## Probability Density Function (pdf)

- non-negative, i.e. $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
- $P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$
- $f_X(x)$ is not a probability!
  - gives probabilities of intervals when integrated

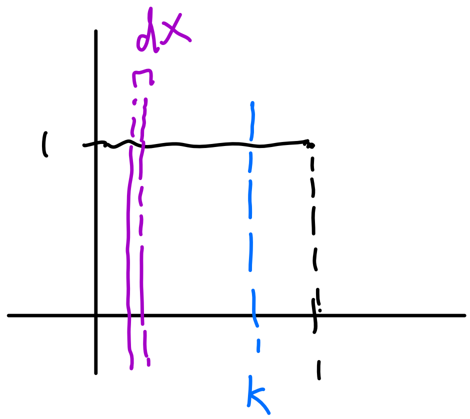
## Cumulative Density Function (cdf)

- $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx$
- $f_X(x) = \dfrac{d}{dx} F_X(x)$ * often find cdf first, then pdf

ex) $X \sim U(0,1)$

$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

$\underline{f_X(x)}$




$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \in [0,1] \\ 1 & \text{if } x > 1 \end{cases}$$

$$P(k \le X \le k+dx) = \int_k^{k+dx} f_X(x)\,dx \approx f_X(k)\,dx$$

# 1 Max of Uniforms

Let $X_1, \ldots X_n$ be independent $U[0,1]$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of $n$.

(a) What is the cdf of $X$?

(b) What is the pdf of $X$?

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

a) $F_X(x) = P(X \le x)$

$$P(X \le x) = P(\max(X_1, \ldots, X_n) \le x)$$

$$= P(X_1 \le x, \ldots, X_n \le x)$$

$$= P(X_1 \le x) \ldots P(X_n \le x)$$

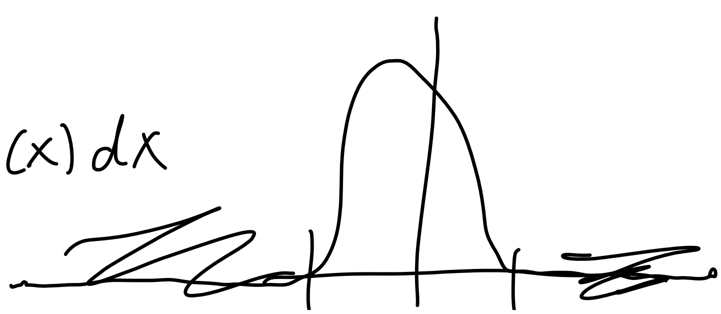
$$= x^n \quad \text{for } x \in [0,1]$$

$F_{X_1}(x) F_{X_2}(x) \ldots F_{X_n}(x)$

b) $f_X(x) = \frac{d}{dx} F_X(x)$

$$= n x^{n-1} \quad \text{for } x \in [0,1]$$

c) $E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx$



$$= \int_0^1 x n x^{n-1} \, dx$$

$$= n \int_0^1 x^n \, dx$$

$$= n \left[ \frac{x^{n+1}}{n+1} \right]_0^1$$

$$= n \left[ \frac{1}{n+1} - 0 \right]$$

$$= \frac{n}{n+1}$$

d) $Var(X) = E[X^2] - E[X]^2$

$$= \int_{-\infty}^{\infty} x^2 f_X(x) \, dx - \left( \frac{n}{n+1} \right)^2$$

$$= \int_0^1 n x^{n+1} \, dx - \left( \frac{n}{n+1} \right)^2$$

$$= n \left[ \frac{x^{n+2}}{n+2} \right]_0^1 - \left( \frac{n}{n+1} \right)^2$$

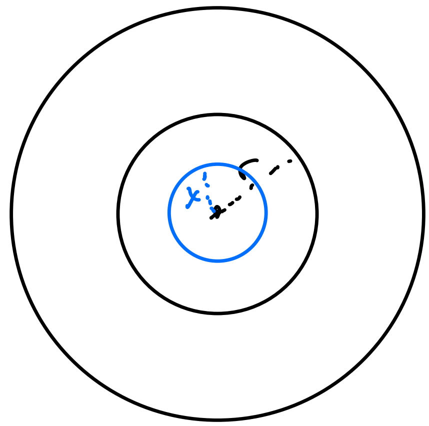
$$= \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2$$

## 2 Darts with Friends

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius $r$ around the center. Alex's aim follows a uniform distribution over a disk of radius $2r$ around the center.

(a) Let the distance of Michelle's throw from the center be denoted by the random variable $X$ and let the distance of Alex's throw from the center be denoted by the random variable $Y$.

- What's the cumulative distribution function of $X$?
- What's the cumulative distribution function of $Y$?
- What's the probability density function of $X$?
- What's the probability density function of $Y$?




- $F_X(x) = P(X \le x) = \frac{\pi x^2}{\pi r^2} = \frac{x^2}{r^2} \quad x \in [0, r]$
- $F_Y(y) = P(Y \le y) = \frac{\pi y^2}{\pi (2r)^2} = \frac{y^2}{4r^2} \quad y \in [0, 2r]$
- $f_X(x) = \frac{d}{dx} F_X(x) = \frac{2x}{r^2} \quad x \in [0, r]$
- $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{2y}{4r^2} = \frac{y}{2r^2} \quad y \in [0, 2r]$

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

$$P(X \leq Y) = \int_0^{2r} P(X \leq Y \mid Y = y)\, f_Y(y)\, dy$$

$$= \int_0^{2r} F_X(y)\, f_Y(y)\, dy$$

$* P(X \leq Y \mid Y = y) = P(X \leq y)$ only when $X, Y$ independent

$$= \int_0^{r} \frac{y^2}{r^2} \cdot \frac{y}{2r^2}\, dy + \int_r^{2r} 1 \cdot \frac{y}{2r^2}\, dy$$

$$= \frac{r^4 - 0}{8r^4} + \frac{4r^2 - r^2}{4r^2}$$

$$= \frac{1}{8} + \frac{3}{4}$$

$$= \frac{7}{8}$$

$$P(Y \leq X) = 1 - P(X \leq Y)$$

$$= \frac{1}{8}$$

(c) What's the cumulative distribution function of $U = \min\{X, Y\}$?

$$F_U(u) = P(U \leq u)$$

$$= 1 - P(U \geq u)$$

$$= 1 - P(X \geq u, Y \geq u)$$

$$= 1 - P(X \geq u)P(Y \geq u)$$

$$= 1 - (1 - F_X(u))(1 - F_Y(u))$$

$$= 1 - \left(1 - \frac{u^2}{r^2}\right)\left(1 - \frac{u^2}{4r^2}\right)$$

$$= \frac{5u^2}{4r^2} - \frac{u^4}{4r^4} \qquad \text{for } u \in [0, r]$$

$$F_U(u) = \begin{cases} 0 & \text{if } u < 0 \\ \frac{5u^2}{4r^2} - \frac{u^4}{4r^4} & \text{if } u \in [0, r] \\ 1 & \text{if } u > r \end{cases}$$

(d) What's the cumulative distribution function of $V = \max\{X, Y\}$?

$F_V(v) = P(V \leq v)$

<u>for $v \in [0, r]$</u>

$P(V \leq v) = P(X \leq v, Y \leq v)$

$= P(X \leq v)\, P(Y \leq v)$

$= \left(\frac{v^2}{r^2}\right)\left(\frac{v^2}{4r^2}\right)$

$= \frac{v^4}{4r^4}$

<u>for $v \in [r, 2r]$</u> $P(X \leq v) = 1$, so

$P(V \leq v) = P(Y \leq v)$

$= \frac{v^2}{4r^2}$

$$F_V(v) = \begin{cases} 0 & \text{if } v < 0 \\ \frac{v^4}{4r^4} & \text{if } v \in [0, r] \\ \frac{v^2}{4r^2} & \text{if } v \in [r, 2r] \\ 1 & \text{if } v > 2r \end{cases}$$

(e) What is the expectation of the absolute difference between Michelle's and Alex's distances from the center, that is, what is $\mathbb{E}[|X-Y|]$? [*Hint*: Use parts (c) and (d), together with the continuous version of the tail sum formula, which states that $\mathbb{E}[Z]=\int_0^\infty P(Z \geq z)dz$.]

let $Z = \max\{X, Y\} - \min\{X, Y\}$

$= V - U$

$$E[Z] = E[V-U]$$

$$= E[V] - E[U]$$

$$= \int_0^{2r} P(V \geq v)\,dv - \int_0^r P(U \geq u)\,du$$

$$= \int_0^r \left(1 - \frac{v^4}{4r^4}\right) dv + \int_r^{2r} \left(1 - \frac{v^2}{4r^2}\right) dv$$

$$- \int_0^r \left(1 - \frac{5u^2}{4r^2} + \frac{u^4}{4r^4}\right) du$$

$$= \frac{19r}{20} + \frac{5r}{12} - \frac{19r}{30}$$

$$= \frac{11r}{15}$$