Counting

- **First rule of counting:** if you have k choices to make and there are n_1 ways of making first choice and for every way of making first choice there are n_2 ways of making second choice etc.
  - total # of choices is $n_1 \times n_2 \times \ldots \times n_k$

  (ordered sampling with replacement)

- **second rule of counting:**

  $$\text{\# of ways to choose when order doesn't matter} = \frac{\text{\# of ways to choose w/ order mattering}}{\text{\# of ordered ways per unordered way}}$$

  - example:

    # of ways to choose k items out of n...

    * without replacement, order matters: $\frac{n!}{(n-k)!}$

    * without replacement, order doesn't matter:

    $$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

1 Clothing Argument

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

- 4 choices, 10 options per choice for every choice

$10 \times 10 \times 10 \times 10 = \boxed{10^4}$

(b) How many outfits are there if we wanted to wear exactly two categories?

- first chose 2 categories, 10 options per category

$\boxed{\binom{4}{2} \cdot 10^2}$

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

$\underline{10}\ \underline{9}\ \underline{8}\ \underline{7}$ $\boxed{\frac{10!}{6!}}$

- first pick 4 hats

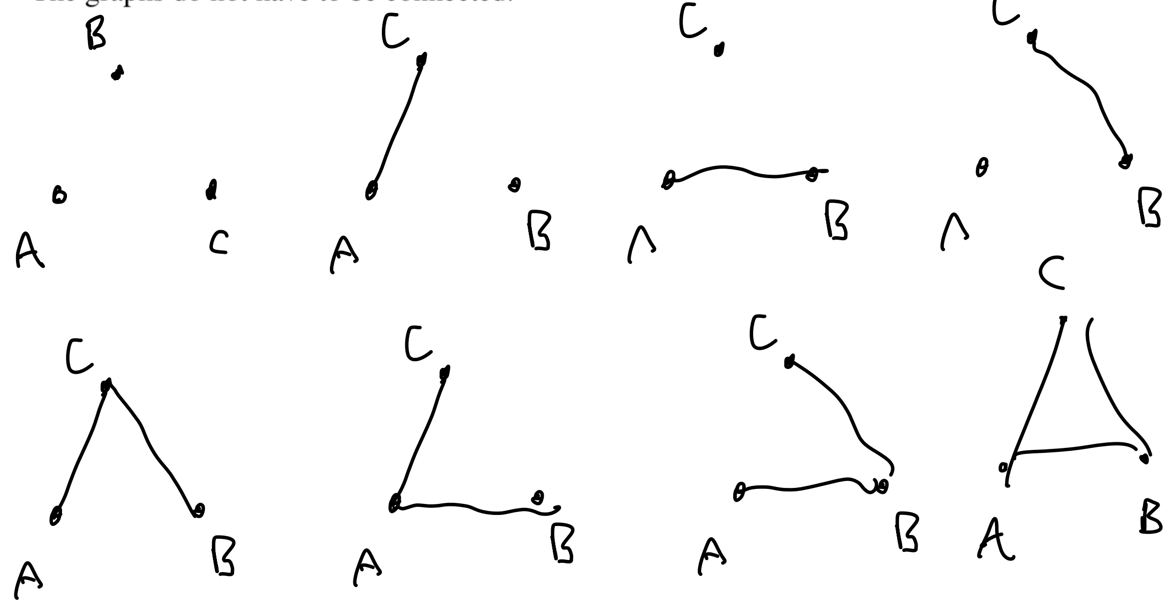
$\binom{10}{4} \cdot 4! = \frac{10!}{6!\,4!} \cdot 4! = \frac{10!}{6!}$

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

$\binom{10}{4}$ $\frac{\text{part c)}}{4!}$

2 Counting on Graphs + Symmetry

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.



There are $\binom{n}{2} = \frac{n(n-1)}{2}$ possible edges

· each edge can either appear or not

$$\boxed{2^{\binom{n}{2}}}$$

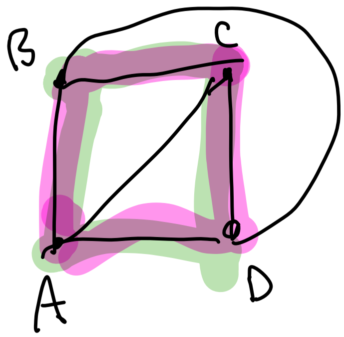
∗ bijection from $\binom{n}{2}$-length bitstrings → graphs on n labeled vertices

$\underline{1}\ \underline{0}\ \underline{1}\ \ldots\ \underline{1}$ length $\binom{n}{2}$

- ith bit is 1 if the ith edge appears in graph
- " 0 otherwise

(b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

- at least 3 vertices, at most n vertices in a cycle



- any permutation of k vertices will result in valid cycle of length k

- so $\frac{n!}{(n-k)!}$ cycles of length $k \Rightarrow \sum_{k=3}^{n} \frac{n!}{(n-k)!}$ cycles?

ABCDA
ADCBA
ABDCA
ACDBA
ACBDA
ADBCA

$\Big]$ Inversions

ABCDA = BCDAB = CDABC = DABCD

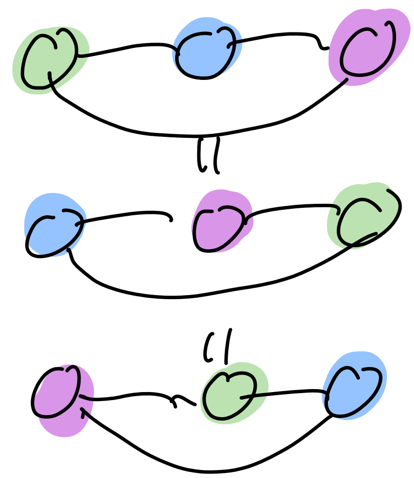
- for every distinct cycle of length k there's $2k$ permutations that look the same to us
  - $2k$ ordered ways for every unordered way!

So: $\boxed{\sum_{k=3}^{n} \frac{n!}{(n-k)!\,2k}}$

(c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

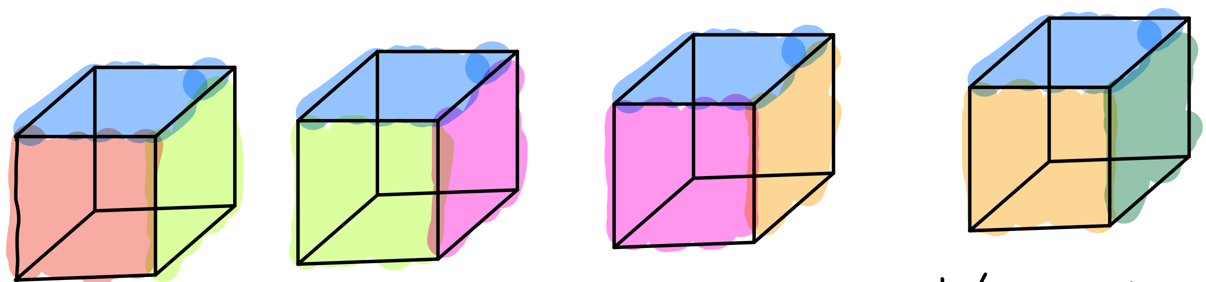
- at first think: $n!$
- accounting for rotations:

$$\frac{n!}{n} = \boxed{(n-1)!}$$



(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

- at first think: $6!$



- rotations w/ top side fixed: 4 permutations all the same coloring
- 6 possible colors for the top side

$$\frac{6!}{4 \cdot 6} = \boxed{\frac{6!}{24}}$$

let 1 = top, 2 = front, 3 = right, 4 = back, 5 = left, 6 = bottom sides of the cube

(think about assigning colors to each fixed side of the cube. Then, there are 24 assignments that are actually the same for every physical coloring of the cube.)

<u>For this one specific coloring of physical cube:</u>

Blue on top:

1	2	3	4	5	6
B	R	G	P	O	Y

=

1	2	3	4	5	6
B	G	P	O	R	Y

=

1	2	3	4	5	6
B	P	O	R	G	Y

=

1	2	3	4	5	6
B	O	R	G	P	Y

Green on top:

1	2	3	4	5	6
G	R	Y	P	B	O

=

1	2	3	4	5	6
G	Y	P	B	R	O

=

1	2	3	4	5	6
G	P	B	R	Y	O

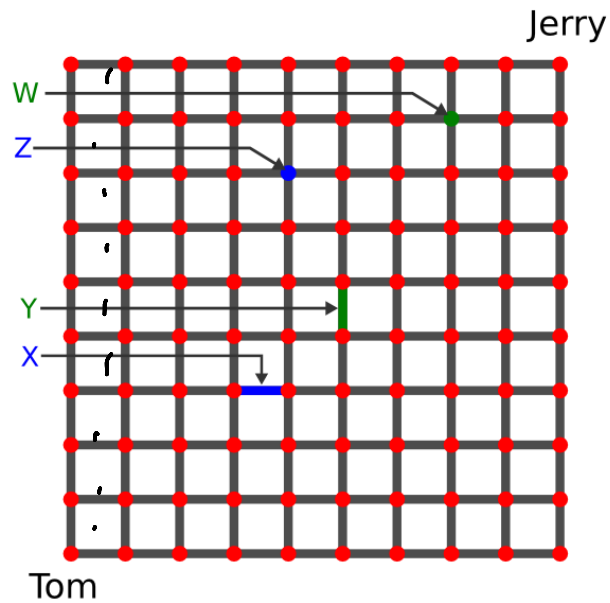
=

1	2	3	4	5	6
G	B	R	Y	P	O

can also have orange, red, yellow, or purple on top!

3 Maze

Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.



(a) How many such shortest paths exist?

see solutions on website

(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

see solutions on website