Joint PDF

For continuous r.v.'s X and Y, $f_{X,Y} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

satisfies

- $P(a \le X \le b, \ c \le Y \le d) = \int_a^b \int_c^d f_{X,Y}(x,y) \, dy \, dx$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy \, dx = 1$
- $f_{X,Y}(x,y) \ge 0 \quad \forall x,y \in \mathbb{R}$

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2 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \le x \le 1, 0 \le y \le 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

$$1 = \underbrace{\int_0^1 \underbrace{\int_0^2 Cxy\,dy}_{y}\,dx}_{x} = \int_0^1 \left[\frac{Cxy^2}{2}\right]_0^2 dx$$

$$= \int_0^1 2Cx\,dx$$

$$= \frac{2Cx^2}{2}\Big|_0^1$$

$$= C$$

$$\boxed{C=1}$$

$$P(X=x) = \sum_y P(X=x, Y=y)$$

(b) Find $f_X(x)$, the marginal distribution of X.

$$f(x,y) = xy$$

$$f_x(x) = \int_0^2 f(x,y)\,dy = \int_0^2 xy\,dy = \frac{xy^2}{2}\Big|_0^2$$

$$= 2x$$

(c) Find the conditional distribution of Y given $X = x$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xy}{2x} = \frac{y}{2}$$

(d) Are X and Y independent?

- yes, $f_{Y|X}(y|x)$ does not depend on x
- $f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 xy\,dx = \frac{x^2 y}{2}\Big|_0^1$

$$= \frac{y}{2}$$

$f_{Y|X}(y|x) = f_Y(y)$ ✓

$f_X(x)\,f_Y(y) = f(x,y)$

1 Uniform Distribution

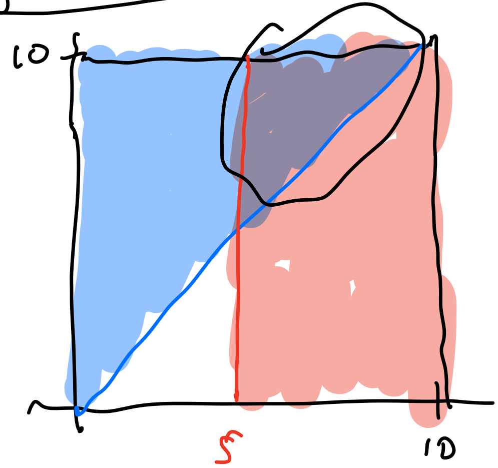
You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

$X \sim unif(0, 10)$ $\quad$ $Y \sim unif(0, 10)$

$$P(X \geq 5 \mid Y \geq X) = \frac{P(X \geq 5, Y \geq X)}{P(Y \geq X)}$$

$$P(X \geq 5, Y \geq 5) = P(5 \leq X \leq Y)$$

joint density domain



$Y \geq X$

$X \geq 5$

$$\boxed{\begin{aligned} 100 \cdot h &= 1 \\ h &= \frac{1}{100} \end{aligned}}$$

integrate over joint density = find volume of the graph of joint density

$$P(X \geq 5, Y \geq X) = \frac{25}{2} \cdot \frac{1}{100} = \frac{1}{8}$$

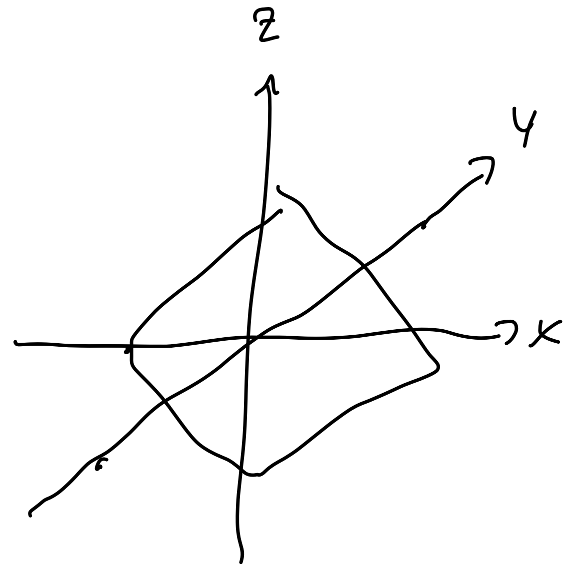
area of triangle $\quad$ height = $f(x, y)$

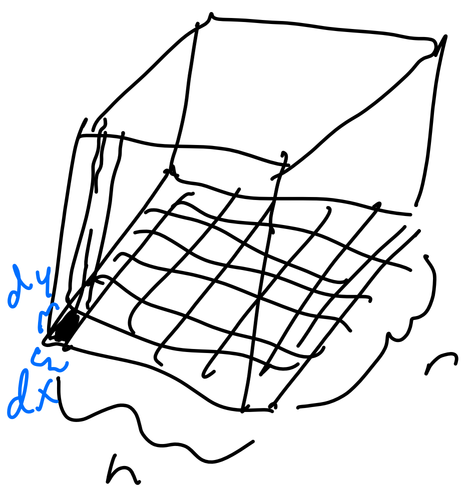
$$P(Y \geq X) = \frac{1}{2}$$

$$P(X \geq 5 \mid Y \geq X) = \frac{P(X \geq 5, Y \geq X)}{P(Y \geq X)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$$P(Y \geq X) = \int_0^{10} \int_x^{10} \underbrace{f(x,y)}_{\frac{1}{100}} \, dy \, dx$$

$$P(5 \leq X \leq Y) = \int_5^{10} \int_x^{10} f(x,y) \, dy \, dx$$





$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i, y_j)\Delta x\Delta y$$

3 Joint Distributions

(a) Give an example of discrete random variables X and Y with the property that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. You should specify the joint distribution of X and Y.

* intuition: if $X = Y$ then just need an X w/ non-zero variance

let $X = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$ let $Y = X$

$$E[X] = 1 \cdot P(X=1) + (-1)P(X=-1)$$
$$= \frac{1}{2} - \frac{1}{2}$$
$$= 0$$

$$E[X^2] = 1^2 \cdot P(X=1) + (-1)^2 P(X=-1)$$
$$= \frac{1}{2} + \frac{1}{2}$$
$$= 1$$

$E[X]E[Y] = 0$ so $E[XY] \neq E[X]E[Y]$

(b) Give an example of discrete random variables X and Y that (i) are *not independent* and (ii) have the property that $\mathbb{E}[XY] = 0$, $\mathbb{E}[X] = 0$, and $\mathbb{E}[Y] = 0$. Again you should specify the joint distribution of X and Y.

joint distribution

	$X=-1$	$X=0$	$X=1$
$Y=\frac{1}{3}$	1/3	0	1/3
$Y=-\frac{2}{3}$	0	1/3	0

$$X = \begin{cases} 1 & \text{w.p. } \frac{1}{3} \\ 0 & \text{w.p. } \frac{1}{3} \\ -1 & \text{w.p. } \frac{1}{3} \end{cases}$$

$$Y = \begin{cases} \frac{1}{3} & \text{w.p. } \frac{2}{3} \\ -\frac{2}{3} & \text{w.p. } \frac{1}{3} \end{cases}$$

$$P(X=-1, Y=\tfrac{1}{3}) \neq P(X=-1)\,P(Y=\tfrac{1}{3})$$

$$E[X] = E[Y] = 0$$

$$E[XY] = \sum_{x,y} xy\,P(X=x, Y=y) \qquad * \sum_{x,y} = \sum_x \sum_y$$

$$= (-1)(\tfrac{1}{3})(\tfrac{1}{3}) + 1(\tfrac{1}{3})(\tfrac{1}{3})$$

$$+ 0(-\tfrac{2}{3})(\tfrac{1}{3})$$

$$= 0$$