36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 3: WED, SEP 9, 2020

<u>SUPPORT OF A MEASURE</u>: $(\Omega, \mathcal{B}, \mu)$ $\mu(\Omega) = a < \infty$

$\mathcal{B}$: BOREL σ-FIELD

SUPPORT OF μ IS $\text{Supp}(\mu) = \bigcap \{ C \in \mathcal{B},\ C \text{ CLOSED AND } \mu(C) = a \}$

SMALLEST CLOSED SET OF FULL μ-MEASURE

ALTERNATIVELY,

$\text{supp}(\mu) = \{ \omega \in \Omega : \mu(N_\omega) > 0 \text{ FOR ALL OPEN NEIGHBORHOODS OF } \omega \}$

IT IS A CLOSED SET!
EXERCISE

<u>EXAMPLE</u> 1) $X \sim \text{Uniform}$ ON $(0,1)$. THEN THE SUPPORT OF THE DISTRIBUTION OF X IS $[0,1]$

2) $X \sim N(\mu, \sigma^2)$, THE CORRESPONDING SUPPORT IS $\mathbb{R}$

CONTINUITY PROPERTIES OF MEASURES

$(\Omega, \mathcal{F}, \mu)$ MEASURE SPACE

UNIVERSE SET (Ω), σ-FIELD ($\mathcal{F}$), MEASURE (μ)

RECALL THAT A SEQUENCE OF (MEASURABLE) SETS $\{A_n\}$ IS MONOTONE IF

i) $A_n \subseteq A_{n+1}$ INCREASING

ii) $A_n \supseteq A_{n+1}$ DECREASING

THEN $A_\infty = \lim_n A_n$
- i) $\bigcup_n A_n$ IF INCREASING
- ii) $\bigcap_n A_n$ IF DECREASING

Lemma IF $\{A_n\}$ IS A MONOTONE SEQUENCE OF SETS, THEN

$$\mu(A_\infty) = \mu(\lim_n A_n) = \lim_n \mu(A_n)$$

IF EITHER OF THE FOLLOWING HOLDS:

i) $\{A_n\}$ IS INCREASING

ii) $\{A_n\}$ IS DECREASING AND $\mu(A_k) < \infty$ FOR SOME k

Pf/ CASE i): SET $B_1 = A_1$ AND $B_n = A_n \cap A_{n-1}^c = A_n \setminus A_{n-1}$

THEN

$$A_n = \biguplus_{k=1}^{n} B_k \quad \text{ALL } n \text{ (INCLUDING } \infty\text{)}.$$

SO $\mu(A_n) = \sum_{k=1}^{n} \mu(B_k)$ BY COUNTABLE ADDITIVITY.

NEXT

$$\mu(A_\infty) = \mu\left(\bigcup_n A_n\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} \mu(B_k)}_{\mu(A_n)}$$

$$= \lim_{n\to\infty} \mu(A_n).$$

AS FOR CASE $ii)$: WLOG ASSUME $\mu(A_1) < \infty$

SINCE $\{A_n\}$ IS DECREASING, FOR EACH $n \geq 2$,

$\{A_1 \setminus A_n\}$ IS INCREASING AND $A_1 \setminus A_n \uparrow A_1 \setminus A_\infty$

THEREFORE, BY PART $i)$,

$$\mu(A_1 \setminus A_n) \uparrow \mu(A_1 \setminus A_\infty) \quad \text{AS } n \to \infty$$

$$\cancel{\mu(A_1)} - \mu(A_n) \uparrow \cancel{\mu(A_1)} - \mu(A_\infty)$$

IT IS OK BECAUSE $\mu(A_1) < \infty$ AND $\mu(A_n) \leq \mu(A_1)$

SO $\mu(A_n) \downarrow \mu(A_\infty)$ ∎

USING THIS RESULT, WE CAN PROVE THE FOLLOWING:

Thm LET $\{A_n\}$ BE A SEQUENCE OF (MEASURABLE!) SETS, THEN μ FINITE.

$i)$ $\mu(\liminf_n A_n) \leq \liminf_n \mu(A_n) \leq \limsup_n \mu(A_n) \leq \mu(\limsup_n A_n)$

$ii)$ IF $A_n \to A_\infty$ THEN

$$\lim_n \mu(A_n) = \mu(A_\infty).$$

Pf/ LET $B_n = \bigcap_{k=n}^{\infty} A_n$ AND $C_n = \bigcup_{k=1}^{n} A_n$. THEN

$B_n \uparrow \liminf A_n$ AND $C_n \downarrow \limsup A_n$

SO

$$\liminf_n \mu(A_n) \geq \liminf_n \mu(B_n) = \lim_n \mu(B_n) = \mu(\liminf_n A_n)$$

AND SIMILARLY

$$\limsup_n \mu(A_n) \leq \mu(\limsup_n A_n)$$ ∎

UNIQUENESS OF MEASURES AND CONSTRUCTION OF MEASURES

EXAMPLE SUPPOSE I TELL YOU THAT I HAVE A SET FUNCTION P DEFINED ON $(\mathbb{R}, \mathcal{B})$, SUCH THAT

$$P((-\infty, a]) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \qquad a \in \mathbb{R}$$

DOES THIS DEFINE A PROBABILITY MEASURE ON $(\mathbb{R}, \mathcal{B})$.

DEF (π-SYSTEM) A COLLECTION OF SUBSETS OF Ω IS A π-SYSTEM IF IT IS CLOSED WRT FINITE INTERSECTIONS (A_1 AND A_2 ARE IN THE COLLECTION, SO IS $A_1 \cap A_2$).

EXAMPLE THE COLLECTION $\mathcal{A} = \{(-\infty, a],\ a \in \mathbb{R}\} \cup \{\emptyset\}$

$$\mathcal{A} = \{(a, b],\ -\infty < a \leq b < \infty\} \cup \{\emptyset\}$$

$$\mathcal{A} = \{(-\infty, a] \times (-\infty, b],\ (a, b) \in \mathbb{R}^2\} \cup \{\emptyset\}$$

Thm (UNIQUENESS): LET μ_1 AND μ_2 BE MEASURES ON $(\Omega, \mathcal{F})$ AND $\mathcal{F} = \sigma(\mathcal{A})$, FOR A π-SYSTEM $\mathcal{A}$. IF μ_1 AND μ_2 ARE σ-FINITE AND THEY AGREE ON $\mathcal{A}$, THEN THEY AGREE ON $\mathcal{F}$ (SO $\mu_1 = \mu_2$).

Remark THIS IS AN APPLICATION OF THE π-λ THEOREM!

HOW DO WE CONSTRUCT A MEASURE? WE WILL CONSIDER ONLY THE CASE OF $(\mathbb{R}, \mathcal{B})$.

LET'S FIRST LOOK AT PROBABILITY MEASURES. LET

$$F: \mathbb{R} \to [0, 1] \quad \text{BE A cdf}$$

OR MORE PRECISELY, ASSUME THAT

i) $\lim_{x \to -\infty} F(x) = 0$ ii) $\lim_{x \to \infty} F(x) = 1$

iii) F IS NON-DECREASING iv) $\lim_{x \downarrow y} F(x)$ RIGHT CONTINUITY v) $\lim_{x \uparrow y} F(x)$ EXISTS

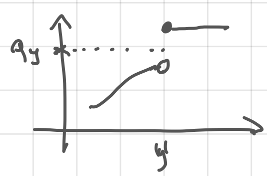
<u>Remark</u> WE ONLY NEED PROPERTIES iii), iv) AND v)

$\downarrow$ CADLAG FUNCTIONS

<u>ASIDE</u>: THE SET OF DISCONTINUITY POINTS OF A cdf IS COUNTABLE

Pf/ LET y BE A POINT OF DISCONTINUITY OF F.

$$\underset{\lim_{x \uparrow y}}{F(y^-)} < F(y^+)$$
$$\lim_{x \downarrow y} F(x) = F(y)$$



$\exists$ RATIONAL NUMBER q_y S.T. $F(y-) < q_y < F(y^+)$

IN THIS WAY WE HAVE ESTABLISHED A ONE-TO-ONE CORRESPONDENCE BETWEEN THE SET OF DISCONTINUITIES OF F AND A SUBSET OF $\mathbb{Q}$

BECAUSE $\mathbb{Q}$ IS COUNTABLE, WE ARE DONE ∎

BACK TO THE CONSTRUCTION OF PROBABILITY. LET $\mathcal{U}$ CONSISTS OF FINITE DISJOINT UNIONS OF SETS OF THE FORM $\begin{cases} (a, b] & -\infty \le a < b < \infty \\ (b, \infty) \\ \emptyset \end{cases}$

NEXT, LET $\mu(A) = \sum_{k=1}^{n} F(b_k) - F(a_k)$ WHEN $A = \biguplus_{k=1}^{n} [a_k, b_k] \in \mathcal{U}$

$\downarrow$

THIS IS A FINITELY ADDITIVE SET FUNCTION ON FIELD $\mathcal{U}$

IN Lemma 21 IN NOTES, IT IS SHOWN IT IS ALSO COUNTABLY ADDITIVE ON $\mathcal{U} \Rightarrow$ THIS DEFINE A PROBABILITY MEASURE ON $\mathcal{U}$.

<u>Thm</u> (CARATHEODORY EXTENSION THEOREM) LET μ BE A σ-FINITE MEASURE ON A FIELD $\mathcal{C}$ OF SUBSETS OF Ω. THEN μ HAS A UNIQUE EXTENSION TO $\sigma(\mathcal{C})$.