Lecture 03: Dynamic Programming

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What is dynamic programming (DP)?

Basic DP definition

- Dynamic: sequential or temporal problem structure
- Programming: mathematical optimization, i.e., numerical solutions

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  - We will focus only on finite MDPs.
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- DP is a collection of algorithms to solve MDPs and neighboring problems.
  - We will focus only on finite MDPs.
  - In case of continuous action/state space: apply quantization.
- Use of value functions to organize and structure the search for an optimal policy.
- Breaks problems into subproblems and solves them.

Requirements for DP

DP can be applied to problems with the following characteristics.

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How is that connected to MDPs?

- MDPs satisfy above's properties:
  - Bellman equation provides recursive decomposition.
  - Value function stores and reuses solutions.

Utility of DP in the RL context

DP is used for iterative model-based prediction and control in an MDP.

- Prediction:
  - Input: MDP $\langle \mathcal{S}, \mathcal{A}, \boldsymbol{\mathcal{P}}, \mathcal{R}, \gamma \rangle$ and policy π
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- Feasibility in real-world engineering applications (model vs. system) is therefore limited.
- But: following DP concepts are largely used in modern data-driven RL algorithms.

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$$\begin{aligned} v_\pi(s) &= \mathbb{E}_\pi \left[G_t | S_t = s \right], \\ &= \mathbb{E}_\pi \left[R_{t+1} + \gamma G_{t+1} | S_t = s \right], \\ &= \mathbb{E}_\pi \left[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s \right]. \end{aligned}$$

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- Or in matrix form:

$$
\boldsymbol{v}_{\mathcal{S}}^{\pi} = \boldsymbol{r}_{\mathcal{S}}^{\pi} + \gamma \boldsymbol{\mathcal{P}}_{ss'}^{\pi} \boldsymbol{v}_{\mathcal{S}}^{\pi},
$$

$$
\begin{bmatrix} v_1^\pi \\ \vdots \\ v_n^\pi \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1^\pi \\ \vdots \\ \mathcal{R}_n^\pi \end{bmatrix} + \gamma \begin{bmatrix} p_{11}^\pi & \cdots & p_{1n}^\pi \\ \vdots & & \vdots \\ p_{n1}^\pi & \cdots & p_{nn}^\pi \end{bmatrix} \begin{bmatrix} v_1^\pi \\ \vdots \\ v_n^\pi \end{bmatrix}.
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- Solving the Bellman expectation equation for v_π requires handling a linear equation system with n unknowns (i.e., number of states).

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$$\left\| v_i(s) - v_\pi(s) \right\|_\infty \to 0 \quad \text{for} \quad i = 1, 2, 3, \ldots \tag{3.1}$$

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- The Bellman equation in matrix form can be rewritten as:

$$\underbrace{\left(\boldsymbol{I} - \gamma \boldsymbol{\mathcal{P}}_{ss'}^{\pi}\right)}_{\boldsymbol{A}} \underbrace{\boldsymbol{v}_{\mathcal{S}}^{\pi}}_{\boldsymbol{x}} = \underbrace{\boldsymbol{r}_{\mathcal{S}}^{\pi}}_{\boldsymbol{b}} . \tag{3.2}$$

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- To iteratively solve this linear equation $\boldsymbol{Ax} = \boldsymbol{b}$, one can apply numerous methods such as
  - General gradient descent,
  - Richardson iteration,
  - Krylov subspace methods.

Richardson iteration (1)

In the MDP context, the Richardson iteration became the default solution approach to iteratively solve:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

The Richardson iteration is

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + \omega(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_i) \tag{3.3}$$

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To evaluate convergence we inspect the following norm:

$$\|\boldsymbol{e}_{i+1}\|_\infty = \|(\boldsymbol{I} - \omega \boldsymbol{A})\, \boldsymbol{e}_i\|_\infty . \tag{3.5}$$

Richardson iteration (2)

Since any induced matrix norm is sub-multiplicative, we can approximate (3.5) by the inequality:

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Since the row elements of $\boldsymbol{\mathcal{P}}^{\pi}_{ss'}$ always sum up to 1,

$$\gamma < 1 \tag{3.10}$$

follows. Hence, when discounting the Richardson iteration always converges for MDPs even if we assume $\omega = 1$.

Iterative policy evaluation by Richardson iteration (1)

Applying the Richardson iteration (3.3) with $w = 1$ to the Bellman equation for any $s \in \mathcal{S}$ at iteration i results in:

$$v_{i+1}(s) = \sum_{a \in \mathcal{A}} \boldsymbol{\pi}(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} p_{ss'}^a v_i(s') \right) . \tag{3.11}$$

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Matrix form then is:

$$\boldsymbol{v}_{\mathcal{S},i+1}^{\pi} = \boldsymbol{r}_{\mathcal{S}}^{\pi} + \gamma \boldsymbol{\mathcal{P}}_{ss'}^{\pi} \boldsymbol{v}_{\mathcal{S},i}^{\pi} . \tag{3.12}$$

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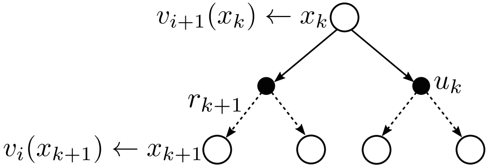

Fig. 3.1: Backup diagram for iterative policy evaluation

Iterative policy evaluation by Richardson iteration (2)

- During one Richardson iteration the 'old' value of s is replaced with a 'new' value from the 'old' values of the successor state s'.
  - Update $v_{i+1}(s)$ from $v_i(s')$, see Fig. 3.1.
  - Updating estimates (v_{i+1}) on the basis of other estimates (v_i) is often called bootstrapping.

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- Also called expected update because it is based on the expectation over all possible next states (utilizing full model knowledge).

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- Also called expected update because it is based on the expectation over all possible next states (utilizing full model knowledge).
- In subsequent lectures, the expected update will be supplemented by data-driven samples from the environment.

Iterative policy evaluation example: forest tree MDP

Let's reuse the forest tree MDP example with *fifty-fifty policy*:

$$\boldsymbol{\mathcal{P}}_{ss'}^{\pi} = \begin{bmatrix} 0 & \frac{1-\alpha}{2} & 0 & \frac{1+\alpha}{2} \\ 0 & 0 & \frac{1-\alpha}{2} & \frac{1+\alpha}{2} \\ 0 & 0 & \frac{1-\alpha}{2} & \frac{1+\alpha}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{r}_{\mathcal{S}}^{\pi} = \begin{bmatrix} 0.5 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

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i	$v_i(s=1)$	$v_i(s=2)$	$v_i(s=3)$	$v_i(s=4)$
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3	1.03	1.85	2.85	0

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1	0.5	1	2	0
2	0.82	1.64	2.64	0
3	1.03	1.85	2.85	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	1.12	1.94	2.94	0

Tab. 3.1: Policy evaluation by Richardson iteration (3.12) for forest tree MDP with $\gamma = 0.8$ and $\alpha = 0.2$

Variant: in-place updates

Instead of applying (3.12) to the entire vector $\boldsymbol{v}^{\pi}_{\mathcal{S},i+1}$ in 'one shot' (synchronous backup), an elementwise in-place version of the policy evaluation can be carried out:

input: full model of the MDP, i.e., $\langle \mathcal{S}, \mathcal{A}, \boldsymbol{\mathcal{P}}, \mathcal{R}, \gamma \rangle$ including policy π
parameter: $\delta > 0$ as accuracy termination threshold
init: $v_0(s)\,\forall\, s \in \mathcal{S}$ arbitrary except $v_0(s) = 0$ if s is terminal
repeat
  $\Delta \leftarrow 0$;
  for $\forall\, s \in \mathcal{S}$ **do**
    $\tilde{v} \leftarrow \hat{v}(s)$;
    $\hat{v}(s) \leftarrow \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}^a_s + \gamma \sum_{s' \in \mathcal{S}} p^a_{ss'} \hat{v}(s')\right)$;
    $\Delta \leftarrow \max\left(\Delta, |\tilde{v} - \hat{v}(s)|\right)$;
until $\Delta < \delta$;

Algo. 3.1: Iterative policy evaluation using in-place updates (output: estimate of $\boldsymbol{v}^{\pi}_{\mathcal{S}}$)

In-place policy evaluation updates for forest tree MDP

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- May converge faster than regular Richardson iteration if state update order is chosen wisely (sweep through state space).
- For forest tree MDP: reverse order, i.e., start with $s = 4$.
- As can be seen in Tab. 3.2 the in-place updates especially converge faster for the 'early states'.

i	$v_i(s=1)$	$v_i(s=2)$	$v_i(s=3)$	$v_i(s=4)$
0	0	0	0	0
1	1.03	1.64	2	0
2	1.09	1.85	2.64	0
3	1.11	1.91	2.85	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
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Tab. 3.2: In-place updates for forest tree MDP

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General idea on policy improvement

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  - Check the action value of this 'new move'. If it is better than the 'old' value, take it:

$$q_\pi(s, a) = \mathbb{E}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s, A_t = a\right] . \tag{3.13}$$

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Theorem 3.1: Policy improvement

If for any deterministic policy pair π and π'

$$q_\pi(s, \pi'(s)) \geq v_\pi(s) \quad \forall s \in \mathcal{S} \tag{3.14}$$

applies, then the policy π' must be as good as or better than π. Hence, it obtains greater or equal expected return

$$v_{\pi'}(s) \geq v_\pi(s) \quad \forall s \in \mathcal{S}. \tag{3.15}$$

Proof of policy improvement theorem

Start with (3.14) and recursively reapply (3.13):

$$v_\pi(s) \leq q_\pi(s, \pi'(s)),$$

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&= \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 v_\pi(S_{t+2}) | S_t = s\right], \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 v_\pi(S_{t+3}) | S_t = s\right], \\
&\vdots \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots | S_t = s\right], \\
&= v_{\pi'}(s).
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\begin{aligned}
\pi'(s) &= \arg\max_{a\in\mathcal{A}} q_\pi(s, a), \\
&= \arg\max_{a\in\mathcal{A}} \mathbb{E}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s, A_t = a\right], \\
&= \arg\max_{a\in\mathcal{A}} \mathcal{R}_s^a + \gamma \sum_{s'\in\mathcal{S}} p_{ss'}^a v_\pi(s') \, .
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- This is the Bellman optimality equation, which guarantees that $\pi' = \pi$ must be optimal policies.
- Although the proof for policy improvement theorem was presented for deterministic policies, transfer to stochastic policies $\pi(a|s)$ is possible.
- Takeaway message: policy improvement theorem guarantees finding optimal policies in finite MDPs (e.g., by DP).

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Concept of policy iteration

- Policy iteration combines the previous policy evaluation and policy improvement in an iterative sequence:

$$\pi_0 \rightarrow v_{\pi_0} \rightarrow \pi_1 \rightarrow v_{\pi_1} \rightarrow \cdots \pi^* \rightarrow v_{\pi^*} \tag{3.19}$$

- Evaluate $\rightarrow$ improve $\rightarrow$ evaluate $\rightarrow$ improve ...

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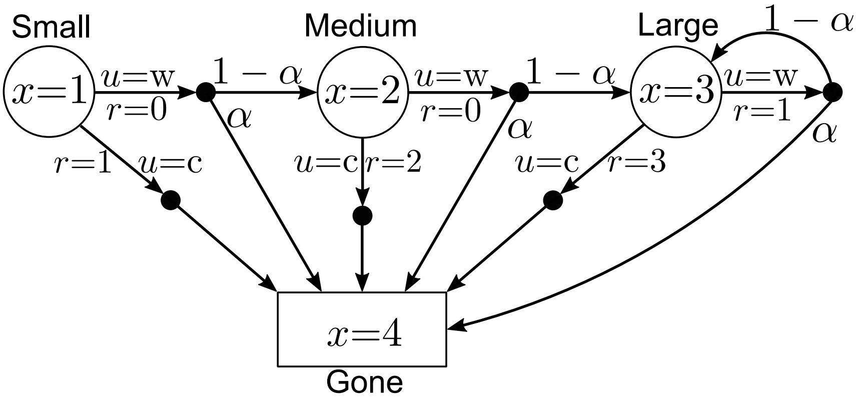
- Evaluate $\rightarrow$ improve $\rightarrow$ evaluate $\rightarrow$ improve ...
- In the 'classic' policy iteration, each policy evaluation step in (3.19) is fully executed, i.e., for each policy π_i an exact estimate of v_{π_i} is provided either by iterative policy evaluation with a sufficiently high number of steps or by any other method that fully solves (3.2).

Policy iteration example: forest tree MDP (1)



- Two actions possible in each state:
  - Wait $a = w$: let the tree grow.
  - Cut $a = c$: gather the wood.

Policy iteration example: forest tree MDP (2)

Assume $\alpha = 0.2$ and $\gamma = 0.8$ and start with 'tree hater' initial policy:

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3. Greedy policy improvement:

$$\begin{aligned}\pi_1(s) &= \arg\max_{a \in \mathcal{A}} \mathbb{E}\left[R_{t+1} + \gamma v_{\pi_0}(S_{t+1}) | S_t = s, A_t = a\right], \\ &= \{\pi(a = \mathsf{w}|s = 1), \pi(a = \mathsf{c}|s = 2), \pi(a = \mathsf{c}|s = 3)\}\end{aligned}$$

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Value iteration (1)

- Policy iteration involves full policy evaluation steps between policy improvements.
- In large state-space MDPs the full policy evaluation may be numerically very costly.

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- Value iteration: One step iterative policy evaluation followed by policy improvement.
- Allows simple update rule which combines policy improvement with truncated policy evaluation in a single step:

$$\begin{aligned} v_{i+1}(s) &= \max_{s \in \mathcal{A}} \mathbb{E}\left[R_{t+1} + \gamma v_i(S_{t+1}) | S_t = s, A_t = a\right], \\ &= \max_{a \in \mathcal{A}} \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} p_{ss'}^a v_i(s'). \end{aligned} \tag{3.20}$$

Value iteration (2)

input: full model of the MDP, i.e., $\langle \mathcal{S}, \mathcal{A}, \boldsymbol{\mathcal{P}}, \mathcal{R}, \gamma \rangle$
parameter: $\delta > 0$ as accuracy termination threshold
init: $v_0(s) \, \forall \, s \in \mathcal{S}$ arbitrary except $v_0(s) = 0$ if s is terminal
repeat
    $\Delta \leftarrow 0$;
    for $\forall \, s \in \mathcal{S}$ **do**
        $\tilde{v} \leftarrow \hat{v}(s)$;
        $\hat{v}(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} p_{ss'}^a \hat{v}(s') \right)$;
        $\Delta \leftarrow \max \left(\Delta, |\tilde{v} - \hat{v}(x_k)| \right)$;
until $\Delta < \delta$;
output: deterministic policy $\pi \approx \pi^*$, such that
$\pi(s) \leftarrow \arg\max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} p_{ss'}^a \hat{v}(s') \right)$;

Algo. 3.2: Value iteration (note: compared to policy iteration, value iteration does not require an initial policy but only a state-value guess)

Value iteration example: forest tree MDP

- Assume again $\alpha = 0.2$ and $\gamma = 0.8$.

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Value iteration example: forest tree MDP

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- Similar to in-place update policy evaluation, reverse order and start value iteration with $s = 4$.
- As shown in Tab. 3.3 value iteration converges in one step (for the given problem) to the optimal state value.

i	$v_i(s=1)$	$v_i(s=2)$	$v_i(s=3)$	$v_i(s=4)$
0	0	0	0	0
1	1.28	2	3	0
*	1.28	2	3	0

Tab. 3.3: Value iteration for forest tree MDP

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Summarizing DP algorithms

- All DP algorithms are based on the state value $v(s)$.
  - Complexity is $\mathcal{O}(m \cdot n^2)$ for m actions and n states.
  - Evaluate all n^2 state transitions while considering up to m actions per state.

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Problem	Relevant Equations	Algorithm
prediction	Bellman expectation eq.	policy evaluation
control	Bellman expectation eq. & greedy policy improvement	policy iteration
control	Bellman optimality eq.	value iteration

Tab. 3.4: Short overview addressing the treated DP algorithms

Curse of dimensionality

- DP is much more efficient than an exhaustive search over all n states and m actions in finite MDPs in order to find an optimal policy.
  - Exhaustive search for deterministic policies: m^n evaluations.
  - DP results in polynomial complexity regarding m and n.

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- Hence, DP is can be effective up to medium-sized MDPs (i.e., million finite states)
- For large problems DP suffers from the curse of dimensionality:
  - Single update step may become computational infeasible.
  - Also: if continuous states need quantization, number of finite states n grows exponentially with the number of state variables (assuming fixed number of discretization levels).

Generalized policy iteration (GPI)

- Almost all RL methods are well-described as GPI.
- Push-pull: Improving the policy will deteriorate value estimation.
- Well balanced trade-off between evaluating and improving is required.

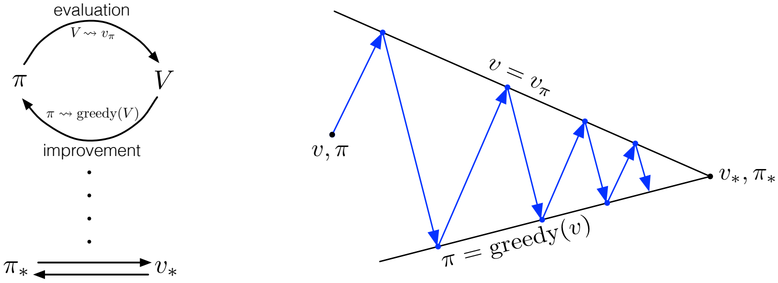

Fig. 3.2: Interpreting generalized policy iteration to switch back and forth between (arbitrary) evaluations and improvement steps

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- Both steps can be combined via value iteration.
- The idea of (generalized) policy iteration is a basic scheme of RL.