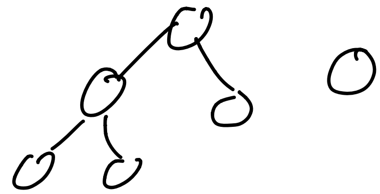
- degree of vertex is # of incident edges
- connected component is a group of vertices that all have a path to each other
- tree is
  - connected & acyclic graph
  - has $|V|-1$ edges

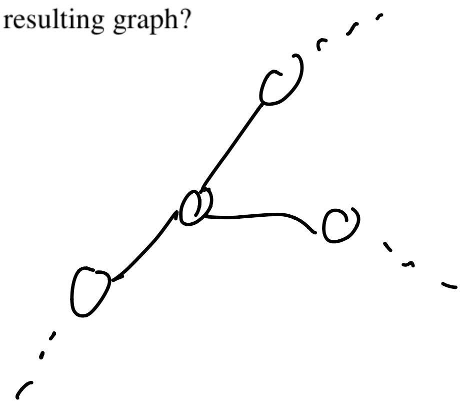


1 Short Answers - Graphs

(a) Bob removed a degree 3 node from an n-vertex tree. How many connected components are there in the resulting graph?

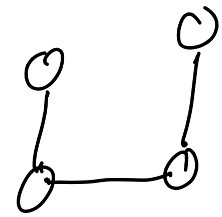
(b) Given an n-vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph?

a) 3



- must be ≥ 3 connected components since each neighbor of the removed vertex must be in a different component
- must be ≤ 3 since the original graph was connected so every vertex had a path to the removed vertex. Each of these paths must go through a neighbor of removed vertex

b)



- n-vertex tree has $n-1$ edges
- now have $n-1+10-5 = n+4$ edges
- let n_1, n_2, n_3 be # of vertices in each component
  - want to have n_1-1, n_2-1, n_3-1 edges per component

  i.e. want $n_1-1+n_2-1+n_3-1 = n_1+n_2+n_3-3$ total edges left

$$n+4-(\underbrace{n_1+n_2+n_3}_{n}-3) = n+4-(n-3)$$

$$= \boxed{7}$$

Euler's formula: $v+f=e+2$ for any connected planar graph

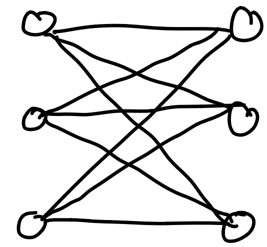
* $e > 3v-6 \Rightarrow$ non-planar

Kuratowski's theorem: graph is non-planar iff it contains $K_{3,3}$ or K_5 (for part b)

2 Planarity

(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.



a) $K_{3,3}$ has $v=6$, $e=9$ $\quad 9 > 3(6)-6 = 12$

Suppose $K_{3,3}$ is planar.

- $6+f=9+2$, $f=5$
- each face must be bounded by ≥ 4 edges (if it weren't the case, we would have an edge b/w 2 vertices in the same set, contradiction)
- count # of face-edge adjacencies 2 ways:
  - must be $\geq 4f$
  - must be $\leq 2e$

of "sides" that a face has, summed over all faces

have $4f \leq 2e \Rightarrow 4(5) \leq 2(9)$

$\Rightarrow 20 \leq 18$ contradiction $\square$

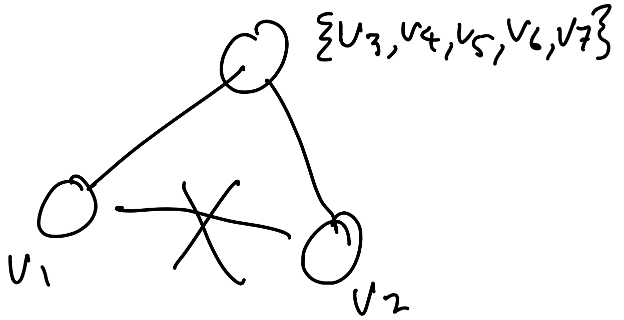
b) suppose G is planar & has 7 vertices. Select any 5 vertices out of the 7, the subgraph formed by these 5 vertices can't form K_5, so have a pair of vertices that must not have an edge b/w them. (v_1, v_2). Remaining 5 vertices also can't form K_5, so another pair (v_3, v_4) must not have an edge b/w them. let v_5, v_6, v_7 be 3 vertices left.

- $\{v_1, v\}$, $\{v_2, v\}$ are edges for $v \in \{v_3, v_4, v_5, v_6, v_7\}$
- $\{v_3, v\}$, $\{v_4, v\}$ are edges for $v \in \{v_1, v_2, v_5, v_6, v_7\}$

now let $C = \{v_1, v_2, v_3\}$, $R = \{v_5, v_6, v_7\}$

This subgraph must contain $K_{3,3}$, contradiction. □

* Any graph w/ ≥ 7 vertices & property T will also be non-planar since it will have a subgraph w/ 7 vertices that has property T.



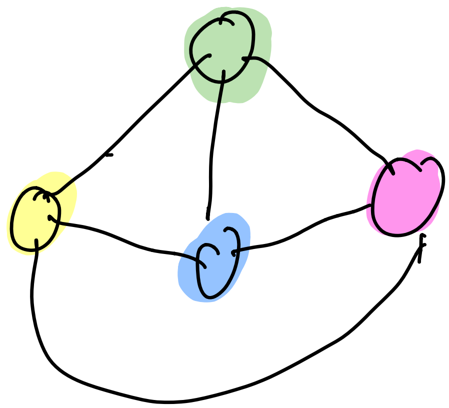
3 Graph Coloring

Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

induction on # of vertices, n

Base case: $n=1$, 1-vertex graph has max degree 0 & is 1-colorable ✓

Inductive Step: Assume statement holds for some n vertex graph w/ max deg. at most k. let G be a $(n+1)$-vertex graph w/ max deg at most k. Remove a vertex v from G, have a n-vertex graph H w/ max deg at most k. So H is $k+1$-colorable by the inductive hypothesis. Now add back v & assign one of $k+1$ colors to v so that v is a different color than all its neighbors. This is possible since there are at most k neighbors of v, and $k+1$ total colors. Thus G is $k+1$ colorable, & any graph w/ max deg at most k is $(k+1)$-colorable by induction $\square$



max deg at most 3

colorable w/ 4 colors