1 Divisibility Induction

Prove that for all $n \in \mathbb{N}$ with $n \geq 1$, the number $n^3 - n$ is divisible by 3. (**Hint**: recall the binomial expansion $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$)

Base case: $n=1$. $1^3 - 1 = 0$, 0 is divisible by 3.

Inductive Hypothesis: suppose that for some $k \geq 1$ $k^3 - k$ is divisible by 3.

Inductive Step: Show $(k+1)^3 - (k+1)$ is divisible by 3.

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$$

$$= \underbrace{k^3 - k}_{\text{divisible by 3 by ind. hyp.}} + \underline{3k^2 + 3k}$$

$$= 3q + 3(k^2 + k), \quad q \in \mathbb{Z}$$

$$= 3(q + k^2 + k)$$

Thus, $3 \mid [(k+1)^3 - (k+1)]$ so by induction,

$\forall n \in \mathbb{Z}^+ \; 3 \mid (n^3 - n)$ $\square$

2 Make It Stronger

Let $x \geq 1$ be a real number. Use induction to prove that for all positive integers n, all of the entries in the matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n$$

are $\leq xn$. (Hint 1: Find a way to strengthen the inductive hypothesis! Hint 2: Try writing out the first few powers.)

Strengthen hypothesis: wanted to prove $\forall_n \in \mathbb{N}\ P(n)$.

Find $Q(n)$ s.t. $Q(n) \Rightarrow P(n)$

Prove $Q(0)$ & $Q(n) \Rightarrow Q(n+1)$ i.e. $\forall n \in \mathbb{N}\ Q(n)$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix}$$

Stronger statement: $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & xn \\ 0 & 1 \end{pmatrix}$ $\forall n \geq 1$

note that this implies all entries of $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n$ are $\leq xn$.

- Base case: $n=1$ $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ ✓
- Ind. hyp.: Assume for some $k \geq 1$, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & xk \\ 0 & 1 \end{pmatrix}$
- ind. step: $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & xk \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ by ind. hyp.}$$

$$= \begin{pmatrix} 1 & x+xk \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x(k+1) \\ 0 & 1 \end{pmatrix}$$

We have shown $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & xn \\ 0 & 1 \end{pmatrix}$ $\forall n \geq 1$

thus all entries of $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n$ are $\leq xn$ $\forall n \geq 1$

$\square$

3 Binary Numbers

Prove that every positive integer n can be written in binary. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0,1\}$.

Strong induction:

- Show $P(0)$
- Assume $P(0) \wedge P(1) \wedge \dots \wedge P(k)$ is true.
- Show $P(k+1)$

Base case: $n=1$ $\quad 1 = 1 \cdot 2^0$ ✓

Inductive hypothesis (take 1): Assume for some $m \geq 1$

$m = C_k \cdot 2^k + C_{k-1} \cdot 2^{k-1} + \dots + C_1 \cdot 2^1 + C_0 \cdot 2^0$

- if m is even: $m = C_k \cdot 2^k + C_{k-1} \cdot 2^{k-1} + \dots + C_1 \cdot 2^1 + \underline{0} \cdot 2^0$

  $m+1 = C_k \cdot 2^k + C_{k-1} \cdot 2^{k-1} + \dots + C_1 \cdot 2^1 + \underline{1} \cdot 2^0$

- if m is odd: ...?
  - have that $\frac{m+1}{2}$ is an integer...

Inductive hypothesis (take 2): Assume for all m s.t. $1 \leq m \leq n$ for arbitrary n,

$m = C_k \cdot 2^k + C_{k-1} \cdot 2^{k-1} + \dots + C_1 \cdot 2^1 + C_0 \cdot 2^0$

Inductive step: (Show that $n+1$ has valid representation)

• if n is even:

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \dots + c_1 \cdot 2^1 + \underline{0} \cdot 2^0$$

$$n+1 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \dots + c_1 \cdot 2^1 + \underline{1} \cdot 2^0$$

• if n is odd: $n+1 = 2\left(\frac{n+1}{2}\right)$

$$\frac{n+1}{2} = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \dots + c_1 \cdot 2^1 + c_0 \cdot 2^0$$

by inductive hypothesis

$$n+1 = 2\left(\frac{n+1}{2}\right) = c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \dots + c_1 \cdot 2^2 + c_0 \cdot 2^1 + 0 \cdot 2^0$$

Thus, by induction $\forall n \geq 1$, n has a binary representation

$\square$