```
(Math 7670 Spring 2019, Handout #1)

Macaulay2, version 1.12

i1 : e = entries (id_(ZZ^4))
o1 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

i2 : A = transpose matrix flatten (
       for i from 0 to 3 list
         for j from 0 to 3 list
           if i ==j then continue else 2*e#i+e#j
       )

o2 = | 2 2 2 1 0 0 1 0 0 1 0 0 |
     | 1 0 0 2 2 2 0 1 0 0 1 0 |
     | 0 1 0 0 1 0 2 2 2 0 0 1 |
     | 0 0 1 0 0 1 0 0 1 2 2 2 |

              4        12
o2 : Matrix ZZ  <--- ZZ

i3 : needsPackage "FourierMotzkin"

i4 : (B, Bh) = fourierMotzkin A

o4 = (| -1 0  0  1  -2 -2 0  -2 |, 0)
      | 0  -1 0  -2 1  -2 0  -2 |
      | 0  0  -1 -2 -2 1  0  -2 |
      | 0  0  0  -2 -2 -2 -1 1  |

o4 : Sequence

i5 : (A', A'h) = fourierMotzkin B

o5 = (| 2 1 2 0 1 0 2 0 0 1 0 0 |, 0)
      | 1 2 0 2 0 1 0 2 0 0 1 0 |
      | 0 0 1 1 2 2 0 0 2 0 0 1 |
      | 0 0 0 0 0 0 0 1 1 1 2 2 2 |

o5 : Sequence

i6 : -- note: A' is equal to A: meaning all the columns of A
     -- are irredundant (as A' has this property)
     -- The columns of B are the rays of (dualCone C).
     (transpose B) * A -- all entries are <= 0.

o6 = | -2 -2 -2 -1 0  0  -1 0  0  -1 0  0  |
     | -1 0  0  -2 -2 -2 0  -1 0  0  -1 0  |
     | 0  -1 0  0  -1 0  -2 -2 -2 0  0  -1 |
     | 0  0  0  -3 -6 -6 -3 -6 -6 -3 -6 -6 |
     | -3 -6 -6 0  0  0  -6 -3 -6 -6 -3 -6 |
     | -6 -3 -6 -6 -3 -6 0  0  0  -6 -6 -3 |
     | 0  0  -1 0  0  -1 0  0  -1 -2 -2 -2 |
     | -6 -6 -3 -6 -6 -3 -6 -6 -3 0  0  0  |

              8        12
o6 : Matrix ZZ  <--- ZZ

i7 : needsPackage "Polyhedra"

i8 : C = coneFromVData A

o8 = C

o8 : Cone

i9 : rays C

o9 = | 2 1 2 0 1 0 2 0 0 1 0 0 |
     | 1 2 0 2 0 1 0 2 0 0 1 0 |
     | 0 0 1 1 2 2 0 0 2 0 0 1 |
     | 0 0 0 0 0 0 0 1 1 1 2 2 2 |

              4        12
o9 : Matrix ZZ  <--- ZZ

i10 : halfspaces C

o10 = | 1  0  0  0  |
      | 0  1  0  0  |
      | 0  0  1  0  |
      | -1 2  2  2  |
      | 2  -1 2  2  |
      | 2  2  -1 2  |
      | 0  0  0  1  |
      | 2  2  2  -1 |

               8        4
o10 : Matrix ZZ  <--- ZZ

i11 : (halfspaces C) * (rays C) -- hmmm, opposite of our definition!

o11 = | 2 1 2 0 1 0 2 0 0 1 0 0 |
      | 1 2 0 2 0 1 0 2 0 0 1 0 |
      | 0 0 1 1 2 2 0 0 2 0 0 1 |
      | 0 3 0 6 3 6 0 6 6 3 6 6 |
      | 3 0 6 0 6 3 6 0 6 6 3 6 |
      | 6 6 3 3 0 0 6 6 0 6 6 3 |
      | 0 0 0 0 0 0 0 1 1 1 2 2 2 |
      | 6 6 6 6 6 6 3 3 3 0 0 0 |
```

```
              8        12
o11 : Matrix ZZ  <--- ZZ

i12 : rays dualCone C -- transpose of (halfspaces C), also not
matching our def!

o12 = | 1 0 0 -1 2  2  0 2  |
      | 0 1 0 2  -1 2  0 2  |
      | 0 0 1 2  2  -1 0 2  |
      | 0 0 0 2  2  2  1 -1 |

              4        8
o12 : Matrix ZZ  <--- ZZ

i13 : -- OUR def of dualCone is the negative of this!!
      dualRays = - rays dualCone C

o13 = | -1 0  0  1  -2 -2 0  -2 |
      | 0  -1 0  -2 1  -2 0  -2 |
      | 0  0  -1 -2 -2 1  0  -2 |
      | 0  0  0  -2 -2 -2 -1 1  |

              4        8
o13 : Matrix ZZ  <--- ZZ

i14 : dualHalf = - halfspaces C

o14 = | -1 0  0  0  |
      | 0  -1 0  0  |
      | 0  0  -1 0  |
      | 1  -2 -2 -2 |
      | -2 1  -2 -2 |
      | -2 -2 1  -2 |
      | 0  0  0  -1 |
      | -2 -2 -2 1  |

              8        4
o14 : Matrix ZZ  <--- ZZ

i15 :
      matrix{hilbertBasis C}

o15 = | 2 0 0 1 1 0 0 1 0 0 1 0 1 1 1 2 2 1 0 1 0 0 |
      | 0 2 0 0 1 1 1 2 2 1 0 0 0 1 0 1 0 1 1 0 1 0 |
      | 0 0 2 0 0 1 0 0 1 2 1 1 2 1 0 0 1 0 1 1 0 1 |
      | 1 1 1 2 0 0 1 0 0 0 0 1 0 0 1 0 0 1 1 1 2 2 |

              4        22
o15 : Matrix ZZ  <--- ZZ
```

```
i16 : matrix{hilbertBasis dualCone C}

o16 = | 2  0 2  0 0 1 -1 2  |
      | 2  1 -1 0 0 0 2  2  |
      | 2  0 2  0 1 0 2  -1 |
      | -1 0 2  1 0 0 2  2  |

              4        8
o16 : Matrix ZZ  <--- ZZ
```