Week 4

Tuesday, April 13, 2021 11:27 PM

RoboWeek
4-1

Chapter 22

Week 4a: Bridge of Doom Work Day

A good time to work on Bridge of Doom and get all the help you need!

Chapter 23

Week 4b: Debrief, Partial Derivatives

Schedule

23.1 Debrief on the Bridge of Doom

To get the most of the bridge of doom challenge, we want you to complete a reflection exercise where you create a cheat-sheet with your breakout room. There is a blank cheat-sheet a few pages below that you will fill in with your group; the left column depicts the major tasks you needed to complete for the challenge, while the other columns have whitespace where you can list equations, matlab functions, and other notes you found useful during the challenge. Make sure to talk with your group; they may have discovered some useful things you'd like to learn about!

Exercise 23.1

1. Which parts of the BOD challenge were confusing? Are there any math or MATLAB tools you can write down to help avoid that confusion in the future?
2. Were any of the MATLAB functions you used surprising? Can you write yourself some notes to avoid being surprised in the future?
3. Were there any expressions / functions that were useful? Can you write yourself some notes to make sure you can rediscover these useful tools in the future? As an example, I partially filled a sheet (Fig. 23.1) You should go much further than this, though!

	Math	MATLAB
Parameterized Curve		
↓	$\vec{\omega} = \hat{T} \times \frac{d\hat{T}}{dt}$ Note: This assumes the robot always faces along its tangent direction	omega_vec = cross(T_vec, dTdt_vec) Note: I found it surprising that cross() will work on both vectors and symbolic expressions... I need to keep track of what objects I'm working with to avoid getting confused!
Wheel Velocities		
↓	your work here!	your work here!
ROS Code		
↓	your work here!	your work here!
Encode Data		
↓	your work here!	your work here!
Reconstructed Curve		

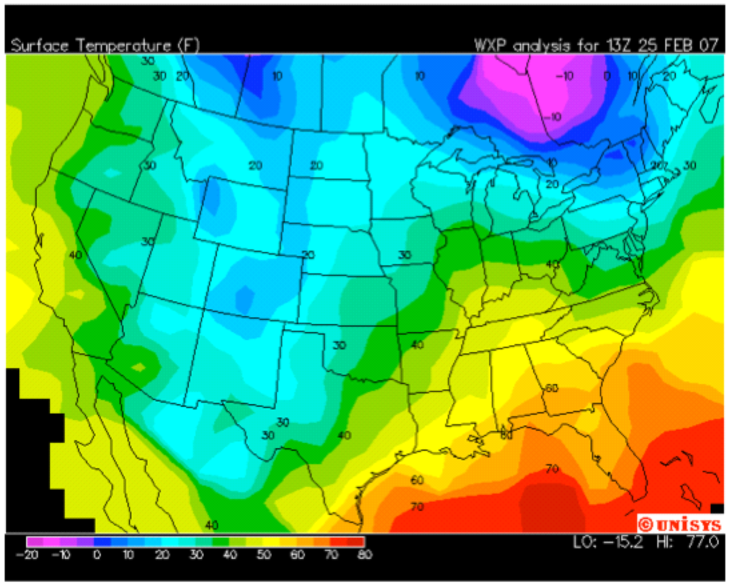
Figure 23.1: Examples of things you might put in your reflection sheet.

The empty cheat-sheet is on the next page.

	Math	MATLAB
Parameterized Curve		
↓		
Wheel Velocities		
↓		
ROS Code		
↓		
Encode Data		
↓		
Reconstructed Curve		

23.2 Conceptual Exercise: The Leisure Seeker

The map below gives the temperature across the United States on a certain winter day. Regions of the same color have the same temperature: violet represents the coldest areas, and temperatures rise as the colors traverse the spectrum from indigo to blue to green to yellow to orange to red.



Exercise 23.2

Locate Chicago on the map and mark it with a dot. The weather in Chicago is freezing in winter, so a resident of the city decides to embark on a journey in search of the sun. From Chicago, she wants to travel in the direction in which the temperature rises most quickly. As her journey proceeds, she decides to keep traveling in the direction in which the weather warms up most quickly: wherever she is at any moment, she moves in the direction of fastest temperature rise.

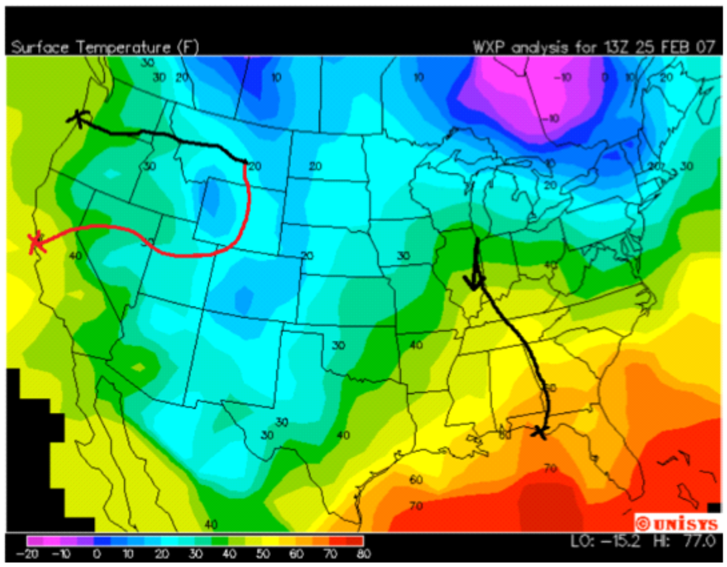
1. Make a rough sketch of the route she takes.
2. Where does she end up, assuming that she doesn't leave the United States?
3. What would happen if her friend started in Billings, Montana? Where would he end up?

Solution 23.1

1. Which parts of the BOD challenge were confusing? Are there any math or MATLAB tools you can write down to help avoid that confusion in the future? Answers will vary.
2. Were any of the MATLAB functions you used surprising? Can you write yourself some notes to avoid being surprised in the future? Answers will vary.
3. Were there any expressions / functions that were useful? Can you write yourself some notes to make sure you can rediscover these useful tools in the future? Answers will vary. As an example, I partially filled a sheet (Fig. 23.1) You should go much further than this, though! There are many ways to tackle this one!

Solution 23.2

1. See Fig. 1.



2. Near Panama City, FL
3. Hard to know exactly without more isotherms (contours of constant temperature), but maybe the SF Bay Area or the Portland area.

Chapter 24

Homework 4: Partial Derivatives, Chain Rule, Max and Min

Learning Objectives

Concepts

- Compute partial derivatives using the chain rule and product rules for differentiation.
- Determine critical points of a continuous function of single variable.
- Use the gradient function to compute critical points of multivariate functions.
- Evaluate whether critical points of a multivariate function are local maxima, minima or otherwise.

24.1 Partial Derivatives and the Chain Rule

We met partial derivatives earlier, and we are now going to return to this idea to reinforce it and to extend it. Work your way through sections 14.3 and 14.5 from the book `Multivariable Calculus by Stewart`—we've included section 14.6 for completeness, but we will discuss that chapter in a future assignment.

Exercise 24.1

1. Please read Section 14.3 from Stewart on Partial Derivatives. Take notes on important concepts and definitions.
2. Please read Section 14.5 from Stewart on The Chain Rule. Take notes on important concepts and definitions.

   This is new material on extending the notion of the chain rule from functions of one variable to functions of many variables. The main results are captured in the pink boxes labeled 1 through 4 - these are various cases of the chain rule. Again, this text is written for a student who doesn't have linear algebra. As you read these rules, think about how you might use matrix notation to make this cleaner and more compact. At this stage you should ignore the section on Implicit Derivatives - it will be too confusing and take too long.

   Do the following exercises (by hand or using a Computer Algebra System)

1) find $\frac{dz}{dt}$ $\quad \frac{dx}{dt} = \cos t \quad \frac{dy}{dt} = e^t$

$z = x^2 + y^2 + xy \quad x = \sin t \quad y = e^t$ $\quad \frac{\partial z}{\partial x} = 2x + y$

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$ $\quad \frac{\partial z}{\partial y} = 2y + x$

$dz = (2x+y)\cos t + (2y+x)e^t$

$dt \quad \partial x \; dt \quad \partial y \; dt$

$$\frac{\partial z}{\partial y} = 2y + x$$

$$\frac{dz}{dt} = (2x+y)\cos t + (2y+x)e^{t}$$

$$= (2\sin(t) + e^{t})\cos t + (2e^{t} + \sin(t))e^{t}$$

3. Complete question 1 from 14.5 Exercises.
4. Complete question 5 from 14.5 Exercises.
5. Complete question 11 from 14.5 Exercises.

24.2 Max and Min of Single Variable Functions

In high school you probably spent a good amount of time thinking about the relative maximum or relative minimum of a function of one variable. You've probably met the following idea before:

> $x = a$ is a critical point of $f(x)$ if $f'(a) = 0$.

This simply means that a critical point is a point where the derivative is zero.

Exercise 24.2

Find the critical point(s) of $f(x) = x^4 - x^2 + 1$ (can be done by hand or computer algebra system).

Once we have the critical points, there is a straightforward test to determine whether any critical point is a relative maximum or relative minimum.

> The critical point $x = a$ is a relative minimum if $f''(a) > 0$. The critical point $x = a$ is a relative maximum if $f''(a) < 0$.

This means that in order for a point to be a relative minimum, the first derivative must be zero and the second derivative must be positive, and vice versa for a relative maximum. If both the first and second derivatives are zero then no conclusion can be drawn without further investigation.

Exercise 24.3

Classify the critical point(s) of $f(x) = x^4 - x^2 + 1$.

If you would like to review this material or have more practice please check out the following sections from Paul's Online Math Notes:

- Critical Points
- Minimum and Maximum Values

24.3 Max and Min of Multivariable Functions

Let's examine the corresponding idea for functions of two variables.

> $(x, y) = (a, b)$ is a critical point of $f(x, y)$ if $\nabla f(a, b) = \mathbf{0}$. Recall that the gradient vector is composed of the first-partial derivatives, $\nabla f = [f_x f_y]$.

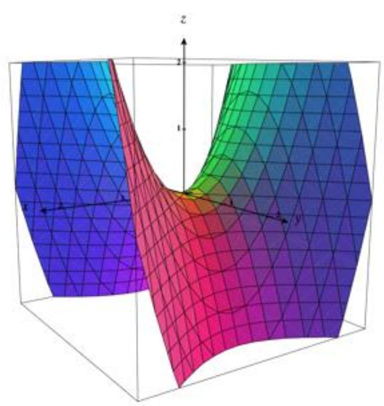
This means that in order for a point (a, b) to be a critical point in two dimensions then both partial derivatives need to be zero there.

Exercise 24.4

Determine the critical points of the following functions (can be done by hand or using a Computer Algebra System.)

1. $f(x,y) = 4 + x^3 + y^3 - 3xy$
2. $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

Once we have the critical points, there is another straightforward test to classify the type of critical point. However, when we have more than a single variable, the critical points can be something other than a simple maximum or minimum. This is illustrated in figure below of the $f(x,y) = x^2 - y^2$. At the critical point, $(0,0)$, $\nabla f = 0$. As you look at $(0,0)$ does it appear to you as a maximum or a minimum point? To see what is unusual about $(0,0)$, imagine you were to take a cut of the yz-plane at $x = 0$. What would you see? Then take a cut of the xz-plane at $y = 0$; what would you see in this case?



This is true b/c:
- Hf includes info abt 1st & 2nd derivatives
- Eigenvals/vecs represent the speed of change/ Variation in a space's natural coordinate axes

Figure 24.1: The plot of a saddle function. Generated with CalcPlot3D.

The following approach to characterizing critical points is not usually discussed in a multi-variable calculus course because students are not expected to know linear algebra.

Suppose (a,b) is a critical point of $f(x,y)$. The Hessian matrix Hf evaluated at the critical point has real eigenvalues λ_1 and λ_2 (Recall that a symmetric matrix has real eigenvalues). Then the following classifications are possible:

1. If $\lambda_1 < 0$ and $\lambda_2 < 0$, then the critical point is a relative maximum.
2. If $\lambda_1 > 0$ and $\lambda_2 > 0$, then the critical point is a relative minimum.
3. If $\lambda_1 > 0$ and $\lambda_2 < 0$ OR $\lambda_2 > 0$ and $\lambda_1 < 0$, then the critical point is a saddle.
4. If $\lambda_1 = 0$ or $\lambda_2 = 0$ (or both equal zero) at the critical point then no conclusions can be drawn without further investigation.

So we see that for a function of two variables that there is a new type of critical point, the saddle - it is a relative maximum in one direction, and a relative minimum in another.

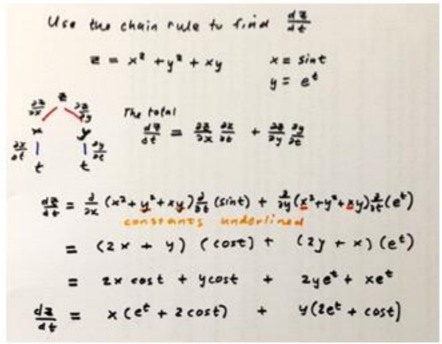
Exercise 24.5

Classify the critical points of the following functions

1. $f(x, y) = 4 + x^3 + y^3 - 3xy$
2. $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

Solution 24.1

- Complete question 1 from 14.5 Exercises. Notice that z is a function of two variables, x and y; x and y vary with t, so to compute the $\frac{dz}{dt}$, we need to sum portions of $\frac{dz}{dt}$ due to x and the portion due to y.



Use the chain rule to find $\frac{dz}{dt}$

$z = x^2 + y^2 + xy \qquad x = \sin t$

$y = e^t$

The total

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{dz}{dt} = \frac{\partial}{\partial x}(x^2 + \underline{y}^2 + x\underline{y})\frac{\partial}{\partial t}(\sin t) + \frac{\partial}{\partial y}(\underline{x}^2 + y^2 + \underline{x}y)\frac{\partial}{\partial t}(e^t)$$

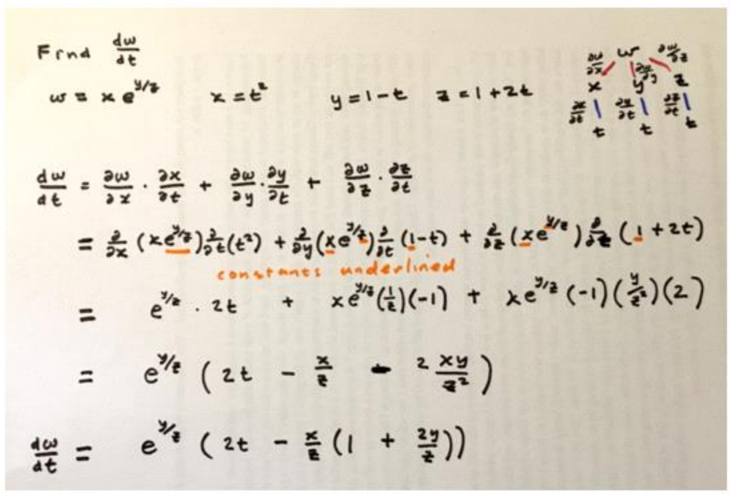
constants underlined

$$= (2x + y)(\cos t) + (2y + x)(e^t)$$

$$= 2x\cos t + y\cos t + 2ye^t + xe^t$$

$$\frac{dz}{dt} = x(e^t + 2\cos t) + y(2e^t + \cos t)$$

- Complete question 5 from 14.5 Exercises. Notice that ω is a function of three variables (x, y and z), each varying with t, so to compute the $\frac{d\omega}{dt}$, we need to sum that parts of the total $\frac{d\omega}{dt}$ due to the partial contributions ($\frac{\partial \omega}{\partial t}$) of x, y and z.



Find $\frac{dw}{dt}$

$w = xe^{y/z} \qquad x = t^2 \qquad y = 1 - t \qquad z = 1 + 2t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= \frac{\partial}{\partial x}(x\underline{e^{y/z}})\frac{\partial}{\partial t}(t^2) + \frac{\partial}{\partial y}(\underline{x}e^{y/z})\frac{\partial}{\partial t}(\underline{1} - t) + \frac{\partial}{\partial z}(\underline{x}e^{y/z})\frac{\partial}{\partial z}(\underline{1} + 2t)$$

constants underlined

$$= e^{y/z}\cdot 2t + xe^{y/z}\left(\tfrac{1}{z}\right)(-1) + xe^{y/z}(-1)\left(\tfrac{y}{z^2}\right)(2)$$

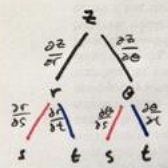
$$= e^{y/z}\left(2t - \frac{x}{z} - 2\frac{xy}{z^2}\right)$$

$$\frac{dw}{dt} = e^{y/z}\left(2t - \frac{x}{z}\left(1 + \frac{2y}{z}\right)\right)$$

- Complete question 11 from 14.5 Exercises. Notice that z is a function of two variables (r and θ), each varying with s and t. To compute $\frac{\partial z}{\partial s}$, for example, we need to sum portions of $\frac{\partial z}{\partial s}$ due to the variation with ∂s in each of r and θ. In this case, $\frac{\partial z}{\partial s}$ are partial derivatives, since the total dz consists of ∂s and ∂t.

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

$z = e^r \cos\theta \qquad r = st, \ \theta = \sqrt{s^2+t^2}$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial}{\partial r}(e^r \underline{\cos\theta})\frac{\partial}{\partial s}(s\underline{t}) + \frac{\partial}{\partial \theta}(\underline{e^r}\cos\theta)\frac{\partial}{\partial s}((s^2+\underline{t^2})^{1/2})$$

constants underlined

$$= \cos\theta \ e^r t \quad + e^r(-\sin\theta)\left(\tfrac{1}{2}\right)(s^2+t^2)^{-1/2}(2s)$$

$$\frac{\partial z}{\partial s} = e^r\left(t\cos\theta - \frac{s\cdot\sin\theta}{\sqrt{s^2+t^2}}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial r}(e^r \underline{\cos\theta})\frac{\partial}{\partial t}(\underline{s}t) + \frac{\partial}{\partial \theta}(\underline{e^r}\cos\theta)\frac{\partial}{\partial t}((\underline{s^2}+t^2)^{1/2})$$

constants

$$= \cos\theta \ e^r s \quad + e^r(-\sin\theta)\left(\tfrac{1}{2}\right)(s^2+t^2)^{-1/2}(2t)$$

$$\frac{\partial z}{\partial t} = e^r\left(s\cos\theta - \frac{t\cdot\sin\theta}{\sqrt{s^2+t^2}}\right)$$

Solution 24.2

We first take the derivative so that $f'(x) = 4x^3 - 2x$ and then we set it equal to zero so that $4x^3 - 2x = 0$ and then we solve for x. Since x is a common factor we see that the solutions are dictated by $x = 0$ and $4x^2 - 2 = 0$ or $x = \pm\sqrt{2}/2$. This function therefore has three critical points.

Solution 24.3

We've already found the critical points. Let's now determine the second-derivative, and evaluate it at each of the critical points. The second derivative is $f''(x) = 12x^2 - 2$ and evaluating at each of the critical points gives:

$$\begin{aligned} f''(0) &= -2 \\ f''(\pm\sqrt{2}/2) &= 4 \end{aligned}$$

The critical point at $x = 0$ is therefore a relative maximum, while the critical points at $\pm\sqrt{2}/2$ are both relative minima. Graphing the function would confirm this very quickly.

Solution 24.4

1. We already met this function in Robo Night 2 and computed its partial derivatives then. The gradient vector is:

$$\nabla f = \begin{bmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{bmatrix}$$

We now determine the values of x and y for which both derivatives are zero simultaneously. Setting the first component to zero leads to $y = x^2$ which we can substitute into the second component and set it equal to zero to obtain $x^4 - x = 0$. The solutions are therefore $x = 0$ and $x = 1$ which leads to the points $(0, 0)$ and $(1, 1)$.

2. We've already met this function in Robo Night 2 and computed its partial derivatives then. The gradient vector is:

$$\nabla f = \begin{bmatrix} 6xy - 6x \\ 3x^2 + 3y^2 - 6y \end{bmatrix}$$

We now determine the values of x and y for which both derivatives are zero simultaneously. Setting the first component to zero leads to $x = 0$ or $y = 1$ which we can substitute into the second component and set it equal to zero to obtain two possibilities: $x = 0$ AND $3y^2 - 6y = 0$ OR $y = 1$ AND $3x^2 - 3 = 0$. The first possibility has $y = 0$ ad $y = 2$ as solutions, while the second possibility has $x = \pm 1$ as solutions. There are therefore 4 solutions: $(0, 0)$,$(0, 2)$,$(1, 1)$, and $(-1, 1)$.

Solution 24.5

1. We already know the critical points are $(0, 0)$ and $(1, 1)$. We need to compute the Hessian matrix to classify it. Fortunately, we already computed the second-derivatives in Robo Night 2 so that

$$Hf = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

   - At $(0, 0)$ we see that

   $$Hf = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

   which has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 3$ so that this critical point is therefore a saddle.

   - At $(1, 1)$ we see that

   $$Hf = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

   which has eigenvalues $\lambda_1 = 9$ and $\lambda_2 = 3$ which implies that this point is a relative minimum.

2. We already know the critical points are $(0, 0)$,$(0, 2)$,$(1, 1)$, and $(-1, 1)$ We need to compute the Hessian matrix to classify it. Fortunately, we already computed the second-derivatives in Robo Night 2 so that

$$Hf = \begin{bmatrix} 6y - 6 & 6x \\ 6x & 6y - 6 \end{bmatrix}$$

   - At $(0, 0)$ we see that

   $$Hf = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$$

   which has eigenvalues $\lambda_1 = -6$ and $\lambda_2 = -6$ which implies that this point is a relative maximum.

- At $(0, 2)$ we see that
$$Hf = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$
which has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 6$ which implies that $(0, 2)$ is a relative minimum.
- At $(1, 1)$ we see that
$$Hf = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$$
which has eigenvalues $\lambda_1 = -6$ and $\lambda_2 = 6$, which implies that this point is a saddle.
- At $(-1, 1)$ we see that
$$Hf = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$$
which has eigenvalues $\lambda_1 = -6$ and $\lambda_2 = 6$, so that this point is also a saddle.

StewartOnPartialDeri...

25–26 Find $h(x, y) = g(f(x, y))$ and the set on which h is continuous.

25. $g(t) = t^2 + \sqrt{t}, \quad f(x, y) = 2x + 3y - 6$

26. $g(t) = t + \ln t, \quad f(x, y) = \dfrac{1 - xy}{1 + x^2y^2}$

27–28 Graph the function and observe where it is discontinuous. Then use the formula to explain what you have observed.

27. $f(x, y) = e^{1/(x-y)}$

28. $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$

29–38 Determine the set of points at which the function is continuous.

29. $F(x, y) = \dfrac{xy}{1 + e^{x-y}}$

30. $F(x, y) = \cos\sqrt{1 + x - y}$

31. $F(x, y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$

32. $H(x, y) = \dfrac{e^x + e^y}{e^{xy} - 1}$

33. $G(x, y) = \ln(x^2 + y^2 - 4)$

34. $G(x, y) = \tan^{-1}\left((x + y)^{-2}\right)$

35. $f(x, y, z) = \arcsin(x^2 + y^2 + z^2)$

36. $f(x, y, z) = \sqrt{y - x^2}\,\ln z$

37. $f(x, y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

38. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

39–41 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geqslant 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

39. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^3 + y^3}{x^2 + y^2}$

40. $\displaystyle\lim_{(x, y) \to (0, 0)} (x^2 + y^2)\ln(x^2 + y^2)$

41. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$

42. At the beginning of this section we considered the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and guessed that $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ on the basis of numerical evidence. Use polar coordinates to confirm the value of the limit. Then graph the function.

43. Graph and discuss the continuity of the function

$$f(x, y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

44. Let

$$f(x, y) = \begin{cases} 0 & \text{if } y \leqslant 0 \quad \text{or} \quad y \geqslant x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

(a) Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any path through $(0, 0)$ of the form $y = mx^a$ with $a < 4$.
(b) Despite part (a), show that f is discontinuous at $(0, 0)$.
(c) Show that f is discontinuous on two entire curves.

45. Show that the function f given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$. [*Hint*: Consider $|\mathbf{x} - \mathbf{a}|^2 = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.]

46. If $\mathbf{c} \in V_n$, show that the function f given by $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ is continuous on $\mathbb{R}^n$.

14.3 Partial Derivatives

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

TABLE 1
Heat index I as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

If we concentrate on the highlighted column of the table, which corresponds to a relative humidity of $H = 70\%$, we are considering the heat index as a function of the single variable T for a fixed value of H. Let's write $g(T) = f(T, 70)$. Then $g(T)$ describes how the heat index I increases as the actual temperature T increases when the relative humidity is 70%. The derivative of g when $T = 96°\text{F}$ is the rate of change of I with respect to T when $T = 96°\text{F}$:

$$g'(96) = \lim_{h \to 0} \frac{g(96 + h) - g(96)}{h} = \lim_{h \to 0} \frac{f(96 + h, 70) - f(96, 70)}{h}$$

We can approximate $g'(96)$ using the values in Table 1 by taking $h = 2$ and -2:

$$g'(96) \approx \frac{g(98) - g(96)}{2} = \frac{f(98, 70) - f(96, 70)}{2} = \frac{133 - 125}{2} = 4$$

$$g'(96) \approx \frac{g(94) - g(96)}{-2} = \frac{f(94, 70) - f(96, 70)}{-2} = \frac{118 - 125}{-2} = 3.5$$

Averaging these values, we can say that the derivative $g'(96)$ is approximately 3.75. This means that, when the actual temperature is 96°F and the relative humidity is 70%, the apparent temperature (heat index) rises by about 3.75°F for every degree that the actual temperature rises!

Now let's look at the highlighted row in Table 1, which corresponds to a fixed temperature of $T = 96°\text{F}$. The numbers in this row are values of the function $G(H) = f(96, H)$, which describes how the heat index increases as the relative humidity H increases when the actual temperature is $T = 96°\text{F}$. The derivative of this function when $H = 70\%$ is the rate of change of I with respect to H when $H = 70\%$:

$$G'(70) = \lim_{h \to 0} \frac{G(70 + h) - G(70)}{h} = \lim_{h \to 0} \frac{f(96, 70 + h) - f(96, 70)}{h}$$

By taking $h = 5$ and -5, we approximate $G'(70)$ using the tabular values:

$$G'(70) \approx \frac{G(75) - G(70)}{5} = \frac{f(96, 75) - f(96, 70)}{5} = \frac{130 - 125}{5} = 1$$

$$G'(70) \approx \frac{G(65) - G(70)}{-5} = \frac{f(96, 65) - f(96, 70)}{-5} = \frac{121 - 125}{-5} = 0.8$$

By averaging these values we get the estimate $G'(70) \approx 0.9$. This says that, when the temperature is 96°F and the relative humidity is 70%, the heat index rises about 0.9°F for every percent that the relative humidity rises.

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of f with respect to x at (a, b)** and denote it by $f_x(a, b)$. Thus

1 $$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

and so Equation 1 becomes

2 $$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, the **partial derivative of f with respect to y at (a, b)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3 $$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

With this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 96$°F and $H = 70$% as follows:

$$f_T(96, 70) \approx 3.75 \qquad f_H(96, 70) \approx 0.9$$

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

There are many alternative notations for partial derivatives. For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f/\partial x$. But here $\partial f/\partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed. Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

EXAMPLE 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

SOLUTION Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S. By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

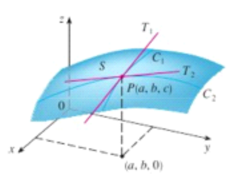


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

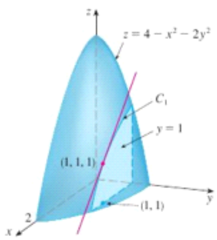


FIGURE 2

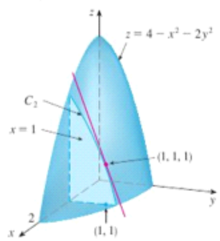


FIGURE 3

As we have seen in the case of the heat index function, partial derivatives can also be interpreted as *rates of change*. If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

SOLUTION We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.) The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$. Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$. (See Figure 3.)

Figure 4 is a computer-drawn counterpart to Figure 2. Part (a) shows the plane $y = 1$ intersecting the surface to form the curve C_1 and part (b) shows C_1 and T_1. [We have used the vector equations $\mathbf{r}(t) = \langle t, 1, 2 - t^2 \rangle$ for C_1 and $\mathbf{r}(t) = \langle 1 + t, 1, 1 - 2t \rangle$ for T_1.] Similarly, Figure 5 corresponds to Figure 3.

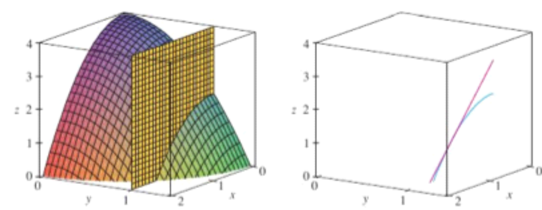


FIGURE 4 (a) (b)

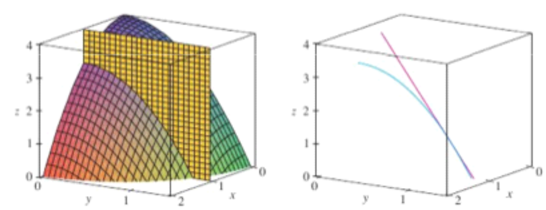


FIGURE 5

V EXAMPLE 3 If $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

SOLUTION Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

Some computer algebra systems can plot surfaces defined by implicit equations in three variables. Figure 6 shows such a plot of the surface defined by the equation in Example 4.

FIGURE 6

V EXAMPLE 4 Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

$3x^2 + \frac{\partial z}{\partial x} + 6yz + 6yx\frac{\partial z}{\partial x}$

SOLUTION To find $\partial z/\partial x$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving this equation for $\partial z/\partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x. If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

43–44 Show that the function is differentiable by finding values of ε_1 and ε_2 that satisfy Definition 7.

43. $f(x, y) = x^2 + y^2$ **44.** $f(x, y) = xy - 5y^2$

45. Prove that if f is a function of two variables that is differentiable at (a, b), then f is continuous at (a, b).
Hint: Show that

$$\lim_{(\Delta x, \Delta y) \to (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b)$$

46. (a) The function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

was graphed in Figure 4. Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but f is not differentiable at $(0, 0)$. [*Hint:* Use the result of Exercise 45.]

(b) Explain why f_x and f_y are not continuous at $(0, 0)$.

14.5 The Chain Rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$y = f(x(t))$

$$\boxed{1} \qquad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function. The first version (Theorem 2) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t. This means that z is indirectly a function of t, $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t. We assume that f is differentiable (Definition 14.4.7). Recall that this is the case when f_x and f_y are continuous (Theorem 14.4.8).

$\boxed{2}$ **The Chain Rule (Case 1)** Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

f is one Var

PROOF A change of Δt in t produces changes of Δx in x and Δy in y. These, in turn, produce a change of Δz in z, and from Definition 14.4.7 we have

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \overbrace{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}^{\text{goes to } 0}$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. [If the functions ε_1 and ε_2 are not defined at $(0, 0)$, we can define them to be 0 there.] Dividing both sides of this equation by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

If we now let $\Delta t \to 0$, then $\Delta x = g(t + \Delta t) - g(t) \to 0$ because g is differentiable and therefore continuous. Similarly, $\Delta y \to 0$. This, in turn, means that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, so

$$\begin{aligned}\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\ &= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_1\right) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_2\right) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\end{aligned}$$

Since we often write $\partial z/\partial x$ in place of $\partial f/\partial x$, we can rewrite the Chain Rule in the form

Notice the similarity to the definition of the differential:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

SOLUTION The Chain Rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)\end{aligned}$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6$$

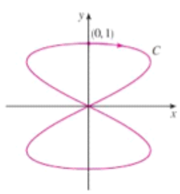


FIGURE 1
The curve $x = \sin 2t$, $y = \cos t$

The derivative in Example 1 can be interpreted as the rate of change of z with respect to t as the point (x, y) moves along the curve C with parametric equations $x = \sin 2t$, $y = \cos t$. (See Figure 1.) In particular, when $t = 0$, the point (x, y) is $(0, 1)$ and $dz/dt = 6$ is the rate of increase as we move along the curve C through $(0, 1)$. If, for instance, $z = T(x, y) = x^2y + 3xy^4$ represents the temperature at the point (x, y), then the composite function $z = T(\sin 2t, \cos t)$ represents the temperature at points on C and the derivative dz/dt represents the rate at which the temperature changes along C.

V EXAMPLE 2 The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

SOLUTION If t represents the time elapsed in seconds, then at the given instant we have $T = 300$, $dT/dt = 0.1$, $V = 100$, $dV/dt = 0.2$. Since

$$P = 8.31\frac{T}{V}$$

the Chain Rule gives

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s.

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t: $x = g(s, t)$, $y = h(s, t)$. Then z is indirectly a function of s and t and we wish to find $\partial z/\partial s$ and $\partial z/\partial t$. Recall that in computing $\partial z/\partial t$ we hold s fixed and compute the ordinary derivative of z with respect to t. Therefore we can apply Theorem 2 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

A similar argument holds for $\partial z/\partial s$ and so we have proved the following version of the Chain Rule.

3 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then z is 2 var

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

SOLUTION Applying Case 2 of the Chain Rule, we get

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2} \cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= 2ste^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t)$$

Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the dependent variable. Notice that Theorem 3 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule in Equation 1.

FIGURE 2

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2. We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then we draw branches from x and y to the independent variables s and t. On each branch we write the corresponding partial derivative. To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly, we find $\partial z/\partial t$ by using the paths from z to t.

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, \dots, x_n$, each of which is, in turn, a function of m independent variables $t_1, \dots, t_m$. Notice that there are n terms, one for each intermediate variable. The proof is similar to that of Case 1.

4 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \dots, t_m$. Then u is a function of $t_1, t_2, \dots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \dots, m$.

V EXAMPLE 4 Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

FIGURE 3

SOLUTION We apply Theorem 4 with $n = 4$ and $m = 2$. Figure 3 shows the tree diagram. Although we haven't written the derivatives on the branches, it's understood that if a branch leads from y to u, then the partial derivative for that branch is $\partial y/\partial u$. With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

V EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

FIGURE 4

SOLUTION With the help of the tree diagram in Figure 4, we have

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} \\ &= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)\end{aligned}$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$

EXAMPLE 6 If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

SOLUTION Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$ and the Chain Rule gives

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Therefore

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0$$

EXAMPLE 7 If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find (a) $\partial z/\partial r$ and (b) $\partial^2 z/\partial r^2$.

SOLUTION

(a) The Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

(b) Applying the Product Rule to the expression in part (a), we get

5
$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)\\ &= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\end{aligned}$$

FIGURE 5

But, using the Chain Rule again (see Figure 5), we have

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y\,\partial x}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\,\partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)$$

Putting these expressions into Equation 5 and using the equality of the mixed second-order derivatives, we obtain

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y\,\partial x}\right) + 2s\left(2r\frac{\partial^2 z}{\partial x\,\partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)\\ &= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x\,\partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Implicit Differentiation

The Chain Rule can be used to give a more complete description of the process of implicit differentiation that was introduced in Sections 2.6 and 14.3. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is,

$y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

But $dx/dx = 1$, so if $\partial F/\partial y \neq 0$ we solve for dy/dx and obtain

6
$$\boxed{\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x. The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid: It states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 6.

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

SOLUTION The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 6 gives

The solution to Example 8 should be compared to the one in Example 2 in Section 2.6.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But $$\frac{\partial}{\partial x}(x) = 1 \qquad \text{and} \qquad \frac{\partial}{\partial x}(y) = 0$$ ← b/c partial means y is const

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain the first formula in Equations 7 on page 954. The formula for $\partial z/\partial y$ is obtained in a similar manner.

$$\boxed{7} \qquad \frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid: If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by $\boxed{7}$.

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, from Equations 7, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

The solution to Example 9 should be compared to the one in Example 4 in Section 14.3.

14.5 Exercises

1–6 Use the Chain Rule to find dz/dt or dw/dt.

1. $z = x^2 + y^2 + xy$, $x = \sin t$, $y = e^t$
2. $z = \cos(x + 4y)$, $x = 5t^4$, $y = 1/t$
3. $z = \sqrt{1 + x^2 + y^2}$, $x = \ln t$, $y = \cos t$
4. $z = \tan^{-1}(y/x)$, $x = e^t$, $y = 1 - e^{-t}$
5. $w = xe^{y/z}$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$
6. $w = \ln\sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$

7–12 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

7. $z = x^2y^3$, $x = s\cos t$, $y = s\sin t$
8. $z = \arcsin(x - y)$, $x = s^2 + t^2$, $y = 1 - 2st$
9. $z = \sin\theta\cos\phi$, $\theta = st^2$, $\phi = s^2t$
10. $z = e^{x+2y}$, $x = s/t$, $y = t/s$
11. $z = e^r\cos\theta$, $r = st$, $\theta = \sqrt{s^2 + t^2}$
12. $z = \tan(u/v)$, $u = 2s + 3t$, $v = 3s - 2t$

13. If $z = f(x, y)$, where f is differentiable, and

$$\begin{array}{ll} x = g(t) & y = h(t) \\ g(3) = 2 & h(3) = 7 \\ g'(3) = 5 & h'(3) = -4 \\ f_x(2, 7) = 6 & f_y(2, 7) = -8 \end{array}$$

find dz/dt when $t = 3$.

14. Let $W(s, t) = F(u(s, t), v(s, t))$, where F, u, and v are differentiable, and

$$\begin{array}{ll} u(1, 0) = 2 & v(1, 0) = 3 \\ u_s(1, 0) = -2 & v_s(1, 0) = 5 \\ u_t(1, 0) = 6 & v_t(1, 0) = 4 \\ F_u(2, 3) = -1 & F_v(2, 3) = 10 \end{array}$$

Find $W_s(1, 0)$ and $W_t(1, 0)$.

15. Suppose f is a differentiable function of x and y, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
(0, 0)	3	6	4	8
(1, 2)	6	3	2	5

16. Suppose f is a differentiable function of x and y, and $g(r, s) = f(2r - s, s^2 - 4r)$. Use the table of values in Exercise 15 to calculate $g_r(1, 2)$ and $g_s(1, 2)$.

1. Homework Hints available at stewartcalculus.com

1) find $\dfrac{dz}{dt}$

$\dfrac{dx}{dt} = \cos t$ $\quad \dfrac{dy}{dt} = e^t$

$z = x^2 + y^2 + xy$ $\quad x = \sin t$ $\quad y = e^t$

$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$\dfrac{\partial z}{\partial x} = 2x + y$

$\dfrac{\partial z}{\partial y} = 2y + x$

$\dfrac{dz}{dt} = (2x+y)\cos t + (2y+x)e^t$

14.6 Directional Derivatives and the Gradient Vector

FIGURE 1

The weather map in Figure 1 shows a contour map of the temperature function $T(x, y)$ for the states of California and Nevada at 3:00 PM on a day in October. The level curves, or isothermals, join locations with the same temperature. The partial derivative T_x at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno; T_y is the rate of change of temperature if we travel north. But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction? In this section we introduce a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction.

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$\boxed{1} \qquad \begin{aligned} f_x(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\ f_y(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \end{aligned}$$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.) To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of $\mathbf{u}$ intersects S in a curve C. (See Figure 3.) The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\mathbf{u}$.

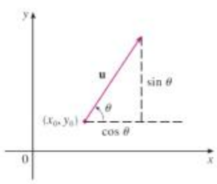


FIGURE 2
A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$

TEC Visual 14.6A animates Figure 3 by rotating $\mathbf{u}$ and therefore T.

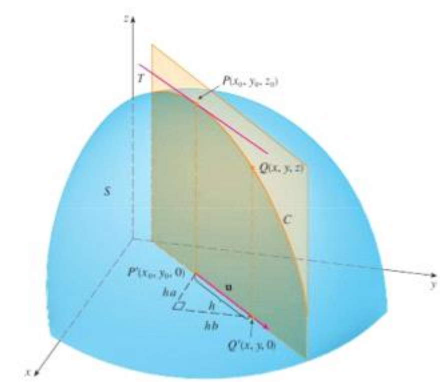


FIGURE 3

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$ and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h. Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of $\mathbf{u}$, which is called the directional derivative of f in the direction of $\mathbf{u}$.

2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing Definition 2 with Equations $\boxed{1}$, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}} f = f_y$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

EXAMPLE 1 Use the weather map in Figure 1 to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

SOLUTION The unit vector directed toward the southeast is $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, but we won't need to use this expression. We start by drawing a line through Reno toward the southeast (see Figure 4).

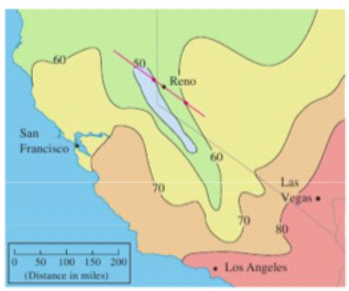


FIGURE 4

We approximate the directional derivative $D_{\mathbf{u}} T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and

$T = 60$. The temperature at the point southeast of Reno is $T = 60°\text{F}$ and the temperature at the point northwest of Reno is $T = 50°\text{F}$. The distance between these points looks to be about 75 miles. So the rate of change of the temperature in the southeasterly direction is

$$D_{\mathbf{u}} T \approx \frac{60 - 50}{75} = \frac{10}{75} \approx 0.13°\text{F/mi}$$

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

3 Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\,a + f_y(x, y)\,b$$

PROOF If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

$$\boxed{4} \qquad g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= D_{\mathbf{u}} f(x_0, y_0)$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 14.5.2) gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)\,a + f_y(x, y)\,b$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$\boxed{5} \qquad g'(0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

$$\boxed{6} \qquad D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

The directional derivative $D_{\mathbf{u}} f(1, 2)$ in Example 2 represents the rate of change of z in the direction of $\mathbf{u}$. This is the slope of the tangent line to the curve of intersection of the surface $z = x^3 - 3xy + 4y^2$ and the vertical plane through $(1, 2, 0)$ in the direction of $\mathbf{u}$ shown in Figure 5.

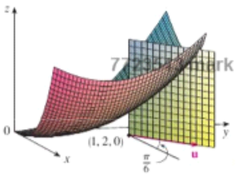


FIGURE 5

SOLUTION Formula 6 gives

$$
\begin{aligned}
D_{\mathbf{u}} f(x, y) &= f_x(x, y) \cos \frac{\pi}{6} + f_y(x, y) \sin \frac{\pi}{6} \\
&= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y) \frac{1}{2} \\
&= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + (8 - 3\sqrt{3})y\right]
\end{aligned}
$$

Therefore

$$
D_{\mathbf{u}} f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + (8 - 3\sqrt{3})(2)\right] = \frac{13 - 3\sqrt{3}}{2}
$$

The Gradient Vector

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

7

$$
\begin{aligned}
D_{\mathbf{u}} f(x, y) &= f_x(x, y)\, a + f_y(x, y)\, b \\
&= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\
&= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}
\end{aligned}
$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f, which is read "del f").

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$
\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}
$$

EXAMPLE 3 If $f(x, y) = \sin x + e^{xy}$, then

$$
\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle
$$

and

$$
\nabla f(0, 1) = \langle 2, 0 \rangle
$$

With this notation for the gradient vector, we can rewrite Equation 7 for the directional derivative of a differentiable function as

9

$$
\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}
$$

This expresses the directional derivative in the direction of a unit vector $\mathbf{u}$ as the scalar projection of the gradient vector onto $\mathbf{u}$.

The gradient vector $\nabla f(2, -1)$ in Example 4 is shown in Figure 6 with initial point $(2, -1)$. Also shown is the vector $\mathbf{v}$ that gives the direction of the directional derivative. Both of these vectors are superimposed on a contour plot of the graph of f.

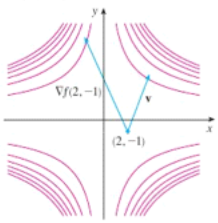


FIGURE 6

V EXAMPLE 4 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\,\mathbf{i} + 5\,\mathbf{j}$.

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3\,\mathbf{i} + (3x^2y^2 - 4)\,\mathbf{j}$$

$$\nabla f(2, -1) = -4\,\mathbf{i} + 8\,\mathbf{j}$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}$$

Therefore, by Equation 9, we have

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\,\mathbf{i} + 8\,\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

Functions of Three Variables

For functions of three variables we can define directional derivatives in a similar manner. Again $D_{\mathbf{u}} f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector $\mathbf{u}$.

10 Definition The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

11
$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$. This is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{u}$ (Equation 12.5.1) and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then the same method that was used to prove Theorem 3 can be used to show that

12 $$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)\,a + f_y(x, y, z)\,b + f_z(x, y, z)\,c$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

13 $$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}}$$

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

14 $$\boxed{D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}}$$

V EXAMPLE 5 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION

(a) The gradient of f is

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle \end{aligned}$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\,\mathbf{i} + \frac{2}{\sqrt{6}}\,\mathbf{j} - \frac{1}{\sqrt{6}}\,\mathbf{k}$$

Therefore Equation 14 gives

$$\begin{aligned} D_{\mathbf{u}} f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\,\mathbf{i} + \frac{2}{\sqrt{6}}\,\mathbf{j} - \frac{1}{\sqrt{6}}\,\mathbf{k} \right) \\ &= 3\left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}} \end{aligned}$$

Maximizing the Directional Derivative

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

TEC Visual 14.6B provides visual confirmation of Theorem 15.

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

PROOF From Equation 9 or 14 we have

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}} f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f.

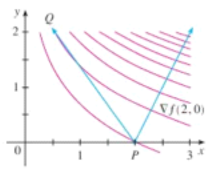


FIGURE 7

At (2, 0) the function in Example 6 increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2\rangle$. Notice from Figure 7 that this vector appears to be perpendicular to the level curve through (2, 0). Figure 8 shows the graph of f and the gradient vector.

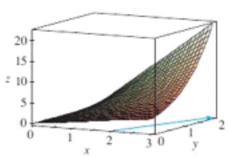


FIGURE 8

EXAMPLE 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

SOLUTION

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y\rangle = \langle e^y, xe^y\rangle$$

$$\nabla f(2, 0) = \langle 1, 2\rangle$$

The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2\rangle$ is $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$, so the rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}} f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2\rangle \cdot \langle -\tfrac{3}{5}, \tfrac{4}{5}\rangle$$

$$= 1(-\tfrac{3}{5}) + 2(\tfrac{4}{5}) = 1$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2\rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2\rangle| = \sqrt{5}$$

EXAMPLE 7 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

SOLUTION The gradient of T is

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

$$= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{k}$$

$$= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\,\mathbf{i} - 2y\,\mathbf{j} - 3z\,\mathbf{k})$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \tfrac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

By Theorem 15 the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \tfrac{5}{8}|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \tfrac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4°\text{C/m}$.

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Recall from Section 13.1 that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$\boxed{16} \qquad F(x(t), y(t), z(t)) = k$$

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

$$\boxed{17} \qquad \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z\rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$, so

$$\boxed{18} \qquad \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

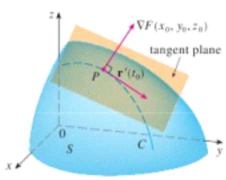


FIGURE 9

Equation 18 says that *the gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P.* (See Figure 9.) If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$** as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane (Equation 12.5.7), we can write the equation of this tangent plane as

$$\boxed{19 \qquad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, by Equation 12.5.3, its symmetric equations are

$$\boxed{20} \qquad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$

$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$

$$F_z(x_0, y_0, z_0) = -1$$

so Equation 19 becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

which is equivalent to Equation 14.4.2. Thus our new, more general, definition of a tangent plane is consistent with the definition that was given for the special case of Section 14.4.

V EXAMPLE 8 Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

SOLUTION The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Figure 10 shows the ellipsoid, tangent plane, and normal line in Example 8.

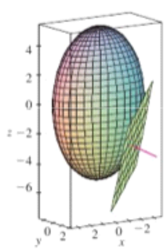


FIGURE 10

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\tfrac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at $(-2, 1, -3)$ as

$$-1(x + 2) + 2(y - 1) - \tfrac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$

Significance of the Gradient Vector

We now summarize the ways in which the gradient vector is significant. We first consider a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain. On the one hand, we know from Theorem 15 that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f. On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P. (Refer to Figure 9.) These two properties are quite compatible intuitively because as we move away from P on the level surface S, the value of f does not change at all. So it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain. Again the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f. Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P. Again this is intuitively plausible because the values of f remain constant as we move along the curve. (See Figure 11.)

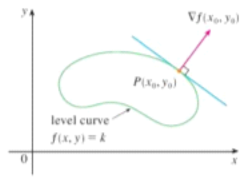


FIGURE 11

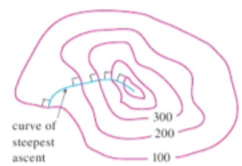


FIGURE 12

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn as in Figure 12 by making it perpendicular to all of the contour lines. This phenomenon can also be noticed in Figure 12 in Section 14.1, where Lonesome Creek follows a curve of steepest descent.

Computer algebra systems have commands that plot sample gradient vectors. Each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b). Figure 13 shows such a plot (called a *gradient vector field*) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f. As expected, the gradient vectors point "uphill" and are perpendicular to the level curves.

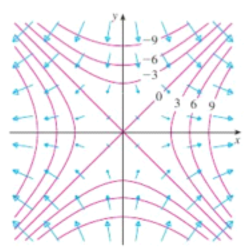
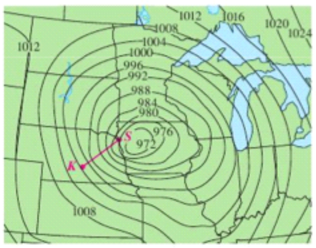


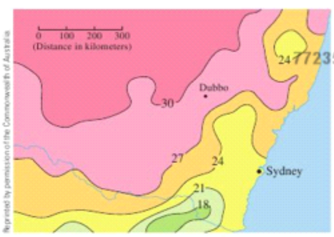
FIGURE 13

14.6 Exercises

1. Level curves for barometric pressure (in millibars) are shown for 6:00 AM on November 10, 1998. A deep low with pressure 972 mb is moving over northeast Iowa. The distance along the red line from K (Kearney, Nebraska) to S (Sioux City, Iowa) is 300 km. Estimate the value of the directional derivative of the pressure function at Kearney in the direction of Sioux City. What are the units of the directional derivative?



2. The contour map shows the average maximum temperature for November 2004 (in °C). Estimate the value of the directional derivative of this temperature function at Dubbo, New South Wales, in the direction of Sydney. What are the units?



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3. A table of values for the wind-chill index $W = f(T, v)$ is given in Exercise 3 on page 935. Use the table to estimate the value of $D_{\mathbf{u}} f(-20, 30)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

4–6 Find the directional derivative of f at the given point in the direction indicated by the angle θ.

4. $f(x, y) = x^3y^4 + x^4y^3$, $(1, 1)$, $\theta = \pi/6$

5. $f(x, y) = ye^{-x}$, $(0, 4)$, $\theta = 2\pi/3$

6. $f(x, y) = e^x \cos y$, $(0, 0)$, $\theta = \pi/4$

7–10

(a) Find the gradient of f.
(b) Evaluate the gradient at the point P.
(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

7. $f(x, y) = \sin(2x + 3y)$, $P(-6, 4)$, $\mathbf{u} = \frac{1}{2}(\sqrt{3}\,\mathbf{i} - \mathbf{j})$

8. $f(x, y) = y^2/x$, $P(1, 2)$, $\mathbf{u} = \frac{1}{3}(2\mathbf{i} + \sqrt{5}\,\mathbf{j})$

9. $f(x, y, z) = x^2yz - xyz^3$, $P(2, -1, 1)$, $\mathbf{u} = \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle$

10. $f(x, y, z) = y^2e^{xyz}$, $P(0, 1, -1)$, $\mathbf{u} = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \rangle$

11–17 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

11. $f(x, y) = e^x \sin y$, $(0, \pi/3)$, $\mathbf{v} = \langle -6, 8 \rangle$

12. $f(x, y) = \dfrac{x}{x^2 + y^2}$, $(1, 2)$, $\mathbf{v} = \langle 3, 5 \rangle$

13. $g(p, q) = p^4 - p^2q^3$, $(2, 1)$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

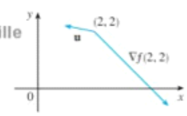
14. $g(r, s) = \tan^{-1}(rs)$, $(1, 2)$, $\mathbf{v} = 5\mathbf{i} + 10\mathbf{j}$

15. $f(x, y, z) = xe^y + ye^z + ze^x$, $(0, 0, 0)$, $\mathbf{v} = \langle 5, 1, -2 \rangle$

16. $f(x, y, z) = \sqrt{xyz}$, $(3, 2, 6)$, $\mathbf{v} = \langle -1, -2, 2 \rangle$

17. $h(r, s, t) = \ln(3r + 6s + 9t)$, $(1, 1, 1)$, $\mathbf{v} = 4\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$

18. Use the figure to estimate $D_{\mathbf{u}} f(2, 2)$.



19. Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

20. Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at $P(1, -1, 3)$ in the direction of $Q(2, 4, 5)$.

21–26 Find the maximum rate of change of f at the given point and the direction in which it occurs.

21. $f(x, y) = 4y\sqrt{x}$, $(4, 1)$

22. $f(s, t) = te^{st}$, $(0, 2)$

23. $f(x, y) = \sin(xy)$, $(1, 0)$

24. $f(x, y, z) = (x + y)/z$, $(1, 1, -1)$

25. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $(3, 6, -2)$

26. $f(p, q, r) = \arctan(pqr)$, $(1, 2, 1)$

Graphing calculator or computer required   1. Homework Hints available at stewartcalculus.com

27. (a) Show that a differentiable function f decreases most rapidly at $\mathbf{x}$ in the direction opposite to the gradient vector, that is, in the direction of $-\nabla f(\mathbf{x})$.
(b) Use the result of part (a) to find the direction in which the function $f(x, y) = x^4y - x^2y^3$ decreases fastest at the point $(2, -3)$.

28. Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point $(0, 2)$ has the value 1.

29. Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

30. Near a buoy, the depth of a lake at the point with coordinates (x, y) is $z = 200 + 0.02x^2 - 0.001y^3$, where x, y, and z are measured in meters. A fisherman in a small boat starts at the point $(80, 60)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when he departs? Explain.

31. The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$.
(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.
(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.

32. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

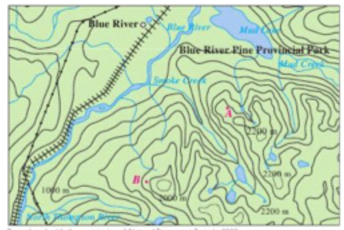
where T is measured in °C and x, y, z in meters.
(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.
(b) In which direction does the temperature increase fastest at P?
(c) Find the maximum rate of increase at P.

33. Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.
(a) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.
(b) In which direction does V change most rapidly at P?
(c) What is the maximum rate of change at P?

34. Suppose you are climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, where x, y, and z are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$. The positive x-axis points east and the positive y-axis points north.
(a) If you walk due south, will you start to ascend or descend? At what rate?
(b) If you walk northwest, will you start to ascend or descend? At what rate?
(c) In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin?

35. Let f be a function of two variables that has continuous partial derivatives and consider the points $A(1, 3)$, $B(3, 3)$, $C(1, 7)$, and $D(6, 15)$. The directional derivative of f at A in the direction of the vector $\overrightarrow{AB}$ is 3 and the directional derivative at A in the direction of $\overrightarrow{AC}$ is 26. Find the directional derivative of f at A in the direction of the vector $\overrightarrow{AD}$.

36. Shown is a topographic map of Blue River Pine Provincial Park in British Columbia. Draw curves of steepest descent from point A (descending to Mud Lake) and from point B.



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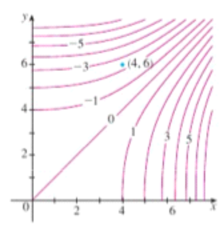
37. Show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and that a, b are constants.

(a) $\nabla(au + bv) = a\nabla u + b\nabla v$ (b) $\nabla(uv) = u\nabla v + v\nabla u$

(c) $\nabla\left(\dfrac{u}{v}\right) = \dfrac{v\nabla u - u\nabla v}{v^2}$ (d) $\nabla u^n = nu^{n-1}\nabla u$

38. Sketch the gradient vector $\nabla f(4, 6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.



39. The **second directional derivative** of $f(x, y)$ is

$$D_{\mathbf{u}}^2 f(x, y) = D_{\mathbf{u}}[D_{\mathbf{u}} f(x, y)]$$

If $f(x, y) = x^3 + 5x^2y + y^3$ and $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, calculate $D_{\mathbf{u}}^2 f(2, 1)$.

40. (a) If $\mathbf{u} = \langle a, b \rangle$ is a unit vector and f has continuous second partial derivatives, show that

$$D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$$

(b) Find the second directional derivative of $f(x, y) = xe^{2y}$ in the direction of $\mathbf{v} = \langle 4, 6 \rangle$.

41–46 Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

41. $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$, $(3, 3, 5)$

42. $y = x^2 - z^2$, $(4, 7, 3)$

43. $xyz^2 = 6$, $(3, 2, 1)$

44. $xy + yz + zx = 5$, $(1, 2, 1)$

45. $x + y + z = e^{xyz}$, $(0, 0, 1)$

46. $x^4 + y^4 + z^4 = 3x^2y^2z^2$, $(1, 1, 1)$

47–48 Use a computer to graph the surface, the tangent plane, and the normal line on the same screen. Choose the domain carefully so that you avoid extraneous vertical planes. Choose the viewpoint so that you get a good view of all three objects.

47. $xy + yz + zx = 3$, $(1, 1, 1)$ **48.** $xyz = 6$, $(1, 2, 3)$

49. If $f(x, y) = xy$, find the gradient vector $\nabla f(3, 2)$ and use it to find the tangent line to the level curve $f(x, y) = 6$ at the point $(3, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

50. If $g(x, y) = x^2 + y^2 - 4x$, find the gradient vector $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

51. Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

52. Find the equation of the tangent plane to the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ at (x_0, y_0, z_0) and express it in a form similar to the one in Exercise 51.

53. Show that the equation of the tangent plane to the elliptic paraboloid $z/c = x^2/a^2 + y^2/b^2$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}$$

54. At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$?

55. Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

56. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at the point.)

57. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

58. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

59. Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1, 1, 2)$ intersect the paraboloid a second time?

60. At what points does the normal line through the point $(1, 2, 1)$ on the ellipsoid $4x^2 + y^2 + 4z^2 = 12$ intersect the sphere $x^2 + y^2 + z^2 = 102$?

61. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

62. Show that the pyramids cut off from the first octant by any tangent planes to the surface $xyz = 1$ at points in the first octant must all have the same volume.

63. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

64. (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.

(b) Graph the cylinder, the plane, and the tangent line on the same screen.

65. (a) Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_xG_x + F_yG_y + F_zG_z = 0 \quad \text{at } P$$

(b) Use part (a) to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Can you see why this is true without using calculus?

66. (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.

(b) Graph f near the origin and comment on how the graph confirms part (a).

67. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find ∇f at this point? If so, how would you do it?

68. Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0 \rangle$, then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$$

[*Hint:* Use Definition 14.4.7 directly.]

hw4

hw4

Homework 4

Exercise 24.1.3 (Stewart 14.5.1)

```
% Did this by hand, just checking work (this exercise only)
syms t z(t) x(t) y(t);

x(t) = sin(t)
```

x(t) = $\sin(t)$

```
y(t) = exp(t)
```

y(t) = e^t

```
z(t) = x(t)^2 + y(t)^2 + (x(t) * y(t))
```

z(t) = $\sin(t)^2 + e^t \sin(t) + e^{2t}$

```
dzdt = simplify(diff(z, t))
```

dzdt(t) = $2 e^{2t} + e^t \cos(t) + 2 \cos(t) \sin(t) + e^t \sin(t)$

Exercise 24.1.4 (Stewart 14.5.5)

```
syms w(t) x(t) y(t) z(t)

x(t) = t^2
```

x(t) = t^2

```
y(t) = 1 - t
```

y(t) = $1 - t$

```
z(t) = 1 + (2 * t)
```

z(t) = $2t + 1$

```
w(t) = x(t) * exp(y(t) / z(t))
```

w(t) =

$$t^2 e^{-\frac{t-1}{2t+1}}$$

```
dwdt = simplify(diff(w, t))
```

dwdt(t) =

$$\frac{t e^{-\frac{t-1}{2t+1}} (8t^2 + 5t + 2)}{(2t+1)^2}$$

1

Exercise 24.1.5 (Stewart 14.5.11)

```
syms s t z r theta
r = s * t
```

r = $s\,t$

```
theta = sqrt(s ^2 + t ^ 2)
```

theta = $\sqrt{s^2+t^2}$

```
z = exp(r) * cos(theta)
```

z = $e^{s\,t}\cos\left(\sqrt{s^2+t^2}\right)$

```
dzds = diff(z, s)
```

dzds =

$$t\,e^{s\,t}\cos\left(\sqrt{s^2+t^2}\right)-\frac{s\,e^{s\,t}\sin\left(\sqrt{s^2+t^2}\right)}{\left(s^2+t^2\right)^{0.5000}}$$

```
dzdt = diff(z, t)
```

dzdt =

$$s\,e^{s\,t}\cos\left(\sqrt{s^2+t^2}\right)-\frac{t\,e^{s\,t}\sin\left(\sqrt{s^2+t^2}\right)}{\left(s^2+t^2\right)^{0.5000}}$$

Exercise 24.2

```
syms x f(x)
f(x) = x^4 - x^2 + 1
```

f(x) = x^4-x^2+1

```
dfdx = diff(f, x)
```

dfdx(x) = $4\,x^3-2\,x$

```
clf; fplot(f); hold on; fplot(dfdx); legend("f(x)", "df/dx", "d^2f/dx^2"); hold off;
```

Warning: Ignoring extra legend entries.

```
critical_pt_x = fzero(dfdx, 0)
```

critical_pt_x = 0

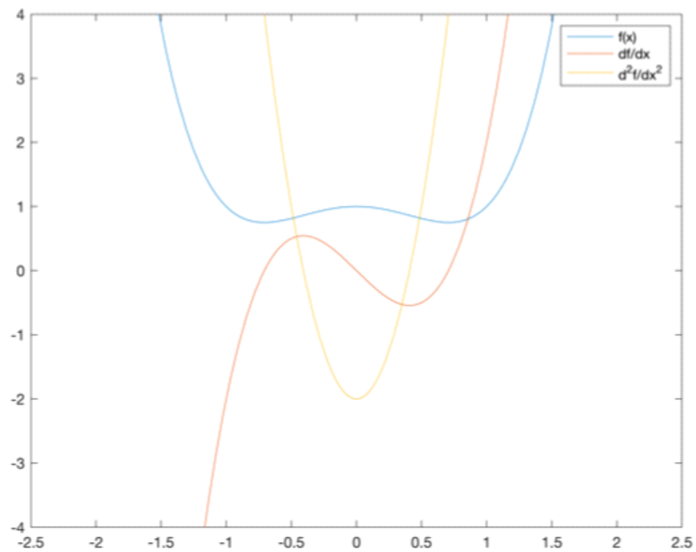
```
%%% Exercise 24.3

d2fdx2 = diff(dfdx, x)
```

d2fdx2(x) = $12\,x^2-2$

```
hold on; fplot(d2fdx2);  legend("f(x)", "df/dx", "d^2f/dx^2");

xlim([-2.5 2.5])
ylim([-4 4])
```




```
d2fdx2fn = matlabFunction(d2fdx2)
```

```
d2fdx2fn = function_handle with value:
    @(x)x.^2.*1.2e+1-2.0
```

```
d2fdx2fn(critical_pt_x)
```

```
ans = -2
```

```
if d2fdx2fn(critical_pt_x) < 0
    disp("Critical Point is a Relative Maximum!")
else
    disp("Critical Point is a Relative Minimum!")
end
```

```
Critical Point is a Relative Maximum!
```

Exercise 24.4.1

```
syms x y f(x,y)
f(x, y) = 4 + x^3 + y^3 - (3 * x * y);
ffn = matlabFunction(f)
```

```
ffn = function_handle with value:
    @(x,y)x.*y.*-3.0+x.^3+y.^3+4.0
```

```
pdfdx = diff(f, x)
```

pdfdx(x, y) = $3\,x^2 - 3\,y$

```
pdfdy = diff(f, y)
```

pdfdy(x, y) = $3\,y^2 - 3\,x$

```
gradient = [pdfdx; pdfdx]
```

gradient(x, y) =

$$\begin{pmatrix} 3\,x^2 - 3\,y \\ 3\,x^2 - 3\,y \end{pmatrix}$$

```
pdfdxfn = matlabFunction(pdfdx)
```

```
pdfdxfn = function_handle with value:
    @(x,y)y.*-3.0+x.^2.*3.0
```

```
pdfdyfn = matlabFunction(pdfdy)
```

```
pdfdyfn = function_handle with value:
    @(x,y)x.*-3.0+y.^2.*3.0
```

```
gradientFn = matlabFunction(gradient)
```

```
gradientFn = function_handle with value:
    @(x,y)[y.*-3.0+x.^2.*3.0;y.*-3.0+x.^2.*3.0]
```

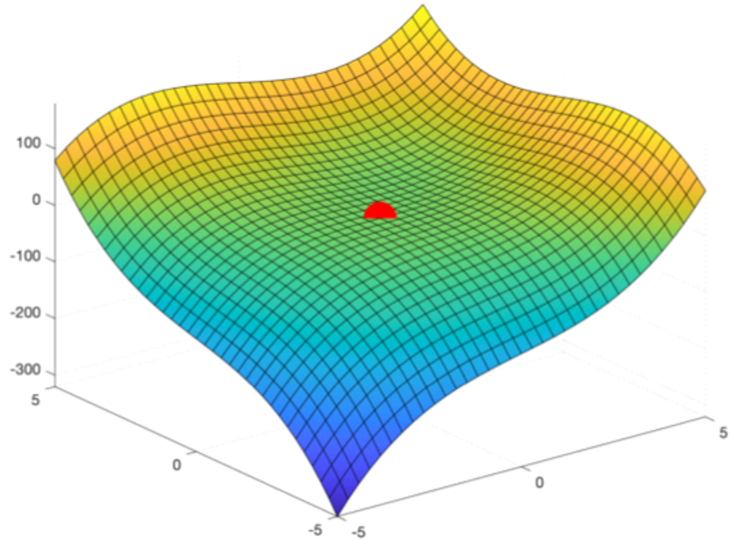
```
cp_x = fzero(@(x) pdfdx(x, 0), 0)
```

```
cp_x = 0
```

```
cp_y = fzero(@(y) pdfdx(cp_x, y), 0)
```

```
cp_y = 0
```

```
clf; fsurf(f); hold on; plot3(cp_x, cp_y, double(f(cp_x, cp_y)), 'r.', 'MarkerSize', 7
```



Exercise 24.4.2

```
syms x y f(x,y)
f(x, y) = (3 * x^2 * y) + y^3 - (3 * x^2) - (3 * y^2) + 2
```

f(x, y) = $3\,x^2\,y - 3\,x^2 + y^3 - 3\,y^2 + 2$

```
ffn = matlabFunction(f)
```

ffn = *function_handle with value:*
    @(x,y)x.^2.*y.*3.0-x.^2.*3.0-y.^2.*3.0+y.^3+2.0

```
pdfdx = diff(f, x)
```

pdfdx(x, y) = $6\,x\,y - 6\,x$

```
pdfdy = diff(f, y)
```

pdfdy(x, y) = $3\,x^2 + 3\,y^2 - 6\,y$

```
gradient = [pdfdx; pdfdx]
```

gradient(x, y) =

$$\begin{pmatrix} 6\,x\,y - 6\,x \\ 6\,x\,y - 6\,x \end{pmatrix}$$

```
pdfdxfn = matlabFunction(pdfdx)
```

```
pdfdxfn = function_handle with value:
    @(x,y)x.*-6.0+x.*y.*6.0
```

```
pdfdyfn = matlabFunction(pdfdy)
```

```
pdfdyfn = function_handle with value:
    @(x,y)y.*-6.0+x.^2.*3.0+y.^2.*3.0
```

```
gradientFn = matlabFunction(gradient)
```

```
gradientFn = function_handle with value:
    @(x,y)[x.*-6.0+x.*y.*6.0;x.*-6.0+x.*y.*6.0]
```

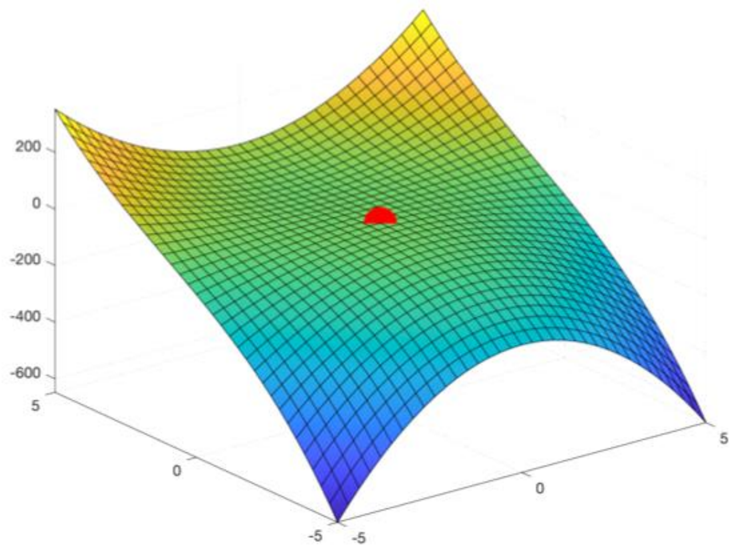
```
cp_x = fzero(@(x) pdfdx(x, 0), 0)
```

```
cp_x = 0
```

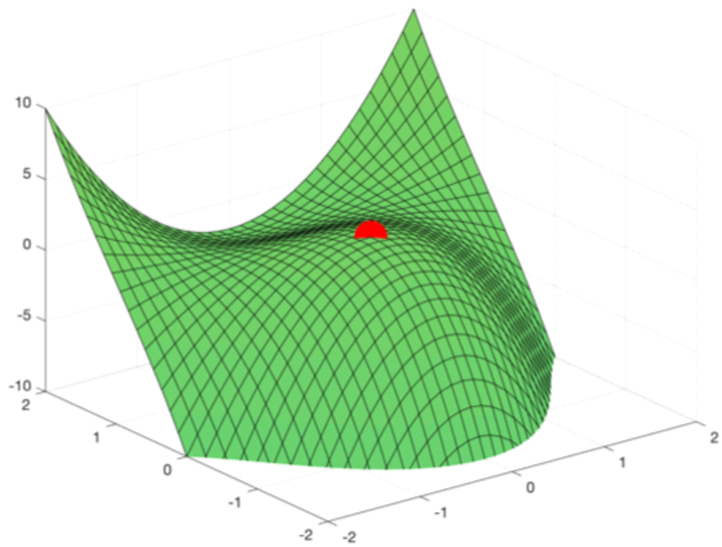
```
cp_y = fzero(@(y) pdfdx(cp_x, y), 0)
```

```
cp_y = 0
```

```
clf; fsurf(f); hold on; plot3(cp_x, cp_y, double(f(cp_x, cp_y)), 'r.', 'MarkerSize', 7!
```




```
figure; clf; fsurf(f); hold on; plot3(cp_x, cp_y, double(f(cp_x, cp_y)), 'r.', 'Marker!
xlim([-2, 2])
ylim([-2, 2])
zlim([-10, 10])
```




```
pd2fdx2(x, y) = diff(pdfdx, x)
```

pd2fdx2(x, y) = $6\,y-6$

```
pd2fdx2(cp_x, cp_y)
```

ans = -6

```
pd2fdy2(x, y) = diff(pdfdy, y)
```

pd2fdy2(x, y) = $6\,y-6$

```
pd2fdy2(cp_x, cp_y)
```

ans = -6

Since the 2nd derivitives are negative in both directions, this is a local maximum

Exercise 24.4.1

```
syms x y f(x,y)
f(x, y) = 4 + x^3 + y^3 - (3 * x * y);
ffn = matlabFunction(f)
```

```
ffn = function_handle with value:
    @(x,y)x.*y.*-3.0+x.^3+y.^3+4.0
```

```
pdfdx = diff(f, x)
```

pdfdx(x, y) = $3x^2 - 3y$

```
pdfdy = diff(f, y)
```

pdfdy(x, y) = $3y^2 - 3x$

```
gradient = [pdfdx; pdfdx]
```

gradient(x, y) =

$$\begin{pmatrix} 3x^2 - 3y \\ 3x^2 - 3y \end{pmatrix}$$

```
pdfdxfn = matlabFunction(pdfdx)
```

```
pdfdxfn = function_handle with value:
    @(x,y)y.*-3.0+x.^2.*3.0
```

```
pdfdyfn = matlabFunction(pdfdy)
```

```
pdfdyfn = function_handle with value:
    @(x,y)x.*-3.0+y.^2.*3.0
```

```
gradientFn = matlabFunction(gradient)
```

```
gradientFn = function_handle with value:
    @(x,y)[y.*-3.0+x.^2.*3.0;y.*-3.0+x.^2.*3.0]
```

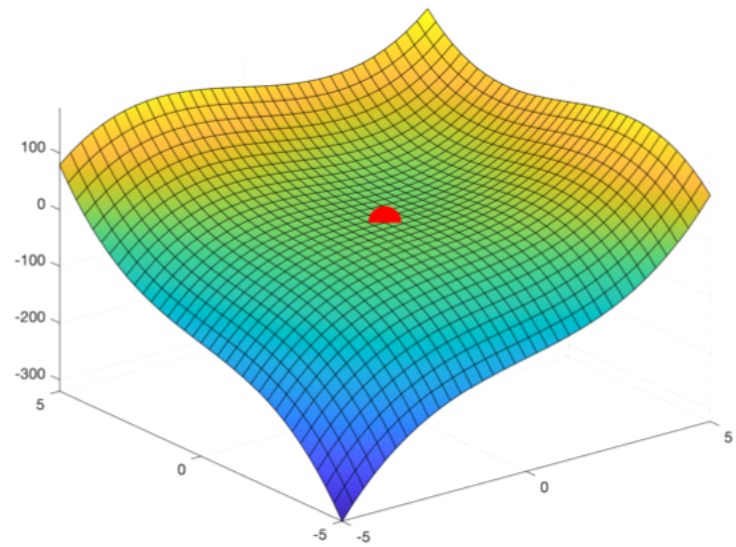
```
cp_x = fzero(@(x) pdfdx(x, 0), 0)
```

```
cp_x = 0
```

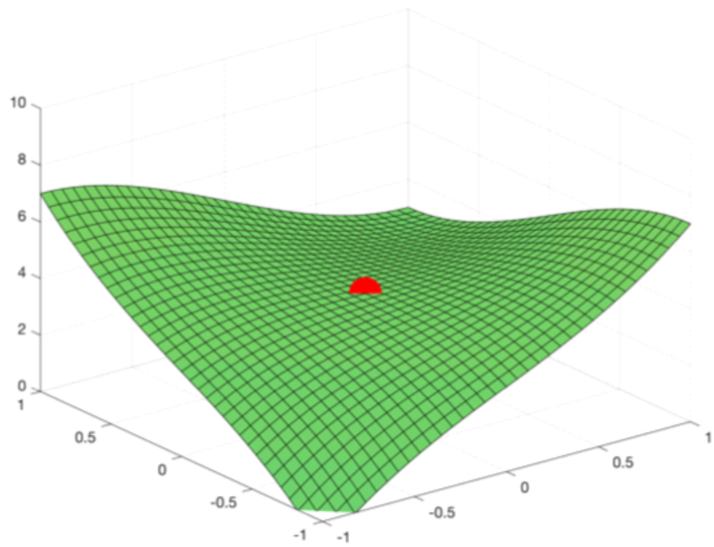
```
cp_y = fzero(@(y) pdfdx(cp_x, y), 0)
```

```
cp_y = 0
```

```
clf; fsurf(f); hold on; plot3(cp_x, cp_y, double(f(cp_x, cp_y)), 'r.', 'MarkerSize', 7
```




```
figure; clf; fsurf(f); hold on; plot3(cp_x, cp_y, double(f(cp_x, cp_y)), 'r.', 'Marker
xlim([-1, 1])
ylim([-1, 1])
zlim([0, 10])
```




```
pd2fdx2(x, y) = diff(pdfdx, x)
```

pd2fdx2(x, y) = $6x$

```
pd2fdx2(cp_x, cp_y)
```

ans = 0

```
pd2fdy2(x, y) = diff(pdfdy, y)
```

pd2fdy2(x, y) = $6y$

```
pd2fdy2(cp_x, cp_y)
```

ans = 0

Since the 2nd derivitives are zero in both directions, this critical point can't be classified.