ETF 5500 Assignment 2

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Question 1

Part A

Here, we have a data matrix Y with a dimension of $n \times 3$ which has a mean of 0.

For a vector Y which has been demeaned, the sample covariance matrix S can be obtained through the following computation.

$$S = \frac{1}{n-1}(Y^T Y) \tag{1}$$

To maintain matrix conformability, the dimension of S will be a 3×3 matrix.

In order to define the covariance matric S as defined by Equation 1, we require the following quanities:

1. A data matrix Y with a mean of 0.
2. The total number of observations n which is given by the number of rows in Y.

Part B

An Eigen Value problem is linear in nature and defined by Equation 2.

$$SW = \lambda W \tag{2}$$

Where,

S = Covariance Matrix
W = Eigen Vector
λ = Eigen Value

As the Eigen Vector is a column vector, hence, to satisfy matrix conformability in Equation 2, the dimension of W must be a 3×1 vector.

Part C

We are given the linear combination as stated through Equation 3.

$$X = \beta Y \tag{3}$$

We know that Y is a $n \times 1$ vector and β is stated as a 3×1 vector. To obtain the Variance-Covariance matrix, we perform the following computations.

$$\text{Cov}(X) = \frac{1}{n-1} X^T X$$

$$\implies \text{Cov}(X) = \sum_{i=1}^{n} \frac{(\text{Y}_i \beta)^2}{n-1}$$

$$\implies \text{Cov}(X) = \beta^T \Big[\sum_{i=1}^{n} \frac{(\text{Y}_i)^2}{n-1}\Big] \beta$$

For a data matrix Y_i with mean $= 0$, the term $\sum_{i=1}^{n} \dfrac{(Y_i)^2}{n-1}$ is considered as the covariance and is denoted by S.

$$\boxed{\implies \text{Cov}(X) = \beta^T S \beta} \tag{4}$$

Equation 4 provides us with the final form of the variance-covariance matrix.

We know, the dimension of β^T is 1×3 and that of $beta$ is 3×1. In order to maintain matrix conformability, the value dimension of the variance covariance matrix $\text{Var}(X)$ will be 1×1. This suggests that the **resultant matrix is a scalar value.**

Exercise 2

The dimensions of the matrices of interest are as follows:

W is a 3×1 matrix
Y is a $n \times 1$ matrix
S is a 3×3 matrix

Part A

W^TY matrix will be conformable **if the matrix Y has a $n = 3$ observations.**

The dimension of this matrix will be 1×3 .

Part B

WW^T will be a conformable matrix with dimension 3×3.

Part C

WV^T will be a conformable matrix with dimension 3×3.

Part D

S^TY matrix will be conformable **if the matrix Y has a $n = 3$ observations.**

The dimension of this new matrix will be 3×3.

Part E

$YW + V^T$ is **not a conformable matrix** as the number of columns for matrix Y and the number of rows for matrix W are not identical.

Exercise 3

We are given with the equation $X = YC^T$

Where

$$C = \begin{bmatrix} W^T \\ U^T \end{bmatrix}$$

and is a 2×3 dimensional matrix.

W is the eigen vector of S corresponding to the largest eigen value while U is the eigen vector of S corresponding to the second-largest eigen value.

Performing a matrix multiplication to obtain X gives us a $n \times 2$ matrix where n is the number of observations.

Content of matrix X

The matrix multiplication performs a linear combination such that the data contained in matrix Y is now projected along the eigen vectors relating to the largest and the second-largest eigen value. **In the context of principal component analysis (PCA), X contains the data which is projected onto the top 2 principal components.**

Derivation for sample covariance of X

We know, the variance-covariance matrix is calculated as follows:

$$\mathrm{Cov}(X) = \frac{(X^T X)}{n-1}$$

$$\implies \mathrm{Cov}(X) = \sum_{i=1}^{n} \frac{(YC^T)^2}{n-1}$$

$$\implies \mathrm{Cov}(X) = C^T \Big[\sum_{i=1}^{n} \frac{(Y_i)^2}{n-1}\Big] C$$

As previously defined in Equation 1, we can use the matrix S in the above equation as follows.

$$\boxed{\implies \mathrm{Cov}(X) = C^T S C}$$