CS 70 Midterm Review Session: Graphs

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1 Warm Up

a) T/F: Any graph where every triple of vertices is a triangle, i.e. $\forall u, v, w \in V, (u, v), (u, w), (v, w) \in E$, is a complete graph.

b) T/F: A graph with k edges and n vertices has a vertex of degree at least $\frac{2k}{n}$.

c) T/F: Any complete graph has a Hamiltonian cycle. (A Hamiltonian cycle is a cycle that visits every vertex exactly once)

d) T/F: Any graph with n vertices and 2n edges is connected.

e) What is the maximum number of edges you could have in a bipartite graph with 2n vertices?

f) Does a graph G with n vertices such that every vertex except 2 has even degree have an Eulerian Tour? Does G have an Eulerian Walk?

2 Short Answer

a) Consider a planar graph G where each face is incident to exactly 5 edges. Derive an expression for the number of edges in G, e, in terms of the number of vertices, v in G.

b) Consider a bipartite graph G with vertex set $V = L \cup R$ and edges across L and R. What is the sum of degrees of vertices in L in terms of $|L|, |R|$, and $|E|$?

3 Proofs

a) Prove that a graph where every vertex has degree at least 2 has a cycle.

b) Prove that a connected graph with n vertices has at least $n - 1$ edges.

c) Prove that every tree is bipartite.

d) Prove that every connected, undirected graph has a tour that uses each edge at least once and at most twice.

e) Prove that if G is a graph with n vertices such that for any two non-adjacent vertices u and v, it holds that deg u + deg $v \geq n - 1$, then G is connected.

f) (Fall 2017 MT1) Consider a directed graph where every pair of vertices u and v are connected by a single directed arc either from u to v or from v to u. Show that every vertex has a directed path of length at most two to the vertex with maximum in-degree.

4 Misc. Questions

a) Suppose there are 29 people in a room and among every three people at least two of them know each other. Prove that someone knows at least 14 people.

b) Suppose an odd number of soldiers are stationed in a field, in such a way that all the pair-wise distances are distinct. Each soldier is told to keep an eye on the nearest other soldier. Prove that at least one soldier is not being watched.