36710 – 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 8 : MON, SEP 28, 2020

NO SYNCHRONOUS CLASS ON WED, SEP 30

I WILL RECORD AND POST LECTURE THIS EVENING

LAST TIME:

<u>Thm (MONOTONE CONVERGENCE THEOREM)</u> (MCT): $\{f_n\}$ SEQUENCE OF NON-NEG., MEAS. FUNCTIONS. LET f BE A MEAS. FUNCTION S.T. $f_n \le f$ ALL n AND $f_n \to f$ AS $n \to \infty$ a.e. $[\mu]$. THEN

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu = \int \lim f_n \, d\mu$$

POSSIBLY INFINITY

- USING MCT WE CAN ESTABLISH LINEARITY OF THE INTEGRAL:

<u>Thm</u> IF $\int f \, d\mu$ AND $\int g \, d\mu$ ARE DEFINED AND NOT BOTH INFINITE AND OF OPPOSITE SIGNS, THEN

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$$

Pf/ PROOF IS AN EXAMPLE OF STANDARD MACHINERY. IF f AND g ARE ≥ 0 a.e. $[\mu]$, THEN WE HAVE SEEN THAT $\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

LET f AND g BE MEAS., NON-NEG. FUNCTIONS. THEN THERE EXIST SEQUENCES $\{f_n\}$ AND $\{g_n\}$ OF NON-NEG. SIMPLE FUNCTIONS S.T.

$f_n \uparrow f$ AND $g_n \uparrow g$ a.e. $[\mu]$. THEN, $(f_n + g_n) \uparrow (f+g)$

AND BY MCT

LINEARITY FOR SIMPLE FUNCTIONS

$$\int (f+g) \, d\mu = \lim_n \int (f_n + g_n) \, d\mu = \lim_n \left(\int f_n \, d\mu + \int g_n \, d\mu \right)$$

$$= \int f \, d\mu + \int g \, d\mu$$

THIS COULD BE INFINITY

IN THE LAST STEP, WE NEED TO VERIFY THAT THIS PROPERTY HOLDS FOR

$(f+g) = (f+g)^+ - (f+g)^-$ WHERE f AND g ARE GENERAL.

USE THE IDENTITY

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

BOTH SIDES CONSIST OF NON-NEG. FUNCTIONS, SO

$$\int (f+g)^+ d\mu + \int f^- d\mu + \int g^- d\mu \underset{\text{BY LINEARITY OF INTEGRALS OF NON-NEG FUNCTIONS}}{=} \int (f+g)^+ + f^- + g^- \, d\mu =$$

$$= \int (f+g)^- + f^+ + g^+ \, d\mu \underset{\text{LINEARITY AGAIN}}{=} \int (f+g)^- d\mu + \int f^+ d\mu + \int g^+ d\mu$$

REARRANGE:

$$\int f+g \, d\mu = \int (f+g)^+ d\mu - \int (f+g)^- d\mu = \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu$$

$$= \int f \, d\mu + \int g \, d\mu$$

∎

APPLICATIONS:

i) CHANGE OF VARIABLE. LET $(\Omega, \mathcal{F}, \mu)$ BE A MEASURE SPACE AND $(S, \mathcal{A})$ A MEASURABLE SPACE. LET $f: \Omega \to S$ BE MEAS. AND ν BE THE INDUCED MEASURE ON $(S, \mathcal{A})$ $[\nu(A) = \mu(f^{-1}(A))$, $A \in \mathcal{A}]$

LET $g: S \to \mathbb{R}$ THAT IS $\mathcal{A}/\mathcal{B}^1$ MEAS. THEN

$$(*) \qquad \int_S g \, d\nu = \int_\Omega \underbrace{g \circ f}_{\omega \in \Omega \mapsto g(f(\omega)) \in \mathbb{R}} \, d\mu$$

IF EITHER INTEGRAL EXISTS.

PF/ ANOTHER APPLICATION OF STANDARD MACHINERY. LET $g = 1_A$, $A \in \mathcal{A}$ THEN EQ. $(*)$ BECOMES $\nu(A) = \mu(f^{-1}(A))$. THEN WE CAN PROVE THAT $(*)$ HOLDS FOR NON-NEG SIMPLE FUNCTIONS. THEN PASS TO NON-NEG $g \geq 0$ USING A LIMIT ARGUMENT AND THE MCT. AND FINALLY EXTEND TO GENERAL $g = g^+ - g^-$. ∎

↓

LAW OF THE UNCONSCIOUS STATISTICIAN!

IF X IS A RV. AND f IS A MEAS. FUNCTION

$$\mathbb{E}[f(X)] = \int f(x) \, \underbrace{d\mu_X(x)}_{\text{PROB. DISTR. OF } X} = \int_\Omega f(X(\omega)) \, \underbrace{d\mu(\omega)}_{\text{PROB. MEAS.}}$$

2) DENSITY FUNCTIONS

Thm LET $(\Omega, \mathcal{F}, \mu)$ BE A MEASURE SPACE AND $f : \Omega \to \bar{\mathbb{R}}_{\geq 0}$ MEAS. THEN THE FUNCTION $A \in \mathcal{F} \mapsto \nu(A) = \int_A f \, d\mu$ IS ALSO A MEASURE.

$$= \int_\Omega 1_A f \, d\mu$$

Pf/ THE ONLY TRICKY POINT IN VERIFYING THAT ν IS A MEASURE IS TO ESTABLISH COUNTABLE ADDITIVITY:

LET $A_1, A_2, \ldots$ BE PAIRWISE DISJOINT SETS IN $\mathcal{F}$. WANT TO SHOW THAT

$$\int_{\uplus A_n} f \, d\mu = \nu\left(\biguplus_n A_n\right) = \sum_n \nu(A_n) = \sum_n \int_{A_n} f \, d\mu$$

LET $f_n = 1_{A_n} f$, SO THAT THE LAST IDENTITY BECOMES

$$\int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu$$

BUT THIS FOLLOWS FROM MCT: IF $\{f_n\}$ IS A SEQUENCE OF NON-NEG. FUNCTIONS THEN

$$\int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu$$

HINT: $\sum_{n=1}^{\infty} f_n = \lim_{k \to \infty} \underbrace{\sum_{n=1}^{k} f_n}_{g_k}$

- THE FUNCTION f IS CALLED THE DENSITY OF ν
- IF μ IS σ-FINITE, THE f IS UNIQUE a.e. $[\mu]$
- IF X IS A R.V. (SAY $X \sim N(0,1)$) THEN WE COMPUTE

$$\Pr(X \leq c) = \mu_X((-\infty, c]) = \int_{-\infty}^{c} \underbrace{\phi(x)}_{\text{pdf of } N(0,1)} \, \underbrace{d\lambda(x)}_{\text{LEBESGUE MEASURE}}$$

IF X IS A DISCRETE R.V. THEN

$$\Pr(X \leq c) = \mu_X((-\infty, c]) = \sum_{\substack{x \text{ IN SUPPORT OF } \mu_X \\ \text{S.T. } x \leq c}} \underbrace{p_X(x)}_{\Pr(X = x)}$$

$$= \int p_X \, d\underbrace{\nu}_{\text{COUNTING MEASURE}}$$

- IF X HAS BOTH A DISCRETE AND CONTINUOUS COMPONENT, WE TAKE INTEGRALS WRT

$$\underbrace{\lambda}_{\text{LEB. MEAS.}} + \underbrace{\nu}_{\text{COUNTING MEASURE}}$$

- DIRAC MEASURE: DEGENERATE RANDOM VARIABLE

IF $\Pr(X = c) = 1$ THEN $\mathbb{E}[f(X)] = f(c)$

$$= \int f \, d\delta_c$$

$$\delta_c(x) = \begin{cases} 1 & x = c \\ 0 & \text{OTH} \end{cases}$$

DOMINATED CONVERGENCE THEOREM

<u>Thm (DCT)</u> LET $\{f_n\}$ BE A SEQUENCE OF MEAS. FUNCTIONS AND LET f AND g (MEAS.) SUCH THAT $f_n \to f$ AND $|f_n| \leq g$ (ALL n) a.e. $[\mu]$ WHERE $\int g \, d\mu < \infty$. THEN

$$\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu = \int f \, d\mu$$

PROOF USES MCT AND FATOU'S LEMMA

<u>REMARK</u>: THE DOMINATED CONDITION IS NECESSARY:

$x \in [0,1]$ $\quad f_n(x) = n \, \mathbb{1}_{(0, 1/n]}(x)$ THEN $f_n(x) \to 0$ ALL x BUT

$$\int f_n \, d\lambda = 1 \neq \int 0 \, d\lambda = 0$$

RESTRICTION OF
LEB MEAS ON $[0,1]$

APPLICATION: CONSIDER THE SEQUENCE $\sum_{k=1}^{n} f_k$ SOME $\{f_k\}$

i) IF $f_n \geq 0$ ALL n THE, AS WE SAW, THE MCT

$$\int \sum_k f_k \, d\mu = \sum_k \int f_k \, d\mu$$

ii) IF $\sum_k f_k$ CONVERGES a.e. $[\mu]$ AND $\left| \sum_{k=1}^{n} f_k \right| \leq g$ FOR ALL n AND SOME INTEGRABLE g, THEN BY DCT

$$\int \sum_k f_k \, d\mu = \sum_k \int f_k \, d\mu$$

iii) SAME CONCLUSION IS TRUE IF $\sum_k \int |f_k| \, d\mu < \infty$

APPLICATION: INTERCHANGING DERIVATIVES AND INTEGRALS

SUPPOSE $f(\omega, t)$ IS A MEAS. AND INTEGRABLE FUNCTION OF FOR EACH $t \in [a,b]$. LET A BE THE SET OF ω S.T. $f(\omega,t)$ HAS A DERIVATIVE $f'(\omega,t)$ IN (a,b) AND $|f'(\omega,t)| \leq g(\omega)$ WHERE g IS INTEGRABLE. ASSUME $\mu(A^c)=0$.

THEN, LETTING $\phi(t) = \int_\Omega f(\omega,t) \, d\mu(\omega)$

$$\phi'(t) = \int_\Omega f'(\omega,t) \, d\mu(\omega).$$

Pf/ ASSUME $\omega \in A$. BY MEAN VALUE THEOREM

$$\frac{f(\omega, t+h) - f(\omega,t)}{h} = f'(\omega,s)$$

SOME s BETWEEN t AND $t+h$, h SMALL ENOUGH. AS $h \to 0$ THE LHS $\to f'(\omega, t)$ AND IS DOMINATE BY $g(\omega)$. g INTEGRABLE. SO

$$\frac{\phi(t+h) - \phi(t)}{h} = \int_{\Omega} \frac{f(\omega, t+h) - f(\omega, t)}{h} \, d\mu(\omega)$$

$$\to \int f'(\omega, t) \, d\mu \qquad \text{AS } h \to 0$$

$\blacksquare$