

# ETF 5500 Assignment 2

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## Question 1

### Part A

Here, we have a data matrix  $Y$  with a dimension of  $n \times 3$  which has a mean of 0.

Equation 12 shows the equation of sample variance for a single variable  $y$ .

$$\text{Var}(y) = \frac{1}{n-1} \sum_{j=1}^n (y_{i,j} - \bar{y}_i) \quad (1)$$

Where,

$y$  are the observations

$\bar{y}$  is the mean of the observations

$n$  is the number of observations

While Equation 12 provides us with a singular value for a single variable, we would like to extend this idea for high dimensions in a matrix form. This matrix is called the variance-covariance matrix where every element in the diagonal provides us with the variance of the variable while every off-diagonal element provides us with the covariance of the variable with respect to another variable. This is given by Equation 2.

$$\text{Cov}(Y) = \frac{1}{n-1} ((Y - \bar{Y})'(Y - \bar{Y})) \quad (2)$$

Where,

$Y$  is the data matrix,

$\bar{Y}$  is the matrix containing the mean of each variable,

$n$  is the number of observations in the data.

For a vector  $Y$  that has been demeaned ( $\bar{Y} = 0$ ), the sample variance-covariance matrix  $S$  can be obtained through the following computation.

$$S = \text{Cov}(Y) = \frac{1}{n-1}(Y^T Y) \quad (3)$$

To maintain matrix conformability, the dimension of  $S$  will be a  $3 \times 3$  matrix.

In order to define the covariance matrix  $S$  as defined by Equation 3, we require the following quantities:

1. A data matrix  $Y$  with a mean of 0.
2. The total number of observations  $n$  which is given by the number of rows in  $Y$ .

## Part B

An Eigen Value problem is linear in nature and defined by Equation 4.

$$Sw = \lambda w \quad (4)$$

Where,

$S$  = Covariance Matrix

$w$  = Eigen Vector

$\lambda$  = Eigen Value

As the Eigen Vector is a column vector, hence, to satisfy matrix conformability in Equation 4, the dimension of  $w$  must be a  $3 \times 1$  vector.

## Part C

We are given the linear combination as stated through Equation 5.

$$X = \beta Y \quad (5)$$

We know that  $Y$  is a  $n \times 1$  vector and  $\beta$  is stated as a  $3 \times 1$  vector. To obtain the Variance-Covariance matrix, we perform the following computations.

$$\text{Cov}(X) = \frac{1}{n-1} X' X$$

$$\begin{aligned}\Rightarrow \text{Cov}(X) &= \sum_{i=1}^n \frac{(Y_i \beta)^2}{n-1} \\ \Rightarrow \text{Cov}(X) &= \beta' \left[ \sum_{i=1}^n \frac{(Y_i)^2}{n-1} \right] \beta\end{aligned}$$

For a data matrix  $Y_i$  with mean = 0, the term  $\sum_{i=1}^n \frac{(Y_i)^2}{n-1}$  is considered as the covariance and is denoted by  $S$ .

$$\boxed{\Rightarrow \text{Cov}(X) = \beta' S \beta} \quad (6)$$

Equation 6 provides us with the final form of the variance-covariance matrix.

We know, the dimension of  $\beta^T$  is  $1 \times 3$  and that of  $\beta$  is  $3 \times 1$ . In order to maintain matrix conformability, the value dimension of the variance covariance matrix  $\text{Var}(X)$  will be  $1 \times 1$ . This suggests that the **resultant matrix is a scalar value**.

## Exercise 2

The dimensions of the matrices of interest are as follows:

$w$  is a  $3 \times 1$  matrix

$Y$  is a  $n \times 3$  matrix

$S$  is a  $3 \times 3$  matrix

$v$  is an arbitrary column vector with a dimension  $n \times 1$

In order to assess matrix multiplication conformability, **the number of columns in the first matrix should be equal to the number of rows in the second matrix**.

### Part A

The matrix  $Y$  is a general matrix with  $n \times 3$  dimension. The only condition when the product  $w'Y$  will be conformable is for a specific condition,  $n = 3$ . However, a data matrix with only 3 dimensions and only 3 observations is highly unlikely.

As a result, this product is expected to be **non-conformable**.

Under the special condition when  $n = 3$ , the dimension of  $w'Y$  matrix will be  $1 \times 3$ .

## Part B

$ww'$  will be a conformable matrix with dimension  $3 \times 3$ .

## Part C

$wv'$  will be a **non-conformable matrix** since  $v$  is an arbitrary column vector with no fixed dimension.

## Part D

$S'Y$  matrix will be a **non-conformable matrix** since  $Y$  is an arbitrary column vector with no fixed dimension.

## Part E

$Yw + v'$  is **not a conformable matrix** as the number of columns for matrix  $Y$  and the number of rows for matrix  $W$  are not identical.

## Exercise 3

We are given with the equation  $X = YC'$

Where

$$C = \begin{bmatrix} w' \\ u' \end{bmatrix}$$

and is a  $2 \times 3$  dimensional matrix.

$w$  is the eigen vector of  $S$  corresponding to the largest eigen value while  $u$  is the eigen vector of  $S$  corresponding to the second-largest eigen value.

Performing a matrix multiplication to obtain  $X$  gives us a  $n \times 2$  matrix where  $n$  is the number of observations.

## Content of matrix X

The matrix multiplication performs a linear combination such that the data contained in matrix  $Y$  is now projected along the eigen vectors relating to the largest and the second-largest eigen value. **In the context of principal component analysis (PCA),  $X$  contains the data which is projected onto the top 2 principal components.**

## Derivation for sample covariance of $X$

We know, the variance-covariance matrix is calculated as follows:

$$\begin{aligned}\text{Cov}(X) &= \frac{(X'X)}{n-1} \\ \Rightarrow \text{Cov}(X) &= \sum_{i=1}^n \frac{(YC')^2}{n-1} \\ \Rightarrow \text{Cov}(X) &= C' \left[ \sum_{i=1}^n \frac{(Y_i)^2}{n-1} \right] C\end{aligned}$$

As previously defined in Equation 3, we can use the matrix  $S$  in the above equation as follows.

$$\boxed{\Rightarrow \text{Cov}(X) = C'SC} \quad (7)$$

Equation 7 is the expression of the sample variance-covariance matrix  $\text{Cov}(X)$  in terms of  $C$  and  $S$ .

## Exercise 4

Vectors  $w$  and  $u$  are said to be orthogonal when they are at complete right angles to each other. This means that performing a matrix dot product,  $w \cdot u$  will **yield a result of 0**.

In the context of PCA, vectors  $w$  and  $u$  (also called principal components) are **uncorrelated** and capture unique signals in the data.

## Proof that $\text{Cov}(X)$ is a diagonal matrix

From Equation 7, we know that  $\text{Cov}(X) = C'SC$ .

Additionally,  $w$  and  $u$  are orthogonal.

Based on the spectral theorem, we can write the variance-covariance matrix  $S$  as follows.

$$S = \sum \lambda_i v_i v_i'$$

Where  $\lambda_i$  is an eigen value and  $v_i$  is an eigen vector corresponding to the eigen value.

So, based on the above information, we can write Equation 7 as follows.

$$\begin{aligned}\text{Cov}(X) &= \frac{1}{n-1} C^T (\lambda_w ww' + \lambda_u uu') C \\ \text{Cov}(X) &= \frac{1}{n-1} \begin{bmatrix} w' \\ u' \end{bmatrix} (\lambda_w ww' + \lambda_u uu') \begin{bmatrix} w' u' \end{bmatrix}\end{aligned}\quad (8)$$

All dot products between  $w$  and  $u$  reduce to 0 due to the orthogonality condition between the two vectors.

Additionally, for normalised vectors,  $vv' = 1$ . As a result, Equation 8 reduces to the following.

$$\text{Cov}(X) = \frac{1}{n-1} (\lambda_w w + \lambda_u u) \quad (9)$$

Converting Equation 9 to the matrix form, we can write it as follows.

$$\text{Cov}(X) = \frac{1}{n-1} \begin{bmatrix} \lambda_w & 0 \\ 0 & \lambda_u \end{bmatrix} \quad (10)$$

The final form of the variance-covariance matrix is diagonal, as shown in Equation 10. This diagonal structure arises from the orthogonality condition, which ensures that all off-diagonal elements representing covariances are zero.

## Exercise 5

Here, we have observations give as follows.

$$a = \frac{x_1}{\sqrt{\lambda_u}} \text{ and } b = \frac{x_2}{\sqrt{\lambda_u}}$$

$Z$  is a datamatrix consisting of the vectors  $a$  and  $b$ .

Now, the sample variance-covariance matrix  $\text{Cov}(Z)$  will be calculated as follows.

$$\begin{aligned}\text{Cov}(Z) &= \frac{1}{n-1} Z' Z \\ \text{Cov}(Z) &= \frac{1}{n-1} \begin{bmatrix} a'a & a'b \\ b'a & b'b \end{bmatrix}\end{aligned}\quad (11)$$

From @covmatfinal, we know that the variances of  $x_1$  and  $x_2$  are given by their eigen values. Using this result, we obtain the variances of the vectors  $a$  and  $b$  as follows.

$$\text{Var}(a) = \text{Var}\left(\frac{x_1}{\sqrt{\lambda_w}}\right)$$

Now, implementing the variance scaling rule  $\text{Var}(cX) = c^2\text{Var}(X)$  and utilising results from Equation 10 to the above equation, we get the result as follows.

$$\begin{aligned}\text{Var}(a) &= \frac{1}{\lambda_w} \text{Var}(x_1) \\ \Rightarrow \text{Var}(a) &= \frac{\lambda_w}{\lambda_w} = 1\end{aligned}$$

Similarly, we obtain the value of  $\text{Var}(b) = 1$ .

For a demeaned vector upon normalisation, we know that,

$$\begin{aligned}\text{Var}(a) &= \frac{1}{n-1} a' a \\ \Rightarrow \frac{1}{n-1} a' a &= 1 \\ \Rightarrow a^T a &= n-1\end{aligned}\tag{12}$$

Similarly,  $b^T b = n-1$

Next, we obtain the covariances between  $a$  and  $b$ .

$$\begin{aligned}\text{Cov}(a, b) &= \text{Cov}\left(\frac{x_1}{\lambda_w}, \frac{x_2}{\lambda_u}\right) \\ \Rightarrow \text{Cov}(a, b) &= \frac{1}{\sqrt{\lambda_w \lambda_u}} \text{Cov}(x_1, x_2)\end{aligned}$$

However, we know that our vectors  $x_1$  and  $x_2$  are the columns in the matrix  $X$ , representing the two principal components PC1 and PC2. Hence,  $x_1$  and  $x_2$  are uncorrelated and will have a covariance  $\text{Cov}(x_1, x_2) = 0$ .

As a result, we obtain the following.

$$\text{Cov}(a, b) = a'b = b'a = 0\tag{13}$$

Replacing the relevant terms in Equation 11 by Equation 12 and Equation 13, we obtain the following result.

$$\begin{aligned}\text{Cov}(Z) &= \frac{1}{n-1} \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix} \\ \implies \text{Cov}(Z) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}\end{aligned}\tag{14}$$

As observed in Equation 14, the sample variance-covariance matrix of  $Z$  is an identity matrix.