Week 4a

Tuesday, September 29, 2020 10:27 AM

10.2) $M = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

1) $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 33 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = M \begin{bmatrix} 13 \\ 33 \end{bmatrix}$

$\underbrace{M^{-1}}_{L} U x = b$

$LUx = b$

$LU = A$

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ ✓

10.4)

$\text{inv}\left(\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\right)$:

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$y_1 = 1 \quad y_2 = -2$

$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$x_1 = 3/2 \quad x_2 = -2$

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$y_1 = 0 \quad y_2 = 1$

$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x_1 = -1/2 \quad x_2 = 1$

$\text{inv} = \begin{bmatrix} 3/2 & -1/2 \\ -2 & 1 \end{bmatrix}$

10.5)

(e.5)

$$a(C_3H_8) + b(O_2) \to c(CO_2) + d(H_2O)$$

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} C \\ H \\ O \end{array} & \begin{bmatrix} 3 & 0 \\ 8 & 0 \\ 6 & 2 \end{bmatrix} \end{array} x = \begin{array}{cc} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \begin{array}{c} C \\ H \\ O \end{array} \end{array}$$

$$\begin{array}{cc} \begin{array}{cc} c & d \end{array} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} & \begin{array}{c} C \\ H \\ O \end{array} \end{array}$$

C: $3a = c$

H: $8a = 2d$

O: $2b = 2c + d$

$3a - c = 0$

$$8a - 2d = 0$$

$$2b - 2c - d = 0$$

$$\begin{array}{cccc} a & b & c & d \end{array}$$

$$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \\ 11 & 2 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Chapter 10

Week 4a: Linear Systems of Algebraic Equations

Schedule

10.1 Debrief [15 mins]

- Please discuss your homework with your breakout room, and resolve any issues with your peers and/or an instructor.

10.2 Synthesis [55 mins]

We will increasingly use a computational tool like MATLAB to compute determinants, matrix inverses, and the solutions to linear systems of algebraic equations. In this synthesis section we will explore the theoretical foundation of these algorithms - the so-called LU decomposition.

Gaussian Elimination

The basic process of *elimination of variables* can be formalized and is known as *Gaussian Elimination*. Here will briefly introduce it but you can consult other sources on the internet.

Rather than writing equations, we can cast a linear system of algebraic equations in matrix form and perform *Gaussian Elimination* on the augmented matrix $[\mathbf{A}\ \mathbf{b}]$.

For example, the linear systems of algebraic equations

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ 4x_1 + 9x_2 &= 15 \end{aligned}$$

can be written as the following augmented matrix

$$\begin{bmatrix} 2 & 3 & 6 \\ 4 & 9 & 15 \end{bmatrix}$$

Thinking now in terms of rows, we replace the second row with (row 2) − (2 × (row 1)) to give

$$\begin{bmatrix} 2 & 3 & 6 \\ 0 & 3 & 3 \end{bmatrix}$$

This matrix is now in so-called *echelon* form: we can find the solution to the original linear system of algebraic equations by first solving the equation implied by the last row and then back-substituting into the equation implied by the previous row. The equation corresponding to the second row is

$$3x_2 = 3$$

which has solution $x_2 = 1$. Replacing into the equation corresponding to the first row we find

$$2x_1 + 3 = 6$$

which has solution $x_1 = 3/2$.

Exercise 10.1

$R_2 - 2R_1$

1. Set up the augmented matrix for the following example (you will recognise this from the last assignment)

$$\begin{aligned} 2x_1 + x_2 &= 13 \\ 4x_1 + 3x_2 &= 33 \end{aligned} \quad \begin{bmatrix} 2 & 1 & 13 \\ 4 & 3 & 33 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 13 \\ 0 & 1 & 7 \end{bmatrix}$$

3

$x_1 =$

$x_2 = 7$

and perform *Gaussian Elimination* to reduce the augmented matrix to *echelon form*. Interpret the resulting system and determine the solution(s).

LU Decomposition

The steps used to solve a linear system of algebraic equations using Gaussian Elimination can also be used to *decompose* a matrix into a product of two matrices: a *lower-triangular* matrix $\mathbf{L}$ and an *upper-triangular* matrix $\mathbf{U}$. Here we will briefly introduce it but you could consult other sources on the internet.

In Gaussian Elimination we execute a set of row operations. In our ongoing example, we replaced row 2 with the result of row 2 - 2 × row 1. This action can be neatly represented in terms of a matrix operation. Let's multiply the original matrix equation $\mathbf{Ax} = \mathbf{b}$ with the transformation matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

to form $\mathbf{MAx} = \mathbf{Mb}$. Note that this transformation leaves row 1 of $\mathbf{A}$ unchanged, and it replaces the row 2 with row 2 - 2 × row 1. The product $\mathbf{MA}$ is an *upper-triangular* matrix $\mathbf{U}$

$$\mathbf{U} = \begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}$$

and the linear system of algebraic equations is now expressed as $\mathbf{Ux} = \mathbf{Mb}$. If we now multiply this expression by $\mathbf{M}^{-1}$ we obtain

$$\mathbf{M}^{-1}\mathbf{Ux} = \mathbf{b}$$

The inverse of $\mathbf{M}$ is straight-forward to write down because it "undoes" the row operations

$$\mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Notice that this matrix is just a *lower-triangular* matrix $\mathbf{L}$. The linear system of algebraic equations now reads

$$\mathbf{LUx} = \mathbf{b}$$

We have therefore *decomposed* the original matrix $\mathbf{A}$ into the product of $\mathbf{L}$ and $\mathbf{U}$,

$$\mathbf{A} = \mathbf{LU}$$

How does this help, you might be asking? First of all, knowing the decomposition of $\mathbf{A}$ into $\mathbf{LU}$ allows us to solve the original linear system of algebraic equations $\mathbf{Ax} = \mathbf{b}$. Here is how.

Let's define a new vector $\mathbf{y} = \mathbf{Ux}$. Then the original lienar system of algebraic equations can be expressed as

$$\mathbf{Ly} = \mathbf{b}$$

which is straight-forward to solve by *forward-substitution* because $\mathbf{L}$ is *lower-triangular*,

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \end{bmatrix}$$

and the solution for $\mathbf{y}$ is $y_1 = 6$, $y_2 = 3$. We can now solve $\mathbf{Ux} = \mathbf{y}$ for $\mathbf{x}$ using *backward-substitution* because $\mathbf{U}$ is *upper-triangular*,

$$\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

and the solution for $\mathbf{x}$ is $x_1 = 1$, $x_2 = 3/2$.

Second of all, and more importantly, knowing the decomposition of $\mathbf{A}$ into $\mathbf{LU}$ allows us to solve any linear system of algebraic equations involving $\mathbf{A}$. Need to solve the linear system of algebraic equations with a different $\mathbf{b}$? No problem, just use the $\mathbf{LU}$ decomposition that you already computed and away you go. No need to redo all the steps of *Gaussian Elimination* just because $\mathbf{b}$ changed. Need to solve a linear system of algebraic equations for lots of different $\mathbf{b}$'s? No problem, just use the $\mathbf{LU}$ decomposition that you already computed and away you go. Finally, if you want to compute the inverse or determinant of a matrix this is easy too using LU decomposition as we show next.

There is an algorithm in MATLAB, *lu*, which does LU decomposition for you, but you should not necessarily expect to get the same $\mathbf{L}$ and $\mathbf{U}$, even for this example. (There are a variety of ways to define the $\mathbf{L}$ and $\mathbf{U}$ matrices, but this is beyond the scope of this section.)

Exercise 10.2

1. Consider the appropriate matrix from the last exercise and perform *LU Decomposition*. Check your answer by confirming that $\mathbf{A} = \mathbf{LU}$. (Please note that you perform LU decomposition on the original matrix $\mathbf{A}$, not the augmented matrix.)

$A = \Big[\begin{matrix} 2 \\ 4 \end{matrix}$

$M = \Big[\begin{matrix} 1 \\ -2 \end{matrix}$

Determinant

The basic algorithm for computing a determinant of $\mathbf{A}$ is to first perform LU decomposition, and make use of the following property:

> The determinant of an upper-triangular or lower-triangular matrix is just the product of the diagonal entries.

$U = \Big[\begin{matrix} 2 \\ 0 \end{matrix}$

We already met another property of determinants, namely that the determinant of a product is just the product of the determinants. Therefore, $det(\mathbf{A}) = det(\mathbf{L})det(\mathbf{U})$, each of which is just the product of the diagonal entries.

$M^{-1} = \Big[\begin{matrix} 1 \\ 2 \end{matrix}$

Exercise 10.3

1. Consider the appropriate matrix from the last exercise and find the determinant using the LU

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \times \Big[$

et(L) × det(U) = 1 × 2 = 2

decomposition previously determined. Check your answer using *det* in MATLAB.

Inverse

The basic algorithm for computing the inverse of $\mathbf{A}$ is to first perform LU decomposition, and make use of the following idea. $\mathbf{B}$ is the inverse of $\mathbf{A}$ if it satisfies the following property

$$\mathbf{AB} = \mathbf{I}$$

The columns of $\mathbf{B}$ are just the solutions of a linear system of algebraic equations with a different $\mathbf{b}$. For example, in the two by two case we can solve

$$\mathbf{Ax} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and then

$$\mathbf{Ax} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and if we fill the columns of $\mathbf{B}$ with the solution to these linear system of algebraic equations we will have constructed the inverse. Since we already have the LU decomposition of $\mathbf{A}$ we simply solve each case using the technique already presented.

For example, the first column of $\mathbf{B}$ is determined as follows: First we solve $\mathbf{Ly} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

to give $y_1 = 1$ and $y_2 = -2$. Now we solve $\mathbf{Ux} = \mathbf{y}$

$$\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

and the solution for $\mathbf{x}$ is $x_1 = 3/2$, $x_2 = -2/3$. This is the entries in the first column of the inverse. Repeating this process for $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ will give the second column of the inverse which now reads

$$\mathbf{A}^{-1} = \begin{bmatrix} 3/2 & -1/2 \\ -2/3 & 1/3 \end{bmatrix}$$

Exercise 10.4

1. Consider the appropriate matrix from the previous exercise and find the inverse using the LU decomposition previously determined. Check your answer using *inv* in MATLAB.

10.3 Applications of Linear Systems of Algebraic Equations [20 mins]

Chemical Analysis

Exercise 10.5

The complete combustion of propane, C_3H_8, with oxygen, O_2 yields carbon dioxide, CO_2, and water, H_2O. Based on conservation of mass, this reaction can be written as

$$a(C_3H_8) + b(O_2) \rightarrow c(CO_2) + d(H_2O)$$

Determine the coefficients in the combustion equation. Note that you will need to learn how to "balance" a chemical reaction.

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}$$

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