Principle of Inclusion - Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

generally: $|A_1 \cup \dots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$

$$- \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l| + \dots$$

Stars and Bars

- How many ways to distribute n indistinguishable balls into m bins?

$$\binom{n+m-1}{m-1}$$

$$\underbrace{\underline{*}\ \ \underline{|}\ \ \underline{*}\ \ \underline{*}\ \ \dots\ \ \underline{*}\ \ \underline{|}\ \ \underline{*}}$$

- $n+m-1$ total slots.
  - n slots for balls
  - $m-1$ slots for $m-1$ bars, or dividers

1 The Count

(a) How many of the first 100 positive integers are divisible by 2, 3, or 5?

- let A be set of integers div. by 2
- let B " div. by 3
- let C " div. by 5

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{6} \right\rfloor - \left\lfloor \frac{100}{10} \right\rfloor - \left\lfloor \frac{100}{15} \right\rfloor + \left\lfloor \frac{100}{30} \right\rfloor$$

$$= 50 + 33 + 20 - 16 - 10 - 6 + 3$$

$$= 74$$

(b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

- The digits can repeat, all 9's come before 8's, all 8's come before 7's etc.

- 7 stars 9 bars where bars dilineate the value of the digits between them

9876543:

9976543

9 9 | | 7 | 6 | 5 | 4 | 3 | | |

9's 8's 7's 6's 5's 4's 3's 2's 1's

$$\binom{7+9}{9} = \binom{16}{9}$$

(c) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

For any choice of 7 digits, exactly one arrangement is strictly decreasing

# of choices of 7 digits = $\binom{10}{7}$

2 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity: $\binom{2n}{2} = 2\binom{n}{2} + n^2$

Story: Edward wants to chose 2 directors out of 2n candidates

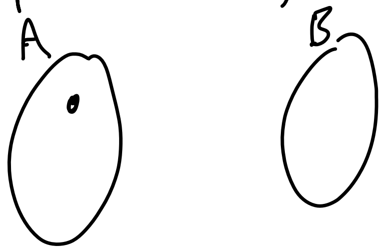
LHS: $\binom{2n}{2}$ is # of ways to choose 2 directors out of 2n candidates

RHS: split 2n candidates into 2 groups A & B

- choose 2 directors from A : $\binom{n}{2}$
- choose 2 directors from B : $\binom{n}{2}$
- chose 1 from A, 1 from B : n^2

cases are mutually exclusive & cover all possibilities, so total # = $\binom{n}{2} + \binom{n}{2} + n^2$

$= 2\binom{n}{2} + n^2$



(b) Edward would now like to select a crew out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (this is called Pascal's Identity)

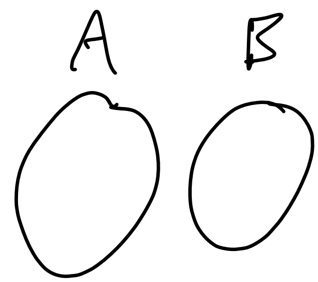
Story: select K crew members

LHS: $\binom{n}{k}$ is # of ways to select k crew members out of n candidates.

RHS: Edward looks at the first candidate he sees:

1) Edward chooses this first candidate, so still needs to choose k-1 more crew members from remaining n-1

2) Edward does not choose this first candidate, still needs to choose k more crew members from remaining n-1



cases are mutually exclusive

$\binom{n-1}{k-1} + \binom{n-1}{k}$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

Story: Edward selects some subset of the n actors, and a lead from among the subset

LHS: cast k actors and choose 1 lead among them: $\binom{n}{k}\binom{k}{1} = k\binom{n}{k}$

sum over all k: $\sum_{k=1}^{n} k\binom{n}{k}$

RHS: From n people, select 1 to be the lead. Decide if each of n-1 remaining people will be in the cast or not.

$\binom{n}{1}2^{n-1} = n2^{n-1}$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$.

Story: Eduard selects some subset of the n actors, and j leads from among the subset

LHS: cast k actors and choose j leads among them: $\binom{n}{k}\binom{k}{j}$

sum over all k, $\sum_{k=1}^{n}\binom{n}{k}\binom{k}{j}$

RHS: From n people, select j to be the leads. Decide if each of n-j remaining people will be in the cast or not.

$\binom{n}{j}2^{n-j}$

3 Bit String

How many bit strings of length 10 contain at least five consecutive 0's?

- There are 6 possible positions where the "run" of 0's begins (between 1st & 6th digit)

- run begins on 1st digit $\Rightarrow 2^5$ choices for remaining 5 digits

- run begins on ith digit $(i > 1) \Rightarrow (i-1)$th digit must be 1, remaining 4 digits chosen arbitrarily

so in total, $2^5 + 5 \cdot 2^4 = 112$

$\underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{*}\ \underline{*}\ \underline{*}\ \underline{*}$

$\uparrow$

$\underline{\ \ }\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$

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