1 Why Is It Gaussian?

Let X be a normally distributed random variable with mean μ and variance σ^2. Let $Y = aX + b$, where $a > 0$ and b are non-zero real numbers. Show explicitly that Y is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$. The PDF for the Gaussian Distribution is $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. One approach is to start with the cumulative distribution function of Y and use it to derive the probability density function of Y.

[1.You can use without proof that the pdf for any gaussian with mean and standard deviation is given by the formula $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where μ is the mean value for X and σ^2 is the variance. 2. The derivative of CDF gives PDF.]

$X \sim N(\mu, \sigma^2) \qquad Y = aX + b \quad \text{for } a > 0$

$$F_Y(x) = P(Y \le x)$$

$$= P(aX + b \le x)$$

$$= P\left(X \le \frac{x-b}{a}\right) \quad \text{since } a > 0$$

$$= F_X\left(\frac{x-b}{a}\right)$$

$\exp(3) = e^3$

$g(x) = \frac{x-b}{a}$

$$f_Y(x) = \frac{d}{dx} F_Y(x)$$

$$= \frac{d}{dx} \underbrace{F_X\left(\frac{x-b}{a}\right)}_{} \Rightarrow F_X(g(x))$$

$\frac{d}{dx} F_X(g(x))$

$= F_X'(g(x)) \cdot g'(x)$

$$= \frac{1}{a} f_X\left(\frac{x-b}{a}\right)$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{\left(\frac{x-b}{a} - \mu\right)^2}{2\sigma^2}\right)$$

want $\frac{\left(\frac{x-b}{a}-\mu\right)^2}{2\sigma^2} = \frac{(x-(a\mu+b))^2}{2a^2\sigma^2}$

$$\frac{x-b}{a}-\mu = \frac{1}{a}(x-b-a\mu)$$

$$\frac{1}{a\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-b-a\mu)^2}{2a^2\sigma^2}\right)$$

$f_Y(x)$ is the pdf of gaussian r.v. w/
mean $a\mu+b$ & variance $\sigma^2 a^2$

Central limit theorem

For $X_1, \dots, X_n$ i.i.d. r.v.'s w/ mean μ & variance σ^2, $\frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges in distribution to $N(\mu, \frac{\sigma^2}{n})$.

$* S_n = \sum_{i=1}^{n} X_i$

$$\frac{\frac{S_n}{n} - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \approx N(0,1)$$

Standard Normal - $N(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\Phi(z) = P(X \leq z)$$

2 Hypothesis testing

We would like to test the hypothesis claiming that a coin is fair, i.e. $P(H) = P(T) = 0.5$. To do this, we flip the coin $n = 100$ times. Let Y be the number of heads in $n = 100$ flips of the coin. We decide to reject the hypothesis if we observe that the number of heads is less than $50 - c$ or larger than $50 + c$. However, we would like to avoid rejecting the hypothesis if it is true; we want to keep the probability of doing so less than 0.05. Please determine c. (*Hints: use the central limit theorem to estimate the probability of rejecting the hypothesis given it is actually true. Table is provided in the appendix.*)

let $X_i = \begin{cases} 1 & \text{if ith flip heads} \\ 0 & \text{o.w.} \end{cases}$ $\quad Y = \sum_{i=1}^{n} X_i$

if the hypothesis were true, $\mu = E[X_i] = \frac{1}{2}$

$\sigma^2 = Var(X_i) = \frac{1}{4}$

$P\left(\frac{Y - n\mu}{\sqrt{n\sigma^2}} \le z\right) \approx \Phi(z)$ since $\frac{Y - n\mu}{\sqrt{n\sigma^2}} \approx N(0,1)$ by the CLT

$$P\left(\frac{Y - 100 \cdot \frac{1}{2}}{\sqrt{100 \cdot \frac{1}{4}}} \le z\right) = P\left(\frac{Y-50}{5} \le z\right) \approx \Phi(z)$$

we reject the hypothesis when $|Y - 50| > c$

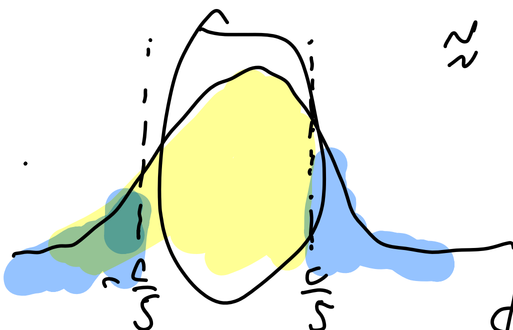
want $P(|Y-50| > c) < 0.05$

$\iff P(|Y-50| \le c) > 0.95$

$$P(|Y-50| \le c) = P\left(\frac{|Y-50|}{5} \le \frac{c}{5}\right)$$

$$= P\left(-\frac{c}{5} \le \frac{Y-50}{5} \le \frac{c}{5}\right)$$

$$\approx \Phi\left(\frac{c}{5}\right) - \Phi\left(-\frac{c}{5}\right)$$



$$= \Phi\left(\frac{c}{5}\right) - \left(1 - \Phi\left(\frac{c}{5}\right)\right)$$

$$= 2\Phi\left(\frac{c}{5}\right) - 1$$

$$2\Phi\left(\frac{c}{5}\right) - 1 = 0.95$$

$$\Phi\left(\frac{c}{5}\right) = 0.975$$

$$\frac{c}{5} = \Phi^{-1}(0.975)$$

$$\frac{c}{5} = 1.96$$

$$c = 9.8$$

if we see at least $50+10=60$ heads or at most $50-10=40$ heads, reject the hypothesis.

$X \sim N(0,1)$

$P(X \leq 1.86) = 0.9686$

Probability Content from -oo to Z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Table 1: Table of the Normal Distribution