Black-hole Perturbation Theory Notes

Arindam Bhunia

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Contents

Chapter 1

Scalar perturbations

In this section, our main aim is to discuss the scalar perturbations. This discussion will be relevant during the discussion of Black-hole perturbation. First, we discuss the equation of motion in curved spacetime. Then, we go for finding the behaviour of massless scalar field in curved background. In the end, we look for the behaviour of massless scalar field in the Schwarzschild background.

1.1 Equation of motion in curved spacetime

In curved spacetime, the action is

$$S = \int d^4x \, \mathcal{L}'(\Phi, \nabla_\mu \Phi) \tag{1.1}$$

where $\mathcal{L}'$ is the Lagrangian density of field in curved spacetime and $g = \det(g^{\mu\nu})$ with $g^{\mu\nu}$ being the background metric.
Then the Euler-Lagrange equation of motion in curved spacetime becomes

$$\nabla_\mu \left(\frac{\partial((-g)^{-1/2}\mathcal{L}')}{\partial(\nabla_\mu \Phi)} \right) = \frac{\partial((-g)^{-1/2}\mathcal{L}')}{\partial \Phi} \tag{1.2}$$

where ∇_μ is the covariant derivative.

1.2 Massless scalar field in curved spacetime

The Lagrangian density of a massless real scalar field in the curved background is given by,

$$\mathcal{L}' = \sqrt{-g} \left(\frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi \right) \tag{1.3}$$

and the action is given by,

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi \right) \tag{1.4}$$

Therefore, using the Eq. 1.3 the equation of motion for massless scalar becomes (ref App. 8.1),

$$\Box \Phi \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \, g^{\mu\nu} \, \partial_\nu \Phi) = 0 \tag{1.5}$$

where $g^{\mu\nu}$ is the background metric and $g = \det(g^{\mu\nu})$ is the metric determinant. This equation is known as the massless Klein-Gordon equation in curved background.

1.2.1 Massless scalar field in Schwarzschild background

Now, we are interested in the behaviour of massless scalar field in Schwarzschild background.
For a Schwarzschild BH, the metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -A(r)c^2 dt^2 + B(r)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.6}$$

where

$$A(r) = 1 - \frac{R_S}{r}, \quad B(r) = \frac{1}{A(r)}, \tag{1.7}$$

and $R_S = 2GM/c^2$ is the Schwarzschild radius.
Using the Schwarzchild metric in Eq. (1.5), we get,

$$\partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu\right] \Phi = -r^2 \sin\theta \cdot \frac{1}{A(r)} \partial_0^2 \Phi + \sin\theta \partial_r \left[r^2 A(r) \partial_r \Phi\right] + \partial_\theta \left[\sin\theta \partial_\theta \Phi\right] + \frac{1}{\sin\theta} \partial_\phi^2 \Phi = 0 \tag{1.8}$$

Since the Schwarzschild metric is static, using the separation of variables, we can write Φ as a radial & time-dependent part and spherical harmonics which capture the angular dependence.

$$\boxed{\Phi(t, r, \theta, \phi) = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{lm}(t, r) Y_{lm}(\theta, \phi)} \tag{1.9}$$

where $Y_{lm}(\theta, \phi)$ are spherical harmonics that follow the differential equation

$$-\left[\frac{1}{\sin\theta} \partial_\theta \left(\sin\theta \partial_\theta Y_{lm}\right) + \frac{1}{\sin^2\theta} Y_{lm}\right] = l(l+1) Y_{lm} \tag{1.10}$$

Plugging Eq. 1.9 & 1.10 in Eq. 1.8 (ref App 8.2), we get

$$\boxed{A \partial_r \left(A \partial_r u_{lm}\right) - \partial_0^2 u_{lm} - V_l(r) u_{lm} = 0,} \tag{1.11}$$

with

$$\boxed{V_l(r) = A(r) \left[\frac{l(l+1)}{r^2} + \frac{R_S}{r^3}\right]} \tag{1.12}$$

Due to the spherical symmetry of the Schwarzschild metric, $V_l(r)$ does not depend on the azimuthal number m.
To simplify the Eq. (1.11) further, we use "tortoise coordinate" r_* defined as

$$r_* = r + R_S \ln\left(\frac{r - R_S}{R_S}\right). \tag{1.13}$$

Here, $-\infty < r_* < \infty$. When $r \to R_S^+$, $r_* \to -\infty$ and when $r \to \infty$, $r_* \to \infty$
We introduce the notation

$$\partial_* \equiv \frac{\partial}{\partial r_*}. \tag{1.14}$$

From the definition of r_*, we get $\partial_* r = A$ and $A\partial_r = \partial_*$ (ref App. (8.3)). Using these in Eq. (1.11), we get

$$\left[\partial_*^2 - \partial_0^2 - V_l(r)\right] u_{lm}(t, r) = 0 \tag{1.15}$$

Performing a Fourier transform with respect to the time gives,

$$u_{lm}(t, r) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{u}_{lm}(\omega, r) e^{-i\omega t}, \tag{1.16}$$

Then $\tilde{u}_{lm}(\omega, r)$ (frequency space) satisfies the equation (ref App. 8.4)

$$\boxed{\left[-\frac{d^2}{dr_*^2} + V_l(r)\right] \tilde{u}_{lm} = \frac{\omega^2}{c^2} \tilde{u}_{lm}} \tag{1.17}$$

This is equivalent to the Schrodinger equation in one dimension, defined on the line $-\infty < r_* < \infty$, for a particle of mass m, written in units $(\hbar^2/2m) = 1$ with $V_l(r)$ playing the role of the potential and ω^2/c^2 that of the energy. In the following section, we will see that how very similar equations are obtained for metric perturbations as well.

Chapter 2

Gravitational perturbations

In this chapter, we are going to discuss the gravitational perturbation. Rather than discussing the BH perturbation theory directly, at first let us focus on the general gravitation perturbation.

Let, a perturbed metric $g_{\mu\nu}$ be written as the sum of well-known metric tensor and perturbation.

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

where $\bar{g}_{\mu\nu}$ is well-known metric and $h_{\mu\nu}$ is the perturbation to the metric.

The Einstein's field equation reads as

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.2}$$

Let, $G_{\mu\nu}$ can be written as sum of $\bar{G}_{\mu\nu}$ (arising due to $\bar{g}_{\mu\nu}$) and $\delta G_{\mu\nu}$ (arising due to $g_{\mu\nu}$ and $h_{\mu\nu}$).

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + \delta G_{\mu\nu} \tag{2.3}$$

Therefore, the Eq. (2.2) becomes,

$$\bar{G}_{\mu\nu} + \delta G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.4}$$

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2.1 Tensor harmonics

Since, $h_{\mu\nu}$ is a symmetric tensor, the no of degrees of freedom is 10. Since the background metric is spherically symmetric, we can separate the radial and angular dependence of $h_{\mu\nu}$, and for this we need a basis of 10 independent tensor spherical harmonics, corresponding to the 10 components of $h_{\mu\nu}$. In the next section, we introduce such basis.

Here, we will use some results from the theory of linearized gravity to construct the bases. In linearized gravity, during the separation of the angular dependence in the spatial components h_{ij} (see Sec 3.5.2 of [1]), a basis for the six components of a symmetric (but not necessarily traceless) matrix h_{ij} is given by the six tensor harmonics,

$$\left(\mathbf{T}^{L0}_{lm}\right)_{ij}, \quad \left(\mathbf{T}^{T0}_{lm}\right)_{ij}, \quad \left(\mathbf{T}^{E1}_{lm}\right)_{ij}, \quad \left(\mathbf{T}^{B1}_{lm}\right)_{ij}, \quad \left(\mathbf{T}^{E2}_{lm}\right)_{ij}, \quad \left(\mathbf{T}^{B2}_{lm}\right)_{ij}, \tag{2.5}$$

given in Eqs. (3.268) - (3.272) and (3.276) of [1]. These tensor harmonics are 3×3 matrices. But we need 4×4 matrices. Therefore, we need to increase the dimension to 4×4 from 3×3. Trivially we can do that to 4×4 matrices $\left(\mathbf{T}^{L0}_{lm}\right)_{\mu\nu}$, etc. by stating that, in the rest frame of the BH, $\left(\mathbf{T}^{L0}_{lm}\right)_{\mu\nu}$ is equal to $\left(\mathbf{T}^{L0}_{lm}\right)_{ij}$ when both μ and ν are spatial indices, and is zero otherwise. Similar things can be done for $\mathbf{T}^{T0}_{lm}, \ldots, \mathbf{T}^{B2}_{lm}$ as well.

Now we next need 4 independent tensor harmonics to express the components $h_{0\mu}$. Since h_{00} is a scalar under rotations, it can be expanded in terms of the ordinary scalar spherical harmonics, so we define the tensor $\left(\mathbf{T}^{tt}_{lm}\right)_{\mu\nu}$ by stating that, in the BH rest frame,

$$\left(\mathbf{T}^{tt}_{lm}\right)_{\mu\nu} = \delta^0_\mu \delta^0_\nu Y_{lm} \tag{2.6}$$

From the point of view of rotations, h_{0i} is a vector, and can be expanded in terms of the vector spherical harmonics $\left(\mathbf{Y}^R_{lm}\right)_i$, $\left(\mathbf{Y}^E_{lm}\right)_i$ and $\left(\mathbf{Y}^B_{lm}\right)_i$ defined in Eqs. (3.254) - (3.256) of [1]. We then define the tensor harmonics $\left(\mathbf{T}^{Rt}_{lm}\right)_{\mu\nu}$, $\left(\mathbf{T}^{Et}_{lm}\right)_{\mu\nu}$ and $\left(\mathbf{T}^{Bt}_{lm}\right)_{\mu\nu}$ by stating that, in the BH rest frame, they are non-vanishing only when $\mu = 0$ and $\nu = i$ (or when $\mu = i$ and $\nu = 0$, with $\left(\mathbf{T}^{Rt}_{lm}\right)_{0i} = \left(\mathbf{T}^{Rt}_{lm}\right)_{i0}$, etc.), and

$$\left(\mathbf{T}^{Rt}_{lm}\right)_{0i} = \frac{1}{\sqrt{2}} \left(\mathbf{Y}^R_{lm}\right)_i, \tag{2.7}$$

$$\left(\mathbf{T}^{Et}_{lm}\right)_{0i} = \frac{1}{\sqrt{2}} \left(\mathbf{Y}^E_{lm}\right)_i, \tag{2.8}$$

$$\left(\mathbf{T}^{Bt}_{lm}\right)_{0i} = \frac{1}{\sqrt{2}} \left(\mathbf{Y}^B_{lm}\right)_i. \tag{2.9}$$

Explicitly,

$$\left(\mathbf{T}^{Rt}_{lm}\right)_{0i} = \frac{1}{\sqrt{2}} n_i Y_{lm}, \tag{2.10}$$

$$\left(\mathbf{T}^{Et}_{lm}\right)_{0i} = [2l(l+1)]^{-1/2} r \partial_i Y_{lm}, \tag{2.11}$$

$$\left(\mathbf{T}^{Bt}_{lm}\right)_{0i} = i[2l(l+1)]^{-1/2} L_i Y_{lm}. \tag{2.12}$$

where $\mathbf{L} = -i\mathbf{r} \times \boldsymbol{\nabla}$, and $\hat{\mathbf{n}}$ is the unit vector in the radial direction, normalized with respect to the flat-space metric, $\delta_{ij} n^i n^j = 1$ (see Sec 3.5.2 of [1]).

The factors i have been inserted in the definition of $\mathbf{T}^{Bt}_{lm}$, $\mathbf{T}^{B1}_{lm}$ and $\mathbf{T}^{B2}_{lm}$ (see Eqs. (3.270), (3.272) and (12.23) of [1]) so that these tensors also satisfy the relation

$$\left(\mathbf{T}^a_{lm}\right)^*_{\mu\nu} = (-1)^m \left(\mathbf{T}^a_{lm}\right)_{\mu\nu} \tag{2.13}$$

to make the quantity $i\mathbf{L} = \mathbf{r} \times \nabla$ real.

From Eqs. (3.254) - (3.256) of [1], $\mathbf{T}^{Rt}_{lm}$ is defined for $l \geq 0$, while $\mathbf{T}^{Et}_{lm}$ and $\mathbf{T}^{Bt}_{lm}$ have $l \geq 1$. Similarly, a complete basis for the expansion of h_{ij} is provided by the six spherical harmonics (Eq. 2.15), where T^{L0}_{lm} and T^{T0}_{lm} have $l \geq 0, T^{E1}_{lm}$ and T^{B1}_{lm} have $l \geq 1$, while T^{E2}_{lm} and T^{B2}_{lm} have $l \geq 2$. Similarly, we saw in Section 3.5.2 that

Common Details of the Tensor Harmonics

These 10 tensor harmonics are normalized so that

$$\int d\Omega \, \eta^{\mu\rho} \eta^{\nu\sigma} \left(\mathbf{T}^a_{l'm'}\right)^*_{\mu\nu} \left(\mathbf{T}^b_{lm}\right)_{\rho\sigma} = \epsilon_a \delta^{ab} \delta_{ll'} \delta_{mm'} \tag{2.14}$$

where the indices a, b run over the set

$$\{L0, T0, E1, B1, E2, B2, tt, Rt, Et, Bt\}. \tag{2.15}$$

The coefficient ϵ_a is equal to -1 for $a = Rt, Et, Bt$, and $+1$ for all the other values of a. There is no sum over a in $\epsilon_a \delta^{ab}$ on the right-hand side, since the index a also appears on the left-hand side. The contractions of the Lorentz indices are performed with the flat Minkowski metric $\eta_{\mu\nu}$. The labels have the following meaning

Label	Mode
$L0$	Longitudinal scalar mode
$T0$	Transverse scalar mode
$E1$	Vector mode with electric-type parity
$B1$	Vector mode with magnetic-type parity
$E2$	Spin-2 mode with electric-type parity
$B2$	Spin-2 mode with magnetic-type parity
tt	Scalar mode corresponding to the time-component h_{00}
Rt	Radial part of the vector h_{0i}
Et	Transverse component of h_{0i} with electric-type parity
Bt	Transverse component of h_{0i} with magnetic-type parity

Table 2.1: Modes and corresponding labels

Let $h_{\mu\nu}$ can be separated into part that depends on the variable (t, r) and a part that depends on the angular variables (θ, ϕ),

$$h_{\mu\nu}(t, \mathbf{x}) = \sum_a \sum_{l,m} H^a_{lm}(t, r)\, (\mathbf{T}^a_{lm})_{\mu\nu}\, (\theta, \phi), \tag{2.16}$$

where the label a runs over the 10 values (Eq. 2.15), and the sum over l runs over $l \geq 0, l \geq 1$ or $l \geq 2$, depending on the tensor spherical harmonic concerned. The $(\mathbf{T}^a_{lm})_{\mu\nu}$ are known as the Zerilli tensor spherical harmonics.

2.1.1 Construction of Zerilli tensor harmonics in Schwarzschild space-time

The tensors $(\mathbf{T}^a_{lm})_{\mu\nu}$ describe the expansion of the metric $h_{\mu\nu}$ in Cartesian coordinates, $x^\mu = (ct, x, y, z)$, i.e. they are a basis for the expansion of $h_{xx}, h_{xy}, \ldots$. Since the background Schwarzschild metric is simpler in polar coordinates (ct, r, θ, ϕ), we need the perturbation with respect to these coordinates, i.e. we need $h_{rr}, h_{r\theta}$, $h_{\theta\theta}, \ldots$. These are related to the components in a Cartesian reference frame, $h_{xx}, h_{xy}, \ldots$ etc.

We use the notation $x^\mu = (ct, x, y, z)$ and $x^\alpha = (ct, r, \theta, \phi)$, and we denote by $h_{\mu\nu}$ the Cartesian components of the metric perturbation, i.e. h_{xx}, h_{xy}, etc., and by $h_{\alpha\beta}$ the polar components, i.e. $h_{rr}, h_{r\theta}$, etc. By definition, the polar components $h_{\alpha\beta}$ are related to the Cartesian components $h_{\mu\nu}$ by

$$h_{\alpha\beta} dx^\alpha dx^\beta = h_{\mu\nu} dx^\mu dx^\nu, \tag{2.17}$$

and similarly we can define the polar components of the Zerilli tensors $(\mathbf{T}^a_{lm})_{\alpha\beta}$ from

$$(\mathbf{T}^a_{lm})_{\alpha\beta}\, dx^\alpha dx^\beta = (\mathbf{T}^a_{lm})_{\mu\nu}\, dx^\mu dx^\nu \tag{2.18}$$

so that

$$h_{\alpha\beta}(t, r, \theta, \phi) = \sum_a \sum_{l,m} H^a_{lm}(t, r)\, (\mathbf{T}^a_{lm})_{\alpha\beta}\,. \tag{2.19}$$

Since the time variable is the same in Cartesian and in polar coordinates, the transformation is only on the spatial variables. Now we define the matrix

$$A_a{}^i = \frac{\partial x^i}{\partial x^a} \tag{2.20}$$

where $x^a = (r, \theta, \phi)$ and $x^i = (x, y, z)$.

Using $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, we get

$$A_a{}^i = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix}, \tag{2.21}$$

Then, for a generic tensor, $T_{ab} = A_a{}^i A_b{}^j T_{ij}$, while $T_{0a} = A_a{}^i T_{0i}$.

At the time of finding expressions for the spatial parts of the Zerilli tensor harmonics we encounter the terms containing $A_a{}^i n_i$, $A_a{}^i \partial_i$, and $A_a{}^i L_i$ (see Eqs. 2.10, 2.11, 2.12). Starting from the expression for n_i in Cartesian coordinates, $n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and, using the explicit form of $A_a{}^i$ (see Eq. 2.21), we get (see Appen. 8.5)

$$n_a \equiv A_a{}^i n_i = (1, 0, 0), \tag{2.22}$$

i.e. $n_r = 1, n_\theta = n_\phi = 0$.

Similarly

$$\partial_a \equiv A_a{}^i \partial_i = (\partial_r, \partial_\theta, \partial_\phi) \tag{2.23}$$

For the angular momentum we get (see Maggiore vol 2)

$$L_a \equiv A_a{}^k L_k = ir\left(0, \frac{1}{\sin\theta}\partial_\phi, -\sin\theta\partial_\theta\right)_a, \tag{2.24}$$

where the components L_a of the angular momentum are with respect to the basis of $1-$forms $dx^a = (dr, d\theta, d\phi)$.

These are related to the Cartesian components L_i by $dx^i L_i = dx^a L_a$, i.e.

$$dxL_x + dyL_y + dzL_z = drL_r + d\theta L_\theta + d\phi L_\phi \tag{2.25}$$

By raising the index a with the metric g^{ab} we get $L^a = g^{ab} L_b$, whose explicit form is given by

$$L^a f = \frac{i}{r\sin\theta}\left(0, \partial_\phi f, -\partial_\theta f\right)^a,$$

Using these results, the expressions for all tensor harmonics in polar coordinates can be found. The result can be written as

$$(\mathbf{T}^a_{lm})_{\alpha\beta} = c^a(r)\,(\mathbf{t}^a_{lm})_{\alpha\beta} \tag{2.26}$$

(no sum over a), where $c^a(r)$ are given by

$$c^{L0} = c^{tt} = 1, \quad c^{T0} = \frac{r^2}{\sqrt{2}}, \quad c^{Rt} = \frac{1}{\sqrt{2}}, \tag{2.27}$$

$$c^{Et} = c^{E1} = -c^{Bt} = -c^{B1} = \frac{r}{[2l(l+1)]^{1/2}}, \tag{2.28}$$

$$c^{E2} = c^{B2} = r^2\left[\frac{1}{2}\frac{(l-2)!}{(l+2)!}\right]^{1/2}, \tag{2.29}$$

and $(\mathbf{t}^a_{lm})_{\alpha\beta}$ are the tensor harmonics (see Sec 12.2 of [2]).

Now the metric perturbation can be written as

$$h_{\alpha\beta}(x) = \sum_a \sum_{lm} h^a_{lm}(t,r)\,(\mathbf{t}^a_{lm})_{\alpha\beta}\,(\theta,\phi), \tag{2.30}$$

where $h^a_{lm}(t,r)$ is defined as

$$h^a_{lm}(t,r) \equiv c^a(r) H^a_{lm}(t,r). \tag{2.31}$$

2.1.2 Polar and axial perturbations

Under a parity transformation, $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$, the tensor harmonics with $a = L0$, $T0$, $E1$, $E2$, tt, Rt and Et pick a factor $(-1)^l$, and are said to be polar, or even. In contrast, those with $a = B1, B2$ and Bt pick a factor $(-1)^{l+1}$, and are said to be axial, or odd. Since the Schwarzschild metric is invariant under parity, when we linearize around it, polar and axial perturbations do not mix. It is therefore useful to separate

$$h_{\alpha\beta}(x) = h^{\mathrm{pol}}_{\alpha\beta}(x) + h^{\mathrm{ax}}_{\alpha\beta}(x),$$

Chapter 3

Black-hole quasi-normal modes

Now we want to solve the RW and Zerilli equations, and we will see that the solutions describe oscillations of space-time that can be interpreted as a characteristic ringing" of black holes.

3.1 General discussion

The RW or Zerilli equation have the general form

$$\Phi''(\omega, x) + \left[\omega^2 - V(x)\right] \Phi(\omega, x) = 0, \tag{3.1}$$

where $x \equiv r_*/c$ ranges from $-\infty$ to $+\infty$, the prime denotes differentiation with respect to $x, \Phi(\omega, x)$ is the Fourier transform with respect to time of either the RW or the Zerilli function, $V(x)$ is the RW or Zerilli potential, respectively, a factor c^2 absorbed into $V(x)$. We will use the qualitative form of $V(x)$ and its asymptotic behavior for $x \to \pm\infty$.

Setting the source term to zero and looking for the evolution of an initial perturbation is the standard method for understanding the intrinsic properties of the system under study.

For BH perturbation, the problem is equivalent to Schrödinger equation, with a potential given by the Zerilli or the RW potentials. But, these potentials do not admit bound states. From eqs. (12.116)-(12.121) of Maggorie Vol 2, we get the appropriate boundary conditions for a BH perturbation $\Phi(\omega, x)$ are

$$\Phi(\omega, x) \propto e^{+i\omega x} \quad (x \to +\infty), \tag{3.2}$$
$$\Phi(\omega, x) \propto e^{-i\omega x} \quad (x \to -\infty), \tag{3.3}$$

or, more compactly,

$$\Phi(\omega, x) \propto e^{+i\omega|x|} \quad (x \to \pm\infty). \tag{3.4}$$

Imposition of these boundary conditions selects some discrete values of ω, and the corresponding solutions are the normal modes of the system. For BHs these normal mode frequencies have both a real and an imaginary part. The corresponding solutions are called quasi-normal modes (QNMs), and the corresponding frequencies are called the quasi-normal mode frequencies (QNM frequencies).

Now we show that the boundary conditions Eq. (3.2) and Eq. (3.3) only take the discrete values of ω. In the time domain, the equivalent equation of Eq. (3.1) can be written as[1]

$$\left[-\frac{\partial}{\partial t^2} + \frac{d^2}{dx^2} - V(x)\right] \Phi(t, x) = 0. \tag{3.6}$$

[1] Here, the Fourier transform is defined as

$$F(t) = \int_{\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{F}(\omega) e^{-i\omega t} \tag{3.5}$$

For $x \to \pm\infty$, $V(x) \to 0$. Therefore, under this limit Eq. 3.7 becomes

$$\left[-\frac{\partial}{\partial t^2} + \frac{d^2}{dx^2}\right]\Phi(t,x) = 0. \tag{3.7}$$

Let, at $x = -\infty$, a right-moving wavepacket takes the form

$$\Phi_0(t,x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A_0(\omega) \exp\{-i\omega(t-x)\}, \quad (x \to -\infty). \tag{3.8}$$

This wavepacket will be partly reflected and partly transmitted by the potential $V(x)$. So, at $x = -\infty$ there will also be a reflected, left-moving, wavepacket

$$\Phi_{\mathrm{r}}(t,x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A_{\mathrm{r}}(\omega) \exp\{-i\omega(t+x)\}, \quad (x \to -\infty), \tag{3.9}$$

At $x = +\infty$ there will be only a right-moving wavepacket,

$$\Phi_{\mathrm{t}}(t,x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A_{\mathrm{t}}(\omega) \exp\{-i\omega(t-x)\}, \quad (x \to +\infty). \tag{3.10}$$

Therefore, the asymptotic solution at $x = -\infty$ will be

$$\Phi(\omega,x) \simeq A_0(\omega)e^{+i\omega x} + A_{\mathrm{r}}(\omega)e^{-i\omega x}, \tag{3.11}$$

and the asymptotitc solution at $x = +\infty$ will be

$$\Phi(\omega,x) \simeq A_{\mathrm{t}}(\omega)e^{+i\omega x}. \tag{3.12}$$

From the conservation of probability we get

$$|A_0(\omega)|^2 = |A_{\mathrm{r}}(\omega)|^2 + |A_{\mathrm{t}}(\omega)|^2. \tag{3.13}$$

Similar to the one-dimensional scattering problem in quantum mechanics, the amplitude for reflection is

$$S(\omega) = \frac{A_{\mathrm{r}}(\omega)}{A_0(\omega)}. \tag{3.14}$$

The boundary conditions Eqs. 3.2 and 3.3 implies $A_0(\omega) = 0$ with $A_{\mathrm{r}}(\omega) \neq 0$, and therefore corresponds to the poles of the scattering amplitude $S(\omega)$. Thus, the boundary conditions Eqs. 3.2 and 3.3 selects some discrete values of ω. In the scattering theory, these special frequencies called the *resonances of the system.*

Since, ω_{QNM} in general is a complex, we can write it as

$$\omega_{\mathrm{QNM}} \equiv \omega_R + i\omega_I \tag{3.15}$$

$$\equiv \omega_R - i\frac{\gamma}{2}. \tag{3.16}$$

Here, an initial BH perturbation decays and disappears either into gravitational radiation at $x = +\infty$ or approaching the horizon at $x = -\infty$. This means that at each fixed x, the perturbation amplitude must eventually go to zero. So for any solution of the form $e^{-i\omega_{\mathrm{QNM}}t}\psi_n(x)$ must have $\omega_I < 0$, or $\gamma > 0$.

Of course, in any realistic macroscopic system the normal-mode frequencies always have an imaginary part, because of dissipation, so in this sense one might think that the difference between BH QNMs and the usual normal modes of elastic bodies is not so much a difference of principle. However, in normal macroscopic bodies the mechanisms responsible for dissipation, and therefore for ω_I, are partly independent of the mechanisms giving rise to rigidity, i.e. to ω_R, and we can tune the parameters of the system, or external parameters such as the temperature, so that $|\omega_I| \ll \omega_R$. We saw for instance in Vol. 1 that the quality factor Q of resonant bars, which is related to ω_R and ω_I by $Q = \omega_R/\left(2\,|\omega_I|\right) = \omega_R/\gamma$, can be made as large as $O\left(10^7\right)$, choosing appropriate materials and working at cryogenic temperatures. In contrast,

3.2 QNMs from Laplace transform

Now we aim for solving the QNM problem using Laplace transfromation. The homogeneous Zerilli or RW equations in the time domain is written as,

$$\left[\partial_x^2 - \partial_t^2 - V(x)\right] \phi(t,x) = 0, \tag{3.17}$$

with $x \equiv r_*/c$. We want to solve the above differential equation using Laplace transformation with the initial conditions

$$\phi(t,x)|_{t=0} \quad \text{and} \quad \partial_t \phi(t,x)|_{t=0}. \tag{3.18}$$

3.2.1 Solution via Laplace transform

Let, $\hat{\phi}(s,x)$ be the Laplace transform of $\phi(t,x)$. Therefore,

$$\hat{\phi}(s,x) = \int_0^\infty dt e^{-st} \phi(t,x) \tag{3.19}$$

where $\mathrm{Re}(s) > 0$.

If the function $\phi(t,x)$ is bounded, $\hat{\phi}(s,x)$ has an analytic continuation into the complex half-plane $\mathrm{Re}(s) > 0$. Equation (12.176) can be inverted to give $\phi(t,x)$ with $t \geqslant 0$, by performing an integral over a contour parallel to the imaginary axis in the complex s-plane, and displaced into the right half-plane by an infinitesimal quantity $\epsilon > 0$, [32]

$$\phi(t,x) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{ds}{2\pi i} e^{st} \hat{\phi}(s,x). \tag{3.20}$$

Taking Laplace transform of Eq. (3.17), we get

$$\int_0^\infty dt e^{-st} \left[\partial_x^2 - \partial_t^2 - V(x)\right] \phi(t,x) = 0 \tag{3.21}$$

$$\text{or, } \int_0^\infty dt e^{-st} \partial_x^2 \phi(t,x) - \int_0^\infty dt e^{-st} \partial_t^2 \phi(t,x) - V(x) \int_0^\infty dt e^{-st} \phi(t,x) = 0 \tag{3.22}$$

$$\text{or, } \partial_x^2 \hat{\phi}(s,x) - \int_0^\infty dt e^{-st} \partial_t^2 \phi(t,x) - V(x) \hat{\phi}(s,x) = 0. \tag{3.23}$$

Finally we obtain

$$\boxed{\left[\partial_x^2 - s^2 - V(x)\right] \hat{\phi}(s,x) = \mathcal{J}(s,x),} \tag{3.24}$$

where

$$\boxed{\mathcal{J}(s,x) = -\left\{s\phi(t,x) + \partial_t \phi(t,x)\right\}|_{t=0}.} \tag{3.25}$$

The source term $\mathcal{J}(s,x)$ therefore depends on the initial conditions at $t = 0$. This is different from the Fourier transform, in which the equation in Fourier space is homogeneous and has no dependence on the initial conditions.

Now we try for solving Eq. (3.24) using method of Green's function. Let $\hat{G}\left(s,x,x'\right)$ be any Green's function of Eq. (3.24). Therefore,

$$\left[\partial_x^2 - s^2 - V(x)\right] \hat{G}\left(s,x,x'\right) = \delta\left(x - x'\right). \tag{3.26}$$

Then the corresponding solution of (3.24) is given by

$$\hat{\phi}(s,x) = \int_{-\infty}^{\infty} dx' \hat{G}\left(s,x,x'\right) \mathcal{J}\left(s,x'\right) \tag{3.27}$$

There are many possible Green's functions. But, once the initial data $\phi(t,x)|_{t=0}$ and $[\partial_t \phi(t,x)]|_{t=0}$ are fixed, the solution of Eq. (3.17) is uniquely determined.

If $\hat{\phi}_+(s,x)$ and $\hat{\phi}_-(s,x)$ are any two linearly independent solutions of the homogeneous equation

$$\left[\partial_x^2 - s^2 - V(x)\right] \hat{\phi}(s,x) = 0, \tag{3.28}$$

the Green's function will be

$$\hat{G}\left(s,x,x'\right) = \frac{1}{W(s)} \Big[\theta\left(x-x'\right) \hat{\phi}_-\left(s,x'\right) \hat{\phi}_+(s,x) + \theta\left(x'-x\right) \hat{\phi}_-(s,x) \hat{\phi}_+\left(s,x'\right)\Big], \tag{3.29}$$

where $\theta(x)$ is the step function and $W(s)$ is the Wronskian of $\hat{\phi}_-(s,x)$ and $\hat{\phi}_+(s,x)$,

$$W(s) \equiv \hat{\phi}_-(s,x)\partial_x\hat{\phi}+_(s,x) - \hat{\phi}_+(s,x)\partial_x\hat{\phi}_-(s,x). \tag{3.30}$$

From this definition, and using the fact that $\hat{\phi}^-$and $\hat{\phi}^+$ are solutions of Eq. (3.28), we can show that Wronskian $W(s)$ is a function of s only.

$$\begin{aligned}
\partial_x W &= \hat{\phi}^- \partial_x^2 \hat{\phi}^+ - \hat{\phi}^+ \partial_x^2 \hat{\phi}^- && (3.31)\\
&= \hat{\phi}^- \left(s^2 + V\right) \hat{\phi}^+ - \hat{\phi}^+ \left(s^2 + V\right) \hat{\phi}^- && (3.32)\\
&= 0. && (3.33)
\end{aligned}$$

Write this portion carefully later

The fact that the function $\hat{G}\left(s,x,x'\right)$ defined by eq. (12.185) gives indeed a Green's function can be verified straightforwardly, by applying the operator $\left[\partial_x^2 - s^2 - V(x)\right]$ to it and using the fact that, for any differentiable function $f(x)$,

$$\begin{aligned}
\partial_x^2\left[\theta\left(x-x'\right) f(x)\right] &= \partial_x\left[\delta\left(x-x'\right) f(x) + \theta\left(x-x'\right) f'(x)\right]\\
&= \delta'\left(x-x'\right) f(x) + 2\delta\left(x-x'\right) f'(x) + \theta\left(x-x'\right) f''(x)\\
&= \delta\left(x-x'\right) f'(x) + \theta\left(x-x'\right) f''(x),
\end{aligned}$$

where in the first line we have used $\partial_x \theta\left(x-x'\right) = \delta\left(x-x'\right)$ and in the last line we have used one of the defining properties of the Dirac delta, $\delta'\left(x-x'\right) f(x) = -\delta\left(x-x'\right) f'(x)$.[33] Similarly

$$\partial_x^2\left[\theta\left(x'-x\right) f(x)\right] = -\delta\left(x-x'\right) f'(x) + \theta\left(x-x'\right) f''(x).$$

Then, applying $\left[\partial_x^2 - s^2 - V(x)\right]$ to the function $\hat{G}\left(s,x,x'\right)$ defined by eq. (12.185), we see that the terms of the generic form $\theta\left(x-x'\right) f''(x)$, i.e. the terms where ∂_x^2 passed throught the step function, cancel using the fact that $\hat{\phi}-$and $\hat{\phi}+$are solutions of the homogeneous equation, while the remaining terms reconstruct $\delta\left(x-x'\right) W(s)$ in the numerator, so that eq. (12.182) is indeed satisfied.

Write the above portion carefully later

Combining Eqs. (3.27) and (3.29), the expression for $\hat{\phi}(s,x)$ can be written as

$$\boxed{\hat{\phi}(s,x) = \frac{1}{W(s)}\left[\hat{\phi}_+(s,x)\int_{-\infty}^{x} dx' \hat{\phi}_-\left(s,x'\right)\mathcal{J}\left(s,x'\right) + \hat{\phi}_-(s,x)\int_{x}^{\infty} dx' \hat{\phi}_+\left(s,x'\right)\mathcal{J}\left(s,x'\right)\right].} \tag{3.34}$$

Now we focus on finding the expressions of $\hat{\phi}_{\pm}$. In the $x \to \pm\infty$ limit, the Zerilli and RW potentials both go to zero and, as long as $\mathrm{Re}(s)$ is non-zero, to leading order in Eq. (3.28) $V(x)$ can be neglected with respect to s^2. So, for $x \to \pm\infty$ Eq. (3.28) becomes

$$\left[\partial_x^2 - s^2\right] \hat{\phi}(s, x) \simeq 0. \tag{3.35}$$

This has the solutions $e^{\pm sx}$. In other words, we can choose the two linearly independent solutions of the second-order differential equation Eq. (3.28) so that, at $x \to +\infty$, one is proportional to e^{+sx} and the other to e^{-sx}. The most general solution of the homogeneous equation therefore has the asymptotic form

$$\hat{\phi}_{\text{hom}}(s, x) \to a_1(s)e^{sx} + a_2(s)e^{-sx} \quad (x \to +\infty). \tag{3.36}$$

$$\hat{\phi}_{\text{hom}}(s, x) \to b_1(s)e^{sx} + b_2(s)e^{-sx} \quad (x \to -\infty). \tag{3.37}$$

Since $\mathrm{Re}(s) > 0$, a generic solution with $a_1(s) \neq 0$ and $b_2(s) \neq 0$ diverges exponentially both at the horizon $(x = -\infty)$ and at spatial infinity $(x = -\infty)$. At the horizon, the special solution with $b_2(s) = 0$ is finite but, at spatial infinity for generic values of s, the solution will be of the form (Eq. (3.36)) with $a_1(s)$ and $a_2(s)$ both non-vanishing.

For all time, linear perturbation of the Schwarzschild metric uniformly remain bounded in space (Wald 1979).

Since linear perturbations of the Schwarzschild space-time are uniformly bounded in x (see Note 27 on page 142), and since $\phi(t, x)$ is obtained, through eq. (12.177), from $\hat{\phi}(s, x)$ with $\mathrm{Re}(s) > 0, \hat{G}(s, x, x')$ must be such that the solution obtained from eq. (12.183) is bounded everywhere and therefore also for $x \to \pm\infty$.

For mathematical simplicity, consider an initial perturbation such that $\mathcal{J}(s, x')$ in Eq. (3.27) is non-vanishing only for $x_L < x' < x_R$.

For $x > x_R$, the second integral in Eq. (3.34) vanishes, and

$$\hat{\phi}(s, x) = \frac{c_-(s)}{W(s)} \hat{\phi}_+(s, x) \quad (x > x_R), \tag{3.38}$$

where

$$c_-(s) = \int_{x_L}^{x_R} dx' \hat{\phi}_-(s, x') \mathcal{J}(s, x'). \tag{3.39}$$

Therefore the solution $\hat{\phi}(s, x)$ [with $\mathrm{Re}(s) > 0$] is bounded as $x \to +\infty$ if, and only if, $\hat{\phi}_+(s, x)$ is bounded as $x \to +\infty$. Similarly, at $x < x_L$ we get

$$\hat{\phi}(s, x) = \frac{c_+(s)}{W(s)} \hat{\phi}_-(s, x) \quad (x < x_L),$$

where

$$c_+(s) = \int_{x_L}^{x_R} dx' \hat{\phi}_+(s, x') \mathcal{J}(s, x').$$

So, in order to have $\hat{\phi}(s, x)$ bounded at $x \to -\infty$, we must require that $\hat{\phi}_-(s, x)$ stays bounded at $x \to -\infty$. Comparing with the asymptotic behaviors (12.192) and (12.193), we see that $\hat{\phi}_+(s, x)$ is fixed uniquely as that particular solution of the homogeneous equation that has $a_1(s) = 0$, and $\hat{\phi}_-(s, x)$ as that with $b_2(s) = 0$. The asymptotic behaviours of $\hat{\phi}_-(s, x)$ and $\hat{\phi}_+(s, x)$ are

$$\hat{\phi}_-(s, x) \simeq \begin{cases} e^{+sx} & (x \to -\infty), \\ a_1(s)e^{sx} + a_2(s)e^{-sx} & (x \to +\infty) \end{cases} \tag{3.40}$$

and

$$\hat{\phi}_+(s,x) \simeq \begin{cases} b_1(s)e^{sx} + b_2(s)e^{-sx} & (x \to -\infty), \\ e^{-sx} & (x \to +\infty). \end{cases} \tag{3.41}$$

$\hat{\phi}_-(s,x)$ is normalized so that, for $x \to -\infty$, it approaches e^{sx} with a proportionality coefficient of unity, and similarly for $\hat{\phi}_+(s,x)$ at $x \to +\infty$.

The solution (Eq. (3.34)) is independent of the normalization of $\hat{\phi}\pm(s,x)$, since a rescaling of $\hat{\phi}_\pm$ in the numerator is compensated by the same rescaling in the Wronskian in the denominator.

Since, Wronskian $W(s)$ is independent of x, we choose $x \to -\infty$ limit for the computation of $W(s)$. Using the normalized asympotic forms of $\hat{\phi}_\pm$ (Eqs. (3.40) and (3.41)), the Wronskian will be

$$W(s) = -2sb_2(s) \tag{3.42}$$

Similarly for $x \to \infty$, the Wronskian will be

$$W(s) = -2sa_1(s) \tag{3.43}$$

Comparing Eqs. (3.42) and (3.43) we get

$$\boxed{a_1(s) = b_2(s)} \tag{3.44}$$

3.2.2 QNMs as poles in the complex s-plane

The foregoing analysis has shown that the Green's function is in principle uniquely determined and therefore, using eq. (12.183), the solution corresponding to given initial data is also in principle fixed. Below we will give the explicit form of the solutions $\hat{\phi}_\pm(s,x)$ for all x. However, it is possible to understand the emergence of QNMs without making reference to the explicit (and rather involved) solutions. To this purpose,

Now, we examine $\phi(t,x)$, which is obtained from $\hat{\phi}(s,x)$ by applying inverse Laplace transform (using Eq. (3.20)). For mathematical simplicity, we consider $\mathcal{J}(s,x')$ is non-vanishing only for $x_L < x' < x_R$, and for

For $x > x_R$, $\phi(t,x)$ becomes

$$\phi(t,x) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} ds\, e^{st}\hat{\phi}(s,x) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} dse^{st}\frac{c_-(s)}{W(s)}\hat{\phi}_+(s,x). \tag{3.45}$$

We now attempt to close the contour in the complex s-plane. We are interested in the response to an initial perturbation specified at $t = 0$, so we want the solution for $t > 0$. The factor e^{st} diverges on a circle at infinity in the right half of the complex plane, where $\mathrm{Re}(s) > 0$, so we can only hope to be able to close the contour in the left half of the complex plane, where $\mathrm{Re}(s) < 0$. We then consider the contour shown in Fig. 12.4 of Maggiore.

Assume that the function $\hat{\phi}+(s,x)$, as well as $\hat{\phi}-(s,x)$ and the source term $\mathcal{J}(s,x)$ that enter in $c_-(s)$, are analytic in s, so there are no essential singularities or branch cuts inside the contour shown in Fig. 12.4. We hasten to add that these assumptions turn out to be correct if the potential $V(x)$ has compact support (such as a square barrier) or goes to zero exponentially at both boundaries, but are wrong for the Zerilli and RW potentials, which have a power-like behavior at infinity. We will therefore come back to these assumptions later. First, however, it is instructive to see what would the result be if these assumptions were fulfilled. We also neglect for the moment the contribution of the semicircle at infinity.

Under these assumptions, the only contributions to the integral can come from the poles of the integrand due to zeros of $W(s)$. Suppose that there is a set of simple zeros in the left half of the complex plane, at complex values $s = s_n$. Near these zeros $W(s)$ can be written as

$$W(s) = W'(s_n)(s - s_n) + O(s - s_n)^2 \tag{3.46}$$

where

$$W'(s_n) \equiv \left.\frac{dW}{ds}\right|_{s=s_n}. \tag{3.47}$$

Picking the residues of the poles (and observing that the poles are encircled counterclockwise), Eq. (3.45) becomes

$$\phi(t,x) = \sum_n e^{s_n t} \frac{c_-(s_n)}{W'(s_n)} \hat{\phi}_+(s_n, x) \tag{3.48}$$

$$= \sum_n c_n u_n(x) e^{s_n t} \quad \text{[Define, } c_n = \frac{c_-(s_n)}{W'(s_n)} \text{ and } u_n(x) \equiv \hat{\phi}_+(s_n,x)\text{]} \tag{3.49}$$

where c_n and $u_n(x)$ are defined as

$$c_n = \frac{c_-(s_n)}{W'(s_n)} = \frac{1}{W'(s_n)} \int_{x_L}^{x_R} dx' \hat{\phi}_-(s_n, x') \mathcal{J}(s_n, x'), \tag{3.50}$$

$$u_n(x) = \hat{\phi}_+(s_n, x) \tag{3.51}$$

We observe that, under the assumptions $\phi(t,x)$ can be expressed as a sum over a discrete set of terms. Using the condition $W(s_n) = 0$ in Eq. (3.30),

$$W(s_n) = 0 \tag{3.52}$$

$$\text{or, } \hat{\phi}_-(s_n,x)\partial_x\hat{\phi}_+(s_n,x) - \hat{\phi}_+(s_n,x)\partial_x\hat{\phi}_-(s_n,x) = 0 \tag{3.53}$$

$$\text{or, } \hat{\phi}_-(s_n,x)\hat{\phi}'_+(s_n,x) - \hat{\phi}_+(s_n,x)\hat{\phi}'_-(s_n,x) = 0 \quad \text{[Prime denotes differentiation with respect to } x\text{]} \tag{3.54}$$

$$\text{or, } \hat{\phi}_-(s_n,x)\hat{\phi}'_+(s_n,x) = \hat{\phi}_+(s_n,x)\hat{\phi}'_-(s_n,x) \tag{3.55}$$

$$\text{or, } \frac{\hat{\phi}'_+(s_n,x)}{\hat{\phi}_+(s_n,x)} = \frac{\hat{\phi}'_-(s_n,x)}{\hat{\phi}_-(s_n,x)} \tag{3.56}$$

$$\text{or, } \int \frac{d\hat{\phi}_+(s_n,x)}{\hat{\phi}_+(s_n,x)} = \int \frac{d\hat{\phi}_-(s_n,x)}{\hat{\phi}_-(s_n,x)} + \ln c(s_n) \quad \text{[Integrating on both sides, } \ln c_n(s) \text{ is integral constant]} \tag{3.57}$$

$$\text{or, } \ln \hat{\phi}_+(s_n,x) = \hat{\phi}_-(s_n,x) + \ln c(s_n) \tag{3.58}$$

$$\text{or, } \hat{\phi}_+(s_n,x) = c(s_n)\hat{\phi}_-(s_n,x). \tag{3.59}$$

This means, for $s = s_n$ the two solutions of the homogeneous equation, $\hat{\phi}_+(s,x)$ and $\hat{\phi}_-(s,x)$, are no longer independent, and we have a single solution

$$\hat{\phi}_+(s_n,x) \propto \hat{\phi}_-(s_n,x) \equiv \hat{\phi}*(s_n,x) \tag{3.60}$$

From Eqs. (3.40) and (3.41), we know $\hat{\phi}_-(s,x)$ approaches e^{+sx} for $x \to -\infty$ and $\hat{\phi}_+(s,x)$ goes to e^{-sx} for $x \to +\infty$. Therefore

$$\hat{\phi}_*(s_n,x) \propto e^{-s_n|x|}, \quad (x \to \pm\infty). \tag{3.61}$$

To relate this with the QNMs, discussed in the previous section in terms of the Fourier transform, and also to relate with the BHS, we write s as

$$s = -i\omega, \tag{3.62}$$

where

$$\omega = \omega_R + i\omega_I \tag{3.63}$$

We introduce the following notation

$$\hat{u}^{\text{in}}(\omega, x) \equiv \hat{\phi}_-(-i\omega, x), \tag{3.64}$$

$$\hat{u}^{\text{up}}(\omega, x) \equiv \hat{\phi}_+(-i\omega, x). \tag{3.65}$$

Using this notation, (3.40) and (3.41) become

$$\hat{u}^{\text{in}}(\omega, x) \simeq \begin{cases} e^{-i\omega x} & (x \to -\infty), \\ A_1(\omega)e^{-i\omega x} + A_2(\omega)e^{i\omega x} & (x \to +\infty), \end{cases} \tag{3.66}$$

where $A_{1,2}(\omega) \equiv a_{1,2}(-i\omega)$, and

$$\hat{u}^{\text{up}}(\omega, x) \simeq \begin{cases} B_1(\omega)e^{-i\omega x} + B_2(\omega)e^{i\omega x} & (x \to -\infty), \\ e^{i\omega x} & (x \to +\infty), \end{cases} \tag{3.67}$$

where $B_{1,2}(\omega) \equiv b_{1,2}(-i\omega)$.

The time dependence is given by $e^{-i\omega t}$, and that $x = r_*/c$, we see that $\hat{u}^{\text{in}}$ is purely ingoing at the horizon, i.e. it describes a wave that, near the horizon, is falling toward the BH, while $\hat{u}^{\text{up}}$ is purely outgoing at future null infinity. In terms of $A_1(\omega)$, Eq. (3.43) can be written as

$$\boxed{W = 2i\omega A_1(\omega),} \tag{3.68}$$

and $A_1(\omega) = B_2(\omega)$. Similarly, computing the Jacobian between the "up" and "out" solutions (see Note 35) at $x \to +\infty$ and comparing it with the value at $x \to -\infty$, we get $B_1(\omega) = -A_2^*(\omega)$.

Writing $s_n = -i\omega_n$ and $u_*(\omega_n, t) \equiv \phi_*(s_n = -i\omega_n, x)$, Eq. (3.61) becomes

$$u_*(\omega_n, t) \simeq e^{+i\omega_n|x|} \quad (x \to \pm\infty). \tag{3.69}$$

These are just the boundary conditions satisfied by the solutions of the Fourier-transformed Zerilli or RW equation which we called the QNMs (see Eq. (3.4)).

Under the replacement $s = -i\omega$, the homogeneous Laplace-transformed equation (Eq. (3.28)) becomes identical to the Fourier-transformed equation (Eq. (3.28)), so both equations should give similar solutions. We observe that the solutions $\hat{\phi}_*(s_n, x)$ are just the QNMs, expressed using the Laplace transform. Writing $\omega_{I,n} = -\gamma_n/2$, we have

$$s_n = \omega_{I,n} - i\omega_{R,n} = -\frac{\gamma_n}{2} - i\omega_{R,n}. \tag{3.70}$$

So Eq. (3.49) can be rewritten as

$$\phi(t, x) = \sum_n c_n u_n(x)e^{-i\omega_n t} = \sum_n c_n u_n(x)e^{i\omega_{R,n}t}e^{-\gamma_n t/2}. \tag{3.71}$$

Since the poles s_n inside the integration contour have $\text{Re}(s_n) < 0$, we have $\gamma_n > 0$, so this solution is a superposition of damped oscillations.

$\text{Re}(s_n) < 0$ also means that the QNM wavefunction $u_n(x)$ diverges exponentially both at spatial infinity and at the horizon as $\exp\{\gamma_n|x|/2\}$, (see Eq. (3.61)), we already found in Eq. (12.171) in Maggorie.

In Section 12.3.4 we will see how to compute the QNM frequencies. Then, the analytic result (12.220) can be compared with the direct numerical integration of the differential equation. Figure 12.5 shows the comparison for the scalar wave equation given by eqs. (12.9) and (12.6), for $l = 2$. Observe that, by including more QNMs, the agreement can be pushed toward earlier times.

3.3 Power-law tails

The RW equation (Eq. (12.102) of Maggiore Vol 2) in the absence of source can be recast in a form known as a generalized spheroidal wave equation, whose solutions have the form (in units $c = G = 2M = 1$)

$$\hat{\phi}_+(s,r) = (2is)^s e^{i\phi_+}(1-1/r)^s \times \sum_{L=-\infty}^{\infty} b_L \left[G_{L+\nu}(-is, isr) + iF_{L+\nu}(-is, isr)\right] \tag{3.72}$$

$$\hat{\phi}_-(s,r) = r^{-2s}(r-1)^s e^{-s(r-2)} \sum_{n=0}^{\infty} a_n (1-1/r)^n \tag{3.73}$$

where $G_{L+\nu}$ and $F_{L+\nu}$ are Coulomb wavefunctions, and the coefficients a_n and b_L, as well as the parameter ν, are defined in terms of a three-term recurrence relation (Eqs. 17 and 18 and Appendix A of [3]). We observe that $\hat{\phi}_-(s,r)$ is analytic in s, while $\hat{\phi}_+(s,r)$ has a branch cut in the complex s-plane, along the negative real axis, because of the factor $s^s = e^{s\log s}$.
There are three distinct contribution to $\phi(t,x)$:

1. the contribution from the poles inside the contour;
2. the contribution from the two quarter-circles at infinity;
3. the contribution from the branch cut.

The sum of these three contributions determines the response of the BH to an initial perturbation.

To find out the properties intrinsic to the BH space-time, rather than using $\phi(t,x)$, which depends on the details of the initial conditions encoded in $\mathcal{J}(s,x)$, we want to work with the Green's function $G(t,x,x')$ in the time domain.

3.3.1 The time-domain Green's function $G(t,x,x')$

When $t \geqslant 0$, define $G(t,x,x')$ by

$$G\left(t,x,x'\right) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds}{2\pi i} e^{st} \hat{G}\left(s,x,x'\right). \tag{3.74}$$

We complete the definition of $G(t,x,x')$ by setting $G(t,x,x') = 0$ for $t < 0$. This defines the retarded Green's function. From Eq. (3.26), we get

$$\left[\partial_x^2 - \partial_t^2 - V(x)\right] G\left(t,x,x'\right) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds}{2\pi i} e^{st} \left[\partial_x^2 - s^2 - V(x)\right] \hat{G}\left(s,x,x'\right) \tag{3.75}$$

$$= \delta\left(x-x'\right) \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds}{2\pi i} e^{st}. \tag{3.76}$$

Using the properties of the Laplace transform (see Note 38 of Maggiore)

$$\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds}{2\pi i} e^{st} = \delta(t) \tag{3.77}$$

Therefore

$$\left[\partial_x^2 - \partial_t^2 - V(x)\right] G\left(t,x,x'\right) = \delta\left(x-x'\right)\delta(t). \tag{3.78}$$

So $G(t,x,x')$ is the (retarded) Green's function of the operator $\left[\partial_x^2 - \partial_t^2 - V(x)\right]$. Performing the inverse Laplace transform of eq. (12.183) of Maggiore and using the expression of $\mathcal{J}(s,x)$ (Eq. (3.25)), we find

$$\phi(t,x) = -\int_{-\infty}^{\infty} dx' \left[\phi(t=0,x) \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds}{2\pi i} s e^{st} \hat{G}\left(s,x,x'\right) + \partial_t\phi(t=0,x) \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{ds}{2\pi i} e^{st} \hat{G}\left(s,x,x'\right)\right]. \tag{3.79}$$

In the first integral, using $se^{st} = \partial_t e^{st}$ we get

$$\phi(t,x) = -\int_{-\infty}^{\infty} dx' \left[\partial_t G\left(t,x,x'\right)\phi(t=0,x) + G\left(t,x,x'\right)\partial_t\phi(t=0,x)\right]. \tag{3.80}$$

Therefore, given the initial data $\phi(t=0,x)$ and $\partial_t\phi(t=0,x)$, we can construct $\phi(t,x)$ from the Green's function $G\left(t,x,x'\right)$.

Assume $\mathcal{J}\left(x'\right)$ is non-vanishing only for $x_L < x' < x_R$, with $R_S/c \ll x_L$ $(x = r_*/c)$. Now we study the response seen by an observer at $x > x_R$ (so also $x \gg R_S/c$). Therefore, we have $x > x'$. Using this Eq. (3.34) becomes

$$\hat{G}\left(s,x,x'\right) = \frac{1}{W(s)}\hat{\phi}_-\left(s,x'\right)\hat{\phi}_+(s,x). \tag{3.81}$$

At $s = -i\omega$, the expression of $\hat{G}\left(s,x,x'\right)$ becomes

$$\hat{G}\left(s=-i\omega,x,x'\right) = \frac{1}{W(s=-i\omega)}\hat{\phi}_-\left(-i\omega,x'\right)\hat{\phi}_+(-i\omega,x) = \frac{1}{W(s=-i\omega)}\hat{u}^{\rm in}(\omega,x')\hat{u}^{\rm up}(\omega,x). \tag{3.82}$$

Since both x and x' are in the asymptotic region, using the asymptotic behaviour of $\hat{u}^{\rm in}(\omega,x)$ and $\hat{u}^{\rm up}(\omega,x')$ at spatial infinity (Eq. (3.66) and (3.67)) and using the expression of Wronskian (Eq. (3.68)), we get

$$\hat{G}\left(s=-i\omega,x,x'\right) \simeq \frac{1}{2i\omega}\left[e^{i\omega(x-x')} + \frac{A_2(\omega)}{A_1(\omega)}e^{i\omega(x+x')}\right]. \tag{3.83}$$

This is called the "asymptotic approximation" to the Green's function. Inserting this into Eq. (12.223) of Maggiore, and rotating the contour from the imaginary to the real axis (see Note 38 of Maggiore) gives

$$G\left(t,x,x'\right) \simeq \int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{d\omega}{2\pi}\frac{1}{2i\omega}\left[e^{i\omega(x-x'-t)} + \frac{A_2(\omega)}{A_1(\omega)}e^{i\omega(x+x'-t)}\right]. \tag{3.84}$$

Now we close the contour. In the complex ω-plane the contour is shown in Fig. (3.1). $G\left(t,x,x'\right)$

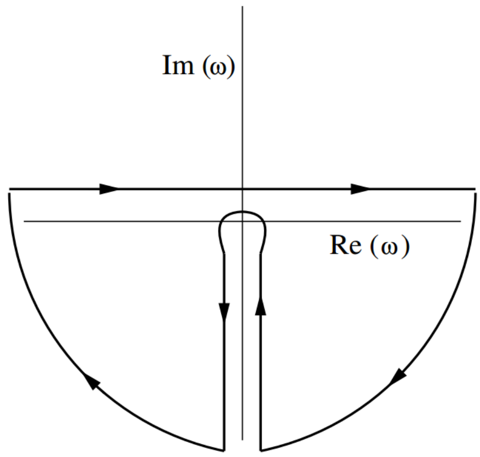

Figure 3.1: The Integration in the ω-plane (Taken from Maggiore Vol 2)

separates into three contributions,

$$G\left(t,x,x'\right) = G_{\rm QNM}\left(t,x,x'\right) + G_{\rm B}\left(t,x,x'\right) + G_{\rm F}\left(t,x,x'\right), \tag{3.85}$$

where $G_{\rm QNM}\left(t,x,x'\right)$ is the contribution from the poles inside the contour, $G_B\left(t,x,x'\right)$ is the contribution from the branch cut, and $G_F\left(t,x,x'\right)$ is the contribution from the two quarter-circles at infinity. Now, we discuss the three contributions one by one.

Contribution from the poles inside the contour

The QNMs correspond to the poles of the Wronskian, i.e. of $A_1(\omega)$. Near a QNM with complex frequency ω_n we can write

$$A_1(\omega) = A_1'(\omega_n)(\omega - \omega_n) + O(\omega - \omega_n)^2 . \tag{3.86}$$

Because of reality, the QNM frequencies appear in pairs with $\omega_n = (\pm\omega_{R,n}) + i\omega_{I,n}$ and $\omega_{I,n} = -\gamma_n < 0$ (see Note 36). Then, in the asymptotic approximation, the expression of $G_{\mathrm{QNM}}(t, x, x')$ becomes (taking into account that the poles are circled clockwise)

$$G_{\mathrm{QNM}}(t, x, x') \simeq \mathrm{Re}\left[\sum_n B_n e^{-i\omega_n(t-x-x')}\right] = \mathrm{Re}\left[\sum_n B_n e^{-i\omega_{R,n}(t-x-x')} e^{-(\gamma_n/2)(t-x-x')}\right], \tag{3.87}$$

where the coefficients B_n are called the *quasi-normal excitation factors* given by

$$B_n = -\frac{A_2(\omega_n)}{\omega_n A_1'(\omega_n)}, \tag{3.88}$$

and the sum runs over all normal modes, with one representative for each pair $(\pm\omega_{R,n}) + i\omega_{I,n}$. They depend only on the BH properties, and not on the initial perturbation.

Contribution from the branch cut

The contribution from the branch cut can be computed using the exact solutions $\hat{\phi}_+(s, r)$ and $\hat{\phi}_+(s, r)$ and putting in Eq. (12.230) of Maggiore, as well as into the Wronskian. For exact expression of $G_{\mathrm{B}}(t, x, x')$ see [4]. We want to observe the behaviour of $G_{\mathrm{B}}(t, x, x')$ n the limit $t \gg x + x'$, since the expression becomes simpler.

Let, the source of disturbance is localized at a distance $\sim cx'$ from the centre of the BH (with $x_L < x' < x_R$ where $x \equiv r_*/c$) much larger than R_S. Then x' is the time needed for a disturbance traveling at the speed of light to go from the source to the near-horizon region, where the potential barrier of $V(x)$ is large. So x' is the time-scale needed for the initial disturbance to propagate to the near-horizon region and excite the quasi-normal modes. x is the time that the signal generated by the QNMs takes to go back from the near-horizon region to the observer. Thus, the signal due to QNMs ringing reaches the observer at time $\sim (x + x')$, and the condition $t \gg x + x'$ means that, at the fixed location x of the observer, the part of the BH signal due to the ringing of the QNMs has already passed through the observer position. Our aim is to find any residual signal for the fixed position x, after the QNM signal fades away.

In the limit $t \gg x + x'$, the integral along the branch cut is dominated by small values of $|\omega|$. While taking $t \gg x + x'$, we can further distinguish two different limits:

1. **Case (i): ($t \to \infty$ with x fixed)**
   In the limit $t \to \infty$ with x fixed, the expression of $G_{\mathrm{B}}(t, x, x')$ is [5]

$$G_{\mathrm{B}}(t, x, x') \simeq (-1)^{l+1}\frac{R_S}{c} \times \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{2^{1-m}(2l+2m+2)!}{n!(m-n)!(2l+2n+1)!!(2l+2m-2n+1)!!} \times \frac{x^{l+2m-2n+1}x'^{l+1+2n}}{t^{2l+3+2m}} \tag{3.89}$$

   In the leading term ($m = 0, n = 0$), the above equation becomes

$$G_{\mathrm{B}}(t, x, x') \simeq (-1)^{l+1}\frac{2(2l+2)!}{[(2l+1)!!]^2}\frac{R_S}{c}\frac{(xx')^{l+1}}{t^{2l+3}}. \tag{3.90}$$

   We observe a power-law tail at spatial infinity, i.e. a non-radiative tail.

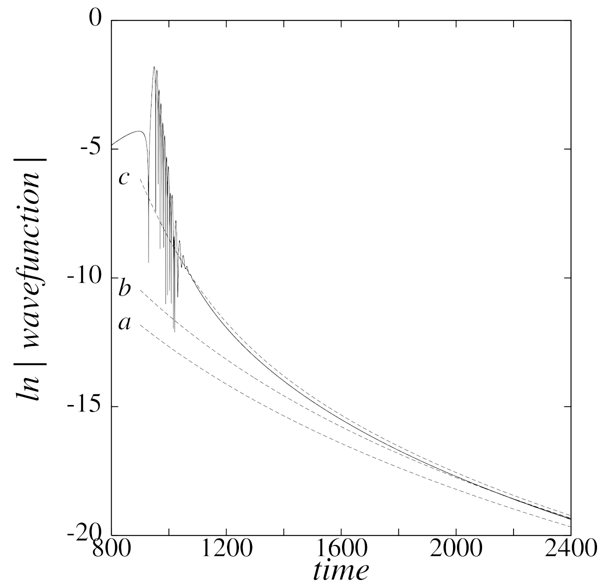

Figure 3.2: The numerical evolution of the line $l = 2$ scalar wave equation (solid line) compared with the power-law tail (dashed line): (a) the leading-order power-law tail, (b) the tail approximation including the first two terms and (c) using the first 11 terms in the tail-sum. [Taken from Andersson (1997)]

2. **Case (ii): ($t \to \infty$ with t/x fixed)**
   In the limit $t \to \infty$ with t/x fixed, we are looking for radiative terms at future null infinity. The expression of $G_{\mathrm{B}}(t, x, x')$ is

$$G_{\mathrm{B}}\left(t, x, x'\right) \simeq (-1)^{l+1} \frac{(l+1)!}{(2l+1)!!} \frac{R_S}{c} \frac{\left(x'\right)^{l+1}}{u^{l+2}}, \tag{3.91}$$

   where $u \equiv t - x$.

   We again get a power-law tail. Since we are at future null infinity, this is a radiative contribution, i.e. a radiative tail.

   Both the overall amplitude of the tail term, and the exponent $l + 2$ of the power-law decay, are independent of the spin of the perturbation, i.e. they are the same for scalar perturbations, and for gravitational perturbations, governed by the RW or Zerilli equation. Same result also holds for electromagnetic, and for higher-spin perturbations.

The nature of initial data determines the effect of the behavior of Green's function on the solution $\phi(t, x)$. For example, if the initial is $\phi(t = 0, x) = 0$, then Eq. 12.229 of Maggiore follows that the time behavior of $\phi(t, x)$ is the same as that of $G(t, x, x')$, i.e. $\sim u^{-l-2}$ at future null infinity and $\sim t^{-2l-3}$ at spatial infinity.

The QNM ringing physically represents the oscillations of space-time in the near-horizon region, actually in the region inside the peak of the RW or Zerilli equation.

Now we want to explain the power-law tails. The presence of tails is depends the form of $V(x)$ as $x \to +\infty$. It is independent of whether or not the metric has a horizon, so the same tails would be generated by a neutron star and by a black hole of the same mass. For example, if the potential behaves asymptotically as $l(l+1)/r^2$ (corresponding to a free propagation in the original three-dimensional problem, since this is just the centrifugal potential that comes from writing the Laplacian in polar coordinates), with no subleading powers, there are no power-law tails. The addition of higher powers such as $1/x^3$, $1/x^4$, etc. to the large- x limit of the potential generates tails. The tails are absent if the potential has a potential barrier, or if it decreases exponentially at infinity.

These results indicate that the tails are generated by back-scattering off the space-time curvature at large distances. They correspond to a part of the signal that was emitted outward from the source at a radial distance cx', reached a large distance, and was then scattered back toward the observer at radial distance cx; see Fig. 12.9 of Maggiore Vol 2. The propagation direction of the tail is not the same as that of the QNM part of the signal, which rather propagates directly from the central object to the observer.

Contribution from the two quarter-circles at infinity

Finally, we have the contribution from the two quarter-circles at infinity. These contributions are evaluated in the limit $|\omega| R_S/c \to \infty$. In this limit, $|\omega|$ is much larger than the height of the peak of the potential, so effectively they can be computed setting $V = 0$. They describe the free propagation of the high-frequency components of that part of the initial perturbation which moves outward, and travels directly from the source to the observer.

Summary of different contributions

If at $t = 0$ we have an initial disturbance localized around a radial distance cx' from the BH, the signal seen by an observer located at a distance $cx > cx'$ (taken for simplicity in the same angular direction) can be separated into the following main contributions:

1. For $t < x - x'$, for causality reasons, no signal has yet reached the observer.
2. At $t \simeq x - x'$ there arrives that part of the initial disturbance that was initially moving outward. This is propagated by $G_F\left(t, x, x'\right)$. Because of dispersion effects, part of this signal will arrive at a later time, so this will give the main contribution until the next epoch, which, as we will see, is at $t \simeq x + x'$. This part of the signal is known as the precursor.
3. At $t \simeq x + x'$ there arrives the signal from the QNMs of the BH. In fact, the part of the signal that traveled toward the BH arrives in a time $\sim x'$ in the near-horizon region, where it excites the QNMs. The resulting excitation takes a time $\sim x$ to propagate outward to the observer. This ringdown phase lasts for a time given by the lifetime of the longest-lived QNM. As we will see in the next section, this is typically of the order of R_S/c, times numerical factors. The relevant part of the Green's function in this phase is G_{QNM}.
4. When the signal from the QNMs has vanished exponentially, we can begin to see the late-time power-law tail (which in principle was present even before, but was masked by the stronger contribution from the QNMs). The relevant part of the Green's function in this phase is G_{B}.

Since an expansion of the form (Eq. (3.71)) does not represent the full signal, we say that QNMs do not form a complete set. This can be also explained by the Green's function in the complex ω-plane. In Green's function, beside the contribution of the poles inside the integration contour, which corresponds to QNMs, we also have a contribution from the branch cut and from the quarter-circles at infinity.

Chapter 4

Perturbative QNM Frequencies

To find the quasi-normal modes we need to solve the following (radial) master equation

$$\boxed{\left(\frac{d^2}{dx^2} + \omega^2 - V(x)\right)\Phi(x) = 0} \tag{4.1}$$

where x is the tortoise coordinate which follows the relation

$$\frac{dx}{dr} = \frac{1}{f(r)} \tag{4.2}$$

where $f(r)$ is a function that has a zero at the event horizon $r = r_H$. Explicit forms of $f(r)$ depends on the the particular problem. The QNM boundary condition is given by

$$\Phi(x) \sim e^{\pm i\omega x} \quad (x \to \pm\infty). \tag{4.3}$$

Now we want to solve Eq. (4.1) perturbatively using the boundary condition Eq. (4.3). But, we cannot apply the well-known formula in quantum mechanics to the computation of perturbative corrections to the QNM spectrum since in the QNM problem, a set of the eigenfunctions is not complete in the usual sense.

Now, we assume that all the quantities in Eq. 4.1 have smooth perturbative expansions in the parameter α

$$V(x) = \sum_{k=0}^{\infty} \alpha^k V_k(x), \quad \omega^2 = \mathcal{E} = \sum_{k=0}^{\infty} \alpha^k \mathcal{E}_k, \quad \Phi(x) = \sum_{k=0}^{\infty} \alpha^k \Phi_k(x). \tag{4.4}$$

Typically the parameter α is the deformation parameter of a black hole or a modified theory.

We also ω as a series of α,

$$\omega = \sum_{k=0}^{\infty} \alpha^k \omega_k, \tag{4.5}$$

the QNM boundary condition (Eq. 4.3) can be written as

$$\Phi \sim \exp\left(\pm ix \sum_{k=0}^{\infty} \alpha^k \omega_k\right) \tag{4.6}$$

$$= e^{i\pm\omega_0 x} e^{\pm i(\alpha\omega_1 + \alpha^2\omega_2 + \cdots)x} \tag{4.7}$$

$$= e^{\pm i\omega_0 x}(1 + \alpha P_1^{\pm}(x) + \alpha^2 P_2^{\pm}(x) + \cdots), \tag{4.8}$$

where $P_1^{\pm}(x)$, $P_2^{\pm}(x)$, $\cdots$ are polynomials of x.

Comparing Eq. 4.8 with the expression

$$\Phi(x) = \sum_{k=0}^{\infty} \alpha^k \Phi_k(x) \tag{4.9}$$

we get the boundary condition for Φ_k to be

$$\Phi_k \sim e^{\pm i\omega_0 x} \tag{4.10}$$

Now we want to solve Eq. 4.1 perturbatively in α.

$$\text{For } \alpha^0: \quad \left(\frac{d^2}{dx^2} + \mathcal{E}_0 - V_0(x)\right)\Phi_0(x) = 0 \tag{4.11}$$

$$\text{For } \alpha^k (k>0): \quad \left(\frac{d^2}{dx^2} + \mathcal{E}_0 - V_0(x)\right)\Phi_k(x) + \sum_{l=1}^{k}(\mathcal{E}_l - V_l(x))\Phi_{k-l}(x) = 0 \tag{4.12}$$

4.1 Zeroth order correction

$$\left(\frac{d^2}{dx^2} + \mathcal{E}_0 - V_0(x)\right)\Phi_0(x) = 0 \tag{4.13}$$

where $\mathcal{E}_0$ denotes the zeroth order eigenvalue in perturbation in α, not the fundamental mode eigenvalue. We solve the zeroth order equation (Eq. 4.13) under the ordinary QNM boundary condition:

$$\Phi_0(x) \sim e^{\pm i\omega_0 x} \quad (x \to \pm\infty) \tag{4.14}$$

There are many techniques to solve Eq. 4.13 numerically.

4.2 First order expression

After obtaining the eigenvalue and the eigenfunction at the zeroth order, we go for solving the first order equation. We need to solve the equation

$$\left(\frac{d^2}{dx^2} + \mathcal{E}_0 - V_0(x)\right)\Phi_1(x) + (\mathcal{E}_1 - V_1(x))\Phi_0(x) = 0 \tag{4.15}$$

The above equation is a homogeneous differential equation for $\Phi_1(x)$ with the unknown constant $\mathcal{E}_1$, with $\Phi_0(x)$ and $\mathcal{E}_0$ are known. The function $\Phi_1(x)$ follows the boundary condition

$$\Phi_1(x) \sim e^{\pm i\omega_0 x} \quad (x \to \pm\infty) \tag{4.16}$$

4.3 kth order correction

We follow similar approach for higher order corrections. After obtaining $\Phi_0(x)$, $\Phi_1(x)$, …, $\Phi_{k-1}(x)$ and $\mathcal{E}_0$, $\mathcal{E}_1$, …, $\mathcal{E}_{k-1}$, we go for solving the kth order equation. We need to solve the equation

$$\left(\frac{d^2}{dx^2} + \mathcal{E}_0 - V_0(x)\right)\Phi_k(x) + \sum_{l=1}^{k}(\mathcal{E}_l - V_l(x))\Phi_{k-l}(x) = 0 \tag{4.17}$$

The above equation is a homogeneous differential equation for $\Phi_k(x)$ with the unknown constant $\mathcal{E}_k$ with $\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_{k-1}$ and $\Phi_0(x)$, $\Phi_1(x)$, …, $\Phi_{k-1}(x)$ being unknown. The function $\Phi_k(x)$ follows the boundary condition

$$\Phi_k(x) \sim e^{\pm i\omega_0 x} \quad (x \to \pm\infty) \tag{4.18}$$

4.4 Dependence of α in $f(r)$

So far we have assumed that the function $f(r)$ does not depend on α. If $f(r)$ depends on α then we also need to expand $f(r)$ in α. This gives a perturbative relation between r and x via Eq. 4.2 Now, we have two options. We can either expand x in terms of r and α or expand r in terms of x and α. Expanding r in terms of x and α,

$$x = x(r, \alpha) = \sum_{k=0}^{\infty} \alpha^k x_k(r), \tag{4.19}$$

where $x_k(r)$ are functions of r.
Expanding r in terms of x and α,

$$r = r(x, \alpha) = \sum_{k=0}^{\infty} \alpha^k r_k(x). \tag{4.20}$$

We can choose either r or x, as fundamental variable in the perturbative expansion. But, here we will choose x as a funmental variable sine the boundary conditions of QNM are in terms of x. We use Eq. (4.20) to eliminate r to expand the potential perturbatively.

But in many cases, it is difficult to explicitly write the tortoise coordinate x as a function of r and also the master equation in Eq. (4.1) as a function of x. Therefore imposing a boundary condition at each order $\Phi_k \sim e^{\pm i\omega_0 x}$ is not trivial.

But, if the function f has a zero at $r = r_H$, and it is close to $1 - r_H/r$, we can write f as [6],

$$f = \left(1 - \frac{r_H}{r}\right) Z(r; \alpha), \tag{4.21}$$

where $Z(r; \alpha)$ is a function of r which contains the small parameter α. Choosing r_H and α as the fundamental parameters, we can write the master equation in the form

$$\left(1 - \frac{r_H}{r}\right) \frac{d}{dr} \left(\left(1 - \frac{r_H}{r}\right) \frac{d\Phi'}{dr} \right) + (\omega^2 - \tilde{V})\Phi' = 0, \tag{4.22}$$

where $\Phi' = \sqrt{Z}\Phi$ and $\tilde{V}$ is the effective potential which depends on α [6].

We treat this equation as the basic master equation and apply our framework to this system because the tortoise coordinate in this system is explicitly written as $r + r_H \ln(1 - r_H/r)$.

4.5 Bender-Wu Method

In this article we will focus on solving different order corrections using Bender-Wu method (BW method) (see App. (8.7)). The necessary condition for using BW method is $V_0(x)$ must have a term quadratic in x [7].

4.5.1 Zeroth order correction

For zeroth order we need to solve

$$\left(-g^4 \frac{d^2}{dx^2} + \mathcal{E}_0 - V_0(x)\right) \Phi_0(x) = 0 \tag{4.23}$$

Expanding $-V_0(x)$ around the minima $\bar{x}$, we get

$$-V_0(x) = V_{00} + \sum_{j=2}^{\infty} V_{0j}(x - \bar{x})^j \tag{4.24}$$

where

$$V_{0j} = \frac{1}{j!}\frac{d^j\left(-V_0(x)\right)}{dx^j}\bigg|_{x=\bar{x}} \tag{4.25}$$

Here, $\bar{x}$ is unique, since $-V_0(x)$ has only minima.

Let,

$$x - \bar{x} = gq \tag{4.26}$$

$$x - \bar{x} = gq \tag{4.27}$$

$$\text{or, } dx = gdq \tag{4.28}$$

$$\text{or, } \frac{d}{dx} = \frac{1}{g}\cdot\frac{d}{dq} \tag{4.29}$$

$$\text{or, } \frac{d^2}{dx^2} = \frac{1}{g^2}\cdot\frac{d^2}{dq^2} \tag{4.30}$$

Eq. 4.24 becomes

$$-V_0(x) = V_{00} + \sum_{j=2}^{\infty} V_{0j}(x-\bar{x})^j = V_{00} + V_{02}g^2q^2 + \sum_{j=3}^{\infty} V_{0j}(gq)^j \tag{4.31}$$

Putting Eqs. (4.30), (4.31) and $\psi_0(q) = \phi_0(\bar{x}+gq)$ in Eq. 4.23,

$$\left(-g^2\frac{d^2}{dq^2} + \mathcal{E}_0 + V_{00} + V_{02}g^2q^2 + \sum_{j=3}^{\infty} V_{0j}(gq)^j\right)\psi_0(q) = 0 \tag{4.32}$$

Dividing both sides by $2g^2$

$$\left(-\frac{1}{2}\frac{d^2}{dq^2} + \frac{\mathcal{E}_0 + V_{00}}{2g^2} + \frac{V_{02}}{2}q^2 + \frac{1}{2g^2}\sum_{j=3}^{\infty} V_{0j}(gq)^j\right)\psi_0(q) = 0 \tag{4.33}$$

Let,

$$\epsilon_0 = -\frac{\mathcal{E}_0 + V_{00}}{2g^2} \tag{4.34}$$

$$\Omega = \sqrt{V_{02}} \tag{4.35}$$

$$v_0(q) = \frac{1}{2g^2}\sum_{j=3}^{\infty} V_{0j}(gq)^j \tag{4.36}$$

Therefore, the Eq. 4.33 becomes,

$$\left(-\frac{1}{2}\frac{d^2}{dq^2} - \epsilon_0 + \frac{1}{2}\Omega^2q^2 + v_0(q)\right)\psi_0(q) = 0 \tag{4.37}$$

$$\text{or, } \left(-\frac{1}{2}\cdot\frac{d^2}{dq^2} + \frac{1}{2}\Omega^2q^2 + v_0(q) - \epsilon_0\right)\psi_0(q) = 0 \tag{4.38}$$

We substitute $\psi_0(q) = e^{-\Omega q^2/2} u_0(q)$,

$$\frac{d\psi_0(q)}{dq} = \frac{d}{dq}\left(e^{-\Omega q^2/2}u_0(q)\right) = e^{-\Omega q^2/2}\left(u_0' - \Omega q u_0\right) \tag{4.39}$$

$$\frac{d^2\psi_0(q)}{dq^2} = \frac{d}{dq}\left[e^{-\Omega q^2/2}\left(u_0' - \Omega q u_0\right)\right] \tag{4.40}$$

$$= (-\Omega q)e^{-\Omega q^2/2}\left(u_0' - \Omega q u_0\right) + e^{-\Omega q^2/2}\left(u_0''(q) - \Omega q u_0'(q) - \Omega u_0(q)\right) \tag{4.41}$$

$$= e^{-\Omega q^2/2}\left[u_0''(q) - 2\Omega q u_0'(q) + \left(\Omega^2 q^2 - \Omega\right)u_0(q)\right] - \frac{1}{2}\frac{d^2\psi_0(q)}{dq^2} + \frac{1}{2}\Omega^2 q^2 \psi_0(q) \tag{4.42}$$

$$= -e^{-\Omega q^2/2}\left[-\frac{1}{2}u_0''(q) + \Omega q u_0'(q) + \left(\frac{\Omega}{2} - \frac{\Omega^2}{2}q^2\right)u_0(q)\right] + e^{-\Omega q^2/2}\frac{1}{2}\Omega^2 q^2 u_0(q) \tag{4.43}$$

$$= e^{-\Omega q^2/2}\left[-\frac{1}{2}u_0''(q) + \Omega q u_0'(q) + \frac{\Omega}{2}u_0(q)\right] \tag{4.44}$$

where the primes denote the derivatives with respect to q.

Therefore,

$$\frac{d\psi_0(q)}{dq} = e^{-\Omega q^2/2}\left(u_0' - \Omega q u_0\right) \tag{4.45}$$

$$\frac{d^2\psi_0}{dq^2} = e^{-\Omega q^2/2}\left[-\frac{1}{2}u_0'' + \Omega q u_0' + \frac{\Omega}{2}u_0\right] \tag{4.46}$$

Putting Eq. 4.45 and 4.46 in Eq. 4.83, we get

$$\boxed{-\frac{1}{2}u_0'' + \Omega q u_0' + \left(\frac{\Omega}{2} + v_0(q) - \epsilon_0\right)u_0 = 0.} \tag{4.47}$$

Solving zeroth order equation

Let,

$$u_0(q) = \sum_{n=0}^{\infty} g^n u_{0n}(q), \qquad \epsilon_0 = \sum_{n=0}^{\infty} g^n \epsilon_{0n}. \tag{4.48}$$

$$v_0(q) = \frac{1}{2g^2}\sum_{j=3}^{\infty} V_{0j}(gq)^j = \frac{1}{2}\sum_{j=3}^{\infty} V_{0j} g^{j-2} q^j = \frac{1}{2}\sum_{j=1}^{\infty} g^j V_{0,j+2} q^{j+2}. \tag{4.49}$$

Using $v_{0j} = \dfrac{V_{0,j+2}}{2}$,

$$v_0(q) = \sum_{j=1}^{\infty} g^j v_{0,j} q^{j+2} \tag{4.50}$$

Putting these expansions into (4.47), we get

$$\sum_{n=0}^{\infty} g^n\left(-\frac{1}{2}u_{0n}''\right) + \sum_{n=0}^{\infty} g^n \Omega q u_{0n}' + \left(\frac{\Omega}{2} + \sum_{j=1}^{\infty} g^j v_{0j} q^{j+2} - \sum_{j=0}^{\infty} g^j \epsilon_{0j}\right)\sum_{n=0}^{\infty} g^n u_{0n} = 0 \tag{4.51}$$

$$\text{or, } \sum_{n=0}^{\infty} g^n\left(-\frac{1}{2}u_{0n}'' + \Omega q u_{0n}' + \frac{\Omega}{2}u_{0n}\right) + \sum_{j=1}^{\infty} g^j v_{0,j+2} q^{j+2}\sum_{n=0}^{\infty} g^n u_{0n} - \sum_{j=0}^{\infty} g^j \epsilon_{0j}\sum_{n=0}^{\infty} g^n u_{0n} = 0 \tag{4.52}$$

The 2nd term of Eq. (4.52) becomes,

$$\sum_{j=1}^{\infty} g^j v_{0,j} q^{j+2} \sum_{n=0}^{\infty} g^n u_{0n} = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} g^{n+j} v_{0j} q^{j+2} u_{0n} = \sum_{n=0}^{\infty} g^n \sum_{j=1}^{\infty} v_{0j} q^{j+2} u_{0,n-j} \tag{4.53}$$

The 3rd term of Eq. (4.52) becomes,

$$\sum_{j=0}^{\infty} g^j \epsilon_{0j} \sum_{n=0}^{\infty} g^n u_{0n} = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} g^{n+j} \epsilon_{0j} u_{0n} = \sum_{n=0}^{\infty} g^n \sum_{j=0}^{\infty} \epsilon_{0j} u_{0,n-j} \tag{4.54}$$

Therefore Eq. (4.52) becomes,

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2} u''_{0n} + \Omega q u'_{0n} + \frac{\Omega}{2} u_{0n} + \sum_{j=1}^{n} v_{0j} q^{j+2} u_{0,n-j} - \sum_{j=0}^{n} \epsilon_{0j} u_{0,n-j} \right) = 0 \tag{4.55}$$

For all $n \geq 0$,

$$\boxed{-\frac{1}{2} u''_{0n} + \Omega q u'_{0n} + \frac{\Omega}{2} u_{0n} + \sum_{j=1}^{n} v_{0j} q^{j+2} u_{0,n-j} - \sum_{j=0}^{n} \epsilon_{0j} u_{0,n-j} = 0.} \tag{4.56}$$

(i) Lowest or Fundamental mode ($\nu = 0$)
Let us focus on the lowest (or fundamental) overtone mode in the QNM problem.

For $n = 0$, we have

$$-\frac{1}{2} u''_{00} + \Omega q u'_{00} + \frac{\Omega}{2} u_{00} - \epsilon_{00} u_{00} = 0 \tag{4.57}$$

$$\text{or,} \quad -\frac{1}{2} u''_{00} + \Omega q u'_{00} = \left(\epsilon_{00} - \frac{\Omega}{2} \right) u_{00} \tag{4.58}$$

Hermite polynomial is one of the solutions of the above equation. Therefore,

$$(u_{00})_\nu(q) = H_\nu(q) \tag{4.59}$$

$$\epsilon_{00} - \frac{\Omega}{2} = \nu\Omega \implies (\epsilon_{00})_\nu = \left(\nu + \frac{1}{2} \right) \Omega \tag{4.60}$$

where ν is the *level number* and $H_\nu(q)$ denotes the Hermite polynomial of order ν.

For $\nu = 0$ (ground state), the solution is

$$\boxed{u_{00}(q) = 1 \quad \text{with} \quad \epsilon_{00} = \frac{\Omega}{2}} \tag{4.61}$$

Using it in Eq. 4.56 for $n \geq 1$, we get

$$-\frac{1}{2} u''_{0n} + \Omega q u'_{0n} + \frac{\Omega}{2} u_{0n} + \sum_{j=1}^{n} v_{0j} q^{j+2} u_{0,n-j} - \sum_{j=1}^{n} \epsilon_{0j} u_{0,n-j} - \epsilon_{00} u_{0n} = 0 \tag{4.62}$$

$$\text{or,} \quad -\frac{1}{2} u''_{0n} + \Omega q u'_{0n} + \frac{\Omega}{2} u_{0n} + \sum_{j=1}^{n} v_{0j} q^{j+2} u_{0,n-j} - \sum_{j=1}^{n} \epsilon_{0j} u_{0,n-j} - \frac{\Omega}{2} u_{0n} = 0 \tag{4.63}$$

$$\text{or,} \quad -\frac{1}{2} u''_{0n} + \Omega q u'_{0n} + \sum_{j=1}^{n} v_{0j} q^{j+2} u_{0,n-j} - \sum_{j=1}^{n} \epsilon_{0j} u_{0,n-j} = 0 \tag{4.64}$$

$$\boxed{-\frac{1}{2}u''_{0n} + \Omega q u'_{0n} + \sum_{j=1}^{n}(v_{0j}q^{j+2} - \epsilon_{0j})u_{0,n-j} = 0, \quad n \geq 1.} \tag{4.65}$$

The solution $u_{0n}(q)$ is a *polynomial of degree* $\nu + 3n$ [7, 8]:

$$u_{0n}(q) = \sum_{k=0}^{\nu+3n} A_{0n}^{k} q^{k}, \quad n \geq 0 \tag{4.66}$$

where the superscript k is just an index in A_{0n}. The differential equation (4.65) determines all the coefficients A_{0n}^{k} and ϵ_{0n} recursively [8]. For detailed calculation see Sec. (4.5.4).

Therefore, for the ground state, we have

$$\mathcal{E}_0 = -V_{00} - 2g^2 \sum_{n=0}^{\infty} g^n \epsilon_{0n}, \tag{4.67}$$

$$\Phi_0(q) = e^{-\Omega q^2/2} \sum_{n=0}^{\infty} g^n u_{0n}(q), \qquad u_{0n}(q) = \sum_{k=1}^{3n} A_{0n}^{k} q^{k}, \tag{4.68}$$

with $\epsilon_{00} = \Omega/2$ and $u_{00}(q) = 1$.

Finally we want to set $g = e^{i\pi/4}$ in the perturbative series. But, in general, the power series (4.48) is not convergent for any $g \neq 0$. So we use Pade'-Borrel resummation of the power series to make it convergent for any $g \neq 0$ and evaluate it at $g = e^{i\pi/4}$. The Pade'-Borel summation of (4.48) correctly reproduces the QNM frequencies [9]. Performing this we find the value of $\mathcal{E}_0$.

From Eqs. (4.4) and (4.5), we get

$$\mathcal{E}_0 = \omega_0^2 \implies \omega_0 = \sqrt{\mathcal{E}_0} \tag{4.69}$$

In this section, we have considered only one root.

4.5.2 First order correction

For first order we need to solve

$$\left(-g^4 \frac{d^2}{dx^2} + \mathcal{E}_0 - V_0(x)\right) \Phi_1(x) + (\mathcal{E}_1 - V_1(x))\, \Phi_0(x) = 0 \tag{4.70}$$

Here zeroth order eigenfunction $\Phi_0(x)$ and eigenvalue $\mathcal{E}_0$ are known from previous section.

Expanding $-V_1(x)$ around the minima $\bar{x}$, we get

$$-V_1(x) = \sum_{j=0}^{\infty} V_{1j}(x - \bar{x})^j \tag{4.71}$$

where

$$V_{1j} = \left.\frac{1}{j!}\frac{d^j\,(-V_1(x))}{dx^j}\right|_{x=\bar{x}} \tag{4.72}$$

Here, $x = \bar{x}$ does not need to extremize $V_1(x)$.

Similar to zeroth order correction, let $x - \bar{x} = gq$ and this gives

$$\frac{d^2}{dx^2} = \frac{1}{g^2}\frac{d^2}{dq^2} \tag{4.73}$$

Eq. (8.4) can be written as

$$-V_1(x) = V_{10} + V_{11}(x-\bar{x}) + \sum_{j=2}^{\infty} V_{1j}(x-\bar{x})^j = V_{10} + V_{11}gq + \sum_{j=2}^{\infty} V_{1j}(gq)^j \tag{4.74}$$

Putting Eqs. (4.73), (4.31), $\psi_0(q) = \Phi_0(\bar{x}+gq)$ and $\psi_1(q) = \Phi_1(\bar{x}+gq)$ in Eq. 4.70,

$$\left(-g^2\frac{d^2}{dq^2} + \mathcal{E}_0 + V_{00} + V_{02}g^2q^2 + \sum_{j=3}^{\infty} V_{0j}(gq)^j\right)\psi_1(q) + \left(\mathcal{E}_1 + V_{10} + V_{11}gq + \sum_{j=2}^{\infty} V_{1j}(gq)^j\right)\psi_0(q) = 0 \tag{4.75}$$

Dividing both sides by $2g^2$,

$$\left(-\frac{1}{2}\frac{d^2}{dq^2} + \frac{\mathcal{E}_0 + V_{00}}{2g^2} + \frac{V_{02}}{2}q^2 + \frac{1}{2g^2}\sum_{j=3}^{\infty} V_{0j}(gq)^j\right)\psi_1(q) + \left(\frac{\mathcal{E}_1 + V_{10}}{2g^2} + \frac{V_{11}}{2g}q + \frac{1}{2g^2}\sum_{j=2}^{\infty} V_{1j}(gq)^j\right)\psi_0(q) = 0 \tag{4.76}$$

Let,

$$\epsilon_0 = -\frac{\mathcal{E}_0 + V_{00}}{2g^2} \tag{4.77}$$

$$\Omega = \sqrt{V_{02}} \tag{4.78}$$

$$v_0(q) = \frac{1}{2g^2}\sum_{j=3}^{\infty} V_{0j}(gq)^j \tag{4.79}$$

$$\epsilon_1 = -\frac{\mathcal{E}_1 + V_{10}}{2g^2} \tag{4.80}$$

$$v_1(q) = \frac{1}{2g^2}\sum_{j=2}^{\infty} V_{0j}(gq)^j \tag{4.81}$$

Therefore, Eq. 4.76 becomes,

$$\left(-\frac{1}{2}\frac{d^2}{dq^2} - \epsilon_0 + \frac{1}{2}\Omega^2q^2 + v_0(q)\right)\psi_1(q) + \left(-\epsilon_1 + \frac{V_{11}}{2g}q + v_1(q)\right)\psi_0(q) = 0 \tag{4.82}$$

$$\text{or,}\quad \left(-\frac{1}{2}\cdot\frac{d^2}{dq^2} + \frac{1}{2}\Omega^2q^2 + v_0(q) - \epsilon_0\right)\psi_1(q) + \left(\frac{V_{11}}{2g}q + v_1(q) - \epsilon_1\right)\psi_0(q) = 0 \tag{4.83}$$

Similar to zeroth order correction, we substitute $\psi_0(q) = e^{-\Omega q^2/2}u_0(q)$ and $\psi_1(q) = e^{-\Omega q^2/2}u_1(q)$. We get

$$\frac{d\psi_0(q)}{dq} = e^{-\Omega q^2/2}\left(u_0' - \Omega q u_0\right) \tag{4.84}$$

$$\frac{d^2\psi_0(q)}{dq^2} = e^{-\Omega q^2/2}\left[-\frac{1}{2}u_0'' + \Omega q u_0' + \frac{\Omega}{2}u_0\right] \tag{4.85}$$

$$\frac{d\psi_1(q)}{dq} = e^{-\Omega q^2/2}\left(u_1' - \Omega q u_1\right) \tag{4.86}$$

$$\frac{d^2\psi_1(q)}{dq^2} = e^{-\Omega q^2/2}\left[-\frac{1}{2}u_1'' + \Omega q u_1' + \frac{\Omega}{2}u_1\right] \tag{4.87}$$

Putting Eqs. (4.84), (4.85), (4.86) and (4.87) in Eq. (4.83), we get

$$\boxed{-\frac{1}{2}u_1''(q) + \Omega q u_1'(q) + \left(\frac{\Omega}{2} + v_0(q) - \epsilon_0\right)u_1(q) + \left(\frac{V_{11}}{2g}q + v_1(q) - \epsilon_1\right)u_0(q) = 0.} \tag{4.88}$$

Solving first order equation

Let,

$$u_0(q) = \sum_{n=0}^{\infty} g^n u_{0n}(q), \qquad \epsilon_0 = \sum_{n=0}^{\infty} g^n \epsilon_{0n} \tag{4.89}$$

$$u_1(q) = \sum_{n=-1}^{\infty} g^n u_{1n}(q), \qquad \epsilon_1 = \sum_{n=0}^{\infty} g^n \epsilon_{1n}. \tag{4.90}$$

The terms in $u_1(g)$ are starting from the order g^{-1} instead of g^0 to maintain the consistency in the powers of g.

$$v_0(q) = \sum_{j=1}^{\infty} g^j v_{0j} q^{j+2} \tag{4.91}$$

$$v_1(q) = \frac{1}{2g^2} \sum_{j=2}^{\infty} V_{1j} (gq)^j = \frac{1}{2} \sum_{j=2}^{\infty} V_{1j} g^{j-2} q^j = \frac{1}{2} \sum_{j=0}^{\infty} g^j V_{1,j+2} q^{j+2}. \tag{4.92}$$

Using $v_{1j} = \dfrac{V_{1,j+2}}{2}$,

$$v_1(q) = \sum_{j=0}^{\infty} g^j v_{1j} q^{j+2} \tag{4.93}$$

Putting these expressions in Eq. 4.88, we get

$$\sum_{n=-1}^{\infty} g^n \left[-\frac{1}{2}u''_{1n} + \Omega q u'_{1n}\right] + \left(\sum_{n=-1}^{\infty} g^n u_{1n}\right) \left(\frac{\Omega}{2} + \sum_{j=1}^{\infty} g^j v_{0j} q^{j+2} - \sum_{j=0}^{\infty} g^j \epsilon_{0j}\right) \tag{4.94}$$

$$+ \left(\sum_{n=0}^{\infty} g^n u_{0n}\right) \left(\frac{V_{11}}{2g} q + \sum_{j=0}^{\infty} g^j v_{1j} q^{j+2} - \sum_{j=0}^{\infty} g^j \epsilon_{1j}\right) = 0. \tag{4.95}$$

Comparing the powers of g^{-1}, we get

$$\boxed{-\frac{1}{2}u''_{1,-1} + \Omega q u'_{1,-1} + \frac{\Omega}{2} u_{1,-1} - \epsilon_{00} u_{1,-1} + \frac{V_{11}}{2} q u_{00} = 0} \tag{4.96}$$

For ground state,

$$u_{00} = 1 \quad \text{and} \quad \epsilon_{00} = \frac{\Omega}{2} \tag{4.97}$$

Therefore, Eq. 4.96 becomes

$$-\frac{1}{2}u''_{1,-1} + \Omega q u'_{1,-1} + \frac{V_{11}}{2} q = 0 \tag{4.98}$$

The general solution of the above equation is

$$u_{1,-1}(q) = c_1 \sqrt{\frac{\pi}{\Omega}}\, \text{erfi}(\sqrt{\Omega} q) + c_2 - \frac{V_{11}}{2\Omega} q \tag{4.99}$$

where c_1 and c_2 are arbitrary constants. erfi is the imaginary error function [10].

For ground state, we assume $u_{1,-1}$ to be

$$\boxed{u_{1,-1}(q) = -\frac{V_{11}}{2\Omega}q} \tag{4.100}$$

Why only single solution is considered here?

Excluding the terms having g^{-1}, Eq. 4.95 becomes

$$\sum_{n=0}^{\infty} g^n\left(-\frac{1}{2}u''_{1n} + \Omega q u'_{1n}\right) + \sum_{n=0}^{\infty}\frac{\Omega}{2}g^n u_{1n} + \left(\sum_{n=-1}^{\infty} g^n u_{1n}\right)\left(\sum_{j=1}^{\infty} g^j v_{0j} q^{j+2}\right)$$
$$-\left(\sum_{n=0}^{\infty} g^n u_{1n}\right)\left(\sum_{j=0}^{\infty} g^j \epsilon_{0j}\right) + \left(\sum_{n=0}^{\infty} g^n u_{0n}\right)\left(\sum_{j=0}^{\infty} g^j v_{1j} q^{j+2} - \sum_{j=0}^{\infty} g^j \epsilon_{1j}\right) = 0 \tag{4.101}$$

The 3rd term in Eq. (4.101) can be written as

$$\left(\sum_{n=-1}^{\infty} g^n u_{1n}\right)\left(\sum_{j=1}^{\infty} g^j v_{0j} q^{j+2}\right) \tag{4.102}$$
$$= \left(\sum_{n=0}^{\infty} g^{n-1} u_{1,n-1}\right)\left(\sum_{j=0}^{\infty} g^{j+1} v_{0,j+1} q^{j+3}\right) \tag{4.103}$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} g^{n+j} v_{0,j+1} q^{j+3} u_{1,n-1} \tag{4.104}$$
$$= \sum_{n=0}^{\infty} g^n \sum_{j=0}^{n} v_{0,j+1} q^{j+3} u_{1,n-j-1} \tag{4.105}$$

The 4th and 5th term together in Eq. (4.101) can be written as

$$-\left(\sum_{n=0}^{\infty} g^n u_{1n}\right)\left(\sum_{j=0}^{\infty} g^j \epsilon_{0j}\right) + \left(\sum_{n=0}^{\infty} g^n u_{0n}\right)\left(\sum_{j=0}^{\infty} g^j v_{1j} q^{j+2} - \sum_{j=0}^{\infty} g^j \epsilon_{1j}\right) \tag{4.106}$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} g^{n+j}\left(-\epsilon_{0j} u_{1n} + \left(v_{1j} q^{j+2} - \epsilon_{1j}\right) u_{0n}\right) \tag{4.107}$$
$$= \sum_{n=0}^{\infty} g^n \sum_{j=0}^{n}\left(-\epsilon_{0j} u_{1,n-j} + \left(v_{1j} q^{j+2} - \epsilon_{1j}\right) u_{0,n-j}\right) \tag{4.108}$$

Using this expression in Eq. (4.101) we get

$$\sum_{n=0}^{\infty} g^n\left(-\frac{1}{2}u''_{1n} + \Omega q u'_{1n} + \frac{\Omega}{2}u_{1n} + \sum_{j=0}^{n}\left(v_{0,j+1} q^{j+3} u_{1,n-j-1} - \epsilon_{0j} u_{1,n-j} + \left(v_{1j} q^{j+2} - \epsilon_{1j}\right) u_{0,n-j}\right)\right) = 0 \tag{4.109}$$

Therefore for $n \geq 0$,

$$\boxed{-\frac{1}{2}u''_{1n} + \Omega q u'_{1n} + \frac{\Omega}{2}u_{1n} + \sum_{j=0}^{n}\left(v_{0,j+1} q^{j+3} u_{1,n-j-1} - \epsilon_{0j} u_{1,n-j} + \left(v_{1j} q^{j+2} - \epsilon_{1j}\right) u_{0,n-j}\right) = 0} \tag{4.110}$$

For ground state, $u_{1n}(q)$ $(n \geq 0)$ is a polynomial of degree $3n+4$ [11].

$$u_{1n}(q) = \sum_{k=0}^{3n+4} A_{1n}^k q^k, \quad n \geq 0. \tag{4.111}$$

Therefore, for the ground state, we have

$$\mathcal{E}_1 = -V_{10} - 2g^2 \sum_{n=0}^{\infty} g^n \epsilon_{1n}, \tag{4.112}$$

$$\Phi_1(q) = e^{-\Omega q^2/2} \left(-\frac{V_{11}}{2\Omega} q + \sum_{n=0}^{\infty} g^n u_{1n}(q) \right) \tag{4.113}$$

with

$$u_{1n}(q) = \sum_{k=0}^{3n+4} A_{1n}^k q^k \quad (\text{For } n \geq 0) \tag{4.114}$$

where the superscript k is an index in A_{1n}. Similar to the zeroth order case, the differential equation (4.110) determines all the coefficients A_{1n}^k and ϵ_{1n} recursively. For detailed calculation see Sec. 4.5.5.

Performing the Pade'-Borel summation of (4.112) at $g = e^{i\pi/4}$, we find the value of $\mathcal{E}_1$. From Eqs. (4.4) and (4.5), we get

$$\mathcal{E}_1 = 2\omega_0\omega_1 \implies \omega_1 = \frac{\mathcal{E}_1}{2\omega_0} \tag{4.115}$$

4.5.3 kth order correction

For k th order correction, we need the equation

$$\left(-g^4 \frac{d^2}{dx^2} + \mathcal{E}_0 - V_0(x) \right) \Phi_k(x) + \sum_{\ell=1}^{k} \left(\mathcal{E}_\ell - V_\ell(x) \right) \Phi_{k-\ell}(x) = 0. \tag{4.116}$$

This leads to

$$-\frac{1}{2} u_k'' + \Omega q u_k' + \left(\frac{\Omega}{2} + v_0 - \epsilon_0 \right) u_k + \sum_{\ell=1}^{k} \left(\frac{V_{\ell 1}}{2g} q + v_\ell - \epsilon_\ell \right) u_{k-\ell} = 0, \tag{4.117}$$

where $\Phi_k(x) = e^{-\Omega q^2/2} u_k(q)$ and

$$\epsilon_\ell := -\frac{\mathcal{E}_\ell + V_{\ell 0}}{2g^2}, \quad v_\ell(q) := \frac{1}{2g^2} \sum_{j=2}^{\infty} V_{\ell j} (gq)^j. \tag{4.118}$$

We observe that the ground state solution, in general, behaves as

$$u_k(q) = \frac{u_{k,-k}(q)}{g^k} + \cdots = \sum_{n=-k}^{\infty} g^n u_{kn}(q), \tag{4.119}$$

$$\epsilon_k = \frac{\epsilon_{k,-2}}{g^2} + \cdots = \sum_{n=-1}^{\infty} g^{2n} \epsilon_{k,2n}, \tag{4.120}$$

where $u_{kn}(q)$ is a polynomial of at most degree $3n+4k$. Under this assumption, we can easily compute ϵ_k perturbatively in g.

4.5.4 Zeroth order correction (advanced)

Now we aim to find zeroth order correction for the ground state ($\nu = 0$) only.

$$u_{0n} = \sum_{k=0}^{3n} A_{0n}^k q^k \tag{4.121}$$

Since u_{0n} is a polynomial of order $3n$, we impose the condition

$$\boxed{A_{0n}^k = 0 \quad \text{when } k > 3n} \tag{4.122}$$

$$u'_{0n} = \sum_{k=1}^{3n} A_{0n}^k k q^{k-1} \tag{4.123}$$

$$qu'_{0n} = \sum_{k=1}^{3n} A_{0n}^k k q^k = \sum_{k=0}^{3n} A_{0n}^k k q^k - A_{0n}^0 .0.q^0 = \sum_{k=0}^{3n} A_{0n}^k k q^k \tag{4.124}$$

$$u''_{0n} = \sum_{k=2}^{3n} A_{0n}^k k(k-1) q^{k-2} = \sum_{k=0}^{3n-2} A_{0n}^{k+2}(k+2)(k+1)q^k = \sum_{k=0}^{3n} A_{0n}^{k+2}(k+2)(k+1)q^k \quad \text{[Using Eq. (4.122)]} \tag{4.125}$$

Putting the above expressions in Eq. (4.56), we get

$$\sum_{k=0}^{3n} q^k \left[-\frac{1}{2}(k+2)(k+1)A_{0n}^{k+2} + \Omega k A_{0n}^k + \frac{\Omega}{2} A_{0n}^k \right] + \sum_{j=1}^{n} v_{0j} q^{j+2} \sum_{k=0}^{3n} A_{0,n-j}^k q^k \tag{4.126}$$

$$- \sum_{j=0}^{n} \epsilon_{0j} \sum_{k=0}^{3n} A_{0,n-j}^k q^k = 0 \tag{4.127}$$

Simplifying 2nd term of Eq. (4.127),

$$\sum_{j=1}^{n} v_{0j} q^{j+2} \sum_{k=0}^{3n} A_{0,n-j}^k q^k = \sum_{k=0}^{3n} \sum_{j=1}^{n} q^{k+j+2} v_{0j} A_{0,n-j}^k = \sum_{k=0}^{3n} q^k \sum_{j=1}^{n} v_{0j} A_{0,n-j}^{k-j-2} \tag{4.128}$$

Simplifying 3rd term of Eq. (4.127),

$$\sum_{j=0}^{n} \epsilon_{0j} \sum_{k=0}^{3n} A_{0,n-j}^k q^k = \sum_{k=0}^{3n} q^k \sum_{j=0}^{n} \epsilon_{0j} A_{0,n-j}^k \tag{4.129}$$

Therefore Eq. (4.127) becomes

$$\sum_{k=0}^{3n} q^k \left[-\frac{1}{2}(k+2)(k+1)A_{0n}^{k+2} + \Omega k A_{0n}^k + \frac{\Omega}{2} A_{0n}^k + \sum_{j=1}^{n} v_{0j} A_{0,n-j}^{k-j-2} - \sum_{j=0}^{n} \epsilon_{0j} A_{0,n-j}^k \right] = 0 \tag{4.130}$$

Collecting the terms with q^k,

$$-\frac{1}{2}(k+2)(k+1)A_{0n}^{k+2} + \Omega k A_{0n}^k + \frac{\Omega}{2} A_{0n}^k + \sum_{j=1}^{n} v_{0j} A_{0,n-j}^{k-j-2} - \sum_{j=0}^{n} \epsilon_{0j} A_{0,n-j}^k = 0 \tag{4.131}$$

$$-\frac{1}{2}(k+2)(k+1)A_{0n}^{k+2} + \Omega k A_{0n}^k + \frac{\Omega}{2} A_{0n}^k + \sum_{j=1}^{n} v_{0j} A_{0,n-j}^{k-j-2} - \sum_{j=1}^{n} \epsilon_{0j} A_{0,n-j}^k - \epsilon_{00} A_{0n}^k = 0 \tag{4.132}$$

Using $\epsilon_{00} = \Omega/2$,

$$-(k+2)(k+1)A_{0n}^{k+2} + 2\Omega k A_{0n}^{k} + 2\sum_{j=1}^{n}\left(v_{0j}A_{0,n-j}^{k-j-2} - \epsilon_{0j}A_{0,n-j}^{k}\right) = 0 \tag{4.133}$$

Now Eq. (4.133) can be solved recursively for A_{0n}^{k} and ϵ_{0n}.

Case I: $k = 0$

For $k = 0$,

$$-2A_{0n}^{2} + 2\sum_{j=1}^{n}\left(v_{0j}A_{0,n-j}^{-j-2} - \epsilon_{0j}A_{0,n-j}^{0}\right) = 0 \tag{4.134}$$

$$\text{or, } -2A_{0n}^{2} + 2\sum_{j=1}^{n}v_{0j}A_{0,n-j}^{-j-2} - 2\sum_{j=1}^{n-1}\epsilon_{0j}A_{0,n-j}^{0} - 2\epsilon_{0n}A_{00}^{0} = 0 \tag{4.135}$$

$$\text{or, } 2\epsilon_{0n}A_{00}^{0} = -2A_{0n}^{2} + 2\sum_{j=1}^{n}v_{0j}A_{0,n-j}^{-j-2} - 2\sum_{j=1}^{n-1}\epsilon_{0j}A_{0,n-j}^{0} \tag{4.136}$$

So we can determine the energy ϵ_{0n} ($n > 0$) as long as we have all A_0-coefficients with order smaller than n and as long as we find the coefficient A_{0n}^{2}. Using $u_{00}(q) = 1$, we find

$$\boxed{A_{00}^{0} = 1 \quad \text{and} \quad A_{00}^{k} = 0 \quad \forall k > 0} \tag{4.137}$$

We also choose the normalization

$$\boxed{A_{00}^{0} = 1 \quad \text{and} \quad A_{0n}^{0} = 0 \quad \forall n > 0} \tag{4.138}$$

Therefore, using Eq. (4.138) and $A_{00}^{0} = 1$, the 3rd term of the right hand side of Eq. (4.136) becomes zero and the equation becomes

$$\epsilon_{0n} = -A_{0n}^{2} + \sum_{j=1}^{n}v_{0j}A_{0,n-j}^{-j-2} \tag{4.139}$$

To avoid unnecessary recursion, we restrict the limits of the summation.

$$\epsilon_{0n} = -A_{0n}^{2} + \sum_{j=1}^{\min\{n,-2\}}v_{0j}A_{0,n-j}^{-j-2} \tag{4.140}$$

$$\text{or, } \epsilon_{0n} = -A_{0n}^{2} \tag{4.141}$$

Case II: $k > 0$

For $k > 0$,

$$A_{0n}^{k} = \frac{1}{2\Omega k}\left[(k+2)(k+1)A_{0n}^{k+2} + 2\sum_{j=1}^{n}\epsilon_{0j}A_{0,n-j}^{k} - 2\sum_{j=1}^{n}v_{0j}A_{0,n-j}^{k-j-2}\right] \tag{4.142}$$

To avoid unnecessary recursion, we restrict the limits of the summation.

$$A_{0n}^k = \frac{1}{2\Omega k}\left[(k+2)(k+1)A_{0n}^{k+2} + 2\sum_{j=1}^{n}\epsilon_{0j}A_{0,n-j}^{k} - 2\sum_{j=1}^{\min\{n,k-2\}} v_{0j}A_{0,n-j}^{k-j-2}\right] \tag{4.143}$$

Now we will solve Eq. (4.141) and Eq. (4.143) for all A_{0n}^k and ϵ_{0n} recursively. The following steps are followed

1. Consider Eq. (4.143) for $k > 0$ and $n > 0$.

$$\boxed{A_{0n}^k = \frac{1}{2\Omega k}\left[(k+2)(k+1)A_{0n}^{k+2} + 2\sum_{j=1}^{n}\epsilon_{0j}A_{0,n-j}^{k} - 2\sum_{j=1}^{\min\{n,k-2\}} v_{0j}A_{0,n-j}^{k-j-2}\right]} \tag{4.144}$$

   We assume $A_{0n}^{3n+2} = A_{0n}^{3n+1} = 0$. If we know all coefficients and energies of order $< n$ by taking $k = 3n$ in Eq. (4.144) we can compute A_{0n}^{3n} and A_{0n}^{3n-1}. Once we know A_{0n}^{3n} and A_{0n}^{3n-1}, we can compute A_{0n}^k with $k = 3n, 3n-1, 3n-2, \ldots, 1$.
   This process is still valid even when the values of k were overestimated. We can also start with $k = 3n$ and solve in sequence for $k = 3n, 3n-1, \ldots, 1$.
2. Now that all A_{0n}^k, $k > 0$ are known, and in particular A_{0n}^2 is known, use Eq. 4.141

$$\boxed{\epsilon_{0n} = -A_{0n}^2} \tag{4.145}$$

   to compute ϵ_{0n}.

Summary of the formulas of zeroth order correction

$$\boxed{A_{0n}^k = \frac{1}{2\Omega k}\left[(k+2)(k+1)A_{0n}^{k+2} + 2\sum_{j=1}^{n}\epsilon_{0j}A_{0,n-j}^{k} - 2\sum_{j=1}^{\min\{n,k-2\}} v_{0j}A_{0,n-j}^{k-j-2}\right]} \tag{4.146}$$

$$\boxed{\epsilon_{0n} = -A_{0n}^2} \tag{4.147}$$

4.5.5 First order correction (advanced)

For finding the first order correction, we will stick to ground state ($\nu = 0$) only.

$$u_{0n} = \sum_{k=0}^{3n} A_{0n}^k q^k \tag{4.148}$$

$$u_{1n} = \sum_{k=0}^{3n+4} A_{1n}^k q^k \quad \text{where } n \geq 0 \tag{4.149}$$

Since u_{1n} is a polynomial of order $3n+4$, we impose the condition

$$\boxed{A_{1n}^k = 0 \quad \text{when } k > 3n+4} \tag{4.150}$$

$$u'_{1n} = \sum_{k=1}^{3n+4} A_{1n}^k k q^{k-1} \tag{4.151}$$

$$qu'_{1n} = \sum_{k=0}^{3n+4} A_{1n}^k k q^k \tag{4.152}$$

$$u''_{1n} = \sum_{k=2}^{3n+4} A_{1n}^k k(k-1) q^{k-2} = \sum_{k=0}^{3n+4} A_{1n}^{k+2}(k+2)(k+1) q^k \tag{4.153}$$

Putting these expressions in Eq. (4.110), we get

$$\begin{aligned}
&\sum_{k=0}^{3n+4} \left[-\frac{1}{2}(k+2)(k+1)A_{1n}^{k+2} + \Omega k A_{1n}^k + \frac{\Omega}{2} A_{1n}^k \right] q^k \\
&+ \sum_{k=0}^{3n+4} \left(\sum_{j=0}^{n} q^{k+j+3} v_{0,j+1} A_{1,n-j-1}^k - q^k \sum_{j=0}^{n} \epsilon_{0j} A_{1,n-j}^k \right) \\
&+ \sum_{k=0}^{3n} \left(\sum_{j=0}^{n} q^{k+j+2} v_{1j} A_{0,n-j}^k - q^k \sum_{j=0}^{n} \epsilon_{1j} A_{0,n-j}^k \right) = 0
\end{aligned} \tag{4.154}$$

$$\begin{aligned}
\text{or,}\ &\sum_{k=0}^{3n+4} q^k \left(-\frac{1}{2}(k+2)(k+1)A_{1n}^{k+2} + \Omega k A_{1n}^k + \frac{\Omega}{2} A_{1n}^k \right) \\
&+ \sum_{k=0}^{3n+4} q^k \left(\sum_{j=0}^{n} v_{0,j+1} A_{1,n-j-1}^{k-j-3} - \sum_{j=0}^{n} \epsilon_{0j} A_{1,n-j}^k \right) \\
&+ \sum_{k=0}^{3n} q^k \left(\sum_{j=0}^{n} v_{1j} A_{0,n-j}^{k-j-2} - \sum_{j=0}^{n} \epsilon_{1j} A_{0,n-j}^k \right) = 0
\end{aligned} \tag{4.155}$$

Comparing the coefficients of q^k with $0 \le k \le 3n$

$$\boxed{\begin{aligned}
&-\frac{1}{2}(k+2)(k+1)A_{1n}^{k+2} + \Omega k A_{1n}^k + \frac{\Omega}{2} A_{1n}^k + \sum_{j=0}^{n} v_{0,j+1} A_{1,n-j-1}^{k-j-3} - \sum_{j=0}^{n} \epsilon_{0j} A_{1,n-j}^k \\
&+ \sum_{j=0}^{n} v_{1j} A_{0,n-j}^{k-j-2} - \sum_{j=0}^{n} \epsilon_{1j} A_{0,n-j}^k = 0
\end{aligned}} \tag{4.156}$$

Comparing the coefficients of q^k with $3n+1 \le k \le 3n+4$

$$\boxed{-\frac{1}{2}(k+2)(k+1)A_{1n}^{k+2} + \Omega k A_{1n}^k + \frac{\Omega}{2} A_{1n}^k + \sum_{j=0}^{n} v_{0,j+1} A_{1,n-j-1}^{k-j-3} - \sum_{j=0}^{n} \epsilon_{0j} A_{1,n-j}^k = 0} \tag{4.157}$$

Case I: ($3n+1 \leq k \leq 3n+4$)

From Eq. (4.157),

$$-\frac{1}{2}(k+2)(k+1)A_{1n}^{k+2} + \Omega k A_{1n}^{k} + \frac{\Omega}{2}A_{1n}^{k} + \sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{k-j-3} - \sum_{j=0}^{n} \epsilon_{0j}A_{1,n-j}^{k} = 0 \quad (4.158)$$

$$\text{or, } -(k+2)(k+1)A_{1n}^{k+2} + 2\Omega k A_{1n}^{k} + \Omega A_{1n}^{k} + 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{k-j-3} - 2\sum_{j=0}^{n} \epsilon_{0j}A_{1,n-j}^{k} = 0 \quad (4.159)$$

$$\text{or, } -(k+2)(k+1)A_{1n}^{k+2} + 2\Omega k A_{1n}^{k} + \Omega A_{1n}^{k} + 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{k-j-3} - 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{k} - 2\epsilon_{00}A_{1n}^{k} = 0 \quad (4.160)$$

$$\text{or, } -(k+2)(k+1)A_{1n}^{k+2} + 2\Omega k A_{1n}^{k} + 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{k-j-3} - 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{k} = 0 \quad \left[\text{Using } \epsilon_{00} = \frac{\Omega}{2}\right] \quad (4.161)$$

$$\text{or, } 2\Omega k A_{1n}^{k} = (k+2)(k+1)A_{1n}^{k+2} + 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{k} - 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{k-j-3} \quad (4.162)$$

$$\text{or, } A_{1n}^{k} = \frac{1}{2\Omega k}\left[(k+2)(k+1)A_{1n}^{k+2} + 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{k} - 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{k-j-3}\right] \quad (4.163)$$

To avoid unnecessary recursion, we restrict the limits of the summations.

$$\boxed{A_{1n}^{k} = \frac{1}{2\Omega k}\left[(k+2)(k+1)A_{1n}^{k+2} + 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{k} - 2\sum_{j=0}^{\min\{n,k-3\}} v_{0,j+1}A_{1,n-j-1}^{k-j-3}\right]} \quad (4.164)$$

We use $A_{1n}^{3n+6} = A_{1n}^{3n+5} = 0$. Taking $k = 3n+4$ in Eq. (4.164) we can compute A_{1n}^{3n+4} and A_{1n}^{3n+3}. Once we know A_{1n}^{3n+4} and A_{1n}^{3n+3}, we can compute A_{1n}^{k} with $k = 3n+4$, $3n+3$, $3n+2$ and $3n+1$.

Case II: $(0 \leq k \leq 3n)$

From Eq. (4.156),

$$-(k+2)(k+1)A_{1n}^{k+2} + 2\Omega k A_{1n}^{k} + \Omega A_{1n}^{k} + 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{k-j-3} - 2\sum_{j=0}^{n} \epsilon_{0j}A_{1,n-j}^{k}$$
$$+ 2\sum_{j=0}^{n} v_{1j}A_{0,n-j}^{k-j-2} - 2\sum_{j=0}^{n} \epsilon_{1j}A_{0,n-j}^{k} = 0 \tag{4.165}$$

or, $$-(k+2)(k+1)A_{1n}^{k+2} + 2\Omega k A_{1n}^{k} + \Omega A_{1n}^{k} + 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{k-j-3} - 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{k} - 2\epsilon_{00}A_{1n}^{k}$$
$$+ 2\sum_{j=0}^{n} v_{1j}A_{0,n-j}^{k-j-2} - 2\sum_{j=0}^{n} \epsilon_{1j}A_{0,n-j}^{k} = 0 \tag{4.166}$$

or, $$-(k+2)(k+1)A_{1n}^{k+2} + 2\Omega k A_{1n}^{k} + 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{k-j-3} - 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{k} + 2\sum_{j=0}^{n} v_{1j}A_{0,n-j}^{k-j-2}$$
$$- 2\sum_{j=0}^{n} \epsilon_{1j}A_{0,n-j}^{k} = 0 \qquad \left[\text{Using } \epsilon_{00} = \frac{\Omega}{2}\right] \tag{4.167}$$

Now Eq. (4.167) can be solved recursively for A_{1n}^{k} and ϵ_{1n}.

Case IIa: $(k = 0)$
For $k = 0$, Eq. (4.167) becomes

$$-2A_{1n}^{2} + 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{-j-3} - 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{0} + 2\sum_{j=0}^{n} v_{1j}A_{0,n-j}^{-j-2} - 2\sum_{j=0}^{n} \epsilon_{1j}A_{0,n-j}^{0} = 0 \tag{4.168}$$

or, $$-2A_{1n}^{2} + 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{-j-3} - 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{0} + 2\sum_{j=0}^{n} v_{1j}A_{0,n-j}^{-j-2} - 2\sum_{j=0}^{n-1} \epsilon_{1j}A_{0,n-j}^{0}$$
$$-2\epsilon_{1n}A_{00}^{0} = 0 \tag{4.169}$$

or, $$2\epsilon_{1n}A_{00}^{0} = -2A_{1n}^{2} + 2\sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{-j-3} - 2\sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{0} + 2\sum_{j=0}^{n} v_{1j}A_{0,n-j}^{-j-2} - 2\sum_{j=0}^{n-1} \epsilon_{1j}A_{0,n-j}^{0} \tag{4.170}$$

Therefore, using Eq. (4.138), the 5th term of the right hand side of Eq. (4.170) becomes zero and the equation becomes

$$\epsilon_{1n} = -A_{1n}^{2} + \sum_{j=0}^{n} v_{0,j+1}A_{1,n-j-1}^{-j-3} - \sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{0} + \sum_{j=0}^{n} v_{1j}A_{0,n-j}^{-j-2} \tag{4.171}$$

To avoid unnecessary recursion, we restrict the limits of the summations.

$$\epsilon_{1n} = -A_{1n}^{2} + \sum_{j=0}^{\min\{n,-3\}} v_{0,j+1}A_{1,n-j-1}^{-j-3} - \sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{0} + \sum_{j=0}^{\min\{n,-2\}} v_{1j}A_{0,n-j}^{-j-2} \tag{4.172}$$

or, $$\epsilon_{1n} = -A_{1n}^{2} - \sum_{j=1}^{n} \epsilon_{0j}A_{1,n-j}^{0} \tag{4.173}$$

$$\boxed{\epsilon_{1n} = -A_{1n}^2 - \sum_{j=1}^{n} \epsilon_{0j} A_{1,n-j}^0} \tag{4.174}$$

So we can determine the energy ϵ_{1n} ($n \geq 0$) as long as we have all the A_1-coefficients with order smaller than n and as long as we find the coefficient A_{1n}^2 and all the A_0 coefficients and ϵ_0 energies.

Case IIb: ($0 < k \leq 3n$)
For $0 < k \leq 3n$, Eq. (4.167) can be written as

$$\begin{aligned}
&-(k+2)(k+1)A_{1n}^{k+2} + 2\Omega k A_{1n}^k + 2\sum_{j=0}^{n} v_{0,j+1} A_{1,n-j-1}^{k-j-3} - 2\sum_{j=1}^{n} \epsilon_{0j} A_{1,n-j}^k + 2\sum_{j=0}^{n} v_{1j} A_{0,n-j}^{k-j-2} \\
&-2\sum_{j=0}^{n} \epsilon_{1j} A_{0,n-j}^k = 0
\end{aligned} \tag{4.175}$$

$$\begin{aligned}
\text{or, } 2\Omega k A_{1n}^k &= (k+2)(k+1)A_{1n}^{k+2} - 2\sum_{j=0}^{n} v_{0,j+1} A_{1,n-j-1}^{k-j-3} + 2\sum_{j=1}^{n} \epsilon_{0j} A_{1,n-j}^k - 2\sum_{j=0}^{n} v_{1j} A_{0,n-j}^{k-j-2} \\
&+2\sum_{j=0}^{n} \epsilon_{1j} A_{0,n-j}^k
\end{aligned} \tag{4.176}$$

$$\begin{aligned}
\text{or, } A_{1n}^k = \frac{1}{2\Omega k} \Bigg[&(k+2)(k+1)A_{1n}^{k+2} - 2\sum_{j=0}^{n} v_{0,j+1} A_{1,n-j-1}^{k-j-3} + 2\sum_{j=1}^{n} \epsilon_{0j} A_{1,n-j}^k \\
&-2\sum_{j=0}^{n} v_{1j} A_{0,n-j}^{k-j-2} + 2\sum_{j=0}^{n} \epsilon_{1j} A_{0,n-j}^k \Bigg]
\end{aligned} \tag{4.177}$$

To avoid unnecessary recursion, we restrict the limits of the summations.

$$\boxed{\begin{aligned}
A_{1n}^k = \frac{1}{2\Omega k} \Bigg[&(k+2)(k+1)A_{1n}^{k+2} - 2\sum_{j=0}^{\min\{n-1,k-3\}} v_{0,j+1} A_{1,n-j-1}^{k-j-3} + 2\sum_{j=1}^{n} \epsilon_{0j} A_{1,n-j}^k \\
&-2\sum_{j=0}^{\min\{n,k-2\}} v_{1j} A_{0,n-j}^{k-j-2} + 2\sum_{j=0}^{n} \epsilon_{1j} A_{0,n-j}^k \Bigg]
\end{aligned}} \tag{4.178}$$

Now we will solve Eqs. (4.174) and (4.178) for all A_{1n}^k and ϵ_{1n} recursively.

Summary of formulas of first order correction

For $3n + 1 \leq k \leq 3n + 4$**,**

$$\boxed{A_{1n}^k = \frac{1}{2\Omega k} \left[(k+2)(k+1)A_{1n}^{k+2} + 2\sum_{j=1}^{n} \epsilon_{0j} A_{1,n-j}^k - 2\sum_{j=0}^{\min\{n-1,k-3\}} v_{0,j+1} A_{1,n-j-1}^{k-j-3} \right]} \tag{4.179}$$

For $0 \leq k \leq 3n$**,**

$$\boxed{\begin{aligned} A_{1n}^{k} = \frac{1}{2\Omega k}\Bigg[(k+2)(k+1)A_{1n}^{k+2} - 2\sum_{j=0}^{\min\{n-1,k-3\}} v_{0,j+1}A_{1,n-j-1}^{k-j-3} + 2\sum_{j=1}^{n}\epsilon_{0j}A_{1,n-j}^{k} \\ -2\sum_{j=0}^{\min\{n,k-2\}} v_{1j}A_{0,n-j}^{k-j-2} + 2\sum_{j=0}^{n}\epsilon_{1j}A_{0,n-j}^{k}\Bigg]\end{aligned}} \tag{4.180}$$

$$\boxed{\epsilon_{1n} = -A_{1n}^{2} - \sum_{j=1}^{n}\epsilon_{0j}A_{1,n-j}^{0}} \tag{4.181}$$

Chapter 5

QNM of Schwarzschild BH on the real axis (summary)

The QNM equation is

$$\frac{d^2\Phi}{dx^2} + (\omega^2 - V(x))\Phi = 0 \tag{5.1}$$

General solutions of this equation are asymptotically given by a superposition of plane waves, $e^{\pm i\omega x}$. Let, scattering under the potential $V(x)$, the phase shift is $\delta(\omega)$. From the graph of $\frac{d\delta}{d\omega}$ vs ω, we can find different frequencies of the quasinormal modes.

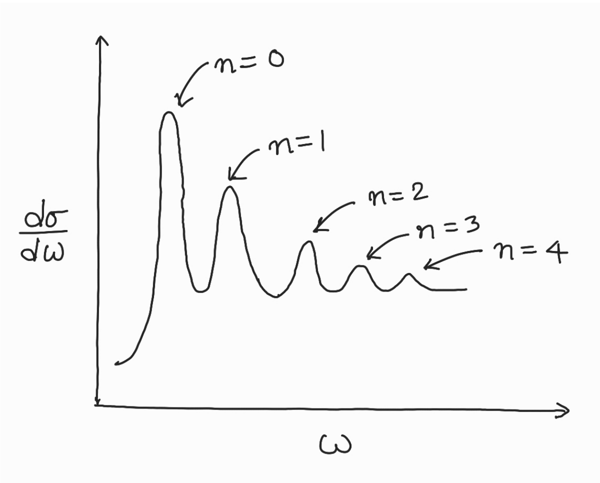

Figure 5.1: Graph of $\frac{d\sigma}{d\omega}$ vs ω

Finding the fundamental QNM frequency ($n = 0$)

1. The lowest value of ω for which the first peak is observed, that value is ω_R.
2. If Γ is FWHM (Full Width Half Maxima) of the first peak then $\omega_I = -\frac{\Gamma}{2}$.

Therefore, the fundamental QNM frequency ($n = 0$),

$$\omega = \omega_R + i\omega_I = \omega_R - i\frac{\Gamma}{2} \tag{5.2}$$

Finding the overtone QNM frequencies ($n > 0$)

Similar to the fundamental QNM frequency, we can find overtone QNM frequency.

5.1 Potential 1 (Regge-Wheeler Potential)

The Regge-Wheeler potential takes the form

$$V_{RW}(x) := \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right) \tag{5.3}$$

In this chapter we will focus on $l = 2$ case only.

Using the scattering method, we get the fundamental QNM frequency $n = 0$ to be [12]

$$M\omega = 0.3737 - 0.08896i \tag{5.4}$$

where M is the mass of the Schwartzschild BH.

For $M = 1$, the value of ω becomes

$$\omega = 0.3737 - 0.08896i \tag{5.5}$$

The Zeroth order QNM frequency

$$\omega = 0.373291 - 0.0889925i \tag{5.6}$$

Relative error (in %)	
ω_R	0.1
ω_I	0.03
ω	0.04

Table 5.1: Relative error in ω_R, ω_I and ω

5.2 Potential 2 (Regge-Wheeler Potential + Poshl-Teller Potential)

Now we consider the potential of the form

$$V(x) = V_{RW}(x) + V_{PT}(x) \tag{5.7}$$

where $V_{PT}(x)$ is the Poshl-Teller potential given by

$$V_{PT}(x;\epsilon,b) := \frac{\epsilon}{(2M)^2\cosh^2\left(\frac{x-b}{2M}\right)} \tag{5.8}$$

$$V_{RW}(x) := \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right) \tag{5.9}$$

Now we fix $l = 2$, $\epsilon = 10^{-3}$ and $b = 100M$.

In the $\frac{d\delta}{d\omega}$ vs ω graph, we only observe one peak. This might be happening due to contamination by higher overtones, multiple poles are likely to be contribute to apparently single peak. From the graph, we get

$$M\omega_R = 0.3758 \tag{5.10}$$

$$M\omega_I = -0.1055i \tag{5.11}$$

Therefore, from the scattering method [13],

$$M\omega = 0.3758 - 0.1055i \tag{5.12}$$

For $M = 1$,

$$\omega = 0.3758 - 0.1055i \tag{5.13}$$

From zeroth order perturbation theory, we get

$$\omega = 0.373671 - 0.0889699i \tag{5.14}$$

Relative error (in %)	
ω_R	0.6
ω_I	15.66
ω	4.27

Table 5.2: Relative error in ω_R, ω_I and ω

Chapter 6

Pöshl - Teller Potential

given by

$$\left(\frac{d^2}{dx^2}+\omega^2-V_{\mathrm{PT}}(x)\right)\phi(x)=0, \tag{6.1}$$

$$V_{\mathrm{PT}}(x)=\frac{1}{2\cosh^2 x}+\mu^2\frac{1+\tanh x}{2}. \tag{6.2}$$

where $V_{\mathrm{PT}}(x)$ is Pöschl-Teller potential.

This system is exactly solvable. We perform a change of variables and a transformation of the wave function by

$$z=\frac{1}{2}(1+\tanh x), \tag{6.3}$$

$$\phi(x)=z^{-i\omega/2}(1-z)^{-i\omega/2}y(z). \tag{6.4}$$

Then, the new function $y(z)$ satisfies the standard hypergeometric equation:

$$z(1-z)y''(z)+[c-(a+b+1)z]y'(z)-aby(z)=0, \tag{6.5}$$

where

$$a=\frac{1}{2}-\frac{i}{2}\left(2\omega+1\right), \tag{6.6}$$

$$b=\frac{1}{2}-\frac{i}{2}\left(2\omega-1\right), \tag{6.7}$$

$$c=1-i\omega. \tag{6.8}$$

We impose the QNM boundary condition:

$$\lim_{x\to-\infty}\phi(x)\sim e^{-i\omega x},\quad \lim_{x\to+\infty}\phi(x)\sim e^{+i\omega x}, \tag{6.9}$$

where we have to choose a branch of the square root so that $\sqrt{z^2}=z$ for $z\in\mathbb{C}$. In terms of $y(z)$, this boundary condition is translated into the regularity condition both at $z=0,1$ simultaneously. The regular solution at $z=0$ is given by the Gauss hypergeometric function

$$y(z)=F(a,b;c;z). \tag{6.10}$$

Using the well-known analytic connection formula of the hypergeometric function:

$$\begin{aligned}F(a,b,c;z)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1;1-z)\\&+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a,c-b,c-a-b+1;1-z),\end{aligned} \tag{6.11}$$

The regularity condition at $z = 1$ requires

$$\frac{1}{\Gamma(a)\Gamma(b)} = 0. \tag{6.12}$$

Therefore we obtain $a = -n$ or $b = -n$ for $n = 0, 1, 2, \ldots$ This condition leads to the following exact spectrum:

$$\omega^{(n,\pm)} = \pm\frac{1}{2} - i\left(n + \frac{1}{2}\right). \tag{6.13}$$

We have two symmetric branches of the spectra. The exact eigenfunction is given by

$$\phi^{(n,\pm)}(x) = \left(\frac{1 + \tanh x}{2}\right)^{-i\omega^{(n,\pm)}} \left(\frac{1 - \tanh x}{2}\right)^{-i\omega^{(n,\pm)}/2} \tag{6.14}$$

$$\times F\left(-n, -n \mp i; 1 - i\omega^{(n,\pm)}; \frac{1 + \tanh x}{2}\right) \tag{6.15}$$

For a non-negative integer n, the hypergeometric function in this equation is a polynomial of degree n. For simplicity, we consider the case of $b = -n$, and abbreviate the upper index in these expressions. For the lowest overtone number $n = 0$, we have

$$\omega = \frac{1 - i}{2}, \tag{6.16}$$

$$\phi(x) = \left(\frac{1}{2\cosh x}\right)^{-i\omega}. \tag{6.17}$$

Chapter 7

Dark matter spike around Schwarzschild Blackhole

Suppose a Schwarzschild blackhole is surrounded by a dark matter spike whose mass is much less than the mass of the blackhole. We want to find the change in the space-time metric due to the dark matter and look for modifications in the Regge-Wheeler (RW) potential. We will also look for quasi-normal modes (QNM) arising due to this modified potential.

In this discussion, we consider the dark matter to be a perfect fluid, but make no other assumption about its nature. We also assume that this DM profile does not consider the self-interactions of DM. Different DM profile may change the QNMs properties [14].

For spherically symmetric BHs, the density peaks near $r \sim 4R_S$ with R_S the Schwarzschild radius and has a steep cutoff at $r = 2R_S$, below which the density of dark matter (DM) vanishes due to annihilation or dropping into the BHs [15].

Here, we will use three dark matter profiles to explore the quasi-normal modes (QNMs) of the perturbed Schwarzschild BHs. Here we investigate the axial gravitational perturbations with the dark matter profiles - .

The structure of the article is as follows. In Sec. 7.1, we discussed the change in geometry due to the dark matter shell and we derive the modified gauge metric of spherically symmetric BHs. In Sec. 7.2.2, we derive the QNMs for axial gravitational perturbations and in Sec. **??** we have discussed the techniques for finding the QNM.

Throughout the article, we have used $c = G = 1$, where c is the speed of light and G is the gravitational constant.

7.1 Black holes surrounded by Dark Matter Shell

7.1.1 Case I: $2M < r \leq 4M$

The Schwarzschild metric remains unaffected by the DM spike

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7.1}$$

where the Schwarzschild radius $R_s = 2M$

7.1.2 Case II: $r > 4M$

Assume that DM can affect the Schwarzschild metric. The most general static and spherically symmetric metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{7.2}$$

where

$$f(4M) = g(4M) = \left(1 - \frac{2M}{r}\right) \tag{7.3}$$

Assume the contribution of the DM comes through the energy-momentum tensor in the Einstein's field equation. We consider dark matter as a perfect fluid, but no other assumption about its nature. Then the energy-momentum tensor takes the form

$$T^{\mu}_{\nu} = \mathrm{diag}\left\{-\rho(r), p(r), p(r), p(r)\right\} \tag{7.4}$$

where $\rho(r)$ depends on the density distribution of DM. The density profile of the DM depends on the DM model we are considering. This affects the spacetime geometry and gives the expressions of $f(r)$ and $g(r)$.

Assume the DM distributes like dust and the pressure $p(r) \ll \rho(r)$ and $p(r) \ll (1 - g(r))/8\pi r^2$. The Einstein equations $G^{\mu}_{\nu} = 8\pi T^{\mu}_{\nu}$ now become (see Appen. 8.9),

$$rg'(r) + g(r) - 1 + 8\pi r^2 \rho(r) = 0, \tag{7.5}$$

$$rg(r)f'(r) + f(r)g(r) - f(r) = 0, \tag{7.6}$$

$$p'(r) = \rho(r)\frac{g(r) - 1}{2rg(r)}. \tag{7.7}$$

Using the expression of $\rho(r)$ in Eq. (7.5) and using the boundary condition in Eq. (7.3), we can find $g(r)$. Then using the expression of $g(r)$ in Eq. (7.6) and using the boundary condition in Eq. (7.3), we can find $f(r)$.

7.2 Axial perturbations of Schwarzschild-like black holes

7.2.1 $r > 4M$

Consider the linear gravitational perturbation for the modified Schwarzschild-like background metric given by Eq. (7.2). We will ignore the perturbation of the DM since its effects are negligible compared to the effects of DM from the modified background geometry.

Perturbed metric $g_{\mu\nu}$ is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{7.8}$$

where $\bar{g}_{\mu\nu}$ is the metric of background spacetime given by Eq. (7.2) and $h_{\mu\nu}$ is the linear perturbation term. All higher-order perturbation terms are neglected.

$$h_{\mu\nu} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \left[\left(h^{\ell m}_{\mu\nu}\right)^{(\text{axial})} + \left(h^{\ell m}_{\mu\nu}\right)^{(\text{polar})}\right] \tag{7.9}$$

The axial gravitational perturbation is intrinsically decoupled with scalar fields and the perturbations of DM is neglected for simplification [16]. But, for the case of the polar perturbations, it always couple with any extra matter fields, therefore this case is difficult to deal with [14]. So, from now we will focus on the axial case. For the expressions of $(h^{\ell m}_{\mu\nu})^{(\text{axial})}$ see [14].

7.2.2 QNM Equations for Axial Perturbation

Case I: $r > 4M$

The QNM equations of the axial gravitational perturbation in the frequency domain are [14, 17]

$$\left[\frac{\partial^2}{\partial r_*^2} + \omega^2 - V_{\text{axial}}(r)\right]\Phi(r) = 0 \tag{7.10}$$

where r_* is defined by

$$dr_* = \frac{dr}{\sqrt{f(r)g(r)}} \tag{7.11}$$

with the effective potential [14]

$$V_{\text{axial}}(r) = \frac{rf'g' + g\left[f' + 2rf''\right]}{2r} - \frac{gf'^2}{2f} + \frac{f\left[rg' + 4g + 2\left(\ell^2 + \ell - 2\right)\right]}{2r^2} \tag{7.12}$$

where the prime denotes the derivative with respect to r.

Case II: $2M < r \leq 4M$

When $2M < r \leq 4M$, the QNM equations are

$$\left[\frac{\partial^2}{\partial r_*^2} + \omega^2 - V_{\text{RW}}\right]\Phi = 0 \tag{7.13}$$

where r_* is the “tortoise coordinate” given by

$$dr_* = \left(1 - \frac{2M}{r}\right)^{-1} dr \tag{7.14}$$

with

$$V_{\text{RW}}(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right] \tag{7.15}$$

which is known as Regge–Wheeler equation [2, 18].

7.3 Different dark matter profiles

Chapter 8

Appendix

8.1 Derivation of equation of motion for massless scalar field

The Lagrangian density for massless scalar field in curved spacetime is given by

$$\mathcal{L}' = \sqrt{-g}\left(\frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi\right) = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi\right) \tag{8.1}$$

$$(-g)^{-1/2}\mathcal{L}' = \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi = \frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi \tag{8.2}$$

$$\frac{\partial((-g)^{-1/2}\mathcal{L}')}{\partial\left(\nabla_\mu\Phi\right)} = \frac{1}{2}g^{\mu\nu}\left[\nabla_\nu\Phi + \delta^\mu_\nu\left(\nabla_\mu\Phi\right)\right] = \frac{1}{2}g^{\mu\nu}\left[\nabla_\nu\Phi + \nabla_\nu\Phi\right] = \nabla^\mu\Phi \tag{8.3}$$

$$\frac{\partial((-g)^{-1/2}\mathcal{L}')}{\partial\Phi} = 0 \tag{8.4}$$

$$\nabla_\mu\left(\frac{\partial((-g)^{-1/2}\mathcal{L}')}{\partial(\nabla_\mu\Phi)}\right) = \frac{\partial((-g)^{-1/2}\mathcal{L}')}{\partial\Phi} \tag{8.5}$$

$$\text{or, } \nabla_\mu\nabla^\mu\Phi = 0 \tag{8.6}$$

Therefore, the equation of motion is

$$\nabla_\mu\nabla^\mu\Phi = 0 \tag{8.7}$$

From general relativity, we know that the covariant derivative of a four-vector V^μ is given by [19]

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}V^\mu)\,, \tag{8.8}$$

Using $V^\mu = \nabla^\mu\Phi$ in the above expression, we get

$$\nabla_\mu\nabla^\mu\Phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\;\nabla^\mu\Phi) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\;g^{\mu\nu}\;\nabla_\nu\Phi) \tag{8.9}$$

For any scalar f, $\nabla_\nu f = \partial_\nu f$. Therefore Eq. 8.9 becomes

$$\nabla_\mu\nabla^\mu\Phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\;g^{\mu\nu}\;\partial_\nu\Phi) \tag{8.10}$$

Using Eq. 8.10, the equation of motion (Eq. 8.7) becomes

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\;g^{\mu\nu}\;\partial_\nu\Phi) = 0 \tag{8.11}$$

8.2 Derivation of equation of motion under polar coordinates

$$
\begin{aligned}
&\partial_\mu \left[r^2 \sin\theta \bar{g}^{\mu\nu}\partial_\nu\right]\Phi && (8.12)\\
=&\partial_0\left[r^2\sin\theta\bar{g}^{00}\partial_0\Phi\right]+\partial_1\left[r^2\sin\theta\bar{g}^{11}\partial_1\Phi\right]+\partial_2[r^2\sin\theta\bar{g}^{22}\partial_2\Phi]+\partial_3\left[r^2\sin\theta\bar{g}^{33}\partial_3\Phi\right] && (8.13)\\
=&\partial_0\left[r^2\sin\theta\bar{g}^{00}\partial_0\Phi\right]+\partial_r\left[r^2\sin\theta\bar{g}^{rr}\partial_r\Phi\right]+\partial_\theta[r^2\sin\theta\bar{g}^{\theta\theta}\partial_\theta\Phi]+\partial_\phi\left[r^2\sin\theta\bar{g}^{\phi\phi}\partial_\phi\Phi\right] && (8.14)\\
=&\partial_0\left[r^2\sin\theta\left(-\frac{1}{A(r)}\right)\partial_0\Phi\right]+\partial_r\left[r^2\sin\theta\cdot A(r)\cdot\partial_r\Phi\right]+\partial_\theta\left[r^2\sin\theta\cdot\frac{1}{r^2}\partial_\theta\Phi\right] && (8.15)\\
&+\partial_\phi\left[r^2\sin\theta\frac{1}{r^2\sin^2\theta}\partial_\phi\Phi\right] && (8.16)\\
=&\underbrace{-r^2\sin\theta\cdot\frac{1}{A(r)}\partial_0^2\Phi}_{\text{1st term}}+\underbrace{\sin\theta\partial_r\left[r^2A(r)\partial_r\Phi\right]}_{\text{2nd term}}+\underbrace{\partial_\theta\left[\sin\theta\partial_\theta\Phi\right]+\frac{1}{\sin\theta}\partial_\phi^2\Phi}_{\text{3rd term}} && (8.17)
\end{aligned}
$$

1st term

$$
\begin{aligned}
&-r^2\sin\theta\cdot\frac{1}{A(r)}\partial_0^2\Phi && (8.18)\\
&=-r^2\sin\theta\cdot\frac{1}{A(r)}\cdot\frac{1}{r}\sum_{l,m}Y_{lm}(\theta,\phi)\partial_0^2u_{lm}(t,r) && (8.19)\\
&=-r\sin\theta\cdot\frac{1}{A(r)}\sum_{l,m}Y_{lm}(\theta,\phi)\partial_0^2u_{lm}(t,r) && (8.20)
\end{aligned}
$$

2nd term
$\sin\theta\partial_r\left[r^2A(r)\partial_r\Phi\right]$

$$
\begin{aligned}
&\partial_r\left[r^2A(r)\partial_r\Phi\right] && (8.21)\\
&=\partial_r\left[r^2A(r)\partial r\left(\frac{1}{r}\sum_{l,m}u_{lm}(t,r)Y_{lm}(\theta,\phi)\right)\right] && (8.22)\\
&=\sum_{l,m}Y_{lm}(\theta,\phi)\partial_r\left[r^2A(r)\partial_r\left(\frac{u_{lm}}{r}\right)\right] && (8.23)\\
&=\sum_{l,m}Y_{lm}(\theta,\phi)\partial_r\left[r^2A(r)\left(-\frac{u_{lm}}{r^2}+\frac{\partial_r u_{lm}}{r}\right)\right] && (8.24)\\
&=\sum_{l,m}Y_{lm}(\theta,\phi)\partial_r\left[A(r)\left(-u_{lm}+r\partial_r u_{lm}\right)\right] && (8.25)\\
&=\sum_{l,m}Y_{lm}(\theta,\phi)\left[-\partial_r\left(A(r)u_{lm}\right)+\partial_r\left(A(r)r\partial_r u_{lm}\right)\right] && (8.26)\\
&=\sum_{l,m}Y_{lm}(\theta,\phi)\left[-A(r)\partial_r u_{lm}-u_{lm}\partial_r A(r)+A(r)\partial_r u_{lm}+\partial_r\left(A(r)\partial_r u_{lm}\right)\right] && (8.27)\\
&=\sum_{l,m}Y_{lm}\left[-u_{lm}\partial_r A(r)+r\partial_r\left(A(r)\partial_r u_{lm}\right)\right] && (8.28)
\end{aligned}
$$

Therefore

$$\sin\theta\partial_r\left[r^2A(r)\partial_r\Phi\right]=\sum_{l,m}\sin\theta Y_{lm}\left[-u_{lm}\partial_r A(r)+r\partial_r\left(A(r)\partial_r u_{lm}\right)\right] \tag{8.29}$$

3rd term

$$
\begin{aligned}
&\partial_\theta \left[\sin\theta \partial_\theta \Phi\right] + \frac{1}{\sin\theta}\partial_\phi^2 \Phi && (8.30)\\
&= \sum_{l,m} \frac{u_{lm}}{r} \sin\theta \left[\frac{1}{\sin\theta}\partial_\theta \left(\sin\theta\partial_\theta Y_{lm}\right) + \frac{1}{\sin^2\theta} Y_{lm}\right] && (8.31)\\
&= \sum_{l,m} \frac{u_{lm}}{r} \sin\theta \left[-l(l+1) Y_{lm}\right] && (8.32)\\
&= -\sum_{l,m} \frac{u_{lm}}{r} \sin\theta\, l(l+1) Y_{lm} && (8.33)
\end{aligned}
$$

Adding all the 3 terms,

$$
\begin{aligned}
&\partial_\mu \left[r^2 \sin\theta \bar{g}^{\mu\nu} \partial_\nu\right] \Phi && (8.34)\\
&= \sum_{l,m} \left[-r\sin\theta \cdot \frac{1}{A(r)} Y_{lm}(\theta,\phi)\partial_0^2 u_{lm} - \sin\theta Y_{lm} u_{lm} \partial_r A(r) + \sin\theta Y_{lm} r \partial_r \left(A(r)\partial_r u_{lm}\right)\right.\\
&\quad \left. - \frac{u_{lm}}{r}\sin\theta\, l(l+1) Y_{lm}\right] && (8.35)\\
&= \sum_{l,m} \sin\theta Y_{lm} \left[-\frac{r}{A(r)}\partial_0^2 u_{lm} - u_{lm}\partial_r A(r) + r\partial_r \left(A(r)\partial_r u_{lm}\right) - \frac{u_{lm}}{r} l(l+1)\right] && (8.36)\\
&= \sum_{l,m} \sin\theta Y_{lm} \left[r\partial_r \left(A(r)\partial_r u_{lm}\right) - \frac{r}{A(r)}\partial_0^2 u_{lm} - \left(\partial_r A(r) + \frac{l(l+1)}{r}\right) u_{lm}\right] && (8.37)\\
&= \sum_{l,m} \sin\theta Y_{lm} \left[r\partial_r \left(A(r)\partial_r u_{lm}\right) - \frac{r}{A(r)}\partial_0^2 u_{lm} - \left(\frac{R_S}{r^2} + \frac{l(l+1)}{r}\right) u_{lm}\right] \quad \left[\text{Using } A(r) = \left(1 - \frac{R_S}{r}\right)\right] && (8.38)\\
&= \sum_{l,m} \sin\theta Y_{lm} \left[r\partial_r \left(A(r)\partial_r u_{lm}\right) - \frac{r}{A(r)}\partial_0^2 u_{lm} - r\left(\frac{R_S}{r^3} + \frac{l(l+1)}{r^2}\right) u_{lm}\right] && (8.39)\\
&= \sum_{l,m} \sin\theta Y_{lm} \left[r\partial_r \left(A(r)\partial_r u_{lm}\right) - \frac{r}{A(r)}\partial_0^2 u_{lm} - \frac{r}{A(r)} V_l(r) u_{lm}\right] \quad \left[\text{Define, } V_l(r) = A(r)\left(\frac{R_S}{r^3} + \frac{l(l+1)}{r^2}\right)\right] && (8.40)
\end{aligned}
$$

Therefore, $\partial_\mu \left[r^2 \sin\theta \bar{g}^{\mu\nu}\partial_\nu\right]\Phi = 0$ gives

$$
\begin{aligned}
&r\partial_r \left(A(r)\partial_r u_{lm}\right) - \frac{r}{A(r)}\partial_0^2 u_{lm} - \frac{r}{A(r)} V_l(r) u_{lm} = 0 && (8.41)\\
&\text{or, } A(r)\partial_r \left(A(r)\partial_r u_{lm}\right) - \partial_0^2 u_{lm} - V_l(r) u_{lm} = 0 \quad \left[\text{Multiplying both sides by } \frac{A(r)}{r}\right] && (8.42)
\end{aligned}
$$

$$
\boxed{A(r)\partial_r \left(A(r)\partial_r u_{lm}\right) - \partial_0^2 u_{lm} - V_l(r) u_{lm} = 0} \tag{8.43}
$$

8.3 Derivation related to tortoise coordinate

$$
r_* = r + R_S \ln\left(\frac{r - R_S}{R_S}\right) \tag{8.44}
$$

$$\frac{\partial f}{\partial r_*} = \frac{dr}{dr_*} \cdot \frac{\partial f}{\partial r} = \left(\frac{dr_*}{dr}\right)^{-1} \frac{\partial f}{\partial r} \tag{8.45}$$

$$\frac{dr_*}{dr} = 1 + R_S \cdot \frac{1}{r - R_S} = \frac{r - R_S + R_S}{r - R_S} = \frac{r}{r - R_S} = \frac{1}{1 - \frac{R_S}{r}} = \frac{1}{A(r)} \tag{8.46}$$

Therefore,

$$\frac{\partial f}{\partial r_*} = A(r)\frac{\partial f}{\partial r} \tag{8.47}$$

The above expression is true for any arbitrary function f. Therefore

$$\frac{\partial}{\partial r_*} = A(r)\frac{\partial}{\partial r} \quad \text{or equivalently} \quad \partial_* = A(r)\partial_r \tag{8.48}$$

8.4 Derivation 4

$$\left[\partial_*^2 - V_l(r)\right] u_{lm}(t,r) = \partial_0^2 u_{lm}(t,r) \tag{8.49}$$

$$\text{or,} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[\partial_*^2 \tilde{u}_{lm}(\omega,r) - V_l(r)\tilde{u}_{lm}(\omega,r)\right] = \frac{1}{c^2} \int \frac{d\omega}{2\pi} e^{-i\omega t} (-i\omega)^2 \tilde{u}_{lm}(\omega,r) \tag{8.50}$$

$$\text{or,} \; \partial_*^2 \tilde{u}_{lm}(\omega,r) - V_l(r)\tilde{u}_{lm}(\omega,r) = -\frac{\omega^2}{c^2}\tilde{u}_{lm}(\omega,r) \tag{8.51}$$

$$\text{or,} \; -\partial_*^2 \tilde{u}_{lm}(\omega,r) + V_l(r)\tilde{u}_{lm}(t,r) = \frac{\omega^2}{c^2}\tilde{u}_{lm}(\omega,r) \tag{8.52}$$

$$\text{or,} \; \left[-\frac{d^2}{dr_*^2} + V_l(r)\right] \tilde{u}_{lm}(\omega,r) = \frac{\omega^2}{c^2}\tilde{u}_{lm}(\omega,r) \tag{8.53}$$

$$\boxed{\left[-\frac{d^2}{dr_*^2} + V_l(r)\right] \tilde{u}_{lm}(\omega,r) = \frac{\omega^2}{c^2}\tilde{u}_{lm}(\omega,r)} \tag{8.54}$$

8.5 Derivation 5

$$n_a^T = (A_a^i n_i)^T = \left[\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}\right]^T \tag{8.55}$$

$$= \begin{pmatrix} \sin^2\theta\cos^2\phi + \sin^2\theta\cos^2\phi + \cos^2\phi \\ r\sin\theta\cos\theta\cos^{\phi} + r\cos\theta\sin\theta\sin^2\phi - r\sin\theta\cos\theta \\ -r\sin^2\theta\sin\phi\cos\phi + r\sin^2\theta\cos\phi\sin\phi + 0 \end{pmatrix}^T \tag{8.56}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \tag{8.57}$$

8.6 Angular Momentum operator in SWBH

To understand the meaning of this expression, and its relation to the more usual form of the angular momentum vector in polar coordinates, observe that eq. (12.36) gives

T where f is any differentiable function. These are the components of the vector $(L^a f)$, expressed in the basis of tangent space which is dual to the basis $(dr, d\theta, d\phi)$ in the space of 1 -forms, i.e. in the basis $(\partial_r, \partial_\theta, \partial_\phi)$, so, as a tangent space vector,

$$\mathbf{L}f = \frac{i}{r\sin\theta}\left(\partial_\phi f\right)\partial_\theta - \frac{i}{r\sin\theta}\left(\partial_\theta f\right)\partial_\phi$$

In polar coordinates, it is more convenient to use as a basis for tangent space

$$\mathbf{e}a = \left(\partial_r, \frac{1}{r}\partial\theta, \frac{1}{r\sin\theta}\partial_\phi\right)_a,$$

which is equivalent to the basis $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$, in the sense that the components of a directional derivative $\partial_{\mathbf{u}}$ with respect to the basis $\mathbf{e}_a$ are the same as the components of the vector $\mathbf{u}$ in the basis $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$. In the basis $\mathbf{e}_a$, the components of the vector $\mathbf{L}f$ become

$$L^a f = i\left(0, \frac{1}{\sin\theta}\partial_\phi f, -\partial_\theta f\right)^a.$$

From quantum mechanics, we know that

$$\mathbf{L}f = -i\mathbf{r} \times \boldsymbol{\nabla} f \tag{8.58}$$

This is the expression that can be obtained directly from, using the usual form of the gradient in polar coordinates.

8.7 Bender-Wu Method

Bender-Wu method (BW method) is one of methods for finding the eigenfunction and energy eigenvalues of non-relativistic time-independent 1-D Schrodinger equation with anharmonic potential.

We want to solve the following equation using BW method

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x) \tag{8.59}$$

where $V(x)$ is the quantum potential and E is the energy eigenvalue with wavefunction $\psi(x)$.

We define reduced potential $\tilde{V}(x)$ and reduced energy $\tilde{E}$ respectively by

$$\tilde{V}(x) = \frac{2m}{\hbar^2}V(x), \quad \tilde{E} = \frac{2m}{\hbar^2}E \tag{8.60}$$

Therefore, Eq. (8.59) becomes

$$-\frac{d^2}{dx^2}\psi(x) + \tilde{V}(x)\psi(x) = \tilde{E}\psi(x) \tag{8.61}$$

Rearranging the above equation, we get

$$\left(-\frac{d^2}{dx^2} + (-\tilde{E} + \tilde{V}(x))\right)\psi(x) = 0 \tag{8.62}$$

Now we introduce a quantity g^4 in Eq. (8.62) where g can be complex with the condition $\mathrm{Re}(1/g) > 0$.

$$\left(-g^4\frac{d^2}{dx^2} + (-\tilde{E} + \tilde{V}(x))\right)\psi(x) = 0 \tag{8.63}$$

Putting $g = 1$, we get the equation Eq. (8.60) back.

Let, $\tilde{V}(x)$ has global minimum at $\bar{x}$. The BW method will only work when

$$\tilde{V}''(x)|_{x=\bar{x}} > 0. \tag{8.64}$$

Expanding $\tilde{V}(x)$ around the stable minimum $\bar{x}$, we get

$$\tilde{V}(x) = V_0 + \sum_{j=2}^{\infty} V_j (x - \bar{x})^j \tag{8.65}$$

where

$$V_j = \left. \frac{1}{j!} \frac{d^j\, (V(x))}{dx^j} \right|_{x=\bar{x}} \tag{8.66}$$

Let,

$$x - \bar{x} = gq \tag{8.67}$$

$$x - \bar{x} = gq \tag{8.68}$$

$$\text{or, } dx = g dq \tag{8.69}$$

$$\text{or, } \frac{d}{dx} = \frac{1}{g} \cdot \frac{d}{dq} \tag{8.70}$$

$$\text{or, } \frac{d^2}{dx^2} = \frac{1}{g^2} \cdot \frac{d^2}{dq^2} \tag{8.71}$$

$$\boxed{\frac{d^2}{dx^2} = \frac{1}{g^2} \cdot \frac{d^2}{dq^2}} \tag{8.72}$$

Then Eq. (8.65) becomes

$$\tilde{V}(x) = V_0 + \sum_{j=2}^{\infty} V_j (x - \bar{x})^j = V_0 + V_2 g^2 q^2 + \sum_{j=3}^{\infty} V_j (gq)^j \tag{8.73}$$

Putting Eqs. (8.72), (8.73) and $\tilde{\psi}(q) = \psi(\bar{x} + gq)$ in Eq. (8.63),

$$\left(-g^2 \frac{d^2}{dq^2} - \tilde{E} + V_0 + V_2 g^2 q^2 + \sum_{j=3}^{\infty} V_j (gq)^j \right) \tilde{\psi}(q) = 0 \tag{8.74}$$

Dividing both sides by $2g^2$,

$$\left(-\frac{1}{2} \frac{d^2}{dq^2} + \frac{-\tilde{E} + V_0}{2g^2} + \frac{V_2}{2} q^2 + \frac{1}{2g^2} \sum_{j=3}^{\infty} V_j (gq)^j \right) \tilde{\psi}(q) = 0 \tag{8.75}$$

Let,

$$\epsilon = -\frac{-\tilde{E} + V_0}{2g^2} \tag{8.76}$$

$$\Omega = \sqrt{V_2} \tag{8.77}$$

$$v(q) = \frac{1}{2g^2} \sum_{j=3}^{\infty} V_j (gq)^j \tag{8.78}$$

Therefore, Eq. (8.75) becomes,

$$\left(-\frac{1}{2}\frac{d^2}{dq^2} - \epsilon + \frac{1}{2}\Omega^2 q^2 + v(q)\right)\tilde{\psi}(q) = 0 \tag{8.79}$$

$$\text{or, } \left(-\frac{1}{2}\cdot\frac{d^2}{dq^2} + \frac{1}{2}\Omega^2 q^2 + v(q) - \epsilon\right)\tilde{\psi}(q) = 0 \tag{8.80}$$

We substitute $\tilde{\psi}(q) = e^{-\Omega q^2/2}u(q)$,

$$\frac{d\tilde{\psi(q)}}{dq} = \frac{d}{dq}\left(e^{-\Omega q^2/2}u(q)\right) = e^{-\Omega q^2/2}\left(u' - \Omega q u\right) \tag{8.81}$$

$$\frac{d^2\tilde{\psi}(q)}{dq^2} = \frac{d}{dq}\left[e^{-\Omega q^2/2}\left(u' - \Omega q u\right)\right] \tag{8.82}$$

$$= (-\Omega q)e^{-\Omega q^2/2}\left(u' - \Omega q u\right) + e^{-\Omega q^2/2}\left(u''(q) - \Omega q u'(q) - \Omega u(q)\right) \tag{8.83}$$

$$= e^{-\Omega q^2/2}\left[u''(q) - 2\Omega q u'(q) + \left(\Omega^2 q^2 - \Omega\right)u(q)\right] - \frac{1}{2}\frac{d^2\tilde{\psi}(q)}{dq^2} + \frac{1}{2}\Omega^2 q^2\tilde{\psi}(q) \tag{8.84}$$

$$= -e^{-\Omega q^2/2}\left[-\frac{1}{2}u''(q) + \Omega q u'(q) + \left(\frac{\Omega}{2} - \frac{\Omega^2}{2}q^2\right)u(q)\right] + e^{-\Omega q^2/2}\frac{1}{2}\Omega^2 q^2 u(q) \tag{8.85}$$

$$= e^{-\Omega q^2/2}\left(-\frac{1}{2}u''(q) + \Omega q u'(q) + \frac{\Omega}{2}u(q)\right) \tag{8.86}$$

Here the prime denotes the derivative with respect to q.

Therefore,

$$\frac{d\tilde{\psi}}{dq} = e^{-\Omega q^2/2}\left(u' - \Omega q u\right) \tag{8.87}$$

$$\frac{d^2\tilde{\psi}}{dq^2} = e^{-\Omega q^2/2}\left(-\frac{1}{2}u'' + \Omega q u' + \frac{\Omega}{2}u\right) \tag{8.88}$$

Putting Eqs. (8.87) and (8.88) in Eq. (8.80), we get

$$\boxed{-\frac{1}{2}u''(q) + \Omega q u'(q) + \left(\frac{\Omega}{2} + v(q) - \epsilon\right)u(q) = 0.} \tag{8.89}$$

Let,

$$u(q) = \sum_{n=0}^{\infty} g^n u_n(q), \qquad \epsilon = \sum_{n=0}^{\infty} g^n \epsilon_n. \tag{8.90}$$

$$v(q) = \frac{1}{2g^2}\sum_{j=3}^{\infty} V_j (gq)^j = \frac{1}{2}\sum_{j=3}^{\infty} V_j g^{j-2} q^j = \frac{1}{2}\sum_{j=1}^{\infty} g^j V_{j+2} q^{j+2}. \tag{8.91}$$

Using $v_j = \dfrac{V_{j+2}}{2}$,

$$v(q) = \sum_{j=1}^{\infty} g^j v_j q^{j+2} \tag{8.92}$$

Putting these expansions into Eq. (8.89), we get

$$\sum_{n=0}^{\infty} g^n\left(-\frac{1}{2}u_n''\right) + \sum_{n=0}^{\infty} g^n \Omega q u_n' + \left(\frac{\Omega}{2} + \sum_{j=1}^{\infty} g^j v_j q^{j+2} - \sum_{j=0}^{\infty} g^j \epsilon_j\right)\sum_{n=0}^{\infty} g^n u_n = 0 \tag{8.93}$$

$$\text{or, } \sum_{n=0}^{\infty} g^n\left(-\frac{1}{2}u_n'' + \Omega q u_n' + \frac{\Omega}{2}u_n\right) + \sum_{j=1}^{\infty} g^j v_{j+2} q^{j+2}\sum_{n=0}^{\infty} g^n u_n - \sum_{j=0}^{\infty} g^j \epsilon_j \sum_{n=0}^{\infty} g^n u_n = 0 \tag{8.94}$$

The 2nd term of Eq. (8.94) becomes,

$$\sum_{j=1}^{\infty} g^j v_j q^{j+2} \sum_{n=0}^{\infty} g^n u_n = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} g^{n+j} v_j q^{j+2} u_n = \sum_{n=0}^{\infty} g^n \sum_{j=1}^{\infty} v_j q^{j+2} u_{n-j} \tag{8.95}$$

The 3rd term of Eq. (8.94) becomes,

$$\sum_{j=0}^{\infty} g^j \epsilon_j \sum_{n=0}^{\infty} g^n u_n = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} g^{n+j} \epsilon_j u_n = \sum_{n=0}^{\infty} g^n \sum_{j=0}^{\infty} \epsilon_j u_{n-j} \tag{8.96}$$

Therefore Eq. (8.94) becomes,

$$\sum_{n=0}^{\infty} g^n \left(-\frac{1}{2}u_n'' + \Omega q u_n' + \frac{\Omega}{2} u_n + \sum_{j=1}^{n} v_j q^{j+2} u_{n-j} - \sum_{j=0}^{n} \epsilon_j u_{n-j} \right) = 0 \tag{8.97}$$

For all $n \geq 0$,

$$\boxed{-\frac{1}{2}u_n'' + \Omega q u_n' + \frac{\Omega}{2} u_n + \sum_{j=1}^{n} v_j q^{j+2} u_{n-j} - \sum_{j=0}^{n} \epsilon_j u_{n-j} = 0.} \tag{8.98}$$

8.7.1 Ground State ($\nu = 0$)

Now we limit our discussion to the ground state ($\nu = 0$).

Case I: ($n = 0$)

For $n = 0$, we have

$$-\frac{1}{2}u_0'' + \Omega q u_0' + \frac{\Omega}{2} u_0 - \epsilon_0 u_0 = 0 \tag{8.99}$$

$$\text{or,} \quad -\frac{1}{2}u_0'' + \Omega q u_0' = \left(\epsilon_0 - \frac{\Omega}{2}\right) u_0 \tag{8.100}$$

Hermite polynomial is one of the solutions of the above equation. Therefore,

$$(u_0)_\nu(q) = H_\nu(q) \tag{8.101}$$

$$\epsilon_0 - \frac{\Omega}{2} = \nu\Omega \implies (\epsilon_0)_\nu = \left(\nu + \frac{1}{2}\right)\Omega \tag{8.102}$$

where ν is the *level number* and $H_\nu(q)$ denotes the Hermite polynomial of order ν.

For $\nu = 0$ (ground state), the solution is

$$\boxed{u_0(q) = 1 \quad \text{with} \quad \epsilon_0 = \frac{\Omega}{2}} \tag{8.103}$$

Case II: ($n \geq 1$)

Using it in Eq. (8.98) for $n \geq 1$, we get

$$-\frac{1}{2}u_n'' + \Omega q u_n' + \frac{\Omega}{2}u_n + \sum_{j=1}^{n} v_j q^{j+2} u_{n-j} - \sum_{j=1}^{n} \epsilon_j u_{n-j} - \epsilon_0 u_n = 0 \tag{8.104}$$

$$\text{or,} \quad -\frac{1}{2}u_n'' + \Omega q u_n' + \frac{\Omega}{2}u_n + \sum_{j=1}^{n} v_j q^{j+2} u_{n-j} - \sum_{j=1}^{n} \epsilon_j u_{n-j} - \frac{\Omega}{2}u_n = 0 \tag{8.105}$$

$$\text{or,} \quad -\frac{1}{2}u_n'' + \Omega q u_n' + \sum_{j=1}^{n} v_j q^{j+2} u_{n-j} - \sum_{j=1}^{n} \epsilon_j u_{n-j} = 0 \tag{8.106}$$

$$\boxed{-\frac{1}{2}u_n'' + \Omega q u_n' + \sum_{j=1}^{n}(v_j q^{j+2} - \epsilon_j)u_{n-j} = 0, \quad n \geq 1.} \tag{8.107}$$

The solution $u_n(q)$ is a *polynomial of degree* $3n$ [7, 8]:

$$u_n(q) = \sum_{k=0}^{3n} A_n^k q^k, \quad n \geq 0 \tag{8.108}$$

where the superscript k is just an index in A_n. The differential equation (8.107) determines all the coefficients A_n^k and ϵ_n recursively [8]. For detailed calculation see Sec. (4.5.4).

Therefore, for the ground state $\nu = 0$, we have

$$\tilde{E} = V_0 + 2g^2 \sum_{n=0}^{\infty} g^n \epsilon_n, \tag{8.109}$$

$$\tilde{\psi}(q) = e^{-\Omega q^2/2} \sum_{n=0}^{\infty} g^n u_n(q), \qquad u_n(q) = \sum_{k=1}^{3n} A_n^k q^k, \tag{8.110}$$

with $\epsilon_0 = \Omega/2$ and $u_0(q) = 1$.

Finally we want to set $g = 1$ in the perturbative series. But, in general, the power series (8.90) is not convergent for any $g \neq 0$ and evaluate it at $g = 1$. So we use the Pade′-Borrel resummation (Appen. (8.8)) of the power series to make it convergent for any $g \neq 0$. Performing this we find the value of $\tilde{E}$.

8.8 Pade′-Borel resummation

Consider an infinite series

$$\sum_{k=0}^{\infty} a_k g^k \tag{8.111}$$

where g can be complex. We want to evaluate the infinite series at a particular value of g. But in general, the above series is not convergent for any $g \neq 0$.

Moreover, if the above series is a perturbation series, it is practically not possible to know all the a_k terms. We can evaluate the a_k terms upto a finite number. Suppose, we are able to evaluate the a_k terms from a_0 to a_N.

Let,

$$\phi(g) = \sum_{k=0}^{N} a_k g^k \tag{8.112}$$

Using Pade′-Borel resummation of the above finite series, we can make the infinite series convergent for any values of g. Pade′-Borel resummation [9, 20] involves 3 steps:

1. Borel transformation
2. Pade′ approximation
3. Borel summation

8.8.1 Borel transformation

The Borel transformation of a finite series $\phi(g)$ is defined as [20]

$$\hat{\phi}(\zeta) := \sum_{n=0}^{N} \frac{a_n}{n!}\zeta^n \tag{8.113}$$

8.8.2 Pade′ approximation

Let, $P^{[l/m]}(\hat{\phi})$ is the Pade′ approximant [20, 21] of the the series $\hat{\phi}(\zeta)$ with an-order-l numerator and with an-order-m denominator where l and m are positive integers, following $l + m \leq N$. Then

$$P_{\hat{\phi}}^{[l/m]}(\zeta) = \frac{p_0 + p_1\zeta + \cdots + p_l\zeta^l}{1 + q_1\zeta + \cdots + q_m\zeta^m} \tag{8.114}$$

The coefficients p_0, p_2, …, p_l, q_1, …, q_m are found by comparing the coefficients with different powers of ζ using the expression given below

$$(p_0 + p_1\zeta + \cdots + p_l\zeta^l) = (1 + q_1\zeta + \cdots + q_m\zeta^m)\hat{\phi}(\zeta) \tag{8.115}$$

8.8.3 Borel summation

Suppose, $P_{\hat{\phi}}^{[l/m]}(\zeta)$ is analytically continuous to a neighbourhood of the positive real axis such that the Laplace transform

$$s(\phi)_{[l/m]}(g) = g^{-1}\int_0^{\infty} e^{-\zeta/g} P_{\hat{\phi}}^{[l/m]}(\zeta)d\zeta \tag{8.116}$$

exists in some region of the complex z-plane.

In the case, $P_{\hat{\phi}}^{[l/m]}(\zeta)$ is called *Borel summable* and $s(\phi)_{[l/m]}(g)$ is called *Borel resummation* of $P_{\hat{\phi}}^{[l/m]}(\zeta)$ [20].

$$\sum_{k=0}^{\infty} a_k g^k \sim s(\phi)_{[l/m]}(g) \tag{8.117}$$

If $P_{\hat{\phi}}^{[l/m]}(\zeta)$ has singularities on the *positive* real axis, then it can be extended along a neighbourhood of positive real axis as meromorphic or multi-valued function which decreases sufficiently at infinity [20]. Then, the integral (Eq. (8.116)) is ill defined in principle. But we can define related integral by deforming the contour (Eq. (8.116)) appropriately.

Consider the contours $C_\pm$ that avoid the singularities and branch cuts by the following paths slightly above or below the positive real axis (see Fig. (8.1)). Then we define the *lateral Borel resummations* by

$$s(\phi)_{[l/m]\pm}(g) = g^{-1}\int_{C_\pm} e^{-\zeta/g} P_{\hat{\phi}}^{[l/m]}(\zeta)d\zeta \tag{8.118}$$

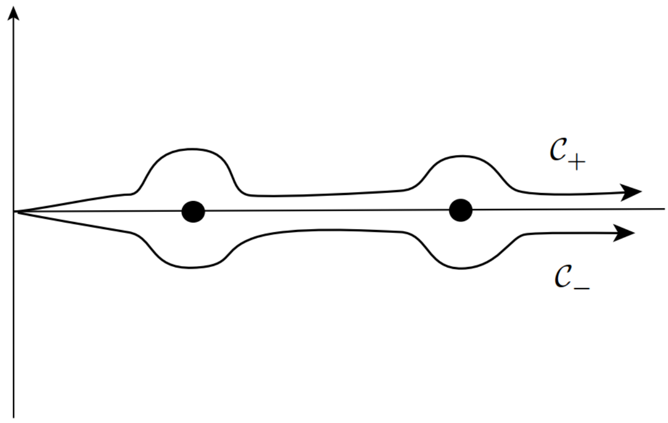

Figure 8.1: The paths $C_\pm$ avoiding the singularities of the Borel transform from above and below respectively (Taken from [20])

Even if all the coefficients of the original series are real, the lateral Borel resummations are in general complex due to the contour deformation. Their difference is purely imaginary and it is encoded in the discontinuity function. The difference is purely imaginary and it is encoded in the discontinuity function

$$\mathrm{disc}(\phi)(g) = s_{[l/m]+}(\phi)(g) - s_{[l/m]-}(\phi)(g) \tag{8.119}$$

This discontinuity gives information on the branch cut structure of the function which is reconstructed by the Borel resummations.

In some cases, some of the singularities of the Borel transform will take place at other points in the complex plane, along rays forming an angle θ with the positive real axis. In this case, we can consider paths $C_{\theta\pm}$ slightly above and below the ray, and define the *generalized lateral Borel resummations* as

$$s(\phi)_{[l/m]\theta\pm}(g) = g^{-1}\int_{C_{\theta\pm}} e^{-\zeta/g} P_{\hat{\phi}}^{[l/m]}(\zeta)d\zeta \tag{8.120}$$

The discontinuity in this case then defined as

$$\mathrm{disc}_\theta(\phi)(z) = s_{\theta+}(\phi)(z) - s_{\theta-}(\phi)(z). \tag{8.121}$$

8.9 Derivation 7

For the metric (Eq. 7.2), the Einstein's tensor G^μ_ν becomes

$$G^\mu_\nu = 0 \text{ when } \mu \neq \nu \tag{8.122}$$

$$G^0_0 = \frac{1}{r^2}\left(-1 + g(r) + rg'(r)\right) \tag{8.123}$$

$$G^1_1 = \frac{1}{r^2 f(r)}\left(-f(r) + f(r)g(r) + rg(r)f'(r)\right) \tag{8.124}$$

$$G^2_2 = G^3_3 = \frac{1}{4r(f(r))^2}\big(2f(r)g(r)f'(r) - rg(r)(f'(r))^2 + 2(f(r))^2 g'(r) + rf(r)f'(r)g'(r) + 2rf(r)g(r)f''(r)\big) \tag{8.125}$$

The Einstein equations

$$G^\mu_\nu = 8\pi T^\mu_\nu \tag{8.126}$$

where

$$T^{\mu}_{\nu} = \text{diag}\left\{-\rho(r), p(r), p(r), p(r)\right\} \tag{8.127}$$

Using $G^0_0 = 8\pi T^0_0$, we get

$$\frac{1}{r^2}\left(-1 + g(r) + rg'(r)\right) = -8\pi\rho(r) \tag{8.128}$$

Using $G^1_1 = 8\pi T^1_1$, we get

$$\frac{1}{r^2 f(r)}\left(-f(r) + f(r)g(r) + rg(r)f'(r)\right) = 8\pi p(r) \tag{8.129}$$

Eq. 8.128 simplifies to

$$\boxed{rg'(r) + g(r) - 1 + 8\pi r^2\rho(r) = 0} \tag{8.130}$$

Eq. 8.129 simplifies to

$$\frac{1}{r^2 f(r)}\left(-f(r) + f(r)g(r) + rg(r)f'(r)\right) = 8\pi p(r) \tag{8.131}$$

$$\text{or, } -\frac{(1-g(r))}{r^2} + \frac{g(r)}{r}\frac{f'(r)}{f(r)} = 8\pi p(r) \tag{8.132}$$

$$\text{or, } -\frac{(1-g(r))}{r^2} + \frac{g(r)}{r}\frac{f'(r)}{f(r)} = 0 \quad \text{[Under the assumption } p(r) \ll (1-g(r))/8\pi r^2\text{]} \tag{8.133}$$

$$\text{or, } rg(r)f'(r) + f(r)g(r) - f(r) = 0 \tag{8.134}$$

$$\boxed{rg(r)f'(r) + f(r)g(r) - f(r) = 0} \tag{8.135}$$

Using the conservation of stress-energy tensor $\nabla_\mu T^{\mu}_{\nu}$, we get

$$\text{diag}\left\{0, \frac{1}{f(r)}\left(f'(r)\cdot(p(r)+\rho(r)) + 2f(r)p'(r)\right), 0, 0\right\} = 0 \tag{8.136}$$

Therefore,

$$f'(r)\cdot(p(r)+\rho(r)) + 2f(r)p'(r) = 0 \tag{8.137}$$

$$\text{or, } f'(r)p(r) + 2f(r)p'(r) = 0 \quad [p(r) \ll \rho(r) \Rightarrow p(r)+\rho(r) \approx \rho(r)] \tag{8.138}$$

$$\text{or, } 2f(r)p'(r) = -f'(r)\rho(r) \tag{8.139}$$

$$\text{or, } p'(r) = -\frac{1}{2}\cdot\frac{f'(r)}{f(r)}\rho(r) \tag{8.140}$$

From Eq. 8.135, we get

$$\frac{f'(r)}{f(r)} = -\frac{g(r)-1}{rg(r)} \tag{8.141}$$

Putting the above expression in the Eq. 8.140, we get

$$\boxed{p'(r) = \frac{1}{2}\frac{g(r)-1}{rg(r)}\rho(r)} \tag{8.142}$$

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