Exponential Distribution

$X \sim exp(\lambda)$ λ: "success rate per unit time"
e.g. a bus comes twice every hour

$P(X > t) = e^{-\lambda t} \Rightarrow F_X(x) = 1 - e^{-\lambda t}$, $f_X(x) = \lambda e^{-\lambda t}$

$E[X] = \frac{1}{\lambda}$ $Var(X) = \frac{1}{\lambda^2}$

memoryless property

$P(X > k + t \mid X > t) = P(X > k)$

"if I already waited at the bus stop for 10 minutes, the probability it takes at least 20 more minutes for a bus to come is the same as if I had started waiting just now"

1 Exponential Distributions: Lightbulbs

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

let $X \sim \exp(1/50)$ be the time until the lightbulb is broken.

$$P(X < 30) = F_X(30) = 1 - e^{-30/50} = 1 - e^{-3/5} \approx 0.451$$

$$\text{"} \int_{-\infty}^{30} f_X(x)\,dx$$

$$P(30 < X < 50) = \int_{30}^{50} f_X(x)\,dx$$

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probability that the new bulb will last at least 30 days?

let $Y \sim \exp(1/50)$ be the time until the new lightbulb is broken.

$$P(Y \geq 30) = 1 - F_Y(30) = e^{-30/50} = e^{-3/5}$$

$$\approx 0.549$$

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

$$P(X > 30+30 \mid X > 30) = P(X > 60 \mid X > 30)$$

$$= P(X > 30)$$

$$= e^{-3/5}$$

$$\approx 0.549$$

*

$P(X \leq k) = P(X < k)$ for continuous distributions

2 Darts Again

Edward and Khalil are playing darts.

Edward's throws are uniformily distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim; the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $1/2$.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}(X < Y)$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0, 10]$. Solve for the distribution of X by first computing the CDF of X, $\mathbb{P}(X < x)$.]

$Y \sim \exp(1/2)$

$$F_X(x) = P(X < x) = \frac{\pi x^2}{\pi 10^2} = \frac{x^2}{100} \quad \text{for } x \in [0, 10]$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{x}{50}$$

$$P(X < Y) = \int_0^\infty P(X < Y \mid Y = y)\, f_Y(y)\, dy$$

$$= \int_0^{10} F_X(y)\, f_Y(y)\, dy + \int_{10}^\infty 1 \cdot f_Y(y)\, dy$$

$$= \int_0^{10} \frac{y^2}{200} \cdot e^{-y/2}\, dy + \int_{10}^\infty \frac{1}{2} e^{-y/2}\, dy$$

$$\approx 0.0767\ldots$$

$$P(Y>X) = \int_0^{10} P(Y>X \mid X=x)\, f_X(x)\, dx$$

$$= \int_0^{10} P(Y>x)\, f_X(x)\, dx$$

$$= \int_0^{10} e^{-\frac{x}{2}} \cdot \frac{x}{50}\, dx$$

$$\approx 0.0767\ldots$$

$$P(Y>X) = \sum_x P(Y>X \mid X=x)\, P(X=x)$$

$$P(k<X<k+dx) = \int_k^{k+dx} f_X(x)\, dx \approx f_X(k)\, dx$$