Homework F1

Sunday, November 22, 2020 11:40 PM

document

Chapter 28

Homework 1: Curves and Surfaces

Contents

Learning Objectives

Concepts

- Distinguish between equations that represent explicit functions, implicit functions, and parametric functions.
- Identify exponential, polynomial and trigonometric relationships by the shape of their curves.
- Describe how changes in parameters affect the shape of curves or surfaces.
- Determine a mathematical approximation to the surface of real physical object.

MATLAB Skills

- Use MATLAB to define and visualize curves and surfaces defined by explicit functions, implicit functions, and parametric functions. These MATLAB functions are **plot, plot3, contour, surf, isosurface**.

28.1 Curves

28.1.1 Curves defined Explicitly

If you recall, single-variable calculus involved explicit functions of a single variable, e.g. $y = t^2$, $y = \sin(t)$, or more generally $y = f(t)$. You spent a lot of time visualizing these functions, solving equations with these functions, and computing related properties like derivatives and integrals.

Let's consider the function, $y = mt + b$, where m and b are parameters. We probably recognise this function, and that it's graph is a straight-line with slope m and intercept b.

In MATLAB we can visualize this function using the **plot** function which you are already familiar with.

```
>> m = 2;
>> b = 1;
>> t = linspace(-10,10,1000);
>> y = m.*t+b;
>> plot(t,y,'red')
```

First, we define a value of m and a value of b as an example. Second, we use the function **linspace** to generate 1000 equally-spaced points between -10 and +10. There is nothing special about this domain, except that the resulting graph captures the behavior of the function. Third, we evaluate the mathematical function at these points and store the result in y. Fourth, we call the **plot** function to generate the curve – we use red in this case because it looks great! Hopefully we recognize the classic straight-line which has the following features:

- y tends to $\pm\infty$ as $t \to \pm\infty$ if $m > 0$.
- y tends to $\mp\infty$ as $t \to \pm\infty$ if $m < 0$.
- The line passes through the point $(0, b)$.

One way to capture these different behaviors is to sketch sample curves in each quadrant of a $b - m$ space.

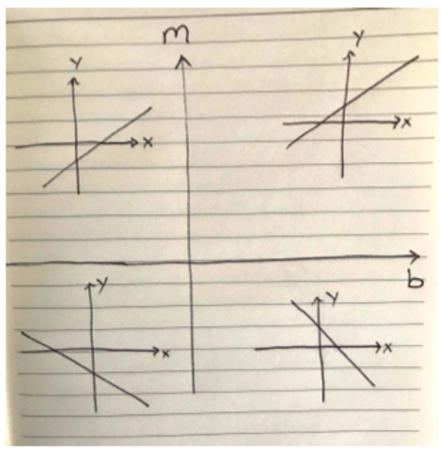
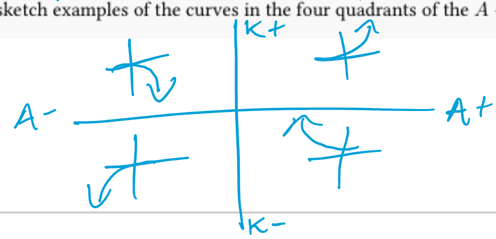


Figure 28.1: Straight-lines with different slopes and intercepts.

Exercise 28.1

Consider the exponential function $y = Ae^{kt}$ where A and k are parameters.

1. What is the value of y when $t = 0$? Y = A @ t = 0
2. What happens to y as $t \to \pm\infty$? How does this limiting behavior depend on the sign of A and k?
   y -> sign(A)∞ as t -> sin(K)∞, y->0 as t -> -sin(K)∞
3. Now sketch examples of the curves in the four quadrants of the $A - k$ space.



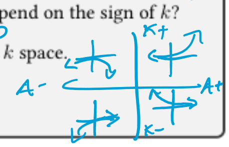
4. What is the effect of the parameters A and k on the curve? k controls if the VA is in $t>0$ or $t<0$, and how steep it is. A controls $+\infty$ VA or $-\infty$ VA.

Exercise 28.2

$y = \frac{A}{1+e^{-kt}}$ ← Basically the same as 28.2: $\frac{1}{y} = \frac{1}{A} + \frac{e^{-kt}}{A} \Rightarrow \frac{1}{y} = \frac{1}{A} + \frac{1}{Ae^{kt}}$

Consider the logistic function $y = A/(1+e^{-kt})$ where A and k are parameters.

1. What is the value of y when $t = 0$? $y = \frac{A}{1+e^0} = \frac{A}{2}$
2. What happens to y as $t \to \pm\infty$? How does this limiting behavior depend on the sign of k?
   $t \to \text{sign}(k)\infty: y \to A$ $\quad t \to -\text{sin}(k)\infty: y \to 0$
   $k>0: t \to \infty: y = A$; $t \to -\infty: y = 0$
   $k<0: t \to \infty: y = 0$; $t \to -\infty: y = A$
3. Now sketch examples of the curves in the four quadrants of the $A - k$ space.
4. What is the effect of the parameters A and k on the curve?



Exercise 28.3

Consider the trigonometric function $y = A\sin(\omega t + \phi)$ with parameters A , ω, and ϕ.

1. Sketch some representative examples of these curves for different values of the parameters.
2. What features of the curve do A, ω, and ϕ control? Use the internet to deepen your understanding of these parameters. A = Amplitude (max = -min = A) ϕ = phase shift (horizontal shift)
   ω = period / 2π (if $\omega = 2\pi$, period = 1)

Exercise 28.4

Consider the quadratic polynomial in vertex form $y = g(x-h)^2 + k$, with parameters g,h,and k.

1. Sketch some representative curves for different parameter values.
2. What features of the curve do g, h, and k control?
   g: steepness: $|g| \to \infty$ = more steep $g<0$ = opens down
   h: horizontal shift = move +x / right
   k: vertical shift = move +y / up
3. What is the relationship between g,h, and k in the vertex form and a, b, and c in the standard form $y = c + bx + ax^2$? (This will require some algebra.)

The quadratic polynomial is probably very familiar to you. There are numerous ways to write this second-order polynomial, and we've used two forms here: the standard form and the vertex form. It is hard to tell the effect of each parameter in standard form. Using the vertex form, however, the effect of each parameter is much easier to interpret.

$g(x-h)^2 + k = c + bx + ax^2$

$g(x^2 - 2xh + h^2) + k = c + bx + ax^2$ $\quad c = gh^2 + k$

$gx^2 - 2xgh + \cancel{gh^2 + k} = \cancel{c} + bx + ax^2$ Is cancelling this valid math? Eh...

$gx^2 - 2xgh = bx + ax^2$ $\quad b = -2gh$

$a = g$

28.1.2 Curves defined Implicitly

Not every curve can be expressed in terms of an explicit function in which there is only one output for each value of the input. Curves can also be expressed implicitly through a relationship between 2 variables. A circle is a good example. For example, the equation for a circle of radius 1, centered at the origin, is

$$x^2 + y^2 - 1 = 0 \tag{28.1}$$

The left-hand side of this equation can be thought of as a function of two variables, $f(x, y) = x^2 + y^2 - 1$ and the set of points (x, y) where $f = 0$ defines a curve that we like to call the unit circle.

In order to visualize such curves in MATLAB we use the **contour** function. We begin by defining a grid of (x, y) points using the **meshgrid** function

```
>> [x,y]=meshgrid(linspace(-2,2,100),linspace(-3,3,200));
```

You will notice that both x and y are 200×100 matrices. There are 200 rows corresponding to the 200 y-values between -3 and 3. There are 100 columns corresponding to the 100 x-values between -2 and 2. There is nothing special about the limits of the domain or the number of points in each direction - we chose values here that would help explain the size of the resulting matrices.

Now that we have the grid defined, we compute the value of the function f at every point. Since x and y are already matrices we can use

```
>> f = x.^2 + y.^2 - 1;
```

Notice that we use the . operator because we want every entry in the x matrix to be squared, and similarly for y. You will also notice in MATLAB that f is a 200×100 matrix. In theory, subtracting "1" (a scalar) from a matrix should not be permitted, but the good people at MathWorks have decided to interpret this for us automatically.

To plot the curve we now use the **contour** function

```
>> contour(x,y,f,[0 0])
>> axis equal
```

which should produce a circle of radius 1 centered at the origin. The last argument to the **contour** function tells it to draw the contour at $f = 0$. Don't ask why you have to put two zeros instead of just one because only MATLAB knows. Without the "axis equal" the curve would look like a ellipse due to the different scaling MATLAB will use in the x and y directions.

There is no end to the functions of two variables that you can define. There is, however, a set of functions that show up again and again, and these are the quadratic functions of two variables. The general form (containing all possible quadratic, linear and constant terms) is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{28.2}$$

where a, b, c, d, e, f are arbitrary parameters, some of which may be zero. The curves defined by this equation are called **conic sections**, and represent the intersection of a double cone and a plane. The non-degenerate cases include circles, parabolas, ellipses, and hyperbolas. See the Wikipedia article on conic section for more information.

Exercise 28.5

Use the internet to find the implicit equation for a circle of radius R, centered at the point (a, b). Visualize the circle in MATLAB for different values of a, b, R.

This is a warm-up question. The implicit equation for a circle centered away from the origin should be easy to find, and you should use the visualization to check that changing the parameters moves the circle and changes its radius in the way you expect. $(x-a)^2 + (y-b)^2 - R^2 = 0$

Conic sections:
- Elipse (incl circle)
- Parabolan
- Hyperbola

Exercise 28.6

Visualize an ellipse using the implicit definition

a: x stretch / x-radius
b: y stretch / y-radius

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{28.4}$$

for different values of a and b. What features of the ellipse do a and b control? Use the internet to deepen your understanding of the parameters a and b—see for example the Wikipedia article on conic section.

This question requires a little modification to the visualization for the circle, and a little internet research to fully understand the parameters. Start with the article on conic section, and then spend a little time exploring after that - don't get lost in the world of the internet, and don't be surprised when you see lots of terminology that you don't understand.

Exercise 28.7

curves are asymptotic to $y = \pm \frac{b}{a} x$

verticies are at $(\pm a, 0)$

Visualize an hyperbola using the implicit definition

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \tag{28.5}$$

for different values of $a > 0$ and $b > 0$. What features of the hyperbola do a and b control? Use the internet to deepen your understanding of the parameters a and b.

This question is similar to the one for the ellipse. In this case, however, interpreting the parameters without some additional reading is much harder because the precise impact of the parameters is not obvious from visualization. Again, start with the Wikipedia article on conic section and take it from there.

$a = c$: circle

sign(a) = sign(c), $a \neq c$: ellipse

$a = 0$ OR $c = 0$: parabola

sign(a) $\neq$ sign(c) : hyperbola

Exercise 28.8

Use the internet to find the conditions under which the solutions of

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{28.6}$$

define an ellipse, a parabola, a hyperbola, and a circle.

This question is meant to broaden your understanding of the possible solutions of this general quadratic polynomial in two variables. Start with the Wikipedia article on conic section.

28.1.3 Curves defined Parametrically

A more general representation of a curve involves expressing it's coordinates in terms of another independent variable or parameter as follows

$$x = f(u), y = g(u), u \in [a, b] \tag{28.7}$$

Each value of u defines a point with coordinates $(f(u), g(u))$. If we collect all the points defined by u in a specific interval, then we get a parametric curve. For example, the definition

$$x = cos(u), y = sin(u), u \in [0, 2\pi] \tag{28.8}$$

defines a unit circle centered at the origin which begins and ends at $(1, 0)$ and is traced out counterclockwise as u increases from 0 to 2π. The parameter u can therefore be thought of as the angle from the x-axis to the current point on the circle.

To visualize parametric curves in MATLAB we still use the **plot** function as follows

```
>> u = linspace(0,2*pi,100);
>> x = cos(u);
>> y = sin(u);
>> plot(x,y,'*')
>> axis equal
```

We first define a set of u points on the interval $[0, 2\pi]$. We then compute the x and y coordinates for every value of u. We finally plot the points, using an * for clarity and an "axis equal" so that we recognise the circle.

How do we "know" that these parametric equations trace out a circle, and not just a curve that looks like a circle? Let's check by substituting the definition of x and y into the equation for a circle of radius 1, centered at the origin,

$$x^2 + y^2 - 1 = cos^2(u) + sin^2(u) - 1 = 0 \tag{28.9}$$

which required the use of the trigonometric identity $\cos^2(u) + \sin^2(u) = 1$.

$x = a\cos t$
$y = b\sin t$
$\forall t \in [0, 2\pi]$

Exercise 28.9

Use the internet to find a set of parametric equations that define an ellipse, and use MATLAB to verify them visually. Show that the parametric equations satisfy the implicit equation for an ellipse. *This is a small change to the parametric equations for a circle, and finding parametric equations on the internet should be straight-forward - try searching on "parametric equations for ellipse" or start with the Wikipedia page on conic section or ellipse.*

a controls size
b controls spiralness
(smaller b = smaller spiral)

Exercise 28.10

A logarithmic spiral can be defined by the parametric equations

$$x = ae^{-bu}\cos(u), y = ae^{-bu}\sin(u), a > 0, b > 0, u \in [0, \infty) \tag{28.12}$$

Visualize the curve in MATLAB for different values of a and b—you won't be able to define an infinite domain but you can define a large one. How does a and b change the curve?
This question involves a curve that has been of interest to mathematicians and scientists for many centuries. Try searching on the internet for the term "logarithmic spiral".

a is the radius of the helix
b is the gap between each ring.

Exercise 28.11

A helix in 3D can be defined by the parametric equations

$$x = a\cos(u), y = a\sin(u), z = bu, a > 0, b > 0, u > 0 \tag{28.13}$$

This is just a circle going up at a constant rate.

Visualize this curve for different values of a and b. How do a and b change the curve? (You will need to use **plot3** in MATLAB)
This question demonstrates that it is relatively simple to define a curve in 3D - just define the x, y, and z coordinates in terms of a single parameter. This curve is a good example, and has been widely studied in modern biology given its connection to the shape of DNA. Use the Wikipedia article on "helix" as a starting point.

28.1.4 Data-Driven Curves

We are often tasked with finding a curve that fits a set of data. You've probably seen informal approaches to this, particularly when finding the best-fit straight-line to a set of data points. Fortunately we have a robust, formal tool at our disposal now - orthogonal projection, often known as linear regression in this context.

Let's start with some data. Consider the 4 points $(0,1),(1,0),(3,2),(5,4)$. We can use MATLAB to plot these points

```
>> x = [0 1 3 5]';
>> y = [1 0 2 4]';
>> plot(x,y,'*')
```

Notice that we placed all of the x-coordinates in a column vector x and all of the y-coordinates in a column vector y. Let's now find the best-fit straight-line through these points, i.e. let's find the parameters m and b so that the straight-line defined by

$$y = mx + b \tag{28.14}$$

fits the points as well as possible.

The approach we take is motivated by our work in linear algebra. If we pack all of the x-coordinates into a vector $\mathbf{x}$ and all of the y-coordinates into a vector $\mathbf{y}$ then we would like to satisfy the vector equation

$$\mathbf{y} = m\mathbf{x} + b \tag{28.15}$$

as well as we can. Notice that there are 4 equations here (one for each point) and only two unknown parameters. An exact solution is impossible (unless the points happen to lie on a line) and so we use orthogonal projection to find the best solution. Let's define a matrix $\mathbf{A}$ and parameter vector $\mathbf{p}$ so that the vector equation for a straight-line becomes

$$\mathbf{A}\mathbf{p} = \mathbf{y} \tag{28.16}$$

where $\mathbf{A}$ and $\mathbf{p}$ are given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{x} & \mathbf{1} \end{bmatrix}, \mathbf{p} = \begin{bmatrix} m \\ b \end{bmatrix}$$

Notice that there is a coefficient of "1" in front of the "b" term so we had to create a column vector and fill it with 1's.

Recall that to find the best solution we multiply by $\mathbf{A}^T$,

$$\mathbf{A}^T\mathbf{A}\mathbf{p} = \mathbf{A}^T\mathbf{y} \tag{28.17}$$

and solve this linear system for $\mathbf{p}$.

```
>> A = [x ones(4,1)]
>> p = A'*A\A'*y
```

For these data points we should find that $m = p(1) = 0.6949$ and $b = p(2) = 0.1864$. You should plot the straight-line defined by this slope and intercept to see how good the fit is.

Exercise 28.12

Find the best-fit parabola for these 4 data points. Recall that a parabola can be defined using the explicit function $y = ax^2 + bx + c$.

Y = aX.^2 + bX + c
M = [x.^2 X 1]
P = [a b c]
MP' = Y

28.2 Surfaces

28.2.1 Surfaces defined Explicitly

If we assign the output of a function of two variables $f(x, y)$ to be a third variable, $z = f(x, y)$, then the set of points in 3D define a surface. For example, $z = x^2 + y^2$ defines a paraboloid. This surface can be visualized in MATLAB using the **surf** function.

```
>> [x,y]=meshgrid(linspace(-2,2,100),linspace(-2,2,100));
>> z = x.^2 + y.^2;
>> surf(x,y,z)
>> shading interp
```

First we lay down a grid of points in the xy-plane using **meshgrid**. Next we compute the value of the function at each of these points and assign the value to z. Finally we pass the x, y, z matrices to **surf** for rendering—we include a shading option to make the surface look nice and smooth.

It is often helpful to visualize a surface by drawing the contours defined by holding one of the variables constant. For example, if we define $z = 1$ in the equation for the paraboloid we obtain $x^2 + y^2 = 1$, which we know to be the equation of a circle of radius 1, centered at the origin. Choosing different values of z will define circles of radius $\sqrt{z}$. We already used the **contour** function in MATLAB earlier—here we will use it to draw the contours at different values of z

```
>> contour(x,y,z,'ShowText','On')
>> axis equal
```

In this case we are allowing MATLAB to pick the contour levels and we are including labels on the contours to show the corresponding value of z. We include the "axis equal" option in order to recognise that the contours are circles.

We can also "slice" the surface along the different coordinate directions. For example, if we wanted to plot the contours in the yz-plane where x is constant we would use

```
>> contour(y,z,x,'ShowText','On')
```

If we define $x = c$ and replace it into the definition of the function we see that

$$z = y^2 + c^2 \tag{28.18}$$

which is the equation of a parabola in the yz-plane and the value of c controls where it crosses the z-axis $(y = 0)$. The contour plot should support that analysis. We could also view the constant y contours in the xz-plane and we would find parabolas again—thus the reason we refer to the surface as a paraboloid.

Exercise 28.13

Visualize the elliptic paraboloid $z = x^2/a^2 + y^2/b^2$ for different values of $a > 0$ and $b > 0$.

1. Describe the contours in the yz-plane defined by $x = c$. squished (short & fat) Parabolas (a b)

Paraboloid:
Parabolas in xz and yz,
circles in xy

2. Describe the contours in the xz-plane defined by $y = c$. Also squished, but less (a,a) Parabolas
3. Describe the contours in the xy-plane defined by $z = c$. ellipses, a and b as expected

This question requires you to combine surface visualization with the curve visualization that we met earlier. To fully understand the parameters you should try to explain why the surface is called an elliptic paraboloid. It's a paraboloid but eliptic instead of circular

28.2.2 Surfaces defined Implicitly

A surface in three dimensions can also be implicitly defined by a function of three variables. For example, the equation for a unit sphere centered at the origin is

$$x^2 + y^2 + z^2 - 1 = 0 \tag{28.19}$$

The left hand side of this equation can be thought of as a function of three variables, $f(x, y, z)$, and the set of points where $f = 0$ defines the unit sphere. We can use the **isosurface** function in MATLAB to visualize:

```
>> [x,y,z] = meshgrid(linspace(-2,2,100),linspace(-2,2,100),linspace(-2,2,100));
>> f = x.^2 + y.^2 + z.^2 - 1;
>> isosurface(x,y,z,f,0)
>> axis equal
```

We first define a set of points in 3D space using the **meshgrid** function. Next we evaluate the function f at all of these points. We then use **isosurface** to render the surface defined by $f = 0$, and we use the "axis equal" option so that the resulting looks like a sphere.

There are lots of implicit surfaces, but a particularly important group is the quadratic (or quadric) surfaces, defined by the equation:

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy + Gx + Hy + Iz + J = 0 \tag{28.20}$$

where A, B, C, D, E, F, G, H, I, and J are all arbitrary constants, some of which may be zero.

Exercise 28.14 a,b,c are x,y, and z radii; respectively.

Visualize the hyperboloid of one sheet defined by

$$x^2/a^2 + y^2/b^2 - z^2/c^2 - 1 = 0 \tag{28.21}$$

for different values of a, b, c. What features of the hyperboloid do a, b, c control?

28.2.3 Surfaces defined Parametrically

Finally, a more general representation of a surface involves expressing it's coordinates in terms of two independent variables as follows

$$x = f(u, v), y = g(u, v), z = h(u, v), u \in [a, b], v \in [c, d] \tag{28.22}$$

Each value of (u, v) defines a point in 3D with coordinates $(f(u, v), g(u, v), h(u, v))$. If we collect all the points defined by (u, v) in the specified domain, then we get a parametric surface. For example, the definition

$$x = sin(u)cos(v), y = sin(u)sin(v), z = cos(u), u \in [0, \pi], v \in [0, 2\pi] \tag{28.23}$$

defines a unit sphere. In MATLAB we visualize a parametric surface using **surf**.

```
>> [u,v] = meshgrid(linspace(0,pi,100),linspace(0,2*pi,100));
>> x = sin(u).*cos(v);
>> y = sin(u).*sin(v);
>> z = cos(u);
>> surf(x,y,z), shading interp
>> axis equal
```

First we lay down a grid of points in the (u, v) space using **meshgrid**. We then compute x, y, z at each of these points, and we render the surface using **surf**.

Exercise 28.15

Lookup the parametric equations that define an ellipsoid, and use MATLAB to visualize. ✓

$X = a\cos u \sin v$
$Y = b \sin u \sin v$
$Z = c \cos v$

Exercise 28.16

Visualize the following parametric surface

This is a torous.
a is outer radius.
r is wall thickness

$$x = (a + r\cos(u))\cos(v), y = (a + r\cos(u))\sin(v), z = r\sin(u) \tag{28.24}$$

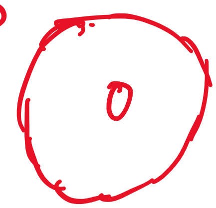
with $r < a$ and $u \in [0, 2\pi]$, $v \in [0, 2\pi]$. Describe the surface and interpret the parameters a and r.

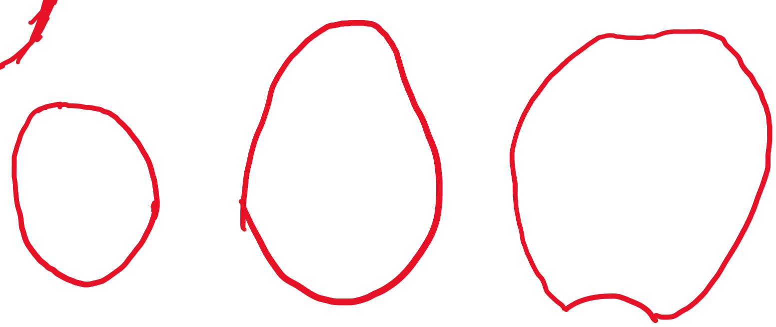
28.3 Designing Curves and Surfaces

Exercise 28.17

1. Pick a fruit or vegetable. Sketch it on paper from a variety of viewpoints. Now slice it in three ways, and sketch the sets of curves defined by each of these sets of slices.
2. Propose and evaluate a mathematical representation that is a good approximation to your fruit or vegetable. You could represent the entire surface, or you could design a set of curves that are good approximations to the slices.

I WAS PROMISED NO ART!





Solution 28.1

1. The value of y at $t = 0$ is A.
2. For negative values of t the value of y tends to zero if $k > 0$ and it tends to $+\infty(A > 0)$ or $-\infty(A < 0)$ if $k < 0$. For positive values of t the value of y tends to zero if $k < 0$ and it tends to $+\infty(A > 0)$ or $-\infty(A < 0)$ if $k > 0$.
3.
4. The sign of the parameter A dictates whether the curve has positive or negative values of y – the curve also passes through the point $(0, A)$. The parameter k dictates whether the curve increases or decreases exponentially.

Solution 28.2

1. The value of y at $t = 0$ is $A/2$.
2. For negative values of t the value of y tends to zero if $k > 0$ and it tends to A if $k < 0$. For positive values of t the value of y tends to A if $k > 0$ and it tends to zero if $k < 0$.
3.
4. The parameter A changes the long-term behavior of the curve while the parameter k changes how quickly the curve tends to this value.

Solution 28.3

1.
2. Since a sin function returns values between 0 and 1, the parameter A controls the height of the function and is usually referred to as the amplitude. Since a sin function is periodic with a period of 2π , the period T of this function is determined by $\omega T = 2\pi$. Increasing ω decreases the period T , and ω is usually referred to as the angular frequency. Since a sin function is 0 when it's argument is 0, the parameter ϕ controls where it crosses the x-axis, and is usually referred to as the phase.

Solution 28.4

1.
2. Graphing the vertex form reveals the effects of the parameters as follows. The vertex of the parabola is located at (h, k). The parabola opens upward if $g > 0$ and downward if $g < 0$. The parabola is narrow and steep for large positive values of g or large negative values of g. Changing h and k simply changes the location of the vertex.
3. Expanding the vertex form of the polynomial leads to $gx^2 - 2ghx + gh^2 + k$. Comparing to the standard form we see that $a = g$, $b = -2gh$, $c = gh^2 + k$. So although a has the same effect as g, the parameter b depends on g and h, and the parameter c depends on g, h, and k. This is why it is difficult to see the effect of the standard-form parameters on the curve.

Solution 28.5

The equation for a circle of radius R, centered at (a, b) is given by

$$(x - a)^2 + (y - b)^2 = R^2 \tag{28.3}$$

Solution 28.6

The parameters a and b determine the axes of the ellipse. The larger one is usually called the major axis and the smaller one is usually called the minor axis. This ellipse is oriented with its major and minor axes along the coordinate axes. Increasing a while holding b fixed results in a vertically-squished ellipse and vice versa.

Solution 28.7

There are two curves that define the hyperbola. Notice that the curves cross the x-axis at $x = -a$ and $x = a$ respectively. The rest of each curve is unbounded, but is asymptotic to the straight lines $y = (b/a)x$ and $y = -(b/a)x$.

Solution 28.8

The type of conic section is determined by the value of $b^2 - 4ac$ as follows:

- If $b^2 - 4ac < 0$ the equation represents an ellipse. In addition, if $a = c$ and $b = 0$ the equation represents a circle.
- If $b^2 - 4ac = 0$ the equation represents a parabola.
- If $b^2 - 4ac > 0$ the equation represents a hyperbola.

Solution 28.9

Although there are lots of parametric equations that trace out an ellipse, the most common are closely related to those for a circle and take the form

$$x = a\cos u, y = b\sin u, u \in [0, 2\pi] \tag{28.10}$$

where a and b represent the ellipses major and minor axes. The ellipse is traced out as u changes from $0to2\pi$, but note that u does not represent the angle between the x-axis and a point on the ellipse - see the Wikipedia page on "Ellipse" for an explanation of this. To confirm that these are valid parametric equations for an ellipse we substitute them into the implicit equation for an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \cos^2 u + \sin^2 u - 1 = 0 \tag{28.11}$$

where again we have used the trigonometric identity $\cos^2(u) + \sin^2(u) = 1$.

Solution 28.10

If $b = 0$ we see that the curve is a circle of radius a, and the parameter u corresponds to the angle of rotation. As you increase b, the circle changes into a spiral which tends to the origin as $u \to \infty$—the larger the value of b the quicker the curve spirals into the origin.

Solution 28.11

If $b = 0$ the curve is a circle of radius a in the x y - plane, and u is the angle of rotation. For $b > 0$ the curve continues to rotate as before when viewed from "above", but its height increases linearly - the resulting curve is a helix. The separation between each rotation of the curve is given by $2\pi b$, which is commonly known as the pitch of the helix.

Solution 28.12

Assuming we have already packed the x-coordinates of the data into $\mathbf{x}$ and the y-coordinates of the data into $\mathbf{y}$ we need to define the matrix $\mathbf{A}$ and solve for a vector $\mathbf{p}$ of unknown parameters. In MATLAB we would use

```
>> A = [x.^2 x ones(4,1)]
>> p = A'*A\A'*y
```

We should find that $a = p(1) = 0.1910$, $b = p(2) = -0.2663$ and $c = p(3) = 0.6784$. You should graph the parabola defined by these parameters and see how good the fit is.

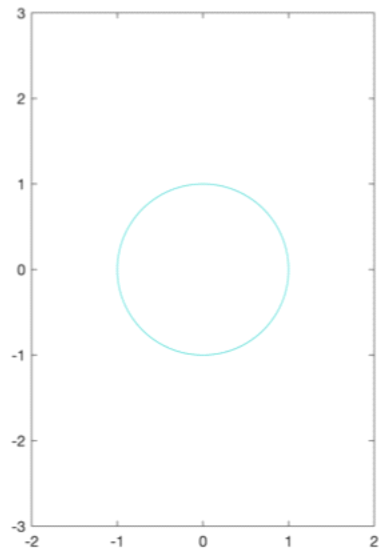
Solution 28.13

Let's take slices through the surface along each of the coordinate axes.

1. If we choose $x = c$ then we obtain $z = c^2/a^2 + y^2/b^2$ which is a parabola in the (y, z)-plane that crosses the z-axis $(y = 0)$ at c^2/a^2.
2. If we choose $y = c$ then we obtain $z = x^2/a^2 + c^2/b^2$ which is a parabola in the (x, z)-plane that crosses the z-axis $(x = 0)$ at c^2/b^2.
3. If we choose $z = c$ then we obtain $c = x^2/a^2 + y^2/b^2$ which is an ellipse in the (x,y)-plane. If we divide both sides by c we get the standard form for an ellipse $1 = x^2/(a\sqrt{c})^2 + y^2/(b\sqrt{c})^2$ so that the major and minor axes are $a\sqrt{c}$ and $b\sqrt{c}$ - increasing the value of c increases the axes of the ellipse.

HW1

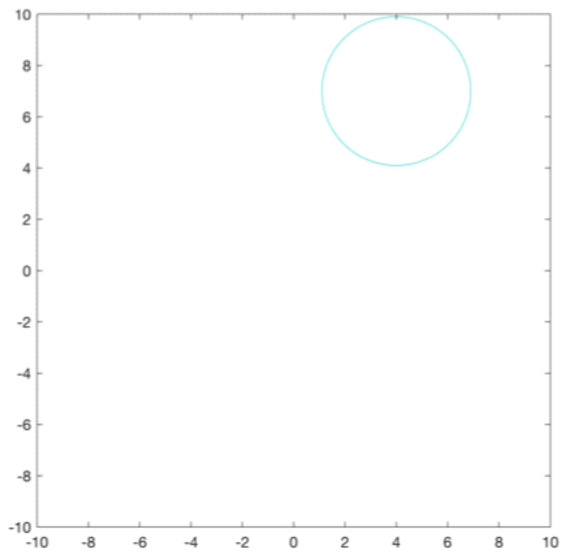
```
%% 28.1.2

[x, y] = meshgrid(linspace(-2, 2, 100), linspace(-3, 3, 200));
f = x.^2 + y.^2 - 1;
contour(x, y, f, [0 0]);
axis equal;
```



Exercise 28.5

Visualization of a circle

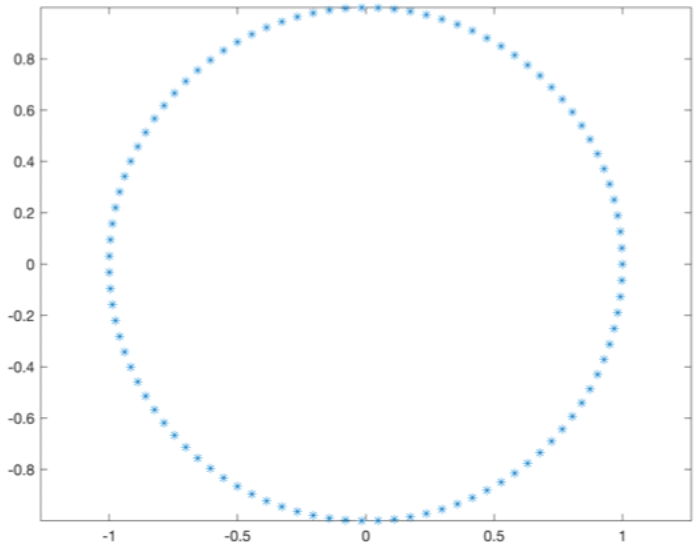
```
R =2.9;
a = 4;
b = 7;

[x, y] = meshgrid(linspace(-10, 10, 1000), linspace(-10, 10, 1000));
f = (x - a).^2 + (y - b).^2 - R^2;
contour(x, y, f, [0 0]);
axis equal;
```



28.1.3

```
u = linspace(0, 2 * pi, 100);
x = cos(u);
y = sin(u);
plot(x, y, '*');
axis equal;
```

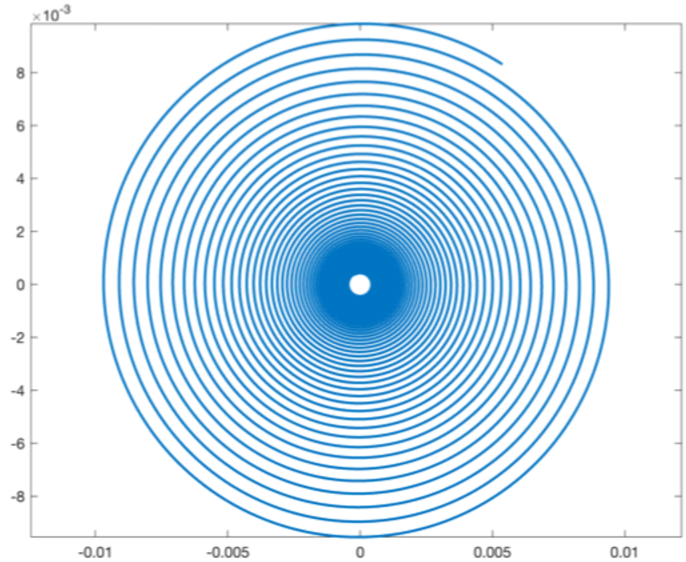
2



Excercise 28.10

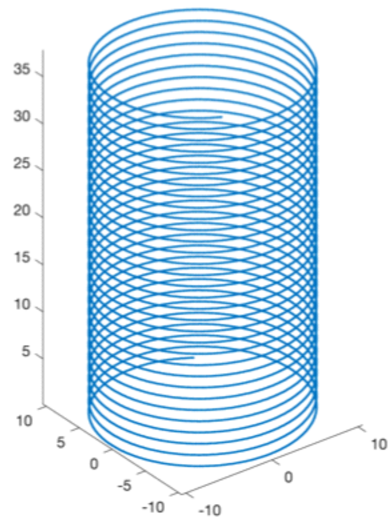
```
e = exp(1);

a = 10 ^-2;
b = 10 ^-2;

min = 10 ^ 0;
max_num = 10 ^ 2.5;
num_points = 10 ^ 6.2;

u = linspace(min, max_num, num_points);

x = a * e.^(-b * u) .* cos(u);
y = a * e.^(-b * u) .* sin(u);

plot(x, y, '.-');
axis equal;
```

3



Exercise 28.11

```
e = exp(1);

a =10.5;
b =0.19;

min = 1;
max_num =199;
num_points = 9369;

u = linspace(min, max_num, num_points);

x = a * cos(u);
y = a * sin(u);
z = b * u;

plot3(x, y, z, '.-');
axis equal;
```



28.1.4

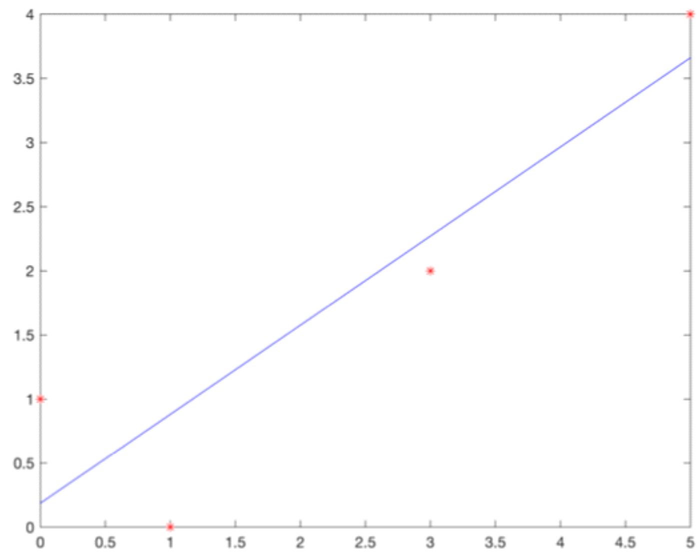
```
x = [0 1 3 5]';
y = [1 0 2 4]';
plot(x, y, 'r*'); hold on;

% Using orthogonal projection to find line of best fit:
% We want to find a line y = mx + b
% What if we used our vectors and so had \vec{y} = m\vec{x} + b
% There's four equations (one for each data point), so we can't get an
% exact answer
% We can rephrase this is a more convinent form (ones column is to multiply by b):
A = [x ones(size(x))]
```

```
A = 4×2
     0     1
     1     1
     3     1
     5     1
```

```
% Need to find: p = [m; b] (parameters): Ap = y
% The best solution is A' * A * p = A' * y
p = (A' * A) \ (A' * y)
```

```
p = 2×1
    0.6949
    0.1864
```

```
xes = 0:max(x);
y = (p(1) * xes) + p(2);
plot(xes, y, 'b-'); hold off;
```




```
%% Exercise 28.12
x = [0 1 3 5]';
y = [1 0 2 4]';
plot(x, y, 'r*'); hold on;
M = [x.^2, x, ones(size(x))]
```

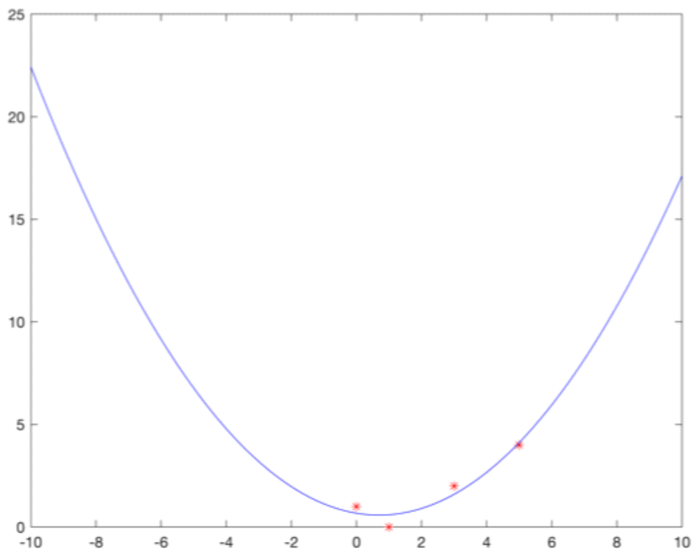
```
M = 4x3
     0     0     1
     1     1     1
     9     3     1
    25     5     1
```

```
p = (M' * M) \ (M' * y)
```

```
p = 3x1
    0.1910
   -0.2663
    0.6784
```

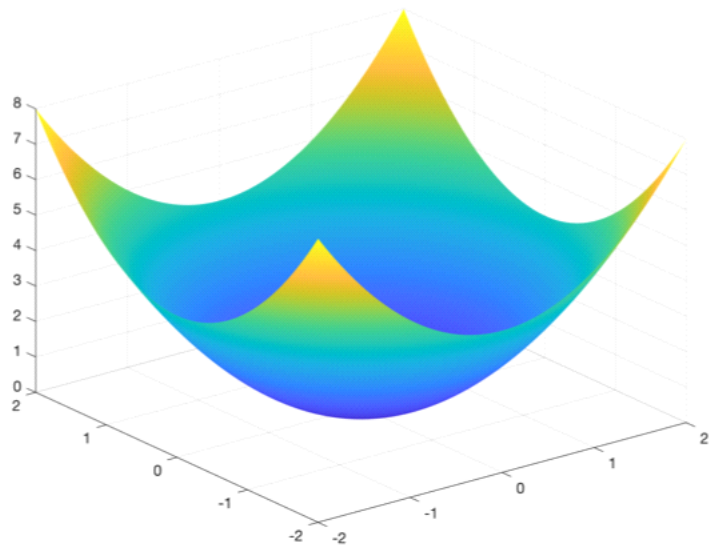
```
xes = -10:0.1:10; %max(x);
y = (p(1) * xes.^2) + (p(2) * xes) + p(3);
plot(xes, y, 'b-')
xlim([-10 10]);
hold off;
```

6

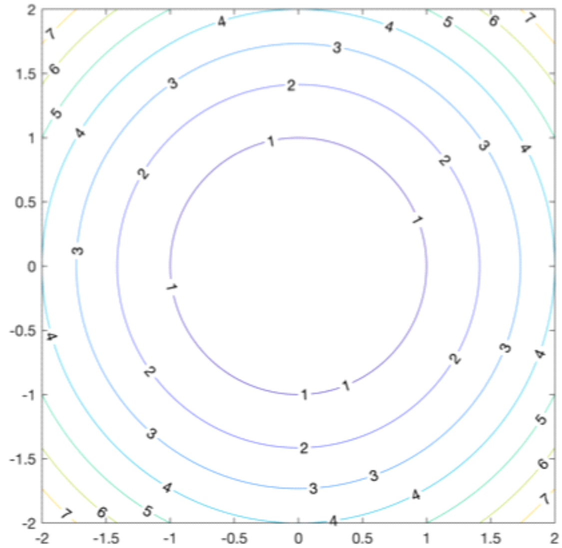


28.2.1

```
[x, y] = meshgrid(linspace(-2, 2, 100), linspace(-2, 2, 100));
z = x.^2 + y.^2;
surf(x, y, z);
shading interp;
```



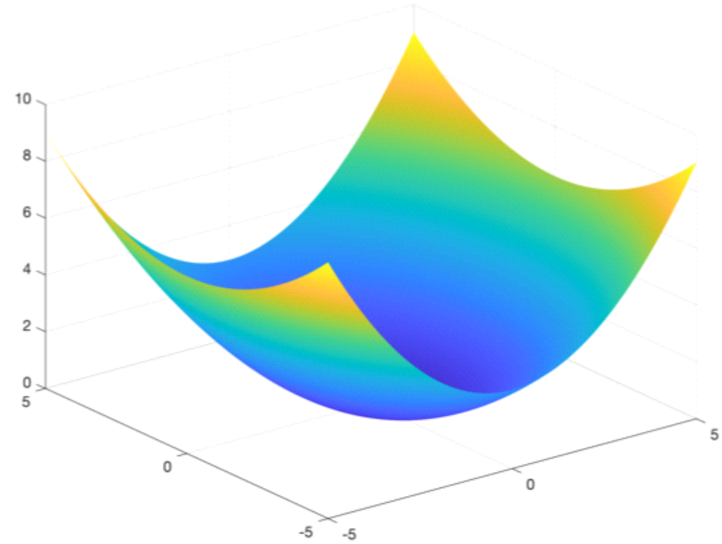

```
figure;
contour(x, y, z, "ShowText","on");
axis equal;
```



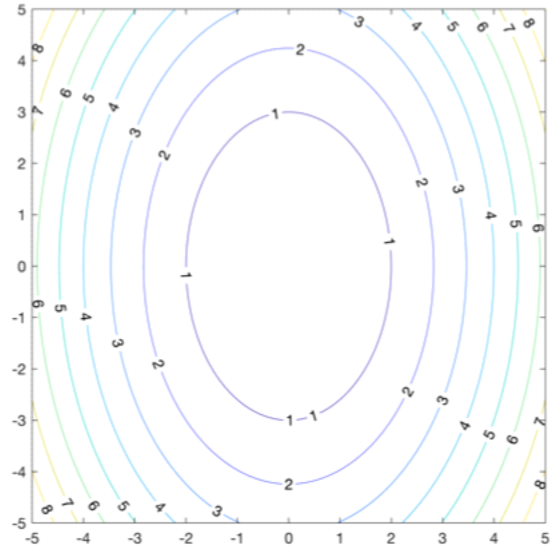
Excercise 28.13

```
a = 2;
b = 3;

[x, y] = meshgrid(linspace(-5, 5, 100), linspace(-5, 5, 100));
z = (x.^2/a^2) + (y.^2/b^2);
surf(x, y, z);
shading interp;
```

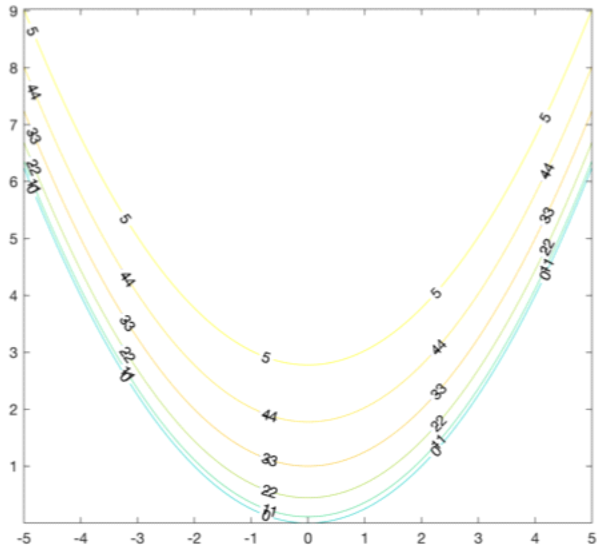
9




```
figure;
contour(x, y, z, "ShowText","on");
axis equal;
```

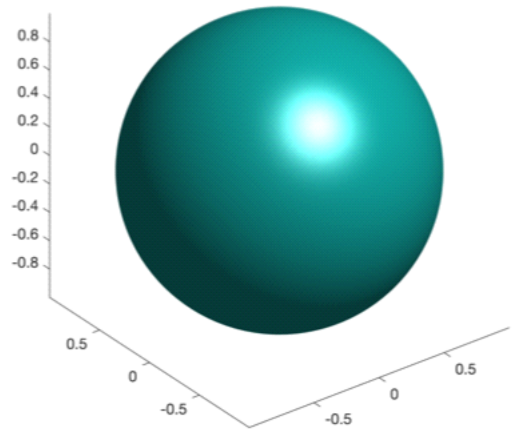


```
figure;
contour(x, z, y, "ShowText","on");
axis equal;
```



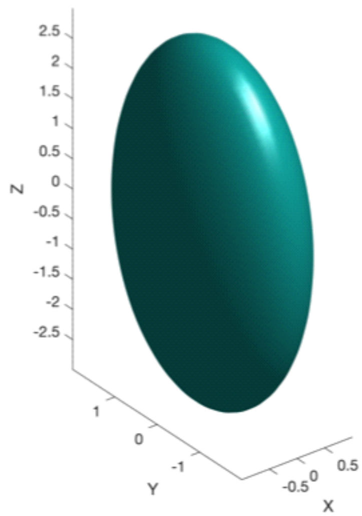
28.2.2

```
[x, y, z] = meshgrid(linspace(-2, 2, 100), linspace(-2, 2, 100), linspace(-2, 2, 100));
f = x .^2 + y .^2 + z .^2 - 1;
clf; isosurface(x, y, z, f, 0);
axis equal;
```



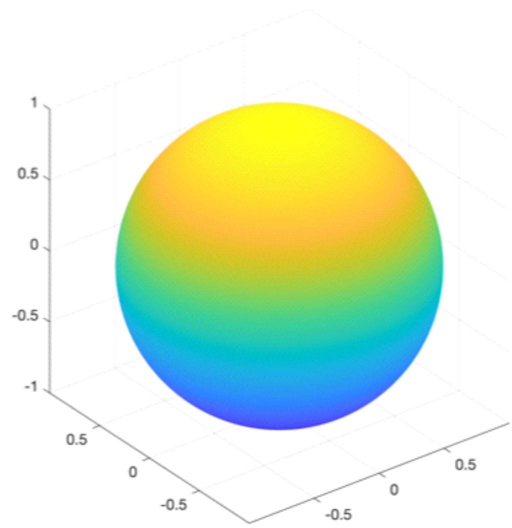
Excercise 28.14

```
a = 1;
b = 2;
c = 3;
[x, y, z] = meshgrid(linspace(-5, 5, 100), linspace(-5, 5, 100), linspace(-5, 5, 100));
f = (x .^2 / a^2) + (y .^2 / b^2) + (z .^2 / c^2) - 1;
isosurface(x, y, z, f, 0);
axis equal;
xlabel("X"); ylabel("Y"); zlabel("Z");
```



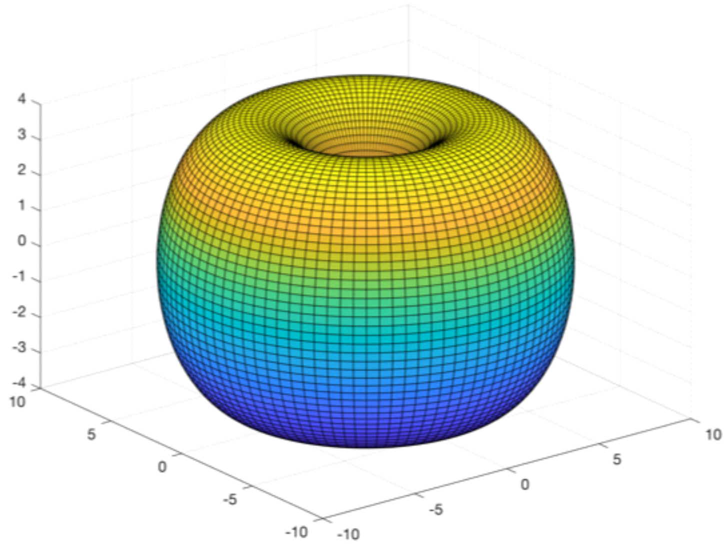
28.2.3

```
[u, v] = meshgrid(linspace(0, pi, 100), linspace(0, 2 * pi, 100));
x = sin(u) .* cos(v);
y = sin(u) .* sin(v);
z = cos(u);
surf(x, y, z); shading interp;
axis equal;
```



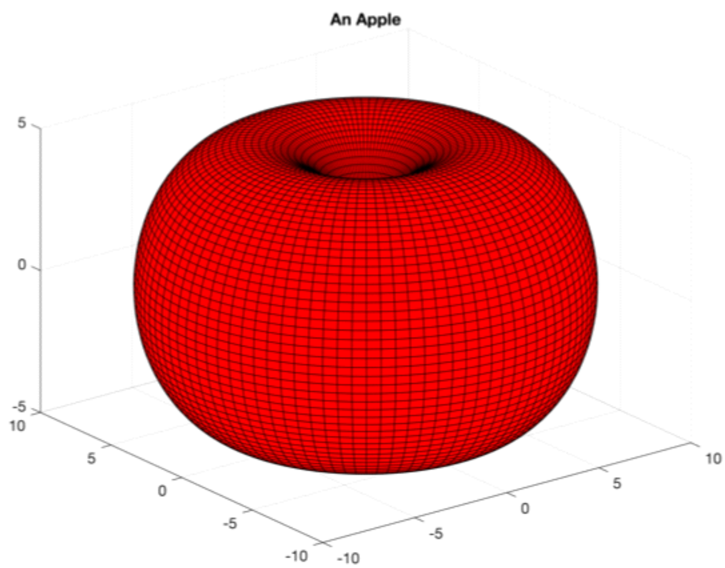
Excercise 28.16

```
[u, v] = meshgrid(linspace(0, 2 * pi, 100), linspace(0, 2 * pi, 100));
a = 5;
r = 4;
x = (a + (r .* cos(u))) .* cos(v);
y = (a + (r .* cos(u))) .* sin(v);
z = r * sin(u);
surf(x, y, z);
```



Excercise 28.17

```
[u, v] = meshgrid(linspace(0, 2 * pi, 100), linspace(0, 2 * pi, 100));
a = 5;
r = 5;
x = (a + (r .* cos(u))) .* cos(v);
y = (a + (r .* cos(u))) .* sin(v);
z = r * sin(u);
surf(x, y, z);
title("An Apple")
colormap([1 0 0]);
```


An Apple
5
0
-5
10
5
0
-5
-10
-10
-5
0
5
10