Integration Techniques

(last tut) Integration by Substitution:

$\longleftrightarrow$ Inverse of Chain Rule

Recall chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

By applying FTC, we get $\boxed{\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C}$

Steps

1. Look if the occurence of a function AND it's derivative can be found
   e.g. $g(x)$ & $g'(x)$ in the above
2. Let $u = g(x)$ s.t. $\frac{du}{dx} = g'(x) \longrightarrow du = g'(x)\,dx$
3. Replace everything until we only depend on u instead of x.
   $$\underbrace{f'(g(x))}_{f'(u)}\ \underbrace{g'(x)\,dx}_{du} = \underbrace{f'(u)\,du}$$
   $f'(u)$ should be ~~a~~ easier to apply integration to
4. $\int f'(u)\,du = f(u) + C$
5. Soln is in terms of u $\longrightarrow$ Make substitution $u = g(x)$ for original answer
   $$\underbrace{\int f'(g(x))\, g'(x)\,dx = f(g(x)) + C}_{\text{final ans}}$$

Integration by Parts

$\longleftrightarrow$ Inverse of product rule

Recall product rule: $\frac{d}{dx}(f(x).g(x)) = f'(x)g(x) + g'(x)f(x)$

$$\int \frac{d}{dx}(f(x).g(x))dx = f(x)g(x) + C \quad [FTC]$$

$$= \int f'(x)g(x)dx + \int g'(x)f(x)dx$$

Hence, $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

Let $u = f(x) \longrightarrow du = f'(x)dx$

$v = g(x) \longrightarrow dv = g'(x)dx$

(*) $\boxed{\int u\,dv = uv - \int v\,du}$, $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

Steps

1. Choose dv & u.

   tip: choose u to be whichever fcn easier to differentiate

   choose dv " integrate

2. Compute v by integrating $\int dv = \int f(x)dx$
3. Compute du by differentiating u.
4. Substitute into (*) appropriately.

Partial Fractions

goal: Break rational functions (quotient of polynomials) into simpler pieces which can integrated more easily

Decomposition

Let p & q be polynomial fcns, $\deg(q(x)) > \deg(p(x))$

Suppose $q(x)$ can be factored into products of linear fcns

s.t. $q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$

* Want $\frac{p(x)}{q(x)} = \frac{c_1}{a_1x+b_1} - \frac{c_2}{a_2x+b_2} + \cdots + \frac{c_n}{a_nx+b_n}$ → for non-repeated lin. factors

where $c_1, c_2, \cdots, c_n \in \mathbb{R}$ (constants)

* If $q(x)$ has a repeated linear factor $(ax+b)^n$, $n > 1$
decompose the factor as follows

$$\frac{c_1}{ax+b} + \frac{c_2}{(ax+b)^2} + \cdots + \frac{c_n}{(ax+b)^n}$$ → for repeated lin. factors

* f $q(x)$ has an irreducible quadratic factor ax^2+bx+c
decomposition must contain the term

$$\frac{Ax+B}{ax^2+bx+c}$$ → for quadratic factors

Note: the common denominator in all these decompositions is the initial polynomial we were trying to decompose

Partial Fractions (continued)

Steps

1. Decompose as above

2. Solve for all constants

   e.g. Supp. $\frac{P(x)}{q(x)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2}$

   (i.e. $q(x) = (a_1x+b_1)(a_2x+b_2)$)

   Want to find A & B s.t.

   $$A(a_2x+b_2) + B(a_1x+b_1) = P(x)$$

3. Integrate resulting decomposition

Q1. Find the indefinite integral $\int \frac{1}{(x-3)(x-2)}\, dx$.

Q2. Find the indefinite integral $\int \frac{1}{x^3 - x}\, dx$.

Improper Integrals

Definite integrals w/ bounds that are infinite or outside the domain of the integrand.

① Supp. f is cont. on $[a, \infty)$.
Then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

② Supp. f is cont. on $(-\infty, b]$.
Then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

Improper integrals in ① & ② converge if the corresponding limits exist & diverge otherwise

③ Supp. f is cont. on $(-\infty, \infty)$
Then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

$$= \lim_{a \to -\infty} \int_a^c f(x)\,dx + \lim_{b \to \infty} \int_c^b f(x)\,dx, \; c \in \mathbb{R}$$

Integral ③ converges if & only if <u>both</u> limits exist

Q3. Determine whether the following integral converges or diverges.

$$\int_e^{\infty} \frac{1}{x \ln x} \, dx$$

Q4. Determine whether the following integral converges or diverges.

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1}\, dx$$

Q5. Determine whether the following integral converges or diverges.

$$\int_0^\infty e^{-x}\,dx$$

Q6. Determine whether the following integral converges or diverges.

$$\int_0^2 \frac{1}{x^3}\, dx$$