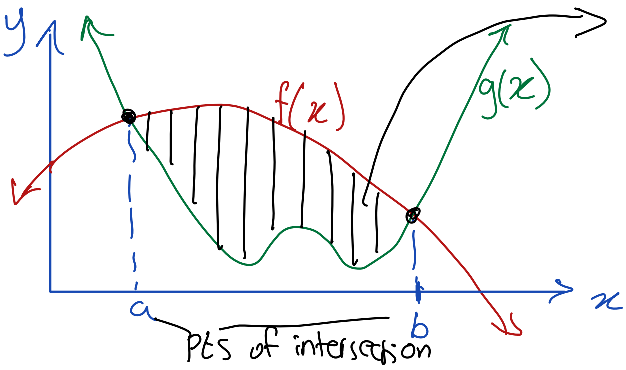
Area btwn Curves

Slicing vertically → Integrating along x-axis (dx)



area of each slice = height$(x)\,\Delta x$

area bounded btwn $f(x)$ & $g(x)$

= sum of area of vertical slices

$$A = \int_{a=\text{smallest } x}^{b=\text{largest } x} \text{height}(x)\,dx$$

"for all"

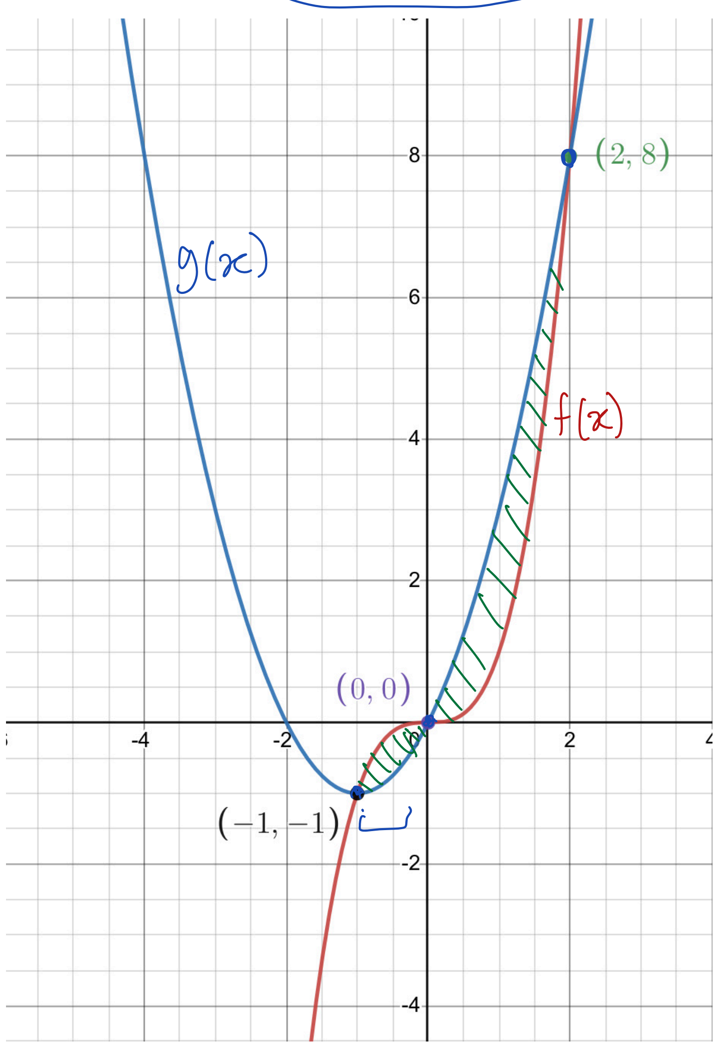
Given 2 curves $y=f(x)$ & $y=g(x)$, If $f(x) > g(x)$ $\forall x \in (a,b)$

Then the area bounded in that region:

$$A = \int_a^b \underbrace{(f(x)-g(x))}_{\text{height}(x)}\,dx .$$

We call $f(x)$ the top curve & $g(x)$ the bottom curve

Q1. Find the area bounded by $f(x) = x^3$ and $g(x) = x^2 + 2x$.



Step 1 Find pts of intersection

i.e. Find x s.t. $f(x) = g(x)$

$$x^3 = x^2 + 2x$$

$$x^3 - x^2 - 2x = 0$$

$$\underbrace{x(x-2)(x+1)}_{f(x)-g(x)} = 0$$

$\therefore$ pts of intersection are at $x = -1, 0, 2$

Step 2 Identify top & bottom curves

i.e. Find intervals where $f(x) > g(x) \Longleftrightarrow f(x)-g(x) > 0$

& intervals where $g(x) > f(x) \Longleftrightarrow f(x)-g(x) < 0$

	$-\infty$		-1		0		2		$+\infty$
x		$-$		$-$	0	$+$		$+$	
$x-2$		$-$		$-$		$-$	0	$+$	
$x+1$		$-$	0	$+$		$+$		$+$	
$f(x)-g(x)=x(x-2)(x+1)$		$-$	0	$+$	0	$-$	0	$+$	

bounded

$f(x)-g(x) < 0$ for $x \in (-\infty,-1) \cup (0,2)$

> 0 for $x \in (-1,0) \cup (2,+\infty)$

$\therefore$ For bounded regions

$g(x)$ is top curve on $x \in (0,2)$

$f(x)$ is top curve on $x \in (-1,0)$

Step 3 Set up appropriate integral for bounded area btwn the curves

f is top curve on $(-1,0)$ g is top curve on $(0,2)$

$$A = \int_{-1}^{0} [f(x)-g(x)]\,dx + \int_{0}^{2} [g(x)-f(x)]$$

$$= \int_{-1}^{0} x^3 - x^2 - 2x\,dx + \int_{0}^{2} x^2 + 2x - x^3\,dx$$

Step 4 Evaluate the integrals

$$A = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} + \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right]_{0}^{2}$$

$$= \left(\frac{0^4}{4} - \frac{0^3}{3} - 0^2 - \frac{(-1)^4}{4} + \frac{(-1)^3}{3} + (-1)^2\right)$$

$$+ \left(\frac{2^3}{3} + 2^2 - \frac{2^4}{4} - \frac{0^3}{3} - 0^2 + \frac{0^4}{4}\right)$$

$$= \frac{37}{12}$$

Q2. Find the area between $f(x) = x^2$ and $g(x) = -x^2 + 18x$.

Step1 Find pts of intersection.

i.e. Find x s.t. $f(x) = g(x)$

$$x^2 = -x^2 + 18x$$

$$2x(x-9) = 0 \longrightarrow f - g = 0$$

$\therefore$ pts of intersection are at $x = 0, 9$

Step2. Identify top & bottom curves

	$-\infty$	0		9	$+\infty$
$\underline{2x(x-9)}$ $f-g$	$+$	ϕ	$-$	ϕ	$+$

$f(x) - g(x) > 0$ on $(-\infty, 0) \cup (9, +\infty)$

$f(x) - g(x) < 0$ on $(0, 9)$ $g(x)$ is top curve on the bounded region

Step3 Set up integrals for bounded area

bc g is the top curve on $(0,9)$

$$A = \int_0^9 [g(x) - f(x)]\,dx = \int_0^9 -x^2 + 18x - x^2\,dx$$

$$= \int_0^9 18x - 2x^2\,dx$$

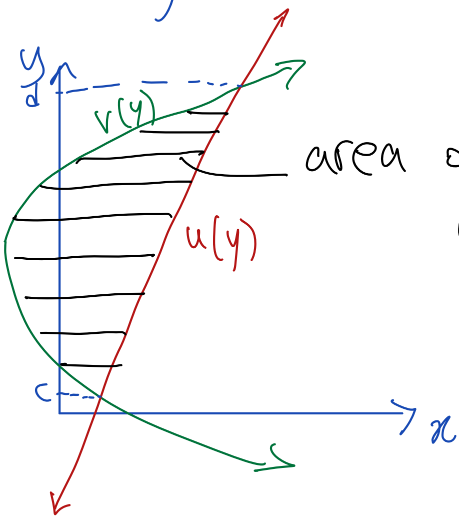
Step4 Evaluate

$$= \left[9x^2 - \frac{2}{3}x^3\right]_0^9$$

$$= 243$$

Area btwn curves (continued)

<u>Slicing horizontally</u> → Integrating along y-axis



area of each slice = width(y) . Δy

area bounded btwn u(y) & v(y)

= sum of area of horizontal slices

$$A = \int_{c = \text{smallest } y\text{-val}}^{d = \text{largest } y\text{-val}} \text{width}(y)\, dy$$

Given 2 curves $x = u(y)$ & $x = v(y)$. If $u(y) > v(y)$ $\forall y \in (c,d)$

Then the area bounded by the region

$$A = \int_c^d \overbrace{(u(y) - v(y))}^{\text{width}(y)}\, dy$$

We call u(y) the right curve

v(y) the left curve

on (c,d)

Q3. Find the area between $x = f(y) = 3y$ and $x = g(y) = y^3 - y$

Step1 Find pts of intersection

i.e. Find y s.t. $f(y) = g(y)$

$$3y = y^3 - y$$

$$y^3 - y - 3y = 0$$

$$\underbrace{y(y-2)(y+2)}_{g(y)-f(y)} = 0$$

$\therefore$ POI's are at $y = 0, \pm 2$

Step2 Identify right & left curves

		-2		0		2	
$g(y)-f(y) = y(y-2)(y+2)$	$-$	0	$+$	0	$-$	0	$+$

$g(y) - f(y) > 0$ on $y \in (-2, 0) \cup (2, +\infty)$

< 0 on $y \in (-\infty, -2) \cup (0, 2)$

$\therefore$ For bounded regions, $g(y)$ is right curve on $(-2, 0)$

$f(y)$ is right curve on $(0, 2)$

Step3 Set up the integrals

$$A = \int_{-2}^{0} [g(y) - f(y)]\,dy + \int_{0}^{2} [f(y) - g(y)]\,dy$$

$$= \int_{-2}^{0} y^3 - 4y\,dy + \int_{0}^{2} 4y - y^3\,dy$$

Step 4 Evaluate

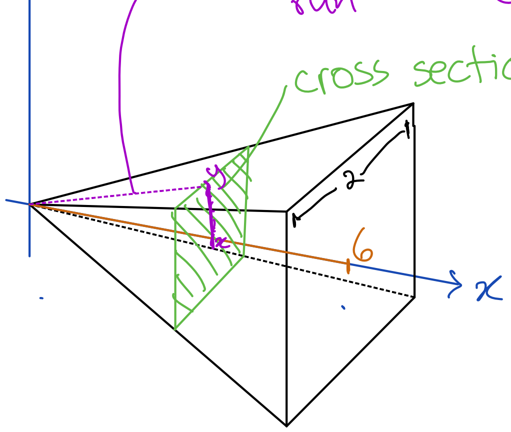
$$A = \left[\frac{y^4}{4} - 2y^2\right]_{-2}^{0} + \left[2y^2 - \frac{y^4}{4}\right]_{0}^{2}$$

$$= 8$$

Q4. Find the volume (integral of areas) of a pyramid with height 6 units and a square base of side 2 units.

Step1 Sketch & assign variables/cross sections

$y = mx = \frac{\text{rise}}{\text{run}} x = \frac{1}{6} x$



Let x be the distance along the line measured from the tip of the pyramid

Let y be half the side length of the square slice of height x (from tip)

Step2 Formulate area of the square cross section $A(x)$

$$A(x) = (\text{sidelength})^2 = \left(\frac{1}{6}x + \frac{1}{6}x\right)^2 = \frac{x^2}{9}$$

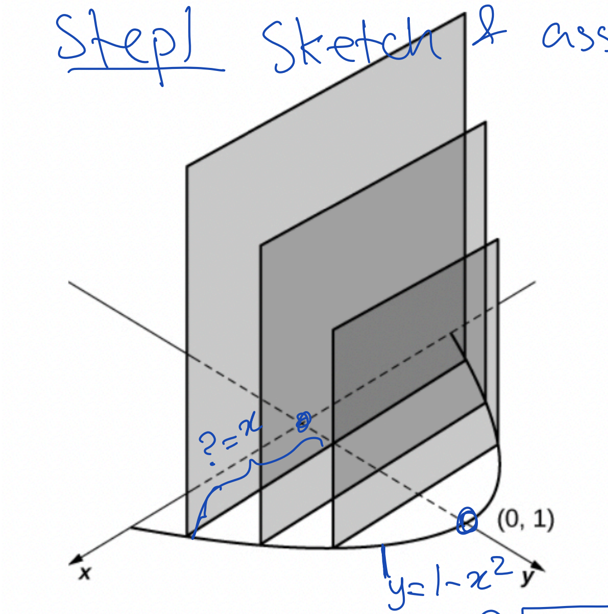
Step3 Set up & evaluate integral for volume

The limits of the integral are $x=0$ & $x=6$

$$\int_0^6 A(x)\,dx = \int_0^6 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_0^6 = 8$$

Q5. The base is the region under the parabola $y = 1 - x^2$ and above the x-axis. Slices perpendicular to the y-axis are squares.

Step1 Sketch & assign cross section/vars



$x = \pm\sqrt{1-y}$ → sidelength of each slice $= 2x$

positive part bc want the side length

$= 2\sqrt{1-y}$

Step2 Formulate area of each cross section/slice

$A(y) = (\text{side length})^2$

$= (2\sqrt{1-y})^2 = 4 - 4y$

Step4 Integrate & evaluate

We're looking at the region btwn ~~under~~ $y = 1 - x^2$ & x-axis

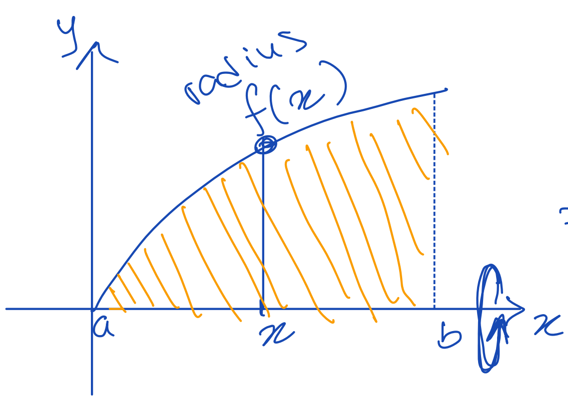
→ The limits are $y = 0$ & $y = 1$

$$\int_0^1 A(y)\,dy = \int_0^1 4 - 4y\,dy = \left[4y - 2y^2\right]_0^1 = 2$$

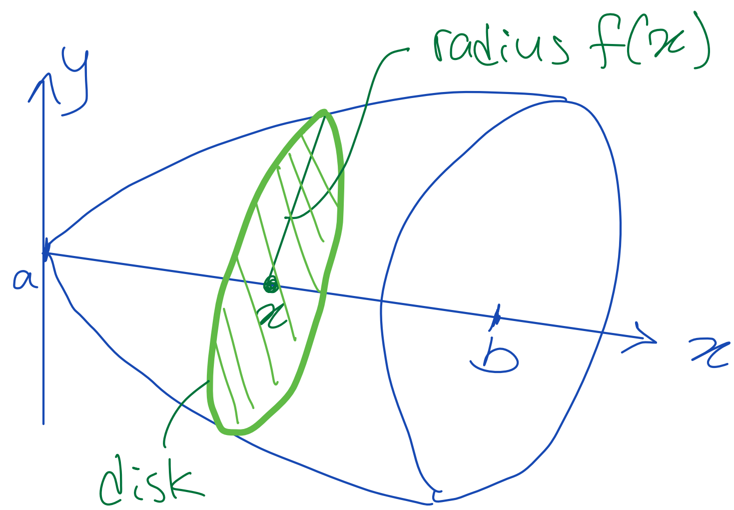
Volume of Solids of Revolution – The Disk Method

Given function $f(x)$ on domain $[a,b]$, we revolve / rotate it around the x-axis to generate a "solid of revolution".

Goal: Compute the volume of such solids



$\Longrightarrow$



Note that since we are rotating around the x-axis, we can have a circular cross section.

$\longrightarrow$ Integrate the circular cross section areas

The Disk Method

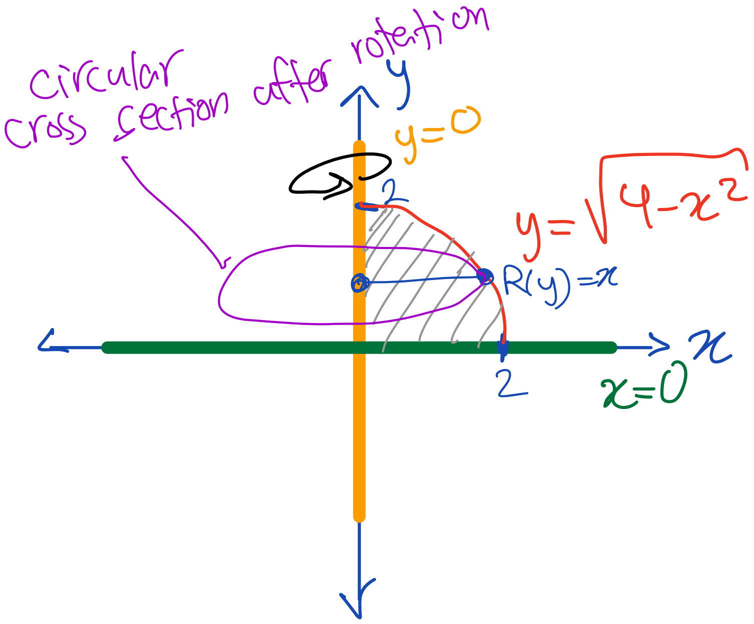
Volume for solid formed by rotating region around x-axis is

$$V = \int_a^b \underbrace{\pi \Big[\overbrace{f(x)}^{\text{radius of cross section}}\Big]^2}_{\text{formula of circle}} dx$$

Q6. Find the volume when the region bounded by $y = \sqrt{4-x^2}$, $y = 0$, and $x = 0$, is rotated around the y-axis.

Step1 Sketch

$y = \sqrt{4-x^2} \longleftrightarrow x^2 + y^2 = 2^2$ where $y > 0$



Step2 Find $R(y)$

$x^2 + y^2 = 2^2 \longleftrightarrow x = \sqrt{4-y^2}$

$\therefore R(y) = \sqrt{4-y^2}$

Step3 Find area of circular cross region

$$A(y) = \pi (R(y))^2$$
$$= \pi (\sqrt{4-y^2})^2$$
$$= \pi (4-y^2)$$

Step4 Integrate area to find volume

$$\int_0^2 A(y)\,dy = \int_0^2 \pi(4-y^2)\,dy$$
$$= \left[\pi\left(4y - \frac{y^3}{3}\right)\right]_0^2 = \frac{16\pi}{3}$$