Week 2a

Wednesday, September 30, 2020 11:36 PM

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Chapter 4

Week 2a: Matrix Transformations

Schedule

4.1 Debrief [15 minutes]

Exercise 4.1

1. In your breakout room, identify a list of key concepts/take home messages/things you learned in the assignment. Try to group them in categories like "Concepts", "Technical Details", "Matlab", etc.

2. Try to resolve your confusions with your breakout room-mates and by talking to an instructor.

4.2 Synthesis [30 minutes]

Exercise 4.2

These are fundamental ideas about matrices and it is important to complete these.

1. What is the difference between a scalar, a vector, a matrix, and an array?

2. What are the rules for adding matrices?

3. When can two matrices be multiplied, and what is the size of the output? $a \times \underbrace{b \quad b}\times c \rightarrow a \times c$

4. What is the distributive property for matrix multiplication? $A(B+C) = AB + A$

5. What is the associative property for matrix multiplication? $ABC = (AB)C = A(BC)$

6. What is the commutative property for matrix multiplication? not a thing!

$+b^2$

Exercise 4.3

These are synthesis problems. It would be helpful to complete these.

1. Use the distribution law to expand $(\mathbf{A}+\mathbf{B})^2$ assuming that $\mathbf{A}$ and $\mathbf{B}$ are matrices of appropriate size. How does this compare to the situation for real numbers? $A^2+\underbrace{AB+BA}_{\text{can't}}+B^2$
2. Show that $\mathbf{D} = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$ satisfies the matrix equation $\mathbf{D}^2 - \mathbf{D} - 6\mathbf{I} = \mathbf{0}$. ne
3. Let $\mathbf{A}$ be a square matrix. Show that $\mathbf{A}^2$ commutes with $\mathbf{A}$.

Exercise 4.4

These are challenge problems. Pick one of them to wrestle with. It is not important to complete these.

1. The matrix exponential is defined by the power series
$$\exp \mathbf{A} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$
Assume $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Find a formula for $\exp \mathbf{A}$.
2. The real number 0 has just one square root: 0. Show, however, that the 2×2 zero matrix has infinitely many square roots by finding all 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{0}$.
3. Use induction to prove that $\mathbf{A}^n$ commutes with $\mathbf{A}$ for any square matrix $\mathbf{A}$ and positive integer n.

P
A^n
$A^{n+1} =$
$A \cdot A^n = A \cdot A$ ✓

4.3 2D Rotation Matrices [45 minutes]

We're going to think about how to use rotation matrices to rotate a geometrical object. In doing so we will solidify fundamental concepts around matrix multiplication and start to explore the notion of "inverse". For clarity we will first work in 2D. Recall that the rotation matrix $\mathbf{R}(\theta)$:

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

will rotate an object counterclockwise **about the origin** through an angle of θ.

Exercise 4.5

This is a hands-on, conceptual problem involving the multiplication of 2D rotation matrices.

1. Place an object on your table, and imagine that the origin of an xy-coordinate system is at the center of your object with $+z$ pointing upwards.
2. Rotate it counterclockwise by 30 degrees. How would you undo this rotation?
3. Starting again, rotate it counterclockwise by 30 degrees, and then again by another 60 degrees. What is it's orientation now? How would you get there in one rotation instead? What does this suggest about the multiplication of rotation matrices?
4. What happens if you first rotate it by 60 degrees, and then by 30 degrees? What does this suggest about the commutative property of 2D rotation matrices?

↳ Rotation is cumulative & commutative

Exercise 4.6

This is an algebra problem involving the multiplication of 2D rotation matrices.

1. Use some algebra to show that 2D rotation matrices commute, i.e. $\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_2)\mathbf{R}(\theta_1)$.
2. Use some algebra to show that $\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2)$. You will need to look up some trig identities.

Exercise 4.7

Now, consider a rectangle of width 2 and height 4, centered at the origin. For clarity, this means that the corners of the rectangle have coordinates $(1, 2)$, $(-1, 2)$, $(-1, -2)$, and $(1, -2)$.

1. Plot these four points by hand and connect them with lines to complete the rectangle.
2. Now, using the appropriate rotation matrix, transform each of the corner points by a rotation through 30 degrees counterclockwise (recall that the sin and cos of 30 degrees can be expressed exactly). Compute and plot the resulting points by hand and connect them with lines. Does the resulting figure look like you'd expect?

Exercise 4.8

Now, let's do it in MATLAB.

1. Create and plot the original 4 points:$(1, 2)$, $(-1, 2)$, $(-1, -2)$, and $(1, -2)$. Then create the matrix that rotates them by 30 degrees counterclockwise, transform each of the four original points using the rotation matrix, and plot the resulting points. Does this look right? *Reminder:* `plot(1,2,'x')` *puts a mark at the point (1,2). Matlab: the functions cos and sin expect radians, while cosd and sind expect degrees.*

2. Operating on individual points with the rotation matrix is cool, but we can be much more efficient by operating on all 4 points at the same time. Write down the matrix whose columns represent the four corners of the rectangle. Then write down the matrix multiplication problem we can solve to transform the rectangle from above all at once. Create these matrices in MATLAB to perform the rotation in a single operation. Plot the resulting matrix to confirm your transformation! *Some MATLAB tips:* `plot(X,Y)` *creates a line plot of the values in the vector Y versus those in the vector X. So if you wanted to plot a line from the origin (0,0) to the point (1,2), you would do this: plot([0 1],[0 2]).* The command `axis([-xlim xlim -ylim ylim])` sets the axes of the current plot to run from `-xlim` to `xlim` and from `-ylim` to `ylim`

3. What is the area of the rectangle before and after the rotation?

4. What matrix should you use to undo this rotation? Define it in MATLAB and check.

5. Show that the product of this matrix (which undoes the rotation) with the original rotation matrix is the identity matrix. For clarity, let's give this matrix the symbol $\mathbf{R}^{-1}$. It is the matrix that undoes or inverts the original operation and is known as the *inverse* of the matrix $\mathbf{R}$.

Solution 4.2

1. Scalars, vectors, and matrices are examples of arrays. A 0-dimensional array can be thought of as a scalar. A 1-dimensional array is a vector. A 2-dimensional array is a matrix.
2. The matrices have to be the same size and addition is element-wise.
3. The matrices have to compatible (inner dimensions agree), and the output is dictated by the outer dimensions, i.e. $(n \times m)(r \times s) = (n \times s)$.
4. Distributive property: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
5. Associative property: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
6. Commutative property: Two matrices commute if $\mathbf{AB} = \mathbf{BA}$ but this is not always true.

Solution 4.3

1. Using the distributive property you can see that $(\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$. In general $\mathbf{AB} \neq \mathbf{BA}$ so no further simplification is possible. Since real numbers always commute the result is the more familiar $(x+y)^2 = x^2 + 2xy + y^2$.
2. If you plug $\mathbf{D}$ and $\mathbf{D}^2$ into the equation you should find that the result is a zero matrix.
3. You need to show that $\mathbf{A}^2\mathbf{A} = \mathbf{A}\mathbf{A}^2$ using already established properties, i.e. $\mathbf{A}^2\mathbf{A} = (\mathbf{AA})(\mathbf{A}) = \mathbf{A}(\mathbf{AA}) = \mathbf{AA}^2$.

Solution 4.4

1. The matrix exponential is defined by the power series $\exp \mathbf{A} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \ldots$. Notice that this $\mathbf{A}$ is diagonal and $\mathbf{A}^2 = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}$ and the exponential becomes $\exp \mathbf{A} = \begin{bmatrix} 1 + 2 + 2^2/2! + \ldots & 0 \\ 0 & 1 + 3 + 3^2/2! + \ldots \end{bmatrix}$. If you have seen power series before then you will recognise that $\exp \mathbf{A} = \begin{bmatrix} \exp 2 & 0 \\ 0 & \exp 3 \end{bmatrix}$.
2. You can define a general two by two matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find $\mathbf{A}^2$, set each of the entries equal to zero and find constraints on the entries a, b, c, d.
3. You need to show that $\mathbf{A}^n\mathbf{A} = \mathbf{A}\mathbf{A}^n$ for any square matrix $\mathbf{A}$ and any positive integer n by induction. First you show it is true for $n = 1$ and $n = 2$. Then assume it is true for some $n = k$, and prove that it must be true for $n = k + 1$. You use the fact that $\mathbf{A}$ commutes with itself and the associative property, i.e $\mathbf{A}^2\mathbf{A} = (\mathbf{AA})\mathbf{A} = \mathbf{A}(\mathbf{AA}) = \mathbf{AA}^2$.

Solution 4.5

1. Okay, I placed my book on the table.
2. You could undo the rotation by rotating **clockwise** by 30 degrees. You could think about this as a counterclockwise rotation of -30 degrees.
3. You could get there by rotating once by 90 degrees. This suggests that the product of two rotation matrices of angles θ_1 and θ_2 is a rotation matrix of $\theta_1 + \theta_2$, i.e. $\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2)$.

4. You end up in the same orientation so it doesn't matter the order. This suggests that the order of multiplication doesn't matter so that two rotation matrices must commute.

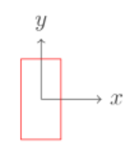
Solution 4.6

1. You could multiply out two rotation matrices with angle θ_1 and θ_2 in the two different orders and you will observe that the output is the same because real numbers commute, i.e. $\cos\theta_1\cos\theta_2 = \cos\theta_2\cos\theta_1$.

2. If you multiply two matrices together you will get the following expression in the first row and first column, $\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$. You will find a trig identity which reduces this to $\cos(\theta_1+\theta_2)$. Similar reductions take place for the other elements.

Solution 4.7

1. The rectangle is



2. The rotation matrix is

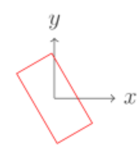
$$\mathbf{R} = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Applying this to each point, we get

$$\mathbf{R}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-2}{2} \\ \frac{1+2\sqrt{3}}{2} \end{bmatrix}, \ \mathbf{R}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+2}{2} \\ \frac{1-2\sqrt{3}}{2} \end{bmatrix},$$

$$\mathbf{R}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{3}-2}{2} \\ \frac{-1+\sqrt{3}}{2} \end{bmatrix}, \ \mathbf{R}\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{3}+2}{2} \\ \frac{-1-\sqrt{3}}{2} \end{bmatrix}.$$

And the rotated figure looks like,



Solution 4.8

1. There are lots of ways to do this point by point. Here is an example of how to transform the bottom right point:

```
>> BR = [1;-2]
>> plot(BR(1,:),BR(2,:),'b*')
>> rotmatrix = [cosd(30) -sind(30); sind(30) cosd(30)]
>> nBR = rotmatrix*BR
>> plot(nBR(1,:),nBR(2,:),'r*')
```

2. There are lots of ways to do this. Here is an example where we include the first point twice so that the points can easily be connected with lines:

```
>> pts = [1 -1 -1 1 1;2 2 -2 -2 2]
>> npts = rotmatrix*pts
>> plot(pts(1,:),pts(2,:),'b'), hold on
>> plot(pts(1,:),pts(2,:),'r')
>> axis([-3 3 -3 3])
>> axis equal
```

3. The area of the rectangle is the same before and after rotation: 8 square units.

4. To undo this rotation you could simply rotate it by 30 degrees clockwise, using the matrix

$$\mathbf{R}^{-1} = \begin{bmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{bmatrix}.$$

5. The product of $\mathbf{R}^{-1}$ and $\mathbf{R}$ is

$$\mathbf{R}^{-1}\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where we have used the trig identity $\cos^2\theta + \sin^2\theta = 1$.