Geometric Distribution: "number of trials until the 1st success" * trials each independently Bernoulli(p)

$X \sim geom(p)$

$P(X=k) = (1-p)^{k-1} p$

$E[X] = \frac{1}{p}$ $Var(X) = \frac{(1-p)}{p^2}$

ex. flip biased coin w/ heads probability 1/3 until see heads.

$X \sim geom(\frac{1}{3})$ $E[X] = 3$ $P(X=10) = (1-\frac{1}{3})^9(\frac{1}{3})$

Poisson Distribution counts the number of occurrences of some rare event in a time period

e.g. # of bus arrivals in an hour

$X \sim pois(\lambda)$ λ is "rate parameter"

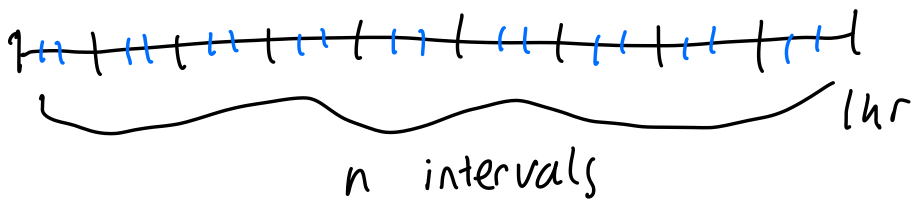
e.g. λ = average # of bus arrivals per hour

$$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$E[X] = \lambda$ $Var(X) = \lambda$

where does Poisson come from?

model the # of bus arrivals in an hour where buses come at a rate of λ buses/hr



suppose probability a bus arrives during an interval is $\frac{\lambda}{n}$, assume independently of other intervals.

let $X \sim Bin(n, \frac{\lambda}{n})$ be # of buses that arrive.

$$P(X=k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\cancel{n(n-1)\dots(n-k+1)}}{k!}\left(\frac{\lambda}{\cancel{n}}\right)^k \overbrace{\left(1-\frac{\lambda}{n}\right)^n}^{\to e^{-\lambda}}\left(1-\cancel{\frac{\lambda}{n}}\right)^{-k}$$

$$= \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{as } n \to \infty$$

1 Warm-up

For each of the following parts, you may leave your answer as an expression.

(a) You throw darts at a board until you hit the center area. Assume that the throws are i.i.d. and the probability of hitting the center area is $p = 0.17$. What is the probability that you hit the center on your eighth throw?

(b) Let $X \sim \text{Geometric}(0.2)$. Calculate the expectation and variance of X.

(c) Suppose the accidents occurring weekly on a particular stretch of a highway is Poisson distributed with average number of accidents equal to 3 cars per week. Calculate the probability that there is at least one accident this week.

(d) Consider an experiment that consists of counting the number of α particles given off in a one-second interval by one gram of radioactive material. If we know from past experience that, on average, 3.2 such α-particles are given off per second, what is a good approximation to the probability that no more that 2 α-particles will appear in a second?

a) let $X \sim geom(0.17)$ be the # of throws until hitting the center

$P(X=8) = (1-0.17)^7 (0.17) \approx 0.0461$

b) $E[X] = \frac{1}{p} = \frac{1}{0.2} = 5$

$Var(X) = \frac{(1-p)}{p^2} = \frac{0.8}{0.2^2} = 20$

c) let $X \sim pois(3)$ be the # of weekly accidents on the stretch of highway.

$P(X \geq 1) = 1 - P(X=0) = 1 - e^{-3}$

d) let $X \sim pois(3.2)$ be the # of α-particles given off in a second.

$$P(X \le 2) = \sum_{k=0}^{2} P(X=k)$$

$$= \sum_{k=0}^{2} \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-3.2} + 3.2e^{-3.2} + \frac{3.2^2 e^{-3.2}}{2}$$

2 Coupon Collector Variance

It's that time of the year again - Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\operatorname{Var}(X) = n^2 \left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}(X)$. *[Hint: Try to break this problem down using indicators as with the coupon collector's problem. Are the indicators independent?]*

let X_i be # of visits made before we have collected the ith unique card given that we have already collected $i-1$ unique cards.

$X_i \sim \text{geom}\left(\frac{n-i+1}{n}\right)$

X_i's are independent b/c each time we collect another unique card, we are "starting over" in trying to collect the next unique card.

$$\operatorname{Var}(X) = \operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i)$$

$$= \sum_{i=1}^{n} \frac{1-(n-i+1)/n}{[(n-i+1)/n]^2} \quad \text{let } j = n-i+1$$

$$= \sum_{j=1}^{n} \frac{1 - j/n}{(j/n)^2}$$

$$= \sum_{j=1}^{n} \frac{n(n-j)}{j^2}$$

$= E[X]$

$$= \sum_{j=1}^{n} \frac{n^2}{j^2} - \sum_{j=1}^{n} \frac{n}{j}$$

$$= n^2 \left(\sum_{j=1}^{n} \frac{1}{j^2} \right) - E[X]$$