Homework F3

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document

Chapter 32

Homework 3: Derivatives, Integrals and Multiple Integrals

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In class, we introduced a few properties of differentiation and integration. Here, we introduce one more integration technique, followed by learning about computing areas between curves. All of these concepts use single-variable calculus. We then transition to problems involving multiple integration starting from Section 32.2.

$\int u\,dv = uv - \int v\,du$

Integration by Parts

$$\int_a^b f(x)g'(x)\,dx \;=\; f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx$$

Exercise 32.1

Use your table of fundamental functions and integration by parts to determine $\int_1^2 x\exp(-x)\,dx$. Verify your answer using **WolframAlpha**.

$\int_1^2 xe^{-x}dx = -xe^{-x}|_1^2 + \int_1^2 e^{-x}dx = -xe^{-x}|_1^2 - e^{-x}|_1^2 = -(2)e^{-2} - e^{-2} - (-1e^{-1} - e^{-1}) = -3e^{-2} + 2e^{-1}$ ✓ ≈ 0.33

General Practice

Exercise 32.2

Find the derivative and integral of the following functions, and verify your answers using **WolframAlpha**. Assume the following are constant values: $A, k, m, a, b, n, \omega, \phi, g, h$.

1. $F(t) = \frac{A}{k}e^{kt} + C$ $\quad f(t) = Ae^{kt}$ $\quad f'(t) = Ake^{kt}$

2. $F(t) = \frac{m}{2}x^2 + bx + c$ $\quad f(x) = mx + b \quad$ $f'(x) = m$
3. $F(t) = \frac{a}{n+1}x^{n+1} + c$ $\quad f(x) = ax^n + b \quad$ $f'(x) = anx^{n-1}$
4. $F(t) = \frac{-A}{w}\cos(wt+\phi) + c$ $\quad f(t) = A\sin(\omega t + \phi) \quad$ $f'(x) = A\cos(wt+\phi)(w)$
5. $F(t) = \frac{g}{3}(x-h)^3 + c$ $\quad f(x) = g(x-h)^2 + k \quad$ $f'(x) = 2g(x-h)$

32.1 Areas enclosed by curves

Additional resources for this section:

- Khan Academy videos and practice problems

Single-variable calculus gives us the tools to compute the area of regions bounded by curves. Consider the region bounded on top by $y = f(x)$, on the bottom by $y = g(x)$, and on the sides by $x = a$ and $x = b$. Appealing to the properties of integrals, the area of this region is

$$\int_a^b (f(x) - g(x))\, dx$$

We should note that integration will return a signed area. For example, the integral will be negative if the value of the function g is greater than that of f.

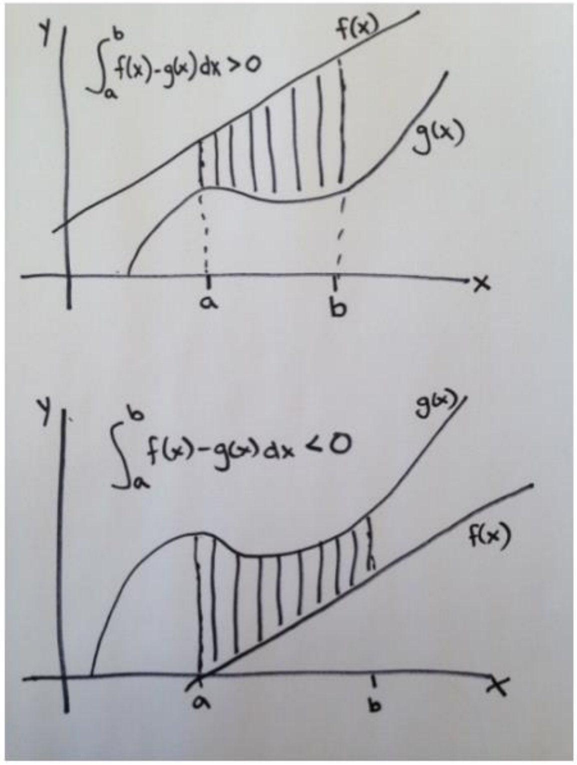


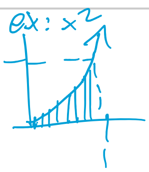
Figure 32.1: The area defined by integration can be positive or negative.

Exercise 32.3

Consider the first four fundamental functions x^n, $\sin(x)$, $\cos(x)$, and $\exp(x)$. For each function, sketch the region which is bounded above by the function, below by the x-axis and between $x = 0$ and $x = 1$. Use an integral to find the area of the region, and use **WolframAlpha** to verify your calculations. To visualize the regions you could type the following in **WolframAlpha**

```
plot 0 < y < x^2 and 0 < x < 1
```

x^n) ex: x^2



$\int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \boxed{\frac{1}{3}}$

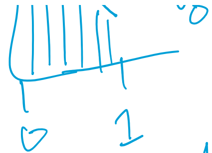
sin(x))

$\int_0^1 \sin x\,dx = -\cos x\Big|_0^1 = -\cos(1)+\cos(0)$
≈ 0.46

cos(x))

$\int_0^1 \cos x\,dx = \sin x\Big|_0^1 = \sin(1)-\sin(0)$
≈ 0.84

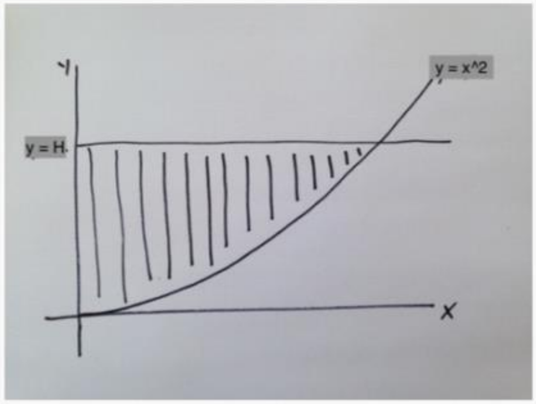
≈ 0.81



$$\int_0^1 e^x\,dx = e^1 - e^0 \approx 1.7$$

Exercise 32.4

Consider the parabola defined by $y = x^2$. Propose an integral that would determine the area enclosed on the top by $y = H, H > 0$, on the bottom by $y = x^2$, on the left by $x = 0$, and on the right by the intersection of the top and bottom functions. Evaluate it by hand and verify the result using **WolframAlpha**.



end bound:

$H = x^2$

$x = \sqrt{H}$ ← we care abt POS

$\int_0^{\sqrt{H}} (H - x^2)\, dx$

$\int_0^{\sqrt{H}} H\, dx - \int_0^{\sqrt{H}} x^2\, dx$

$Hx - \frac{1}{3}x^3 \Big|_0^{\sqrt{H}}$

$H\sqrt{H} - \frac{1}{3}H\sqrt{H}$

$\boxed{\frac{2}{3}H\sqrt{H}}$ ✓

32.2 Volumes enclosed by surfaces defined explicitly

Additional resources for this section:

- Khan Academy: Volume with cross sections – video series

Consider the volume enclosed by the surfaces defined by $z = f(x, y)$, $z = 0$, $x = a$, $x = b$, $y = c$, and $y = d$. How would we compute the volume of this region? One option would be to slice the surface up in sections parallel to one of the coordinate planes. For example, if we make a slice at $x = x_1$, then each planar region is bounded by $z = f(x_1, y)$, $y = c$, and $y = d$. The area of the enclosed region is therefore

$$\int_c^d f(x_1, y)\, dy$$

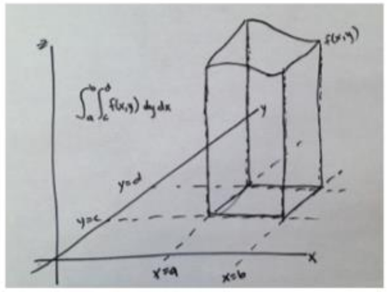


Figure 32.2: Volume defined as a double integral over a rectangle in the plane.

If we repeat this for different values of x, then we could compute the area of each cross-section as a function of x

$$area(x) = \int_c^d f(x, y)\, dy$$

What happens if we now integrate the area function from $x = a$ to $x = b$? We should get a volume, and it should be the volume of the original enclosed region,

$$Volume = \int_a^b area(x)\, dx = \int_a^b \int_c^d f(x, y)\, dydx$$

As we saw earlier, it shouldn't matter whether we change the order of integration.

Notice that the region of integration is the rectangle in the plane defined by $(x, y) \in [a, b] \times [c, d]$. There is no reason that the integral can't be computed over more general regions. In general we will use the coordinate-free notation

$$\int_D f\, dA$$

to define the integral of a function over a general region D in the plane.

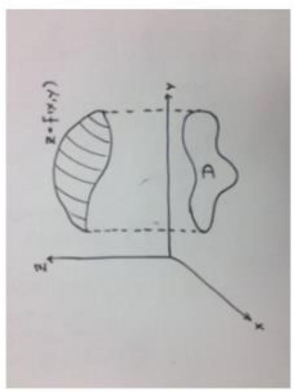


Figure 32.3: Volume defined as a double integral over a general region in the plane.

For example, consider the following integral

$$\int_D \frac{4y}{x^3+2}\,dA, \quad D = \{(x,y) | 1 \le x \le 2, 0 \le y \le 2x\}$$

To sketch this region we draw a line at $y = 0$ and a line defined by $y = 2x$. We add lines at $x = 1$ and $x = 2$. The region of integration is enclosed by these definitions. We can evaluate this integral using **WolframAlpha** by issuing the following request

```
integral of 4y/(x^3+2) from y = 0 to y = 2x and x = 1 to x = 2
```

and we find that the result is approximately 3.21059.

Exercise 32.5

Sketch the following region of integration in the plane, and evaluate the integral using **WolframAlpha**

$$\int_D x \cos y \, dA,$$

where D is bounded by $y = 0, y = x^2, x = 0, x = 1$.

$$\int_0^1 \int_0^{x^2} x\cos y \, dy\, dx$$

$$\int_0^1 x(\sin x^2 - \cancel{\sin 0})\,dx$$

$$\int_0^1 x \sin x^2 \,dx = -\frac{\cos x^2}{2}\Big|_0^1$$

$$= \frac{\cos(1)}{2} + \frac{\cos 0}{2}$$

$$= \sim 0.77$$

!!?

usub!

32.3 Applications of Double Integrals

So far we have been thinking exclusively in terms of geometry. In the same way that single integrals are used widely to compute quantities that are not areas per se, we can use double integrals to compute physically-relevant quantities like mass, center of mass, etc.

As an example, consider a thin plate (thickness $H cm$) in 2D with variable mass density $\rho gm/cm^3$, i.e. the plate could be made of different material with a mass density that varies from location to location. The total mass M of the plate is represented by a double integral of the mass density over the plate. In coordinate-free notation, we can write

is this supposed to be "grams per cubic cm?"

$$M = H \iint_D \rho \, dA$$

where D is the region in the plane occupied by the plate, and we evaluate the double integral depending on how we describe the plate. Likewise, the center of mass can be expressed as a double integral, and the relevant expressions are

$$x_{com} = \frac{H}{M} \iint_D x\rho \, dA$$

$$y_{com} = \frac{H}{M} \iint_D y\rho \, dA$$

This seems dumb if uniform density

Exercise 32.6

Find the total mass and center of mass of the 1 cm thin aluminum plate bounded by the parabola $y = x^2$, $y = 10$, and $x = 0$. Assume x and y are measured in centimeters. Use **WolframAlpha** to confirm your answer.

Mass $D = y = x^2 \; y = 10 \; x = 0 \; x = \sqrt{10}$ ← I see it

$$M = H \iint_D \rho \, dA = H \int_0^{\sqrt{10}} \int_{x^2}^{10} \rho \, dy\, dx$$

$$= H \int_0^{\sqrt{10}} \rho y \Big|_{x^2}^{10} dx$$

$$= H \int_0^{\sqrt{10}} (10\rho - \rho x^2)\, dx$$

$$= H\rho \int_0^{\sqrt{10}} (10 - x^2)\, dx$$

$$= H\rho \left[\int_0^{\sqrt{10}} 10\, dx - \int_0^{\sqrt{10}} x^2\, dx\right]$$

$$= H\rho\left(\int_0^{\sqrt{10}} 10\,dx - \int_0^{\sqrt{10}} x^2\,dx\right)$$

$$= H\rho\left(10x - \tfrac{1}{3}x^3 \Big|_0^{\sqrt{10}}\right)$$

$$= H\rho\left(10\sqrt{10} - \tfrac{1}{3}10\sqrt{10}\right)$$

$$= H\rho(10\sqrt{10})\left(\tfrac{2}{3}\right)$$

$$M = H\rho\,\tfrac{20}{3}\sqrt{10}$$

COM

let $u = x$ or y

$$u_{com} = \frac{H}{M}\iint_D u\rho\,dA = \frac{H}{M}\int_0^{\sqrt{10}}\int_{x^2}^{10} u\rho\,dy\,dx$$

Nope! doesn't work! B/C x/y need to be integrated

$$= \frac{H}{M}\int_0^{\sqrt{10}}\left(u\rho y\Big|_{x^2}^{10}\right)dx$$

$$= \frac{H}{M}\int_0^{\sqrt{10}}\left(10u\rho - u\rho x^2\right)dx$$

$$= \frac{Hu\rho}{M}\int_0^{\sqrt{10}}(10 - x^2)\,dx$$

$$= \frac{Hu\rho}{M}\int_0^{\sqrt{10}} 10 - \int_0^{\sqrt{10}} x^2\,dx$$

$$= \frac{Hu\rho}{M}\left(10x - \tfrac{1}{3}x^3\Big|_0^{\sqrt{10}}\right)$$

$$= \frac{Hu\rho}{M}\left(10\sqrt{10} - \tfrac{1}{3}10\sqrt{10}\right)$$

$$= \frac{Hu\rho}{M}(10\sqrt{10})\left(\tfrac{2}{3}\right)$$

$$= u\,\frac{H\rho}{M}\,\frac{20\sqrt{10}}{3}$$

$$= u\,H\rho\,3 \quad 20\sqrt{10}$$

$$= U \frac{\cancel{HP} \cdot \cancel{3}}{\cancel{HP} \cdot 2\cancel{9010}} \cdot \frac{2\cancel{9010}}{3}$$

$$U_{com} = U\ 21$$

32.4 3D Physical Simulations using Matlab

In this exercise you will be installing some software that will allow you to run 3D physical simulations through MATLAB. These simulations will allow you to explore some of the key ideas in the boats module in an interactive fashion. Additionally, we will be using the same setup when we come to the robot module in QEA2 next semester. In this exercise all we want you to do is attempt to get the software installed and running on your computer. If you run into any issues, please make sure to send an e-mail to paul.ruvolo@olin.edu describing the problem you are facing (Paul is the instructor in charge of helping troubleshoot the software). If you are unable to get the software fully setup by the end of this assignment, that is totally fine. We will work with you to make sure you get things up and running.

We realize there is not a lot of motivation for the steps you need to do to install the software. There's actually a lot of really cool technology under the hood, but we didn't want to burden the class with a bunch of unnecessary information. If you are interested in learning more, e-mail paul.ruvolo@olin.edu, and if enough people are interested I'll make a video explaining the setup in more depth.

In order to get the software setup, you should go through the instructions in the Meeto Your Neato document. Specifically, you should go through the following sections of that document.

- Purpose of this how-to
- Docker Setup
- Downloading Required MATLAB Toolboxes
- Connecting to the Simulated Robot

Solution 32.1

To use the formula above for integration by parts, let $f(x) = x$ and $g(x) = -\exp(-x)$. Then

$$\begin{aligned}
\int_1^2 x\exp(-x)\,dx &= \int_1^2 f(x)g'(x)\,dx \\
&= f(x)g(x)|_1^2 - \int_1^2 g(x)f'(x)\,dx \\
&= -x\exp(-x)|_1^2 - \int_1^2 -\exp(-x)\,dx \\
&= -x\exp(-x)|_1^2 - \exp(-x)|_1^2 \\
&= -2\exp(-2) + \exp(-1) - \exp(2) + \exp(-1) \\
&= -3\exp(-2) + 2\exp(-1)
\end{aligned}$$

Solution 32.2

f	f'	$\int f$
Ae^{kt}	Ake^{kt}	$\frac{A}{k}e^{kt}$
$mx+b$	m	$\frac{1}{2}mx^2 + bx$
$ax^n + b$	anx^{n-1}	$\frac{a}{n+1}x^{n+1} + bx$
$A\sin(\omega t + \phi)$	$A\omega\cos(\omega t + \phi)$	$-\frac{A}{\omega}\cos(\omega t + \phi)$
$g(x-h)^2 + k$	$2g(x-h)$	$\frac{1}{3}g(x-h)^3 + kx$

Solution 32.3

$$\int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}|_0^1 = \frac{1}{n+1}$$

$$\int_0^1 \sin x\,dx = -\cos x|_0^1 = 1 - \cos 1$$

$$\int_0^1 \cos x\,dx = \sin x|_0^1 = \sin 1$$

$$\int_0^1 \exp x\,dx = \exp x|_0^1 = \exp(1) - 1$$

Solution 32.4

The top function is $f(x) = H$ and the bottom function is $g(x) = x^2$. The left limit is $x = 0$ and the right limit is $x = \sqrt{H}$, since this is where the top and bottom functions meet. To find the area of the region described in the problem we need to solve the integral

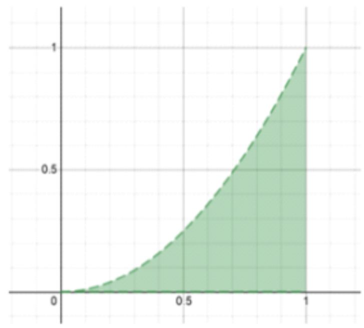
$$\int_0^{\sqrt{H}} (H - x^2)\,dx.$$

This gives

$$\begin{aligned}
\int_0^{\sqrt{H}} (H - x^2)\,dx &= (Hx - \frac{1}{3}x^3)|_0^{\sqrt{H}} \\
&= H^{\frac{3}{2}} - \frac{H^{\frac{3}{2}}}{3} \\
&= \frac{2}{3}H^{\frac{3}{2}}
\end{aligned}$$

Solution 32.5

The region of integration is



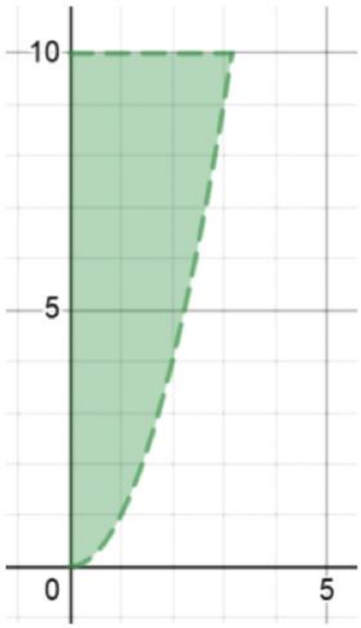
and so the integral will be

$$\int_0^1 \int_0^{x^2} x \cos y \; dy \; dx$$

which we evaluate

$$\begin{aligned}
\int_0^1 \int_0^{x^2} x \cos y \; dy \; dx &= \int_0^1 x \sin y|_0^{x^2} \; dx \\
&= \int_0^1 x \sin(x^2) \; dx \\
&= \frac{1}{2} \int_0^1 \sin(u) \; du \\
&= \frac{1}{2}(-\cos u)|_0^1 \\
&= \frac{1}{2}(1 - \cos(1))
\end{aligned}$$

Solution 32.6



To find the total mass we use the expression from above with $H = 1cm$

$$
\begin{aligned}
M &= \int_0^{10}\int_0^{\sqrt{y}} \rho \, dx \, dy \\
&= \rho \int_0^{10} \sqrt{y} \, dy \\
&= \rho \frac{y^{3/2}}{3/2}|_0^{10} \\
&= \rho \frac{2}{3} 10^{3/2} gm
\end{aligned}
$$

where ρ is the density of Aluminum in gm/cm^3 (about 2.7).

To find the x center of mass we compute

$$\begin{aligned}
x_{com} &= \frac{3}{\rho 2(10^{3/2})}\int_0^{10}\int_0^{\sqrt{y}} \rho x\, dx\, dy \\
&= \frac{3}{\rho 2(10^{3/2})}\rho\int_0^{10} \frac{x^2}{2}|_0^{\sqrt{y}}\, dy \\
&= \frac{3}{2(10^{3/2})}\int_0^{10} \frac{y}{2}\, dy \\
&= \frac{3}{2(10^{3/2})}\left(\frac{y^2}{4}|_0^{10}\right) \\
&= \frac{3}{2(10^{3/2})}\left(\frac{10^2}{4}\right) \\
&= \frac{3}{8}10^{1/2}cm \\
&\approx 1.186cm
\end{aligned}$$

Notice that ρ cancels out, assuming it is uniform. In other words, the center of mass for an object of uniform density only depends on its geometry!
To find the y center of mass we compute

$$\begin{aligned}
y_{com} &= \frac{3}{\rho 2(10^{3/2})}\int_0^{10}\int_0^{\sqrt{y}} \rho y\, dx\, dy \\
&= \frac{3}{\rho 2(10^{3/2})}\rho\int_0^{10} xy|_{x=0}^{x=\sqrt{y}}\, dy \\
&= \frac{3}{2(10^{3/2})}\int_0^{10} y^{3/2}\, dy \\
&= \frac{3}{2(10^{3/2})}\left(\frac{y^{5/2}}{5/2}|_0^{10}\right) \\
&= \frac{3}{2(10^{3/2})}\left(\frac{10^{5/2}}{5/2}\right) \\
&= 6cm
\end{aligned}$$