Derivation of Gravitational Time Dilation

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November 2025

1 Introduction

Einstein's General Theory of Relativity (GR) predicts that near bodies of mass, a clock will tick slower than a clock placed far away. In this document, we derive the equation describing this phenomena and use it to calculate a concrete example, that of time dilation near a black hole's event horizon.

2 Review of Concepts needed

To begin the derivation, let us take a moment to review a few concepts that will be important. Firstly, the difference between Proper Time (τ) and Coordinate Time (t).

2.1 Proper Time

Proper Time (denoted by τ) is the time passed by a clock that moves along with an object, representing in an interval of time experienced by that object. It is invariant under changes of coordinates.

2.2 Coordinate Time

Coordinate Time (denoted by t) is the time measured according to a chosen reference frame or coordinate system. Unlike proper time, it can differ for different observers and depends on the spacetime metric and coordinate choice.

2.3 The Schwarzschild Metric

The Schwarzschild metric represents the spacetime geometry in a universe with a single spherically symmetric, nonrotating body. This is a very good approximation of most space objects (which in general rotate with a negligible angular velocity).

The metric can be written as:

$$g_{\mu\nu} = \text{diag}(-(1 - \frac{2GM}{rc^2})c^2, (1 - \frac{2GM}{rc^2})^{-1}, r^2, r^2 \sin^2\theta) \tag{1}$$

where M is the mass of the body, r is the distance from the body, θ is the polar angle.

In order to respect the symmetry of the spherically symmetric body, the Schwarzschild metric is nearly always used with 3D polar coordinates:

$$x^\mu = (t, r, \theta, \phi) \tag{2}$$

3 The Derivation

We begin with the definition for the spacetime interval:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3}$$

The generalization of the Pythagorean theorem to curved spacetime. We also recall that

$$d\tau^2 = \frac{-ds^2}{c^2} \tag{4}$$

In this case, lets assume that the person near the black hole is stationary relative to the body–As in, they have a rocket that accelerates away from the black hole with the exact force counteract gravity. As such, we can state that:

$$dx^r = dx^\theta = dx^\phi = 0 \tag{5}$$

Now we can expand substitute (4) with the definition of the spacetime interval

$$d\tau^2 c^2 = -g_{\mu\nu} dx^\mu dx^\nu \tag{6}$$

And due to most of the dx^μ being zero, this simplifies out as

$$d\tau^2 c^2 = -g_{tt}(dx^t)^2 \tag{7}$$

Then substituting in the Schwarzschild metric from (1)

$$d\tau^2 = (1 - \frac{2GM}{rc^2})dt^2 \tag{8}$$

Which simplifies to

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2}} \tag{9}$$

Lets analyze this result. Recall that proper time ($d\tau$) is the time ticked by a clock near the object while coordinate time (dt) is the time ticked from someone infinitely far. If M was very large, and r relatively small, $\frac{d\tau}{dt}$ would begin to converge to zero, essentially for a unit of time passed far away from the object, less time would pass near the body, so time can be thought of as slowing down near massive bodies (though this effect is extremely small, as we will soon see).

Now lets see what happens in an extreme case, that of being near the event horizon of a black hole. The distance for the event horizon of a black hole is denoted as r_s (and can be calculated by calculating the escape velocity of light for a body using Newtonian mechanics). Lets calculate the time dilation at a distance $r_s + \epsilon$, a distance slightly larger than the event horizon.

$$r_s = \frac{2GM}{c^2} \tag{10}$$

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{r_s}{r_s + \epsilon}} \tag{11}$$

$$= \sqrt{\frac{\epsilon}{r_s + \epsilon}} \tag{12}$$

Since $\epsilon << r_s$

$$\approx \sqrt{\frac{\epsilon}{r_s}} \tag{13}$$

For the black hole at the center of our galaxy, Sag A*, $r_s \approx 10^{10}m$ and lets take $\epsilon = 1m$

$$\frac{d\tau}{dt} = \sqrt{\frac{1}{10^{10}}} = 10^{-5} \tag{14}$$

So for every 1 second passing far away from the black hole, $10^{-5}s$ passes for the person near the earth. Now imagine if you were the person near the event horizon, just one year spent 1 meter from the event horizon would be 100,000 years passed, for context, 100,000 years ago, humans were just beginning to leave Africa.

Lets take the interesting case of $r = r_s$ or $\epsilon = 0$. In that case,

$$\frac{d\tau}{dt} = \sqrt{\frac{0}{10^{10}}} = 0 \tag{15}$$

So to an outside observer, time 'stops' at the event horizon, viewing an object being pulled into a black hole, the object would appear to stop at the event horizon. It is important to say that from the perspective of someone being pulled into the black hole, time passes by normally for them, time does not stop for them–it is not accurate to say

$$\frac{dt}{d\tau} \to \frac{r_s}{0} \to \infty \tag{16}$$

As the Schwarzschild metric is not valid for the perspective of an observer at an event horizon, to properly describe the local point of view, a different metric such as Eddington–Finkelstein or Kruskal coordinates is needed.

4 Conclusion

To summarize, we began with the definition of the Schwarzschild metric and the spacetime interval and derived the equation for time dilation purely due to Gravity:

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2}} \tag{17}$$

It is important to realize this is only in the special case that the clock near the body is not moving relative to the body, as other terms would otherwise be required.