CS 70 Discrete Mathematics and Probability Theory
Summer 2021 **Discussion 07C**

1 Markov Chain Basics

A Markov chain is a sequence of random variables X_n, $n = 0, 1, 2, \ldots$. Here is one interpretation of a Markov chain: X_n is the state of a particle at time n. At each time step, the particle can jump to another state. Formally, a Markov chain satisfies the Markov property:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i), \quad (1)$$

for all n, and for all sequences of states $i_0, \ldots, i_{n-1}, i, j$. In other words, the Markov chain does not have any memory; the transition probability only depends on the current state, and not the history of states that have been visited in the past.

(a) In lecture, we learned that we can specify Markov chains by providing three ingredients: $\mathscr{X}$, P, and π_0. What do these represent, and what properties must they satisfy?

- $\mathscr{X}$ is the set of states of the Markov chain
- P is the matrix of transition probabilities where P_{ij} is probility of transitioning from state i to j
  - entries of P non-negative, rows must sum to 1
- π_0 is the initial distribution where $\pi_0(i) = P(X_0 = i)$
  - $\sum_i \pi_0(i) = 1$

(b) If we specify $\mathscr{X}$, P, and π_0, we are implicitly defining a sequence of random variables X_n, $n = 0, 1, 2, \ldots$, that satisfies (1). Explain why this is true.

- X_0 has distribution π_0 i.e. $P(X_0 = i) = \pi_0(i)$
- X_1 has (conditional) distribution $P(X_1 = j \mid X_0 = i) = P_{ij}$
- X_{n+1} has " distribution $P(X_{n+1} = j \mid X_n = i_n, \ldots, X_0 = i_0)$

$= P(X_{n+1} = j \mid X_n = i_n)$

$= P_{ij}$

these are all valid distributions and all we needed was $\mathcal{X}$, P, π_0

(c) Calculate $\mathbb{P}(X_1 = j)$ in terms of π_0 and P. Then, express your answer in matrix notation. What is the formula for $\mathbb{P}(X_n = j)$ in matrix form?

$$P(X_1 = j) = \sum_{i \in \mathcal{X}} P(X_1 = j \mid X_0 = i)\, P(X_0 = i)$$

$$= \sum_{i \in \mathcal{X}} P_{ij}\, \pi_0(i)$$

$$\pi_1(j) = \sum_{i \in \mathcal{X}} \pi_0(i) P_{ij}$$

let π_0, π_1 be row vectors i.e. $\pi_0 = [\downarrow \downarrow \downarrow]$

$\Rightarrow$ matrix notation $\pi_1 = \pi_0 P$

$$P(X_1 = j) = \pi_1(j) = (\pi_0 P)_j$$

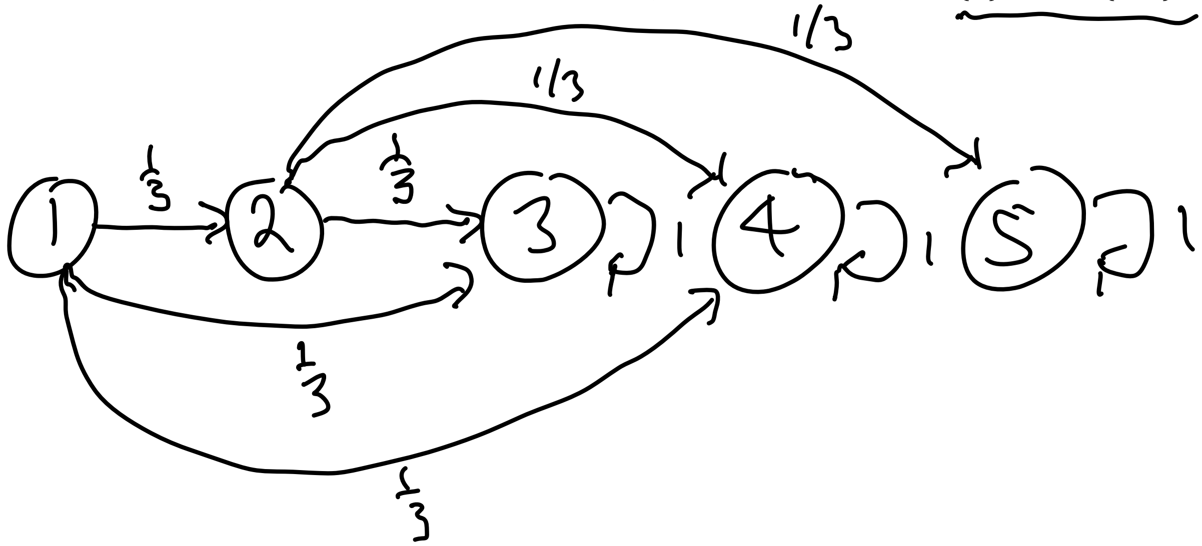
$$\pi_n = \pi_0 P^n \rightarrow P(X_n = j) = \pi_n(j) = (\pi_0 P^n)_j$$

$$[\pi_0(1) \;\; \pi_0(2)] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$$= [\underbrace{\pi_0(1) P_{11} + \pi_0(2) P_{21}}_{\pi_1(1)} \;\; \underbrace{\pi_0(1) P_{12} + \pi_0(2) P_{22}}_{\pi_1(2)}]$$

2 Skipping Stones

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathscr{X} = \{1,2,3,4,5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4,5\}$.



let $\alpha(i)$ be probability of reaching target before overshooting, given that starting at state i

$\alpha(5) = 0$

$\alpha(4) = 0$

$\alpha(3) = 1$

$\alpha(2) = \frac{1}{3}\alpha(3) + \frac{1}{3}\alpha(4) + \frac{1}{3}\alpha(5) = \frac{1}{3}$

$\alpha(1) = \frac{1}{3}\alpha(2) + \frac{1}{3}\alpha(3) + \frac{1}{3}\alpha(4) = \frac{1}{9} + \frac{1}{3} = \frac{4}{9}$

$\boxed{\frac{4}{9}}$

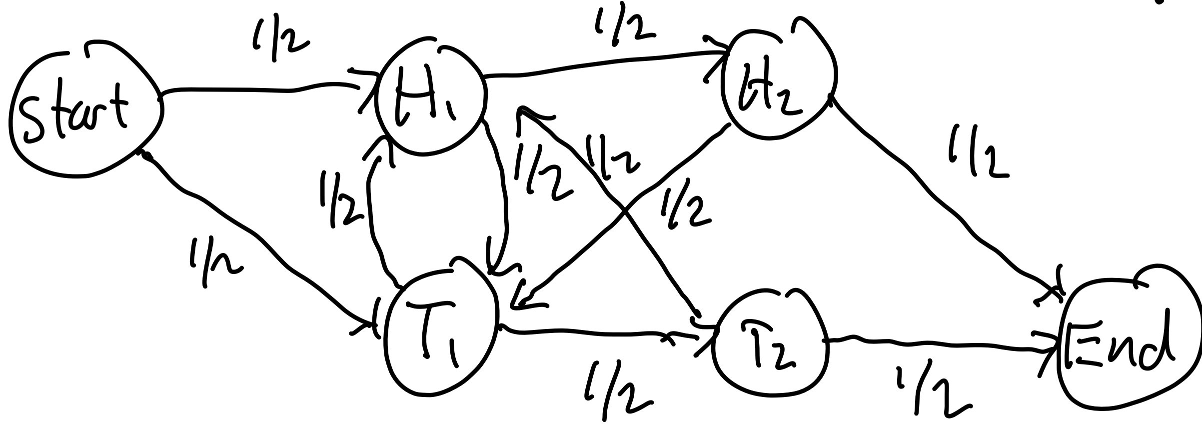
3 Consecutive Flips

Suppose you are flipping a fair coin (one Head and one Tail) until you get the same side 3 times (Heads, Heads, Heads) or (Tails, Tails, Tails) in a row.

(a) Construct an Markov chain that describes the situation with a start state and end state.

$\mathcal{X} = \{\text{start}, \text{end}, H_1, T_1, H_2, T_2\}$

H_2 = 2 heads in a row



(b) Given that you have flipped a (Tails, Heads) so far, what is the expected number of flips to see the same side three times?

Heads

currently at H_1

Let $\beta(i)$ be the expected number of flips to reach end state, given that starting from state i.

$\beta(H_1) = 1 + \frac{1}{2}\beta(T_1) + \frac{1}{2}\beta(H_2)$

$\beta(H_2) = 1 + \frac{1}{2}\beta(T_1) + \frac{1}{2}\beta(End)$

$\beta(T_1) = 1 + \frac{1}{2}\beta(T_2) + \frac{1}{2}\beta(H_1)$

$\beta(T_2) = 1 + \frac{1}{2}\beta(End) + \frac{1}{2}\beta(H_1)$

$\boxed{\beta(H_1) = 6}$

$\beta(End) = 0$

(c) What is the expected number of flips to see the same side three times, beginning at the start state?

$$\beta(\text{start}) = 1 + \frac{1}{2}\beta(H_1) + \frac{1}{2}\beta(T_1)$$

$$= 1 + \frac{1}{2}(6) + \frac{1}{2}(6)$$

$$= 7$$