Spectral Impact of Graph Topology Perturbations: Variance Analysis of Normalized Laplacian Eigenvalues

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Abstract

This work explores the impact of structural changes on the eigenvalue spectrum of the normalized Laplacian matrix in graphs. By analyzing the variance of eigenvalues under three specific scenarios—addition of a single edge, deletion of a single edge, and consecutive addition of multiple edges—the study provides a quantitative framework to evaluate graph perturbations. A focus is placed on deriving mathematical relationships governing the change in variance, highlighting the interplay between graph structure and spectral properties. The results offer insights applicable to graph-based learning, network optimization, and spectral clustering.

Keywords Spectral Graph Theory, Laplacian Matrix, Graph-Based Learning

1 Introduction

Graphs serve as a fundamental representation for complex systems across diverse domains such as social networks, biology, and communication networks. The spectral properties of graphs, particularly the eigenvalues of the normalized Laplacian matrix, play a critical role in understanding their structural and dynamic characteristics. Changes in graph topology, such as the addition or deletion of edges, can significantly alter these spectral properties, impacting applications that rely on them, such as graph partitioning, ranking, and diffusion processes.

This study investigates how perturbations in graph structure affect the variance of the eigenvalues of the normalized Laplacian matrix. Starting from fundamental spectral graph theory principles, we analyze three scenarios: adding a single edge, removing a single edge, and consecutively adding multiple edges. By deriving precise mathematical formulations for these changes, this work aims to bridge the gap between graph perturbations and their spectral impact. The findings offer theoretical contributions and practical tools for fields like machine learning, where graph-based methods such as Graph Neural Networks are increasingly prominent.

This paper is structured as follows: Section 2 introduces the mathematical background, Section 3 presents the derived formulations for each scenario, and Section 4 discusses the implications and potential applications.

2 Variance of Eigenvalues of Normalized Laplacian Matrix

Let us suppose, the normalized Laplacian matrix has eigenvalues $\lambda_{N_1}, \lambda_{N_2}, ..., \lambda_{N_n}$

Now, the sum of these eigenvalues will be equal to the trace of the matrix. Hence,

$$\sum_{i=1}^{n} \lambda_{N_i} = Tr\left\{L_N\right\}$$

So,

$$\overline{\lambda_N} = \frac{1}{n}\sum_{i=1}^{n} \lambda_{N_i}$$

$$\Rightarrow \overline{\lambda_N} = \frac{1}{n} Tr\left\{L_N\right\}$$

and,

$$Var(\lambda_N) = \frac{1}{n}\sum_{i=1}^{n} \left(\lambda_{N_i} - \overline{\lambda_N}\right)^2$$

$$\Rightarrow Var(\lambda_N) = \frac{1}{n}\sum_{i=1}^{n} \left(\lambda_{N_i} - \overline{\lambda_N}\right)^2$$

$$\Rightarrow Var(\lambda_N) = \frac{1}{n}\sum_{i=1}^{n} \left(\lambda_{N_i}^2 - 2\lambda_{N_i}\overline{\lambda_N} + \overline{\lambda_N}^2\right)$$

$$\Rightarrow Var(\lambda_N) = \frac{1}{n}\sum_{i=1}^{n} \lambda_{N_i}^2 - \frac{2\overline{\lambda_N}}{n}\sum_{i=1}^{n} \lambda_{N_i} + \frac{\overline{\lambda_N}^2}{n}\sum_{i=1}^{n} 1$$

$$\Rightarrow Var(\lambda_N) = \frac{1}{n}\sum_{i=1}^{n} \lambda_{N_i}^2 - 2\overline{\lambda_N}^2 + \overline{\lambda_N}^2$$

$$\Rightarrow Var(\lambda_N) = \frac{1}{n}\sum_{i=1}^{n} \lambda_{N_i}^2 - \overline{\lambda_N}^2$$

Now, from the properties of eigenvalues we can say $\lambda_{N_i}^2$ is eigenvalues of L_N^2. So,

$$\sum_{i=1}^{n} \lambda_{N_i}^2 = Tr\left\{L_N^2\right\}$$

$$\sum_{i=1}^{n} \lambda_{N_i}^2 = Tr\left\{L_N L_N^T\right\}$$

$$\sum_{i=1}^{n} \lambda_{N_i}^2 = \left\|L_N\right\|_F^2$$

Hence, we can write,

$$Var(\lambda_N) = \frac{1}{n}\left\|L_N\right\|_F^2 - \overline{\lambda_N}^2$$

2.1 Addition of a Single Edge

Let us consider a random graph G where self-loops and parallel edges are not allowed. Hence,

$$a_{ii} = 0 \quad for\ all\ i = \in [1, n]$$

Hence, the elements of the normalized Laplacian matrix are as -

$$l_{N_{ij}} = \begin{cases} 1 & \text{if}\, i = j \\ -\frac{a_{ij}}{\sqrt{d_i d_j}} & \text{if}\, i \neq j \end{cases}$$

Now, suppose we have added an edge node u and node v. Due to this edge, addition the graph has been modified to $G^{'}$ and suppose the corresponding degree matrix, adjacency matrix, and normalized Laplacian matrix are $D^{'}$, $A^{'}$ and $L_N^{'}$ respectively.

Following the expression of $l_{N_{ij}}$ we can write $l^{'}_{N_{ij}}$ as,

$$l^{'}_{N_{ij}} = \begin{cases} 1 & \text{if}\, i = j \\ -\frac{a^{'}_{ij}}{\sqrt{d^{'}_i d^{'}_j}} & \text{if}\, i \neq j \end{cases}$$

Now, after the addition of an edge between node u and node v, we can say that,

$$a^{'}_{ij} = \begin{cases} a_{uv} + 1 = 1 & \text{if}\, i = u\,\&\, j = v \\ a_{vu} + 1 = 1 & \text{if}\, i = v\,\&\, j = u \\ a_{ij} & \text{else} \end{cases}$$

and,

$$d'_i = \begin{cases} d_u + 1 & \text{if}\, i = u \\ d_v + 1 & \text{if}\, i = v \\ d_i & \text{else} \end{cases}$$

After a close observation on the expression of $l^{'}_{N_{ij}}$ we can conclude that $l^{'}_{N_{ij}}$ will differ from $l_{N_{ij}}$ only when $i = u\, or\, i = v$ and $j = v\, or\, j = v$. In all other cases $l^{'}_{N_{ij}}$ will be same as $l_{N_{ij}}$. So,

$$\|L_N\|_F^2 = \sum_{\substack{i=1,\\ i\neq u,v}}^{n} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 + \sum_{j=u,v} \sum_{\substack{i=1,\\ i\neq u,v}}^{n} |l_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{j=u,v} |l_{N_{ij}}|^2$$

Now, the Laplacian matrix is a symmetric matrix. Hence,

$$\|L_N\|_F^2 = \sum_{\substack{i=1,\\ i\neq u,v}}^{n} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 + 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{j=u,v} |l_{N_{ij}}|^2$$

Similarly, we can write,

$$\|L^{'}_N\|_F^2 = \sum_{\substack{i=1,\\ i\neq u,v}}^{n} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l^{'}_{N_{ij}}|^2 + 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l^{'}_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{j=u,v} |l^{'}_{N_{ij}}|^2$$

Now,

$$\|L^{'}_N\|_F^2 - \|L_N\|_F^2 = 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l^{'}_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{j=u,v} |l^{'}_{N_{ij}}|^2 - 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 - \sum_{i=u,v} \sum_{j=u,v} |l_{N_{ij}}|^2 \quad \begin{matrix}[As, for\, i\neq u,v\, and\, j\neq u,v \\ l^{'}_{N_{ij}} = l_{N_{ij}}]\end{matrix}$$

$$\Rightarrow \|L^{'}_N\|_F^2 - \|L_N\|_F^2 = 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} \left(|l^{'}_{N_{ij}}|^2 - |l_{N_{ij}}|^2\right) + \sum_{i=u,v} \sum_{j=u,v} \left(|l^{'}_{N_{ij}}|^2 - |l_{N_{ij}}|^2\right)$$

Now,

$$\sum_{i=u,v}\sum_{\substack{j=1,\\ j\neq u,v}}^{n}\left(|l'_{N_{ij}}|^2-|l_{N_{ij}}|^2\right)=\sum_{i=u,v}\sum_{\substack{j=1,\\ j\neq u,v}}^{n}\left(|-\frac{a'_{ij}}{\sqrt{d'_i d'_j}}|^2-|-\frac{a_{ij}}{\sqrt{d_i d_j}}|^2\right)$$

$$=\sum_{i=u,v}\sum_{\substack{j=1,\\ j\neq u,v}}^{n}\left(|\frac{a_{ij}}{\sqrt{(d_i+1)d_j}}|^2-|\frac{a_{ij}}{\sqrt{d_i d_j}}|^2\right)$$

$$=\sum_{i=u,v}\sum_{\substack{j=1,\\ j\neq u,v}}^{n}\frac{a_{ij}^2}{d_j}\left(\frac{1}{d_i+1}-\frac{1}{d_i}\right)$$

$$=\sum_{i=u,v}\sum_{\substack{j=1,\\ j\neq u,v}}^{n}\frac{a_{ij}^2}{d_j}h_i\quad [Let\ us\ write, \left(\tfrac{1}{d_i+1}-\tfrac{1}{d_i}\right) as\, h_i]$$

$$=\sum_{\substack{j=1,\\ j\neq u,v}}^{n}\frac{a_{uj}^2}{d_j}h_u+\sum_{\substack{j=1,\\ j\neq u,v}}^{n}\frac{a_{vj}^2}{d_j}h_v$$

$$=\left(\sum_{j=1}^{n}\frac{a_{uj}^2}{d_j}-\frac{a_{uu}^2}{d_u}-\frac{a_{uv}^2}{d_v}\right)h_u+\left(\sum_{j=1}^{n}\frac{a_{vj}^2}{d_j}-\frac{a_{vu}^2}{d_u}-\frac{a_{vv}^2}{d_v}\right)h_v$$

$$=\sum_{j=1}^{n}\frac{a_{uj}^2}{d_j}h_u+\sum_{j=1}^{n}\frac{a_{vj}^2}{d_j}h_v$$

$$=\sum_{i=u,v}\sum_{j=1}^{n}\frac{a_{ij}^2}{d_j}h_i$$

$$=\sum_{i=u,v}c_i h_i\quad [Let\ us\ write \sum_{j=1}^{n}\frac{a_{ij}^2}{d_j}\ as\ c_i]$$

$$=c_u h_u+c_v h_v$$

On the other hand,

$$\sum_{i=u,v}\sum_{j=u,v}\left(|l'_{N_{ij}}|^2-|l_{N_{ij}}|^2\right)=|l'_{N_{uu}}|^2-|l_{N_{uu}}|^2+|l'_{N_{uv}}|^2-|l_{N_{uv}}|^2+|l'_{N_{vu}}|^2-|l_{N_{vu}}|^2+|l'_{N_{vv}}|^2-|l_{N_{vv}}|^2$$

$$=1-1+|l'_{N_{uv}}|^2-|l_{N_{uv}}|^2+|l'_{N_{vu}}|^2-|l_{N_{vu}}|^2+1-1$$

$$=2\left(|l'_{N_{uv}}|^2-|l_{N_{uv}}|^2\right)$$

$$=2\left(|-\frac{a'_{uv}}{\sqrt{d'_u d'_v}}|^2-|\frac{a_{uv}}{\sqrt{d_u d_v}}|^2\right)$$

$$=2\left(\frac{1}{(d_u+1)(d_v+1)}-0\right)$$

$$=\frac{2}{(d_u+1)(d_v+1)}$$

$$= \frac{2d_u d_v}{d_u d_v (d_u+1)(d_v+1)}$$

$$= 2d_u d_v \left(\frac{1}{d_u(d_u+1)}\right)\left(\frac{1}{d_v(d_v+1)}\right)$$

$$= 2d_u d_v \left(\frac{1}{d_u} - \frac{1}{d_u+1}\right)\left(\frac{1}{d_v} - \frac{1}{d_v+1}\right)$$

$$= 2d_u d_v h_u h_v$$

Finally, we can write,

$$\|L_N^{'}\|_F^2 - \|L_N\|_F^2 = 2(c_u h_u + c_v h_v) + 2d_u d_v h_u h_v$$

$$\Rightarrow \|L_N^{'}\|_F^2 - \|L_N\|_F^2 = 2\left(\sum_{i=u,v} c_i h_i + \Pi_{i=u,v} d_i h_i\right)$$

Now the mean of eigenvalues of the normalized Laplacian matrix is defined as the trace of the matrix. And we have noticed that the diagonal elements remain the same after the addition of an edge in the graph. So, the trace of the matrix will also be the same. Hence, the mean of eigenvalues of the normalized Laplacian matrix will not be affected or mathematically we can write,

$$\overline{\lambda_N^{'}} = \overline{\lambda_N}$$

So,

$$Var(\lambda_N^{'}) - Var(\lambda_N) = \frac{1}{n}\|L_N^{'}\|_F^2 - \overline{\lambda_N^{'}}^2 - \frac{1}{n}\|L_N\|_F^2 + \overline{\lambda_N}^2$$

$$= \frac{1}{n}\left[2(c_u h_u + c_v h_v) + 2d_u d_v h_u h_v\right]$$

$$= \frac{2}{n}\left(c_u h_u + c_v h_v + d_u d_v h_u h_v\right)$$

$$where, c_i = \sum_{j=1}^{n} \frac{a_{ij}^2}{d_j} \ and \ h_i = \frac{1}{d_i+1} - \frac{1}{d_i}$$

2.2 Deletion of a Single Edge

Let us consider a random graph G where self-loops and parallel edges are not allowed. Hence,

$$a_{ii} = 0 \quad for\ all\ i = \in [1, n]$$

Hence, the elements of the normalized Laplacian matrix are as -

$$l_{N_{ij}} = \begin{cases} 1 & \text{if } i = j \\ -\frac{a_{ij}}{\sqrt{d_i d_j}} & \text{if } i \neq j \end{cases}$$

Now, suppose we have deleted an edge node u and node v. Due to this edge, deletion the graph has been modified to $G^{'}$ and suppose the corresponding degree matrix, adjacency matrix, and normalized Laplacian matrix are $D^{'}$, $A^{'}$ and $L_N^{'}$ respectively.

Following the expression of $l_{N_{ij}}$ we can write $l_{N_{ij}}^{'}$ as,

$$l_{N_{ij}}^{'} = \begin{cases} 1 & \text{if } i = j \\ -\frac{a_{ij}^{'}}{\sqrt{d_i^{'} d_j^{'}}} & \text{if } i \neq j \end{cases}$$

Now, after the deletion of an edge between node u and node v, we can say that,

$$a'_{ij} = \begin{cases} a_{uv} - 1 = 0 & \text{if}\, i = u \,\&\, j = v \\ a_{vu} - 1 = 0 & \text{if}\, i = v \,\&\, j = u \\ a_{ij} & \text{else} \end{cases}$$

and,

$$d'_i = \begin{cases} d_u - 1 & \text{if}\, i = u \\ d_v - 1 & \text{if}\, i = v \\ d_i & \text{else} \end{cases}$$

After a close observation on the expression of $l'_{N_{ij}}$ we can conclude that $l'_{N_{ij}}$ will differ from $l_{N_{ij}}$ only when $i = u\, or\, i = v$ and $j = v\, or\, j = v$. In all other cases $l'_{N_{ij}}$ will be same as $l_{N_{ij}}$. So,

$$\|L_N\|_F^2 = \sum_{\substack{i=1,\\ i\neq u,v}}^{n} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 + \sum_{j=u,v} \sum_{\substack{i=1,\\ i\neq u,v}}^{n} |l_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{j=u,v} |l_{N_{ij}}|^2$$

Now, the Laplacian matrix is a symmetric matrix. Hence,

$$\|L_N\|_F^2 = \sum_{\substack{i=1,\\ i\neq u,v}}^{n} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 + 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{j=u,v} |l_{N_{ij}}|^2$$

Similarly, we can write,

$$\|L'_N\|_F^2 = \sum_{\substack{i=1,\\ i\neq u,v}}^{n} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l'_{N_{ij}}|^2 + 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l'_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{j=u,v} |l'_{N_{ij}}|^2$$

Now,

$$\|L'_N\|_F^2 - \|L_N\|_F^2 = 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l'_{N_{ij}}|^2 + \sum_{i=u,v} \sum_{j=u,v} |l'_{N_{ij}}|^2 - 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} |l_{N_{ij}}|^2 - \sum_{i=u,v} \sum_{j=u,v} |l_{N_{ij}}|^2 \quad \left[\substack{As,\, for\, i\neq u,v\, and\, j\neq u,v \\ l'_{N_{ij}} = l_{N_{ij}}}\right]$$

$$\Rightarrow \|L'_N\|_F^2 - \|L_N\|_F^2 = 2 \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} \left(|l'_{N_{ij}}|^2 - |l_{N_{ij}}|^2\right) + \sum_{i=u,v} \sum_{j=u,v} \left(|l'_{N_{ij}}|^2 - |l_{N_{ij}}|^2\right)$$

Now,

$$\sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} \left(|l'_{N_{ij}}|^2 - |l_{N_{ij}}|^2\right) = \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} \left(|-\frac{a'_{ij}}{\sqrt{d'_i d'_j}}|^2 - |-\frac{a_{ij}}{\sqrt{d_i d_j}}|^2\right)$$

$$= \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} \left(|\frac{a_{ij}}{\sqrt{(d_i - 1) d_j}}|^2 - |\frac{a_{ij}}{\sqrt{d_i d_j}}|^2\right)$$

$$= \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} \frac{a_{ij}^2}{d_j} \left(\frac{1}{d_i - 1} - \frac{1}{d_i}\right)$$

$$= \sum_{i=u,v} \sum_{\substack{j=1,\\ j\neq u,v}}^{n} \frac{a_{ij}^2}{d_j} h_i \quad \left[Let\ us\ write, \left(\tfrac{1}{d_i - 1} - \tfrac{1}{d_i}\right) as\, h_i\right]$$

$$= \sum_{\substack{j=1,\\ j\neq u,v}}^{n} \frac{a_{uj}^2}{d_j} h_u + \sum_{\substack{j=1,\\ j\neq u,v}}^{n} \frac{a_{vj}^2}{d_j} h_v$$

$$= \left(\sum_{j=1}^{n} \frac{a_{uj}^2}{d_j} - \frac{a_{uu}^2}{d_u} - \frac{a_{uv}^2}{d_v} \right) h_u + \left(\sum_{j=1}^{n} \frac{a_{vj}^2}{d_j} - \frac{a_{vu}^2}{d_u} - \frac{a_{vv}^2}{d_v} \right) h_v$$

$$= \sum_{j=1}^{n} \frac{a_{uj}^2}{d_j} h_u + \sum_{j=1}^{n} \frac{a_{vj}^2}{d_j} h_v$$

$$= \sum_{i=u,v} \sum_{j=1}^{n} \frac{a_{ij}^2}{d_j} h_i$$

$$= \sum_{i=u,v} c_i h_i \quad [Let\ us\ write \sum_{j=1}^{n} \frac{a_{ij}^2}{d_j}\ as\ c_i]$$

$$= c_u h_u + c_v h_v$$

On the other hand,

$$\sum_{i=u,v} \sum_{j=u,v} \left(|l'_{N_{ij}}|^2 - |l_{N_{ij}}|^2 \right) = |l'_{N_{uu}}|^2 - |l_{N_{uu}}|^2 + |l'_{N_{uv}}|^2 - |l_{N_{uv}}|^2 + |l'_{N_{vu}}|^2 - |l_{N_{vu}}|^2 + |l'_{N_{vv}}|^2 - |l_{N_{vv}}|^2$$

$$= 1 - 1 + |l'_{N_{uv}}|^2 - |l_{N_{uv}}|^2 + |l'_{N_{vu}}|^2 - |l_{N_{vu}}|^2 + 1 - 1$$
$$= 2 \left(|l'_{N_{uv}}|^2 - |l_{N_{uv}}|^2 \right)$$
$$= 2 \left(|-\frac{a'_{uv}}{\sqrt{d'_u d'_v}}|^2 - |\frac{a_{uv}}{\sqrt{d_u d_v}}|^2 \right)$$
$$= 2 \left(0 - \frac{1}{(d_u - 1)(d_v - 1)} \right)$$
$$= \frac{2}{(d_u - 1)(d_v - 1)}$$
$$= \frac{2 d_u d_v}{d_u d_v (d_u - 1)(d_v - 1)}$$
$$= 2 d_u d_v \left(\frac{1}{d_u(d_u - 1)} \right) \left(\frac{1}{d_v(d_v - 1)} \right)$$
$$= 2 d_u d_v \left(\frac{1}{d_u - 1} - \frac{1}{d_u} \right) \left(\frac{1}{d_v - 1} - \frac{1}{d_v} \right)$$
$$= 2 d_u d_v h_u h_v$$

Finally, we can write,

$$\|L'_N\|_F^2 - \|L_N\|_F^2 = 2(c_u h_u + c_v h_v) + 2 d_u d_v h_u h_v$$

$$\Rightarrow \|L'_N\|_F^2 - \|L_N\|_F^2 = 2 \left(\sum_{i=u,v} c_i h_i + \Pi_{i=u,v} d_i h_i \right)$$

Now the mean of eigenvalues of the normalized Laplacian matrix is defined as the trace of the matrix. And we have noticed that the diagonal elements remain the same after the addition of an edge in the graph. So, the trace of the matrix will also be the same. Hence, the mean of eigenvalues of the normalized Laplacian matrix will not be affected or mathematically we can write,

$$\overline{\lambda_N^{'}} = \overline{\lambda_N}$$

So,

$$Var(\lambda_N^{'}) - Var(\lambda_N) = \frac{1}{n}\|L_N^{'}\|_F^2 - \overline{\lambda_N^{'}}^2 - \frac{1}{n}\|L_N\|_F^2 + \overline{\lambda_N}^2$$
$$= \frac{1}{n}\left[2(c_u h_u + c_v h_v) + 2d_u d_v h_u h_v\right]$$
$$= \frac{2}{n}\left(c_u h_u + c_v h_v + d_u d_v h_u h_v\right)$$
$$where, c_i = \sum_{j=1}^{n} \frac{a_{ij}^2}{d_j} \ and \ h_i = \frac{1}{d_i - 1} - \frac{1}{d_i}$$

2.3 Addition of multiple edges consecutively

Let us, suppose we have a graph G with a corresponding adjacency matrix, degree matrix, and normalized Laplacian matrix as $A, D, \ and \ L_N$. Suppose the eigenvalues are denoted as $\lambda_{N_1}, \lambda_{N_2}, ..., \lambda_{N_n}$ We are going to add k_1 edge consecutively one by one between any random node u and node v. Let us suppose after adding $k\ -th$ edge, the graph has been modified to G_k with the corresponding adjacency matrix, degree matrix, and normalized Laplacian matrix as $A_k, D_k, \ and \ L_{Nk}$. Assume that the eigenvalues of the normalized Laplacian matrix are denoted as $\lambda_{Nk_1}, \lambda_{Nk_2}, ..., \lambda_{Nk_n}$

Now after the addition of an edge between a set of random nodes $u \ and \ v$ of the graph G will be modified to G_1, the difference in the variance of eigenvalues of the normalized Laplacian matrix of G_1 and G will be,

$$Var(\lambda_{N1}) - Var(\lambda_N) = \frac{2}{n}(h_{1_u} c_{1_u} + h_{1_v} c_{1_v} + d_{1_u} d_{1_v} h_{1_u} h_{1_v})$$

After the addition of a second edge between any random node u and v of the graph G_1, it will be modified to G_2. Hence, we can write,

$$Var(\lambda_{N2}) - Var(\lambda_{N1}) = \frac{2}{n}(h_{2_u} c_{2_u} + h_{2_v} c_{2_v} + d_{2_u} d_{2_v} h_{2_u} h_{2_v})$$

similarly,

$$Var(\lambda_{N3}) - Var(\lambda_{N2}) = \frac{2}{n}(h_{3_u} c_{3_u} + h_{3_v} c_{3_v} + d_{3_u} d_{3_v} h_{3_u} h_{3_v})$$

and,

$$Var(\lambda_{Nk_1}) - Var(\lambda_{N(k_1-1)}) = \frac{2}{n}(h_{k_{1u}} c_{k_{1u}} + h_{k_{1v}} c_{k_{1v}} + d_{k_{1u}} d_{k_{1v}} h_{k_{1u}} h_{k_{1v}})$$

Now,

$$Var(\lambda_{N1}) - Var(\lambda_N) + Var(\lambda_{N2}) - Var(\lambda_{N1}) + Var(\lambda_{N3}) - Var(\lambda_{N2}) + ... + Var(\lambda_{Nk_1}) - Var(\lambda_{N(k_1-1)})$$
$$= \frac{2}{n}(h_{1_u} c_{1_u} + h_{1_v} c_{1_v} + d_{1_u} d_{1_v} h_{1_u} h_{1_v}) + \frac{2}{n}(h_{2_u} c_{2_u} + h_{2_v} c_{2_v} + d_{2_u} d_{2_v} h_{2_u} h_{2_v}) + \frac{2}{n}(h_{3_u} c_{3_u} + h_{3_v} c_{3_v} + d_{3_u} d_{3_v} h_{3_u} h_{3_v}) + ...$$
$$+ \frac{2}{n}(h_{k_{1u}} c_{k_{1u}} + h_{k_{1v}} c_{k_{1v}} + d_{k_{1u}} d_{k_{1v}} h_{k_{1u}} h_{k_{1v}})$$
$$\Rightarrow Var(\lambda_{Nk}) - Var(\lambda_N) = \frac{2}{n}\sum_{k=1}^{k_1}(h_{k_u} c_{k_u} + h_{k_v} c_{k_v} + d_{k_u} d_{k_v} h_{k_u} h_{k_v})$$
$$where, c_{k_i} = \sum_{j=1}^{n} \frac{a_{k_{ij}}^2}{d_{kj}} \ and \ h_{k_i} = \frac{1}{d_{k_i} + 1} - \frac{1}{d_{k_i}}$$

3 Conclusion & Future Plans

This work provides a comprehensive analysis of how changes in graph topology affect the variance of eigenvalues of the normalized Laplacian matrix. By addressing three key scenarios—addition of a single edge, deletion of a single edge, and consecutive addition of multiple edges—the study establishes mathematical frameworks to quantify these effects. The derived formulations demonstrate that the structural changes in a graph directly influence its spectral properties, offering valuable insights for applications in spectral clustering, graph neural networks, and dynamic network analysis. These findings emphasize the importance of understanding spectral behavior for designing and optimizing graph-based systems and algorithms. Future work could extend this analysis to weighted and directed graphs or explore applications in evolving real-world networks.