1 Prove or Disprove

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

a) <u>True</u>

suppose n is odd $\Rightarrow n = 2k+1$, $k \in \mathbb{N}$

$$\Rightarrow n^2 + 4n = (2k+1)^2 + 4(2k+1)$$
$$= 4k^2 + 4k + 1 + 8k + 4$$
$$= 4k^2 + 12k + 5$$
$$= 2\underbrace{(2k^2 + 6k + 2)}_{\in \mathbb{N}} + 1$$

$\Rightarrow n^2 + 4n$ is odd $\square$

b) <u>True</u>

Contrapositive: Suppose $a > 11$ and $b > 4$. Then $a + b > 15$. $\square$

c) <u>True</u>

contrapositive: suppose r is rational

$\Rightarrow r = \frac{a}{b}$ s.t. $a, b \in \mathbb{Z}$ with $b \neq 0$

$\Rightarrow r^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$; $\overset{*}{a^2}, b^2 \in \mathbb{Z}$

$\Rightarrow r^2$ is rational $\square$

d) <u>False</u>

let $n = 7$ $\quad 5 \times 7^3 = 1715$

$7! = 5040$

$5 \times 7^3 < 7!$ so $(\forall n \in \mathbb{Z}^+)\, 5n^3 > n!$ is false $\square$

* $\mathbb{Z}$ closed under addition & multiplication

i.e. $\forall a, b \in \mathbb{Z}, a + b \in \mathbb{Z}$

$\forall a, b \in \mathbb{Z}, a \cdot b \in \mathbb{Z}$

2 Pigeonhole Principle

Prove the following statement: If you put $n+1$ balls into n bins, however you want, then at least one bin must contain at least two balls. This is known as the *pigeonhole principle*.

Contrapositive : suppose every bin contains at most one ball. The maximum number of balls is n. so we have shown the negation of starting with $n+1$ balls. □

$\neg$(# of balls $= n+1$) $=$ # of balls $< n+1$ $\vee$ # of balls $> n+1$

* Every proof by contrapositive can be written as proof by contradiction.

suppose we want to prove $P \Rightarrow Q$. Proving $\neg Q \Rightarrow \neg P$ means we can start w/ $\neg Q$ and arrive at $\neg P$. so for contradiction, $P \wedge \neg Q \Rightarrow P \wedge \neg P$

$P \Rightarrow Q$ assume $P \wedge \neg Q$ ..... $\neg P$

$\neg Q \Rightarrow \neg P$

3 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

- "objects" can be the people
- "containers" can be the # of friends that each person has

- Start w/ n containers labeled $0, 1, 2, \ldots, n-1$

$$\underset{0}{\underline{\quad}}\ \ \underset{1}{\underline{\quad}}\ \ \underset{2}{\underline{\quad}}\ \ \underset{3}{\underline{\quad}}\ \ \cdots\ \ \underset{n-1}{\underline{\quad}}$$

- if we assign each person to a different container, then someone has 0 friends and someone has $n-1$ friends: doesn't work! So at least one of containers 0 and $n-1$ must be empty.

So we have n objects and at most $n-1$ containers. By the Pigeonhole Principle, at least 2 people must have the same number of friends.