Contents

Chapter 1

Truth-Functional Logic

1.1 Descartes's *Meditations on First Philosophy*

"Meditation One: Concerning Those Things That Can Be Called into Doubt"

Several years have now elapsed since I first became aware that I had accepted, even from my youth, many false opinions for true, and that consequently what I afterward based on such principles was highly doubtful; and from that time I was convinced of the necessity of undertaking once in my life to rid myself of all the opinions I had adopted, and of commencing anew the work of building from the foundation, if I desired to establish a firm and abiding superstructure in the sciences. But as this enterprise appeared to me to be one of great magnitude, I waited until I had attained an age so mature as to leave me no hope that at any stage of life more advanced I should be better able to execute my design. On this account, I have delayed so long that I should henceforth consider I was doing wrong were I still to consume in deliberation any of the time that now remains for action. To-day, then, since I have opportunely freed my mind from all cares and am happily disturbed by no passions, and since I am in the secure possession of leisure in a peaceable retirement, I will at length apply myself earnestly and freely to the general overthrow of all my former opinions.

But, to this end, it will not be necessary for me to show that the whole of these are false—a point, perhaps, which I shall never reach; but as even now my reason convinces me that I ought not the less carefully to withhold belief from what is not entirely certain and indubitable, than from what is manifestly false, it will be sufficient to justify the rejection of the whole if I shall find in each some ground for doubt. Nor for this purpose will it be necessary even to deal with each belief individually, which would be truly an endless labor; but, as the removal from below of the foundation necessarily involves the downfall of the whole edifice, I will at once approach the criticism of the principles on which all my former beliefs rested.

All that I have, up to this moment, accepted as possessed of the highest truth and certainty, I received either from or through the senses. I observed, however, that these sometimes misled us; and it is the part of prudence not to place absolute confidence in that by which we have even once been deceived.

But it may be said, perhaps, that, although the senses occasionally mislead us respecting minute objects, and such as are so far removed from us as to be beyond the reach of close observation, there are yet many other of their informations (presentations), of the truth of which it is manifestly impossible to doubt; as for example, that I am in this place, seated by the fire, clothed in a winter dressing gown, that I hold in my hands this piece of paper, with other intimations of the same nature. But how could I deny that I possess these hands and this body, and withal escape being classed with persons in a state of insanity, whose brains are so disordered and clouded by dark bilious vapors as to cause them pertinaciously to assert that they are monarchs when they are in the greatest poverty; or clothed in gold and purple when destitute of any covering; or that their head is made of clay, their body of glass, or that they are gourds? I should certainly be not less insane than they, were I to regulate my procedure according to examples so extravagant.

Though this be true, I must nevertheless here consider that I am a man, and that, consequently, I am in the habit of sleeping, and representing to myself in dreams those same things, or even sometimes others less probable, which the insane think are presented to them in their waking moments. How often have I dreamt that I was in these familiar circumstances, that I was dressed, and occupied this place by the fire, when I was lying undressed in bed? At the present moment, however, I certainly look upon this paper with eyes wide awake; the head which I now move is not asleep; I extend this hand consciously and with express purpose, and I perceive it; the occurrences in sleep are not so distinct as all this. But I cannot forget that, at other times I have been deceived in sleep by similar illusions; and, attentively considering those cases, I perceive so clearly that there exist no certain marks by which the state of waking can ever be distinguished from sleep, that I feel greatly astonished; and in amazement I almost persuade myself that I am now dreaming.

Let us suppose, then, that we are dreaming, and that all these particulars—namely, the opening of the eyes, the motion of the head, the forth-putting of the hands—are merely illusions; and even that we really possess neither an entire body nor hands such as we see. Nevertheless it must be admitted at least that the objects which appear to us in sleep are, as it were, painted representations which could not have been formed unless in the likeness of realities; and, therefore, that those general objects, at all events, namely, eyes, a head, hands, and an entire body, are not simply imaginary, but really existent. For, in truth, painters themselves, even when they study to represent sirens and satyrs by forms the most fantastic and extraordinary, cannot bestow upon them natures absolutely new, but can only make a certain medley of the members of different animals; or if they chance to imagine something so novel that nothing at all similar has ever been seen before, and such as is, therefore, purely fictitious and absolutely false, it is at least certain that the colors of which this is composed are real. And on the same principle, although these general objects, viz. a body, eyes, a head, hands, and the like, be imaginary, we are nevertheless absolutely necessitated to admit the reality at least of some other objects still more simple and universal than these, of which, just as of certain real colors, all those images of things, whether true and real, or false and fantastic, that are found in our consciousness (*cogitatio*), are formed.

To this class of objects seem to belong corporeal nature in general and its extension; the figure of extended things, their quantity or magnitude, and their number, as also the place in, and the time during, which they exist, and other things of the same sort.

We will not, therefore, perhaps reason illegitimately if we conclude from this that Physics, Astronomy, Medicine, and all the other sciences that have for their end the consideration of composite objects, are indeed of a doubtful character; but that Arithmetic, Geometry, and the other sciences of the same class, which regard merely the simplest and most general objects, and scarcely inquire whether or not these are really existent, contain somewhat that is certain and indubitable: for whether I am awake or dreaming, it remains true that two and three make five, and that a square has but four sides; nor does it seem possible that truths so apparent can ever fall under a suspicion of falsity or incertitude.

Nevertheless, the belief that there is a God who is all powerful, and who created me, such as I am, has, for a long time, obtained steady possession of my mind. How, then, do I know that he has not arranged that there should be neither earth, nor sky, nor any extended thing, nor figure, nor magnitude, nor place, providing at the same time, however, for the rise in me of the perceptions of all these objects, and] the persuasion that these do not exist otherwise than as I perceive them? And further, as I sometimes think that others are in error respecting matters of which they believe themselves to possess a perfect knowledge, how do I know that I am not also deceived each time I add together two and three, or number the sides of a square, or form some judgment still more simple, if more simple indeed can be imagined? But perhaps Deity has not been willing that I should be thus deceived, for he is said to be supremely good. If, however, it were repugnant to the goodness of Deity to have created me subject to constant deception, it would seem likewise to be contrary to his goodness to allow me to be occasionally deceived; and yet it is clear that this is permitted.

Some, indeed, might perhaps be found who would be disposed rather to deny the existence of a Being so powerful than to believe that there is nothing certain. But let us for the present refrain from opposing this opinion, and grant that all which is here said of a Deity is fabulous: nevertheless, in whatever way it be supposed that I reach the state in which I exist, whether by fate, or chance, or by an endless series of antecedents and consequents, or by any other means, it is clear (since to be deceived and to err is a certain defect) that the probability of my being so imperfect as to be the constant victim of deception, will be increased exactly in proportion as the power possessed by the cause, to which they assign my origin, is lessened. To these reasonings I have assuredly nothing to reply, but am constrained at last to avow that there is nothing of all that I formerly believed to be true of which it is impossible to doubt, and that not through thoughtlessness or levity, but from cogent and maturely considered reasons; so that henceforward, if I desire to discover anything certain, I ought not the less carefully to refrain from assenting to those same opinions than to what might be shown to be manifestly false.

But it is not sufficient to have made these observations; care must be taken likewise to keep them in remembrance. For those old and customary opinions perpetually recur—long and familiar usage giving them the right of occupying my mind, even almost against my will, and subduing my belief; nor will I lose the habit of deferring to them and confiding in them so long as I shall consider them to be what in truth they are, viz, opinions to some extent doubtful, as I have already shown, but still highly probable, and such as it is much more reasonable to believe than deny. It is for this reason I am persuaded that I shall not be doing wrong, if, taking an opposite judgment of deliberate design, I become my own deceiver, by supposing, for a time, that all those opinions are entirely false and imaginary, until at length, having thus balanced my old by my new prejudices, my judgment shall no longer be turned aside by perverted usage from the path that may conduct to the perception of truth. For I am assured that, meanwhile, there will arise neither peril nor error from this course, and that I cannot for the present yield too much to distrust, since the end I now seek is not action but knowledge.

I will suppose, then, not that Deity, who is sovereignly good and the fountain of truth, but that some malignant demon, who is at once exceedingly potent and deceitful, has employed all his artifice to deceive me; I will suppose that the sky, the air, the earth, colors, figures, sounds, and all external things, are nothing better than the illusions of dreams, by means of which this being has laid snares for my credulity; I will consider myself as without hands, eyes, flesh, blood, or any of the senses, and as falsely believing that I am possessed of these; I will continue resolutely fixed in this belief, and if indeed by this means it be not in my power to arrive at the knowledge of truth, I shall at least do what is in my power, viz, suspend my judgment, and guard with settled purpose against giving my assent to what is false, and being imposed upon by this deceiver, whatever be his power and artifice. But this undertaking is arduous, and a certain indolence insensibly leads me back to my ordinary course of life; and just as the captive, who, perchance, was enjoying in his dreams an imaginary liberty, when he begins to suspect that it is but a vision, dreads awakening, and conspires with the agreeable illusions that the deception may be prolonged; so I, of my own accord, fall back into the train of my former beliefs, and fear to arouse myself from my slumber, lest the time of laborious wakefulness that would succeed this quiet rest, in place of bringing any light of day, should prove inadequate to dispel the darkness that will arise from the difficulties that have now been raised.

"Meditation Two: Concerning the Nature of the Human Mind: That It Is Better Known Than the Body"

The Meditation of yesterday has filled my mind with so many doubts, that it is no longer in my power to forget them. Nor do I see, meanwhile, any principle on which they can be resolved; and, just as if I had fallen all of a sudden into very deep water, I am so greatly disconcerted as to be unable either to plant my feet firmly on the bottom or sustain myself by swimming on the surface. I will, nevertheless, make an effort, and try anew the same path on which I had entered yesterday, that is, proceed by casting aside all that admits of the slightest doubt, not less than if I had discovered it to be absolutely false; and I will continue always in this track until I shall find something that is certain, or at least, if I can do nothing more, until I shall know with certainty that there is nothing certain. Archimedes, that he might transport the entire globe from the place it occupied to another, demanded only a point that was firm and immovable; so, also, I shall be entitled to entertain the highest expectations, if I am fortunate enough to discover only one thing that is certain and indubitable.

I suppose, accordingly, that all the things which I see are false (fictitious); I believe that none of those objects which my fallacious memory represents ever existed; I suppose that I possess no senses; I believe that body, figure, extension, motion, and place are merely fictions of my mind. What is there, then, that can be esteemed true? Perhaps this only, that there is absolutely nothing certain.

But how do I know that there is not something different altogether from the objects I have now enumerated, of which it is impossible to entertain the slightest doubt? Is there not a God, or some being, by whatever name I may designate him, who causes these thoughts to arise in my mind? But why suppose such a being, for it may be I myself am capable of producing them? Am I, then, at least not something? But I before denied that I possessed senses or a body; I hesitate, however, for what follows from that? Am I so dependent on the body and the senses that without these I cannot exist? But I had the persuasion that there was absolutely nothing in the world, that there was no sky and no earth, neither minds nor bodies; was I not, therefore, at the same time, persuaded that I did not exist? Far from it; I assuredly existed, since I was persuaded. But there is I know not what being, who is possessed at once of the highest power and the deepest cunning, who is constantly employing all his ingenuity in deceiving me. Doubtless, then, I exist, since I am deceived; and, let him deceive me as he may, he can never bring it about that I am nothing, so long as I shall be conscious that I am something. So that it must, in fine, be maintained, all things being maturely and carefully considered, that this proposition (*pronunciatum*) I am, I exist, is necessarily true each time it is expressed by me, or conceived in my mind.

But I do not yet know with sufficient clearness what I am, though assured that I am; and hence, in the next place, I must take care, lest perchance I inconsiderately substitute some other object in room of what is properly myself, and thus wander from truth, even in that knowledge (cognition) which I hold to be of all others the most certain and evident. For this reason, I will now consider anew what I formerly believed myself to be, before I entered on the present train of thought; and of my previous opinion I will retrench all that can in the least be invalidated by the grounds of doubt I have adduced, in order that there may at length remain nothing but what is certain and indubitable.

What then did I formerly think I was? Undoubtedly I judged that I was a man. But what is a man? Shall I say a rational animal? Assuredly not; for it would be necessary forthwith to inquire into what is meant by animal, and what by rational, and thus, from a single question, I should insensibly glide into others, and these more difficult than the first; nor do I now possess enough of leisure to warrant me in wasting my time amid subtleties of this sort. I prefer here to attend to the thoughts that sprung up of themselves in my mind, and were inspired by my own nature alone, when I applied myself to the consideration of what I was. In the first place, then, I thought that I possessed a countenance, hands, arms, and all the fabric of members that appears in a corpse, and which I called by the name of body. It further occurred to me that I was nourished, that I walked, perceived, and thought, and all those actions I referred to the soul; but what the soul itself was I either did not stay to consider, or, if I did, I imagined that it was something extremely rare and subtile, like wind, or flame, or ether, spread through my grosser parts. As regarded the body, I did not even doubt of its nature, but thought I distinctly knew it, and if I had wished to describe it according to the notions I then entertained, I should have explained myself in this manner: By body I understand all that can be terminated by a certain figure; that can be comprised in a certain place, and so fill a certain space as therefrom to exclude every other body; that can be perceived either by touch, sight, hearing, taste, or smell; that can be moved in different ways, not indeed of itself, but by something foreign to it by which it is touched and from which it receives the impression; for the power of self-motion, as likewise that of perceiving and thinking, I held as by no means pertaining to the nature of body; on the contrary, I was somewhat astonished to find such faculties existing in some bodies.

But as to myself, what can I now say that I am, since I suppose there exists an extremely powerful, and, if I may so speak, malignant being, whose whole endeavors are directed toward deceiving me? Can I affirm that I possess any one of all those attributes of which I have lately spoken as belonging to the nature of body? After attentively considering them in my own mind, I find none of them that can properly be said to belong to myself. To recount them were idle and tedious. Let us pass, then, to the attributes of the soul. The first mentioned were the powers of nutrition and walking; but, if it be true that I have no body, it is true likewise that I am capable neither of walking nor of being nourished. Perception is another attribute of the soul; but perception too is impossible without the body; besides, I have frequently, during sleep, believed that I perceived objects which I afterward observed I did not in reality perceive. Thinking is another attribute of the soul; and here I discover what properly belongs to myself. This alone is inseparable from me. I am—I exist: this is certain; but how often? As often as I think; for perhaps it would even happen, if I should wholly cease to think, that I should at the same time altogether cease to be. I now admit nothing that is not necessarily true. I am therefore, precisely speaking, only a thinking thing, that is, a mind (*mens sive animus*), understanding, or reason, terms whose signification was before unknown to me. I am, however, a real thing, and really existent; but what thing? The answer was, a thinking thing.

The question now arises, am I aught besides? I will stimulate my imagination with a view to discover whether I am not still something more than a thinking being. Now it is plain I am not the assemblage of members called the human body; I am not a thin and penetrating air diffused through all these members, or wind, or flame, or vapor, or breath, or any of all the things I can imagine; for I supposed that all these were not, and, without changing the supposition, I find that I still feel assured of my existence. But it is true, perhaps, that those very things which I suppose to be non-existent, because they are unknown to me, are not in truth different from myself whom I know. This is a point I cannot determine, and do not now enter into any dispute regarding it. I can only judge of things that are known to me: I am conscious that I exist, and I who know that I exist inquire into what I am. It is, however, perfectly certain that the knowledge of my existence, thus precisely taken, is not dependent on things, the existence of which is as yet unknown to me: and consequently it is not dependent on any of the things I can feign in imagination. Moreover, the phrase itself, I frame an image (*effingo*), reminds me of my error; for I should in truth frame one if I were to imagine myself to be anything, since to imagine is nothing more than to contemplate the figure or image of a corporeal thing; but I already know that I exist, and that it is possible at the same time that all those images, and in general all that relates to the nature of body, are merely dreams or chimeras. From this I discover that it is not more reasonable to say, I will excite my imagination that I may know more distinctly what I am, than to express myself as follows: I am now awake, and perceive something real; but because my perception is not sufficiently clear, I will of express purpose go to sleep that my dreams may represent to me the object of my perception with more truth and clearness. And, therefore, I know that nothing of all that I can embrace in imagination belongs to the knowledge which I have of myself, and that there is need to recall with the utmost care the mind from this mode of thinking, that it may be able to know its own nature with perfect distinctness.

But what, then, am I? A thinking thing, it has been said. But what is a thinking thing? It is a thing that doubts, understands, conceives, affirms, denies, wills, refuses; that imagines also, and perceives.

Assuredly it is not little, if all these properties belong to my nature. But why should they not belong to it? Am I not that very being who now doubts of almost everything; who, for all that, understands and conceives certain things; who affirms one alone as true, and denies the others; who desires to know more of them, and does not wish to be deceived; who imagines many things, sometimes even despite his will; and is likewise percipient of many, as if through the medium of the senses. Is there nothing of all this as true as that I am, even although I should be always dreaming, and although he who gave me being employed all his ingenuity to deceive me? Is there also any one of these attributes that can be properly distinguished from my thought, or that can be said to be separate from myself? For it is of itself so evident that it is I who doubt, I who understand, and I who desire, that it is here unnecessary to add anything by way of rendering it more clear. And I am as certainly the same being who imagines; for although it may be (as I before supposed) that nothing I imagine is true, still the power of imagination does not cease really to exist in me and to form part of my thought. In fine, I am the same being who perceives, that is, who apprehends certain objects as by the organs of sense, since, in truth, I see light, hear a noise, and feel heat. But it will be said that these presentations are false, and that I am dreaming. Let it be so. At all events it is certain that I seem to see light, hear a noise, and feel heat; this cannot be false, and this is what in me is properly called perceiving (*sentire*), which is nothing else than thinking.

From this I begin to know what I am with somewhat greater clearness and distinctness than heretofore. But, nevertheless, it still seems to me, and I cannot help believing, that corporeal things, whose images are formed by thought which fall under the senses, and are examined by the same, are known with much greater distinctness than that I know not what part of myself which is not imaginable; although, in truth, it may seem strange to say that I know and comprehend with greater distinctness things whose existence appears to me doubtful, that are unknown, and do not belong to me, than others of whose reality I am persuaded, that are known to me, and appertain to my proper nature; in a word, than myself. But I see clearly what is the state of the case. My mind is apt to wander, and will not yet submit to be restrained within the limits of truth. Let us therefore leave the mind to itself once more, and, according to it every kind of liberty permit it to consider the objects that appear to it from without, in order that, having afterward withdrawn it from these gently and opportunely and fixed it on the consideration of its being and the properties it finds in itself, it may then be the more easily controlled.

Let us now accordingly consider the objects that are commonly thought to be the most easily, and likewise the most distinctly known, viz, the bodies we touch and see; not, indeed, bodies in general, for these general notions are usually somewhat more confused, but one body in particular. Take, for example, this piece of wax; it is quite fresh, having been but recently taken from the beehive; it has not yet lost the sweetness of the honey it contained; it still retains somewhat of the odor of the flowers from which it was gathered; its color, figure, size, are apparent (to the sight); it is hard, cold, easily handled; and sounds when struck upon with the finger. In fine, all that contributes to make a body as distinctly known as possible, is found in the one before us. But, while I am speaking, let it be placed near the fire—what remained of the taste exhales, the smell evaporates, the color changes, its figure is destroyed, its size increases, it becomes liquid, it grows hot, it can hardly be handled, and, although struck upon, it emits no sound. Does the same wax still remain after this change? It must be admitted that it does remain; no one doubts it, or judges otherwise. What, then, was it I knew with so much distinctness in the piece of wax? Assuredly, it could be nothing of all that I observed by means of the senses, since all the things that fell under taste, smell, sight, touch, and hearing are changed, and yet the same wax remains.

1It was perhaps what I now think, viz, that this wax was neither the sweetness of honey, the pleasant odor of flowers, the whiteness, the figure, nor the sound, but only a body that a little before appeared to me conspicuous under these forms, and which is now perceived under others. But, to speak precisely, what is it that I imagine when I think of it in this way? Let it be attentively considered, and, retrenching all that does not belong to the wax, let us see what remains. There certainly remains nothing, except something extended, flexible, and movable. But what is meant by flexible and movable? Is it not that I imagine that the piece of wax, being round, is capable of becoming square, or of passing from a square into a triangular figure? Assuredly such is not the case, because I conceive that it admits of an infinity of similar changes; and I am, moreover, unable to compass this infinity by imagination, and consequently this conception which I have of the wax is not the product of the faculty of imagination. But what now is this extension? Is it not also unknown? for it becomes greater when the wax is melted, greater when it is boiled, and greater still when the heat increases; and I should not conceive clearly and] according to truth, the wax as it is, if I did not suppose that the piece we are considering admitted even of a wider variety of extension than I ever imagined, I must, therefore, admit that I cannot even comprehend by imagination what the piece of wax is, and that it is the mind alone (mens, Lat., entendement, F.) which perceives it. I speak of one piece in particular; for as to wax in general, this is still more evident. But what is the piece of wax that can be perceived only by the understanding or mind? It is certainly the same which I see, touch, imagine; and, in fine, it is the same which, from the beginning, I believed it to be. But (and this it is of moment to observe) the perception of it is neither an act of sight, of touch, nor of imagination, and never was either of these, though it might formerly seem so, but is simply an intuition (*inspectio*) of the mind, which may be imperfect and confused, as it formerly was, or very clear and distinct, as it is at present, according as the attention is more or less directed to the elements which it contains, and of which it is composed.

But, meanwhile, I feel greatly astonished when I observe the weakness of my mind, and its proneness to error. For although, without at all giving expression to what I think, I consider all this in my own mind, words yet occasionally impede my progress, and I am almost led into error by the terms of ordinary language. We say, for example, that we see the same wax when it is before us, and not that we judge it to be the same from its retaining the same color and figure: whence I should forthwith be disposed to conclude that the wax is known by the act of sight, and not by the intuition of the mind alone, were it not for the analogous instance of human beings passing on in the street below, as observed from a window. In this case I do not fail to say that I see the men themselves, just as I say that I see the wax; and yet what do I see from the window beyond hats and cloaks that might cover artificial machines, whose motions might be determined by springs? But I judge that there are human beings from these appearances, and thus I comprehend, by the faculty of judgment alone which is in the mind, what I believed I saw with my eyes.

The man who makes it his aim to rise to knowledge superior to the common, ought to be ashamed to seek occasions of doubting from the vulgar forms of speech: instead, therefore, of doing this, I shall proceed with the matter in hand, and inquire whether I had a clearer and more perfect perception of the piece of wax when I first saw it, and when I thought I knew it by means of the external sense itself, or, at all events, by the common sense (*sensus communis*), as it is called, that is, by the imaginative faculty; or whether I rather apprehend it more clearly at present, after having examined with greater care, both what it is, and in what way it can be known. It would certainly be ridiculous to entertain any doubt on this point. For what, in that first perception, was there distinct? What did I perceive which any animal might not have perceived? But when I distinguish the wax from its exterior forms, and when, as if I had stripped it of its vestments, I consider it quite naked, it is certain, although some error may still be found in my judgment, that I cannot, nevertheless, thus apprehend it without possessing a human mind.

But finally, what shall I say of the mind itself, that is, of myself? for as yet I do not admit that I am anything but mind. What, then! I who seem to possess so distinct an apprehension of the piece of wax, do I not know myself, both with greater truth and certitude, and also much more distinctly and clearly? For if I judge that the wax exists because I see it, it assuredly follows, much more evidently, that I myself am or exist, for the same reason: for it is possible that what I see may not in truth be wax, and that I do not even possess eyes with which to see anything; but it cannot be that when I see, or, which comes to the same thing, when I think I see, I myself who think am nothing. So likewise, if I judge that the wax exists because I touch it, it will still also follow that I am; and if I determine that my imagination, or any other cause, whatever it be, persuades me of the existence of the wax, I will still draw the same conclusion. And what is here remarked of the piece of wax, is applicable to all the other things that are external to me. And further, if the notion or perception of wax appeared to me more precise and distinct, after that not only sight and touch, but many other causes besides, rendered it manifest to my apprehension, with how much greater distinctness must I now know myself, since all the reasons that contribute to the knowledge of the nature of wax, or of any body whatever, manifest still better the nature of my mind? And there are besides so many other things in the mind itself that contribute to the illustration of its nature, that those dependent on the body, to which I have here referred, scarcely merit to be taken into account.

But, in conclusion, I find I have insensibly reverted to the point I desired; for, since it is now manifest to me that bodies themselves are not properly perceived by the senses nor by the faculty of imagination, but by the intellect alone; and since they are not perceived because they are seen and touched, but only because they are understood or rightly comprehended by thought, I readily discover that there is nothing more easily or clearly apprehended than my own mind. But because it is difficult to rid one's self so promptly of an opinion to which one has been long accustomed, it will be desirable to tarry for some time at this stage, that, by long continued meditation, I may more deeply impress upon my memory this new knowledge.

1.2 Abstraction and truth-functional logic

Consider the sentences: "Pigs fly" and "The Pope is Catholic". In our world, the sentence "Pigs fly, and the Pope is Catholic" is false, since pigs don't fly in our world.

Logic, however, knows not the particulars of the world that we live in. It is not a fact-checker. It is a tool for evaluating some of our reasoning. In fact, Logic doesn't know anything about equine aviation or Catholicism whatsoever. We might as well just call "Pigs fly" A and "The Pope is Catholic" B. To introduce the vernacular of *truth-functional logic*, a particular logical language, A is called an *atomic sentence* and "Pigs fly" is called the *extension* of A. An atomic sentence should be the simplest possible atom of information that still forms a complete sentence, and they are always denoted by capital letters.

Logic will tell us that "A and B" is false in the event that A is false (which we know—but Logic does not—is the case in our world). It will also tell us about all of the other possibilities—not just those limited to our world. That is:

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

These are all of the possibilities, which we'll call *cases*: both A and B are true (the first case), only one of the two is true (the second and third case), and both A and B are not true (the last case).

Definition 1.1 (Case)**.** A case is a particular assignment of truth values to atomic sentences.

By delineating every possible assignment of true (T) and false (F) to the n atomic sentences in question (as in the table above), you've examined all of the 2^n cases.

In truth-functional logic, a sentence such as "Pigs fly" must be either true or false (but not both!) in a case.

Axiom 1.1 (Principle of the Excluded Middle)**.** Given any sentence, either it or its negation must be true in every case.

Axiom 1.2 (Principle of Non-Contradiction)**.** A sentence and its negation cannot both be true in the same case.

The Principle of the Excluded Middle may seem obvious and fundamentally sound, but a few examples might shake our conviction.

Example 1.1. Mr. Barnes is six feet tall. Is the sentence "Mr. Barnes is tall" true or false? We don't really want to commit to either because we know that there are a lot of shorter people and a lot of taller people.

Example 1.2. Is the sentence "The present king of the United States is clothed" true or false. If it's true, then there *is* a king of the United States (absurd!). If it's false, then Principle of the Excluded Middle requires its negation to be true. But its negation is "The present king of the United States is not clothed". So there's still a king of the United States (and he's nude)!

Nevertheless, truth-functional logic abides according to the Principles of the Excluded Middle and Non-Contradiction, and so our two truth values in the language are "true" and "false".

Lastly, we abstract English negation and connectives to our five *logical operators*: $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. Their meanings are as follows:

Operator	Name	English equivalent(s)
$\neg$	negation	**not**
$\wedge$	conjunction	**and**, although, but
$\vee$	disjunction	**or**
$\rightarrow$	conditional	**if-then**, only if
$\leftrightarrow$	biconditional	**if and only if**, necessary and sufficient, exactly when

Our task is to begin with sentences in English, identify and label the atomic sentences within it, and combine those atomic sentences with the appropriate logical operators in a way that preserves as much meaning as possible. We call this *parsing* English into truth-functional logic.

Example 1.3. Translate "Pigs fly, and the Pope is Catholic" into truth-functional logic.
A: Pigs fly.
B: The Pope is Catholic.

$$A \wedge B$$

Example 1.4. Translate "If it ain't the hogs, it's the windmill" into truth-functional logic.
H: It's the hogs.
W: It's the windmill.

$$\neg H \rightarrow W$$

We'll also need parentheses for more complex sentences.

Example 1.5. Translate the following verse from James Taylor's "You've Got a Friend" into truth-functional logic.

When you're down and troubled
And you need a helping hand,
And nothing, oh nothing is going right,
Just close your eyes and think of me,
and soon I will be there
to brighten up even your darkest nights.

D: You're down.
T: You're troubled.
H: You need a helping hand.
N: Nothing is going right.
C: You close your eyes.
M: You think of me.
A: Soon I will be there.
B: I'll brighten up even your darkest nights.

$$((D \wedge T \wedge H \wedge N) \rightarrow (C \wedge M)) \rightarrow (A \wedge B)$$

I think...

The table below summarizes the complete list of symbols used in truth-functional logic.

Atomic sentences	Logical operators	Parentheses
A	$\neg$	(
B	$\wedge$	)
C	$\vee$	
$\vdots$	$\rightarrow$	
Z	$\leftrightarrow$	

If a symbol is not listed in this table, then it does not belong in a sentence written in truth-functional logic.

1.3 Determining truth or falsity in a case

When we were speaking of the truth or falsity of the sentence "Pigs fly, and the Pope is Catholic" in a given case, the following *truth table* seemed pretty obvious.

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

That is, we're very accustomed to the idea that a conjunction is true if and only if both of its *conjuncts* are true.

This disjunction is more ambiguous. We use the same word, "or", when we ask "Would you like coffee or tea?" as we do when we ask "Would you like milk or sugar?", but "or" means something different in each context: the first is *exclusive*, and the second is *inclusive*. That is, "Both" would be an inappropriate answer to "Coffee or tea?", but it would be a perfectly appropriate answer to "Milk or sugar?".

When we use "$\vee$" in truth-functional logic, it must be *truth-functional*. That is, when A and B are both true, then $A \vee B$ must have a well-defined truth value. So a choice must be made, and, in logic, we *define* "$\vee$" to be the inclusive or. The following truth-table, therefore, is a complete definition of "$\vee$".

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

This should be read to conclude that $A \vee B$ is false only in the case that A and B are both false. Since we're using "$\vee$" to denote an inclusive or, we must be careful to preserve accurate meaning when we translate from English to truth-functional logic.

Example 1.6. Translate "He might take milk or sugar with his tea" into truth-functional logic.
M: He might take milk with his tea.
S: He might take sugar with his tea.

$$M \vee S$$

Example 1.7. Translate "I'm going to fly or take the train" into truth-functional logic.
F: I'm going to fly.
T: I'm going to take the train.

$$(F \vee T) \wedge \neg(F \wedge T)$$

The remainder of the logical operators need to be similarly defined.

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

It's important to remember that we've fixed the definitions of these symbols as summarized in the table above, and they do not always exactly mirror English language usages of "not", "if-then", etc. As in the plane versus train example, parsing the English sentence into truth-functional logic by substituting the "$\vee$" for the word "or" does not completely convey the meaning of the English sentence—a "not both" statement must be included.

The following terms will be useful to classify sentences and sets of sentences.

Definition 1.2 (Tautology)**.** A sentence is a tautology if and only if it is true in all cases.

Definition 1.3 (Contingent)**.** A sentence is contingent if and only if it is true in some cases and false in others.

Definition 1.4 (Contradiction)**.** A sentence is called a contradiction if and only if it is of the form $A \wedge \neg A$.

The sentences A and $\neg A$ are called *contradictory*.

Definition 1.5 (Consistent)**.** A **set** of sentences is consistent if and only if there is a case in which all of the sentences are true.

1.4 Arguments

The purpose of developing this logical language is to evaluate *arguments*. Arguments consist of a non-negative number of *premises* and exactly one *conclusion*. A premise, like a conclusion, is just a sentence.

Example 1.8. Every student should take precalculus because you need precalculus to take calculus, and calculus is beautiful. Precalculus also teaches you how to problem solve.

It's useful to write arguments in *standard form*, which lists the premises first, followed by a horizontal line which denotes "therefore", and then the conclusion.

Example 1.9. The argument from the previous example, when written in standard form, looks like:

You need precalculus to take calculus.
Calculus is beautiful.
Precalculus teaches you how to problem solve.
___
Every student should take precalculus.

Logic will be a tool for assessing the *validity* of arguments. What we usually mean when we call an argument a "good" one, however, is that the argument is *sound*. Logic is *not* a tool that can help us determine if an argument is sound.

Definition 1.6 (Valid)**.** An argument is valid if and only if in every case in which all of its premises are true, its conclusion is also true (i.e., there is no case in which all the premises are true and the conclusion is false).

If an argument such as "A, therefore B" is valid, then we say A *implies* B, or B *follows* from A.

Definition 1.7 (Sound)**.** An argument is sound if and only if it is valid, and its premises are all actually true (in real life).

Definition 1.8 (Counterexample)**.** A case in which an argument's premises are all true but its conclusion is false is called a counterexample.

So an argument is invalid if and only if there exists a counterexample to it. The conditions for validity, then, are more relaxed than one might expect. All kinds of terrible arguments are, by this definition, considered valid. For example,

The world is round.
The world is not round.
———
We should legalize marijuana in the United States.

is a valid argument because there is no case in which both premises are true, so there is no counterexample.

Connecting arguments with theorems

In logic and philosophy, we evaluate arguments, while in mathematics, we evaluate propositions. We usually call the propositions that are "true" (i.e., true in every case; i.e., tautologies) *theorems*. Theorems tend to be written as conditional sentences.

Example 1.10. The Pythagorean Theorem, as it's usually stated reads: "If $\triangle ABC$ is a triangle and $m\angle ACB = 90°$, then $c^2 = a^2 + b^2$." This could, however, easily be stated as an argument.

$$\begin{array}{l} \triangle ABC \text{ is a triangle.} \\ m\angle ACB = 90° \\ \hline c^2 = a^2 + b^2 \end{array}$$

This leads us to an extremely important meta-theorem.

Theorem 1.1 (Theorem-argument exchange)**.** The argument

$$\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline C \end{array}$$

is valid if and only if the sentence (or theorem) $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to C$ is a tautology.

1.5 Informal proof

This book will cover a variety of different methods of proof—some *formal*, some *informal*. Calling a proof formal or informal in our context means something very specific.

Definition 1.9 (Formal proof)**.** A proof technique is called formal if and only if it can be carried out by a computer.

That is, "formality" does not have to do with rigor, neatness, or formatting, but rather that the proof can be executed by a programmable *algorithm*.

Informal proofs are actually the standard in mathematics. Typically, they look like one or more paragraphs written in English. The following proof techniques are modeled for proving the validity of argument, but remember that they similarly apply to proving theorems as well (in accordance with the theorem-argument exchange theorem).

Conditional proof

A conditional proof proves the validity of a valid argument by first assuming that all of the argument's premises are true, and then showing that, in such a case, the conclusion must also be true. Note that this sufficiently proves validity because the truth or falsity of an argument's conclusion is not of interest to a validity test in cases with a false premise.

Example 1.11. The following is a valid argument.

$$\begin{array}{l} A \to B \\ \neg A \to B \\ \hline B \end{array}$$

Proof. Let us examine a case in which the premises, $A \to B$ and $\neg A \to B$, are both true. In a case in which $A \to B$ is true, A must be false or B must be true (by the definition of "$\to$"). We must now verify that in the case that A is false, it follows that B is true. If A is false, then $\neg A$ is true (by the definition of "$\neg$"). If $\neg A$ is true and $\neg A \to B$ is true, then B must also be true (by the definition of "$\to$"). This proves that in a case in which the premises, $A \to B$ and $\neg A \to B$, are both true, the conclusion, B, must also be true. Therefore the argument is valid. □

Proof by contradiction

A proof by contradiction proves the validity of a valid argument by first assuming that there exists a counterexample to the argument, and then showing that this assumption leads to a contradiction. We conclude that the existence of a counterexample must be impossible, and hence the argument is valid.

Example 1.12. The following is a valid argument.

$$\begin{array}{l} A \to B \\ \neg A \to B \\ \hline B \end{array}$$

Proof. Assume there is a counterexample; that is, assume there is a case in which the premises, $A \to B$ and $\neg A \to B$, are both true but the conclusion, B, is false. If $A \to B$ is true and B is false, then A must be false (by the definition of "$\to$"). But if A is false and $\neg A \to B$ is true, then B must be true (by the definitions of "$\neg$" and "$\to$"). We have a case in which B is both true and false—a contradiction! Hence the existence of a counterexample is impossible, and the argument is valid. □

1.6 Formal proof

Truth tables

We've already used truth tables to define the truth-functional logical operators, but they're useful for evaluating the validity of arguments as well. We begin by listing an argument's atomic sentences (let's say there are n of them), and then determining the truth values of the premises and conclusion in all 2^n cases. If an argument is invalid, its truth table makes it easy to identify a counterexample.

Example 1.13. The following is a valid argument.

$$\begin{array}{l} A \rightarrow B \\ \neg A \rightarrow B \\ \hline B \end{array}$$

Proof.

A	B	$A \rightarrow B$	$\neg A \rightarrow B$	B
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

In every case where the premises are both true, the conclusion is also true. So the argument is valid. □

Natural deduction

Natural deduction is a formal proof technique that is more akin to the way we actually reason about things than proofs by truth tables. A natural deduction proof of validity begins by listing an argument's premises, and then applying a series of useful equivalences or rules of inference to the premises until arriving at the conclusion.

Useful equivalences

Name	Tautology	"Code"
Conditional Disjunction	$(A \to B) \leftrightarrow (\neg A \vee B)$	CDis
Contraposition	$(A \to B) \leftrightarrow (\neg B \to \neg A)$	ContraPos
Definition of Equivalence	$(A \leftrightarrow B) \leftrightarrow ((A \to B) \wedge (B \to A))$	Equiv
DeMorgan's Law (1)	$\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$	DeM
DeMorgan's Law (2)	$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	DeM
Distribution of And over Or	$(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$	Distr
Distribution of Or over And	$(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$	Distr
Double Negation	$\neg\neg A \leftrightarrow A$	DN
Repetition	$(A \vee A) \leftrightarrow A$	Rep
Also, you may commute sentences over $\wedge$, $\vee$, and $\leftrightarrow$.		

Rules for inference

Name	Inference	"Code"
Addition	From A, infer $A \vee B$ (B may be any sentence)	Add
Conjunction	From $\{A, B\}$, infer $A \wedge B$	Conj
Constructive Dilemma	From $\{A \vee B, A \to C, B \to D\}$, infer $C \vee D$	CD
Contradictory Premises	From $\{A, \neg A\}$, infer B (B may be any sentence)	ContraPrm
Disjunctive Syllogism	From $\{A \vee B, \neg A\}$, infer B	DS
Hypothetical Syllogism	From $\{A \to B, B \to C\}$, infer $A \to C$	HS
Modus Ponens	From $\{A \to B, A\}$, infer B	MP
Simplification	From $A \wedge B$, infer A	Simp
Tautology	Infer $A \vee \neg A$ (A may be any sentence)	Taut

Note that the rules of natural deduction only allow you to make valid deductions from a list of sentences. Hence, a natural deduction computer program would never halt given an input of an invalid argument. So, while natural deduction is more sleek and interesting than the truth table, it is not as powerful.

Example 1.14. The following is a valid argument.

$$\begin{array}{l} A \to B \\ \neg A \to B \\ \hline B \end{array}$$

Proof. The argument is proven valid by natural deduction.

1.	$A \to B$	Premise 1
2.	$\neg A \to B$	Premise 2
3.	$A \vee \neg A$	Taut
4.	$B \vee B$	CD(1,2,3)
5.	B	Rep(4)

□

1.7 Summary

This unit on truth-functional logic begins with reading Descartes's first and second meditations, where he presents his famous *cogito* argument. We begin by boiling the argument down to its essential components:

Either I am deceived or I am not deceived.
If I am deceived, then I exist.
If I am not deceived, then I exist.
―――――――――――――――――――
I exist.

This is a fruitful launching point for this unit for several reasons. First, it is an engaging argument that students are naturally interested in analyzing. Second, we recognize this argument as a "good" one, although maybe not as a convincing one. Third, this is a special case of a constructive dilemma, which we continue to refer to in natural deduction even through Unit 2.

That second property—that we recognize this as a "good" argument—motivates clarifying exactly what we mean when we say "this argument is good". Some arguments, we find, are bad, and others are even worse. Bad arguments rely on myths or falsehoods, but there is nothing wrong with their structure. Others are worse because their structure is nonsense, never mind their content. Here we formalize the notions of validity and soundness. Logic, we emphasize, is well equipped to deal with validity but not with soundness. The definition of invalidity is formalized first, which is much easier for students to understand than validity is.

Once students develop an awareness that the structure of the argument determines its validity, not necessarily its factual correctness, we come to discover that the ideas of deception and existence are not essential to the validity of the *cogito* argument. For example, we could write

Either I am purple or I am not purple.
If I am purple, then I exist.
If I am not purple, then I exist.
―――――――――――――――――――
I exist.

and recognize it as a valid argument (though certainly unsound, which is not so much our concern in this course). Since the particulars of the atoms of information in the sentences are not important to the validity of the argument, we recognize that any argument of the form

A or not A.
If A, then B.
If not A, then B.

B.

is valid. So, again, we abstract to atomic sentences and logical operators.

Next, we examine the first sentence of the argument: "Either I am deceived or I am not deceived." Is it really necessary? This motivates a discussion about the principle of the excluded middle and the first project opportunity drop: three-or-more-valued logics.

We also recognize "either I am deceived or I am not deceived" as a tautology—something that is always true—and between this realization and our definition of invalidity, we have to start talking about truth values. Here, truth tables are introduced as definitions for logical operators and as a formal proof technique.

Students then transition to writing informal proofs of the validity or invalidity of the same arguments that they analyzed using truth tables. (Throughout, only arguments in English are presented to the students, so they are forced to abstract the underlying logical structure, determine validity, and then return to the context of the original statement of the argument to declare its validity.)

This focus on writing and proof writing motivates the ability to write more clearly. For example, we look at the bumper sticker that says, "Don't drink and drive." It's ambiguous and wanting of parenthesis. Students are tasked with rewriting the bumper sticker. From this, we formalize the notion of a logical equivalence. We abstract more equivalences from concrete examples, and they become part of our toolbox for natural deduction. The second Project 1 opportunity—the LSAT—is discussed at this point. Also, we evaluate the strengths and weaknesses of our three proof techniques thus far. We find that natural deduction is incapable of identifying invalid arguments. We drop the tree test as the final Project 1 opportunity as a more powerful validity-testing algorithm.

The unit concludes with the administration of Quiz 1 and completion of Project 1.

1.8 Submission 1.1

1. Knaves always lie, knights always tell the truth, and in Transylvania, where everybody is one or the other (but you can't tell which by looking), you encounter two people, one of whom says, "He's a knight or I'm a knave." What are they? (J 20)

Tasks that may be assigned to this section (problems 2-12) are as follows:

(a) *Write the argument in standard form.*

(b) *Claim whether the argument is valid or sound. In some cases, soundness will be difficult to determine, so "soundness is difficult to determine" is an appropriate answer.*

(c) *Translate the argument into truth-functional logic. Be sure to delineate the extensions you give to atomic sentences and write the argument (now in truth-functional logic) in standard form.*

(d) *Prove that the argument is valid or invalid (as appropriate) with a proof by truth table.*

(e) *Prove that the argument is valid or invalid (as appropriate) with an informal proof.*

2. I have already said that he must have gone to King's Pyland or to Mapleton. He is not at King's Pyland, therefore he is at Mapleton. (Sir Arthur Conan Doyle) (B 20)

3. The patient will die unless we operate. We will operate. Therefore the patient will not die. (B 20)

4. If I'm right, then I'm a fool. But if I'm a fool, I'm not right. Therefore, I'm no fool. (B 70)

5. If I'm right, then I'm a fool. But if I'm a fool, I'm not right. Therefore, I'm not right. (B 70)

6. If Einstein's theory of relativity is correct, light bends in the vicinity of the sun. Light does indeed bend at the vicinity of the sun. It follows that Einstein's theory of relativity is correct. (B 70)

7. Congress will agree to the cut only if the President announces his support first. The President won't announce his support first, so Congress won't agree to the cut. (B 20)

8. If you are ambitious, you'll never achieve all your goals. But life has meaning only if you have ambition. Thus, if you achieve all your goals, life has no meaning. (B 132)

9. If Adams wins the election, Brown will retire to private life. If Brown dies before the election, Adams will win it. Therefore, if Brown dies before the election, he will retire to private life. (Is this evidence that English conditionals aren't truth-functional?) (J 31)

10. "Thin is guilty," observed Watson, "because either Holmes is right and the vile Moriarty is guilty or he is wrong and the scurrilous Thin did the job; but those scoundrels are either both guilty or both innocent; and, as usual, Holmes is right." (J 18)

11. Mittens meows exactly when she is hungry. Mittens is meowing, but she isn't hungry. Therefore the end of the Earth is at hand. (B 70)

12. God is omnipotent if and only if He can do everything. If He can't make a stone so heavy that He can't lift it, then he can't do everything. But if He can make a stone so heavy that He can't lift it, He can't do everything. Therefore, either God is not omnipotent or God does not exist. (B 132)

For the following problems (13-20):

- *If the claim that's made is correct, prove that it's correct.*
- *If the claim that's made is incorrect, prove that it's incorrect.*
- *If the problem asks a question, answer it with a correct claim, and prove that your claim is correct.*

13. Consider the tic-tac-toe grid with the squares labeled as this:

1	2	3
4	5	6
7	8	9

Suppose that X moves first to square 5, and O moves next to square 4. Prove that X can guarantee a win. (Cline)

14. You can't make a valid argument invalid by adding premises. (J 19)
15. You can't make an invalid argument valid by removing premises. (J 19)
16. Suppose $(A \wedge B) \to C$ is contingent. What can you say about the argument $\begin{array}{c} A \\ \underline{B} \\ C \end{array}$? (M 46)
17. Suppose the argument $\begin{array}{c} A \\ \underline{B} \\ C \end{array}$ is valid. What can you say about $(A \wedge B) \to C$?
18. Some arguments with contradictory premises aren't valid. (B 59)
19. Suppose that $\{A, B, C\}$ is inconsistent. What can you say about $A \wedge B \wedge C$? (M 46)
20. Some arguments whose conclusions are contradictions are valid.

1.9 Submission 1.2

1. Jones, feeling upset about the insecurity of the Social Security system, sighs that he faces a dilemma: “If taxes aren’t raised, I’ll have no money when I’m old. If taxes are raised, I’ll have no money now.” Smith, ever the even-tempered one, reasons that neither of Jones’s contentions is true. Jones answers, “Aha! You’ve contradicted yourself!” Show that Smith’s assertion that both Jones’s claims are false is indeed contradictory. (B 133)

2. Consider the statement: *If a fetus is a person, it has a right to life.* Which of the following sentences follow from this? (B 26)

   (a) A fetus is a person only if it has a right to life.

   (b) If a fetus isn’t a person, it doesn’t have a right to life.

   (c) If a fetus doesn’t have a right to life, it isn’t a person.

   (d) A fetus has a right to life.

   (e) A fetus isn’t a person only if it doesn’t have a right to life.

3. Consider this statement from IRS publication 17: *Your Federal Income Tax: If you are single, you must file a return if you had gross income of $3,560 or more for the year.* What follows from this, together with the information listed? (B 26)

   (a) You are single with an income of $2,500.

   (b) You are married with an income of $2,500.

   (c) You are single with an income of $25,000.

   (d) You are single but do not have to file a return.

   (e) You are married but do not have to file a return.

Tasks that may be assigned to this section (problems 4-9) are as follows:

(a) Prove that the argument is valid (if applicable) with a proof by natural deduction.

4. I have already said that he must have gone to King’s Pyland or to Mapleton. He is not at King’s Pyland, therefore he is at Mapleton. (Sir Arthur Conan Doyle) (B 20)

5. If I’m right, then I’m a fool. But if I’m a fool, I’m not right. Therefore, I’m not right. (B 70)

6. Congress will agree to the cut only if the President announces his support first. The President won’t announce his support first, so Congress won’t agree to the cut. (B 20)

7. If you are ambitious, you’ll never achieve all your goals. But life has meaning only if you have ambition. Thus, if you achieve all your goals, life has no meaning. (B 132)

8. Mittens meows exactly when she is hungry. Mittens is meowing, but she isn’t hungry. Therefore the end of the Earth is at hand. (B 70)

9. God is omnipotent if and only if He can do everything. If He can’t make a stone so heavy that He can’t lift it, then he can’t do everything. But if He can make a stone so heavy that

He can't lift it, He can't do everything. Therefore, either God is not omnipotent or God does not exist. (B 132)

For the following problems (10-15):

- *If the claim that's made is correct, prove that it's correct.*
- *If the claim that's made is incorrect, prove that it's incorrect.*
- *If the problem asks a question, answer it with a correct claim, and prove that your claim is correct.*

10. A two-place connective, $\circ$, is called *associative* if $(A \circ B) \circ C$ is logically equivalent to $A \circ (B \circ C)$. Which of $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ are associative? (J 20)

11. Suppose C is a tautology. What can you say about the argument $\begin{array}{c} A \\ \underline{B} \\ C \end{array}$? (M 46)

12. Suppose that A and B are logically equivalent. What can you say about $A \vee B$? (M 46)

13. Suppose that A and B are *not* logically equivalent. What can you say about $A \vee B$? (M 46)

14. There are a number of languages with only two operators that are equivalent to truth-functional logic. Show that it is sufficient to have only the negation ($\neg$) and the conditional ($\rightarrow$) by writing sentences (containing only the operators $\neg$ and $\rightarrow$) that are logically equivalent to the following: (M 46-7)

    - $A \vee B$
    - $A \wedge B$
    - $A \leftrightarrow B$

15. Show that there is a language containing only two truth-functional operators, the negation ($\neg$) and the disjunction ($\vee$), that is equivalent to truth-functional logic. (M 47)

1.10 Unassigned Problems

1. Consider the statement: *If a fetus is a person, it has a right to life.* Which of the following sentences follow from this? Which imply it? (B 26)

   (a) A fetus is a person.

   (b) If a fetus has a right to life, then it's a person.

   (c) A fetus has a right to life only if it's a person.

   (d) A fetus doesn't have a right to life only if it isn't a person.

   (e) A fetus doesn't have a right to life unless it's a person.

   (f) A fetus isn't a person unless it has a right to life.

   (g) A fetus is a person unless it doesn't have a right to life.

   (h) A fetus has a right to life unless it isn't a person.

Tasks that may be assigned to this section problems 2-6) are as follows:

(a) Write the argument in standard form.

(b) Claim whether the argument is valid or sound. In some cases, soundness will be difficult to determine, so "soundness is difficult to determine" is an appropriate answer.

(c) Translate the argument into truth-functional logic. Be sure to delineate the extensions you give to atomic sentences and write the argument (now in truth-functional logic) in standard form.

(d) Prove that the argument is valid or invalid (as appropriate) with a proof by truth table.

(e) Prove that the argument is valid or invalid (as appropriate) with an informal proof.

(f) Prove that the argument is valid (if applicable) with a proof by natural deduction.

2. If the objects of mathematics are material things, then mathematics can't consist entirely of necessary truths. Mathematical objects are immaterial only if the mind has access to a realm beyond the reach of the senses. Mathematics does consist of necessary truths, although the mind has no access to any realm beyond the reach of the senses. Therefore the objects of mathematics are neither material nor immaterial. (B 132)

3. If the President pursues arms limitations talks, then if he gets the foreign policy mechanism working more harmoniously, the European Left will acquiesce to the placement of additional nuclear weapons in Europe. But the European Left will never acquiesce to that. So either the President won't get the foreign policy mechanism working more harmoniously, or he won't pursue arms limitations talks. (B 132)

4. If we continue to run a large trade deficit, then the government will yield to calls for protectionism. We won't continue to run a large deficit only if our economy slows down or foreign economies recover. So, if foreign economies don't recover, then the government will resist calls for protectionism only if our economy slows down. (B 133)

5. We cannot both maintain high educational standards and accept almost every high school graduate unless we fail large numbers of students when (and only when) many students do poorly. We will continue to maintain high standards; furthermore, we will placate the legislature and admit almost all high school graduates. Of course, we can't both placate the legislature and fail large numbers of students. Therefore, not many students will do poorly.

6. God is that, the greater than which cannot be conceived. If the idea of God exists in our understanding, but God does not exist in reality, then something is conceivable as greater than God. If the idea of God exists in our understanding, therefore, God exists in reality. (B 132)

For the following problems (7-14):

- *If the claim that's made is correct, prove that it's correct.*
- *If the claim that's made is incorrect, prove that it's incorrect.*
- *If the problem asks a question, answer it with a correct claim, and prove that your claim is correct.*

7. All contradictory sentences imply one another. (B 59)

8. No sentence implies its own negation. (B 59)

9. Any sentence that implies its own negation is a contradiction. (B 59)

10. Suppose A is a contradiction. What can you say about the argument $\begin{array}{c} A \\ B \\ \hline C \end{array}$? (M 46)

11. All tautologies imply one another. (B 59)

12. Any sentence that follows from a tautology is itself a tautology. (B 59)

13. Any sentence that follows from its own negation is a tautology. (B 59)

14. Under what condition is a sentence whose sole connective is $\leftrightarrow$ a tautology? (J 20)

Chapter 2

Quantificational Logic

2.1 Frege's *Begriffsschrift*

"Preface"

In apprehending a scientific truth we pass, as a rule, through various degrees of certitude. Perhaps first conjectured on the bases of an insufficient number of particular cases, a general proposition comes to be more and more securely established by being connected with other truths through chains of inferences, whether consequences are derived from it that are confirmed in some other way or whether, conversely, it is seen to be a consequence of propositions already established. Hence we can inquire, on the one hand, how we have gradually arrived at a given proposition and, on the other, how we can finally provide it with the most secure foundation. The first question may have to be answered differently for different persons; the second is more definite, and the answer to it is connected with the inner nature of the proposition considered. The most reliable way of carrying out a proof, obviously, is to follow pure logic, a way that, disregarding the particular characteristics of the objects, depends solely on those laws upon which all knowledge rests. Accordingly, we divide all truths that require justification into two kinds, those for which the proof can be carried out purely by means of logic and those for which it mush be supported by facts of experience. But that a proposition is of the first kind is surely compatible with the fact that it could nevertheless not have come to consciousness in a human mind without any activity of the senses.[1] Hence it is not the psychological genesis but the best method of proof that is at the basis of the classification. Now, when I came to consider the question to which of these two kinds the judgments of arithmetic belong, I first had to ascertain how far one could proceed in arithmetic by means of inferences alone, with the sole support of those laws of thought that transcend all particulars. My initial step was to attempt to reduce the concept of ordering in a sequence to that of *logical* consequence, so as to proceed from there to the concept of number. To prevent anything intuitive from penetrating here unnoticed, I had to bend every effort to keep the chain of inferences free of gaps. In attempting to comply with this requirement in the strictest possible way I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more complex, to attain the precision that my purpose required. This deficiency led me to the idea of the present ideography. Its first purpose, therefore, is to

[1]Since without sensory experience no mental development is possible in the beings known to us, that holds of all judgments.

provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated. That is why I decided to forgo expressing anything that is without significance for the *inferential sequence*. In §3 I called what alone mattered to me the *conceptual content*. Hence this definition must always be kept in mind if one wishes to gain a proper understanding of what my formula language is. That, too, is what led me to the name "Begriffsschrift". Since I confined myself for the time being to expressing relations that are independent of the particular characteristics of the objects, I was also able to use the expression "formula language for pure thought". That it is modeled upon the formula language of arithmetic, as I indicated in the title, has to do with fundamental ideas rather than with details of execution. Any effort to create an artificial similarity by regarding a concept as the sum of its marks was entirely alien to my thought. The most immediate point of contact between my formula language and that of arithmetic is the way in which the letters are employed.

I believe that I can best make the relation of my ideography to ordinary language clear if I compare it to that which the microscope has to the eye. Because of the range of its possible uses and the versatility with which it can adapt to the most diverse circumstances, the eye is far superior to the microscope. Considered as an optical instrument, to be sure, it exhibits many imperfections, which ordinarily remain unnoticed only on account of its intimate connection with out mental life. But, as soon as scientific goals demand great sharpness of resolution, the eye proves to be insufficient. The microscope, on the other hand, is perfectly suited to precisely such goals, but that is just why it is useless for all others.

This ideography, likewise, is a device invented for certain scientific purposes, and one must not condemn it because it is not suited to others. If it answers to these purposes in some degree, one should not mind the fact that there are no new truths in my work. I would console myself on this point with the realization that a development of method, too, furthers science. Bacon, after all, thought it better to invent a means by which everything could be easily discovered than to discover particular truths, and all great steps of scientific progress in recent times have had their origin in an improvement of method.

Leibniz, too, recognized—and perhaps overrated—the advantages of an adequate system of notation. His idea of a universal characteristic, of a *calculus philosophicus* or *ratiocinator*, was so gigantic that the attempt to realize it could not go beyond the bare preliminaries. The enthusiasm that seized its originator when he contemplated the immense increase in the intellectual power of mankind that a system of notation directly appropriate to objects themselves would bring about led him to underestimate the difficulties that stand in the way of such an enterprise. But, even if this worthy goal cannot be reached in one leap, we need not despair of a slow, step-by-step approximation. When a problem appears to be unsolvable in its full generality, one should temporarily restrict it; perhaps it can then be conquered by a gradual advance. It is possible to view the signs of arithmetic, geometry, and chemistry as realizations, for specific fields, of Leibniz's idea. The ideography proposed here adds a new one to these fields, indeed the central one, which borders on all the others. If we take our departure from there, we can with the greatest expectation of success proceed to fill the gaps in the existing formula languages, connect their hitherto separated fields into a single domain, and extend this domain to include fields that up to now have lacked such a language.

I am confident that my ideography can be successfully used wherever special value must be placed on the validity of proofs, as for example when the foundations of the differential and integral calculus are established.

It seems to me to be easier still to extend the domain of this formula language to include geometry. We would only have to add a few signs for the intuitive relations that occur there. In this way we would obtain a kind of *analysis situs*.

The transition to the pure theory of motion and then to mechanics and physics could follow at this point. The latter two fields, in which besides rational necessity empirical necessity asserts itself, are the first for which we can predict a further development of the notation as knowledge progresses. That is no reason, however, for waiting until such progress appears to have become impossible.

If it is one of the tasks of philosophy to break the domination of the word over the human spirit by laying bare the misconceptions that through the use of language often almost unavoidably arise concerning the relations between concepts and by freeing thought from that with which only the means of expression of ordinary language, constituted as they are, saddle it, then my ideography, further developed for theses purposes, can become a useful tool for the philosopher. To be sure, it too will fail to reproduce ideas in a pure form, and this is probably inevitable when ideas are represented by concrete means; but, on the one hand, we can restrict the discrepancies to those that are unavoidable and harmless, and, on the other, the fact that they are of a completely different kind from those peculiar to ordinary language already efforts protection against the specific influence that a particular means of expression might exercise.

The mere invention of this ideography has, it seems to me, advanced logic. I hope that logicians, if they do not allow themselves to be frightened off by an initial impression of strangeness, will not withhold their as send from the innovations that, by a necessity inherent in the subject matter itself, I was driven to make. These deviations from what is traditional find their justification in the fact that logic has hitherto always followed ordinary language and grammar too closely. In particular, I believe that the replacement of the concepts *subject* and *predicate* by *argument* and *function*, respectively, will stand the test of time. It is easy to see how regarding a content as a function of an argument leads to the formation of concepts. Furthermore, the demonstration of the connection between the meanings of the words *if*, *and*, *not*, *or*, *there is*, *some*, *all*, and so forth, deserves attention. . . .

As I remarked at the beginning, arithmetic was the point of departure for the train of thought that led me to my ideography. And that is why I intend to apply it first of all to that science, attempting to provide a more detailed analysis of the concepts of arithmetic and a deeper foundation for its theorems. For the present I have reported in the third chapter some of the developments in this direction. To proceed farther along the path indicated, to elucidate the concepts of number, magnitude, and so forth—all this will be the object of further investigations which I shall publish immediately after this booklet.

Jena, 18 December 1878.

2.2 Truth-functional logic versus quantificational logic

Quantificational logic, instead of using atomic sentences, breaks up sentences into subjects and predicates. Quantificational logic is distinct from truth-functional logic because of the addition of *variables* and *quantifiers* to the language.

Truth-functional logic	**Quantificational logic**
Parentheses (,)	Parentheses (,)
Logical operators $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$	Logical operators $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
Atomic sentences $A, B, C, \ldots$	Predicates $P, Q, R, \ldots$
	SUBJECTS Names $a, b, c, \ldots$
	Variables $x, y, z, \ldots$
	Quantifiers $\forall, \exists$

The two quantifiers are:

- the *universal quantifier*, $\forall$, which is usually read "for all", or, when paired with a variable as in $\forall x$, read "every individual x is such that ..."; and

- the *existential quantifier*, $\exists$, which is usually read "there exists", or, when paired with a variable as in $\exists x$, read "some individual x is such that ...".

Example 2.1. If we translate the argument

$$\begin{array}{l} \text{All men are mortal.} \\ \text{Socrates is a man.} \\ \hline \text{Socrates is mortal.} \end{array}$$

into truth-functional logic, we are forced to express it as something like:
A: All men are mortal.
M: Socrates is a man.
S: Socrates is mortal.

$$\begin{array}{c} \text{A} \\ \text{M} \\ \hline \text{S} \end{array}$$

which is an invalid argument. In quantificational logic, however, we can be more expressive.
Mx: x is a man.
Dx: x is mortal.
s: Socrates

$$\begin{array}{l} \forall x(Mx \rightarrow Dx) \\ Ms \\ \hline Ds \end{array}$$

And this, we'll learn, is valid.

Every variable subject must be quantified. Subjects and predicates must combine to make complete sentences.

Example 2.2. Let Px denote "x has property P", let Qx denote "x has property Q", and let a name some particular individual (Alfred, let's say).

Incomplete sentences in QL	Complete sentences in QL
a	
P	
$\forall$	
$\exists x$	
Px	Pa
$\forall x Pa$	$\forall x Px$
$P \to Q$	$Pa \to Qa$
$\forall x Px \lor Qx$	$\forall x(Px \lor Qx)$
$\forall x(Px \land Qy)$	$\forall x Px \land \exists y Qy$

2.3 Multiple variables and multi-place predicates

So far, we've only seen one-place predicates like Px to denote something like "x has property P". We may, however, also define two-place predicates, three-place predicates, and so on.

Example 2.3. Let Bxy denote "x is the brother of y", and let $Pxyz$ denote "x put y on z".

If we introduce more than one variable in a sentence, it is important to remember that each variable must be quantified individually.

Example 2.4. Translate "we are brothers, all of us" into quantificational logic.
Bxy: x is the brother of y

$$\forall x \forall y Bxy$$

"$\forall x \forall y$" should be read "every pair of individuals, x and y, are such that …" Similarly, "$\forall x \forall y \forall z$" should be read "every trio of individuals, x, y, and z, are such that…"; and "$\exists x \exists y$" should be read "there is (or there exists) a pair of individuals, x and y, such that…"

Finally, it is fine if, within the same sentence, there are two different variables with different quantifiers.

Example 2.5. Translate "everybody likes somebody" into quantificational logic.
Lxy: x likes y

$$\forall x \exists y Lxy$$

2.4 The identity predicate

We need a predicate to indicate that two subjects are the same (i.e., identical or equal). All other predicates are written as a capital letter, and while we could do the same with the identity predicate (e.g., Exy), we choose to follow the convention of using the "=" sign. (We'll also adopt the conventional notation for non-identity: "$\neq$".) That is, it is understood that the extension for the identity predicate is, at all times,

$$x = y\text{: } x \text{ is identical to } y.$$

In logic, x and y are called identical if and only if x and y are the very same thing. For example, two distinct copies of the *Begriffsschrift*, although they may be identical in appearance and content, are not themselves identical because they are not the very same book.

In this strict interpretation of identity, it seems clear that identical things have all the same properties.

Axiom 2.1 (Leibniz's Law)**.** For all predicates P, $\forall x \forall y(x = y \rightarrow (Px \leftrightarrow Py))$.

It's tempting to write Leibniz's Law (or the "Indiscernibility of Identicals") as $\forall P[\forall x \forall y(x = y \rightarrow (Px \leftrightarrow Py))]$, but quantificational (or "1st order") logic does not allow quantification over predicates. Second order logic is required to express Leibniz's Law as such.

Describing particular quantities

In quantificational logic, the identity predicate is essential for denoting the specific number of individuals that have a property.

Example 2.6. There is exactly one god.
Gx: x is a god.

$$\exists x(Gx \wedge \forall y(y \neq x \rightarrow \neg Gy))$$

Notice that the second conjunct says something like "if y is not identical to x, then y is not a god". The contrapositive—"if y is a god, then y is identical to x"—would have worked just as well.

Example 2.7. There are at least two gods.
Gx: x is a god.

$$\exists x \exists y(Gx \wedge Gy \wedge x \neq y)$$

2.5 Interpretations

In truth-functional logic, we spoke of the truth or falsity of a sentence in a case. Cases were defined by assignments of truth values to atomic sentences.

In quantificational logic, we no longer have atomic sentences. Instead of speaking about the truth or falsity of a sentence in a case, we'll speak about the truth or falsity of a sentence in an *interpretation*. An interpretation is simply a *universe of discourse* (a set containing all of the things we're talking about) and extensions for all of the names and predicates used.

Example 2.8. In the interpretation

$UD = \{1, 2, 3\}$
Gxy: x is strictly greater than y
$a : 1$
$b : 2$

Gba is true, and $\forall x \exists y Gyx$ is false.

Example 2.9. In the interpretation

$UD = \{1, 2, 3, \dots\}$
Gxy: x is strictly greater than y
$a : 1$
$b : 2$

Gba and $\forall x \exists y Gyx$ are both true, but $\exists x \forall y Gxy$ and $\forall x \exists y (y \neq x \wedge \neg Gyx)$ are both false.

2.6 Natural deduction

All of the rules for natural deduction established in truth-functional logic still apply in quantificational logic.

Additional equivalences for quantificational logic

Name	Tautology	"Code"
Quantifier Exchange	$\neg \forall x(\dots x \dots) \leftrightarrow \exists x \neg(\dots x \dots)$	QEx
Quantifier Exchange	$\neg \exists x(\dots x \dots) \leftrightarrow \forall x \neg(\dots x \dots)$	QEx

Additional rules of inference for quantificational logic

Name	Inference	"Code"
Existential Instantiation	From $\exists x(\dots x \dots)$, infer $\dots a \dots$ (a must be a **new** name)	EI
Universal Instantiation	From $\forall x(\dots x \dots)$, infer $\dots a \dots$ (a may be any name)	UI
Existential Generalization	From $\dots a \dots$, infer $\exists x(\dots x \dots)$	EG
Universal Generalization	From $\dots a \dots$, infer $\forall x(\dots x \dots)$ (iff UI alone deduced a)	UG
Leibniz's Law	Infer $\forall x \forall y(x = y \to (Px \leftrightarrow Py))$ (P may be any predicate)	Leibniz

Example 2.10. *Claim*: The argument $\dfrac{\begin{array}{c}\forall x(Hx \to Gx)\\ \exists x(\neg Gx \wedge Fx)\end{array}}{\exists x(Fx \wedge \neg Hx)}$ is valid.

Proof. The argument is proven valid by natural deduction.

1.	$\forall x(Hx \to Gx)$	Premise 1
2.	$\exists x(\neg Gx \wedge Fx)$	Premise 2
3.	$\neg Ga \wedge Fa$	EI(2)
4.	$\neg Ga$	Simp(3)
5.	$Ha \to Ga$	UI(1)
6.	$\neg Ga \to \neg Ha$	ContraPos(5)
7.	$\neg Ha$	MP(4,6)
8.	Fa	Simp(3)
9.	$Fa \wedge \neg Ha$	Conj(7,8)
10.	$\exists x(Fx \wedge \neg Hx)$	EG(9)

□

Notice that existential instantiation was performed prior to universal instantiation in this proof. Existential instantiation results in an sentence about an arbitrary individual, a. Since a is an individual in the universe of discourse, the universally quantified Premise 1 must be true for that particular individual, a. Instantiating in the opposite order would have resulted in an invalid inference.

2.7 Summary

This unit on quantificational logic begins with reading Gottlob Frege's Preface to his *Begriffsschrift*, wherein he expounds on the need for a language with greater powers of expression than that of truth-functional logic. At the end of Unit 1, we frustrate students by demonstrating their inability to show the validity of one of the most famous arguments there is:

$$\dfrac{\begin{array}{l}\text{All men are mortal.}\\ \text{Socrates is a man.}\end{array}}{\text{Socrates is mortal.}}$$

In truth-functional logic, we are forced to express this as something like

$$\dfrac{\begin{array}{c}\mathrm{A}\\ \mathrm{M}\end{array}}{\mathrm{S}}$$

which is obviously invalid. We know, however, that this is a good argument—that it ought to be valid. The problem, in truth-functional logic, is that we're not able to talk at once about the set of all men and a particular member of that set. This motivates the need for our language to include separate characters for subjects, predicates, and quantifiers.

We show how frequently this language is used in mathematics, incorporating the identity predicate and even some epsilon-delta calculus to tap into and extend from students' prior knowledge. Just as students come to believe that this is the "correct" logic, however, we introduce Leibniz's Law—that if two individuals are identical, then they have all the same properties—as a valid rule of inference. Here, however, we point out two problems that arise. First, this simply is not true of Superman and Clark Kent. They are clearly the same individual, but, while Lois Lane loves Superman, she does not love Clark Kent. Second, Leibniz's Law cannot itself be expressed in the language of quantificational logic, as quantificational logic does not permit quantification over predicates. At this point, the opportunity to choose to investigate 2nd order logic for Project 2 is extended to students.

We continue to use informal proof theories and natural deduction to prove the validity of arguments and various claims. Again, the majority of the arguments presented to students are relatable and concrete. Included are some famous arguments, including the Omnipotence Paradox.

Also, we continue to investigate logical fallacies and fallacious reasoning. We end with a reading on the Ontological argument, which solidifies the need to carefully distinguish between things that are properties (and ought to be made a predicate) and things that are individuals (and ought to be given a name), and it also challenges students to identify subtle mistakes in instantiation. We leave the Ontological argument reminded that logic has no bearing on the soundness of arguments and that the validity of arguments in quantificational logic is affected by our judgment and decisions when we translate from the English to quantificational logic. Students are offered the opportunity to select to study arguments concerning a god's existence as a Project 2 topic.

Lastly, we reflect on the powers of natural deduction in quantificational logic. We encounter several arguments where it is necessary to instantiate a universal multiple times to prove the validity of an argument. If our Universe of Discourse is infinite and we can instantiate a universal for every member of the Universe of Discourse, is it possible that Natural Deduction fails to be a decision procedure for proving the validity of valid arguments? What about the more elegant tree test? Students are offered the opportunity to study the undecidability of quantificational logic for Project 2.

2.8 Submission 2.1

1. Translate into smooth English:

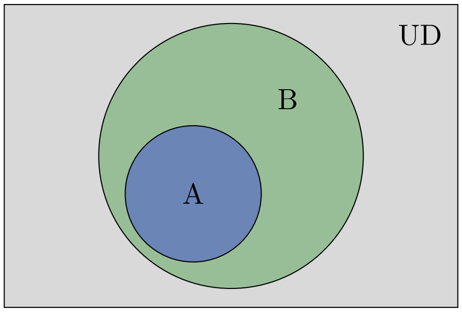
$$\forall x \forall y((Px \wedge Ty \wedge Dxy \wedge Oxy) \rightarrow \neg\exists z(Pz \wedge Kzxy)).$$

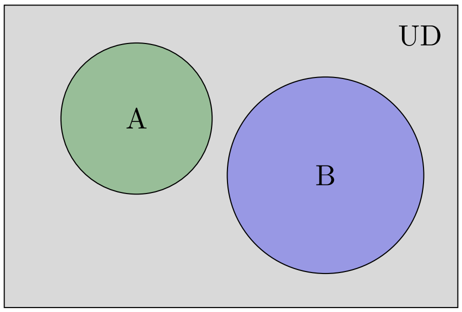
   Let "Px" mean "x is a person", "Tx" mean "x is a time", "Dxy" mean "x is down at time y", "Oxy" mean "x is out at time y", and "$Kxyz$" mean "x knows y at time z".

For problems 2-15, write a sentence in quantificational logic that captures as much of the given information as possible. Remember to delineate the extensions you assign to names and predicates.

2. All's well that ends well. (Shakespeare) (B 148)

3. . . . the things which are seen are temporal; but the things which are not seen are eternal. (II Corinthians 4:18) (B 169)

4. 

5. 

6. If you don't love yourself, you can't love anybody else.
7. 'N Sync is the best band ever.
8. Somebody loves everybody.
9. There is someone for everybody.
10. Scrooge doesn't love anybody.
11. Only the shallow know themselves. (Oscar Wilde) (B 169)
12. Everybody has a mother.
13. There are at least two pigs.
14. There are exactly two pigs.
15. There are at most two pigs.

2.9 Submission 2.2

1. In school, a *curve* is a function, usually from $[0, 100]$ to $[0, 100]$. The following are some made-up definitions:

   - A curve f is called *fair* if
   $$\forall x \forall y\, (x \geq y \rightarrow f(x) \geq f(y))\,.$$
   - A curve f is called *totally unfair* if
   $$\forall x \forall y\, (x > y \rightarrow f(x) < f(y))\,.$$
   - A curve f is called *progressive* if
   $$\forall x \forall y \big(x < y \rightarrow f(x) - x > f(y) - y\big).$$

   (a) Say (both in English and in Quantificational Logic) what it means for a curve to be unfair.

   (b) Say (both in English and in Quantificational Logic) what it means for a curve to be not totally unfair.

   (c) Conjure an example of a curve that is unfair but not totally unfair.

   (d) Say in English what it means for a curve to be progressive. Say (both in English and in Quantificational Logic) what it means for a curve to be not progressive.

   (e) Classify the curve "everyone gets 5 points" as fair or unfair and progressive or not progressive.

2. The limit of a function, $f(x)$, at a point c is l if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, if the distance between x and c is less than δ, then the distance between $f(x)$ and l is less than ε. Suppose f is a function, and the limit of f at 0 is not 1. Given this definition of a limit, how can you prove that the limit of f at 0 is not 1? That is, what evidence would you need to provide in order to prove that the limit of f at 0 is not 1?

3. A function, f, is continuous at a point c if and only if for all sequences of real numbers, x_n, that converge to c, the sequence $f(x_n)$ converges to $f(c)$. Suppose f is a function, and $f(0) = 1$. Given this definition of pointwise continuity above, what would you need to do to prove that f is discontinuous at 0? That is, what evidence would you need to provide in order to prove that f is discontinuous at 0?

Tasks that may be assigned to this section (problems 4-12) are as follows:

(a) Translate the argument into quantificational logic. Be sure to delineate the extensions you give to predicates and names, and write the argument (now in quantificational logic) in standard form.

(b) Claim whether the argument is valid or sound. In some cases, soundness will be difficult to determine, so "soundness is difficult to determine" is an appropriate answer.

(c) Prove that the argument is valid or invalid (as appropriate) with an informal proof.

(d) *Prove that the argument is valid (if applicable) with a proof by natural deduction.*

4. Everything has a cause. Therefore something is the cause of everything. (Some people think St. Thomas Aquinas advocated this.) (B 183)

5. Fred hates everyone who hates Al. Al hates everyone. So Al and Fred hate each other. (B 213)

6. All insects in this house are large and hostile. Some insects in this house are impervious to pesticides. Thus, some large, hostile insects are impervious to pesticides. (B 213)

7. Some students cannot succeed at the university. All students who are bright and mature can succeed. It follows that some students are either not bright or immature. (B 213)

8. There are at least 3 pigs. So there are at least two pigs.

9. Popeye and Olive Oyl like each other since Popeye likes everyone who likes Olive Oyl, and Olive Oyl likes everyone. (B 218)

10. This argument is unsound, for its conclusion is false, and no sound argument has a false conclusion. (J 49)

11. Everyone likes Mandy. Mandy likes nobody but Andy. Therefore Mandy and Andy are the same person. (B 238)

12. Everyone is afraid of Mr. Hyde. Mr. Hyde is afraid only of Dr. Jekyll. Therefore, Dr. Jekyll is Mr. Hyde. (B 234)

Tasks that may be assigned to this section (problems 13-15) are as follows:

(a) *Claim whether the argument is valid or invalid.*

(b) *Prove that the argument is valid or invalid (as appropriate) with an informal proof.*

13. $$\begin{array}{l} \forall x(Fx \to Gx) \\ \underline{\forall x(Fx \to Hx)} \\ \forall x(Gx \to Hx) \end{array}$$ (B 204)

14. $$\begin{array}{l} \underline{\forall x(Fx \to Gx)} \\ \forall x(\neg Gx \to \neg Fx) \end{array}$$ (B 204)

15. $$\begin{array}{l} \neg\exists x(Fx \wedge Gx) \\ \underline{\forall x(Gx \to Hx)} \\ \forall x(Fx \to \neg Hx) \end{array}$$ (B 204)

2.10 Unassigned Problems

1. The Quadratic Formula Theorem states: If $f(x) = ax^2 + bx + c$ is a quadratic function and $f(x_0) = 0$, then $x_0 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x_0 = \frac{-b-\sqrt{b^2-4ac}}{2a}$. Suppose $f(x) = ax^2 + bx + c$ $(a, b, c \in \mathbb{R})$, and $\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \pm 1$. According to this statement of the Quadratic Formula Theorem above, what can you say about $f(1)$?

2. Let our universe of discourse be the natural numbers and "Gxy" mean "x is strictly greater than y". Translate into smooth English and determine the truth value of the following:

   (a) $\forall x \exists y Gyx$

   (b) $\exists x \forall y Gxy$

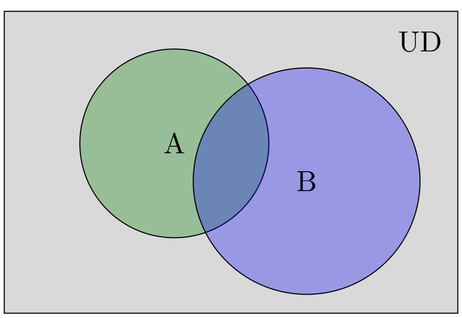
   (c) $\exists x \forall y \neg Gxy$

   (d) $\forall x \exists y (y \neq x \wedge \neg Gyx)$

3. A function, f, is continuous at a point c if and only if for all sequences of real numbers, x_n, that converge to c, the sequence $f(x_n)$ converges to $f(c)$. Write out a sketch of an informal proof that $f(x) = x^2 - 1$ is continuous at 3.

For problems 4-10, write a sentence in quantificational logic that captures as much of the given information as possible. Remember to delineate the extensions you assign to names and predicates.

4. Every day is better than the next. (*There's Something About Mary*)

5. 

6. Every neighbor of a is unsafe.

7. If you've seen one episode, you've seen 'em all.

8. If the only tool you have is a hammer, you tend to see every problem as a nail. (Abraham Maslow) (B 171)

9. Philosophy is the highest music. (Plato) (B 230)

10. An executive organization ... is no stronger than its weakest link. (Robert Patterson) (B 230)

Tasks that may be assigned to this section (problems 11-12) are as follows:

   (a) Translate the argument into quantificational logic. Be sure to delineate the extensions you give to predicates and names, and write the argument (now in quantificational logic) in standard form.

(b) *Claim whether the argument is valid or sound. In some cases, soundness will be difficult to determine, so "soundness is difficult to determine" is an appropriate answer.*

(c) *Prove that the argument is valid or invalid (as appropriate) with an informal proof.*

(d) *Prove that the argument is valid (if applicable) with a proof by natural deduction.*

11. A person is valedictorian of a class if and only if s/he has the highest GPA in the class. So there can be at most one valedictorian in any class. (B 238)

12. Jones's killer weighed over 200 pounds. Smith weighs less than 200 pounds. So, Smith isn't Jones's killer. (B 238)

Tasks that may be assigned to this section (problems 13-14) are as follows:

(a) *Claim whether the argument is valid or invalid.*

(b) *Prove that the argument is valid or invalid (as appropriate) with an informal proof.*

(c) *Prove that the argument is valid (if applicable) with a proof by natural deduction.*

13. $\dfrac{\forall x(Fx \rightarrow Gx)}{\forall x(\neg Fx \rightarrow \neg Gx)}$ (B 204)

14. $\begin{array}{l} \exists x \forall y(\exists z Fyz \rightarrow Fyx) \\ \forall x \exists y Fxy \\ \hline \exists x \forall y Fyx \end{array}$ (B 223)

Chapter 3

Functions

3.1 Functions are special relations

Definition 3.1 (Cross Product)**.** Let A and B be two (non-empty) sets. Then the cross product of A and B is

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Definition 3.2 (Relation)**.** A relation from a non-empty set A to a non-empty set B is a subset of $A \times B$.

Definition 3.3 (Function)**.** A function f from a non-empty set A to a non-empty set B (which we write $f : A \to B$) is a relation in which every element of A is paired with one and only one element of B.

A note on notation: if $(a, b) \in f$, we say $f(a) = b$.

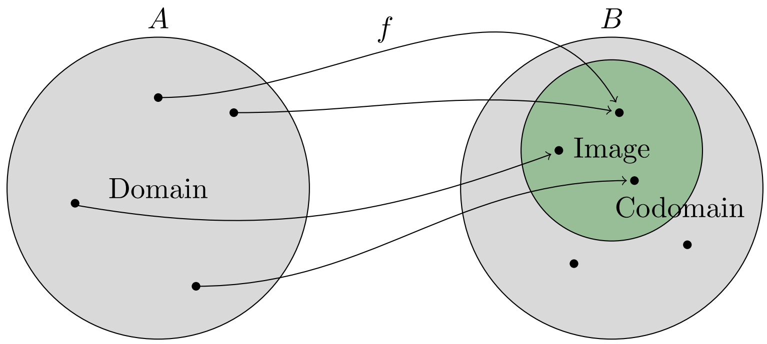
We can write an equivalent definition of a function in the language of quantificational logic.

Definition 3.4 (Function)**.** $f : A \to B$ is a function if and only if

$$\forall a \in A\, [\exists b \in B (f(a) = b) \ \wedge \ \forall c \in B\ \forall d \in B ((f(a) = c \wedge f(a) = d) \to c = d)].$$

3.2 Properties of functions

A function, f, from A to B



Definition 3.5 (Domain)**.** If $f : A \to B$ is a function, then the set A is the domain of f, $Dom(f)$.

Definition 3.6 (Codomain)**.** If $f : A \to B$ is a function, then the set B is the codomain of f, $Cod(f)$.

Definition 3.7 (Image)**.** If $f : A \to B$ is a function, then the image of f, $Im(f)$, is the set

$$\{b \in B \mid \exists a \in A((a, b) \in f)\}.$$

Notice that the codomain is not completely inherent to the function. The author may choose any codomain he likes, so long as it contains the *image* of the function. That is, we could correctly write "let $f : \mathbb{Z} \to \mathbb{Z}$ be defined so that $f(n) = n + 1$", "let $f : \mathbb{Z} \to \mathbb{R}$ be defined so that $f(n) = n + 1$", or "let $f : \mathbb{Z} \to \mathbb{C} \cup \{\text{puppies born in 1962}\}$ be defined so that $f(n) = n + 1$" because $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C} \cup \{\text{puppies born in 1962}\}$ all contain the image of f, namely $\mathbb{Z}$.

Definition 3.8 (Surjective)**.** A function $f : A \to B$ is surjective if and only if

$$\forall b \in B\ \exists a \in A(f(a) = b)$$

or, equivalently, $Im(f) = Cod(f)$.

Definition 3.9 (Injective)**.** A function $f : A \to B$ is injective if and only if

$$\forall a \in A\ \forall a' \in A(f(a) = f(a') \to a = a')$$

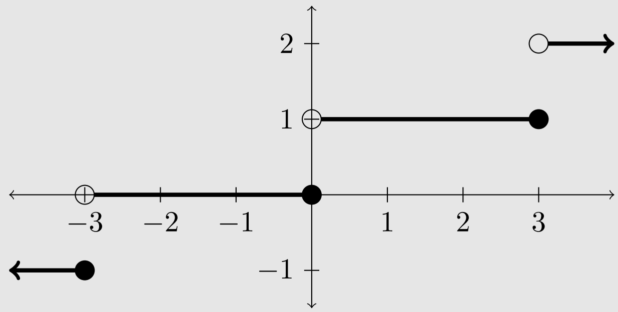
or, equivalently,

$$\forall a \in A\ \forall a' \in A(a \neq a' \to f(a) \neq f(a'))$$

Definition 3.10 (Bijective)**.** A function $f : A \to B$ is bijective if and only if it is both injective and surjective.

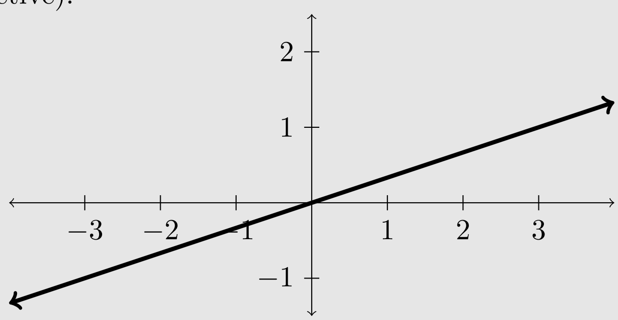
Notice that the function f in the diagram above is neither surjective nor injective.

Example 3.1. Claim: The function $g : \mathbb{R} \to \mathbb{R}$ defined such that $g(x) = \lceil \frac{x}{3} \rceil$ is neither injective nor surjective.



Proof. Consider $1, 2 \in \mathbb{R} = Dom(g)$. $g(1) = \lceil \frac{1}{3} \rceil = 1$ and $g(2) = \lceil \frac{2}{3} \rceil = 1$. So $g(1) = g(2)$, but $1 \neq 2$. Therefore g is not injective.
Consider $1.5 \in \mathbb{R} = Cod(g)$. $1.5 \notin Im(g)$ because, by the definition of the ceiling function, the image contains only integers. That is, there is no element, x, in the domain of g such that $g(x) = 1.5$. Therefore g is not surjective. $\square$

Example 3.2. Claim: The function $h : \mathbb{R} \to \mathbb{R}$ defined such that $h(x) = \frac{x}{3}$ is both injective and surjective (i.e., h is bijective).



Proof. Let $x, v \in \mathbb{R} = Dom(f)$ such that $h(x) = h(v)$. If $h(x) = h(v)$, then $\frac{x}{3} = \frac{v}{3}$ and $x = v$. Therefore h is injective.
Let $y \in \mathbb{R} = Cod(h)$. Then $3y \in \mathbb{R} = Dom(h)$ and $h(3y) = \frac{3y}{3} = y$. Therefore h is surjective. $\square$

3.3 Summary

Functions come as a natural extension to quantificational logic. We begin by looking at two-place predicates. Their extensions are sets of ordered pairs of elements of the Universe of Discourse. We define cross products and relations and recognize that all two-place predicates are relations from the UD to the UD. Only some relations are functions. We generalize from this point. We needn't only look at functions from the UD to itself, or from the all too familiar real line to itself.

We introduce the terms domain, codomain, image, surjective, and injective. Students explore various functions and prove claims about their surjectivity, injectivity, etc. Eventually, this chapter's problem set guides students to realize that functions are useful tools for comparing the sizes of sets. Students finish the problem set having shown that the tangent function is a bijection from a small interval of the real line to the entire real line and having realized that the existence of a bijection from one set to another demonstrates that the sets have the same number of elements. We drop the opportunity to study the cardinality of infinite sets and Georg Cantor's work for Project 2.

3.4 Problem set

For problems 1-6, make a claim about whether or not the relation is a function, and justify your claim by appealing to the definition of a function. For those relations that are valid functions from one set to another, be sure to denote sets the functions map to and from.

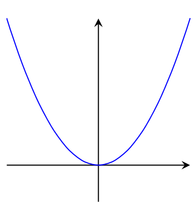
1. Lxy: x loves y
2. Mxy: x is the mother of y
3. Cxy: x is the child of y
4. Hxy: x is the height of y
5. Hxy: y is the height of x
6. Fxy: y is the first (legal) wife of x

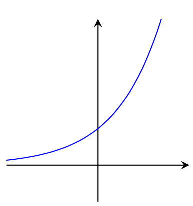
For problems 7 and 8, suppose there are functions f and g that simultaneously satisfy the following properties:

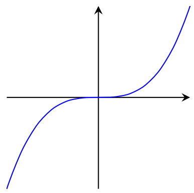
1. $\forall x f(x, 0) = x$.
2. $\forall x f(x, g(x)) = 0$, and
3. $\forall x \forall y f(x, y) = f(y, x)$.

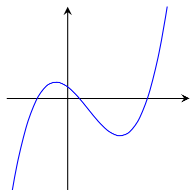
7. Use natural deduction to show that $\exists x(x = g(x))$ and that $\exists x(f(x, x) = x)$. (B 247)
8. Specify a universe of discourse and functions f and g that are consistent with the properties above.

For problems 9-12, make claims about the injectivity and surjectivity of each relation from $\mathbb{R}$ to $\mathbb{R}$. You don't need to prove your claim.

9. 

10. 

11. 

12. 

In problems 13-22, $f : A \to B$. Make claims about the injectivity and surjectivity of the function, and prove your claims.

13. $A = \{1, 2, 3, \dots\}$, $B = \{i, ii, iii, iv, \dots\}$, $f = \{(1, i), (2, ii), \dots\}$
14. $A = \{1, 2, 3, \dots\}$, $B = \{a, b, c, \dots, z\}$, $f = \{(1, a), (2, b), \dots, (27, a), (28, b), \dots\}$
15. $A = \mathbb{Z}$, $B = \mathbb{Z}$, $f(x) = 2x$
16. $A = \mathbb{Z}$, $B = \mathbb{Z}$, $f(x) = x + 1$
17. $A = \mathbb{R}$, $B = \mathbb{R}$, $f(x) = x^2$
18. $A = \mathbb{R}$, $B = \mathbb{R}$, $f(x) = x^3$
19. $A = \mathbb{R}$, $B = \mathbb{R}$, $f(x) = e^x$
20. $A = \mathbb{R}$, $B = \mathbb{R}$, $f(x) = \sin(x)$
21. $A = \mathbb{R}$, $B = [-1, 1]$, $f(x) = \sin(x)$
22. $A = \mathbb{R}$, $B = \mathbb{Z}$, $f(x) = \lfloor x \rfloor$

Let $|A|$ denote the number of elements in A. This is called the *cardinality* of A. For problems 23-25, say whether f is possibly surjective, injective, or bijective based on the cardinalities of the domain and codomain. Justify your claim.

23. $|A| = 10$, $|B| = 5$, $f : A \to B$
24. $|A| = 5$, $|B| = 10$, $f : A \to B$
25. $|A| = 10$, $|B| = 10$, $f : A \to B$

26. Name a function from $\mathbb{Z}$ to $2\mathbb{Z}$ that is both injective and surjective.
27. Name a function from $(\frac{-\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$ that is both injective and surjective.
28. Name a function from $\mathbb{R}$ to $(\frac{-\pi}{2}, \frac{\pi}{2})$ that is both injective and surjective.
29. $f : A \to B$ is both injective and surjective. What can you deduce about the number of elements in A and the number of elements in B.
30. Disprove with a counterexample: If $f : \mathbb{R} \to \mathbb{R}$ is injective, then f^2 is injective. (R 25)
31. Disprove with a counterexample: If $f : \mathbb{R} \to \mathbb{R}$ is injective and bounded, then f^{-1} is injective and bounded. (R25)

32. Given sets A, B, and C and functions $f : A \to B$ and $g : B \to C$, prove that if $g(f(x))$ is injective, then $f(x)$ is injective.

33. Given sets A, B, and C and functions $f : A \to B$ and $g : B \to C$, prove that if $g(f(x))$ is surjective, then $g(x)$ is surjective.

Chapter 4

Induction

4.1 The blue eyes riddle

A group of peculiar people live on a peculiar island. They are all perfect logicians; if a conclusion can be logically deduced, they will do it instantly. Each person on the island has eyes of a single color, but no one knows the color of his own eyes. Every night at midnight, a ferry stops at the island. Any islanders who have figured out the color of their own eyes then leave the island, and the rest stay. Everyone can see everyone else at all times and keeps a count of the number of people they see with each eye color (excluding themselves), but they cannot otherwise communicate. Everyone on the island knows and abides by all the rules in this paragraph.

On this island there are 100 blue-eyed people, 100 brown-eyed people, and a Guru (who happens to have green eyes). So any given blue-eyed person can see 100 people with brown eyes and 99 people with blue eyes (and one with green), but that does not tell him his own eye color; as far as he knows the totals could be 101 brown and 99 blue. Or 100 brown, 99 blue, and he could have red eyes.

The Guru is allowed to speak once (let's say at noon), on one day in all their endless years on the island. Standing before the islanders, she says, "I can see someone who has blue eyes."

Who leaves the island, and on what night?

There are no mirrors or reflecting surfaces—nothing dumb. This is an exercise in serious logic, not a lateral thinking riddle. It doesn't depend on tricky wording or anyone lying or guessing, and it doesn't involve people doing something silly like creating a sign language or doing genetics. The Guru is not making eye contact with anyone in particular; she's simply saying "I count at least one blue-eyed person on this island who isn't me."

And, lastly, the answer is not "no one leaves."

4.2 The induction axiom

Axiom 4.1. $\forall P[(P_1 \wedge \forall k \in \mathbb{N}(P_k \to P_{k+1})) \to \forall n \in \mathbb{N}\ P_n]$ is a tautology.

Notice that this is true for all properties P. That is, the induction axiom is written in 2nd order logic.

There are some names we use for specific parts of the axiom that also outline the format of an induction proof. The goal is to prove that some property P is true for all natural numbers, i.e., $\forall n \in \mathbb{N}\ P_n$.

1. P_1 is called the *base case*.
2. Assuming P_k is true (for some arbitrary natural number, k) is called the *induction hypothesis*.
3. We make this assumption in order to make the *inductive step*; i.e., that $\forall k \in \mathbb{N}(P_k \rightarrow P_{k+1})$ is true.

Example 4.1. Claim: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Proof. Let P_x denote that $1 + 2 + 3 + \cdots + x = \frac{x(x+1)}{2}$.

1. Base case: P_1 is true since $1 = \frac{1(1+1)}{2}$.
2. Induction hypothesis: Assume that P_k is true for some $k \in \mathbb{N}$. That is, for some arbitrary natural number, k, $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$.
3. Inductive step: Observe that

$$\begin{aligned}
1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \quad \text{by the induction hypothesis} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}
\end{aligned}$$

So $\forall k \in \mathbb{N}(P_k \rightarrow P_{k+1})$.

Hence, by the induction axiom, $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$. That is, $\forall n \in \mathbb{N}\ P_n$. $\square$

4.3 Inductive reasoning versus proofs by induction

Inductive reasoning is the process of forming hypotheses based on specific, empirical observations. If, in my lifetime, every swan I ever saw was white, I might (by inductive reasoning) form the hypothesis that all swans are white. That hypothesis, though there is some empirical evidence to support it, has not been *proven* by my observations.

Inductive proofs also begin by making a particular observation, but they are *not* a form of inductive reasoning. Inductive proofs are *proofs*, and they are actually employing deductive reasoning. From a particular observation and an understanding of how each case relates to its successor, we *deduce* that a conclusion is true. The induction axiom says that the argument

$$\begin{array}{l} P_1 \\ \forall k \in \mathbb{N}(P_k \to P_{k+1}) \\ \hline \forall n \in \mathbb{N}\ P_n \end{array}$$

is valid, and an inductive proof demonstrates, given we can show P_1 and $\forall k \in \mathbb{N}(P_k \to P_{k+1})$ are true, that we may deduce $\forall n \in \mathbb{N}\ P_n$.

4.4 Summary

We begin studying induction as a proof theory with a problem-based learning activity. Students are given the blue eyes riddle, which is scaffolded by another card-guessing game that motivates the use of induction. We abstract our reasoning, thereby outlining the structure of a proof by induction. The instructor models a typical number theoretical proof by induction, and the proof theory is formalized in the induction axiom. Here, we recognize that we once again need to quantify over all predicates, and 2nd order logic is required to express the axiom fully.

A theme throughout the course has been to remain aware, reflective, and critical of the discipline. What are the flaws, limitations, and inaccuracies in our symbolic logics? A final Project 2 opportunity is to simultaneously investigate cognitive studies on when and why we (humans) don't reason logically (that is, a very restricted interpretation of "logically", which refers to being in accordance with the logical languages of 0th, 1st, or 2nd order logic) and philosophical arguments for why we ought not reason logically.

4.5 Problem set

1. Suppose we have a group of n students who are all logically omniscient (they immediately infer all logical consequences of everything they learn). Every student gets a card, red or black, and can see all others but not his own. Suppose n students get black cards, and the rest get red. Charlie (not a student) announces, truly, "I see a black card". Then he asks everyone "Can you infer the color of your card?", and everyone announces their answer at once, so that everyone else hears the answer. If everyone says "No", then the question-answer process repeats. Prove that, for n students holding black cards, it takes exactly n repetitions of the question/answer process to get a "Yes" from the black card-holders.

2. Prove that if n is a positive integer, then $n^3 + 5n$ is divisible by 6.

3. Prove that postage of 6 cents or more can be achieved (exactly) using 2- and 7-cent stamps.

4. Prove that for $n \geq 2$ real numbers, $x_1, x_2, \ldots, x_n$,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

5. Let $x > -1$ and $n \in \mathbb{N}$. Prove Bernoulli's inequality:

$$(1 + x)^n \geq 1 + nx.$$

6. Fact: $e = \sum_{i=0}^{\infty} \frac{1}{i!}$.

   Let's define the partial sum, $s(k) \equiv \sum_{i=0}^{k} \frac{1}{i!}$. We're going to prove that e is irrational by assuming that it is rational ($e = \frac{m}{n}$, where m and n are integers) and showing that leads to a contradiction.

   (a) Is $s(k)$ a function? What is its domain and codomain?

   (b) Prove that for all $k \geq 0$

   $$e - s(k) < \frac{1}{(k+1)!} \cdot \left(1 + \frac{1}{k+1} + \frac{1}{(k+1)^2} + \dots\right)$$

   (c) Prove that $e - s(k) < \frac{1}{k \cdot k!}$ for all $k \in \mathbb{N}$.

   (d) If $e = \frac{m}{n}$, prove that $n! \cdot e$ and $n! \cdot s(n)$ are integers.

   (e) If $e = \frac{m}{n}$, prove that $n! \cdot (e - s(n))$ is an integer between 0 and 1, which is absurd.