1 Let's Talk Probability

(a) When is $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ true? What is the general expression for $\mathbb{P}(A \cup B)$ that is always true.

(b) When is $\mathbb{P}(A \cap B) = \mathbb{P}(A) * \mathbb{P}(B)$ true? What is the general expression for $\mathbb{P}(A \cap B)$ that is always true.

(c) If A and B are disjoint, does that imply they're independent?

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ by inclusion-exclusion.

when A, B are disjoint, $P(A \cap B) = 0$ so

$P(A \cup B) = P(A) + P(B)$.

b) $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

$P(A \cap B) = P(A)P(B)$ when A, B are independent.

c) No. consider one roll of a fair 6-sided die. let A be the event we roll 1, B be the event we roll 2. $P(A \cap B) = 0$, $P(A) = P(B) = \frac{1}{6}$.

we've shown $P(A \cap B) \neq P(A)P(B)$.

2 Balls and Bins

union bound: $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

Throw n balls into n labeled bins one at a time.

(a) What is the probability that the first bin is empty?

P(1st is empty) = P(all balls don't land in 1st bin)

$$= \left(\frac{n-1}{n}\right)^n$$

(b) What is the probability that the first k bins are empty?

P(1st k bins empty) = P(all balls don't land in 1st k bins)

$$= \left(\frac{n-k}{n}\right)^n$$

(c) Let A be the event that at least k bins are empty. Notice that there are $m = \binom{n}{k}$ sets of k bins out of the total n bins. If we assume A_i is the event that the i^{th} set of k bins is empty. Then we can write A as the union of A_i's.

$$A = \bigcup_{i=1}^{m} A_i.$$

Write the union bound for the probability A.

$$P(A) = P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i)$$

(d) Use the union bound to give an upper bound on the probability A from part (c).

$$P(A_i) = \left(\frac{n-k}{n}\right)^n$$

$$P(A) \leq \sum_{i=1}^{m} \left(\frac{n-k}{n}\right)^n = m\left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n$$

(e) What is the probability that the second bin is empty given that the first one is empty?

$P(\text{2nd bin empty} \mid \text{1st bin empty}) =$

$$\frac{P(\text{2nd bin empty} \cap \text{1st bin empty})}{P(\text{1st bin empty})}$$

$$= \frac{\left(\frac{n-2}{n}\right)^n}{\left(\frac{n-1}{n}\right)^n}$$

$$= \left(\frac{n-2}{n-1}\right)^n$$

(f) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

No $P(\text{1st bin empty} \mid \text{1st 2 bins empty}) = 1$

(g) Are the events that "the first bin is empty" and "the second bin is empty" independent?

No $P(\text{2nd bin empty} \mid \text{1st bin empty}) \neq P(\text{2nd bin empty})$

since $\left(\frac{n-2}{n-1}\right)^n \neq \left(\frac{n-1}{n}\right)^n$

3 Pairs of Beads

Sinho has a set of $2n$ beads ($n \geq 2$) of n different colors, such that there are two beads of each color. He wants to give out pairs of beads as gifts to all the other $n-1$ TAs, and plans on keeping the final pair for himself (since he is, after all, also a TA). To do so, he first chooses two beads at random to give to the first TA he sees. Then he chooses two beads at random from those remaining to give to the second TA he sees. He continues giving each TA he sees two beads chosen at random from his remaining beads until he has seen all $n-1$ TAs, leaving him with just the two beads he plans to keep for himself. Prove that the probability that at least one of the other TAs (*not* including Sinho himself) gets two beads of the same color is at most $\frac{1}{2}$.

let A_i = event ith TA gets 2 beads of same color

P(at least 1 of the other TA's gets 2 beads of same color)

$= P(A_1 \cup A_2 \cup \dots \cup A_{n-1})$

$P\left(\bigcup_{i=1}^{n-1} A_i\right) \leq \sum_{i=1}^{n-1} P(A_i)$ by union bound

$P(A_1) = \frac{1}{2n-1}$

- pick a random bead. exactly one out of remaining $2n-1$ beads that match the color
- $\frac{n}{\binom{2n}{2}} = \frac{n}{\frac{2n!}{(2n-2)!\,2!}} = \frac{n}{\frac{\cancel{2}n(2n-1)}{\cancel{2}}} = \frac{1}{2n-1}$

Notice $P(A_i) = \frac{1}{2n-1}$ $\forall i$ since each TA has no information on what colors the other TA's got.

$$P\left(\bigcup_{i=1}^{n-1} A_i\right) \leq \sum_{i=1}^{n-1} \frac{1}{2n-1} = \frac{n-1}{2n-1} \leq \frac{n-1}{2n-2} = \frac{1}{2}$$