Announcements / Reminders

- No discussion next Monday (6/28)
  - no lecture/OH
- No discussion next Tuesday for Tues-Fri discussions
- computability not in scope
- Almost 1/8 done w/ CS70!

1 Set Operations

- $\mathbb{R}$, the set of real numbers
- $\mathbb{Q}$, the set of rational numbers: $\{a/b : a, b \in \mathbb{Z} \wedge b \neq 0\}$
- $\mathbb{Z}$, the set of integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{N}$, the set of natural numbers: $\{0, 1, 2, 3, \ldots\}$

- Power set of A is set of all subsets of A
- $x \in A \cup B$ means $x \in A$ or $x \in B$
- $x \in A \cap B$ means $x \in A$ and $x \in B$
- $x \in A \setminus B$ means $x \in A$ and $x \notin B$
- $A \subseteq B$ means if $x \in A \Rightarrow x \in B$

(a) Given a set $A = \{1, 2, 3, 4\}$, what is $\mathscr{P}(A)$ (Power Set)?

(b) Given a generic set B, how do you describe $\mathscr{P}(B)$ using set comprehension notation? (Set Comprehension is $\{x \mid x \in A\}$.)

(c) What is $\mathbb{R} \cap \mathscr{P}(A)$?

(d) What is $\mathbb{R} \cap \mathbb{Z}$?

(e) What is $\mathbb{N} \cup \mathbb{Q}$?

(f) What kind of numbers are in $\mathbb{R} \setminus \mathbb{Q}$?

(g) If $S \subseteq T$, what is $S \setminus T$?

a) $\mathscr{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$

b) $\mathscr{P}(B) = \{T \mid T \subseteq B\}$

c) $\emptyset$ (empty set) since no element of $\mathbb{R}$ can be in $\mathscr{P}(A)$ i.e. no real number is a set. No element of $\mathscr{P}(A)$ can be in $\mathbb{R}$ i.e. no set is a real number.

d) $\mathbb{R} \cap \mathbb{Z} = \{x \mid x \in \mathbb{R}, x \in \mathbb{Z}\}$. All integers are real numbers, but not all real numbers are integers.

So $\mathbb{Z} \subseteq \mathbb{R}$ so $\mathbb{R} \cap \mathbb{Z} = \mathbb{Z}$

e) $\mathbb{N} \cup \mathbb{Q} = \{x \mid x \in \mathbb{N} \text{ or } x \in \mathbb{Q}\}$

All natural numbers are rational numbers $(n = \frac{n}{1} \ \forall n)$

so $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$

f) $\mathbb{R} \setminus \mathbb{Q} = \{x \mid x \in \mathbb{R}, x \notin \mathbb{Q}\}$

irrational numbers

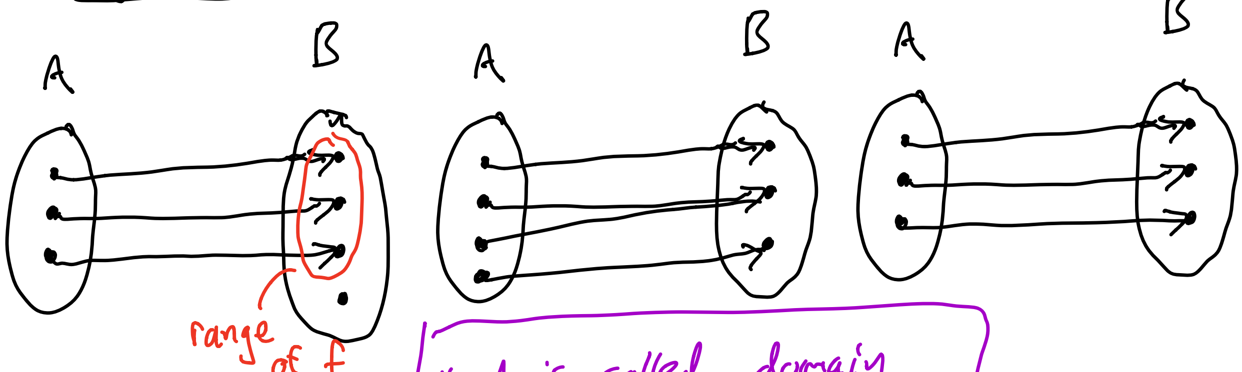
g) $S \subseteq T$ so $S \setminus T = \{x \mid x \in S, x \notin T\}$

$= \emptyset$ empty set

A function $f : A \to B$ is

- **injective**: f maps distinct inputs to distinct outputs
  (one to one)
- **surjective**: for every element $y \in B$, there is some element $x \in A$ s.t. $f(x) = y$.
  (onto)
- **Bijective**: f is injective and surjective.

Injective — Surjective — Bijective



* A is called domain
  B is called codomain

2 Bijections

Consider the function

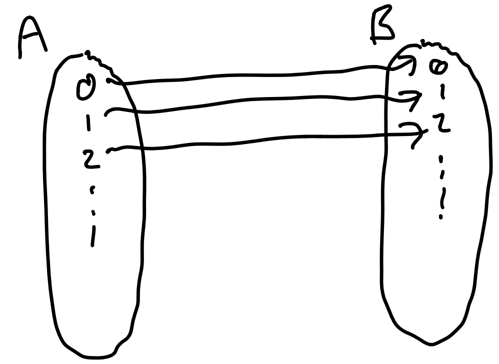
$$f(x) = \begin{cases} x, & \text{if } x \geq 1; \\ x^2, & \text{if } -1 \leq x < 1; \\ 2x+3, & \text{if } x < -1. \end{cases}$$

(a) If the domain and ~~range~~ codomain of f are $\mathbb{N}$, is f injective (one-to-one), surjective (onto), bijective?

(b) If the domain and range of f are $\mathbb{Z}$, is f injective (one-to-one), surjective (onto), bijective?

(c) If the domain and range of f are $\mathbb{R}$, is f injective (one-to-one), surjective (onto), bijective?

a) $\forall x \in \mathbb{N} \quad f(x) = x$



So f is injective, surjective thus bijective as well.

b) f is not injective since $f(1) = f(-1) = 1$

f is not surjective since there is no element in domain that maps to -2

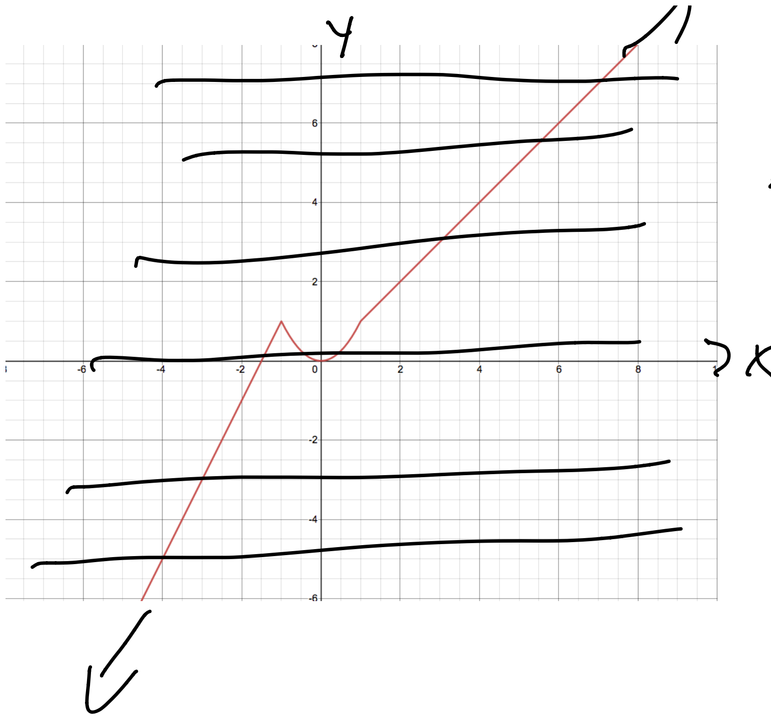
(in general negative integers that are even have no pre-image)

f is not bijective

c) f is not injective since $f(1) = f(-1) = 1$

f is surjective, every value can be attained!

f not bijective



*every y is "hit" by at least one x

3 Unions and Intersections

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $A \cap B$, where A is countable, and B is uncountable

(b) $A \cup B$, where A is countable, and B is uncountable

(c) $\bigcap_{i \in A} S_i$ where A is a countable set of indices and each S_i is an uncountable set.

a) $A \cap B \subseteq A$ & since A is contable this means $A \cap B$ is always countable.

b) $A \cup B$ is a superset of B, i.e. B is a subset of $A \cup B$. B is uncontable so $A \cup B$ is always uncountable.

c) $\bigcap_{i \in A} S_i$ is sometimes contable, sometimes uncontable:

contable: let $A = \mathbb{N}$, $S_i = \{x \mid x \in \mathbb{R}, i \leq x \, i+1\}$

so $\bigcap_{i \in A} S_i$ looks like

$[0,1] \cap [1,2] \cap [2,3] \ldots$

$\bigcap_{i \in A} S_i = \emptyset$ (empty set) which is contable.

uncountable: All S_i are identical

let $A = \mathbb{N}$, $S_i = \mathbb{R}$ for all i

$\bigcap_{i \in A} S_i = \mathbb{R}$ $\square$