Week 5b

Wednesday, October 14, 2020 9:57 PM

Week 5b

Chapter 14

Week 5b: Orthogonal Projection

Schedule

14.1 Debrief [15 mins]

- Please discuss your work with the folks in your breakout-room, and get help with the ideas that you are still confused by.

In the next set of exercises we will explore a common method for finding "the" solution of a linear system of algebraic equations ($\mathbf{A}\mathbf{x} = \mathbf{b}$) in the case where there are more equations that unknowns (more rows than columns). We will first need to synthesise some previous ideas about the span of vectors.

14.2 Range of A [15 mins]

We discussed earlier the concept of the **span** of a collection of vectors. Recall that the span of a collection of vectors is the set of all linear combinations of the vectors. Now we will apply this concept to the columns of a matrix:

Definition: The Range of a matrix $\mathbf{A}$ is the span of its columns.

Exercise 14.1

Describe in words the Range of the following matrices:

1. $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

14.3 Exact Solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ [15 mins]

When does a linear system of algebraic equations, $\mathbf{A}\mathbf{x} = \mathbf{b}$, have a solution? Since the product $\mathbf{A}\mathbf{x}$ is a linear combination of the columns of $\mathbf{A}$, then $\mathbf{A}\mathbf{x} = \mathbf{b}$ will have a solution if and only if $\mathbf{b}$ is in the Range of $\mathbf{A}$. Think about that, and complete the following exercise.

Exercise 14.2

Which of the following linear systems of algebraic equations will have a solution? Think about it from an equation perspective and the Range of $\mathbf{A}$ perspective.

1. $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$
2. $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
3. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}$
4. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$

14.4 Approximate solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ [30 mins]

You should have found that some of these systems do not have a solution in the usual sense, i.e. there is no vector $\mathbf{x}$ which makes the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ true. We might refer to such a solution as an **exact** solution. We will now consider an **approximate** solution, i.e. a vector $\mathbf{x}$ which approximately satisfies $\mathbf{A}\mathbf{x} = \mathbf{b}$. We will consider a particular approximation based on **orthogonal projection** now, and later in QEA we will look at this approximation from a different perspective where it is known as the **Least-Squares** approximation. We met orthogonal projection earlier in the module when we spoke about vector components and basis vectors.

Exercise 14.3

Hold your hand up in front of you, and think about it as occupying a location in 3D.

1. Point to the location on each of the walls surrounding you that is closest to your hand.
2. Point to the location on the floor that is closest to your hand.

3. In your other hand, hold a flat object (like a piece of paper or a book) at some angle. Now imagine extending the surface of this object so that it is larger than the room you are in. Now point to the location on the extended flat surface that is closest to your hand.
4. What do you notice about the relationship between the "pointing" vector and the surface being pointed at?

Now let's put this in the context of solving $\mathbf{A}\mathbf{x} = \mathbf{b}$.

- If $\mathbf{b}$ is not in the Range of $\mathbf{A}$ then we will define an approximate solution by orthogonal projection of $\mathbf{b}$ onto the Range of $\mathbf{A}$.
- The "pointing" vector from $\mathbf{b}$ to the relevant point in the Range of $\mathbf{A}$ is $\mathbf{A}\mathbf{x} - \mathbf{b}$.
- Since the Range of $\mathbf{A}$ is defined by the span of the columns of $\mathbf{A}$ then the "pointing" vector must be orthogonal to **every** column of $\mathbf{A}$.
- This implies that $\mathbf{A}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{0}$. (Think about why this must be true).
- Re-arranging this equation leads to $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. The matrix $\mathbf{A}^T\mathbf{A}$ is a square matrix (which we will meet again and again this module).
- This is a linear system with equal numbers of equations and unknowns and can therefore be solved using our usual techniques. Did you get that? You should re-read this paragraph a few times. To summarize:

  The approximate solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ based on orthogonal projection can be obtained by solving

  $$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

  This solution is also known as the **least-squares** solution because it minimises the distance between $\mathbf{b}$ and the Range of $\mathbf{A}$ (more about this later).

  Warning: Do not think about $\mathbf{x}$ defining a coordinate system that $\mathbf{b}$ lives in! When you draw a picture you should think about the space that the columns of $\mathbf{A}$ live in. We are projecting $\mathbf{b}$ onto a basis defined by the columns of $\mathbf{A}$. The solution vector $\mathbf{x}$ is better thought of as a set of "weights" or "coordinates" with respect to this basis.

Exercise 14.4

1. Consider the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. (You've already thought about this earlier).
   (a) Sketch the Range of $\mathbf{A}$ and locate the point in the Range that is closest to $\mathbf{b}$.
   (b) Multiply both sides of $\mathbf{A}\mathbf{x} = \mathbf{b}$ by $\mathbf{A}^T$ and solve the resulting linear system.
2. Consider the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$. (You've already thought about this earlier).
   (a) Sketch the Range of $\mathbf{A}$ and locate the point in the Range that is closest to $\mathbf{b}$.

(b) Multiply both sides of $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^T$ and solve the resulting linear system.

14.5 Solving $\mathrm{Ax} = \mathrm{b}$ in Matlab [10 mins]

In many ways Matlab makes life easy for us. There is a single command in order to solve a linear system $\mathbf{Ax} = \mathbf{b}$

```
>> x = A\b
```

although it can also be used by typing

```
>> x = mldivide(A,b)
```

If there are more rows than columns then Matlab finds the approximate solution we discussed above. If there are equal numbers of rows and columns then Matlab computes a solution by LU decomposition. If there are less rows than columns then Matlab computes one of the infinite number of solutions - the solution it computes is not an approximation but it does select the solution that minimizes the length of the solution vector.

Exercise 14.5

For each of the linear systems in Exercise 14.4 please find the solution in Matlab using A \b.

Solution 14.1

1. The column is a two-dimensional vector. The span is a line (slope = 1) in 2D space.
2. The columns are linearly-independent three-dimensional vectors. Their span is therefore a plane in 3D space. Since all the z-entries are zero, the plane is actually the xy-plane.
3. The columns are linearly-independent three-dimensional vectors. Their span is therefore a plane in 3D space. The plane is defined by the column vectors.

Solution 14.2

1. The Range of $\mathbf{A}$ is all multiples of $\begin{bmatrix}1\\1\end{bmatrix}$. Since $\mathbf{b}$ is a multiple of this vector then there is a solution. From an equation point of view, the solution is simply $x = 5$.
2. The Range of $\mathbf{A}$ is all multiples of $\begin{bmatrix}1\\1\end{bmatrix}$. Since $\mathbf{b}$ is not a multiple of this vector then there is no solution. From an equation point of view this makes sense because we are demanding that $x = 2$ and $x = 3$ at the same time.
3. The Range of $\mathbf{A}$ is a plane in 3D. Since $\mathbf{b}$ is the sum of the columns it must be in the Range of $\mathbf{A}$ and so there is a solution. From an equation point of view there are two linearly-independent equations in two unknowns.
4. The Range of $\mathbf{A}$ is a plane in 3D. Since $\mathbf{b}$ is not in this plane there is no solution. From an equation point of view this makes sense because trying to solve the equations results in an inconsistency.

Solution 14.3

In each case the "closest" point is the location where the "pointing" vector meets the surface at right angles, i.e. they are orthogonal.

Solution 14.4

1. Consider the linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}2\\3\end{bmatrix}$. (You've already thought about this earlier).
   (a) Sketch the Range of $\mathbf{A}$ and locate the point in the Range that is closest to $\mathbf{b}$. (The Range is a straight line and the point is the orthogonal projection onto this line.)
   (b) Multiply both sides of $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^T$ and solve the resulting linear system. (You should find that $x = 5/2$).
2. Consider the linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}3\\7\\5\end{bmatrix}$. (You've already thought about this earlier).
   (a) Sketch the Range of $\mathbf{A}$ and locate the point in the Range that is closest to $\mathbf{b}$. (The Range is a plane in 3D and the point is the orthogonal projection onto this line.)
   (b) Multiply both sides of $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^T$ and solve the resulting linear system. (You should find that $x = -3$ and $y = 7/2$.)

Solution 14.5

1.
```
>> A = [1;1]

A =

     1
     1

>> b = [2;3]

b =

     2
     3

>> A\b

ans =

    2.5000
```

2.
```
>> A = [1 2;3 4;5 6]

A =

     1     2
     3     4
     5     6

>> b = [3;7;5]

b =

     3
     7
     5

>> A\b

ans =

   -3.0000
    3.5000
```