Integration Techniques

(last tut) Integration by Substitution:

$\longleftrightarrow$ Inverse of Chain Rule

Recall chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

By applying FTC, we get $\boxed{\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C}$

Steps

1. Look if the occurence of a function AND it's derivative can be found e.g. $g(x)$ & $g'(x)$ in the above
2. Let $u = g(x)$ s.t. $\frac{du}{dx} = g'(x) \longrightarrow du = g'(x)\,dx$
3. Replace everything until we only depend on u instead of x.

$$\underbrace{f'(g(x))}_{f'(u)}\, \underbrace{g'(x)\, dx}_{du} = \underbrace{f'(u)\, du}$$

$f'(u)$ should be ~~as~~ easier to apply integration to

4. $\int f'(u)\, du = f(u) + C$
5. Soln is in terms of u $\longrightarrow$ Make substitution $u = g(x)$ for original answer

$$\underbrace{\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C}_{\text{final ans}}$$

Integration by Parts

$\longleftrightarrow$ Inverse of product rule

Recall product rule: $\frac{d}{dx}(f(x).g(x)) = \underline{f'(x)g(x)} + g'(x)f(x)$

$$\int \frac{d}{dx}(f(x).g(x))dx = f(x)g(x) + C \quad [\text{FTC}]$$

$$= \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx$$

Hence, $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

Let $u = f(x) \longrightarrow du = f'(x)dx$

$v = g(x) \longrightarrow dv = g'(x)\,dx$

(*) $\boxed{\int u\,dv = uv - \int v\,du}$, $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

Steps

1. Choose dv & u.

   tip: choose u to be whichever fcn easier to differentiate
   choose dv " integrate

2. Compute v by integrating $\int dv = \int f(x)\,dx$
3. Compute du by differentiating u.
4. Substitute into (*) appropriately.

$\int \frac{dx}{x} = \ln|x| + C$

Partial Fractions

goal: Break rational functions (quotient of polynomials) into simpler pieces which can integrated more easily

Decomposition

Let p & q be polynomial fcns, $\deg(q(x)) > \deg(p(x))$

Suppose $q(x)$ can be factored into products of linear fcns

s.t. $q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$

* Want $\frac{p(x)}{q(x)} = \frac{c_1}{a_1x+b_1} + \frac{c_2}{a_2x+b_2} + \cdots + \frac{c_n}{a_nx+b_n}$ → for non-repeated lin. factors

common denomin. of these terms $= q(x)$

where $c_1, c_2, \cdots, c_n \in \mathbb{R}$ (constants)

* If $q(x)$ has a repeated linear factor $(ax+b)^n$, $n > 1$

decompose the factor as follows

$$\frac{c_1}{ax+b} + \frac{c_2}{(ax+b)^2} + \cdots + \frac{c_n}{(ax+b)^n}$$ → for repeated lin. factors

* f $q(x)$ has an irreducible quadratic factor ax^2+bx+c

decomposition must contain the term

$$\frac{Ax+B}{ax^2+bx+c}$$ → for quadratic factors

Note: the common denominator in all these decompositions is the initial polynomial we were trying to decompose

Partial Fractions (continued)

Steps

1. Decompose as above

2. Solve for all constants

   e.g. Supp. $\dfrac{p(x)}{q(x)} = \dfrac{A}{a_1x+b_1} + \dfrac{B}{a_2x+b_2}$

   $$= \frac{\overbrace{A(a_2x+b_2)+B(a_1x+b_1)}^{p(x)}}{\underbrace{(a_1x+b_1)(a_2x+b_2)}_{q(x)}}$$

   (i.e. $q(x) = (a_1x+b_1)(a_2x+b_2)$)

   Want to find A & B s.t.

   $$A(a_2x+b_2) + B(a_1x+b_1) = p(x)$$

3. Integrate resulting decomposition

Q1. Find the indefinite integral $\int \frac{1}{(x-3)(x-2)}\,dx$. apply partial fraction

Decompose:

$$\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{1}{x-3} - \frac{1}{x-2}$$

Solve for constants (A & B)

$P(x) = 1$

$$A(x-2) + B(x-3) = 1$$

$$Ax - 2A + Bx - 3B = 1$$

$$(A+B)x - 2A - 3B = 1$$

$\therefore$
$$\begin{cases} A + B = 0 & \rightarrow B = -A \\ -2A - 3B = 1 & \rightarrow -2A - 3(-A) = 1 \end{cases}$$

$$-2A + 3A = 1$$

$$\boxed{A = 1}$$

$B = -A$

$\boxed{B = -1}$

Integrate the resulting decomposition

$\boxed{\int \frac{dx}{x} = \ln|x| + C}$

$$\int \frac{1}{x-3} - \frac{1}{x-2}\,dx = \int \frac{dx}{x-3} - \int \frac{dx}{x-2}$$

$$= \ln|x-3| - \ln|x-2| + C$$

Q2. Find the indefinite integral $\int \frac{1}{x^3 - x}\, dx$. $\rightarrow$ apply partial fractions

$$x^3 - x = x(x^2-1) = x(x+1)(x-1)$$

Decompose:

$$\frac{1}{x^3-x} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{-1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$$

Solve for constants A, B, C

$$A\overbrace{(x-1)(x+1)}^{x^2-1} + B\overbrace{(x)(x+1)}^{x^2+x} + C\overbrace{(x)(x-1)}^{x^2-x} = 1$$

$$Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx = 1$$

$$\underbrace{(A+B+C)}_{=0}x^2 + \underbrace{(B-C)}_{=0}x \underbrace{-A}_{1} = 1$$

$$\begin{cases} A+B+C=0 & \longrightarrow (-1)+B+B=0 \longrightarrow \boxed{B=\frac{1}{2}} \\ B-C=0 & \longrightarrow B=C \longrightarrow \boxed{C=\frac{1}{2}} \\ -A=1 & \longrightarrow \boxed{A=-1} \end{cases}$$

Integrate resulting decomposition:

$$\int \frac{-1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}\, dx$$

$$= -\int \frac{dx}{x} + \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$$

$$= -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

Improper Integrals

Definite integrals w/ bounds that are infinite or outside the domain of the integrand.

① Supp. f is cont. on $[a, \infty)$.
Then

$$\int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$\lim_{b\to\infty}(F(b)-F(a))$

② Supp. f is cont. on $(-\infty, b]$.
Then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to -\infty} \int_a^b f(x)\,dx$$

Improper integrals in ① & ② converge if the corresponding limits exist & diverge otherwise

③ Supp. f is cont. on $(-\infty, \infty)$
Then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

$$= \lim_{a\to -\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx \quad, c\in\mathbb{R}$$

Integral ③ converges if & only if both limits exist
approaching finite divergence → not convergence

Q3. Determine whether the following integral converges or diverges.

$$\int_e^\infty \frac{1}{x \ln x}\, dx$$

$\frac{d}{dx} \ln x = \frac{1}{x}$

→ Integrate by substitution

$u = f(x)$

$\int_a^b f(x)\,dx = \int_{f(a)}^{f(b)} u\,du$

Let $u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x} \rightarrow du = \frac{dx}{x}$

$$\int_e^\infty \frac{dx}{x \ln x} = \int_{\ln(e)}^{\ln(\infty)} \frac{du}{u}$$

$$= \int_1^\infty \frac{du}{u}$$

$$= \lim_{b\to\infty} \int_1^b \frac{du}{u}$$

$$= \lim_{b\to\infty} \Big[\ln|u|\Big]_{u=1}^{u=b}$$

$$= \underbrace{\lim_{b\to\infty} \Big(\underbrace{\ln|b|}_{\to\infty} - \cancel{\ln|1|}^{\,0}\Big)}_{\text{as } b\to\infty}$$

∴ the integral diverges

Q4. Determine whether the following integral converges or diverges.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \quad a \le b \le c$$

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$$

Let $u = x^2+1$

$du = 2x\,dx \longrightarrow x\,dx = \frac{du}{2}$

$$\int_{-\infty}^{\infty} \frac{x\,dx}{x^2+1} = \int_{-\infty}^{0} \frac{x\,dx}{x^2+1} + \int_{0}^{\infty} \frac{x\,dx}{x^2+1}$$

$$= \lim_{t\to-\infty} \int_t^0 \frac{x}{x^2+1}\,dx + \lim_{t\to\infty} \int_0^t \frac{x}{x^2+1}\,dx$$

The integral converges iff both of the limit terms converge.

It diverges if at least one of the limit terms diverge

Let $u = x^2+1$

$du = 2x\,dx \longrightarrow x\,dx = \frac{du}{2}$

$$\lim_{t\to\infty} \int_0^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_{0^2+1}^{t^2+1} \frac{du}{2u}$$

$$= \lim_{t\to\infty} \frac{1}{2} \int_1^{t^2+1} \frac{du}{u}$$

$$= \lim_{t\to\infty} \Big[\ln|u| \Big]_{u=1}^{u=t^2+1}$$

$$= \lim_{t\to\infty} \Big[\underbrace{\ln|t^2+1|}_{} - \cancel{\ln|1|}^{\,0} \Big]$$

$t^2+1 \to \infty$ as $t \to \infty$

$\ln|t^2+1| \to \infty$

$\therefore$ This limit diverges

$\therefore$ The integral diverges

Q5. Determine whether the following integral converges or diverges.

$$\int_0^{\infty} e^{-x}\,dx$$

$$\frac{d}{dx} e^{x} = e^{x}$$

$$\int e^{-x}\,dx = -e^{-x}$$

$$\int_0^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_0^{t} e^{-x}\,dx$$

$$= \lim_{t\to\infty} \left[-e^{-x}\right]_{x=0}^{x=t}$$

$$= \lim_{t\to\infty} \left(-e^{-t} + \cancel{e^{-0}}\ 1\right)$$

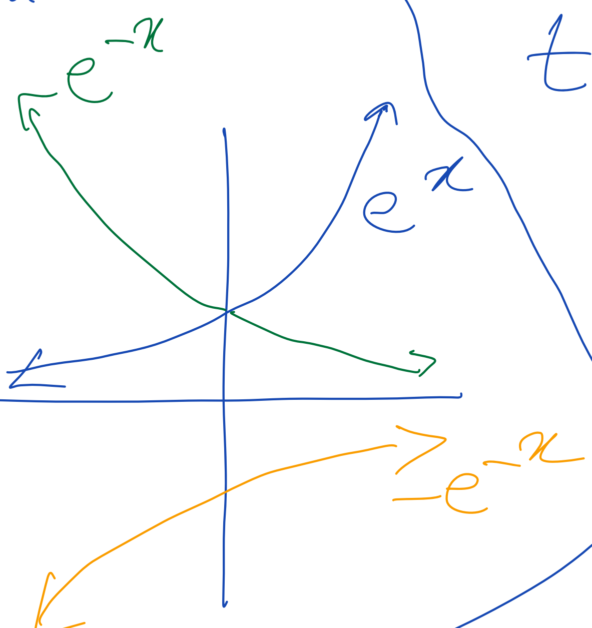
$-e^{-t} \to 0$ as $t \to \infty$

$$= 0 + 1$$

$$= 1$$

$\therefore$ The integral converges to 1

function transform:



Q6. Determine whether the following integral converges or diverges.

$$\int_0^2 \frac{1}{x^3}\,dx$$

$\to 0 \notin \mathrm{dom}\left(\frac{1}{x^3}\right)$

$\to$ impooper

$$\int \frac{dx}{x^3} = \int x^{-3}\,dx$$

$$\int_0^2 \frac{dx}{x^3} = \lim_{t\to 0} \int_t^2 \frac{dx}{x^3}$$

$$= \lim_{t\to 0} \left[\frac{x^{-3+1}}{-3+1}\right]_{x=t}^{x=2}$$

$$= \lim_{t\to 0} \left[-\frac{1}{2x^2}\right]_{x=t}^{x=2}$$

$$= \lim_{t\to 0} \left[\cancel{\frac{-1}{2(2)^2}}^{=-\frac{1}{8}} + \underbrace{\frac{1}{2t^2}}_{\to\infty \text{ as } t\to 0}\right]$$

$\to$ ∴ The limit diverges

$\to$ ∴ The integral diverges