36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 4 : MON, SEP 14, 2020

Thm (CARATHEODORY EXTENSION THEOREM) LET μ BE A σ-FINITE MEASURE ON A FIELD $\underline{\mathcal{C}}$ OF SUBSETS OF Ω. THEN μ HAS A UNIQUE EXTENSION TO $\underline{\sigma(\mathcal{C})}$.

- $(\Omega, \mathcal{F})$. WE WOULD LIKE TO CONSTRUCT A MEASURE ON IT.

  THEN $\mu : \mathcal{F} \to \mathbb{R}_{\geq 0}$ S.T.

  - $\mu(\emptyset) = 0$
  - IF $\{A_n\}$ IS A SEQUENCE OF MUTUALLY DISJOINT MEASURABLE SETS,

    THEN $\mu\left(\biguplus_n A_n\right) = \sum_n \mu(A_n)$ (COUNTABLE ADDITIVITY)

- IF $\mathcal{U}$ IS A FIELD, A MEASURE μ ON $\mathcal{U}$ IS A FUNCTION $\mu : \mathcal{U} \to \mathbb{R}_{\geq 0}$ THAT SATISFIES THE ABOVE PROPERTIES, PROVIDED THAT $\biguplus_n A_n \in \mathcal{U}$

<u>EXAMPLE</u>

$\Omega = \mathbb{R}$ $\quad \mathcal{F} = \mathcal{B}$ (BOREL σ-FIELD) $\quad \mathcal{U}$ = FIELD OF INTERVALS OF THE FORM
$(a, b]$ $\quad -\infty \leq a < b < \infty$
(b, ∞)
$\emptyset$

- IF WE HAVE A cdf F, WE CAN DEFINE A PROB. MEASURE ON $\mathcal{V}$ BY SETTING

$$\mu((a,b]) = F(b) - F(a)$$

$\downarrow$

BY THE EXTENSION THEOREM, THIS PROB. MEASURE IS WELL-DEFINED ON $\mathcal{B}$.

- SO IF
$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

THEN THE CORRESPONDING MEASURE IS THE STANDARD NORMAL DISTRIBUTION!

- WE ARE NOT LOOKING AT PROOF!

LEBESGUE MEASURE ON $\mathbb{R}$

- MEASURE λ ON $(\mathbb{R}, \mathcal{B})$ S.T. $\lambda((a,b]) = \underbrace{b-a}_{\text{LENGTH OF INTERVAL}}$

WE DO NOT NEED F TO SATISFY $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

- LET $\underbrace{F(x)}_{\text{cadlag}} = x$. THEN, LET λ BE A σ-FINITE MEASURE ON $\mathcal{V}$ SATISFYING $\lambda((a,b]) = F(b) - F(a) = b - a$

- THEN, BY EXTENSION THEOREM, λ IS WELL-DEFINED AND σ-FINITE ON $(\mathbb{R}, \mathcal{B})$.

- THIS IS CALLED THE <u>LEBESGUE MEASURE</u>.

- OF COURSE, $\lambda(\{a\}) = 0$ FOR ANY $a \in \mathbb{R}$.

SO, $\lambda((a,b]) = \lambda((a,b)) = \lambda([a,b)) = \lambda([a,b])$

IN PARTICULAR $\lambda(\underbrace{\mathbb{Q}}_{\text{SET OF RATIONALS}}) = 0$

$\downarrow$

$$\lambda((a,b]) = \lambda((a,b] \cap \mathbb{Q}^c) \;!!$$

REMARK (COMPLETION OF MEASURE)

A MEASURE μ ON $(\Omega, \mathcal{F})$ IS COMPLETE WHEN $A \in \mathcal{F}$ $\mu(A) = 0$ IMPLIES THAT $\mu(B) = 0$ FOR ALL $B \subseteq A$. $B \in \mathcal{F}$

- IF μ IS COMPLETE AND $A_1 \Delta A_2 = B$ S.T. $\mu(B) = 0$, $A_1, A_2 \in \mathcal{F}$ THEN $\mu(A_1) = \mu(A_2)$.

- COMPLETION OF A SPACE. LET $(\Omega, \mathcal{F}, \mu)$ BE A MEASURE SPACE.

LET $\bar{\mathcal{F}} = \{ A \cup N,\ A \in \mathcal{F},\ N \subseteq B \text{ FOR SOME } B \in \mathcal{F} \text{ ST } \mu(B) = 0 \}$

$\downarrow$

THIS IS ALSO A σ-FIELD!

LET $\bar{\mu}(A \cup N) = \mu(A)$

THEN $(\Omega, \bar{\mathcal{F}}, \bar{\mu})$ IS A COMPLETE MEASURE SPACE. IT IS THE COMPLETION OF $(\Omega, \mathcal{F}, \mu)$.

- THE LEBESGUE SPACE IS THE COMPLETION OF THE LEBESGUE MEASURE OVER THE BOREL σ-FIELD.

MEASURES ON $(\mathbb{R}^k, \mathcal{B}^k)$ $\hookrightarrow$ BOREL σ-FIELD

NOT THE "RIGHT" DEFINITION OF MONOTONICITY IN $\mathbb{R}^k$

LET $F: \mathbb{R}^k \to \mathbb{R}$ BE SUCH THAT

1) F IS NON DECREASING: IF $x, y \in \mathbb{R}^k$ S.T. $x \leq y$ $\left(\begin{array}{l} x_n \leq y_n \\ \text{ALL} \\ n = 1, \dots, k \end{array} \right)$

THEN $F(x) \leq F(y)$

2) F IS RIGHT CONTINUOUS $\lim_{x \downarrow y} F(x) = F(y)$ $\to$ $(x_n \downarrow y_n \text{ ALL } n)$

AND HAS LEFT LIMIT

"RIGHT" DEFINITION OF MONOTONICITY IN $\mathbb{R}^k$

3) LET $A = (a_1, b_1] \times \dots \times (a_k, b_k]$ HYPER-RECTANGLE

HYPER-RECTANGLE

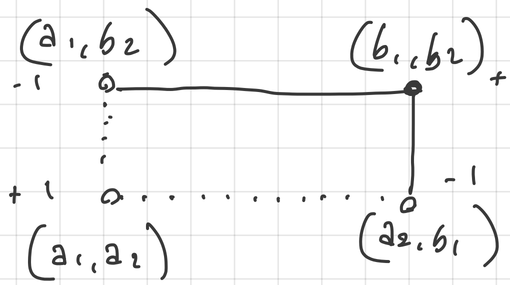
$-\infty < a_i < b_i < \infty$ ALL i

[$\mathcal{B}^k$ IS THE σ-FIELD GENERATED BY SETS OF THIS FORM]

LET $V = V_A = \{a_1, b_1\} \times \dots \times \{a_k, b_k\}$

↳ SET OF VERTICES OF HYPER-RECTANGLE

$k = 2$



FOR ANY $v \in V_A$ LET $\text{sgn}(v) = (-1)^{\#a\text{'s IN } v}$ (SIGN)

PROPERTY 3) REQUIRES THAT, FOR ANY SUCH HYPER-RECTANGLE A,

$$\Delta_A F = \sum_{v \in V_A} \text{sgn}(v)\, F(v) \geq 0$$

ANY F SATISFYING 2) AND 3) DEFINES A UNIQUE MEASURE μ ON $(\mathbb{R}^k, \mathcal{B}^k)$ S.T.

$\mu(A) = \Delta_A F$ FOR ALL HYPER-RECTANGLES A.

- IF $x = (x_1, \dots, x_k) \in \mathbb{R}^k \longmapsto F(x) = \prod_{i=1}^{k} x_i$ IS THIS CHOICE

WE OBTAIN THE LEBESGUE MEASURE IN $(\mathbb{R}^k, \mathcal{B}^k)$.

IN PARTICULAR, IF $A = (a_1, b_1] \times \dots \times (a_k, b_k]$, ITS LEBESGUE MEASURE IS

$$\prod_{i=1}^{k} (b_i - a_i).$$

EXAMPLE IN $\mathbb{R}^2$, THE FUNCTION

$$F(x,y) = \begin{cases} 0 & \max\{x,y\} < 1/2 \\ 3 & \min\{x,y\} \geq 1/2 \\ 2 & \text{OTHERWISE} \end{cases}$$

DOES NOT SATISFY 3)

[redacted] MEASURABLE FUNCTIONS

<u>Def</u>: LET $(\Omega, \mathcal{F})$ AND $(S, \mathcal{A})$ BE TWO MEASURE SPACES.

LET $f : \Omega \to S$. f IS $\mathcal{F}/\mathcal{A}$ - MEASURABLE (OR JUST MEASURABLE, FOR BREVITY) WHEN $f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in A\}$

↳ PRE-IMAGE OF A

IS IN $\mathcal{F}$, FOR ALL $A \in \mathcal{A}$.

<u>EXAMPLE</u> $(S, \mathcal{A}) = (\mathbb{R}, \mathcal{B})$ LET P BE A PROB. MEASURE ON $(\Omega, \mathcal{F})$ AND $f : \Omega \to \mathbb{R}$

WE ARE INTERESTED IN PROBABILITY THAT f IS IN $(a, b]$:

$$P\Big(\underbrace{\{\omega : f(\omega) \in (a, b]\}}_{f^{-1}((a,b])}\Big)$$