Conditional Probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Bayes Rule $P(A|B) = \dfrac{P(B|A)\,P(A)}{P(B)}$

Total Probability Rule

$P(A) = P(A \cap B) + P(A \cap \bar{B})$

$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$ where $B_1, \dots, B_n$ partition Ω

Independence A & B are independent:

$P(A|B) = P(A)$

$P(B|A) = P(B)$

$P(A \cap B) = P(A)\,P(B)$

1 Rain and Wind

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in EECS 70, you are curious to play around with these numbers. Find the probability that:

(a) A given day is both windy and rainy.

$P(\text{rainy} \mid \text{windy}) = 0.3 \quad P(\text{rainy} \mid \text{non-windy}) = 0.8$

$P(\text{windy}) = 0.2$

$$P(\text{rainy} \cap \text{windy}) = P(\text{rainy} \mid \text{windy})\, P(\text{windy})$$
$$= 0.3(0.2)$$
$$= 0.06$$

(b) A given day is rainy.

$$P(\text{rainy}) = P(\text{windy} \cap \text{rainy}) + P(\text{non-windy} \cap \text{rainy})$$
$$= 0.06 + P(\text{rainy} \mid \text{non-windy})\, P(\text{non-windy})$$
$$= 0.06 + 0.8(0.8)$$
$$= 0.06 + 0.64$$
$$= 0.7$$

(c) For a given pair of days, exactly one of the two days is rainy. (You may assume that the weather on the first day does not affect the weather on the second.)

let R_1 be event that it rained on day 1

let R_2 " day 2

$P(R_1) = P(R_2) = 0.7$

want $P(R_1 \cap \overline{R_2}) + P(\overline{R_1} \cap R_2)$

$= P(R_1)P(\overline{R_2}) + P(\overline{R_1})P(R_2)$

$= 0.7(0.3) + 0.3(0.7)$

$= 0.42$

2 Poisoned Smarties

Supposed there are 3 men who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the men, produces considerably more Smarties than his competitors and has a commanding 45% of the market share. Yousef See, who inherited his riches, lags behind Burr and produces 35% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through his investigations, the inspector found that one Smarty out of every 100 at Kelly's factory was poisonous. At See's factory, 1.5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.02.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

let S be event a Smarty is safe to eat

let BK be event a Smarty is from Burr Kelly's factory

let YS " Yousef See

let SF " Stan Furd

$$P(S) = P(S \cap BK) + P(S \cap YS) + P(S \cap SF)$$

$$= P(S|BK)P(BK) + P(S|YS)P(YS) + P(S|SF)P(SF)$$

$$P(\bar{S} \mid BK) = \frac{1}{100} \Rightarrow P(S|BK) = 0.99$$

$$= 0.99(0.45) + 0.985(0.35) + 0.98(0.2)$$

$$= 0.98625$$

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

let P is event Smarty is poisonous

$$P(P \mid \overline{BK}) = \frac{P(P \cap \overline{BK})}{P(\overline{BK})}$$

$$= \frac{P(P \cap SF) + P(P \cap YS)}{P(\overline{BK})}$$

$$= \frac{P(P \mid SF)P(SF) + P(P \mid YS)P(YS)}{P(\overline{BK})}$$

$$= \frac{0.02(0.2) + 0.015(0.35)}{0.55}$$

$$= 0.0168$$

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

$$P(SF|P) = \frac{P(P|SF)P(SF)}{P(P)}$$

$$= \frac{P(P|SF)P(SF)}{1-P(S)}$$

$$= \frac{0.02(0.2)}{1-0.98625}$$

$$= 0.29$$

3 Bag of Coins

Your friend Forrest has a bag of n coins. You know that k are biased with probability p (i.e. these coins have probability p of being heads). Let F be the event that Forrest picks a fair coin, and let B be the event that Forrest picks a biased coin. Forrest draws three coins from the bag, but he does not know which are biased and which are fair.

(a) What is the probability of three coins being pulled in the order FFB?

$$P(FFB) = \left(\frac{n-k}{n}\right)\left(\frac{n-k-1}{n-1}\right)\left(\frac{k}{n-2}\right)$$

BB BF FB FF

(b) What is the probability that the third coin he draws is biased?

$\frac{k}{n}$ by symmetry. In general, probability that the ith coin is biased is the same for all coins, given no other information about other coins.

(c) What is the probability of picking at least two fair coins?

$$P(\text{at least 2 fair coins}) = P(\text{exactly 2 fair coins}) + P(\text{exactly 3 fair coins})$$

$$\underline{P(FFB) = P(FBF) = P(BFF)}$$

$$P(\text{exactly 2 fair coins}) = (3)\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}$$

$$P(\text{exactly 3 fair coins}) = \left(\frac{n-k}{n}\right)\left(\frac{n-k-1}{n-1}\right)\left(\frac{n-k-2}{n-2}\right)$$

$$= \frac{(n-k)!\,(n-3)!}{n!\,(n-k-3)!}$$

$$P(\text{at least 2 fair coins}) =$$

$$\boxed{(3)\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)} + \frac{(n-k)!\,(n-3)!}{n!\,(n-k-3)!}}$$

(d) Given that Forrest flips the second coin and sees heads, what is the probability that this coin is biased?

let H be event Forrest sees heads

$$P(B|H) = \frac{P(H|B)P(B)}{P(H)} \quad P(H|B) = p, \; P(B) = k/n$$

$$P(H) = P(H|B)P(B) + P(H|F)P(F)$$

$$= p\frac{k}{n} + \frac{1}{2}\left(\frac{n-k}{n}\right)$$

$$= \frac{2pk + n - k}{2n}$$

$$P(B|H) = \frac{p \cdot (k/n)}{(2pk+n-k)/2n} = \boxed{\frac{2pk}{2pk + n - k}}$$