Munkres 1.2

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1. Let $f : A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$.
   a. Show that $A_0 \subset f^{-1}(f(A_0))$ and that equality holds if f is injective.

   Let there be $x \in A_0$ and $y = f(x)$ so $y \in f(A_0)$, $f(x) = y \in f(A_0)$. So, $x \in f^{-1}(f(A_0))$. Thus, $A_0 \subset f^{-1}(f(A_0))$. If f is injective, $y = f(x) \in f(A_0)$. Then $x' \in A_0$ where $f(x') = y = f(x)$. Since f is injective, $x = x' \in A_0$. Therefore, $f^{-1}(f(A_0)) \subset A_0$ follows.

   b. Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

   Given y is an element in $f(f^{-1}(B_0))$ and $x \in f^{-1}(B_0)$ where $f(x) = y$. If $x \in f^{-1}(B_0)$, $y = f(x) \in B_0$. Thus, $f(f^{-1}(B_0)) \subset B_0$. If f is surjective and $y \in B_0$, $y \in B$ as $B_0 \subset B$. Since f is surjective, $x \in A$ where $f(x) = y$. Then $x \in f^{-1}(B_0)$ since $f(x) = y \in B_0$. Therefore, $y = f(x) \in f(f^{-1}(B_0))$ so $B_0 \subset f(f^{-1}(B_0))$ and the equality holds.

2. Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets:

   a. $B_0 \subset B_1 \to f^{-1}(B_0) \subset f^{-1}(B_1)$
   Given $B_0 \subset B_1$, let there be $x \in f^{-1}(B_0)$. There is $f(x) \in B_0$ so $f(x) \in B_1$ as $B_0 \subset B_1$. Therefore, $x \in f^{-1}(B_1)$ and $f^{-1}(B_0) \subset f^{-1}(B_1)$.

   b. $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$
   $x \in f^{-1}(B_0 \cup B_1) \iff f(x) \in B_0 \cup B_1$
   $\iff f(x) \in B_0 \vee f(x) \in B_1$
   $\iff x \in f^{-1}(B_0) \vee x \in f^{-1}(B_1)$
   $\iff x \in f^{-1}(B_0) \cup f^{-1}(B_1)$

   c. $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$
   $x \in f^{-1}(B_0 \cap B_1) \iff f(x) \in B_0 \cap B_1$
   $\iff f(x) \in B_0 \wedge f(x) \in B_1$
   $\iff x \in f^{-1}(B_0) \wedge x \in f^{-1}(B_1)$
   $\iff x \in f^{-1}(B_0) \cap x \in f^{-1}(B_1)$

d. $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$
$x \in f^{-1}(B_0 - B_1) \iff f(x) \in B_0 - B_1$
$\iff f(x) \in B_0 \wedge f(x) \notin B_1$
$\iff x \in f^{-1}(B_0) \wedge x \notin f^{-1}(B_1)$
$\iff x \in f^{-1}(B_0) - f^{-1}(B_1)$

e. $A_0 \subset A_1 \to f(A_0) \subset f(A_1)$
Given $A_0 \subset A_1$, let there be $y \in f(A_0)$. There is an $x \in A_0$ where $y = f(x)$. Since $A_0 \subset A_1$, $x \in A_1$ and thus $y = f(x) \in f(A_1)$. Therefore, $f(A_0) \subset f(A_1)$.

f. $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$
$y \in f(A_0 \cup A_1) \iff (x \in A_0 \cup A_1) \wedge (y = f(x))$
$\iff (x \in A_0 \vee x \in A_1) \wedge (y = f(x))$
$\iff (x \in A_0 \wedge y = f(x)) \vee (x \in A_1 \wedge y = f(x))$
$\iff y \in f(A_0) \vee y \in f(A_1)$
$\iff y \in f(A_0) \cup f(A_1)$

g. $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that equality holds if f is injective.
Let there be $y \in f(A_0 \cap A_1)$, there is an $x \in A_0 \cap A_1$ where $y = f(x)$ and $x \in A_0$ and $x \in A_1$. Since $y = f(x)$, $y \in f(A_0)$ and $y \in f(A_1)$ and therefore $y \in f(A_0) \cap f(A_1)$. Given that f is injective and $y \in f(A_0) \cap f(A_1)$, $y \in f(A_0)$ and $y \in f(A_1)$. Then, $f(x_0) = y = f(x_1)$ so $x_0 = x_1$ since f is injective. Thus $x_0 \in A_0$ and $x_0 = x_1 \in A_1$ so $x_0 \in A_0 \cap A_1$. Since $y = f(x_0)$, $y \in f(A_0 \cap A_1)$. Therefore $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$.

h. $f(A_0 - A_1) \supset f(A_0) - f(A_1)$; show that equality holds if f is injective.
Let there be $y \in f(A_0) - f(A_1)$ so $y \in f(A_0)$ and $y \notin f(A_1)$. Then, let there be $x \in A_0$ where $y = f(x)$. There is also no $x' \in A_1$ given $y = f(x')$. Since $y = f(x)$, $x \notin A_1$. Thus, $x \in A_0 - A_1$ so $y \in f(A_0 - A_1)$ since $y = f(x)$. Therefore, $f(A_0 - A_1) \supset f(A_0) - f(A_1)$.

Given that f is injective and $y \in f(A_0) - f(A_1)$, $x \in A_0 - A_1$ where $y = f(x)$. Also, $x \in A_0$ but $x \notin A_1$. Then, $y \in f(A_0)$ since $y = f(x)$ and $x \in A_0$. Given $x' \in A_1$, $y = f(x')$ cannot happen because if it happened, $f(x) = y = f(x')$ so $x = x'$ since f is injective. This would be a contradiction since $x' = x \notin A_1$. Therefore, $y \notin f(A_1)$ and $y \in f(A_0) - f(A_1)$ so $f(A_0 - A_1) \subset f(A_0) - f(A_1)$.

3. Show that b, c, f, and g of Exercise 2 hold for arbitrary unions and intersections.

   b. Arbitrary unions for b. Prove that $f^{-1}(\bigcup_{B' \in B} B') = \bigcup_{B' \in B} f^{-1}(B')$

$x \in f^{-1}(\bigcup_{B' \in B} B') \iff f(x) \in \bigcup_{B' \in B} B'$
$\iff B' \in B(f(x) \in B')$
$\iff B' \in B(x \in f^{-1}(B'))$
$\iff x \in \bigcup_{B' \in B} f^{-1}(B')$

c. Arbitrary intersections for c. Prove that $f^{-1}(\bigcap_{B' \in B} B') = \bigcap_{B' \in B} f^{-1}(B')$
$x \in f^{-1}(\bigcap_{B' \in B} B') \iff f(x) \in \bigcap_{B' \in B} B'$
$\iff B' \in B(f(x) \in B')$
$\iff B' \in B(x \in f^{-1}(B'))$
$\iff x \in \bigcap_{B' \in B} f^{-1}(B')$

f. Arbitrary unions for f. Prove that $f(\bigcup_{A' \in A} A') = \bigcup_{A' \in A} f(A')$
$y \in f(\bigcup_{A' \in A} A') \iff x \in (\bigcup_{A' \in A} A') \wedge (y = f(x))$
$\iff A' \in A(x \in A') \wedge (f(x))$
$\iff A' \in A(y \in f(A'))$
$y \in \bigcup_{A' \in A} f(A')$

g. Arbitrary intersections for g. Prove that $f(\bigcap_{A' \in A} A') \subset \bigcap_{A' \in A} f(A')$ while the equality holds if f is injective.
Let there be $y \in f(\bigcap_{A' \in A} A')$ so $x \in \bigcap_{A' \in A} A'$ where $y = f(x)$. Then $x \in A'$ for every $A' \in A$. For $A' \in A$, $x \in A'$ and $y = f(x)$ so $y \in f(A')$. Therefore, $y \in \bigcap_{A' \in A} f(A')$.

Given that f is injective and $y \in \bigcap_{A' \in A} f(A')$. Then $y \in f(A')$ for every $A' \in A$. For $A_0 \in A$, $y \in f(A_0)$ so there is a $x_0 \in A_0$ where $y = f(x_0)$. Given that $x_0 \notin \bigcap_{A' \in A} A'$ so $A_1 \in A$ where $x_0 \notin A_1$. Since $A_1 \in A$, $y \in f(A_1)$, and $x_1 \in A_1$ where $y = f(x_1)$. Then, $f(x_0) = y = f(x_1)$ so $x_0 = x_1$ since f is injective and $x_0 \notin A_1$ and $x_0 = x_1 \in A_1$. This is a contradiction so $x_0 \in \bigcap_{A' \in A} A'$. Since $y = f(x_0)$, $y \in f(\bigcap_{A' \in A} A')$. Therefore, $f(\bigcap_{A' \in A} A') \supset \bigcap_{A' \in A} f(A')$.

4. Let $f : A \to B$ and $g : B \to C$.

a. If $C_0 \subset C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$.
$x \in (g \circ f)^{-1}(C_0) \iff (g \circ f)(x) \in C_0$
$\iff g(f(x)) \in C_0$
$\iff f(x) \in g^{-1}(C_0)$
$\iff x \in f^{-1}(g^{-1}(C_0))$

b. If f and g are injective, show that $g \circ f$ is injective.
Let there be $x, y \in A$ and $x \neq y$. If f is injective, $f(x) \neq f(y)$. So, $(g \circ f)(x) = g(f(x)) \neq g(f(y)) = (g \circ f)(y)$ as $f(x) \neq f(y)$ and g is injective. Therefore, $g \circ f$ is injective.

c. If $g \circ f$ is injective, what can you say about injectivity of f and g?
If $g \circ f$ is injective, f is injective but g isn't.

Given that $g \circ f$ is injective but that f isn't. Let there be $x, y \in A$ where $x \neq y$ but $f(x) = f(y)$. This would mean that $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$. This contradicts $g \circ f$ being injective since $x \neq y$ so f must be injective.

Proof that g isn't injective:
Let there be sets $A = \{0, 1\}, B = \{0, 1, 2\}, C = \{a, b\}$ and $f = \{(0, 0), (1, 1)\}, g = \{(0, a), (1, b), (2, b)\}$. From the given sets, we can see that $f : A \to B$ is injective as well as $g \circ f = \{(0, a), (1, b)\}$. However, $g : B \to C$ isn't as $g(1) = b = g(2)$

d. If f and g are surjective, show that $g \circ f$ is surjective.
Given that f and g are surjective and let there be $z \in C$. Then $y \in B$ where $z = g(y)$ since g is surjective. Since f is surjective, $x \in A$ where $y = f(x)$. Therefore, $(g \circ f)(x) = g(f(x)) = g(y) = z$.

e. If $g \circ f$ is surjective, what can you say about surjectivity of f and g?
If $g \circ f$ is surjective, g is surjective but f isn't.

Given that $g \circ f$ is surjective and let there be $z \in C$, $x \in A$ where $(g \circ f)(x) = z$. Then, $g(f(x)) = z$ so $y = f(x)$ is part of set B where $g(y) = z$. Therefore, g is surjective.

Proof that f isn't surjective:
Using the sets from part c, we can see that $g \circ f$ and g are surjective but f isn't since no element in A corresponds to $2 \in B$

f. Summarize your answers to b-e in the form of a theorem.
1. If f and g are injective then $g \circ f$ is injective.
2. If $g \circ f$ is injective then f is injective
3. If f and g are surjective then $g \circ f$ is surjective.
4. If $g \circ f$ is surjective then g is surjective.

5. In general, let us denote the identity function for a set C by i_C. That is, define $i_C : C \to C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f : A \to B$, we say that a function $g : B \to A$ is a left inverse for f if $g \circ f = i_A$; and we say that $h : B \to A$ is a right inverse for f if $f \circ h = i_B$.

   a. Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective.

   Given that f has a left inverse $g : B \to A$ so $g \circ f = i_A$ and let there be $x, y \in A$ where $f(x) = f(y)$. Then, $x = i_A(x) = (g \circ f)(x) = g(f(x)) =$

$g(f(y)) = (g \circ f)(y) = i_A(y) = y$ and f is injective.

Given that f has a right inverse $h : B \to A$ so $f \circ h = i_B$ and $y \in B$. Then, $y = i_B(y) = (f \circ h)(y) = f(h(y))$. So, $x = h(y)$ is an element of A where $f(x) = y$ and therefore, f is surjective.

b. Give an example of a function that has a left inverse but no right inverse.

Given the sets $A = \{0, 1\}$, $B = \{a, b, c\}$, and $f = \{(0, a), (1, b)\}$. Then, from the function $g : B \to A$, $g = \{(a, 0), (b, 1), (c, 1)\}$. Therefore, this is a left inverse of f since $(g \circ f)(0) = g(f(0)) = g(a) = 0$ and $(g \circ f)(2) = g(f(2)) = g(b) = 2$ so $g \circ f = i_A$ so $g \circ f = i_A$.

Also, f isn't surjective since no element of A that maps to $c \in B$ so f can't have a right inverse.

c. Give an example of a function that has a right inverse but no left inverse.

Given the sets $A = \{0, 1, 2\}$, $B = \{a, b\}$, and $f = \{(0, a), (1, b), (2, a)$. Then, from the function $h : B \to A$, $h = \{(a, 0), (b, 1)\}$. $(f \circ h)(a) = f(h(a)) = f(0) = a$ and $(f \circ h)(b) = f(h(b)) = f(1) = b$. Therefore, $f \circ h = i_B$ and h is a right inverse of f. Also, f isn't injective since $f(1) = a = f(2)$ so f can't have a left inverse.

d. Can a function have more than one left inverse? More than one right inverse?

A function can have more than one left inverse or right inverse.
Using the example given in part b, use sets $A = \{0, 1\}$, $B = \{a, b, c\}$, and $f = \{(0, a), (1, b)\}$. In this example, $g_1 = \{(a, 0), (b, 1), (c, 1)\}$ was proved to be a left inverse. Given that $g_2 = \{(a, 0), (b, 1), (c, 0)$ so $g_1 \neq g_2$. g_2 is also a left inverse of f so therefore, there can be more than one left inverse for f.

Using the example given in part c, $A = \{0, 1, 2\}$, $B = \{a, b\}$, and $f = \{(0, a), (1, b), (2, a)\}$. In this example, $h_1 = \{(a, 0), (b, 1)\}$ was proved to be a right inverse. Given that $h_2 = \{(a, 2), (b, 1)\}$ so $h_1 \neq h_2$. h_2 is also a right inverse of f so therefore, there can be more than one right inverse for f.

e. Show that if f has both a left inverse g and a right inverse h, f is bijective and $g = h = f^{-1}$.

Consider that f has left inverse g and right inverse h. f must be both injective and surjective to prove that it is bijective. Then, $f^{-1} : B \to A$. Let there be $y \in B$ and $x = f^{-1}(y)$, $y = f(x)$. Then, $g(y) = g(f(x)) = (g \circ f)(x) = i_A(x) = x$ since g is a left inverse of f. Also, $f(h(y)) = (f \circ h)(y) = i_B(y) = y$ so $h(y) = f^{-1}(f(h(y))) = f^{-1}(y) = x$. Therefore, $x = f^{-1}(y) = g(y) = h(y)$ which is $f^{-1} = g = h$.

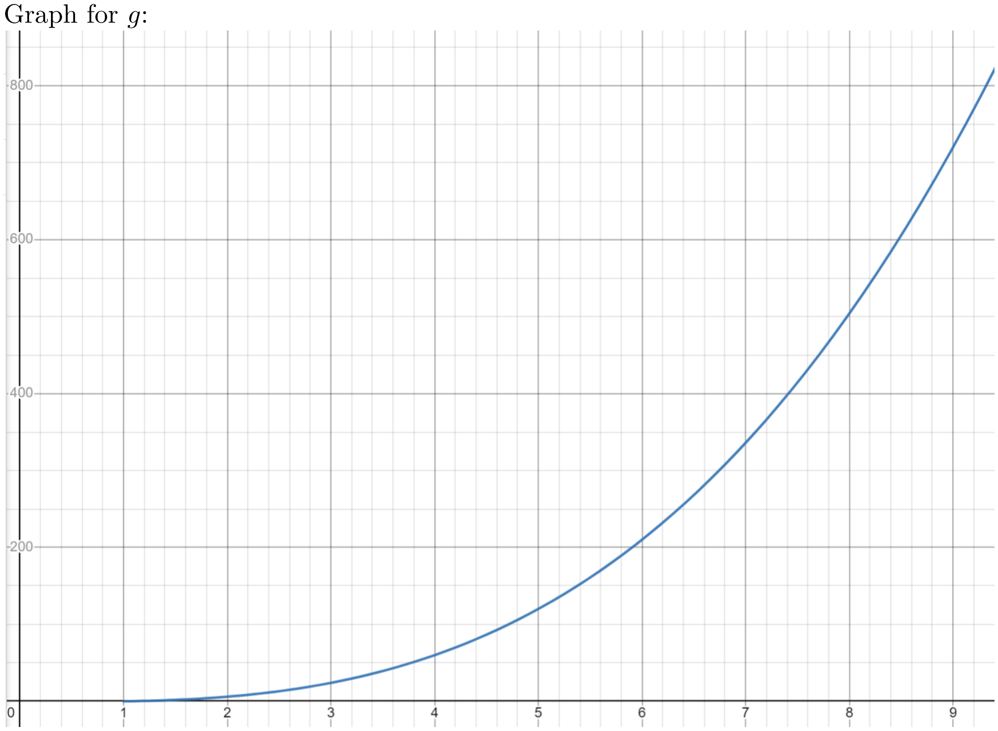
6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^3 - x$. By restricting the domain and range of f appropriately, obtain from f a bijective function g. Draw the graphs of g and g^{-1}.

   Using the subsets $A = [1, \infty)$ and $B = [0, \infty)$. The function $g : A \to B$ when $g(x) = f(x) = x^3 - x$ for $x \in A$ is bijective.

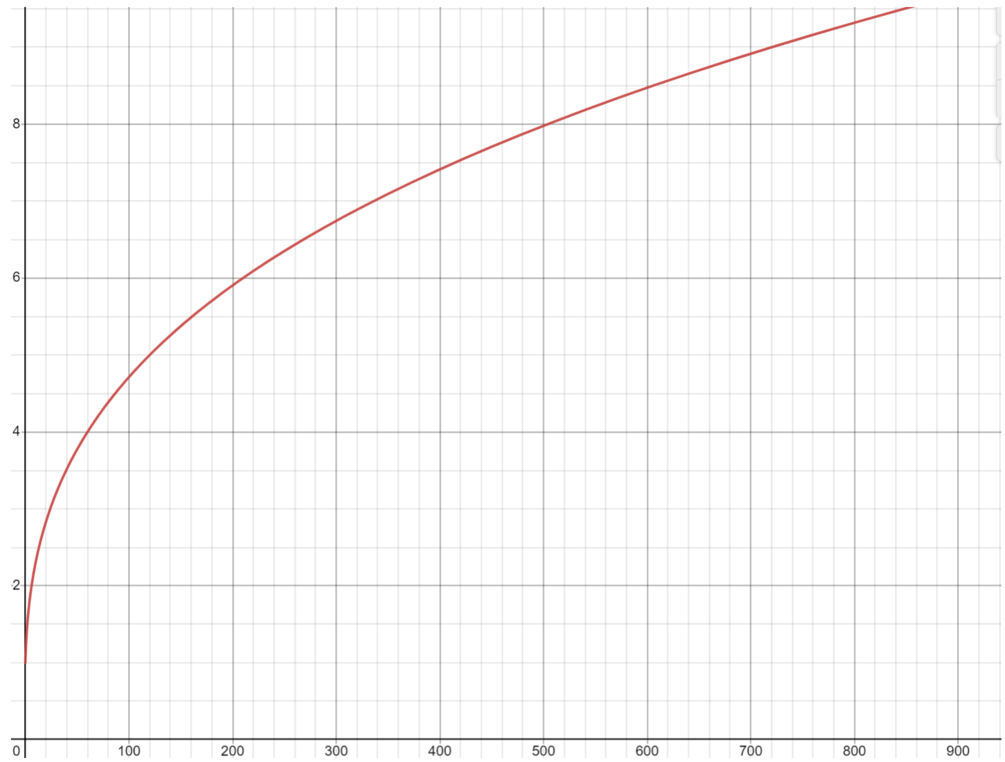
   First, I'll prove that B can be a range for g so $g(x) \in B$ for $x \in A$. Given $x \in A$, $x \geq 1$ and $x^2 \geq 1$ also. Then, $x^2 - 1 \geq 0$ so $x(x^2 - 1) \geq 0$ since $x \geq 1 > 0$. Therefore, $g(x) = f(x) = x^3 - x = x(x^2 - 1) \geq 0$ so $g(x) \in B$.

   For graph g, the domain will be $[1, \infty)$ and range will be $[0, \infty)$. For graph g^{-1}, the domain will be $[0, \infty)$ and range will be $[1, \infty)$.

   Graph for g:



   Graph for g^{-1}: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