Announcements

- last week of new material :O
- Fill out course evaluations!
  - one more HW drop for everyone if $\geq 80\%$ students fill it out :D

Markov's Inequality

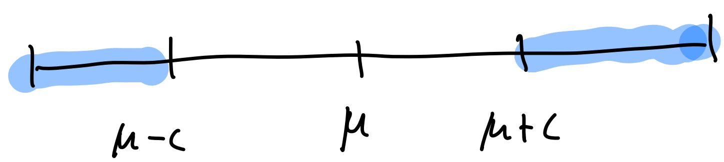
If X is a non-negative r.v. w/ finite mean,

$$P(X \geq c) \leq \frac{E[X]}{c} \quad \text{for } c > 0$$

Chebychev's Inequality

For any r.v. X w/ finite mean,

$$P(|X-\mu| \geq c) \leq \frac{Var(X)}{c^2}$$



- Probability X deviates by more than c away from its mean is small if its variance is small

1 Probabilistic Bounds

A random variable X has variance $\mathrm{Var}(X) = 9$ and expectation $\mathbb{E}[X] = 2$. Furthermore, the value of X is never greater than 10. Given this information, provide either a proof or a counterexample for the following statements.

(a) $\mathbb{E}[X^2] = 13$.

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$9 = E[X^2] - 2^2$$

$$E[X^2] = 13$$

(b) $\mathbb{P}[X = 2] > 0$.

False. want a r.v. X s.t. $E[X] = 2$, $E[X^2] = 13$

$$X = \begin{cases} a & \text{w.p. } p \\ b & \text{w.p. } 1-p \end{cases}$$

$$\left.\begin{aligned} E[X] &= ap + b(1-p) = 2 \\ E[X^2] &= a^2 p + b^2(1-p) = 13 \end{aligned}\right\} \quad \begin{aligned} \tfrac{1}{2}a + \tfrac{1}{2}b &= 2 \\ \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 &= 13 \end{aligned}$$

$$X = \begin{cases} 5 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases} \quad \begin{aligned} E[X] &= \tfrac{1}{2}(5) + \tfrac{1}{2}(-1) = 2 \\ E[X^2] &= \tfrac{1}{2}(25) + \tfrac{1}{2}(1) = 13 \quad \checkmark \end{aligned}$$

(c) $\mathbb{P}[X \geq 2] = \mathbb{P}[X \leq 2]$.

False.

$ap + b(1-p) = 2$

$a^2p + b^2(1-p) = 13$

let $b=0$:

$$\left.\begin{aligned} 2 &= ap \\ 13 &= a^2p \end{aligned}\right\} \quad p = \frac{2}{a} \Rightarrow 13 = a^2\left(\frac{2}{a}\right)$$

$$a = \frac{13}{2}, \; p = \frac{4}{13}$$

$$X = \begin{cases} \frac{13}{2} & \text{w.p. } \frac{4}{13} \\ 0 & \text{w.p. } \frac{9}{13} \end{cases}$$

(d) $\mathbb{P}[X \le 1] \le 8/9$.

True. let $Y = 10 - X$. X never exceeds 10

so Y is never negative.

$$P(10-X \ge a) = P(Y \ge a) \le \frac{E[Y]}{a} = \frac{E[10-X]}{a} = \frac{8}{a}$$

want $10 - a = 1$ so that $P(10-X \ge a) = P(X \le 1)$

set $a = 9$

$$P(X \le 1) = P(10 - X \ge 9) \le \frac{8}{9}$$

(e) $\mathbb{P}[X \ge 6] \le 9/16$.

True. $P(|X-\mu| \ge a) \le \frac{Var(X)}{a^2}$ by Chebychev's inequality

set $a = 4$:

$$P(|X-2| \ge 4) \le \frac{9}{16}$$

$$P(X \ge 6) \le P(|X-2| \ge 4) = P(X \ge 6 \cup X \le -2)$$
$$= P(X \ge 6) + P(X \le -2)$$

$$\therefore P(X \ge 6) \le \frac{9}{16}$$

2 Easy As

A friend tells you about a course called "Laziness in Modern Society" that requires almost no work. You hope to take this course next semester to give yourself a well-deserved break after working hard in CS 70. At the first lecture, the professor announces that grades will depend only on two homework assignments. Homework 1 will consist of three questions, each worth 10 points, and Homework 2 will consist of four questions, also each worth 10 points. He will give an A to any student who gets at least 60 of the possible 70 points.

However, speaking with the professor in office hours you hear some very disturbing news. He tells you that, in the spirit of the class, the GSIs are very lazy, and to save time the grading will be done as follows. For each student's Homework 1, the GSIs will choose an integer randomly from a distribution with mean $\mu = 5$ and variance $\sigma^2 = 1$. They'll mark each of the three questions with that score. To grade Homework 2, they'll again choose a random number from the same distribution, independently of the first number, and will mark all four questions with that score.

If you take the class, what will the mean and variance of your total class score be? Use Chebyshev's inequality to conclude that you have less than a 5% chance of getting an A when the grades are randomly chosen this way.

let X be the total # of points you get

let X_1 be # of points from HW1

let X_2 be " HW2

so $X = X_1 + X_2$

$X_1 = 3Y_1$ where Y_1 is integer GSI chose for HW1

$X_2 = 4Y_2$

$E[Y_1] = E[Y_2] = 5$, $Var(Y_1) = Var(Y_2) = 1$

$E[X] = E[X_1] + E[X_2] = 3E[Y_1] + 3E[Y_2]$

$$Var(X) = Var(X_1) + Var(X_2)$$

$$= 9\,Var(Y_1) + 16\,Var(Y_2)$$

$$= 9 + 16$$

$$= 25$$

$$P(X \geq 60) \leq P(|X-35| \geq 25) \leq \frac{Var(X)}{25^2} = \frac{1}{25}$$

Any student will have at most 4% chance of getting an A ☹

3 Tightest Bounds

A random variable X has expectation $\mathbb{E}[X] = 12$ and variance $\text{Var}[X] = 4$.

(a) For $I \sim \text{Bernoulli}(p)$ and $Y = aI + b$, come up with values for a, b, and p such that $\mathbb{E}[Y] = 12$ and $\text{Var}[Y] = 4$. What are the possible values of your Y?

$$E[aI + b] = ap + p = 12$$

$$Var(aI+b) = a^2 Var(I) = a^2 p(1-p) = 4$$

$$b = 12 - ap$$

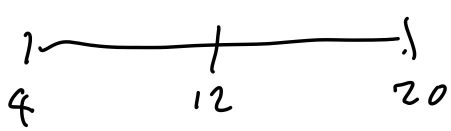
$$a^2 = \frac{4}{p(1-p)} \Rightarrow a = \frac{2}{\sqrt{p(1-p)}}$$

$$Y = \begin{cases} 10 & \text{w.p. } \frac{1}{2} \\ 14 & \text{w.p. } \frac{1}{2} \end{cases}$$

$$\underline{\text{let } p = \tfrac{1}{2}: \quad Y = 4I + 10}$$

(b) Find the tightest bounds you can for $\mathbb{P}[4 \le X \le 20]$.

Notice this is a symmetric interval about $E[X]$



$$P(4 \le X \le 20) = P(-8 \le X - 12 \le 8)$$

$$= P(|X - 12| \le 8)$$

$$= 1 - P(|X-12| > 8)$$

$$\geq 1 - P(|X-12| \geq 8)$$

$$\geq 1 - \frac{Var(X)}{8^2}$$

$$= 1 - \frac{4}{8^2}$$

tightest upper bound is 1 $(X = 4)$

$$\boxed{1 - \frac{4}{8^2} \leq P(4 \leq X \leq 20) \leq 1}$$

(c) Find the tightest bounds you can for $\mathbb{P}[9 \leq X \leq 20]$.

$$P(9 \leq X \leq 20) \geq P(9 \leq X \leq 15)$$

$$= P(-3 \leq X-12 \leq 3)$$

$$= P(|X-12| \leq 3)$$

$$= 1 - P(|X-12| > 3)$$

$$\geq 1 - P(|X-12| \geq 3)$$

$$\geq 1 - \frac{4}{3^2}$$

tightest upper bound 1 when $X=4$

$$\boxed{1-\frac{4}{3^2} \le P(9 \le X \le 20) \le 1}$$

(d) Find the tightest bounds you can for $\mathbb{P}[X \geq 16]$.

$$P(X \ge 16) \le P(X \le 8 \cup X \ge 16)$$
$$= P(|X-12| \ge 4)$$
$$\le \frac{4}{4^2}$$

tightest lower bound is 0 $(X=4)$

$$\boxed{0 \le P(X \ge 16) \le \frac{1}{4}}$$

(e) Find the tightest bounds you can for $\mathbb{P}[X^2 \geq 225]$.

• X^2 is non-negative

Markov's

$E[X^2] = Var(X) + E[X]^2 = 4 + 144 = 148$

$P(X^2 \geq 225) \leq \frac{148}{225}$

Chebyshev's

$P(X^2 \geq 225) = P(X \leq -15 \cup X \geq 15)$

$\leq P(X \leq 9 \cup X \geq 15)$

$= P(|X-12| \geq 3)$

$\leq \frac{4}{3^2}$

tightest lower bound is 0 $(X=4)$

$\boxed{0 \leq P(X^2 \geq 225) \leq \frac{4}{3^2}}$