Sample space: set of outcomes of some random experiment (Ω)

Sample point: An element of sample space (outcome)

probability space: $(\Omega, P(\cdot))$

$P(\cdot)$ assigns a value in $[0,1]$ to every $\omega \in \Omega$

s.t. $\sum_{\omega \in \Omega} P(\omega) = 1$

Events: An event A is a subset of Ω

$$P(A) = \sum_{\omega \in A} P(\omega)$$

1 Sample Space and Events

Consider the sample space Ω of all outcomes from flipping a coin 3 times.

(a) List all the outcomes in Ω. How many are there?

each flip H or T $\Rightarrow |\Omega| = 2^3 = 8$

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(b) Let A be the event that the first flip is a heads. List all the outcomes in A. How many are there?

$A = \{HHH, HHT, HTH, HTT\}$ 4

(c) Let B be the event that the third flip is a heads. List all the outcomes in B. How many are there?

$B = \{HHH, HTH, THH, TTH\}$ 4

(d) Let C be the event that the first and third flip are heads. List all outcomes in C. How many are there?

$C = \{HHH, HTH\}$ 2

(e) Let D be the event that the first or the third flip is heads. List all outcomes in D. How many are there?

$D = \{HHH, HTH, HHT, HTT, THH, TTH\}$ 6

(f) Are the events A and B disjoint? Express C in terms of A and B. Express D in terms of A and B.

$C = A \cap B$ $D = A \cup B$

(g) Suppose now the coin is flipped $n \geq 3$ times instead of 3 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

$|\Omega| = 2^n$ $|A| = 2^{n-1}$ $|B| = 2^{n-1}$ $|C| = 2^{n-2}$

$|D| = |A| + |B| - |C| = 2^n - 2^{n-2} = \boxed{3 \cdot 2^{n-2}}$

(h) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work. [*Hint*: The answer is NOT $1/2$.]

6 possible sample points: 3 coins & for each coin, we either see first or second side.

$\Omega = \{(HH, 1), (HH, 2), (HT, 1), (HT, 2), (TT, 1), (TT, 2)\}$

possible outcomes w/ info that side showing is heads:

$(HH, 1), (HH, 2), (HT, 1)$

event that other side is heads: $\{(HH, 1), (HH, 2)\}$

$\boxed{\frac{2}{3}}$

$P(\text{other side heads} \mid \text{see heads})$

1) see side 1 facing up => other side is heads ✓

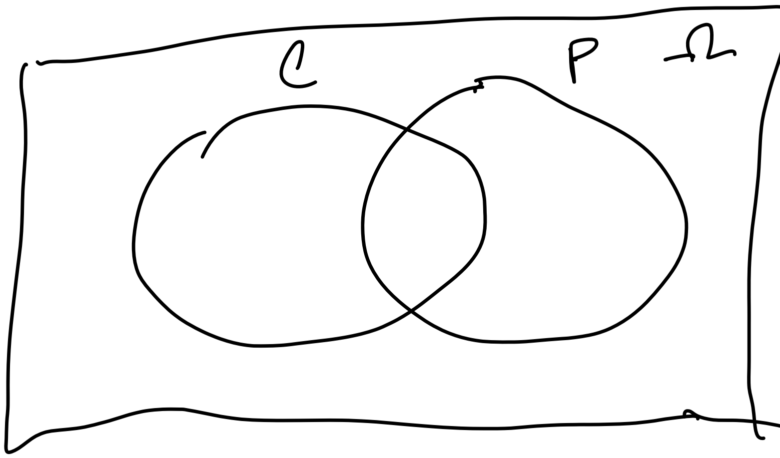
2) see side 2 facing up => other side is heads ✓

3) see side 1 facing up as heads => other side must be tails

2 Venn Diagram

Out of 1,000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.



(b) What is the probability that the student belongs to a club?

$$P(C) = \frac{|C|}{|\Omega|} = \frac{400}{1000} = \boxed{\frac{2}{5}}$$

(c) What is the probability that the student works part time?

$$P(P) = \frac{|P|}{|\Omega|} = \frac{500}{1000} = \boxed{\frac{1}{2}}$$

(d) What is the probability that the student belongs to a club AND works part time?

$$P(P \cap C) = \frac{|P \cap C|}{|\Omega|} = \frac{50}{1000} = \boxed{\frac{1}{20}}$$

(e) What is the probability that the student belongs to a club OR works part time?

$$P(P \cup C) = P(P) + P(C) - P(P \cap C)$$

$$= \frac{1}{2} + \frac{2}{5} - \frac{1}{20}$$

$$= \boxed{\frac{17}{20}}$$

Bonus

We have a room with m people, none of them were born in a leap year. Assume that each person's birthday is uniformly and identically distributed among the set of 365 possibilities.

What is the probability that there are exactly 3 distinct birthdays?

$$\frac{\#\text{ of ways of assigning birthdays to } m \text{ ppl s.t. everyone's bday falls on exactly 3 days}}{\#\text{ of total ways of assigning birthdays to } m \text{ people}}$$

denominator : 365^m

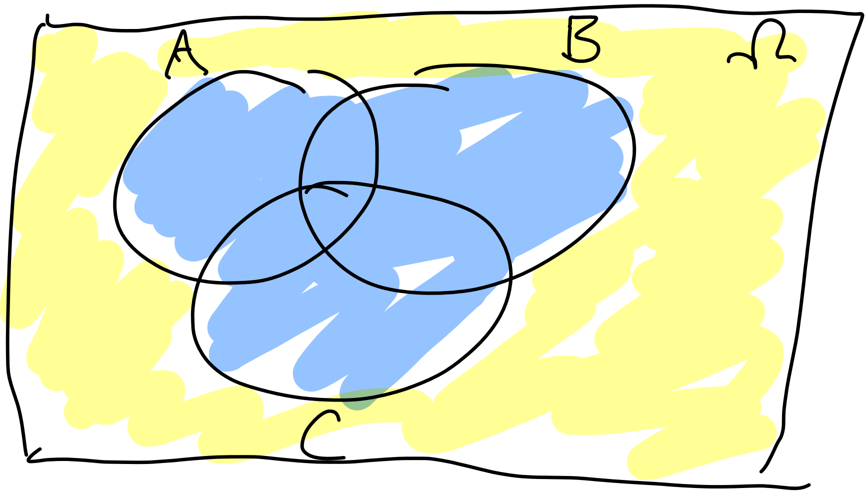
day 1 day 2 day 3

numerator: $\binom{365}{3}\left(3^m - 3 \cdot 2^m + 3\right)$

let A = no one's birthday is on day 1

let B = " day 2

let C = " day 3



_ _ _

1 2 3

want $3^m - |A \cup B \cup C|$

1

$$= 3^m - \left[|A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C| \right]$$

$$= 3^m - \left[3\cdot 2^m - 3\cdot 1 + 0 \right]$$

$$= 3^m - 3\cdot 2^m + 3$$

$$\boxed{\frac{\binom{365}{3}\left(3^m - 3\cdot 2^m + 3\right)}{365^m}}$$