Week 7

Monday, April 26, 2021 10:35 AM

RoboWeek 7

Chapter 28

Week 7: Scalar and Vector Fields, The Gauntlet

Schedule

Learning Objectives

Concepts

1. Differentiate between scalar and vector fields.
2. Determine the Jacobian, the divergence, and the curl of a vector field.
3. Build a scalar potential field using sources and sinks.

28.1 Scalar and Vector Fields - Definitions

Scalar and vector fields are mathematical objects which are ubiquitous across many disciplines. You have already been working with both of these concepts already, but now is a good time to formalize the definitions and deepen our understanding of these ideas.

A **scalar field** is a function which is defined across a space of inputs and outputs a scalar value. The input space can be of any dimensionality. As an example, a function which defines the temperature profile at all points in the air in the QEA classroom, $T(x, y, z)$ is a scalar field with a three dimensional input and a one dimensional output. The word "field" tells us that the input is a vector, and the word "scalar" tells us that the output is, well, a scalar.

Another example of a scalar field is the elevation of a mountain as a function of two-dimensional position, $H(x, y)$. A scalar field then is a function f which accepts a vector as an input and outputs a scalar—we can either use the notation $f(x, y, z)$ which specifies the components of the input, or the more general notation $f(\mathbf{r})$. If we want to be even more specific, we can write $f : \mathbf{R}^N \to \mathbf{R}$, which is shorthand notation for f takes real values in N dimensions as input and outputs a real value.

A **vector field** is a function which is defined across a space of inputs and outputs a vector. An example of this would be the velocity of the air currents at all points in the QEA classroom $\mathbf{v}(x, y, z) = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, where v_x is the component of the velocity in the x-direction etc. (Although this is a widely used notation, it can be confusing because v_x is also the notation used to define the partial derivative of v with respect to x!) Another example is the gradient function of the surface of a mountain $\nabla H = \frac{\partial H}{\partial x}\hat{\mathbf{i}} + \frac{\partial H}{\partial y}\hat{\mathbf{j}}$. A vector field then is a function $\mathbf{F}$ which accepts a vector as an input and outputs a vector of the same number of dimensions—again we can either use the notation $\mathbf{F}(x, y, z)$, or the more general notation $\mathbf{F}(\mathbf{r})$, where $\mathbf{F} : \mathbf{R}^N \to \mathbf{R}^N$.

Exercise 28.1

Which of the following quantities could be considered a scalar field or vector field (or neither).

1. The temperature at all points in your room right now. Scalar field
2. The temperature gradient at all points in your room right now. Vector field
3. The velocity of a bird in flight over the course of a day. neither: scalar in vec out
4. The population density at all points in the USA right now. Scalar field
5. The wind velocity at all points in the USA right now. vector field

Exercise 28.2

Generate a list of at least 3 quantities and classify them as a scalar field or a vector field.

Scalar: density of an obj, avg age by zip code
Vector: Nearest Restaurant ∀ points in USA

28.2 Visualizing Scalar and Vector Fields

We've already spent quite a bit of time visualizing scalar and vector fields. Recall that for scalar fields we often plot contours, and for vector fields we plot, well, a set of vectors defined at lots of points in space.

Exercise 28.3

For each of the functions below:

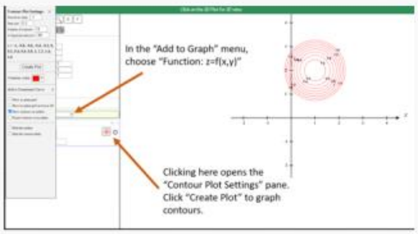
a Determine if the function is a scalar or vector field.

b Predict what the scalar or vector field will look like based on the equation.

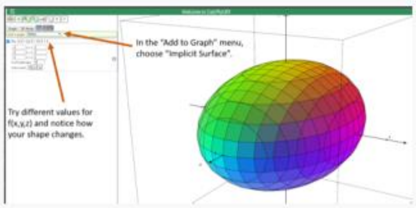
c Visualize the scalar or vector field using CalcPlot3D and compare to your prediction.

Visualizing each of the functions in CalcPlot3D requires slightly different settings. We will provide a starting point, but feel free to play with the plotting to achieved the desired visualization.

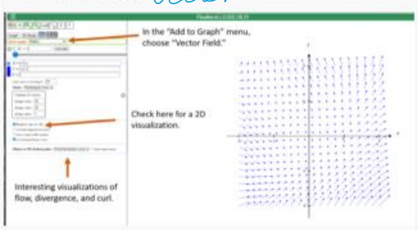
1. $f(x, y) = (x-1)^2 + (y-2)^2$ Scalar Paraboloid



2. $f(x, y, z) = x^2 + 2y^2 + 3z^2$ Scalar orb?



3. $\mathbf{F}(x, y) = (x^2 - y)\mathbf{i} + (x + y^2)\mathbf{j}$ vector



4. $\mathbf{F}(x, y, z) = (2y - z)\hat{\mathbf{i}} + (x + y^2 - z)\hat{\mathbf{j}} + (4y - 3x)\hat{\mathbf{k}}$ vector (3D)

Use the "Vector Field" setting as above, but include all three dimensions.

28.3 Jacobian, Divergence, and Curl of Vector Fields

We've already met partial derivatives, and the gradient operator $\nabla = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix}$, which is the multi-dimensional equivalent of the derivative. We can apply the gradient to vector fields in a number of ways. The material here is a very brief introduction and is focused on introducing terminology, not interpretation - that will come later!

Given a vector field $\mathbf{F}(x, y, z) = X(x, y, z)\hat{\mathbf{i}} + Y(x, y, z)\hat{\mathbf{j}} + Z(x, y, z)\hat{\mathbf{k}}$ its **Jacobian** is the matrix of partial derivatives

$$J\mathbf{F} = \begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{bmatrix} = \begin{bmatrix} \partial x & \cdots \\ \partial y & \cdots \\ \partial z & \cdots \end{bmatrix}$$ (just grad for vec field)

For example, the derivative of the vector field $\mathbf{F}(x, y, z) = x^2\hat{\mathbf{i}} + xyz\hat{\mathbf{j}} + (x^2 - z^2)\hat{\mathbf{k}}$ is the matrix

$$J\mathbf{F} = \begin{bmatrix} 2x & 0 & 0 \\ yz & xz & xy \\ 2x & 0 & -2z \end{bmatrix}$$

. We often simply refer to the Jacobian matrix as the "derivative" of the vector field, but that can be confusing.

Exercise 28.4

Find the Jacobian of the following vector fields.

1. $\mathbf{F}(x, y, z) = (3x + 4y)\hat{\mathbf{i}} + (4y - 5z)\hat{\mathbf{j}} + (x + y - z)\hat{\mathbf{k}}$
2. $\mathbf{F}(x, y, z) = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$

1) $\begin{bmatrix} 3 & 4 & 0 \\ 0 & 4 & 5 \\ 1 & 1 & 1 \end{bmatrix}$

2 $\begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}$

Consider a vector field given in Cartesian coordinates as $\mathbf{F}(x, y, z) = X(x, y, z)\hat{\mathbf{i}} + Y(x, y, z)\hat{\mathbf{j}} + Z(x, y, z)\hat{\mathbf{k}}$. The **divergence** of $\mathbf{F}$ is the scalar field

$$\mathrm{div}\mathbf{F} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$$

For example, the divergence of $\mathbf{F} = \sin(x)\cos(y)\hat{\mathbf{i}} + x^2yz\hat{\mathbf{j}} + (\sin(z) - x^2)\hat{\mathbf{k}}$ is the quantity $\mathrm{div}\mathbf{F}(x, y) = \cos(x)\cos(y) + x^2z + \cos(z)$.

Exercise 28.5

Find the divergence of the following vector fields

1. $\mathbf{F}(x, y) = \cos(x)\sin(y)\hat{\mathbf{i}} - \sin(x)\cos(y)\hat{\mathbf{j}}$
2. $\mathbf{F}(x, y, z) = y\sin(z)\hat{\mathbf{i}} - x\sin(z)\hat{\mathbf{j}} + \cos(z)\hat{\mathbf{k}}$

1) $\sin x \sin y - \sin x \sin y = 0$

2) $0 + 0 - \sin z = -\sin z$

Exercise 28.6

Show that the divergence of **F** is the dot product of the gradient operator with the vector field, i.e.

$$\mathrm{div}\mathbf{F} = \nabla \cdot \mathbf{F}$$

Consider a vector field given in Cartesian coordinates as $\mathbf{F}(x,y,z) = X(x,y,z)\hat{\mathbf{i}} + Y(x,y,z)\hat{\mathbf{j}} + Z(x,y,z)\hat{\mathbf{k}}$. The **curl** of **F** is the vector field

$$\mathrm{curl}\mathbf{F} = (\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z})\hat{\mathbf{i}} - (\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z})\hat{\mathbf{j}} + (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\hat{\mathbf{k}}.$$

Exercise 28.7

Compute the curl of the following vector fields For 2D, infer $z=0$

1. $\mathbf{F}(x,y) = \cos(x)\sin(y)\hat{\mathbf{i}} - \sin(x)\cos(y)\hat{\mathbf{j}}$
2. $\mathbf{F}(x,y,z) = y\sin(z)\hat{\mathbf{i}} - x\sin(z)\hat{\mathbf{j}} + \cos(z)\hat{\mathbf{k}}$

Exercise 28.8

Show that the curl of **F** is the cross product of the gradient operator with the vector field, i.e.

$$\mathrm{curl}\mathbf{F} = \nabla \times \mathbf{F}$$

28.4 Introduction to Potential Fields

In an earlier assignment we used the method of steepest ascent to navigate the NEATO to the top of a "mountain", which we defined using a scalar field $H(x,y)$.

In the Gauntlet challenge we are going to use targets (bucket of benevolence) and objects (walls, boxes, etc) to create an artificial mountain landscape, and we will navigate the NEATO using the **method of steepest descent** so as to avoid the obstacles and hone in on the target. (We use steepest descent in order to connect this with ideas from physics.)

To accomplish this task, we are going to introduce a particular scalar field $V(x,y)$ which can be used to create useful mountain landscapes. It is known as a **potential** field.

Exercise 28.9

Let $V(x,y) = \ln\sqrt{x^2+y^2}$. Note that V is not defined at the origin!

$\mathrm{div}F = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$ $F = [X, Y, Z]$

$\nabla = [\frac{1}{\partial_x}, \frac{1}{\partial_y}, \frac{1}{\partial_z}]$ $\nabla \cdot F = X \cdot \frac{1}{\partial x} + Y \cdot \frac{1}{\partial y} + Z \cdot \frac{1}{\partial z}$

1) $(-\cos(x)\cos(y) - \cos(x)\cos(y))\hat{k}$

2) $x\cos(z)\hat{i} - y\cos(z)\hat{j} + (\sin(x) - \sin(z))\hat{k}$

$\mathrm{curl}\,F = \nabla \times F = |\nabla||F|\sin\theta$ ← Always 90°

$= \sqrt{\partial_x^2 + \partial_y^2 + \partial_z^2} \cdot \sqrt{x^2+y^2+z^2} \cdot 1$

$=$

1. What is V in polar coordinates (note: this question is just here to make you realize that it's actually a pretty function, and to make it easier to visualize)? $V(r,\theta) = \ln r$
2. Create a contour plot of V.
3. Determine the gradient field defined by ∇V, and visualize it.
4. How would a NEATO performing gradient descent behave if you put it on a Flatland defined by V and started it at $(1,2)$?

Exercise 28.10

Now consider $V(x,y) = \ln\sqrt{x^2+y^2} - \ln\sqrt{(x-1)^2+y^2}$. Notice that there is a strong "trough" at $(0,0)$ and a strong "peak" at $(1,0)$. The sign in front of the logarithm determined whether it is a peak or a trough.

1. Create a contour plot of V.
2. Determine the gradient field defined by ∇V, and visualize it.
3. How would a NEATO performing gradient descent behave if you put it on a Flatland that looked like V and started it at $(1,2)$?

Exercise 28.11

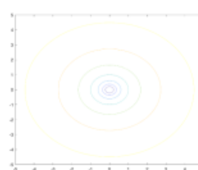
In the last question, a NEATO would be attracted by the trough at $(0,0)$ but repelled by the peak at $(1,0)$. We will refer to the point at $(0,0)$ as a "sink" and the point at $(1,0)$ as a "source".

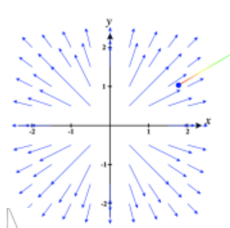
1. Define a scalar field $V(x,y)$ that has a sink at $(0,0)$, and sources at $(1,0)$ and $(2,3)$.
2. Visualize the scalar field V and the gradient field ∇V.
3. How would a NEATO performing gradient descent behave if you put it on a Flatland that looked like V and started it at $(1,2)$?

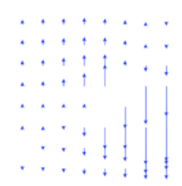
28.5 Introduction to The Gauntlet Challenge

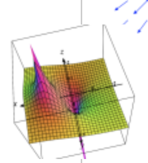
The last challenge of the Robotics Module is The Gauntlet. In this challenge, you will be using the a combination of potential fields and gradient descent to help your robot navigate through a cluttered environment to the Barrel of Benevolence, as seen in Figure 28.1.

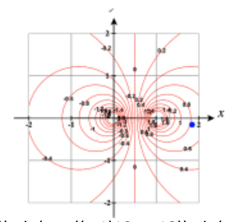
V(x, y) = $\log(\sqrt{x^2+y^2})$



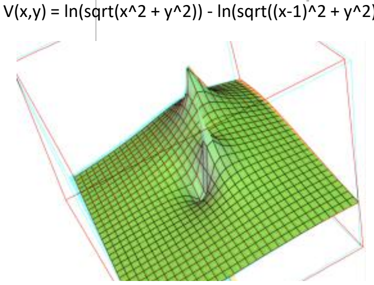






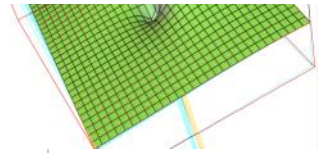


V(x,y) = ln(sqrt(x^2 + y^2)) - ln(sqrt((x-1)^2 + y^2)) - ln(sqrt((x-2)^2 + (y-3)^2))



28.5 Introduction to The Gauntlet Challenge

The last challenge of the Robotics Module is The Gauntlet. In this challenge, you will be using the a combination of potential fields and gradient descent to help your robot navigate through a cluttered environment to the Barrel of Benevolence, as seen in Figure 28.1.



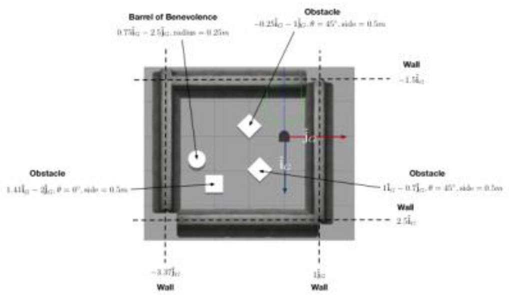


Figure 28.1: In the Gauntlet challenge you will navigate your robot from the starting point to the Barrel of Benevolence, avoiding obstacles as you go.

Exercise 28.12

The Gauntlet Challenge brings together many topics from the robotics module, so it is valuable to take a moment to map out the challenge before diving in. In particular, the Gauntlet Challenge has three levels, and levels 2 and 3 will require learning some additional material beyond what we have covered in class.

1. Read the description of The Gauntlet Challenge in Chapter 30. Pay particular attention to the three levels of the challenge, and note the differences between the levels. Resolve any questions you have with your group and/or an instructor.
2. Create a concept map of the challenge. Identify relevant math concepts, skills, tools, past assignments, scripts you've already written, etc. that are relevant to this challenge. List anything new you will need to learn (especially for challenge levels 2 and 3).
3. At a high level, outline (5-ish) major steps in solving the challenge. What issues do you anticipate at each step?
4. For each step, plan how you will verify that what you're doing is working. This plan should involve making some plots.
5. Answer this survey by the end of class or we will pair you with your nemesis.
6. If you finish early, consider looking at the option material on the RANSAC algorithm in Chapter 31.

Solution 28.1

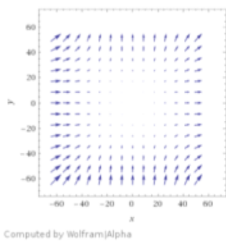
1. $T(x, y, z)$: scalar field because it takes in a vector, e.g., (x, y, z),and puts out a scalar, T, at (x, y, z).
2. $\nabla T(x, y, z)$: vector field because it takes in a vector (x, y, z) and outputs a vector, ∇T at (x, y, z).
3. $\mathbf{v}(t)$: neither because scalar in and vector out
4. $\rho(x, y)$: scalar field because vector in and scalar out
5. $\mathbf{V}(x, y)$: vector field because vector in and vector out

Solution 28.2

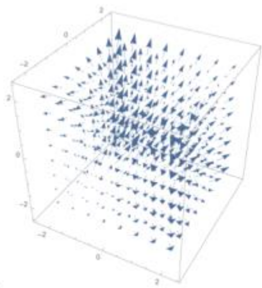
Hopefully lots of examples here!

Solution 28.3

1. The contours will be circles centered at $(1, 2)$.
2. The contours will be ellipsoids (axes 1,2,3) centered at the origin.



3.



4.

Solution 28.4

1. $J\mathbf{F} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 4 & -5 \\ 1 & 1 & -1 \end{bmatrix}$

2. $J\mathbf{F} = \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}$

Solution 28.5

1. $\operatorname{div}\mathbf{F} = -\sin(x)\sin(y) + \sin(x)\sin(y) = 0$

2. $\operatorname{div}\mathbf{F} = 0 + 0 - \sin(z) = -\sin(z)$

Solution 28.6

$$\nabla \cdot \mathbf{F} = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} \cdot \begin{bmatrix} X & Y & Z \end{bmatrix} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$$

Solution 28.7

1. $\operatorname{curl}\mathbf{F} = -2\cos(x)\cos(y)\hat{\mathbf{k}}$

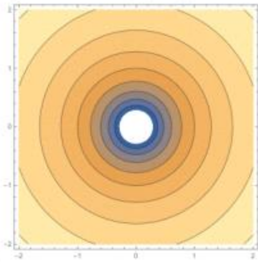
2. $\operatorname{curl}\mathbf{F} = x\cos(z)\mathbf{i} + y\cos(z)\mathbf{j} - 2\sin(z)\hat{\mathbf{k}}$

Solution 28.8

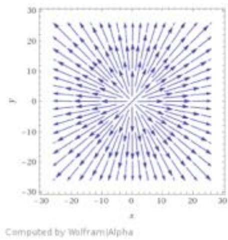
$$\nabla \times \mathbf{F} = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} \times \begin{bmatrix} X & Y & Z \end{bmatrix} = (\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z})\mathbf{i} - (\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z})\mathbf{j} + (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y})\hat{\mathbf{k}}$$

Solution 28.9

1. In polar coordinates $V(r, \theta) = \ln(r)$, so the contours of V should be circles.



2.

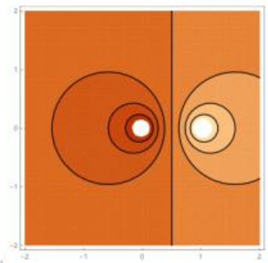


3.

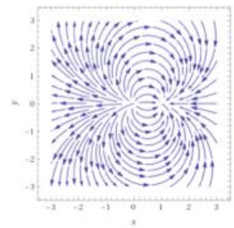
This is not actually the gradient field - we've produced instead the streamlines using the "streamplot" function in WolframAlpha. The streamlines are the path that the Neato would take if it continuously followed the direction of steepest ascent.

4. It would move toward the origin (gradient descent)!

Solution 28.10



1.

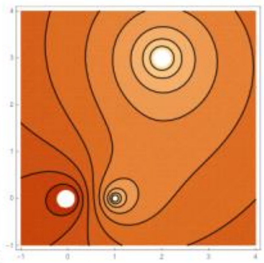


2.

3. It would move away from $(1,2)$ and eventually find its way to $(0,0)$.

Solution 28.11

1. $V(x,y) = \ln\sqrt{x^2+y^2} - \ln\sqrt{(x-1)^2+y^2} - \ln\sqrt{(x-2)^2+(y-3)^2}$



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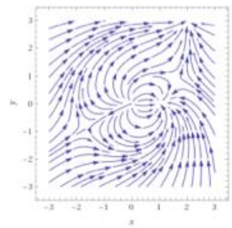


Computed by Wolfram|Alpha

3.

4. It would move away from $(1, 2)$ and eventually find its way to $(0, 0)$.

Chapter 29

Robo Homework 7: Potential Fields and The Gauntlet

Schedule

29.1 Potential Fields

This week you worked on the concepts of scalar fields and vector fields, and you met the concept of "sources" and "sinks" of potential fields. Let's start by clarifying and synthesising some of this work. Recall that a **sink** at the origin is represented by the potential field

$$V(x, y) = \ln \sqrt{x^2 + y^2}$$

which has the following gradient vector field

$$\nabla V = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j}$$

Exercise 29.1

1. What is the potential field and gradient vector field for a **source** at the origin?
2. What is the potential field and gradient vector field for a **sink** at the location (a,b).

$V(x,y) = -\ln\sqrt{x^2+y^2}$

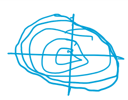
1) $\nabla V = -\frac{x}{x^2+y^2}\hat{i} - \frac{y}{x^2+y^2}\hat{j}$

2) $V(x,y) = \ln\sqrt{(x-a)^2+(y-b)^2}$

$\nabla V = \frac{x-a}{(x-a)^2+(y-b)^2}\hat{i} + \frac{y-b}{(x-a)^2+(y-b)^2}\hat{j}$

Exercise 29.2

1. Sketch the contour levels for a **sink** at the origin.



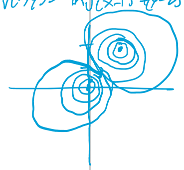
2. Read the following MATLAB script and make sure you understand the purpose of every statement. Then implement it and compare the result with your sketch.

```
[x,y]=meshgrid(-3:0.05:3,-3:0.05:3);
v = log(sqrt(x.^2+y.^2));
contour(x,y,v,'k','ShowText','On')
axis equal
```

I got it backward: contours are closer together near the origin, which makes sense

3. Write down the potential field for a **source** at $(1, 2)$.
4. Sketch the contour levels with a **sink** at the origin and a **source** at (1,2).
5. Modify the MATLAB script above to confirm your prediction.

3) $V(x,y) = -\ln\sqrt{(x-1)^2+(y-2)^2}$



Exercise 29.3

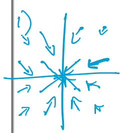
1. Sketch the gradient vector field for a **sink** at the origin.
2. Read the following MATLAB script and make sure you understand the purpose of every statement. Then implement it and compare the result with your sketch.

```
[x,y]=meshgrid(-3:0.3:3,-3:0.3:3);
fx = x./(x.^2+y.^2);
fy = y./(x.^2+y.^2);
quiver(x,y,fx,fy)
axis equal
```

Signs reversed

3. Write down the gradient vector field for a **source** at $(1, 2)$. $\nabla V = \frac{-(x-1)}{(x-1)^2+(y-2)^2}\hat{i} + \frac{-(y-2)}{(x-1)^2+(y-2)^2}\hat{j}$
4. Sketch the gradient vector field with a **sink** at the origin and a **source** at $(1, 2)$.
5. Modify the MATLAB script above to confirm your prediction.

1)



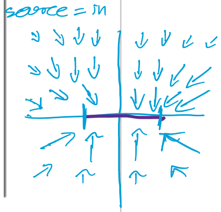
Signs
sink = out
source = in

Exercise 29.4

1. Consider a source "line" sitting on the x-axis between x=-1 and x=+1. Sketch the contours of the potential field and the gradient vector field. (Hint: Start really far away, and then move in closer to the line.)
2. Read the following MATLAB script and make sure you understand the purpose of every statement. Then implement it and compare the result with the contours on your sketch.

```
[x,y]=meshgrid(-3:0.05:3,-3:0.05:3);
v = 0;
for a = -1:0.01:1
    v = v - log(sqrt((x-a).^2 + y.^2));
```

Series of point sources along the line

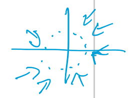


```
end
contour(x,y,v,'k','ShowText','On')
axis equal
```

3. How would you compute and visualize the gradient vector field for this line source? Go ahead and do so.
4. How would you change the source to a line from x=-2 to x=-1 at y=2? Go ahead and do so.

Exercise 29.5

1. Consider a source "circle" of radius 1 centered at the origin. Sketch the contours of the potential field and the gradient vector field (outside of the circle).



2. Read the following MATLAB script and make sure you understand the purpose of every statement. Then implement it and compare the result with the contours on your sketch.

```
[x,y]=meshgrid(-3:0.01:3,-3:0.01:3);
v = 0;
for theta = 0:0.1:2*pi
    a = cos(theta);
    b = sin(theta);
    v = v - log(sqrt((x-a).^2 + (y-b).^2));
end
contour(x,y,v,'k','ShowText','On')
axis equal
```

3. How would you compute and visualize the gradient vector field for this circle source? Go ahead and do so.
4. How would you edit the script to make the source be a circle of radius 2, centered at (-1,2)? Go ahead and do so.

Exercise 29.6

How would you define a general source (or sink) curve in the plane? Hint: use parametric equations!

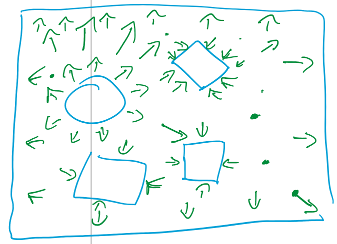
$\ln\sqrt{(f(x,y)-\text{[illegible]})^2+(f(x,y)-\text{[illegible]})^2}$

29.2 Applying these ideas to the Gauntlet - Conceptual

Exercise 29.7

Now that you've thought about potential fields for sources and sinks (including lines and circles), we'd like you to conceptually think about how this might apply to helping a NEATO navigate the Gauntlet.

1. Review the Gauntlet—it can be found in Chapter 30.
2. Thinking just about the walls, sketch the potential field and the gradient vector field if we treat the walls as line sources.
3. Thinking just about the obstacles, sketch the potential field and the gradient vector field if we treat the obstacles as being composed of line sources.
4. Thinking just about the barrel of benevolence, sketch the potential field and the gradient vector field if we treat the BoB as a circle sink.
5. Can you imagine how you would add all of the above sketches to build a sketch for the entire Gauntlet?
6. How would you use the computational tools from the earlier exercises to determine the actual potential field and gradient vector field for the Gauntlet?



29.3 Optional Extension: RANSAC

We did not have time this year to investigate the RANSAC algorithm in class, but it is needed if you want to tackle level 2 or 3 of the Gauntlet challenge. We have included material about the RANSAC algorithm in Chapter 31, and we will also be briefly discussing the algorithm during class next week. If you would like to do level 2 or 3 of the Gauntlet challenge, we recommend workign through these RANSAC exercises.

29.4 Optional Extension: the Best-Fit Circle

If you plan to identify and locate the Bucket of Benevolence in The Gauntlet challenge, you will need to be able to find and fit a circle. The steps outlined in chapter 32 will guide you through the process of creating a circle-fitting algorithm of your own design.

Solution 29.1

1. Include a negative in front to turn a sink into a source.
2. The potential field for a sink at (a,b) is

$$V(x,y) = \ln\sqrt{(x-a)^2 + (y-b)^2}$$

   which has the following gradient vector field

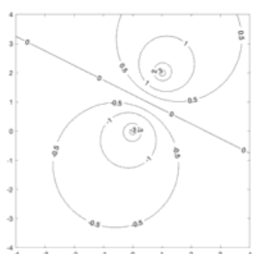
$$\nabla V = \frac{x-a}{(x-a)^2+(y-b)^2}\mathbf{i} + \frac{y-b}{(x-a)^2+(y-b)^2}\mathbf{j}$$

Solution 29.2

1. Circles surrounding the origin.
2.
```
%build a mesh
[x,y]=meshgrid(-3:0.05:3,-3:0.05:3);
%define the potential at all mesh points
v = log(sqrt(x.^2+y.^2));
%plot contours with levels marked
contour(x,y,v,'k','ShowText','On')
%so that circles look like circles
axis equal
```
3.

$$V(x,y) = -\ln\sqrt{(x-1)^2 + (y-2)^2}$$

4. Circles when close to the sink or source. As you move away there are still circles but their center shifts. There is a line of constant potential that bisects the line that connects the sink and the source.



5.

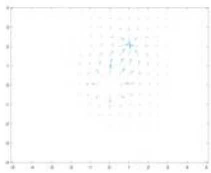
Solution 29.3

1. Arrows pointing radially outward.

2.
```
%create the mesh, less points so vectors visible
[x,y]=meshgrid(-3:0.3:3,-3:0.3:3);
%define the components of the gradient
fx = x./(x.^2+y.^2);
fy = y./(x.^2+y.^2);
%draw the arrows at the mesh points
quiver(x,y,fx,fy)
%so circles look like circles
axis equal
```

3.
$$\nabla V = -\frac{x-1}{(x-1)^2+(y-2)^2}\mathbf{i} - \frac{y-2}{(x-1)^2+(y-2)^2}$$

4. Arrows points away from the sink at the origin, and converging on the source at (1,2).

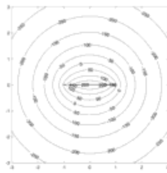


5.

Solution 29.4

1. Far away the contours look like circles and the gradient field radiates inward to the source. Close to the line the contours have to be squeezed and will look more elliptical, with the gradient field radiating inward.

2.
```
%create a mesh
[x,y]=meshgrid(-3:0.05:3,-3:0.05:3);
%initialize the potential to zero
v = 0;
%loop through source points from -1 to +1
for a = -1:0.01:1
    v = v - log(sqrt((x-a).^2 + y.^2));%add the sources
end
%draw the contours
contour(x,y,v,'k','ShowText','On')
%so circles look like circles
axis equal
```



3.

4. We would sum up the gradient field from all of the sources on the line.

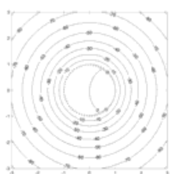
5. We would change the "for loop" as follows:

```
for a = -2:0.01:-1
    v = v - log(sqrt((x-a).^2 + (y-2).^2));
end
```

Solution 29.5

1. The contours will be circles, but they will decrease more slowly than the case of a single source at the origin. The gradient vectors will be radiating inwards.

2.
```
%create a mesh
[x,y]=meshgrid(-3:0.01:3,-3:0.01:3);
%initialize the potential to zero
v = 0;
%loop through angles from 0 to 2 pi
for theta = 0:0.1:2*pi
%sources have coordinates (a,b)
    a = cos(theta);
    b = sin(theta);
    v = v - log(sqrt((x-a).^2 + (y-b).^2));
end
%visualize the contours
contour(x,y,v,'k','ShowText','On')
%make circles look like circles
axis equal
```



3. We would sum up the gradient field from all the sources on the circle.
4. We would change the source coordinates as follows

```
a = -1+2.*cos(theta);
b = +2+2.*sin(theta);
```

Solution 29.6

How would you define a general source (or sink) curve in the plane? Hint: use parametric equations!

Solution 29.7

Chapter 30

The Gauntlet

30.1 Overview

You have mastered the Bridge of Doom and navigated Flatland. Now you face the most challenging challenge you've ever been challenged with. In this final challenge, you will help your robot to dodge through obstacles on your way to the ultimate prize – knowledge.

Throughout this semester, you have applied many quantitative engineering analysis tools. For this challenge, you'll use a powerful new sensor (the laser scanner), explore some algorithms for optimization, and use the mathematics of potential functions and vector fields.

If you'd like you can work in a group (up to 3 people total per group) to complete the challenge and the write-up.

Learning Objectives

This challenge has varying levels of difficulty. The learning objectives vary based on the level you choose.

1. Fit models of lines and circles to laser scan data (level 2 and 3).
2. Implement outlier resistant optimization via the RANSAC algorithm (level 2 and 3).
3. Create suitable potential functions and vector fields to guid a robot to a goal while avoiding obstacles (all levels)
4. Transforming between multiple coordinate systems (all levels)

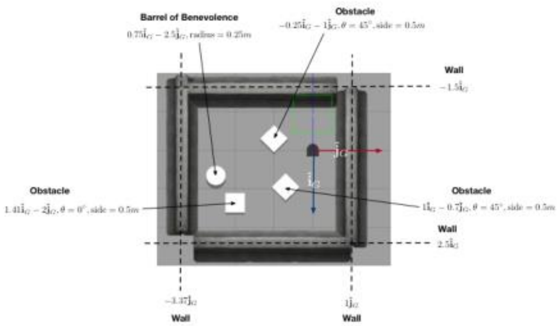


Figure 30.1: A top-down view of The Gauntlet. You should write a program to guide the Neato past the obstacles (boxes) to the goal (the Barrel of Benevolence).

30.2 The Challenge

Your goal is to write a program to pilot your (virtual) Neato through a series of obstacles (concrete barriers and boxes). You will detect these obstacles using your robot's onboard laser scanner, which can detect obstacles within a 5m range (see Figure 30.1).

This final QEA-I challenge has a hierarchy of missions, which get progressively more difficult. For all missions, the goal is to gently tap the Barrel of Benevolence (BoB). For all missions, the Neato starts at coordinate $0\ m\mathbf{i} + 0\ m\mathbf{j}$, facing along the $+\mathbf{i}$ direction.

Here are the conditions for the missions:

Level 1 You are given the coordinates of the BoB and the obstacle and wall locations (see Figure 30.1). You will use these locations to create a potential field and use gradient descent to move your robot through the Gauntlet to the BoB. The path can be entirely pre-planned.

Level 2 You tackle the same challenge as level 1, but you must use the LIDAR to detect and avoid obstacles (i.e., you should not use the obstacle and wall locations in Figure 30.1, but instead determine them from the robot's sensors). You can plan your entire path based on your initial LIDAR scan or dynamically update your path as you go. The location of the BoB is known. To tackle this level of the challenge, you will need the RANSAC material in Chapter 31.

Level 3 You tackle the same challenge as level 2, but you are given the radius of the BoB (see Figure 30.1), but not the coordinates (i.e., you must use LIDAR scans to identify and locate the BoB). You can plan your entire path based on your initial LIDAR scan or dynamically update your path as you go. To identify the BoB, you will need to differentiate the distinctive, circular shape of the goal from the linear shape of the obstacles. If you choose this mission, we recommend you look at the circle-fitting extension in Chapter 32.

To achieve whichever mission(s) you choose to accept, you will apply what you have learned about potential functions and vector fields. You may also choose to extend any of the above missions by applying or creating another goal-seeking/obstacle-avoiding algorithm, not taking the BoB's radius as an input, or trying to reach the target as soon as possible.

30.2.1 Loading the Gauntlet

To start the Gauntlet world, you can use the `qeasim` script by running the following script in MATLAB.

```
qeasim start gauntlet_final
```

30.3 Deliverables

Don't think of these deliverables as only items to check off, but rather think of them as scaffolding to get maximum learning from the task at hand. By creating these intermediate deliverables, you will be able to more easily debug your code as well as your understanding of the algorithms you are utilizing.

Note that all of these deliverables can be done with your group (if you are working with others for the challenge). Make sure to clearly specify who is in your group for the challenge on each deliverable. Unless otherwise noted, the components are required for all levels of the challenge.

Mapping and Path Planning (due 5/6, but suggested completion date is 5/3) [6 pts]: We suggest that you make significant progress on the math side of this project before class on Monday. Create a document with the following plots:

1. A map of the Gauntlet using LIDAR scan data. Please see details of this map for each level of challenge. [2 pts]
   (a) **Level 1:** The map should be an ensemble of data points from laser scans collected at various positions around the Gauntlet and translated/rotated to the global coordinate frame. The points from each scan should be shown in a different color, and all walls, obstacles, and the BoB should be captured by the scans. The plot should have appropriate labels and units.
   (b) **Level 2:** A map of the Gauntlet created using laser scan data as above, and with walls and obstacles fitted as line segments using your RANSAC algorithm.
   (c) **Level 3:** A map with fitted lines as above, with the addition of a fit for the BoB.
2. An equation and a contour (or 3-D) plot of the potential field you developed for the pen. [2 pts]
   (a) **Level 1:** The position of the walls, obstacles, and BoB can be based on the positions of the features shown in Figure 30.1.
   (b) **Level 2 and 3:** The potential field should be generated based on the locations of obstacles and the BoB identified using the laser scans and line/circle fitting algorithms.
3. A quiver plot of the gradient of your potential field. [1 pt]
4. A path of gradient descent from the starting point to the BoB. [1 pt]

These plots should be clearly labeled and readable (decent size fonts!). **You will turn these in on Canvas through the Mapping and Path Planning assignment.**

Navigating the Gauntlet (due 5/6) [8 pts]: Prepare a video and a **brief** writeup of your work on this challenge.

Your final writeup should center around critical information and informative figures. It can be as short as you want, but it must contain the following components:

1. A short introduction stating the mission you chose and a brief summary of the strategy you used to solve the challenge. You might find it helpful to refer to the decomposition exercise from Week 7. [1 pt]
2. Some experimental data that shows, quantitatively, how well your system worked, plus a few sentences of explanation. Note: you could make use of the collectDataset_sim.m function for this.

(a) The time it took your robot to get to the BoB (for whichever mission you choose) and the distance it traveled. [1 pt]

(b) A plot with: a map of the relevant features of the Gauntlet based on a LIDAR scan (levels 2 and 3) or the features given in Figure 30.1 (level 1), the intended path of gradient descent, and the path your robot took calculated from the wheel encoder data. This should be a legible plot with axis labels, a legend, and a caption...the works. [3 pts]

3. A link to a video of your robot in action. [2 pts]

In addition to the writeup, you should also turn in your (commented and readable!) code. [1 pt]