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Sec- 09.

Final examination

MAT216

Ans-01 (a).

Given,

$$S = \text{Span} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}}_{v_1}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Finding null spaces of the matrix

$$S = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 2 \end{bmatrix}$$

$$\text{rrf of } S = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 3 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1 \times 2$$

$$S = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad (R_2 = -R_2/3)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad R_1 = R_1 - R_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad R_1 = R_1 - R_2$$

$$S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here, $x_3 = t$, $x_4 = S$, $x_1 = -S$, $x_2 = t$

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \times S$$

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Here, Basis will be

$$\left\{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}, \left\{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\},$$

$$V_1 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \bar{e}_1 = \frac{\bar{v}_1}{|v_1|}$$

$$= \begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}$$

$$\bar{v}_2 = \bar{V}_2 - \mathrm{Proj}(\bar{v}_1) \to (\bar{V}_2)$$

$$= \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

$$\bar{e}_2 = \frac{\bar{v}_1}{|\bar{v}_2|} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

S'(orthonormal basis) =

$$\left\{ \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

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1 (b)

$w_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $w_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

~~Using~~ Finding ortho noramal basis for $S^{\perp}$

$x = w_1 \Rightarrow \|x\| = \sqrt{2}$

$y_1 = \frac{1}{\sqrt{2}} w_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$x_2 = w_2 - \frac{\langle w_2, x_1 \rangle}{\langle x_1, x_1 \rangle} x_1$

$\langle w_2, w_1 \rangle = 0$,

$x_2 = w_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \|x\| = \sqrt{2}$

$$y_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$\therefore$ orthonormal basis for $S^\perp$

$$= \left\{ y_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, y_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

To write $u = v + w$, such that $v \in S$ and $w \in S^\perp$,

Now, above $\{w_1, w_2\}$ are orthogonal basis for $S^\perp$ ($\because \langle w_1, w_2 \rangle = 0$)

we will find $Proj_{S}(w)$.

$$\text{Proj}(u) = \frac{\langle u, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle u, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$\langle u, w_1 \rangle = 0, \ \langle u, w_2 \rangle = 1,$

$\langle w_2, w_2 \rangle = 2$

$$\therefore \text{Proj}_{S^\perp}(u) = \frac{1}{2} w_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \in S^\perp$$

$$u - \text{Proj}_S(u) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ -1 \\ 5/2 \end{bmatrix}$$

Question 2.

Given,

$$A = \begin{bmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{bmatrix}$$

The characteristics equation,

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 5-\lambda & -2 & 2 \\ 4 & -3-\lambda & 4 \\ 4 & -6 & 7-\lambda \end{vmatrix} = 0$$

$$= \cancel{(5-\lambda)(-3-\lambda)(7-\lambda) - 4\times 6} - (4\times 7-\lambda - 4\times 4)$$
$$+ \cancel{(4\times -6) - (4\times -3)}$$

$$= (5-\lambda)(-3-\lambda)(7-\lambda) + (-2\times 4\times 4) +$$
$$(2\times 4\times -6) - 4)(-3-\lambda)\cdot 2 -$$
$$(-6\times 4)(5-\lambda) - (7-\lambda)(4\times -2)$$

$$= -\lambda^3 + 9\lambda^2 - 23\lambda + 15$$

$$= -(\lambda - 1)(\lambda^2 - 8\lambda + 15)$$

$$= -(\lambda - 1)(\lambda - 3)(\lambda - 5) = 0$$

~~Therefore~~

Therefore, the eigen values are,

$$\lambda = 1, 3, 5.$$

Now, when,

$\lambda = 1$,

$$\begin{bmatrix} 5-1 & -2 & 2 \\ 4 & -\lambda-3 & 4 \\ 4 & -6 & 7-\lambda \end{bmatrix} = \begin{bmatrix} 4 & -2 & 2 \\ 4 & -4 & 4 \\ 4 & -6 & 6 \end{bmatrix}$$

Using rrf →

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 4 & -4 & 4 \\ 4 & -6 & 6 \end{bmatrix} \quad R_1 \to R_1/4$$

$$= \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & -2 & 2 \\ 4 & -6 & 6 \end{bmatrix} \quad R_2 \to R_2 - 4R_1$$

$$= \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & -2 & 2 \\ 0 & -4 & 4 \end{bmatrix} \quad R_3 \to R_3 - 4R_1$$

$$= \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{bmatrix} \quad R_2 \to -R_2/2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{bmatrix} \quad R_1 \to R_1 + R_2/2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 4R_2$$

Now, $\overset{(A-\lambda I)}{\lambda_1} * x = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

If $x_3 = t$, then $x_1 = 0$, $x_2 = t$

$$\therefore x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t$$

Now, $\lambda = 3$,

$$\begin{bmatrix} 5-3 & -2 & 2 \\ 4 & -\lambda-3 & 4 \\ 4 & -6 & 7-\lambda \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 4 & -6 & 4 \\ 4 & -6 & 4 \end{bmatrix}$$

Using rref $\rightarrow$

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & -6 & 4 \\ 4 & -6 & 4 \end{bmatrix} \quad R_1 \rightarrow R_1/2$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \quad \begin{matrix} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 4R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \quad R_2 = -R_2/2$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \quad R_1 = R_1 + R_2$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 = R_3 + 2R_1$$

$\underline{(A - \lambda I)\lambda_2 \cdot x = 0}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $x_3 = t$, $x_1 = -t$, $x_2 = 0$

~~Again,~~ $\vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \times t$

~~when $x =$~~

Again,

when, $\lambda_3 = 5$,

$$\begin{bmatrix} 5-5 & -2 & 2 \\ 4 & -3-5 & 4 \\ 4 & -6-\cancel{5} & \cancel{0}2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 4 & -8 & 4 \\ 4 & -6 & 2 \end{bmatrix}$$

~~R~~ Using rref:

$$\begin{bmatrix} 4 & -8 & 4 \\ 0 & -2 & 2 \\ 4 & 6 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & -2 & 2 \\ 4 & -6 & 2 \end{bmatrix} \quad R_1 \to R_1/4$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \quad \begin{matrix} R_3 = R_3 - 4R_1 \\ R_2 = -R_2/2. \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \quad R_1 \to R_1 + 2R_1$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$(A - \lambda I)\, x_3 - x = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $x_3 = t$, $x_1 = t$, $x_2 = t$

$$\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t.$$

~~The dimension of A, is 3~~

$\dim(A) = 3$

8 Ans - 2(b)

from (a)

~~a/b~~ 1,

$\lambda_3 = 5, \ \vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3, \ \vec{x_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\cancel{a} \cdot \lambda_1 = 1, \ \vec{x_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Therefore,

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$D = P^{-1} \cdot A \cdot P$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2(c)

let, X be the eignvector of B corresponding ~~value~~ eigen value λ

then, $B\cancel{A}x = \lambda x$

again, $(B+CI)x$

$= Bx + CIx$

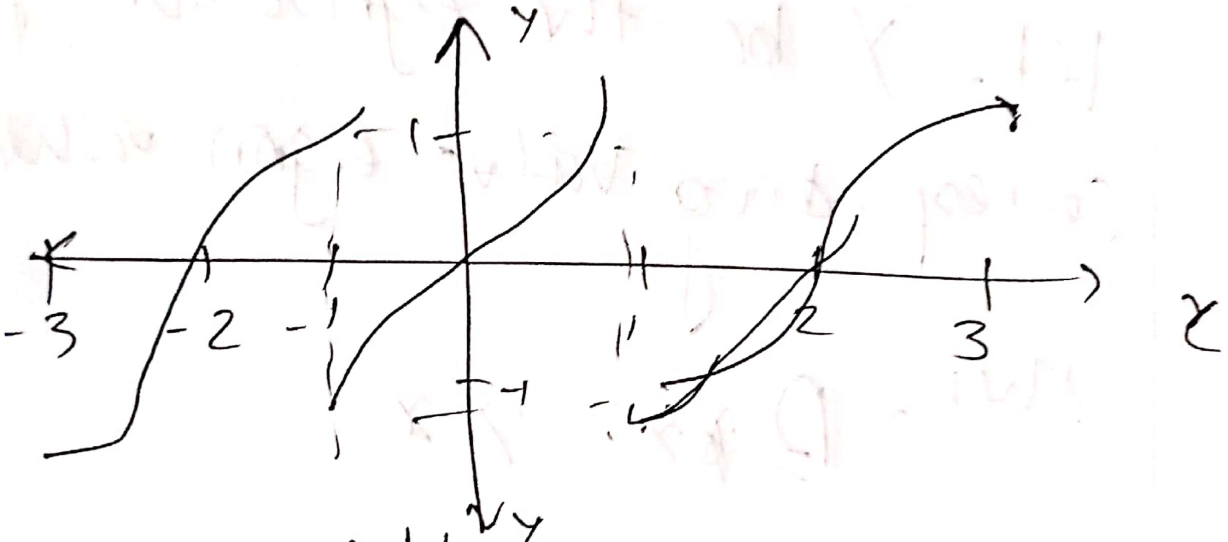
$= \lambda x + Cx \quad [\because Bx = \lambda x]$

$\Rightarrow (B+CI)X = \lambda x + Cx$

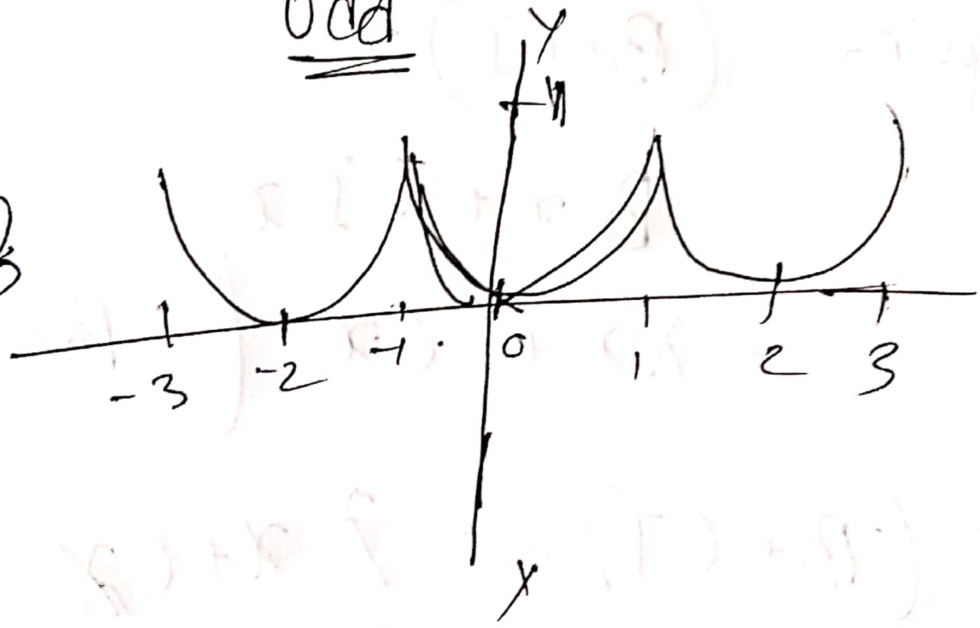
$= (\lambda + C)x$

Hence, x is the eigen vector of $B+CI$ and the eigen value is $(\lambda + C)$.

3(a) $f(x) = x^2, \quad 0 \leq x \leq 1,$



Odd



even

even

Am- 3(b)

For odd extension,

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin(n\pi x)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{(n\pi x)}{L} dx$$

$\therefore 2L = 2, L = 1.$

$$b_n = 2 \int_0^1 \underset{u}{x^2} \cdot \underset{v}{\sin(n\pi(x))}\, dx$$ [using uv rule]

$$= 2 \left[x^2 \int \sin(n\pi(x)) \cdot dx - \int \frac{d}{dx} x^2 \left(\int \sin(n\pi x) dx \right) \right.$$

$$= 2 \left[- x^2 \frac{\cos(n\pi(x))}{n\pi} + 2 \int \frac{x \cdot \cos(n\pi x)}{n\pi} dx \right]_0^1$$

$$= \left[-\frac{2x^2 \cos n\pi x}{n\pi} + \frac{4}{n\pi} \left\{ \frac{x \sin n\pi x}{n\pi} - \left(\frac{\sin n\pi x}{n\pi} \right) \right\} \right]_0^1$$

$$= \left[-\frac{2x^2 \cos n\pi x}{n\pi} + \frac{4}{n^2\pi^2} \left\{ x \sin n\pi x + \frac{\cos n\pi x}{n\pi} \right\} \right]_0^1$$

$$= \left[\frac{-2 \cdot 1 \cos n\pi}{n\pi} + \frac{4}{n^2\pi^2} \left\{ 1 \cdot \sin n\pi x + \cos n\pi x \right\} \right]_0^1$$

$$= \frac{-2 \cdot 1 \cos n\pi}{n\pi} + \frac{4}{n^2\pi^2} \left\{ 1 \cdot \sin(n\pi) + \frac{\cos \pi n}{n\pi} \right\}$$

$$- 0 + 0 + \frac{4}{n^3\pi^3}$$

$$= -\frac{2\cos n\pi}{n\pi} + \frac{4}{n^2\pi^2} \frac{\cos \pi n}{\pi n} + \frac{4}{n^3\pi^3}$$

$$= -\frac{2\cos n\pi}{n\pi} + \frac{4}{n^3\pi^3}(\cos n\pi + 1)$$

$\therefore b_n = \frac{4}{n^3\pi^3}(\cos n\pi + 1) - \frac{2\cos n\pi}{n\pi}$.

$$f_s(x) = \sum_{n=1}^{\infty}\left[\frac{4}{n^3\pi^3}(\cos n\pi + 1) - \frac{2\cos n\pi}{n\pi}\right] \sin nx.$$

Ans-3(c)

From b,

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{4}{n^3\pi^3}(\cos n\pi + 1) - \frac{2\cos n\pi}{n\pi}\right] \sin nx$$

$$\therefore F_s(1) = \sum_{n=1}^{\infty}\left[\frac{4}{n^3\pi^3}(\cos \pi + 1) - \frac{2\cos \pi}{n\pi}\right] \sin(1)$$

$$= 0.$$

For $x = 1$, and all the values of $n = 0$.