Munkres 1.3

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1. Define two points (x_0, y_0) and (x_1, y_1) of the plane to be equivalent if $y_0 - x_0^2 = y_1 - x_1^2$. Check that this is an equivalence relation and describe the equivalence classes.

   To show that this is an equivalence relation, I will show that it is reflexive, symmetric, and transitive.

   Reflexive: Using any point (x, y), $y - x^2 = y - x^2$ is reflexive since every element in the relation is related to itself.

   Symmetry: If $y_0 - x_0^2 = y_1 - x_1^2$ then $y_1 - x_1^2 = y_0 - x_0^2$ based on how equality is defined as left side will be equal to right side and vice versa.

   Transitive: Use a third point (x_2, y_2). If (x_0, y_0) and (x_1, y_1) are related so $y_0 - x_0^2 = y_1 - x_1^2$ and (x_1, y_1) and (x_2, y_2) are related so $y_1 - x_1^2 = y_2 - x_2^2$. Then, $y_0 - x_0^2 = y_2 - x_2^2$ and is transitive.

   Equivalence classes: The points of the plane follow the expression: $\forall_a \forall_b (E(a, b) | y - x^2 = b - a^2)$. For example, $E(0, 0) = \{(x, y) | y - x^2 = 0\}$. Therefore, the equivalence classes formed by this relation are the infinitely many parabolas defined by $y = x^2 + k$ and shifted up and down on the y-axis with vertex at $x = 0$.

2. Let C be a relation on a set A. If $A_0 \subset A$, define the restriction of C to A_0 to be the relation $C \cap (A_0 \times A_0)$. Show that the restriction of an equivalence relation is an equivalence relation.

   C is an equivalence relation.

   Reflexive: $(x, y) \in C$ and $(x, y) \in A_0$, then we have $(x, x) \in C \cap A_0 \times A_0$ since x is in A_0 and in C because C is an equivalence relation.

   Symmetry: $(x, y) \in C \cap (A_0 \times A_0)$. So, $(x, y) \in C$ and then $(y, x) \in C$. Also, $(x, y) \in A_0 \times A_0$ and since $x \in A_0 \wedge y \in A_0$, $(y, x) \in A_0 \times A_0$. Therefore, $(y, x) \in C \cap (A_0 \times A_0)$.

Transitive:

$(x, y) \in C \cap (A_0 \times A_0)$ and $(y, z) \in C \cap (A_0 \times A_0)$.

(x, y) and (y, z) are elements of C.

So (x, z) is in C by transitivity.

$(x, y, z) \in A_0$ so (x, z) is an element of $A_0 \times A_0$.

Therefore, $(x, z) \in C \cap (A_0 \times A_0)$.

This shows that restriction of an equivalence relation is an equivalence relation.

3. Here is a "proof" that every relation C that is both symmetric and transitive is also reflexive: "Since C is symmetric, aCb implies bCa. Since C is transitive, aCb and bCa together imply aCa, as desired." Find the flaw in this argument.

   Reflexivity requires aCa to hold for every $a \in A$.

   We can use a counterexample to prove that C isn't reflexive. Using $A = \{0, 1, 2\}$, and $C = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Since $(2, 2) \notin C$, C isn't reflexive.

4. Let $f : A \Rightarrow B$ be a surjective function. Let us define a relation on A by setting $a_0 \sim a_1$ if $f(a_0) = f(a_1)$.

   a. Show that this is an equivalence relation.

   To show that this is an equivalence relation, I will show that it is reflexive, symmetric, and transitive.

   Reflexive: $=$ is reflexive. If $a_0 \sim a_0$, $f(a_0) = f(a_0)$. Also, the reverse will be true so $f(a_0) = f(a_0)$ and $a_0 \sim a_0$ will be reflexive.

   Symmetry: $=$ is symmetric. If $a_0 \sim a_1$ $f(a_0) = f(a_1)$ then $f(a_1) = f(a_0)$ and $a_1 \sim a_0$.

   Transitive: Using points a_0, a_1, a_2. If $a_0 \sim a_1$ so $f(a_0) = f(a_1)$ and $a_1 \sim a_2$ so $f(a_1) = f(a_2)$. Then, $f(a_0) = f(a_2)$ and $a_0 \sim a_2$ since $=$ is transitive.

   b. Let A^* be the set of equivalence classes. Show there is a bijective correspondence of A^* with B.

First, define new function $f^* : A^* \Rightarrow B$. To prove that f^* has bijective correspondence we just need to prove that it is surjective and injective.

f was surjective since $a_0 \in A$ and $b_0 \in B$ means that $f(a_0) = b_0$. Similarly, f^* is surjective if $a_0 \in A^*$ and $b_0 \in B$ and $f^*(a_0) = b_0$ for at least one $a_0 \in A^*$. Since $b_0 = f(a_0) = f^*(a_0)$, f^* is surjective.

For any equivalence class $a_0^* \in A^*$ and element $a_0 \in a_0^*$. Using $f^*(a_0^*) = f^*(a_1^*)$, I will prove that f^* is injective. If $f(a_0) = f(a_1)$ and $a_0 \sim a_1$, $f^*(a_0^*) = f(a_0)$ and $f^*(a_1^*) = f(a_1)$. Since $f(a_0) = f^*(a_0^*) = f^*(a_1^*) = f(a_1)$, $f(a_0) = f(a_1)$. Therefore, since $a_0 \sim a_1$, $a_0^* = a_1^*$. This means that a_0 and a_1 are in the same equivalence class so that f^* is injective.

5. Let S and S' be the following subsets of the plane:

   $S = \{(x, y)|y = x + 1 \text{ and } 0 < x < 2\}$
   $S' = \{(x, y)|y - x \text{ is an integer }\}$

   a. Show that S' is an equivalence relation on the real line and $S' \supset S$. Describe the equivalence classes of S'.

   S' is an equivalence relation on $\mathbb{R}$. Let $x \in \mathbb{R}$. Since $x - x = 0 \in \mathbb{Z}$, $(x, x) \in S'$ for all x. Thus, S' is reflexive.

   Let $x, y \in \mathbb{R}$. If $y - x \in \mathbb{Z}$, $x - y = -(y - x) \in \mathbb{Z}$. Thus, because $(x, y) \in S'$, $(y, x) \in S'$ and S' is symmetric.

   Let $x, y, z \in \mathbb{R}$. If $y-x \in \mathbb{Z}$ and $z-y \in \mathbb{Z}$ then $z-x = (z-y)-(y-x) \in \mathbb{Z}$. Thus, because $(x, y) \in S' \wedge (y, x) \in S'$, $(x, z) \in S'$ and S' is transitive.

   Next, I will show that $S' \supset S$.

   Let $(x, y) \in S$(which means $y = x + 1$). Then, $y - x = 1 \in \mathbb{Z}$. Thus, $(x, y) \in S'$, and $S' \supset S$.

   Finally, I will describe the equivalence class of S'. If $(x, y) \in S'$, then $y - x = a \in \mathbb{Z}$. Then, $y = a + x$. Therefore, $S'' = \{a + x|a \in \mathbb{Z}\}$ is an equivalence class of S'.

   b. Show that given any collection of equivalence relations on a set A, their intersection is an equivalence relation on A.

   Each equivalence relation on a set A partitions A which means that no matter how specific a partition, their elements are all of A.

For example, consider $x_0, x_1 \in \mathbb{R}$ and $x_0, x_1 \in A$ in the following relations: $E_{(x_0)} = \{(x,y)|y - x = x_0\}$, $E_{(x_1)} = \{(x,y)|y = x * x_1\}$. While both equivalence relations have distinct partitions of A, they both share intersections such as $x_0, x_1 = 2$ at $(2,4)$. Even at an intersection, since this intersection occurs between partitions of A, an intersection $\in A$.

Also, if reflexivity, symmetry, and transitivity hold for a relation in the collection, then they hold for any relation in the intersection too. For example, if xCx for any $C \in \mathcal{A}$, then $(x,x) \in C$ for any $C \in \mathcal{A}$, and $(x,x) \in \bigcap_{C \in \mathcal{A}} C$ which means $x \bigcap_{C \in \mathcal{A}} C_x$. This is true for the properties of symmetry and transitivity too.

c. Describe the equivalence relation T on the real line that is the intersection of all equivalence relations on the real line that contain S. Describe the equivalence classes of T.

If T is an equivalence relation that contains S, it has to be reflexive, symmetric, and transitive. An equivalence relation that contains S will also have $\{(x,y)|y = x\}$ by reflexivity, $\{(x,y)|y = x + 1 \text{ and } 0 < x < 2\}$ and $\{(x,y)|y = x - 1 \text{ and } 1 < x < 3\}$ by symmetry, $\{(x,y)|y = x + 2 \text{ and } 0 < x < 1\}$ and $\{(x,y)|y = x - 2 \text{ and } 2 < x < 3\}$ by transitivity and symmetry. If these pairs are part of T, then T is the intersection and from 5b, we know that the intersection of an equivalence relation is an equivalence relation.

The equivalence classes of T will be formed based on the sets in the intersection. The equivalence class will be $\{x\}$ for $x \leq 0$ and $x \geq 3$. The classes between $0 \leq x \leq 3$ will be different based on value of x.