{-}(\omega) = -\min\{f(\omega), 0\}$$

NOW, f^+ AND f^- ARE NON-NEGATIVE AND CAN BE APPROXIMATED MONOTONICALLY USING SIMPLE FUNCTIONS.

↳ CANONICAL REPRESENTATION

Def (INTEGRAL OF SIMPLE FUNCTIONS) LET $f(\cdot) = \sum_{i=1}^{n} a_i 1_{A_i}(\cdot)$ BE A SIMPLE FUNCTION FROM $(\Omega, \mathcal{F}, \mu)$ INTO $(\mathbb{R}, \mathcal{B})$. THE INTEGRAL OF f WITH RESPECT TO μ IS DEFINED TO BE

$$\sum_{i=1}^{n} a_i \, \mu(A_i)$$

WRITTEN AS

$$\int f \, d\mu, \quad \int_\Omega f(\omega)\, d\mu(\omega) \text{ OR } \int_\Omega f(\omega)\, \mu(d\omega)$$

IT IS POSSIBLE THAT $\int f \, d\mu = +\infty$ OR $-\infty$

HOWEVER, IT IS POSSIBLE THAT IN OUR DEFINITION OF $\int f \, d\mu$ WE MAY END UP WITH $-\infty + \infty$, WHICH IS UNDEFINED.

TO AVOID THIS, LET $f = f^+ - f^-$. WE SAY THAT f IS INTEGRABLE IF $\int f^+ \, d\mu < \infty$ AND $\int f^- \, d\mu < \infty$. OR

$\int f \, d\mu$ IS INFINITY IF ONLY ONE OF f^+ OR f^- HAVE INFINITE INTEGRAL.

$\downarrow$

f IS INTEGRABLE IF $|f| = f^+ + f^-$ IS INTEGRABLE

- CONVENTION FOR HANDLING ∞:

$$0 \times \infty = 0$$
$$c \times \infty = \operatorname{sign}(c) \times \infty$$
$$c \pm \infty = \pm\infty$$
$$\infty - \infty \quad \text{UNDEFINED}$$