CS 70 Midterm Review Session: Graphs

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1 Warm Up

a) T/F: Any graph where every triple of vertices is a triangle, i.e. $\forall u, v, w \in V, (u, v), (u, w), (v, w) \in E$, is a complete graph.
True, we can show that every vertex has an edge to every other vertex.

b) T/F: A graph with k edges and n vertices has a vertex of degree at least $\frac{2k}{n}$.
True, there must be a vertex with degree at least the average degree.

c) T/F: Any complete graph has a Hamiltonian cycle. (A Hamiltonian cycle is a cycle that visits every vertex exactly once)
True, any permutation of all the vertices is a Hamiltonian cycle.

d) T/F: Any graph with n vertices and 2n edges is connected.
False, consider the graph with 2 connected components, each of which are K_5.

e) What is the maximum number of edges you could have in a bipartite graph with 2n vertices?
n^2

f) Does a graph G with n vertices such that every vertex except 2 has even degree have an Eulerian Tour? Does G have an Eulerian Walk?
No Eulerian Tour, Yes Eulerian walk: add a vertex w connecting it with an edge to each odd degree vertex. The new graph has all even degree vertices, since connecting w to each odd degree vertex flips its parity, w is degree 2, while all other vertices are unaffected. Since this graph has all even degree vertices, there is an Eulerian Tour. Run this tour and remove w along with its edges. The path traversed during the tour on the original graph is an Eulerian walk.

2 Short Answer

a) Consider a planar graph G where each face is incident to exactly 5 edges. Derive an expression for the number of edges in G, e, in terms of the number of vertices, v in G.
We have $5f = 2e$, so plugging this into Euler's formula gives $e = \frac{1}{3}(5v - 10)$.

b) Consider a bipartite graph G with vertex set $V = L \cup R$ and edges across L and R. What is the sum of degrees of vertices in L in terms of $|L|, |R|$, and $|E|$?
$|E|$. Every edges connects one vertex in L to one vertex in R, so the sum of degrees in L is exactly half the total sum of degrees.

3 Proofs

a) Prove that a graph where every vertex has degree at least 2 has a cycle.
Approach 1: Take a walk starting from some vertex v_0 using only edges not visited before. We must run out of edges to use at some point and get stuck at a vertex with degree at least 2.

Approach 2: We have that $\sum_{v \in V} deg(v) = 2|E| \geq 2|V| \implies |E| \geq |V|$.
Case 1: Graph was connected to begin with. Since $|E| \geq |V|$, the graph is not acyclic.
Case 2: Graph was not connected to begin with. Suppose there is some k connected components and there are $n_1, n_2, .., n_k$ vertices in each component. If each component is acyclic, then $|E| = n_1 - 1 + n_2 - 1 + ... + n_k - 1 = |V| - k$. So we have $|E| < |V|$ and $|E| \geq |V|$, a contradiction.

b) Prove that a connected graph with n vertices has at least $n - 1$ edges.
We prove by strong induction on the number of vertices n.
Base case: $n = 1$. A single vertex graph has 0 edges and is its own connected component.
Inductive step: Remove an arbitrary vertex v from a k vertex graph G leaving some c connected components. Let $n_1, n_2, .., n_c$ be the number of vertices in each component. By the inductive hypothesis, each component has at least $n_i - 1$ edges, so the sum of edges in the components is at least $n_1 - 1 + n_2 - 1 + ... + n_c - 1 = k - 1 - c$. Since the graph was connected to begin with, the vertex v that we removed must have had at least one edge to each component, so adding back v yields at least $k - 1$ edges total.

c) Prove that every tree is bipartite.
Approach 1: strong induction on the number of vertices n.
Base Case: $n = 2$. Assign one vertex to set L and the other to set R.
Inductive Step: Suppose we have a graph with some $k \geq 2$ vertices. Remove an arbitrary vertex v_0 leaving c connected components. By the inductive hypothesis, each connected component is bipartite, so let $(L_1, R_1), ..., (L_c, R_c)$ be the sets of vertices. Let $L_k = L_1 \cup L_2 \cup ... \cup L_c$, $R_k = R_1 \cup R_2 \cup ... \cup R_c$ and WLOG assign v_0 to L_k. We can rotate the assignments of vertices in each (L_i, R_i) such that the vertices connected to v_0 are in R_k. Thus, we have assigned every vertex to either L_k or R_k such that there are no edges between vertices in L_k or R_k.

Approach 2: construct L and R directly: take an arbitrary vertex v_0 and WLOG assign it to L. Assign all vertices that are an even number of edges away from v_0 to L and assign all other vertices to R. It remains to show no 2 vertices in L (resp. R) are connected: suppose 2 vertices v_1 and v_2 in L are connected. Then we can make a path v_0, v_1, v_2, v_0 which means the original graph was not a tree, a contradiction.

d) Prove that every connected, undirected graph has a tour that uses each edge at least once and at most twice.
Duplicate every edge in the graph and call this new graph G'. Every vertex in G' has even degree, so G' has an Eulerian Tour. Running Eulerian Tour on G' corresponds to using each edge twice in a tour on G.

e) Prove that if G is a graph with n vertices such that for any two non-adjacent vertices u and v, it holds that deg u + deg $v \geq n - 1$, then G is connected.
Approach 1: Pigeonhole Principle. We show there must exist a vertex w incident to both u and v

for every non-adjacent vertices u and v (if for every non-adjacent u and v there exists a w "between" them, the graph must be connected). Suppose there doesn't exist such a w. Then, u and v are collectively incident to $n-1$ distinct vertices, meaning that there are $n+1$ vertices in total, a contradiction.

Approach 2: Contrapositive. Suppose G is disconnected and has k connected components. Suppose the two components with the most vertices have n_1, n_2 vertices respectively. The vertex with max degree in each component has degree at most $n_1 - 1, n_2 - 1$ respectively, meaning their sum of degrees is at most $n_1 - 1 + n_2 - 1 < n - 1$. We have shown there exists two vertices such that deg u + deg $v < n - 1$.

f) (Fall 2017 MT1) Consider a directed graph where every pair of vertices u and v are connected by a single directed arc either from u to v or from v to u. Show that every vertex has a directed path of length at most two to the vertex with maximum in-degree.
The total in-degree is the number of edges which is $\frac{n(n-1)}{2}$, thus the vertex v with maximum in-degree must have in-degree d at least $\frac{n-1}{2}$, as the maximum must be at least the average. These d vertices with incoming edges to v have a directed path of length 1. The other $n-1-d$ vertices each have in-degree at most d (otherwise v wouldn't have the maximum in-degree) and thus each have out-degree at least $n-1-d$, since each vertex has a total of $n-1$ directed edges. Thus, each of the $n-1-d$ vertices must have a directed edge to one of the d vertices that are connected to v.

4 Misc. Questions

a) Suppose there are 29 people in a room and among every three people at least two of them know each other. Prove that someone knows at least 14 people.

Make a graph with 29 vertices where each vertex represents a person. Let there be an undirected edge between two vertices if those two people know each other. We show the existence of a vertex with degree at least 14: pick any arbitrary vertex v_0. Either it has degree at least 14 or not. If it does, we're done. Otherwise, let A be the set of vertices that have an edge to v_0 and B be the set of vertices that don't have an edge to v_0. We know that $|A| < 14$ since v_0 has degree less than 14, so $|B| \geq 15$. Fix some vertex v_1 in B and consider all triples (v_0, v_1, b) for $b \in B$. (v_1, b) must be an edge for every $b \in B$ where $b \neq v_1$. Thus, v_1 has an edge to 14 vertices so it is degree 14.

b) Suppose an odd number of soldiers are stationed in a field, in such a way that all the pairwise distances are distinct. Each soldier is told to keep an eye on the nearest other soldier. Prove that at least one soldier is not being watched.

The idea is that if we consider the two soldiers with smallest pairwise distances, they must be watching each other. If someone else is watching one of these two soldiers then there is a soldier being watched twice which means that there is another soldier not being watched. If no one is watching one of the two soldiers with smallest pairwise distance, these two can be removed without affecting the others. If we keep performing this procedure of removing the two soldiers with smallest pairwise distance, we are eventually left with one soldier that is not being watched.

We can formulate the solution inductively as follows:

<u>Base Case:</u> Suppose $n = 3$, where n is the number of soldiers. Let the two soldiers with smallest pairwise distance be labeled A and B and the other soldier be labeled C. A and B must be watching each other, which leaves C to watch either A or B. There is no one watching C.

<u>Inductive Step:</u> Suppose there are $2k + 1$ soldiers for some $k \geq 1$. Remove the two soldiers with closest pairwise distance. Then by the inductive hypothesis, there was a soldier s not being watched among the remaining $2k - 1$ soldiers. Adding back the two soldiers, we know that the two soldiers will watch each other, and it is possible that someone will choose to now watch one of these two soldiers because their pairwise distance is smaller than the previous person they were watching. However, none of the soldiers will choose to watch s thus s is still unwatched.