36710 - 36752

ADVANCED PROBABILITY OVERVIEW

FALL 2020

LECTURE 7 : WED, SEP 23, 2020

→ A WEEK FROM TODAY

IMPORTANT ANNOUNCEMENT: I WILL BE OUT OF TOWN NEXT WED, SEP 30 AND WE WILL NOT MEET. I WILL RECORD THE LECTURE AND POST IT ON THE CLASS WEBSITE. ALSO, I WILL NOT HOLD OH!

- LET'S CONTINUE WITH INTEGRATION: MEASURE μ AND MEAS. f, WE WOULD LIKE TO DEFINE $\int f \, d\mu$. IF μ IS A PROB. MEAS AND f IS A RANDOM VARIABLE, THEN $\int f \, d\mu$ WOULD BE THE EXPECTED VALUE

- RECALL: $(\Omega, \mathcal{F}, \mu)$ MEASURE SPACE

  SIMPLE FUNCTION $\omega \longmapsto f(\omega) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(\omega)$

  $A_1, A_2, \ldots, A_n$ ARE DISJOINT

  $a_i \in \mathbb{R}$ $\quad i=1,\ldots,n$

  OR

  $a_i \in \overline{\mathbb{R}}$ (BE CAREFUL...)

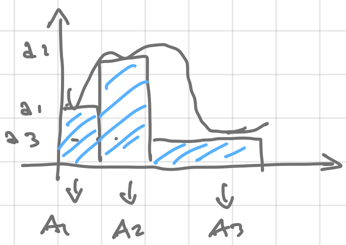
- USUALLY, SIMPLE FUNCTIONS ARE DEFINED TO BE NON-NEGATIVE. IF f IS A GENERAL FUNCTION WE CAN WRITE

TRUE FOR ALL FUNCTIONS ← $f = f^+ - f^-$ $\qquad f^+ \geq 0, \; f^- \geq 0$

- THE INTEGRAL OF A NON-NEGATIVE SIMPLE FUNCTION $f = \sum_{i=1}^{n} a_i 1_{A_i}$

IS

$$\int_\Omega f(\omega)\, d\mu(\omega) = \sum_{i=1}^{n} a_i \,\mu(A_i)$$



IF f CAN BE WRITTEN AS $f = \sum_{j=1}^{m} b_j 1_{B_j}$

ITS INTEGRAL IS

$$\sum_{j=1}^{m} b_j \,\mu(B_j)$$

THE TWO EXPRESSIONS ARE THE SAME (SO THIS DEFINITION IS WELL POSED):

$$f = \sum_{i,j} c_{i,j} 1_{A_i \cap B_j} \qquad \text{WHERE} \quad c_{i,j} = a_i = b_j \qquad \sum_j \mu(A_i \cap B_j)$$

$$\Downarrow$$

$$\int f\, d\mu = \sum_{i,j} c_{i,j}\, \mu(A_i \cap B_j) = \begin{cases} \sum_{i,j} a_i\, \mu(A_i \cap B_j) = \sum_i a_i\, \mu(A_i) \\ \sum_{i,j} b_j\, \mu(A_i \cap B_j) = \sum_j b_j\, \mu(B_j) \end{cases}$$

- IF f IS SIMPLE AND TAKES ON POSITIVE OR NEG. VALUES

$$f = f^+ - f^- \qquad \text{AND}$$

$$\int f\, d\mu = \int f^+ d\mu - \int f^- d\mu$$

BECAUSE IF f AND g ARE SIMPLE FUNCTIONS, THEN FOR ANY $c \in \mathbb{R}$

$$\int c f\, d\mu = c \int f\, d\mu \qquad \text{AND} \qquad \int f + g\, d\mu = \int f\, d\mu + \int g\, d\mu$$

- BE CAREFUL: WE SAY THAT f (SIMPLE FUNCTION) IS INTEGRABLE

IF $\int f^+ d\mu < \infty$ AND $\int f^- d\mu < \infty$

$$\Updownarrow$$

$$\int (f^+ + f^-)\, d\mu = \int |f|\, d\mu < \infty$$

IF ONE OF f^+ OR f^- HAVE INFINITE INTEGRAL

THEN $\int f\, d\mu = \pm\infty$ AND IS NOT INTEGRABLE

HOWEVER IF BOTH f^+ AND f^- HAVE INFINITE INTEGRALS

THEN $\int f \, d\mu$ IS UNDEFINED. (IT DOES NOT EXIST!)

- WHAT ABOUT INTEGRATING MEASURABLE FUNCTIONS f THAT ARE NOT SIMPLE?

LET'S FOCUS ON NON-NEGATIVE FUNCTIONS.

<u>Def</u>. IF $f \geq 0$ IS A MEAS. FUNCTION ON $(\Omega, \mathcal{F})$ AND μ A MEASURE ON THIS SPACE, THEN

$$\int_\Omega f \, d\mu = \sup_{\substack{g \text{, NON-NEGATIVE} \\ \text{SIMPLE AND} \\ g \leq f}} \int_\Omega g \, d\mu$$

$$= \sup_{\substack{\text{ALL FINITE} \\ \text{PARTITIONS } \{A_n\} \text{ OF } \Omega}} \sum_n \mu(A_n) \times \left[\inf_{\omega \in A_n} f(\omega) \right] \longleftarrow$$

- FOR GENERAL $f = f^+ - f^-$, $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu$

PROVIDED THAT AT MOST ONE OF $\int f^+ d\mu$ OR $\int f^- d\mu$ IS INFINITE.

- SOME PROPERTIES OF THE INTEGRAL:

1) IF $c \in \mathbb{R}$ AND $\int f \, d\mu$ EXISTS, $\int_\Omega c f \, d\mu = c \int_\Omega f \, d\mu$

Pf/ ASSUME $f \geq 0$ AND $c > 0$

$$\int c f \, d\mu = \sup \left\{ \int g \, d\mu \,,\ 0 \leq g \leq c f \,,\ g \text{ SIMPLE} \right\}$$

$$= c \sup \left\{ \int \frac{g}{c} \, d\mu \quad 0 \leq \frac{g}{c} \leq f \quad \frac{g}{c} \text{ SIMPLE} \right\}$$

$$= c \int f \, d\mu$$

THE CASE OF GENERAL f AND $c<0$ IS AN EXERCISE ... $\blacksquare$

2) IF $f \leq g$ a.e. $[\mu]$ $\left(\mu(\{\omega : g(\omega) < f(\omega)\}) = 0 \right)$

THEN $\int f \, d\mu \leq \int g \, d\mu$

PF/ ASSUME $f \geq 0$, $g \geq 0$. LET $A_1, \dots, A_n$ BE A PARTITION OF Ω. THEN

$$\sum_{i=1}^{n} \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i) = \sum_{i=1}^{n} \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i \cap G)$$

$G = \{\omega : f(\omega) \leq g(\omega)\}$, $\mu(G^c) = 0$

$$\leq \sum_{i=1}^{n} \left[\inf_{\omega \in A_i \cap G} f(\omega) \right] \mu(A_i \cap G)$$

$$\leq \sum_{i=1}^{n} \left[\inf_{\omega \in A_i \cap G} g(\omega) \right] \mu(A_i \cap G)$$

$$\leq \int g \, d\mu$$

TAKING SUP OVER ALL FINITE DECOMPOSITION OF Ω GIVES THAT

$$\int f \, d\mu \leq \int g \, d\mu$$

IF f AND g ARE GENERAL, THEN $f \leq g \Rightarrow f^+ \leq g^+$, $f^- \geq g^-$

AND REPEAT THE ARGUMENTS... $\blacksquare$

3) IF $A \in \mathcal{F}$ THEN $\int_A f \, d\mu = \int_\Omega 1_A(\omega) f(\omega) \, d\mu(\omega)$

THEN IF $f \geq 0$ IS SUCH THAT $\int f \, d\mu < \infty$ THEN $f < \infty$ a.e. $[\mu]$. TO SEE THIS LET $A = \{\omega : f(\omega) = \infty\}$

WE WANT TO SHOW $\mu(A) = 0$. ASSUME THE OPPOSITE, $\mu(A) > 0$.

THEN $\int f \, d\mu \geq \int 1_A f \, d\mu = \int_A f \, d\mu = \infty$, WHICH IS

A CONTRADICTION.

4) IF $\int f \, d\mu < \infty$ THEN $\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$

EASY TO SEE: $-|f| \leq f \leq |f|$ FOR EACH ω

↳ USING 3)

$$-\int |f| \, d\mu \leq \int f \, d\mu \leq \int |f| \, d\mu$$

AND RESULT FOLLOWS.

WE STILL NEED TO ESTABLISH ONE FUNDAMENTAL PROPERTY:

$$\int (f+g) \, d\mu \stackrel{?}{=} \int f \, d\mu + \int g \, d\mu$$

TO ESTABLISH THIS, WE NEED TO LEARN ABOUT BASIC LIMIT THEOREMS.

BASIC LIMIT THEOREMS

<u>Thm</u> (FATOU'S LEMMA) LET $\{f_n\}$ BE A SEQUENCE OF NON-NEGATIVE MEAS. REAL VALUE FUNCTIONS. THEN

$$\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n \, d\mu$$

THIS IS GOING TO GIVE US THIS FIRST IMPORTANT LIMIT THEOREM:

<u>Thm (MONOTONE CONVERGENCE THEOREM)</u>: $\{f_n\}$ SEQUENCE OF NON-NEG., MEAS. FUNCTIONS. LET f BE A MEAS. FUNCTION S.T. $f_n \leq f$ ALL n AND $f_n \to f$ AS $n \to \infty$ a.e. $[\mu]$. THEN

$$\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu = \int \lim f_n \, d\mu$$

POSSIBLY INFINITY

Pf/ SINCE $f_n \le f$ FOR ALL n, $\int f_n \, d\mu \le \int f \, d\mu$ FOR ALL n. SO

$$\liminf_n \int f_n \, d\mu \le \limsup_n \int f_n \, d\mu \le \int f \, d\mu$$

$$= \int \liminf_n f_n \, d\mu$$

BY FATOU'S LEMMA

$$\le \liminf_n \int f_n \, d\mu$$

SO THE INEQUALITIES ARE IDENTITIES, $\lim \int f_n \, d\mu = \int f \, d\mu$ ∎

Remark: USUALLY THE ASSUMPTION IN THE MCT IS THAT

$$0 \le f_n \uparrow f \quad \text{AS } n \to \infty$$

AS A RESULT, WE CAN PROVE LINEARITY OF THE INTEGRAL FOR NON-NEG. FUNCTIONS

Thm IF $\int f \, d\mu$ AND $\int g \, d\mu$ ARE DEFINED AND NOT BOTH INFINITE AND OF OPPOSITE SIGNS, THEN

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$$

PROOF USES STANDARD MACHINERY: NEXT TIME