Week 3a

Tuesday, September 22, 2020 10:28 AM

Three Main Concepts:
- Linear Independence
- Span
- ???

...

Linear Independence

$\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n$

are linearly independent if the only way for

$\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \ldots \alpha_n\vec{x}_n = \vec{0}$

Is that

$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$

NB: Any set of 3 2D vectors is linearly dependent

Conceptually: A set of vectors is linearly dependent when you could create at least one of the vectors out of the other(s) (i.e. there is redundancy)

Span: All the vectors that can be made with various values for each α

$\vec{x} = \alpha_1\vec{x}_1 + \alpha\ \vec{x} + \ \ + \alpha\ \vec{x}$

$$\vec{X}_a = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \ldots + \alpha_n \vec{x}_n$$

Not Alpha

Basis: A set of vectors forms a basis for a space if that set of vectors <u>spans</u> that space

Example:

Given: $\vec{x}_1, \vec{x}_2, \vec{x}_3$

If it's possible to find alphas such that

$\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \alpha \vec{x}_3$

is equal to any vector in a space,

then $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ are the basis for the space.

Exercise 7.2.2:

$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

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Chapter 7

Week 3a: Linear Independence, Span, Basis, and Decomposition

Schedule

7.1 Debrief and Dancing Animal Demos [30 mins]

- Please discuss your overnight work with your breakout-room mates, create a set of key concepts, and a set of ideas that you are still confused by.
- Be prepared to demo your dancing animal to your breakout room.

7.2 Synthesis [20 mins]

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Exercise 7.1

You should do all of these.

1. Assume the matrix **D** represents a geometrical object. What is the correct matrix expression if we want to rotate it first (**R**), then scale it (**S**), and finally translate (**T**) it?
   A. **DRST**
   B. **TSRD**
   C. **RSTD**
   D. **DTSR**
2. What would be the correct expression in order to undo the transformation in the previous problem?
3. **A** and **B** are square, invertible matrices of the same size. Which of the following are **always** true (no matter the entries in **A** and **B**?

A. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
B. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
C. $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$
D. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
E. $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$
F. $\mathbf{AB} = \mathbf{BA}$
G. $\det(\mathbf{AB}) = \det(\mathbf{A}) + \det(\mathbf{B})$
H. $(\mathbf{AB})^T = \mathbf{A}^T\mathbf{B}^T$
I. $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

7.3 Mini Lecture Linear Independence, Span, Basis [20 mins]

7.4 Linear Independence [20 mins]

A set of non-zero vectors is linearly independent if it is not possible to scale and sum them to make the all zeros vector, except when the scale factors are all zero.

If 3-dimensional vectors $\mathbf{x}_1, \mathbf{x}_2\mathbf{x}_3$ are linearly independent, it means that it is *not* possible to find scale factors c_1, c_2, c_3 so that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0} \tag{7.1}$$

except when c_1, c_2, c_3 are all zero.

This property also implies that if you have n linearly independent, n-dimensional vectors, you can express any other n-dimensional vector by scaling and summing those linearly independent vectors.

If 3-dimensional vectors $\mathbf{x}_1, \mathbf{x}_2\mathbf{x}_3$ are linearly independent, it means that for any 3-dimensional vector $\mathbf{x}_d$, it is possible to find scale factors d_1, d_2, d_3 so that

$$d_1\mathbf{x}_1 + d_2\mathbf{x}_2 + d_3\mathbf{x}_3 = \mathbf{x}_d\,. \tag{7.2}$$

Exercise 7.2

1. Determine which of the following sets of vectors are linearly independent.

   (a) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\end{bmatrix}$ ✓

   (b) $\begin{bmatrix}1\\3\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ ✗

   (c) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\4\\3\end{bmatrix}$ ✗

   (d) $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$, where the vectors are all 3-dimensional. ✗

   (e) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}$ ✓

2. Consider two column vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \tag{7.3}$$

Both these vectors lie on the xy-plane since their z components are zero. Define a new vector $\mathbf{a}_3 = c_1\mathbf{a}_1 + c_2\mathbf{a}_2$, where c_1 and c_2 are arbitrary variables. Therefore $\mathbf{a}_3$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

(a) Does $\mathbf{a}_3$ also lie on the xy-plane? Yes

(b) Next, define a 3×3 matrix $\mathbf{A}$ whose columns are $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. Show that the product of $\mathbf{A}$ and any 3×1 vector always lies on the xy-plane.

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Solution 7.2

1. (a) They are linearly independent since they span $\mathbf{R}^3$.
   (b) They are linearly dependent since the first vector is equal to the second vector plus two times the third vector.
   (c) They are linearly dependent since the third vector is equal to the first vector plus two times the second vector.
   (d) They are linearly dependent. You can have a maximum of n linearly independent vectors in $\mathbf{R}^n$.
   (e) They are linearly independent since they do not lie on the same line.
2. (a) Yes, a linear combination of two vectors which lie in the xy-plane will also lie in the xy-plane.
   (b) Let $\mathbf{A}$ be the matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & c_1 + c_2 \\ 1 & 2 & c_1 + 2c_2 \\ 0 & 0 & 0 \end{bmatrix}$$
   and let $\mathbf{v}$ be an arbitrary 3×1 vector
$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$
   Then the product
$$\mathbf{Av} = \begin{bmatrix} x + y + (c_1 + c_2)z \\ x + 2y + (c_1 + 2c_2)z \\ 0 \end{bmatrix}$$
   lies in the xy-plane