36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 12: MON, OCT 12, 2020

- LAST TIME: $(\Omega, \mathcal{F}, P)$ PROBABILITY SPACE. T ARBITRARY SET. FOR EACH $t \in T$ $\exists$ $(\mathcal{X}_t, \mathcal{F}_t)$. $\mathcal{X} = \prod_{t \in T} \mathcal{X}_t$. PRODUCT σ-FIELD $\bigotimes_{t \in T} \mathcal{F}_t$ GENERATED BY ONE-DIM. CYLINDERS, WHICH IS THE SMALLEST σ-FIELD S.T. ALL COORDINATE PROJECTIONS ARE MEAS.

STOCHASTIC PROCESS: $\{X_t, t \in T\}$ INDEXED COLLECTION OF RV'S ON $(\Omega, \mathcal{F}, P)$ S.T. X_t TAKES VALUES IN $\mathcal{X}_t$, ALL $t \in T$.

IT IS CONVENIENT TO THINK OF $\{X_t, t \in T\}$ AS A RANDOM FUNCTION

$$\omega \in \Omega \longmapsto X(\omega)$$

WHERE $X \in \mathcal{X}$, A FUNCTION ON T, DEFINED BY $f(t) = X_t(\omega)$

$X(\omega)$, FOR FIXED ω, IS CALLED A REALIZATION OR PATH OF THE PROCESS.

<u>Q</u>: HOW DO WE CONSTRUCT A PROB. DISTR. FOR THE PROCESS X?

<u>A</u>: KOLMOGOROV'S EXTENSION THEOREM

RECALL THE NOTION OF FINITE DIMENSIONAL PROJECTIONS OF PROB. MEASURES ON $(\mathcal{X}, \bigotimes_{t \in T} \mathcal{F}_t)$

LET $V \subset T$ OF FINITE CARDINALITY. LET $U \subset V$. IF P_V IS A PROB. DISTR. ON $\left(\mathcal{X}_V, \bigotimes_{t \in V} \mathcal{F}_t\right)$ THEN THE PROJECTION OF P_V ON

$\mathcal{X}_V \to \prod_{t \in V} \mathcal{X}_t$

$\left(\mathcal{X}_U, \mathcal{F}_U\right)$ IS $\pi_U(P_V)$ GIVEN BY

$\mathcal{F}_U \to \bigotimes_{t \in U} \mathcal{F}_t$

U - SUB-VECTOR OF x

$$B \in \mathcal{F}_U \mapsto \pi_U(P_V)(B) = P_V\left(\left\{x \in \mathcal{X}_V : x_U \in B\right\}\right)$$

SIMILARLY IF Q IS A PROB. DISTR. ON $\left(\mathcal{X}, \bigotimes_{t \in T} \mathcal{F}_t\right)$, THE PROJECTION OF Q ON $\left(\mathcal{X}_V, \mathcal{F}_V\right)$ IS

$$\pi_V(Q)(B) = Q\left(\left\{x \in \mathcal{X}, \ x_V \in B\right\}\right) \qquad B \in \mathcal{F}_V.$$

<u>Thm</u> (KOLMOGOROV EXTENSION). FOR EACH t, $\mathcal{X}_t = \mathbb{R}$ AND $\mathcal{F}_t = \mathcal{B}^1$

$\hookrightarrow$ $\mathcal{X} = \mathbb{R}^T = \{$SET OF ALL REAL-VALUED FUNCTIONS ON $T\}$

$\mathcal{B}^T = \bigotimes_{t \in T} \mathcal{B}^1$

ASSUME THAT FOR EACH FINITE NON-EMPTY SUBSET V OF T, THERE EXISTS A PROB. DISTR. ON $\left(\mathbb{R}^V, \mathcal{B}^V\right)$. ASSUME ALSO THAT THIS FAMILY OF DISTRIBUTIONS ARE <u>CONSISTENT</u>:

$\forall V$ AND $\forall U \subset V$ $\quad \pi_U(P_V) = P_U$

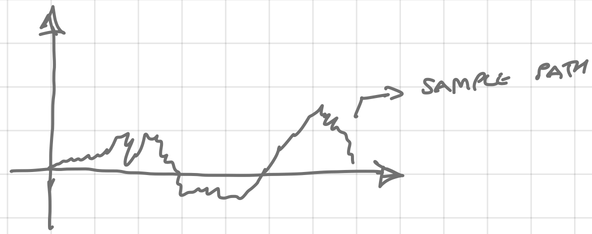
(U) NON-EMPTY

THEN, THERE EXISTS A UNIQUE PROB. DISTR. ON $\left(\mathbb{R}^T, \mathcal{B}^T\right)$ Q S.T. $\forall V \subset T$ FINITE

$$\pi_V(Q) = P_V.$$

EXAMPLE: BROWNIAN MOTION. $T = [0, \infty)$ $\mathcal{X}_t = \mathbb{R}$

$$\omega \longmapsto \{W_t, t \geq 0\}(\omega)$$



BROWNIAN MOTION IS A STOCHASTIC PROCESS OVER $\left(\mathbb{R}^{[0,\infty)}, \mathcal{B}^{[0,\infty)}\right)$

S.T.

1) $W_0 = 0$ WITH PROB 1.

2) IF $0 \leq t_0 < t_1 < \dots < t_k$ THEN

$W_{t_i} - W_{t_{i-1}}$ $i = 1, \dots, k$ ARE INDEPENDENT RV'S

INDEPENDENT INCREMENTS

ALL k

3) FOR ALL $0 \leq s < t$ $W_t - W_s \sim N(0, t-s)$

THIS IS ENOUGH TO PRESCRIBE A PROB. DISTR. FOR $\{W_t, t \geq 0\}$

Pf/ THE JOINT DISTRIBUTION OF W_s AND W_t ARE EASY TO OBTAIN $\forall$ $0 \leq s < t$. IN FACT $W_t = W_t - W_0 \sim N(0, t)$ AND

$$\mathbb{E}[W_s W_t] = \underbrace{\mathbb{E}[W_s(W_t - W_s)]}_{=0} + \underbrace{\mathbb{E}[W_s^2]}_{s} = \min\{s, t\}$$

BECAUSE $W_s \perp\!\!\!\perp W_t - W_s$ AND $\mathbb{E}[W_s] = 0$

SO TO FIND THE PROB. DISTR. OF, SAY, $W_{t_1}, \dots, W_{t_k}$ $(t_1 < t_2 < \dots < t_k)$

NOTICE THAT

$\left(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_k} - W_{t_{k-1}}\right)$ HAS A JOINT MULTIVARIATE GAUSSIAN DISTRIBUTION WITH MEAN 0 AND DIAGONAL COV. MATRIX WITH (i,i)th ENTRY GIVEN BY $(t_i - t_{i-1})$.

K-DIM

SO, BY CHANGE OF VARIABLES, $(W_{t_1}, \dots, W_{t_k})$ HAS LEBESGUE DENSITY IN $\mathbb{R}^k$ GIVEN BY

$$x = (x_1, \dots, x_k) \longmapsto \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi (t_i - t_{i-1})}} \exp\left\{ - \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right\}$$

$$x_0 = t_0 = 0$$

THIS IS THE SAME AS THE DISTRIBUTION OF $(S_1, S_2, \dots, S_k)$

WHERE FOR $i = 1, \dots, k$

$$S_i = \sum_{j=1}^{i} X_j \qquad X_j \sim \begin{cases} N(0, t_1) & j = 1 \\ N(0, t_j - t_{j-1}) & j \geq 2 \end{cases}$$

AND BECAUSE

$$X_i + X_{i+1} \sim N(0, t_{i+1} - t_{i-1})$$

THIS IS A CONSISTENST FAMILY, IN THE SENSE THAT THE MARGINAL DISTRIBUTION OF $(S_1, \dots, S_k)$ OVER ITS k COORDINATE IS

$$\downarrow \qquad (\underbrace{S_1, \dots, S_{k-1}, S_{k+1}, \dots S_k}_{k-1})$$

APPLY KOLMOGORV EXTENSION THEOREM.

- REMARK: THIS RESULT DOES NOT SAY ANYTHING ELSE ABOUT $\{W_t, t \geq 0\}$. IT DOES NOT SAY THAT THE SAMPLE PATHS ARE CONTINUOUS. IN FACT, THE CLASS OF CONTINUOUS REAL VALUED FUNCTIONS ON $[0, \infty)$ IS NOT MEAS. WRT $\mathcal{B}^T$. THERE EXISTS AN EXTENSION OF THIS CONSTRUCTION THAT WOULD GUARANTEE THAT $\{W_t, t \geq 0\}$ IS CONTINUOUS. [SEPARABILITY]

  ↳ REGULARITY INVOLVING T

  ↓ BUT NOWHERE DIFFERENTIABLE

L^p SPACES (5th SET OF LECTURE NOTES)

- LET $(\Omega, \mathcal{F}, \mu)$ BE A MEASURE SPACE.
- FOR $p \geq 1$ LET $\mathcal{L}^p = \{ f : \Omega \to \mathbb{R}$, MEAS. S.T. $\int |f|^p d\mu < \infty \}$
- ON $\mathcal{L}^p$ LET $\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}$. THEN

  $\|f\|_p \geq 0$, $\|af\|_p = |a| \cdot \|f\|_p$ AND $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

  WHENEVER $f, g \in \mathcal{L}^p$ AND $a \in \mathbb{R}$

- HOWEVER $\|\cdot\|_p$ IS NOT A NORM BECAUSE $\|f\|_p = 0$ DOES NOT IMPLY THAT $f = 0$. SO WE MODIFY $\mathcal{L}^p$ TO CREATE A NEW SPACE L^p CONSISTING OF EQUIVALENCE CLASSES OF ELEMENTS OF $\mathcal{L}^p$ S.T.

  f IS EQUIVALENT TO g

  $f \sim g$ WHEN $f = g$ a.e. $[\mu]$. DEFINE $\|f\|_p = \|[f]\|_p$

  WHERE $[f]$ IS THE EQUIVALENCE CLASS CONTAINING f.

  $\|\cdot\|_p$ IS CALLED L^p NORM OF f.

- $\|\cdot\|_p$ IS A NORM ON L^p.

- THE CASE OF $p < 1$ IS OFTEN NOT CONSIDERED, BECAUSE WHEN $p < 1$, $\|\cdot\|_p$ IS NOT A NORM (DOES NOT SATISFY TRIANGLE INEQUALITY).

  $a, b > 0$ $\quad (a+b)^p < a^p + b^p$ IF $p \in (0,1)$.

- THE CASE OF $p = \infty$:

ESSENTIAL SUPREMUM

$$\text{ess sup}(f) = \sup \{ t : \mu(\{\omega : |f(\omega)| \geq t\}) > 0 \}$$

$$= \inf \{ a : \mu(\{\omega : |f(\omega)| > a\}) = 0 \}$$

$$\bigcup_n \{ |f| \geq a + \tfrac{1}{n} \}$$

IF $\text{ess sup}(f) < \infty$ THEN f IS ESSENTIALLY BOUNDED

$$\|f\|_\infty = \text{ess sup}(f).$$

- IF $\Omega = \{0, 1, 2, \dots\}$, $\mathcal{F} = 2^\Omega$, μ IS THE COUNTING MEASURE

THE CORRESPONDING SPACE IS ℓ^p.

<u>HÖLDER'S INEQUALITY</u>: FOR EACH $p \in (1, \infty)$ LET q BE THE UNIQUE VALUE S.T. $\frac{1}{p} + \frac{1}{q} = 1$. IF $p = 1$, THEN $q = \infty$
$p = \infty$ $q = 1$

p, q ARE CONJUGATE INDEXES.

<u>Thm</u> IF $f \in L^p$ AND $g \in L^q$, p, q CONJUGATE, THEN

$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q$$

GENERALIZATION: $f_1, \dots f_k$ ARE S.T. $f_i \in L^{p_i}$ AND $\sum_{i=1}^{k} \frac{1}{p_i} = 1$ THEN

$$\left\| \prod_i |f_i| \right\|_1 \leq \prod_{i=1}^{k} \|f_i\|_{p_i}.$$