CS 70 Discrete Mathematics and Probability Theory
Summer 2021

Discussion 07D

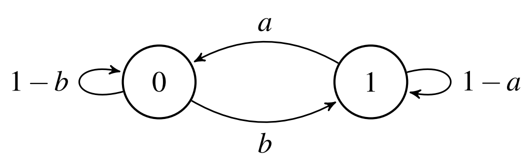
1 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.
2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathscr{X}$. If $d(i) = 1 \; \forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i,j) with probability $P(i,j)$.
4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.

$\pi_1 = \pi_0 P$

$\pi_n = \pi_{n-1} P = \pi_0 P^n$



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

The Markov chain is irreducible if a,b are both non-zero

reducible if at least one of a or b is zero

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

$d(0) = \gcd\{2, 4, 6, 8 \ldots\} = 2$

$d(1) = d(0) = 2$

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

$d(0) = \gcd\{1, 2, 3, \dots\} = 1$

$d(1) = d(0) = 1$

(d) Construct a transition probability matrix using the above Markov chain.

$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

$\pi = \pi P$

$\pi^T = (\pi P)^T = P^T \pi^T$

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

$\pi = \pi P$

$$\begin{bmatrix} \pi(0) & \pi(1) \end{bmatrix} = \begin{bmatrix} \pi(0) & \pi(1) \end{bmatrix} \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

$$\left.\begin{array}{l} \pi(0) = (1-b)\pi(0) + a\pi(1) \\ \pi(1) = b\pi(0) + (1-a)\pi(1) \end{array}\right\} \quad b\pi(0) = a\pi(1)$$

$\pi(0) + \pi(1) = 1$

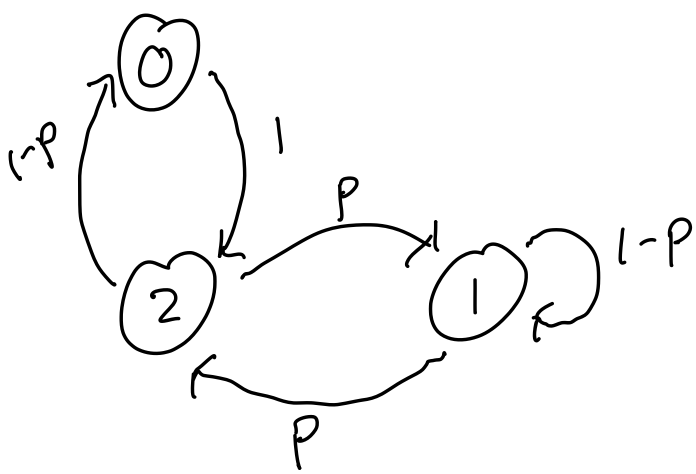
$$\boxed{\pi = \frac{1}{a+b}[a \quad b]}$$

2 Allen's Umbrella Setup

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix.

let states be $\mathcal{X} = \{0, 1, 2\}$ denoting the # of umbrellas Allen has at current location



$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

(b) What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

Transition matrix are P^2 and P^n

Notice the Markov Chain is irreducible and aperiodic, so it converges to its unique stationary distribution (regardless of initial distribution)

$$[\pi(0)\ \pi(1)\ \pi(2)] = [\pi(0)\ \pi(1)\ \pi(2)] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

$\pi(0) = (1-p)\pi(2)$

$\pi(1) = (1-p)\pi(1) + p\pi(2) \rightarrow \pi(1) = \pi(2)$

$\pi(2) = \pi(0) + p\pi(1)$ $\qquad \cancel{p}\pi(1) = \cancel{p}\pi(2)$

$\pi(0) + \pi(1) + \pi(2) = 1$

$(1-p)\pi(2) + 2\pi(2) = 1$

$$\pi(2) = \frac{1}{3-p}$$

$$\pi = [\pi(0) \quad \pi(1) \quad \pi(2)] = \frac{1}{3-p}[1-p \quad 1 \quad 1]$$

long term fraction of time for no umbrella

is $\pi(0) = \dfrac{1-p}{3-p}$

long term fraction of time its raining and

no umbrella is $\dfrac{p(1-p)}{3-p}$