Data Structures & Algorithms

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Balanced Binary Search Trees

Outline

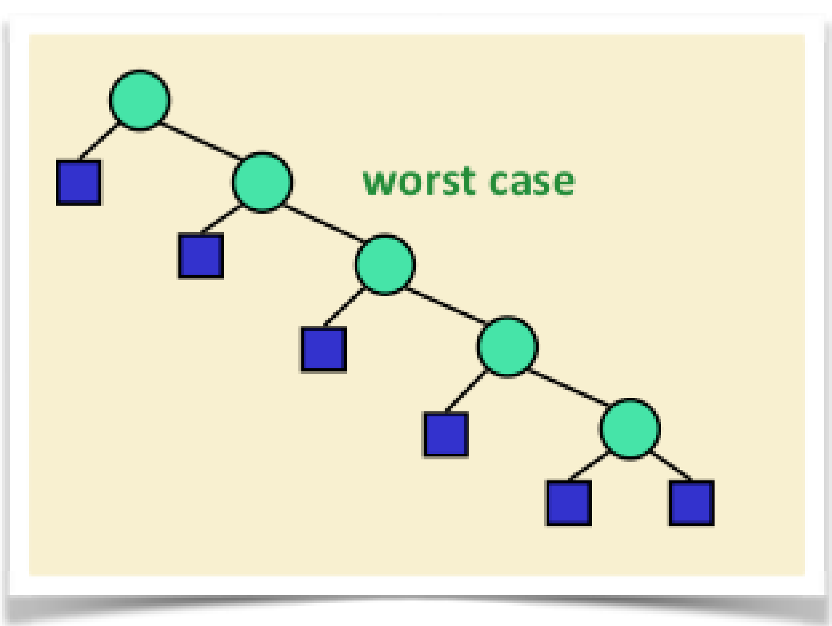
- Motivation
- AVL Trees
- Red-Black Trees

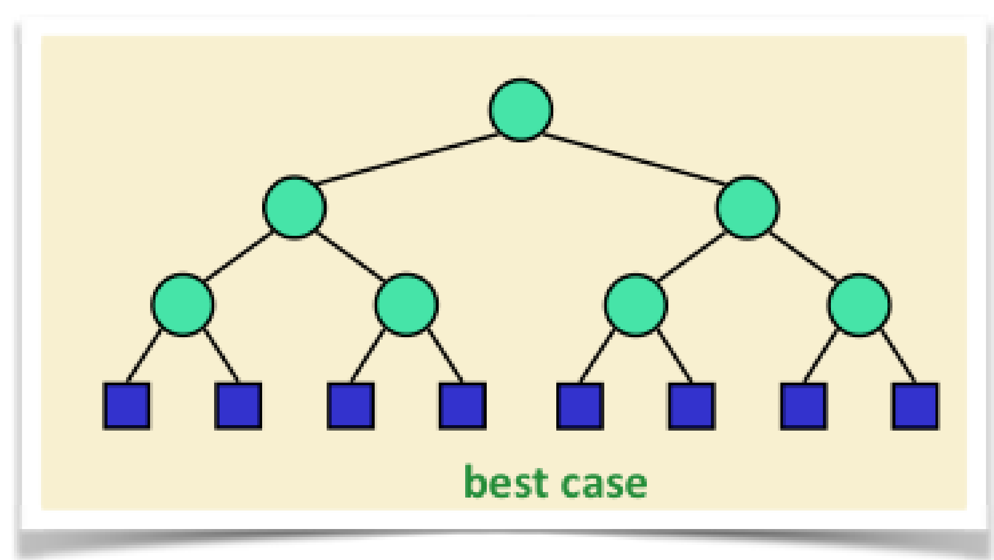
Binary Search Tree

- For a binary search tree with *n* nodes
  - average search and insertion time is O(log *n*)
  - It will take $log_2 n$ comparisons to find a particular node of find out that it isn't
- However, this is only true is the tree is "balanced"
- That is, the "height" of the tree is balanced

Binary Search Tree

- In the worst case, insertion and searching time becomes O(n)
- Because the height is O(n)





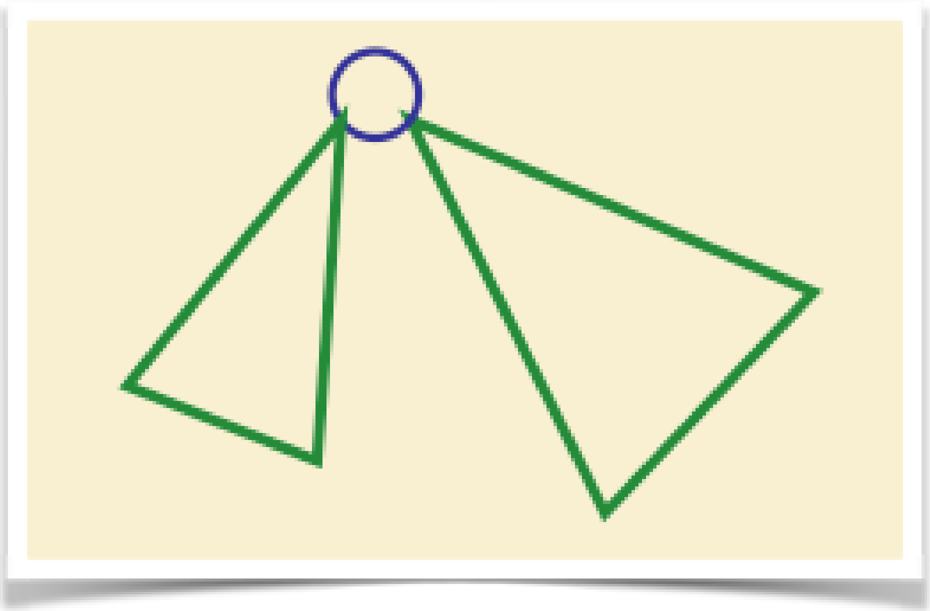
Binary Search Tree

- In a dynamic tree, nodes are inserted and deleted over time

- So we must find a way to keep the height of a binary search tree always O(log *n*)

- To achieve this, the tree must always be balanced

“for any node, its left subtree should not be much higher then its right subtree, and vice-versa”



AVL Trees

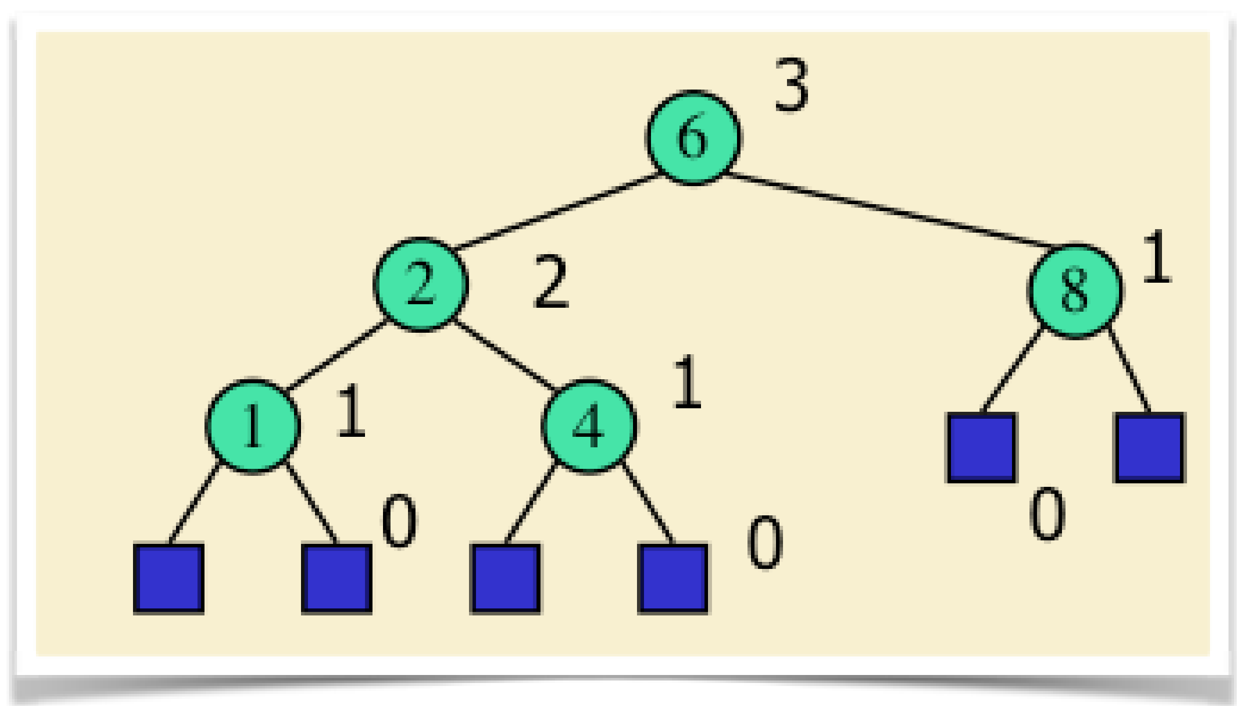
AVL Trees

- Adelson-Velskii and Landis in 1962 introduced a binary tree structure that is balanced with respect to the heights of its subtree

- Insertions and deletions are made such that the tree always remain height-balanced

Height of a Node

- Height of a tree is the maximum over all node depths, or, equivalently, the longest path in the tree
- The height of a tree node **v** is defined as the height of the subtree rooted at node **v**



AVL Trees

- Definition
- An empty tree is height-balanced
- If **T** is non-empty binary tree with left and right sub-trees **T_1** and **T_2**

T is balanced if and only if

  - **T_1** and **T_2** are balanced, and
  - |height(**T_1**) - height(**T_2**)| <= 1

Recall: Binary Tree Terminology

- Height of a tree **T**
- 0 (for convenience) if **T** is empty and

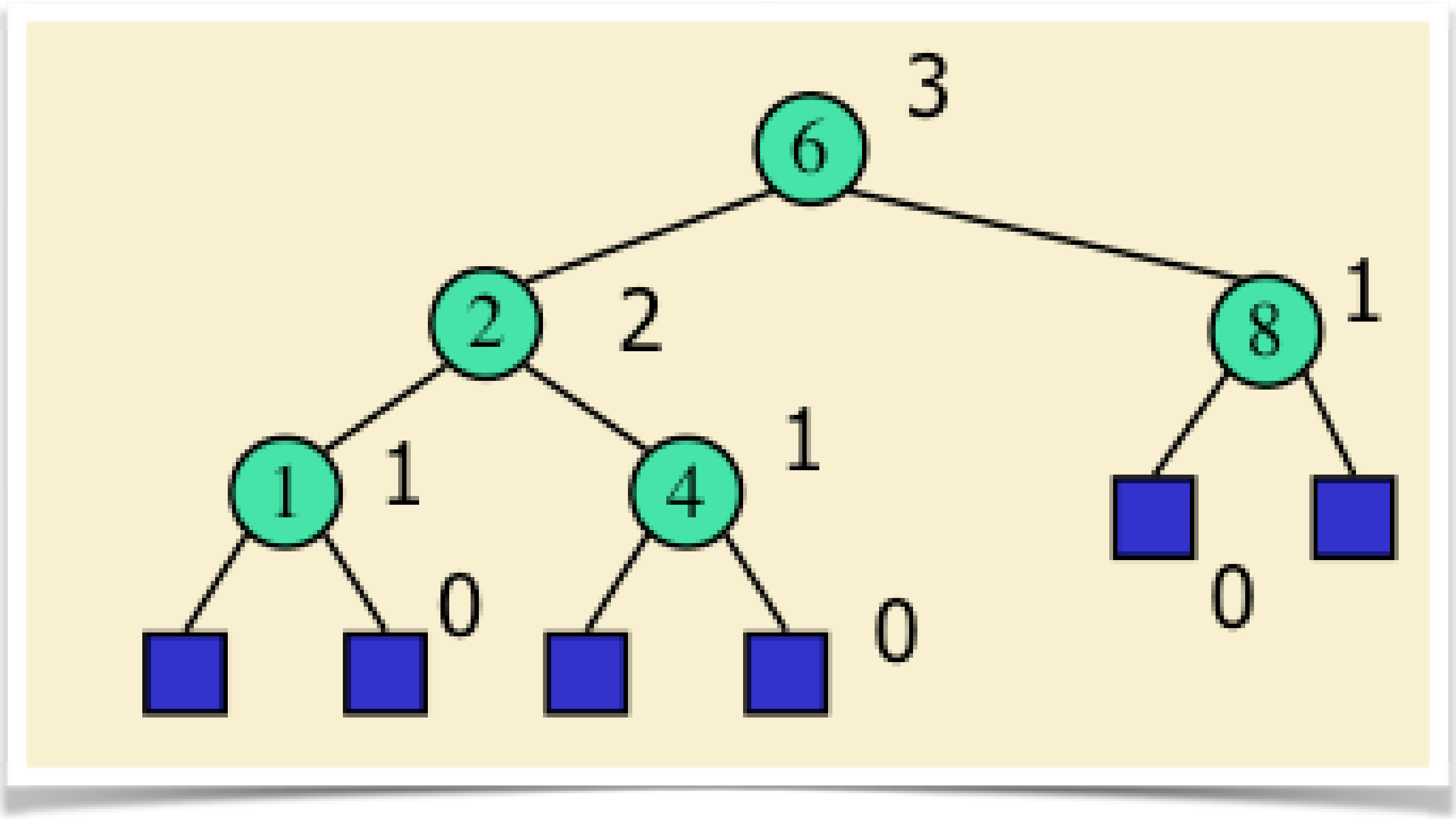
  1 + max(height(**T_1**), height(**T_2**)) otherwise,

  where **T_1** and **T_2** are the subtrees of the root

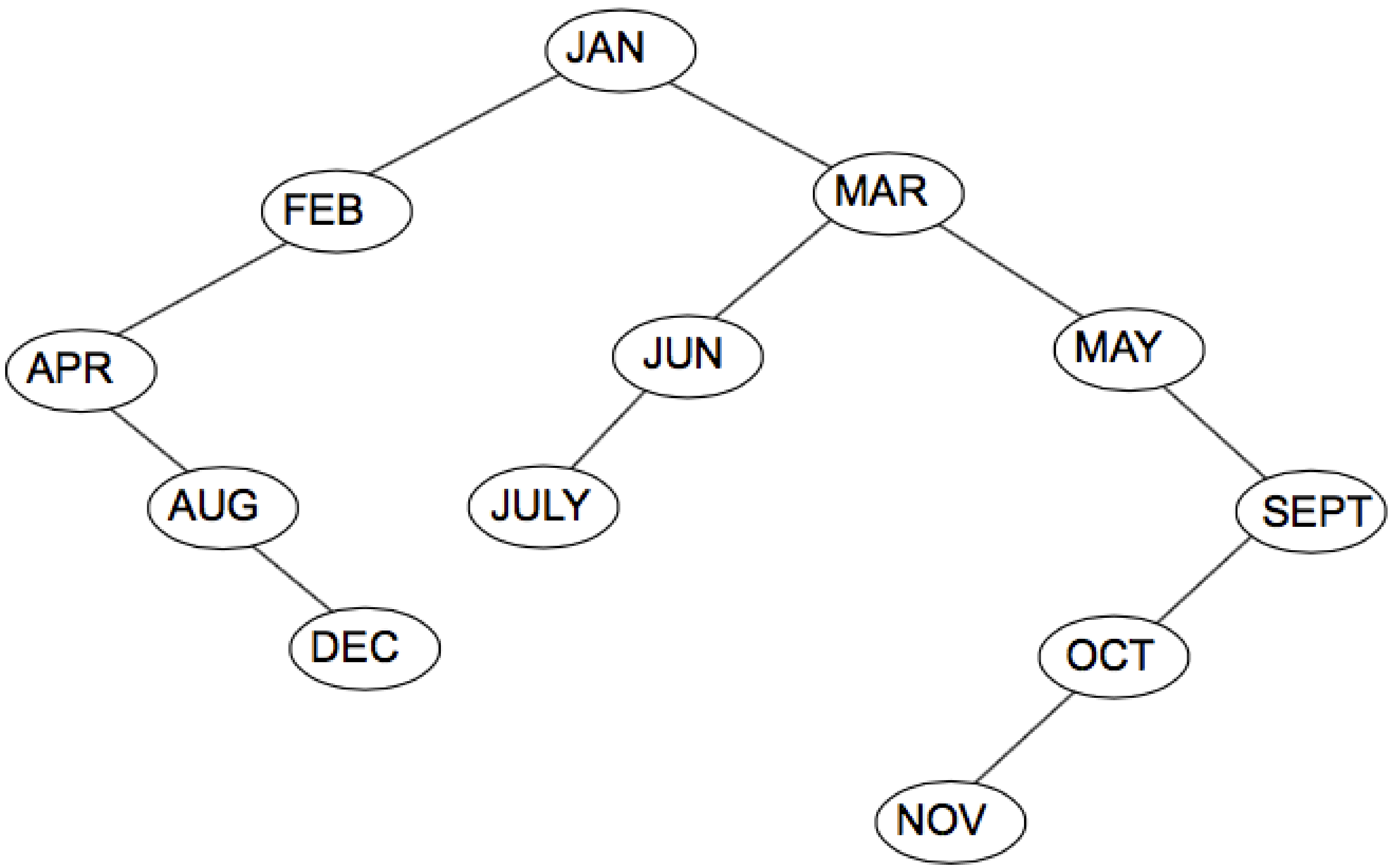
Recall: Binary Tree Terminology

- Height Numbering
  - Number all external (leaf) nodes 0
  - Number each internal node to be one more than the maximum of the numbers of its children
  - Then number of the root node is the height of **T**
- The height of a node **u** in **T** is the height of the subtree rooted at **u**

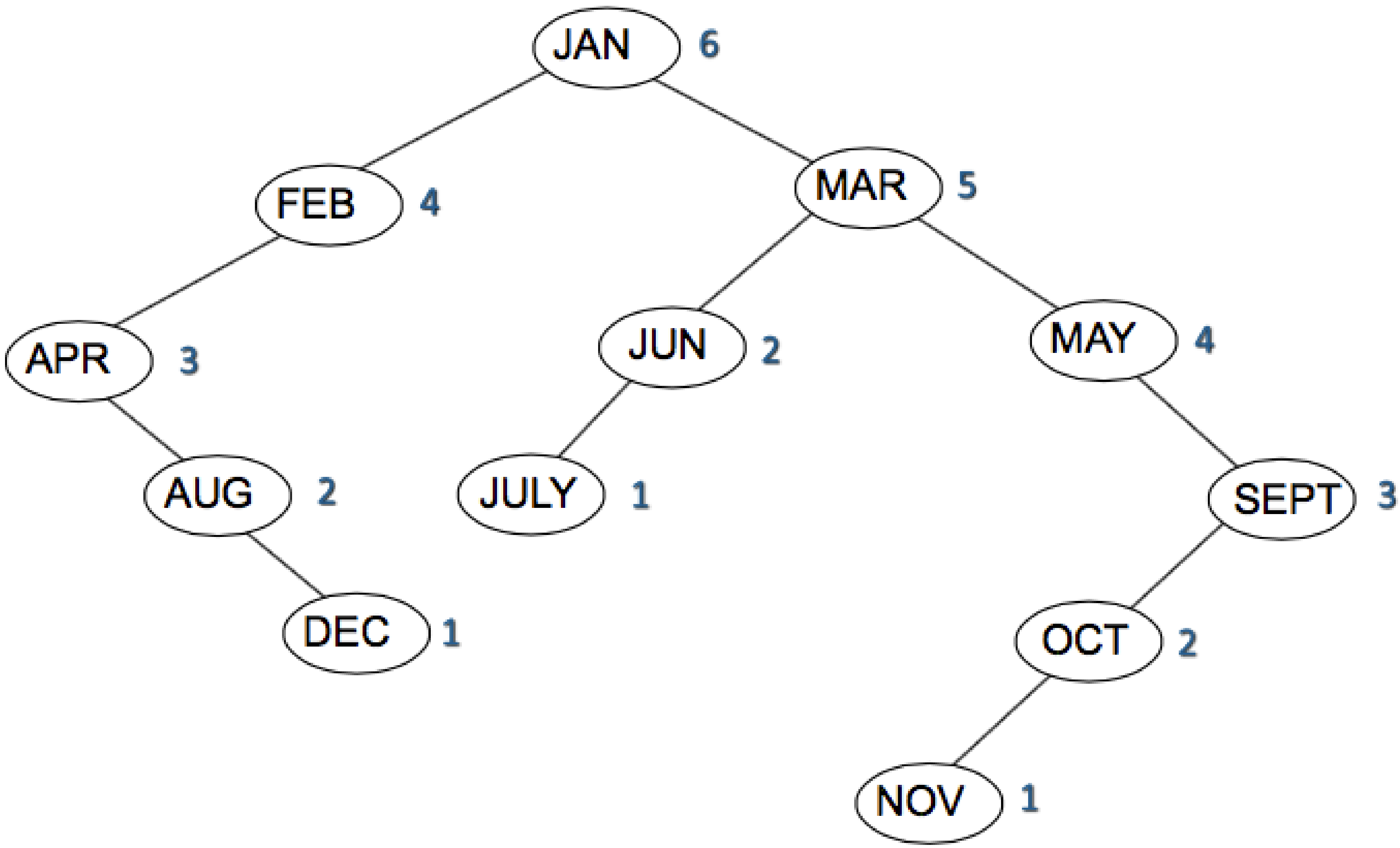
AVL Trees



AVL Trees



AVL Trees



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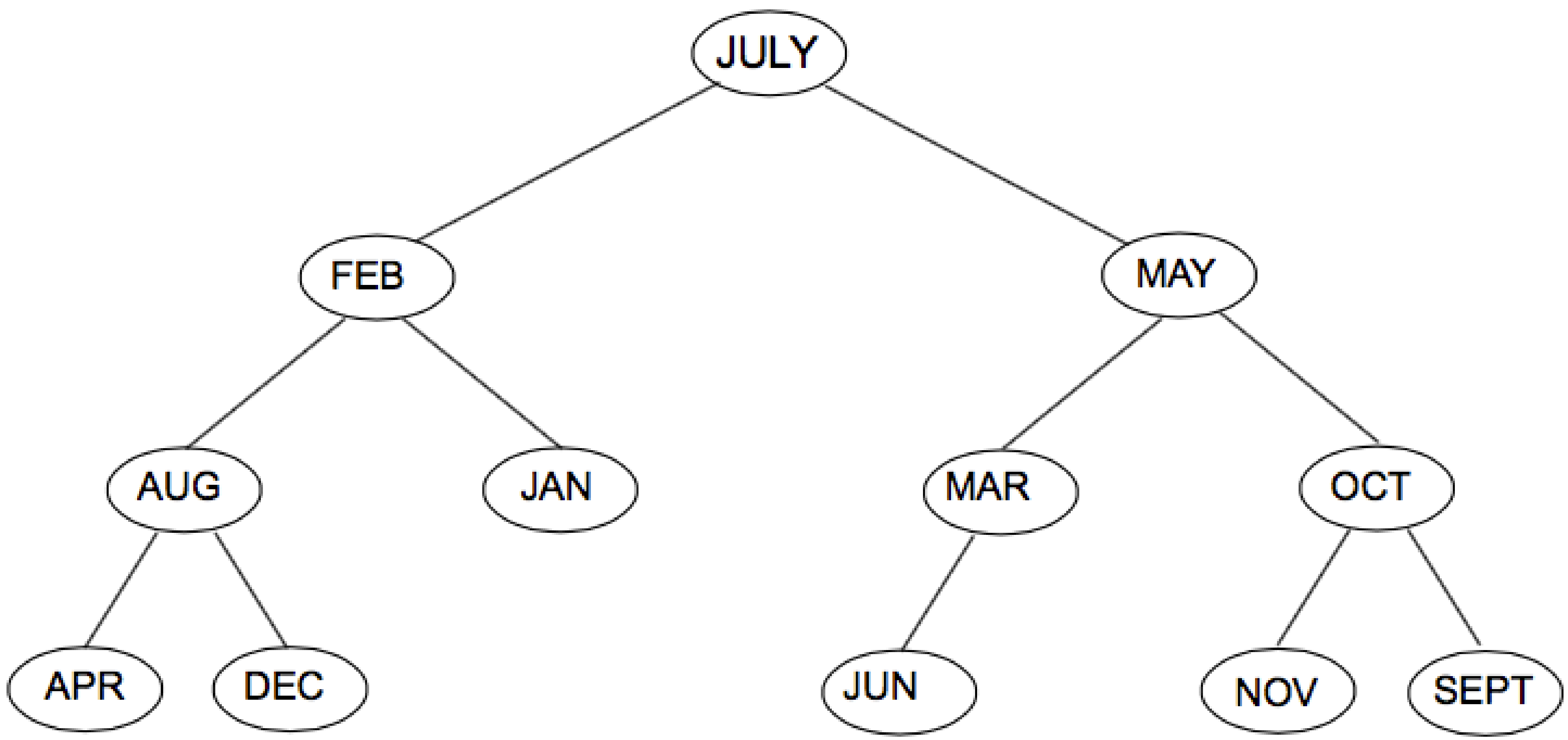
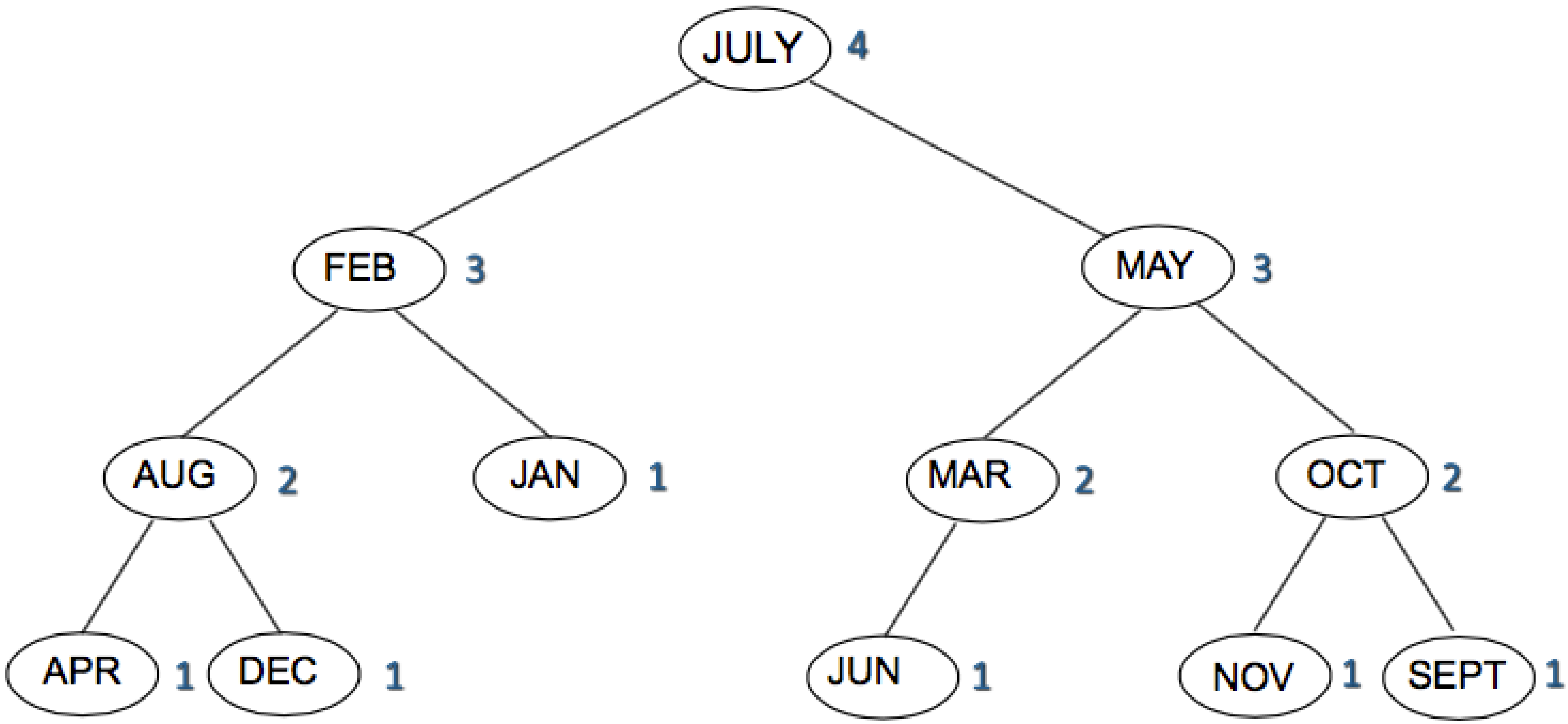
AVL Trees



A Balanced Tree for the Months of the Year

AVL Trees



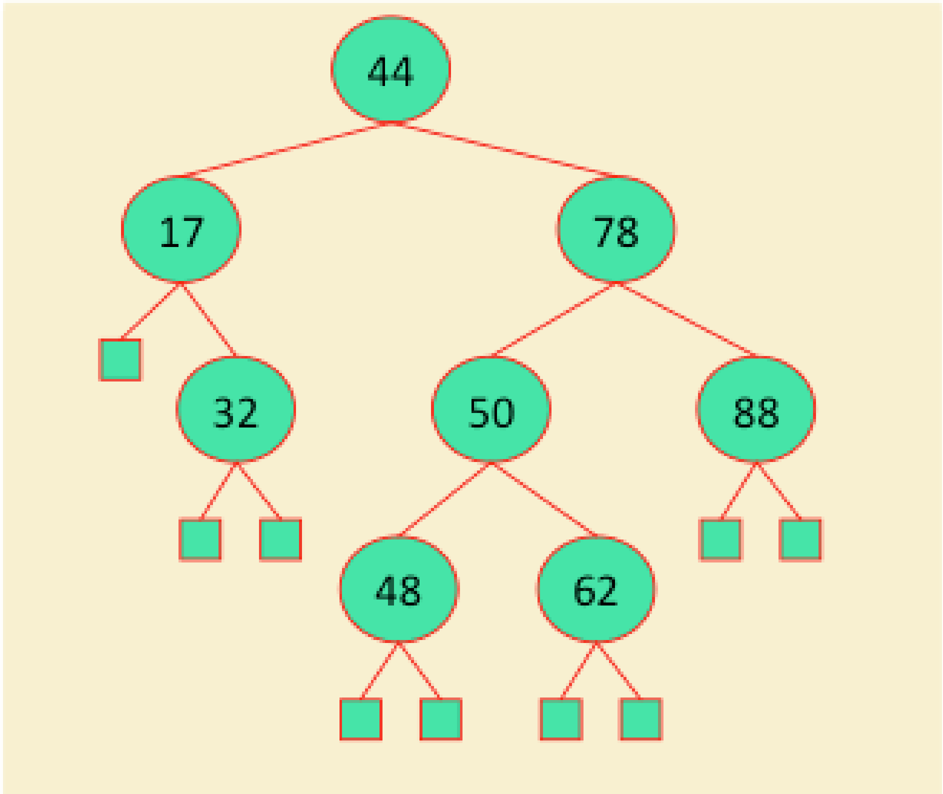
A Balanced Tree for the Months of the Year

Operations in an AVL Tree

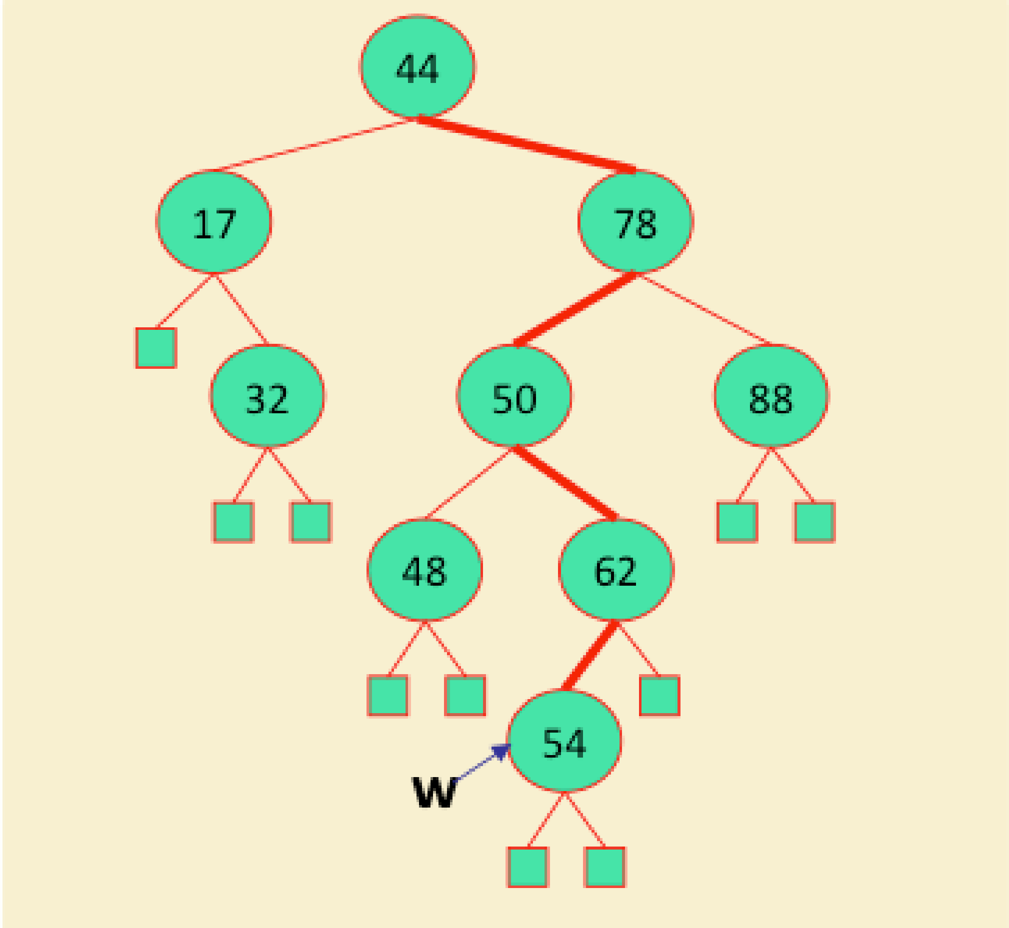
- The height of an AVL tree is O(log n)
- Thus the **search** operation takes O(log n)
    - Performed just like in a binary search tree since AVL tree is a binary search tree
- What we need to show is how to **insert** and **remove** in AVL trees while maintaining
    - the height balanced property
    - the binary search tree order

Insertion in an AVL Tree

- Starts as in a binary search tree
- Always done by expanding an external node



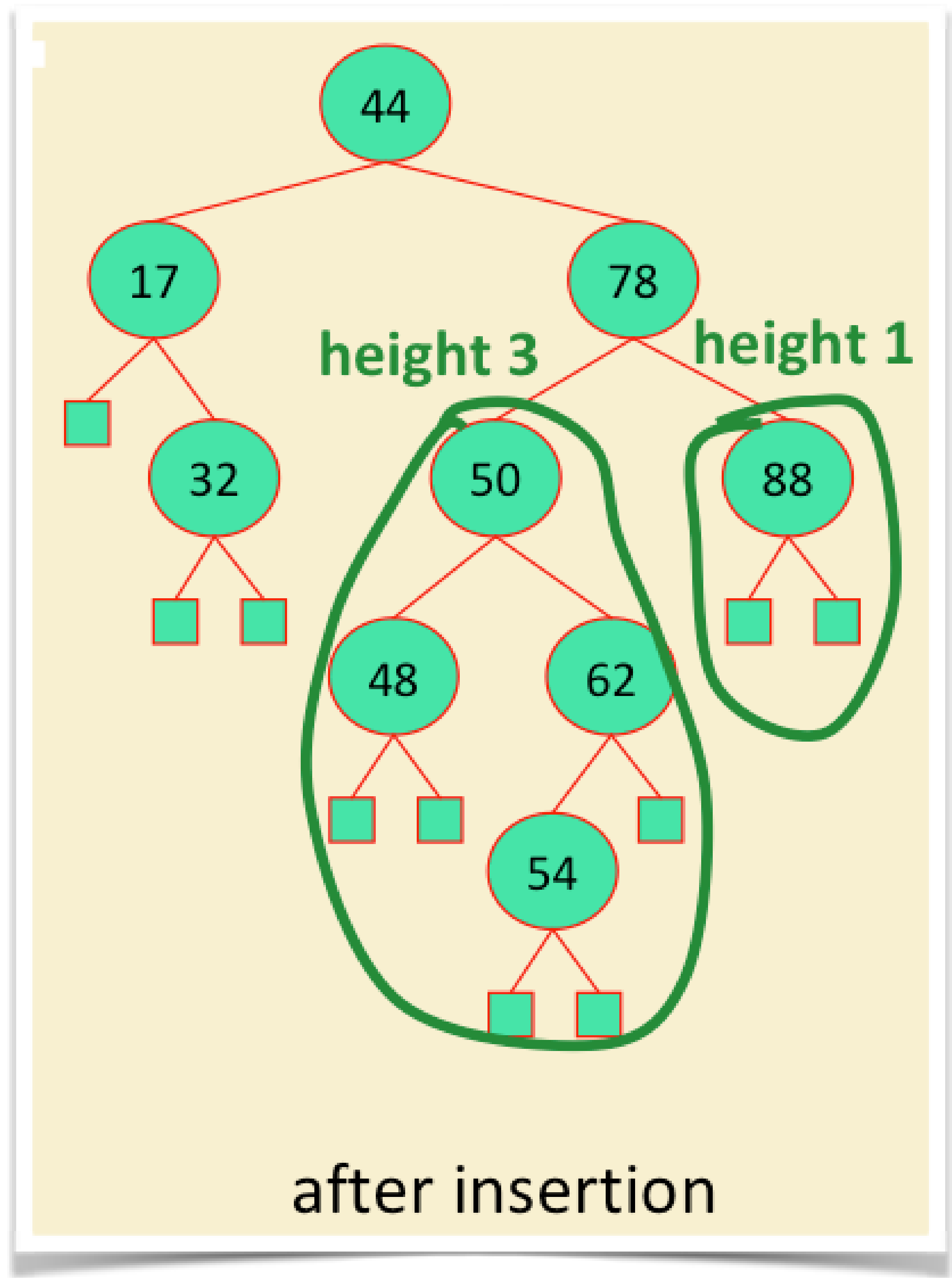
before inserting 54



after insertion

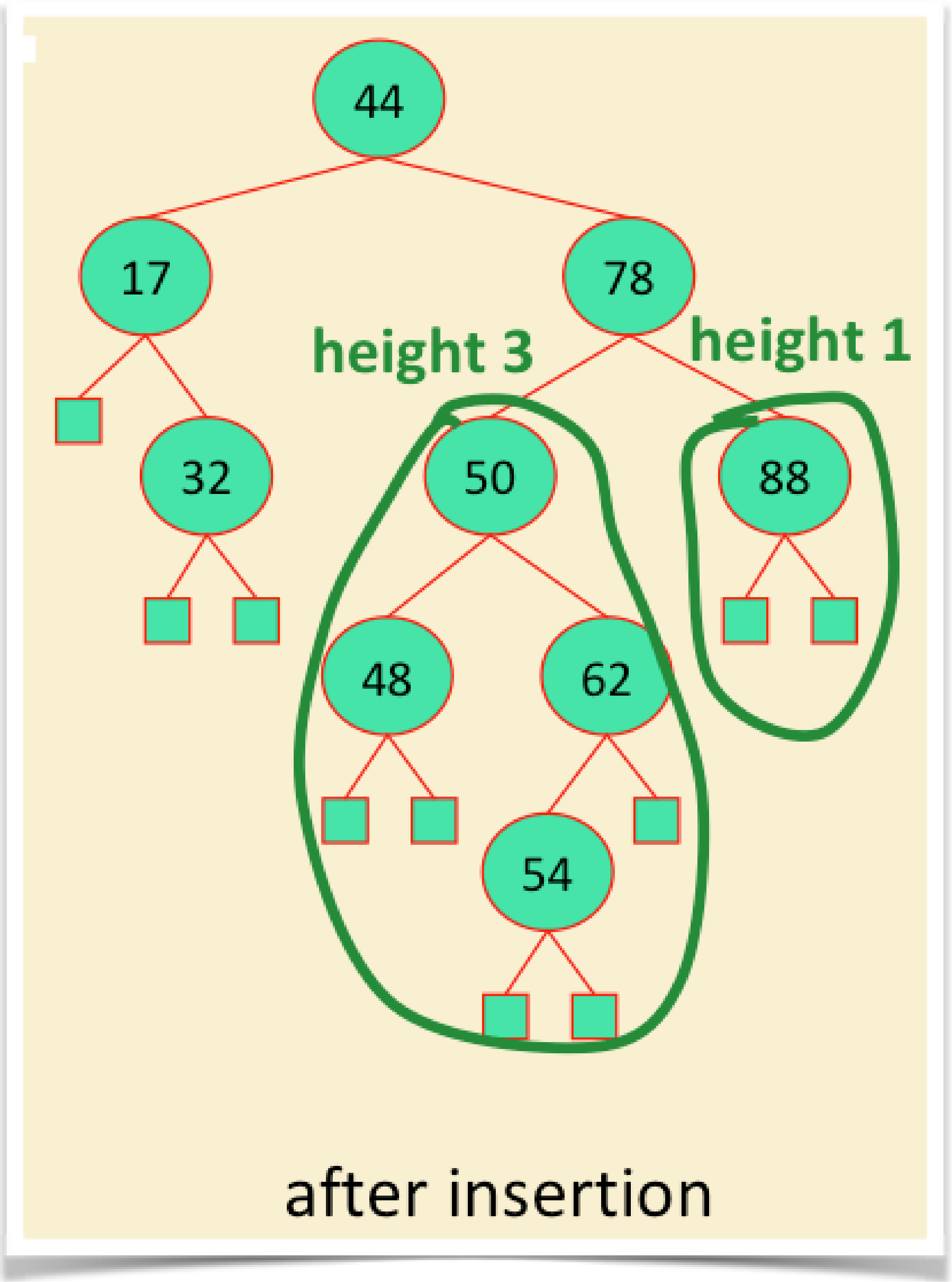
Insertion in an AVL Tree

- After inserting a new node into an AVL tree, the height-balanced property of the AVL tree is very likely lost

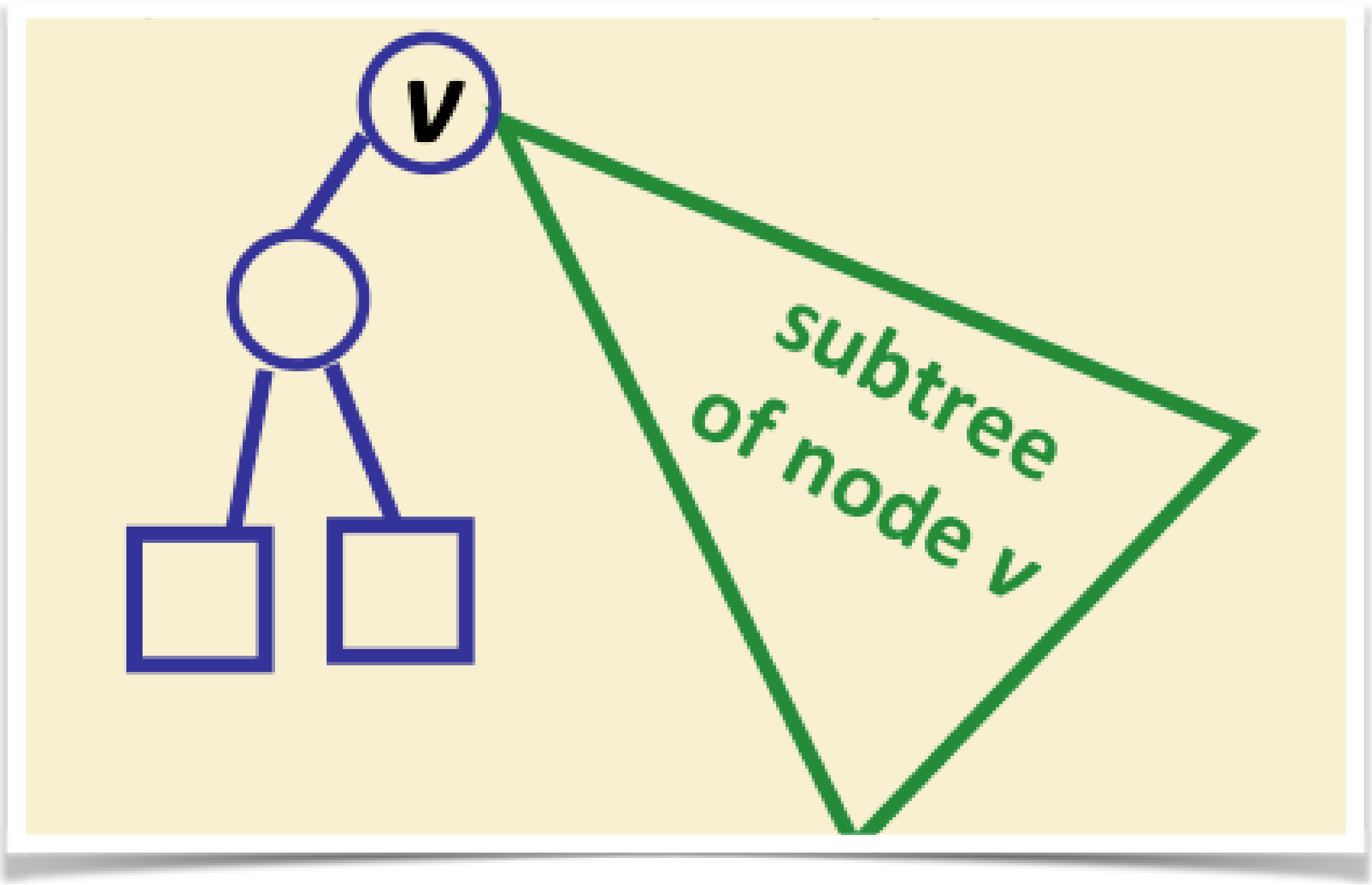


Insertion in an AVL Tree

- Thus, to make it an AVL tree again, we need to **restore the balance** by **restructuring the tree**

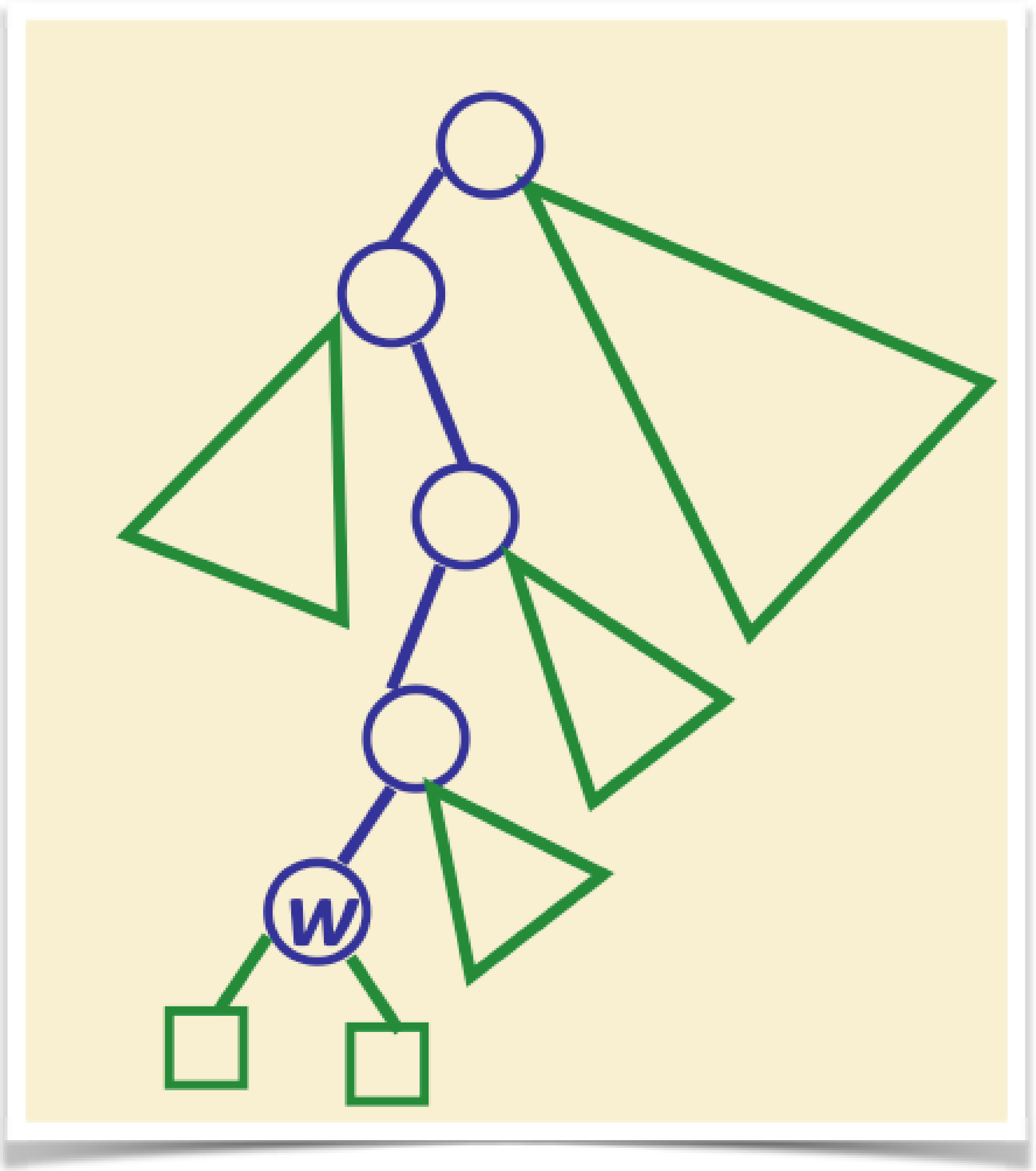


Pictorial Notation



Restructuring

- Let w be the new node, just inserted into an AVL tree

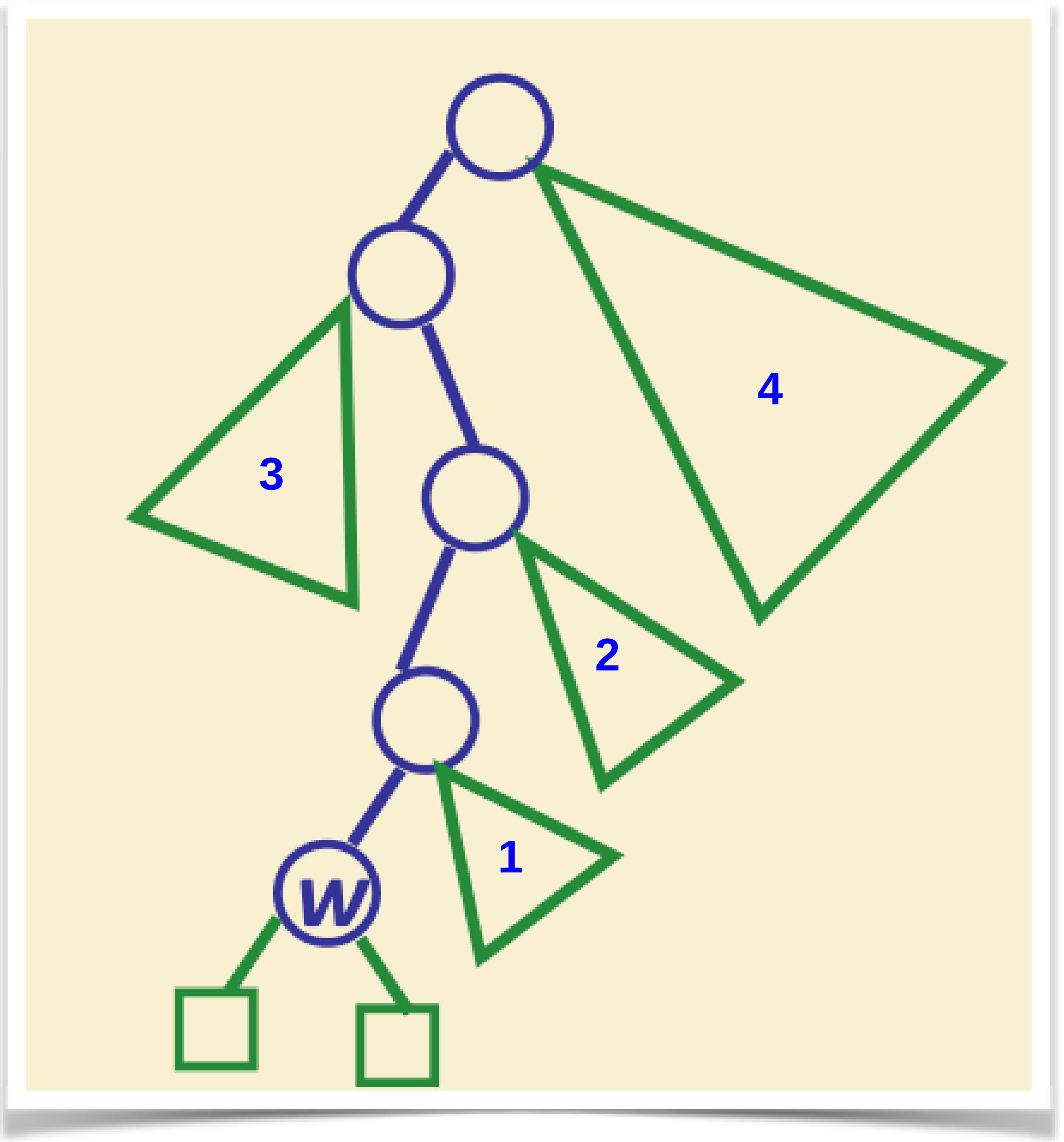


Restructuring

- The next step is to search for the unbalanced node(s)
- Check each node in the tree to see if it is balanced.
- Do you think this approach is efficient?
- Can we make it efficient?

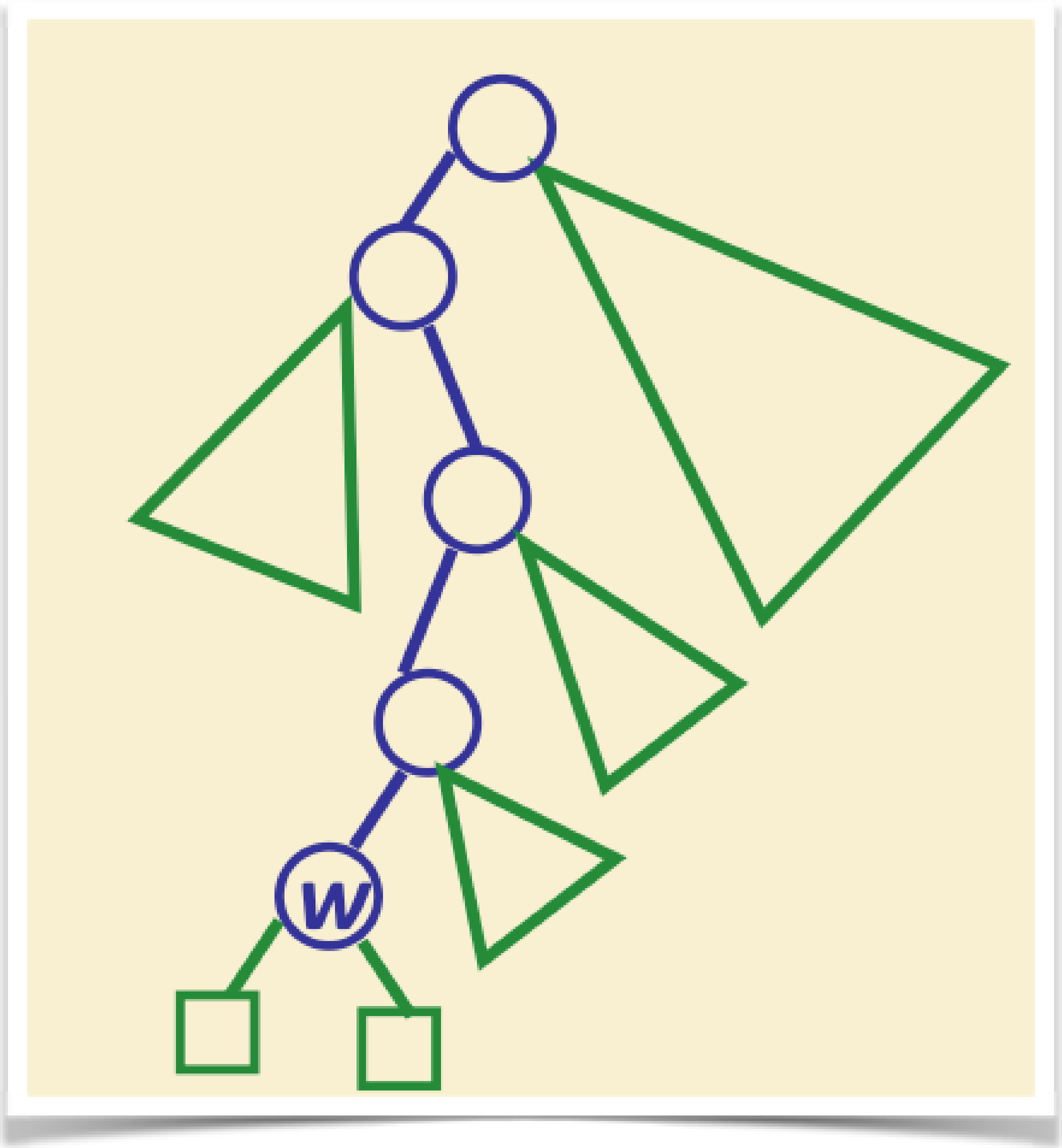
Restructuring

- Tip: after insertion of **w**, heights could change (increase) only for the ancestors of **w**
- Thus only ancestors of w could be unbalanced
- Search up the tree from w checking and correcting any unbalanced node



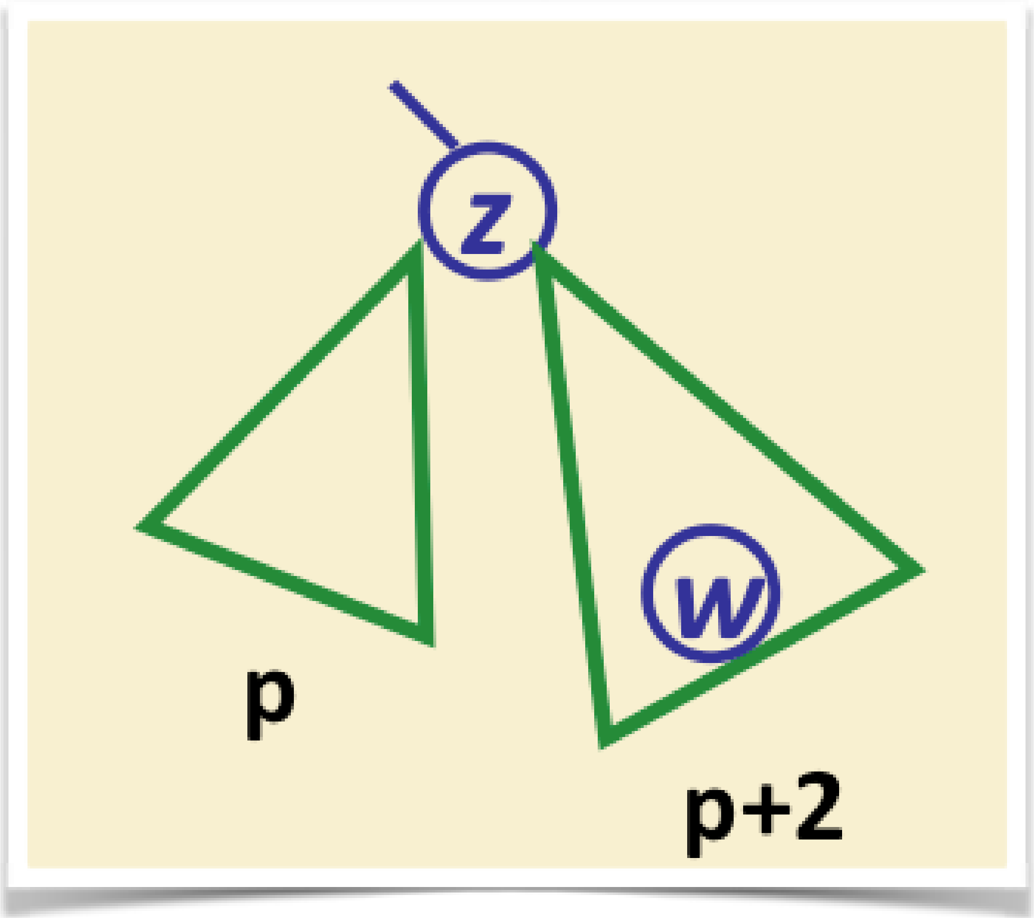
Restructuring

- Follow the path from **w** to the root



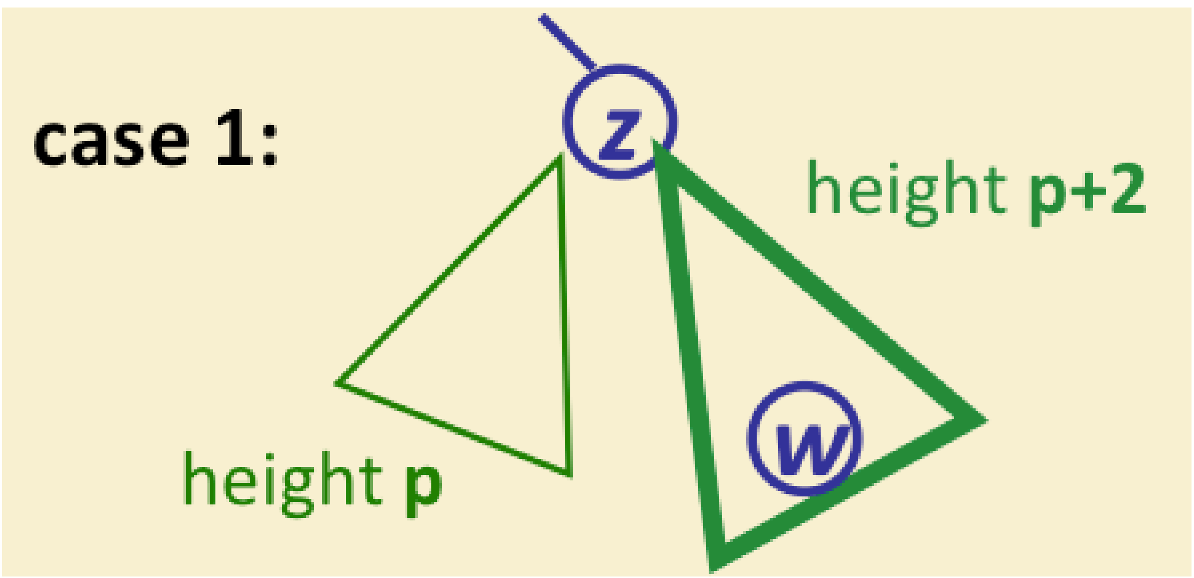
Restructuring

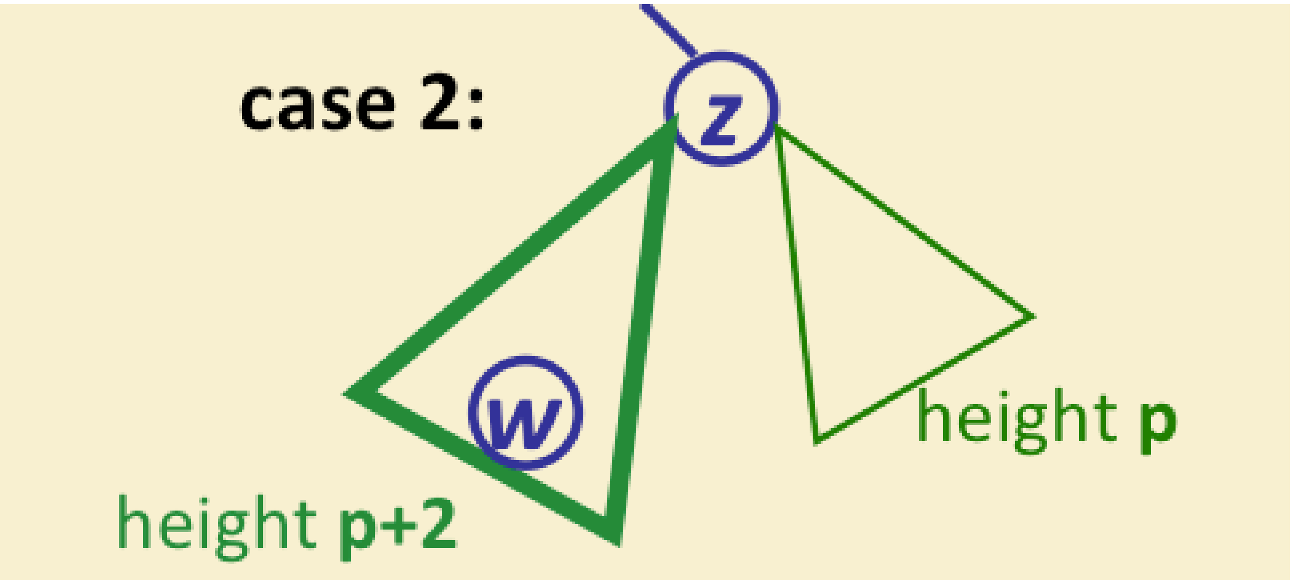
- Suppose the first unbalanced node is at position **z**
- Height difference between the left and the right subtree of **z** is more than 1
- In fact, it is **exactly** 2
  - tree was balanced before insertion
  - each insertion can change height only by a factor of 1
- **w** is in the higher subtree



Restructuring

- Two cases:
  - Right subtree is higher
  - Or, left subtree is higher



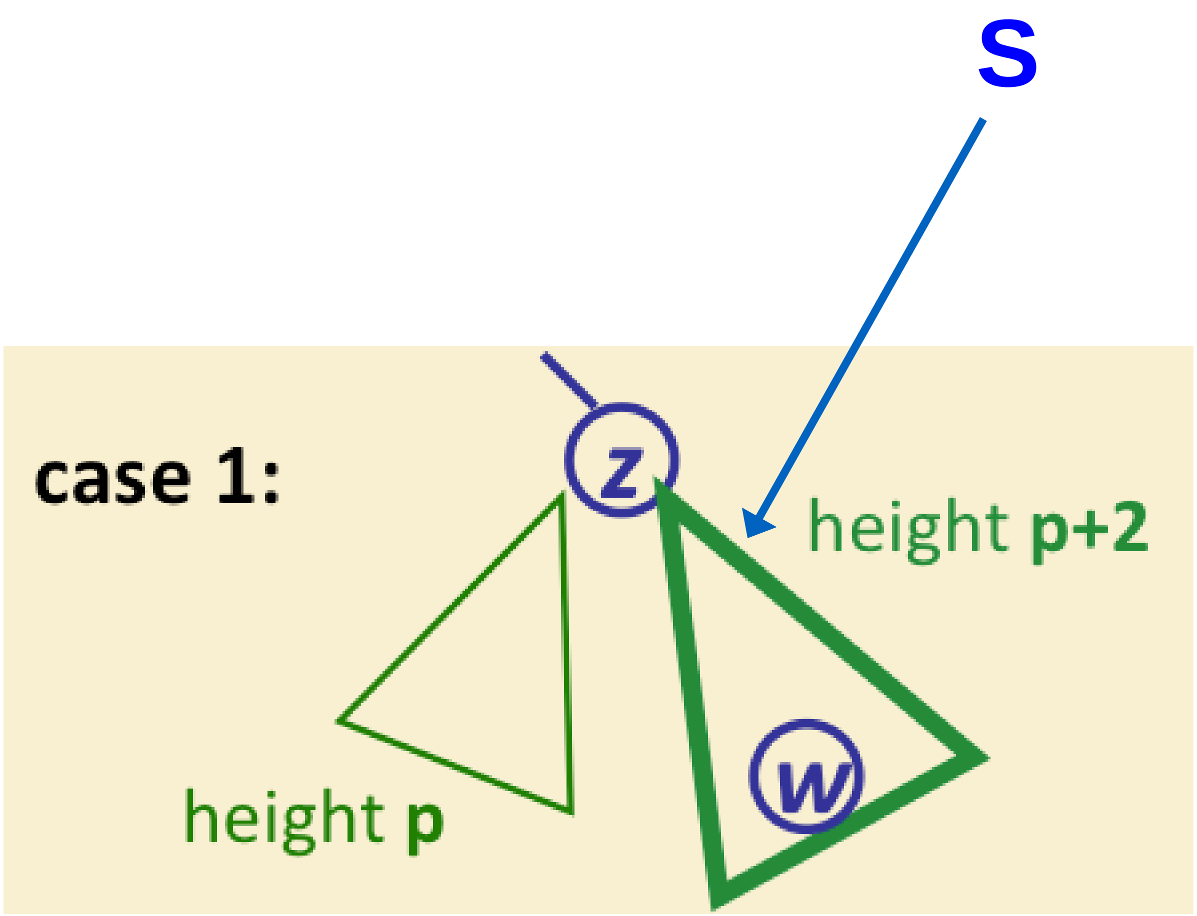


Restructuring

- Let **S** be the higher subtree, with height p+2
- Let **y** be the root of **S**

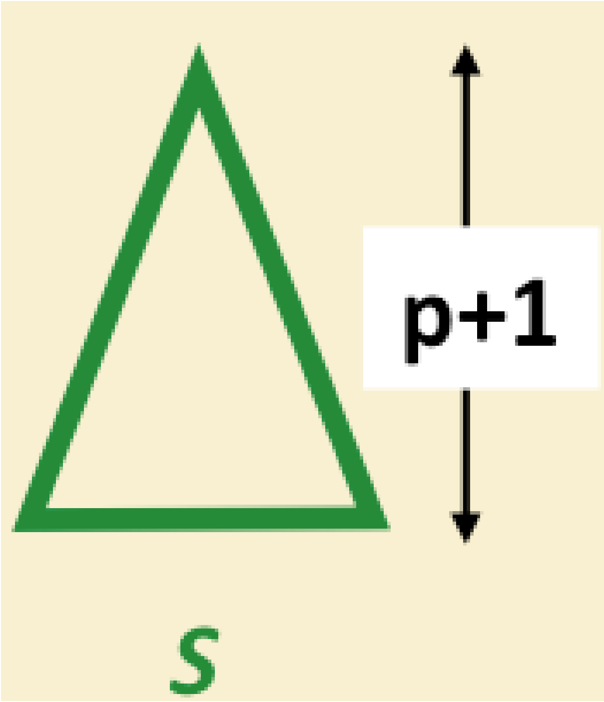
Refresher

- **w** is the new node
- **z** (ancestor of **w**) is the first unbalanced node
- **S** is the higher subtree of **z**
- **y** is the root of **S**



Restructuring

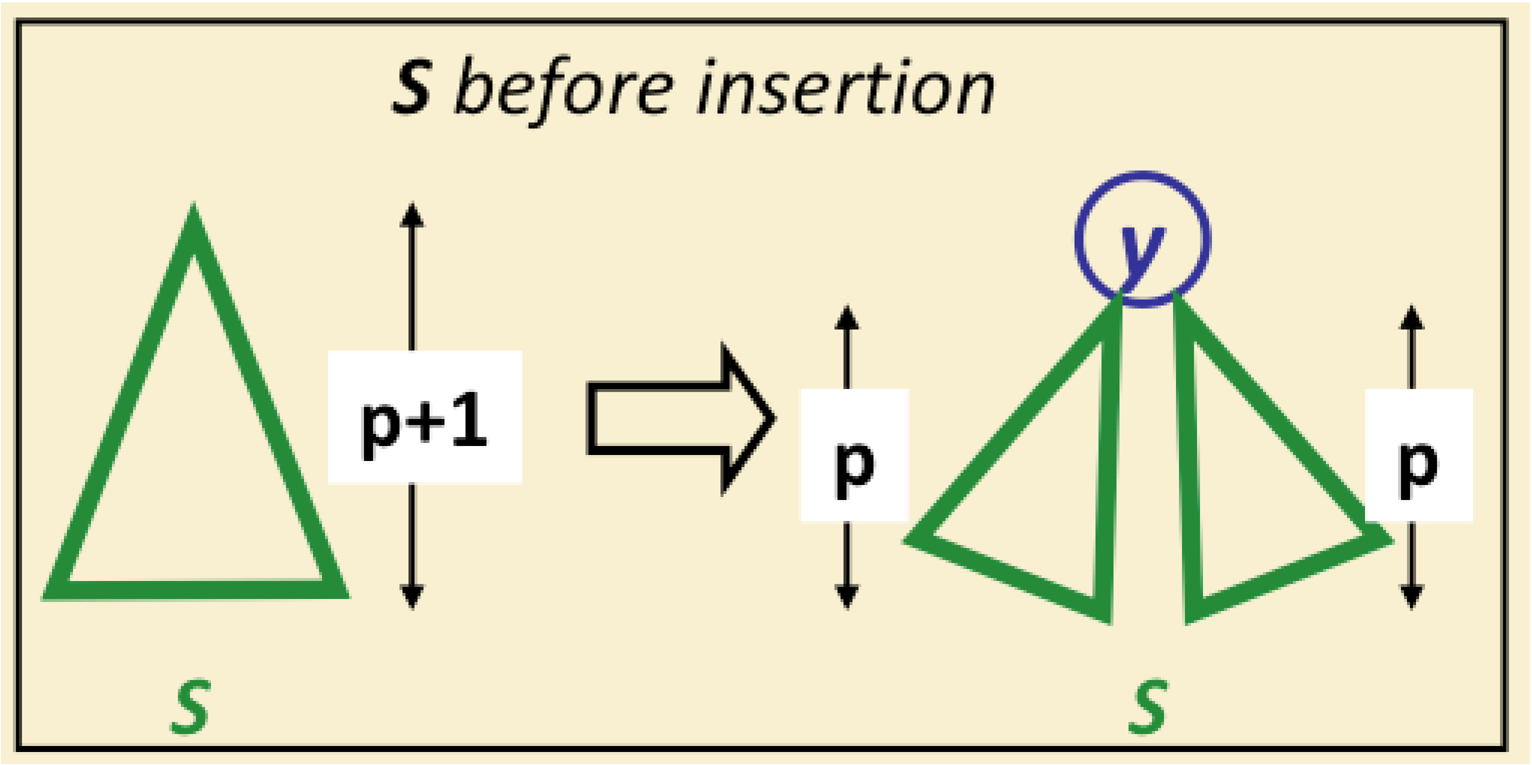
- Remember, the tree was balanced before insertion
- So, the height of **S** was p+1 **before insertion**



Restructuring

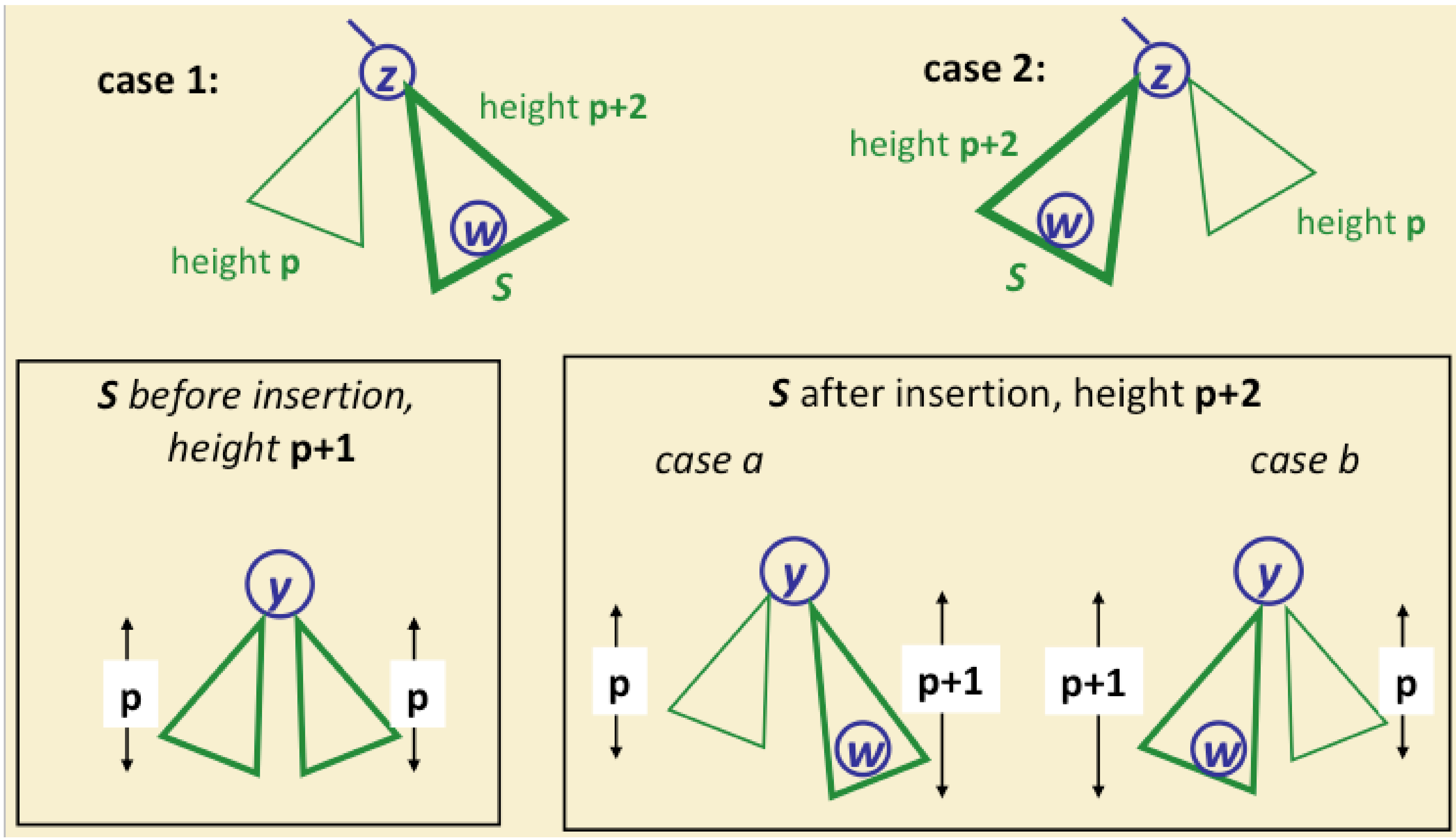
- **y** is balanced after insertion, and **z** is not

- So, both subtrees of **S** had height exactly **p** before insertion



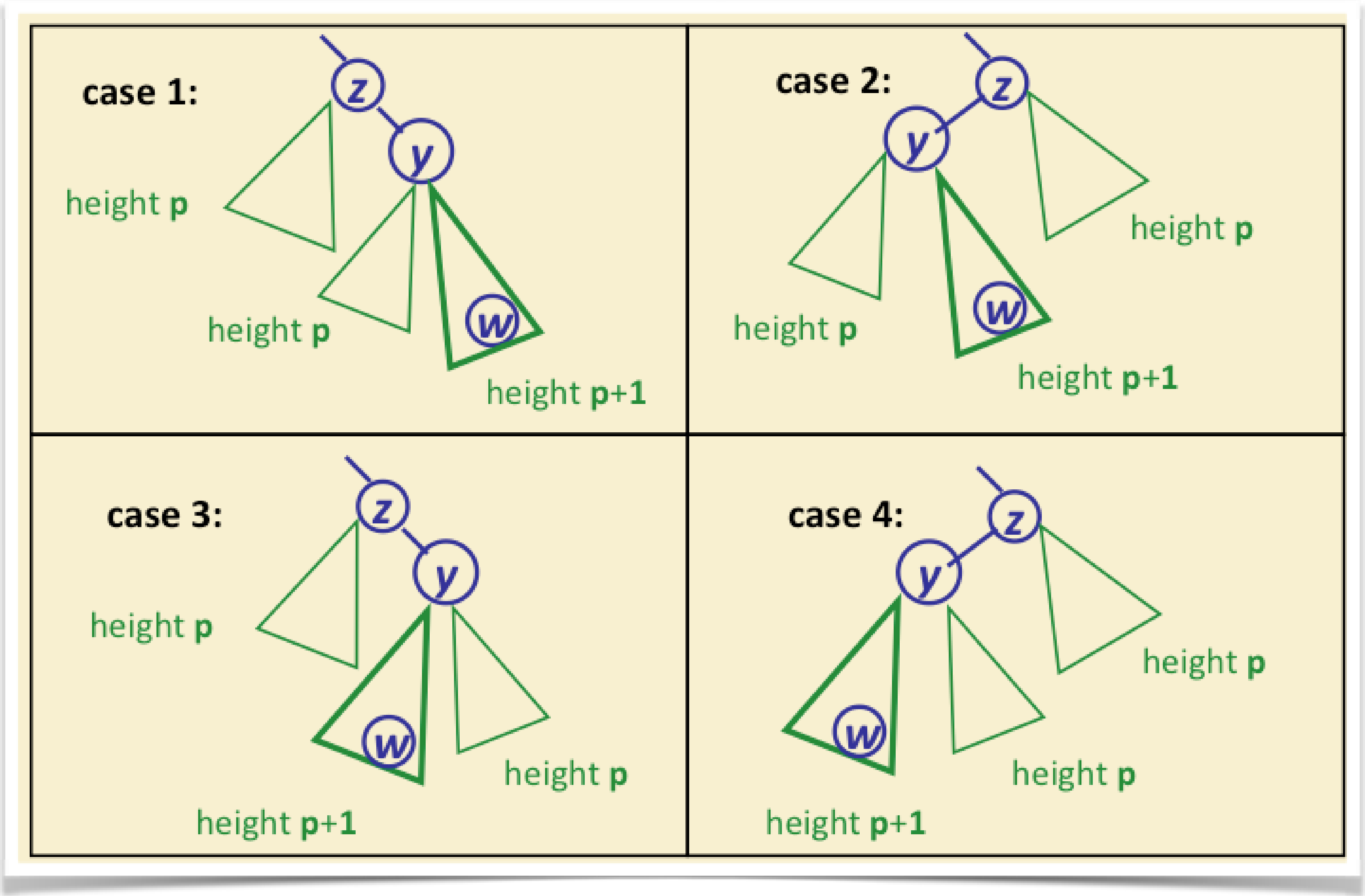
Restructuring

- Complete Picture



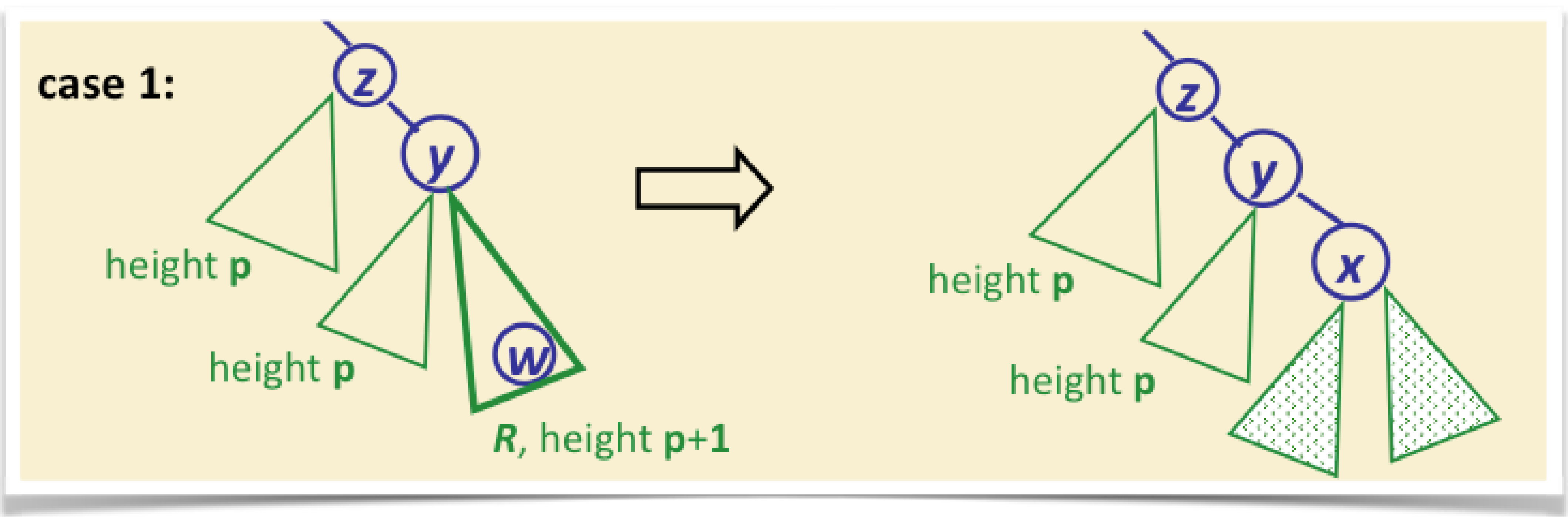
Restructuring

- More Complete Picture :)

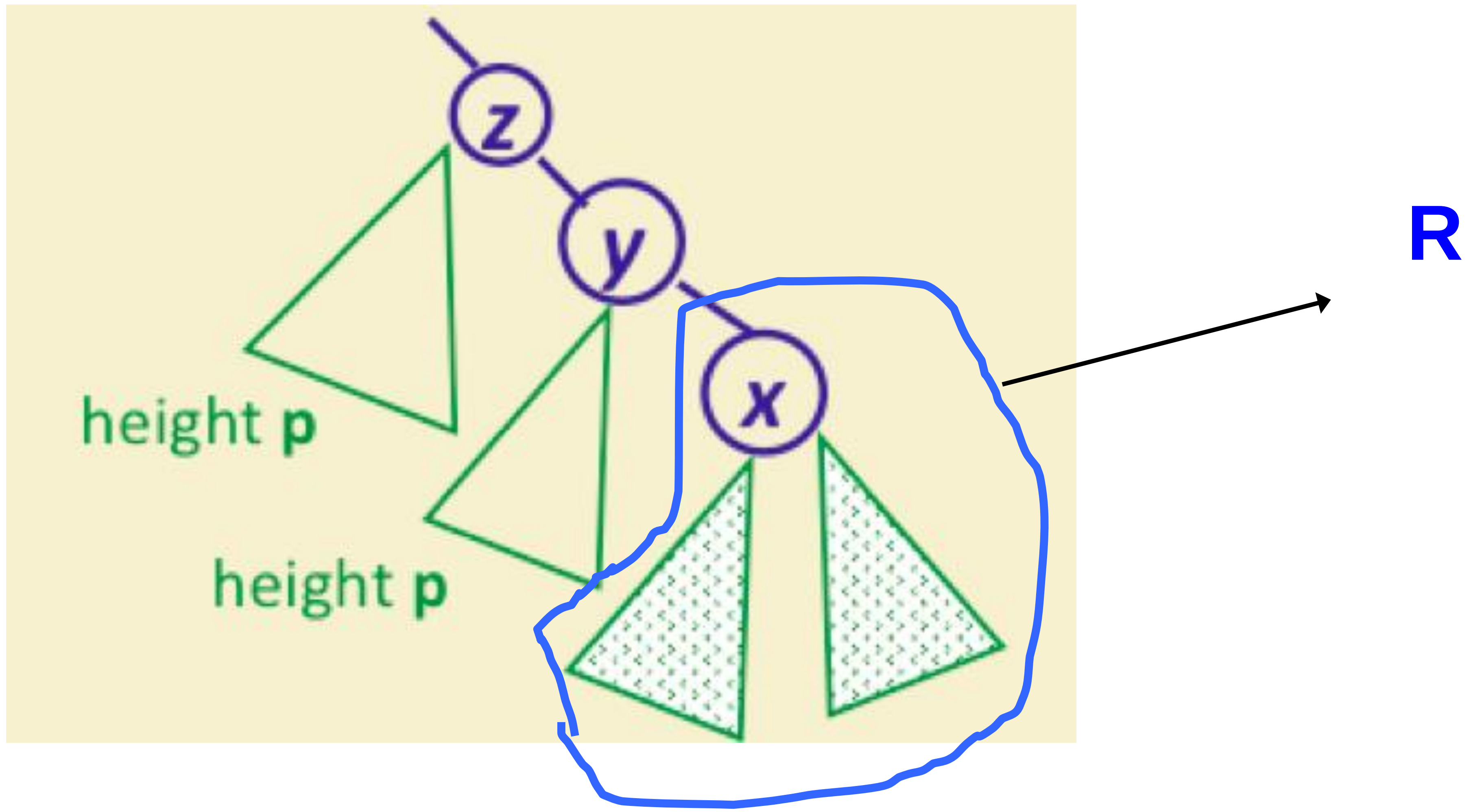


Restructuring

- Let’s consider case 1
- Let R be the right subtree of y
- Let x be the root of R



Restructuring

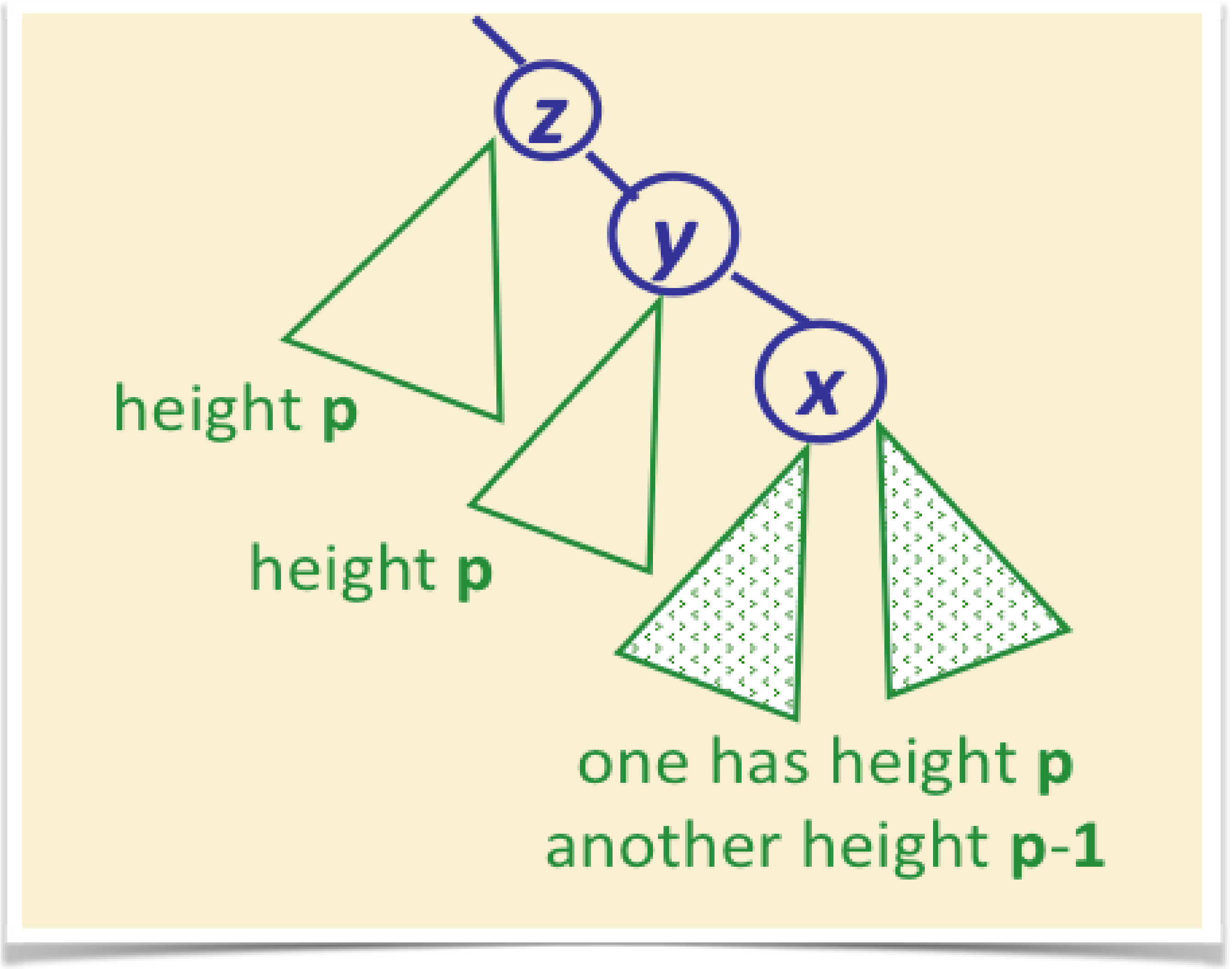


Restructuring

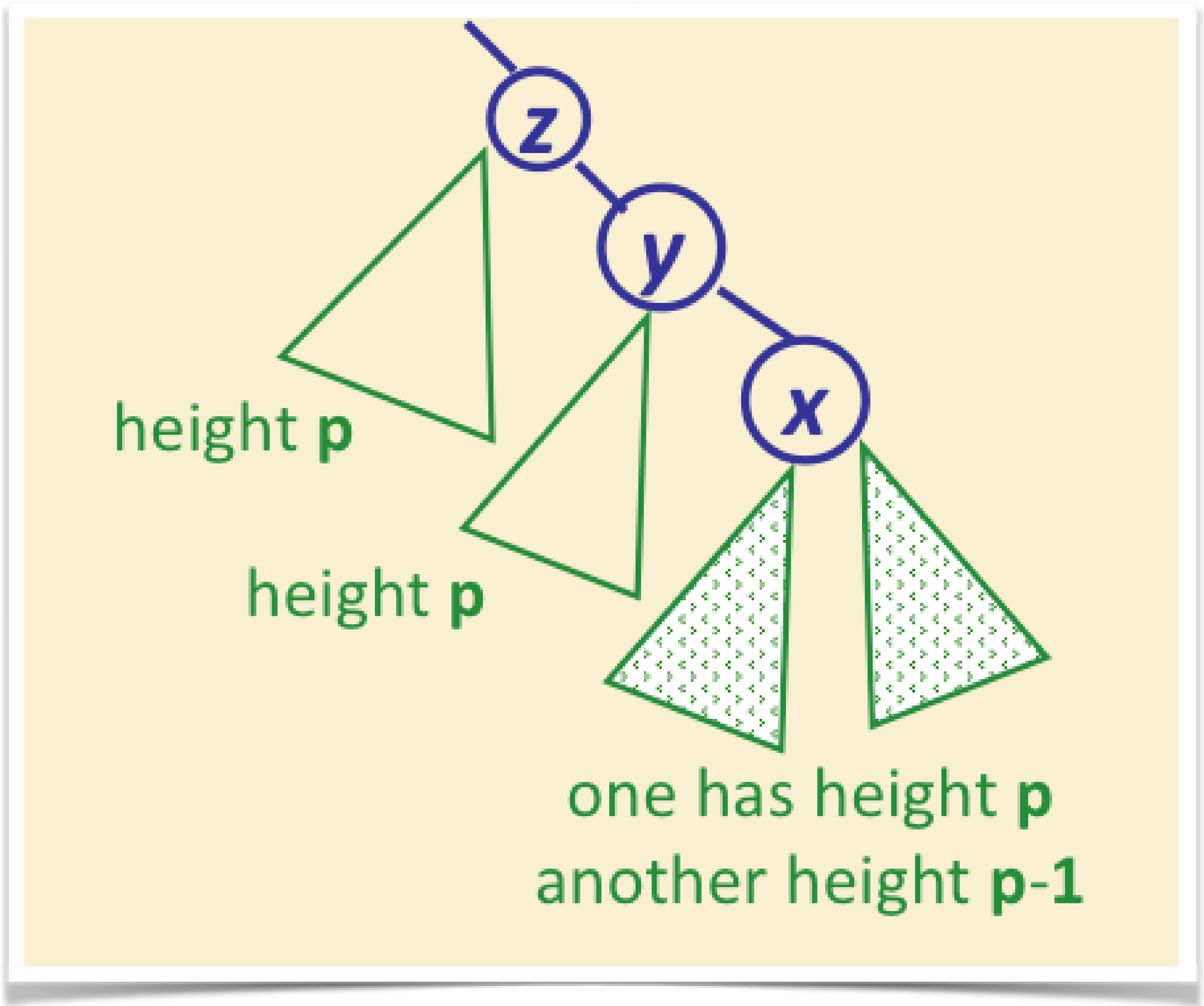
- R contains w, which is an internal node
- Therefore, R has at least one internal node
- There are two cases:
  - Case A: $x = w$ in which case $p = 0$ and both subtrees of w are leaf nodes

Restructuring

- Case B:
  - height of R went from p (before insertion) to p+1 after insertion
  - x was balanced before insertion and is balanced after insertion
  - both subtrees of x had height p-1 before insertion
  - After insertion, one subtree of x has height p, the other p-1

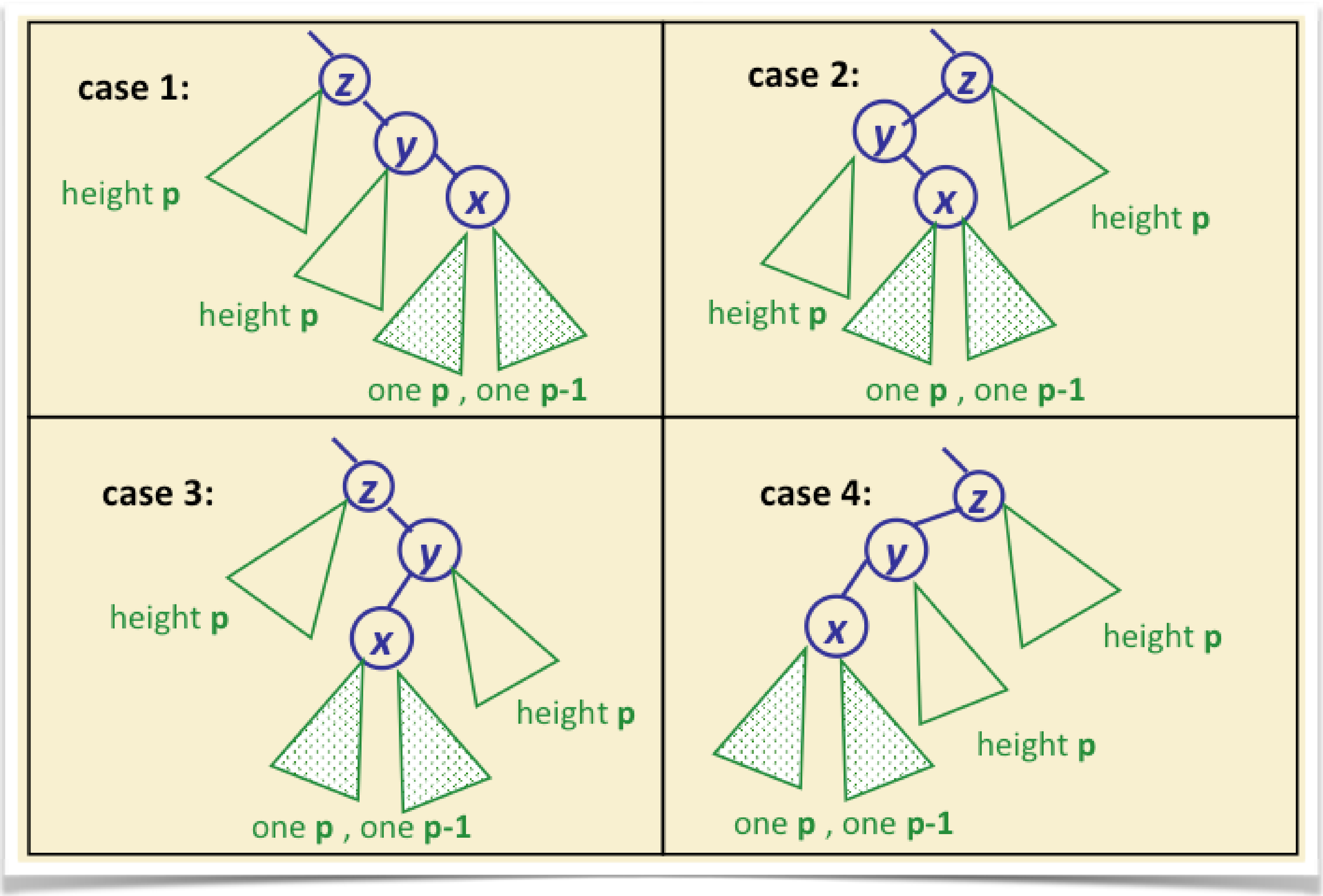


Restructuring



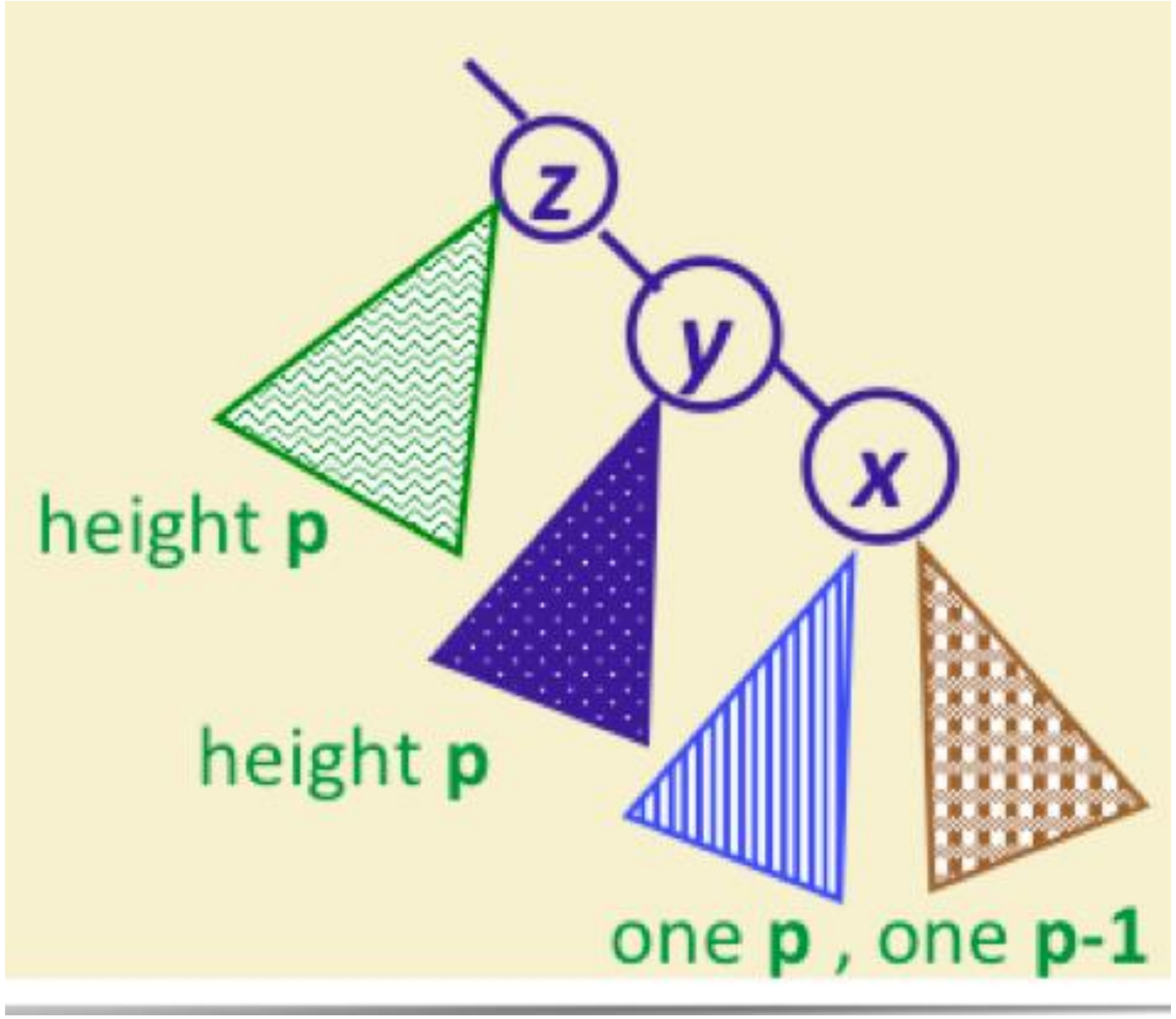
Restructuring

- Even More Complete Picture :)



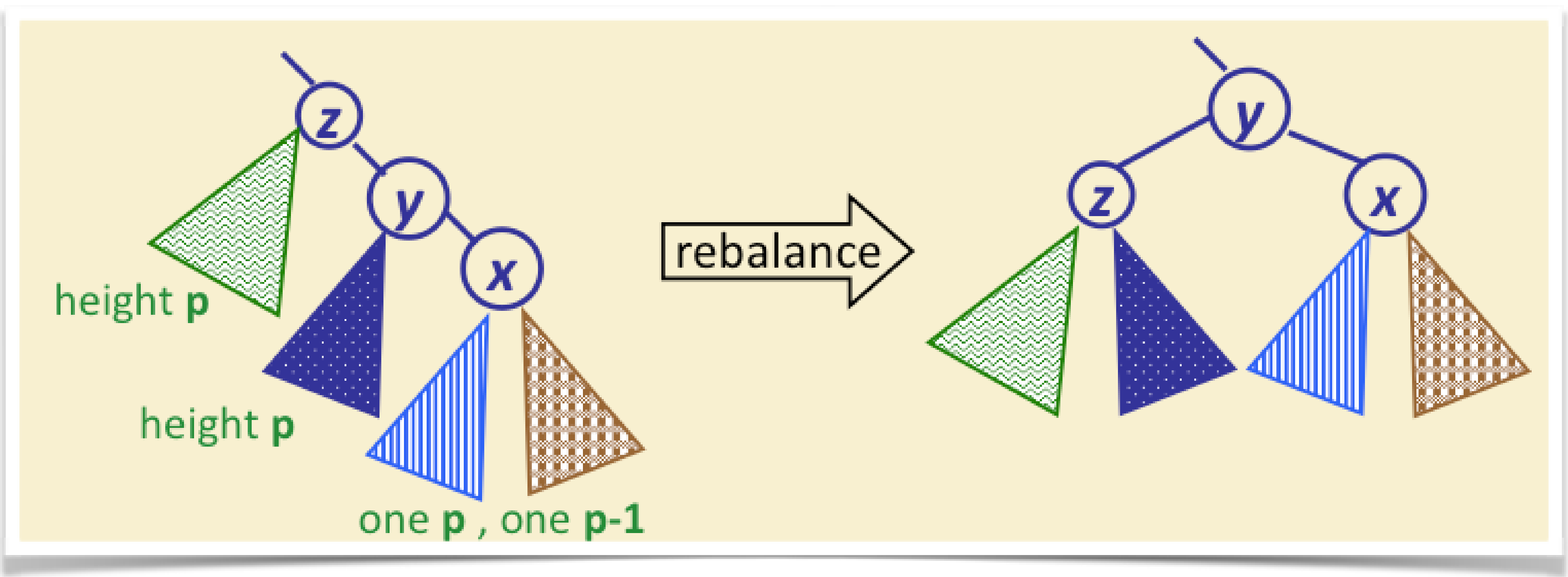
Restructuring

- Finally, let's restructure the tree
- Case 1:



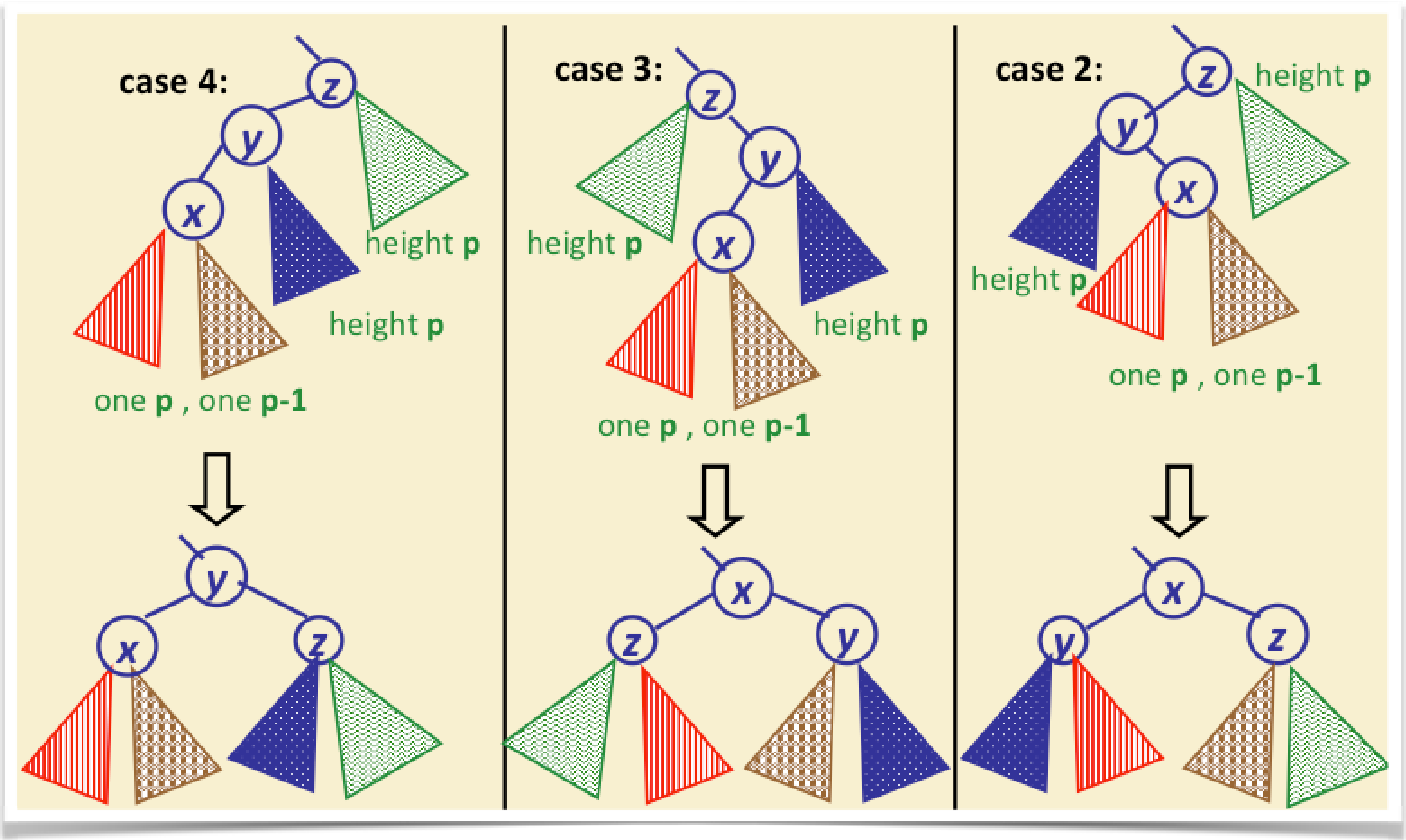
Restructuring

- Finally, let's restructure the tree
- Case 1:



What's the height differences at nodes **x**, **y** and **z** after restructuring?

Restructuring



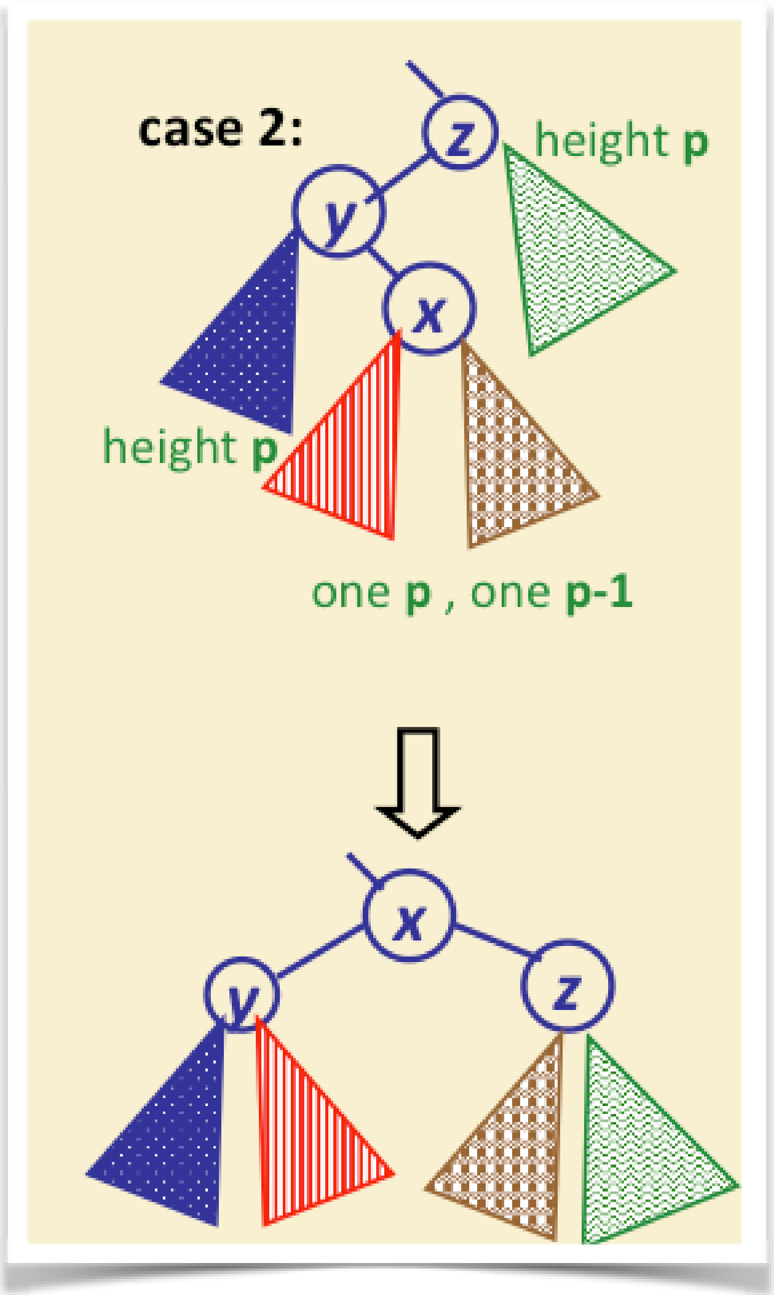
Restructuring

- All four cases can be coded with the same algorithm called: **Trinode Restructuring**

- Trinode because there are three nodes

Trinode Restructuring

- In all four cases, out of the three nodes x, y, and z, make:
  - node with the middle key the new parent
  - smallest key node its left child
  - largest key node its right child
  - for the new parent, the previous subtree (if present) must be put in appropriate positions
    - Left subtree (if present) goes with the new left child
    - Right subtree (if present) goes with the new right child



Trinode Restructuring

- Takes O(logn) + O(1)
- No loops, no recursive calls, constant number of comparisons, and changes in parent-child relationships
- Only 1 trinode restructuring is needed per insertion to restore the height balance property

Deletion

- Deletion from an AVL tress may violate the height-balance property, too
- In this case, procedure for restructuring the tree to restore the balance is the same as in the case of insertion, with some changes
  - how to choose x, y, and z
  - repeated restructuring might be needed, max O(log n)
- For further details, please read section 11.3.1 of your text-books

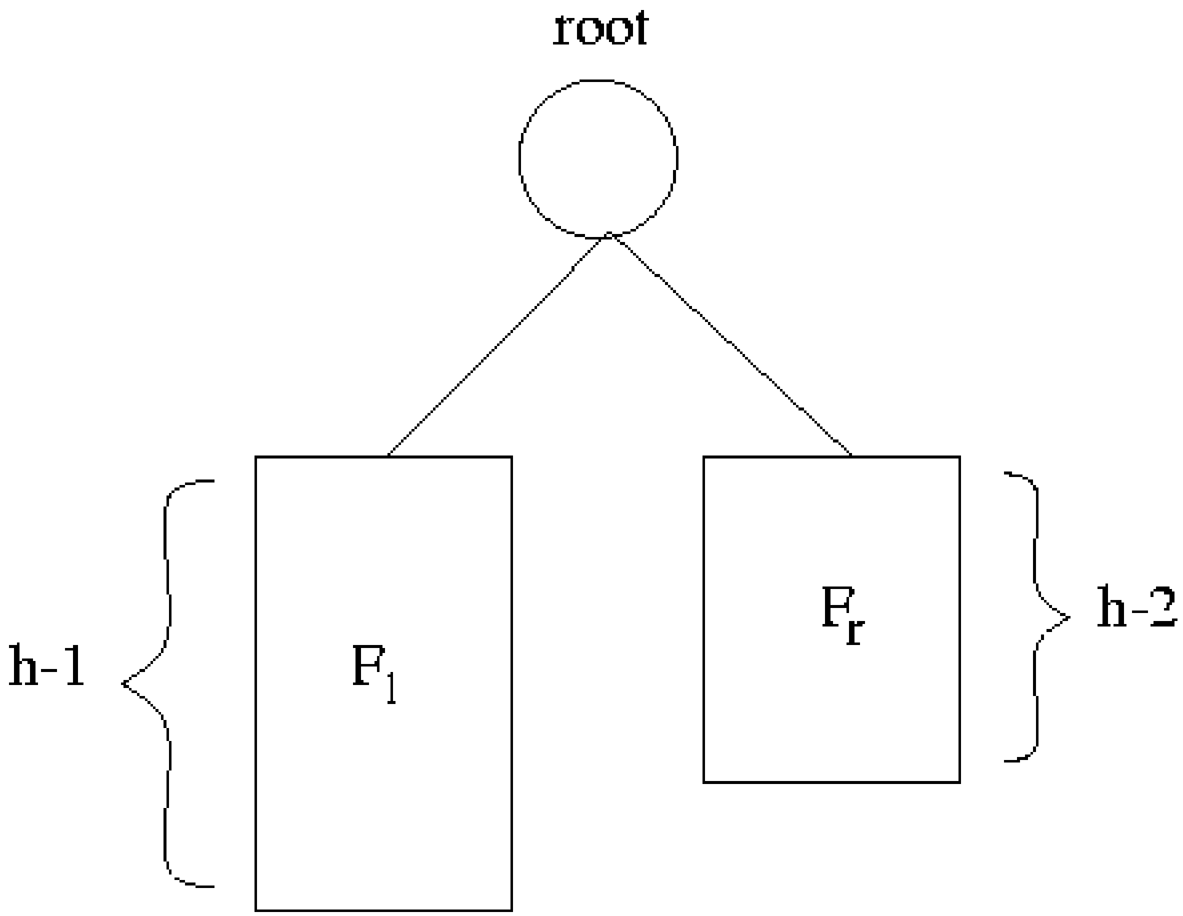
Analysis

- Ok, so we have learned how to keep a binary search tree always balanced (AVL tree) after insertions and deletions
- But why is this important?
- Recall that all we wanted was a way to make and keep the height of a tree with n nodes O(log n)
- Is the height of an AVL tree O(log n)?

Proof

- Proposition: The height of an AVL tree storing **n** entries is O(log n)

Let T be an AVL tree of height h. T can be visualized as



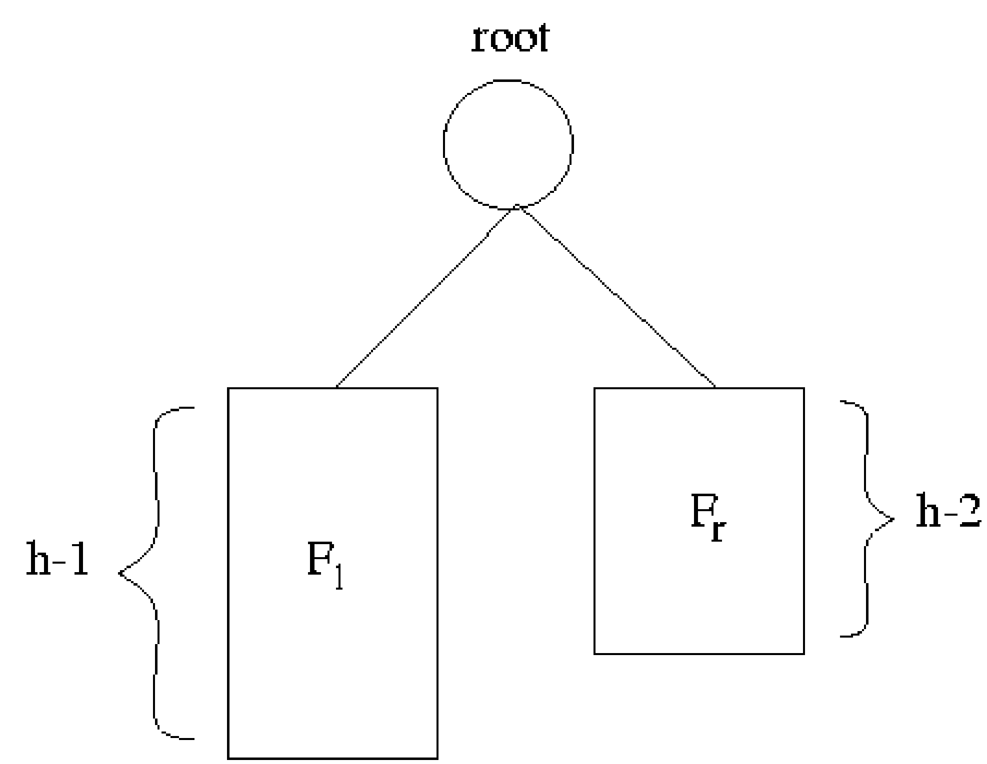
Proof

Let n(h) be the minimum number of **internal nodes** in an AVL tree of height h

we know, n(1) = 1 and n(2) = 2

For **h>=3**

n(h) = 1 + n(h-1) + n(h-2)



Proof

$$n(h) = 1 + n(h-1) + n(h-2)$$

Now that we know this, the rest is just algebra

According to the properties of Fibonacci progressions

$n(h) > n(h-1)$, so $n(h-1) > n(h-2)$

By replacing $n(h-1)$ with $n(h-2)$ and dropping the 1, we get

$$\mathbf{n(h) > 2\, n(h-2)}$$

Proof

$$n(h) > 2\, n(h-2)$$

We can stop at this point. We have shown that **n(h)** at least **doubles** when **h** goes **up by 2**. This says that **n(h)** is **exponential** in **h**, and hence **h** is **logarithmic** in **n**

But let's continue

$$n(h) > 2\,(2\, n(h-4)) = 2^2\, n(h-4)$$

Proof

Thus,

$$\text{For any } i > 0,\ n(h) > 2^i\, n(h-2i)$$

Let's **get rid of i by expressing it in terms of h**, but choose a value that results in making h-2i either 1 or 2. It is because we know the values for n(1) and n(2)

That is, let

$$i = h/2-1$$

Proof

$$n(h) > 2^{h/2-1} n(h-2i) = 2^{h/2-1}$$

By taking logarithmic of both sides

$$\log(n(h)) > (h/2)-1 \text{ or}$$

$$h < 2\log(n(h))+2$$

Or

$$h < \log(n)$$