Q1. Find the following anti-derivative: $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

$= \int x^{\frac{1}{2}}\,dx + \int x^{-\frac{1}{2}}\,dx$

$= \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + C$ [power rule]

$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

Q2. Compute the following definite integral: $\int_0^\pi (\sin x - \cos x)dx$.

$= \left[-\cos x - \sin x + C\right]_{x=0}^{\pi}$

$= (-\overset{-1}{\cos(\pi)} - \overset{0}{\sin(\pi)} + C) - (-\overset{1}{\cos(0)} - \overset{0}{\sin(0)} + \cancel{C})$

$= 2$

THE NET CHANGE THEOREM

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

"The new value at b of a changing quantity $F(x)$ equals the initial value $F(a)$ plus the integral from a to b of the rate of change $F'(x)$."

Q3. Write an integral that expresses the increase in perimeter $P(s)$ of a square when its side length s increases from 2 units to 4 units and evaluate the integral. $\longrightarrow P$

a b

Perimeter of square w/ sidelength s:

$P(s) = 4s$

rate of change: $P'(s) = 4$

Apply net change thm:

$$\int_a^b P'(s)\,ds = \int_2^4 4\,ds$$

$$= 4(4) - 4(2)$$

$$= 8$$

Q4. Re-write the integral in the form $\int f(u)du$ using $u = x - 1$ and $du = dx$.

$$\int \frac{x^2}{\sqrt{x-1}}dx$$

$u = x - 1$

$x = u + 1$

$$\longrightarrow \frac{x^2}{\sqrt{x-1}} = \frac{(u+1)^2}{\sqrt{u+\cancel{1}-\cancel{1}}}$$

$$\int \frac{x^2}{\sqrt{x-1}}dx = \int \frac{(u+1)^2}{\sqrt{u}}du$$

$$\frac{u^2+2u+1}{u^{\frac{1}{2}}} = \cdots + \cdots + \cdots$$

Integration by Substitution

$\longleftrightarrow$ Inverse of Chain Rule

Recall chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

By applying FTC, we get $\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$

Steps

1. Look if the occurence of a function AND it's derivative can be found
   e.g. $g(x)$ & $g'(x)$ in the above
2. Let $\underline{u} = g(x)$ s.t. $\frac{du}{dx} = g'(x) \longrightarrow du = g'(x)\,dx$
3. Replace everything until we only depend on u instead of x.

$$\underbrace{f'(g(x))}_{f'(u)} \underbrace{g'(x)\,dx}_{du} = \underline{f'(u)\,du}$$

f'(u) should be ~~is~~ easier to apply integration to

4. $\int f'(u)\,du = f(u) + C$
5. Soln is in terms of u $\longrightarrow$ Make substitution $u = g(x)$ for original answer

$$\underbrace{\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C}_{\text{final ans}}$$

Q5. Find the indefinite integral: $\int \frac{x}{\sqrt{x^2+1}}\,dx.$ $\longrightarrow$ Integrate by substitution

Let $u = x^2 + 1 \Rightarrow du = 2x\,dx$

$x\,dx = \frac{du}{2}$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}} \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\left(\frac{1}{-\frac{1}{2}+1}\,u^{-\frac{1}{2}+1}\right) + C$$

$$= \sqrt{u} + C$$

$= \sqrt{x^2+1} + C$ [* undo substitution]

Q6. Find the indefinite integral: $\int \underbrace{\cos^3\theta}_{(\cos\theta)^3}\, d\theta$.

$$= \int \cos^2\theta \cdot \cos\theta \, d\theta$$

$$= \int (1 - \overset{(\sin\theta)^2}{\sin^2\theta}) \underbrace{\cos\theta \, d\theta} \longrightarrow \frac{d}{d\theta}(\sin\theta) = \cos\theta$$

[Integrate by substitution

Let $u = \sin\theta \implies du = \cos\theta \, d\theta$

$$= \int (1 - u^2) \, du$$

$$= u - \frac{u^{2+1}}{2+1} + C$$

$$= u - \frac{u^3}{3} + C$$

$$= \sin\theta - \frac{\sin^3\theta}{3} + C$$ [undo substitution]

Q7. Find the indefinite integral: $\int t\sin(t^2)\cos(t^2)dt$. $\rightarrow \frac{d}{dt}\left(\sin(t^2)\right)$

$= \underline{2t\cos(t^2)}$

Chain rule

Let $u = \sin(t^2) \longrightarrow du = 2t\cos(t^2)\,dt$

$t\cos(t^2)dt = \frac{du}{2}$

$$\int \overbrace{\sin(t^2)}^{u}\,\overbrace{t\cos(t^2)dt}^{du/2} = \int u\cdot\frac{du}{2}$$

$$= \frac{1}{2}\int u\,du$$

$$= \left(\frac{u^{1+1}}{1+1}\right)\frac{1}{2} + C$$

$$= \frac{1}{4}u^2 + C$$

undo substitution: $= \frac{1}{4}(\sin(t^2))^2 + C$