Online Learning and Online Algorithms: Chasing Convex Bodies

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26/8/2024

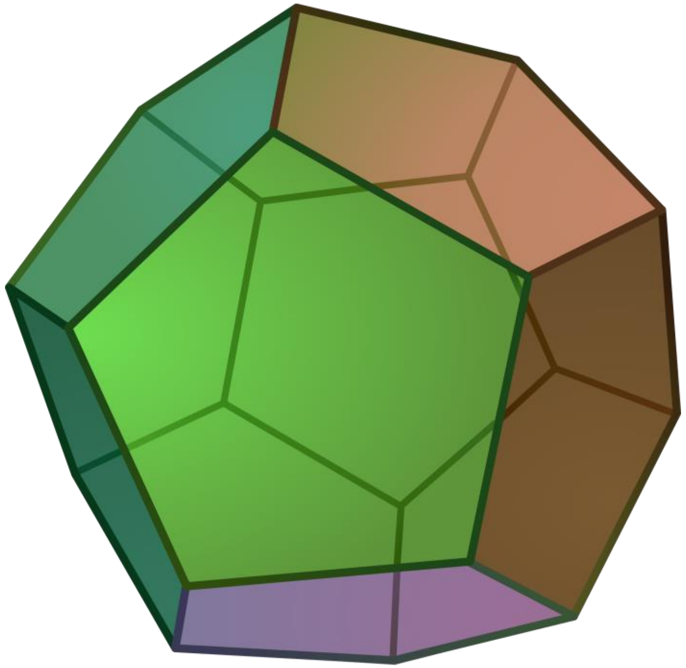
Contents

1. Online Learning and Online Algorithms

- Online learning refers to a setting where an algorithm learns and makes decisions sequentially.
- The algorithm receives data points one by one and must make decisions or predictions based on the current and past observations, without knowing future data.
- Application: Used in environments where data is received incrementally, such as financial markets, adaptive routing in networks, and dynamic pricing.

2. The Chasing Convex Bodies Problem

Convex Body: A compact (closed and bounded) convex set with non-empty interior



2. The Chasing Convex Bodies Problem

For each timestep $t \in \mathbb{N}$, a convex body $K_t \subseteq \mathbb{R}^d$ is given

The player picks a point $x_t \in K_t$

Total Movement Cost: $ALG = \sum_{t=0}^{T-1} ||x_{t+1} - x_t||$

Our goal is to minimize the competitive ratio: $cr(ALG) = \max_{\sigma}\{\frac{ALG(\sigma)}{OPT(\sigma)}\}$

σ an instance of the problem and $OPT(\sigma)$ is the optimal offline cost

Lower bound for competitive ratio: $\Omega(\sqrt{d})$

2. The Chasing Convex Bodies Problem

- Application: Load Balancing in Cloud Computing
- Dynamically allocate computational resources while minimizing energy consumption and latency.
- Varying demand patterns, adapting server allocations in real-time, maintaining system performance within constraints
- Convex Bodies($\boldsymbol{K_t}$) represents feasible regions of server allocations constrained by current system demands.
- At each time step, select an allocation (x_t) that lies within the current convex body and optimizes the competitive ratio

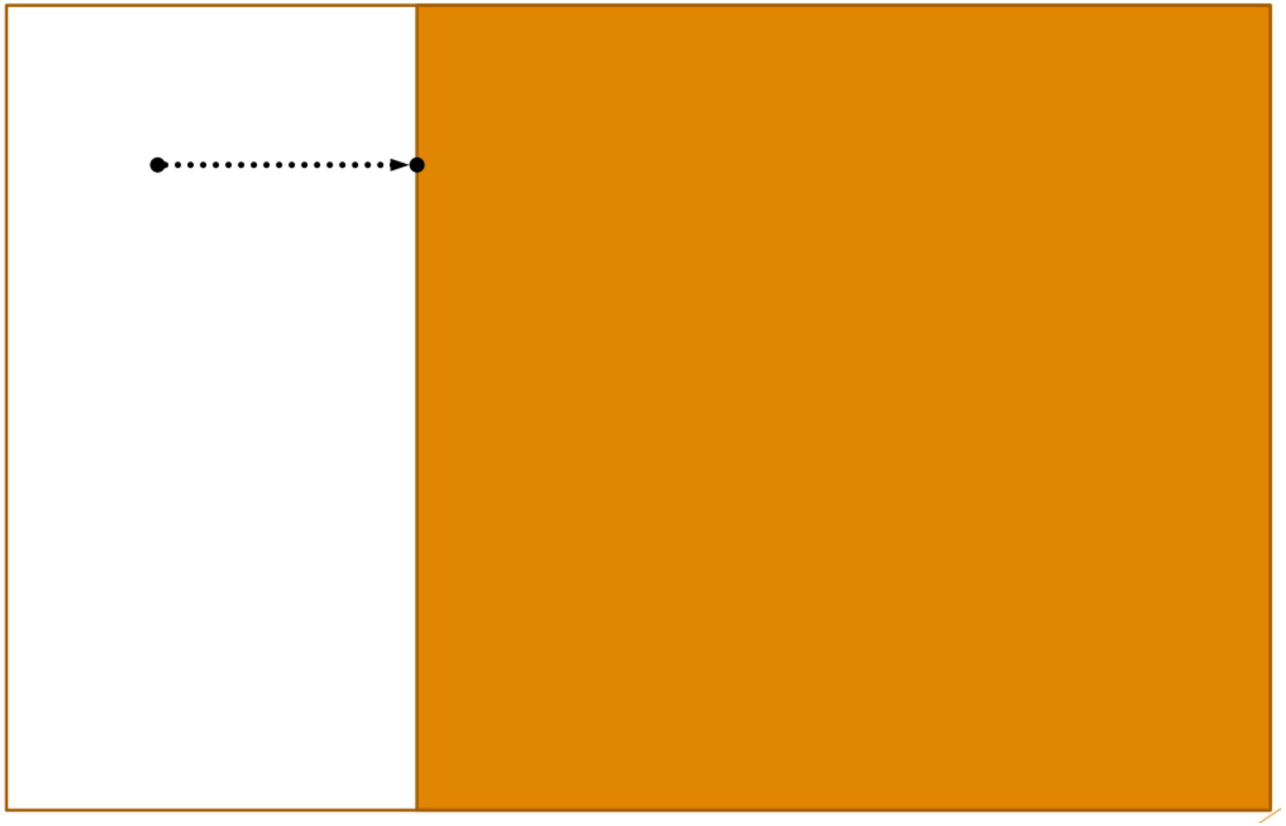
2. The Chasing Convex Bodies Problem

We will focus on the nested version of the problem.

$$K_1 \supseteq K_2 \supseteq \cdots \supseteq K_T$$

In the nested version, each convex body contains the next one, forming a hierarchy of containment.

2. The Chasing Convex Bodies Problem



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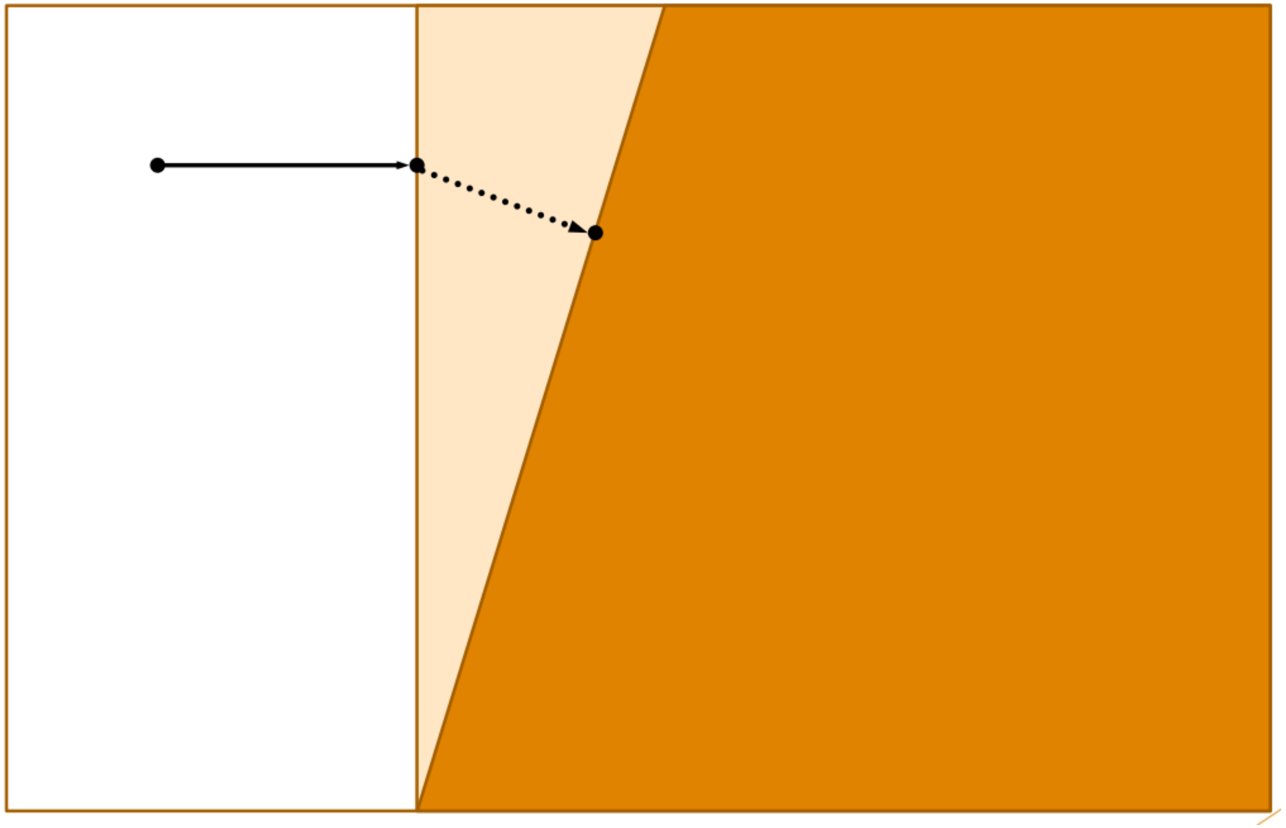

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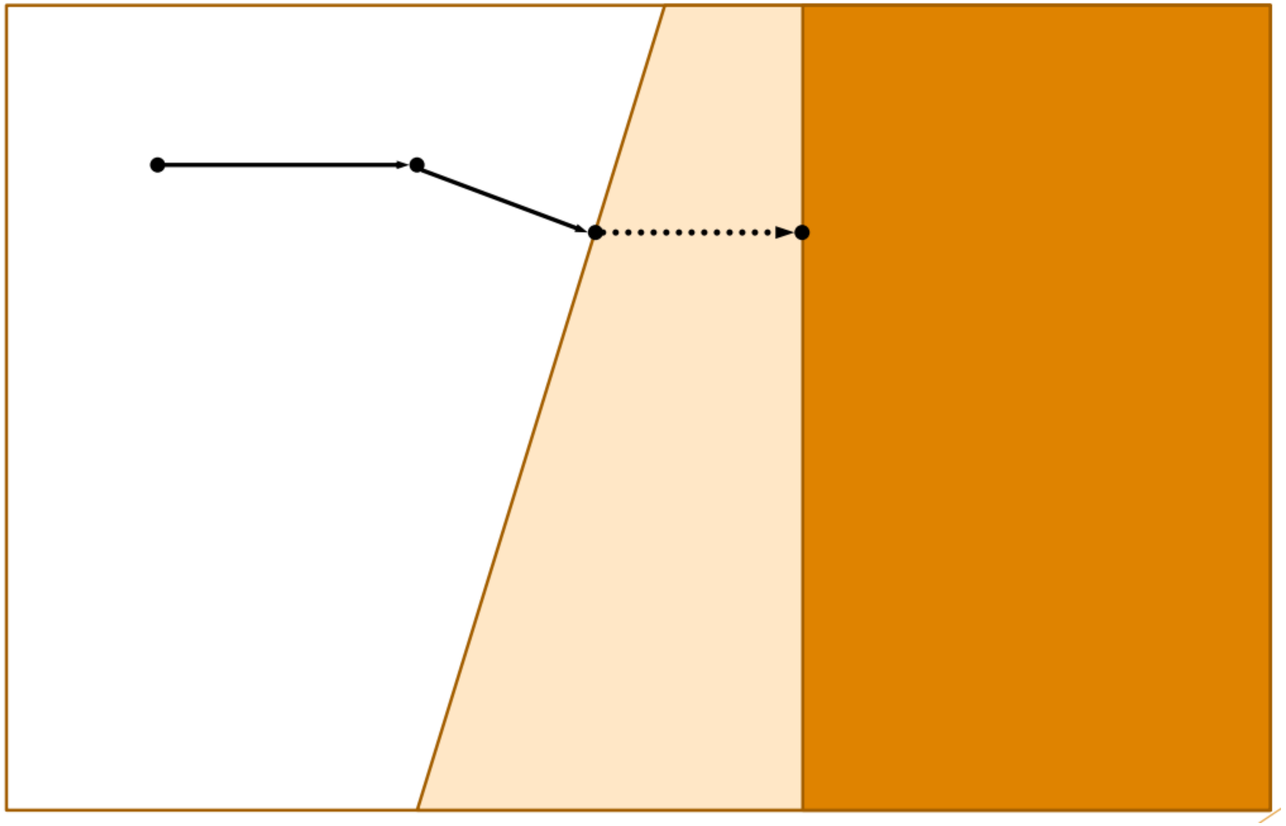

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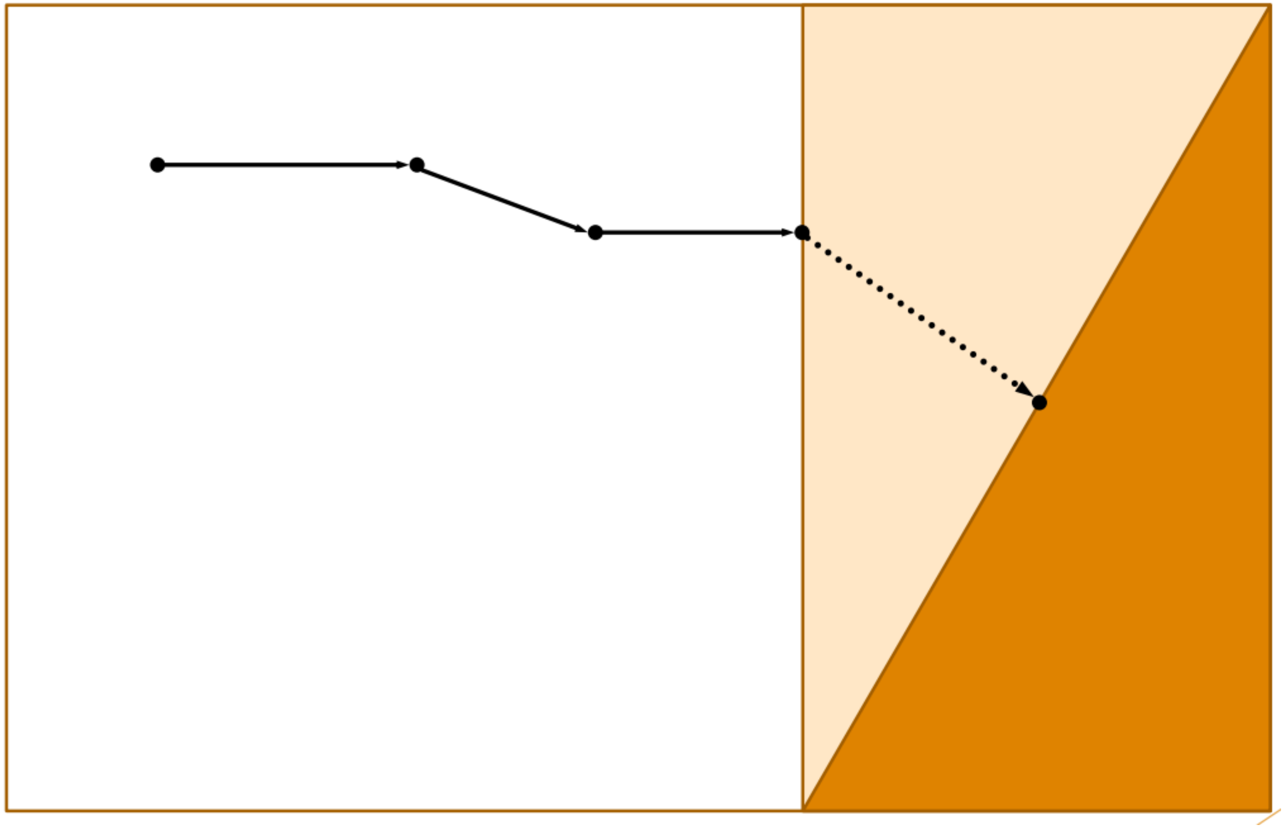

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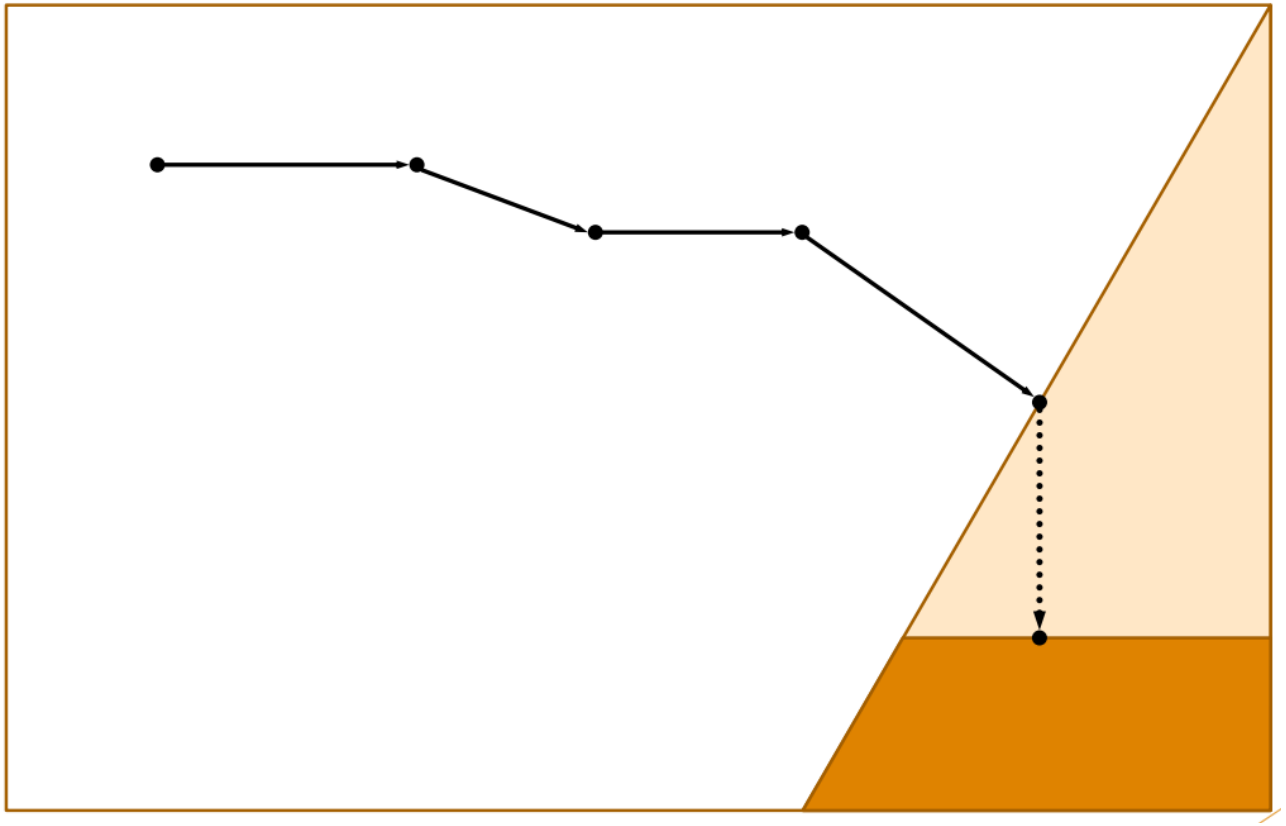

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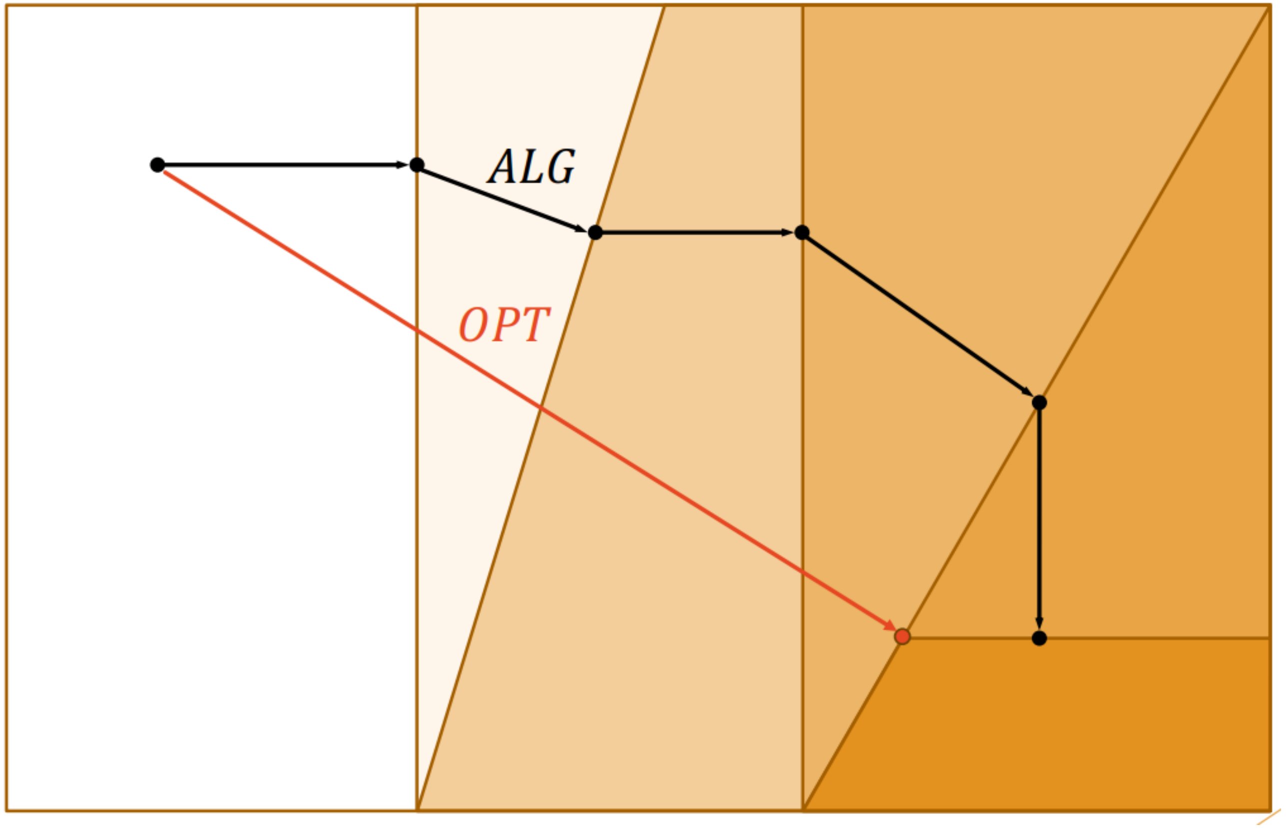


3. Naive Centroid Approach

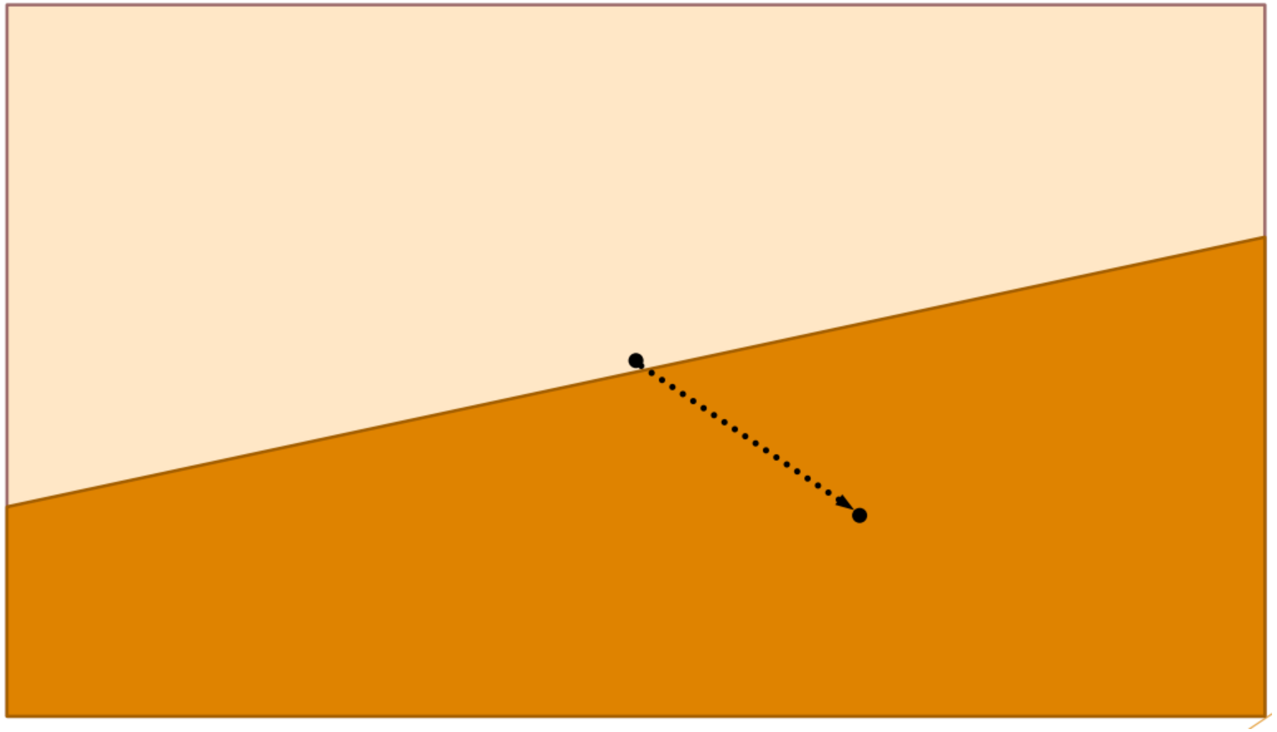
K_t a convex body

Centroid(K_t):=$\frac{1}{Vol(K_T)}\int_{K_T} x\, dx$

Algorithm: At each timestep t move to the centroid(K_t)

This approach is intuitive and ensures a move to a central point within the convex body, but it provides an unbounded competitive ratio,

3. Naive Centroid Approach



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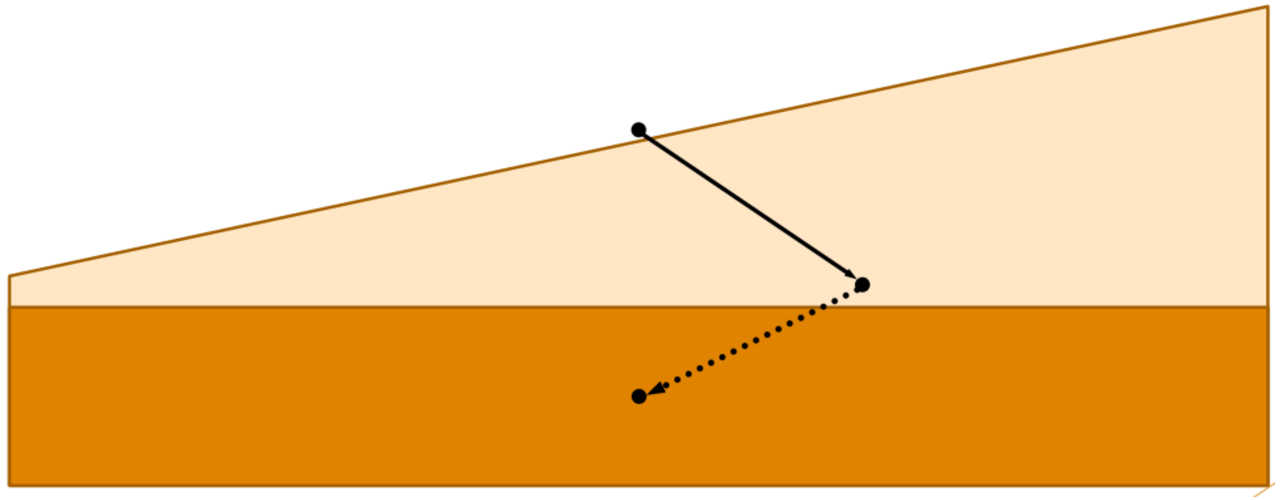

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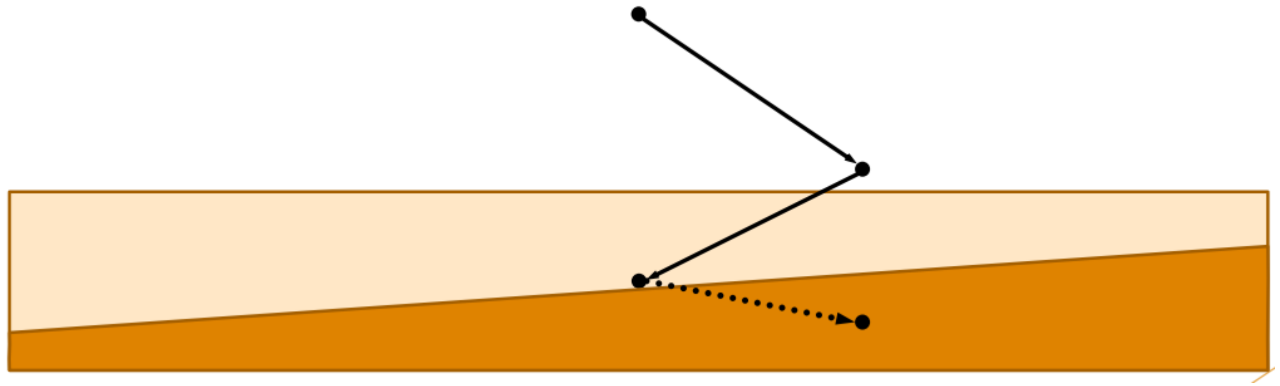
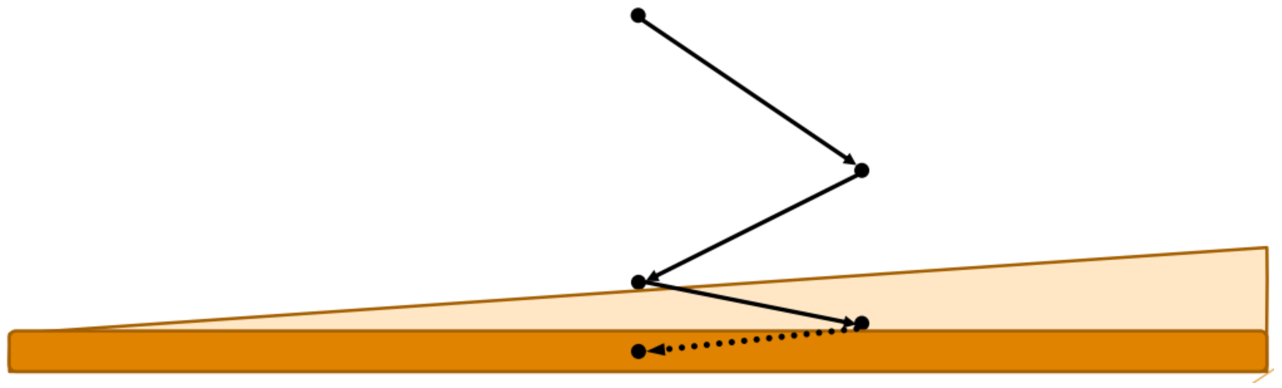


3. Naive Centroid Approach

Not competitive
☹

Diameter constant



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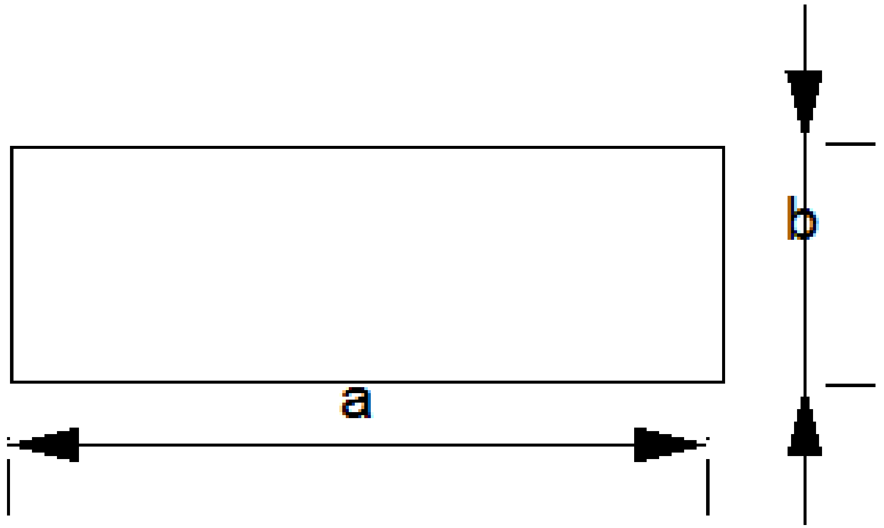
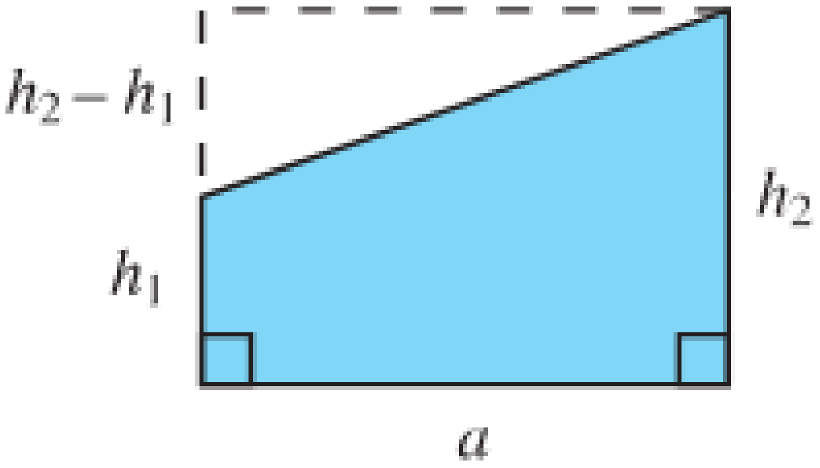




$$x_{centroid} = \frac{a}{2}$$

$$x_{centroid} = \frac{a(h_1+2h_2)}{3(h_1+h_2)}$$

3. Naive Centroid Approach

- Lines: $x = -1,\ x = -1,\ y = 0.$
- For $t = 1, 3, 5 \dots :\ y = (\frac{1}{2})^t$, rectangle, x-coordinate of centroid is $\mathbf{0}$
- For $t = 2, 4, 6, \dots :\ y = \left(\frac{1}{2}\right)^{t+1}(x + 3)$, trapezium, x-coordinate of centroid is $\frac{1}{9}$
- The x-coordinate of the centroid will oscillate between $\mathbf{0}$ and $\frac{1}{9}$ showing that the total distance traveled by the algorithm can be made arbitrarily large

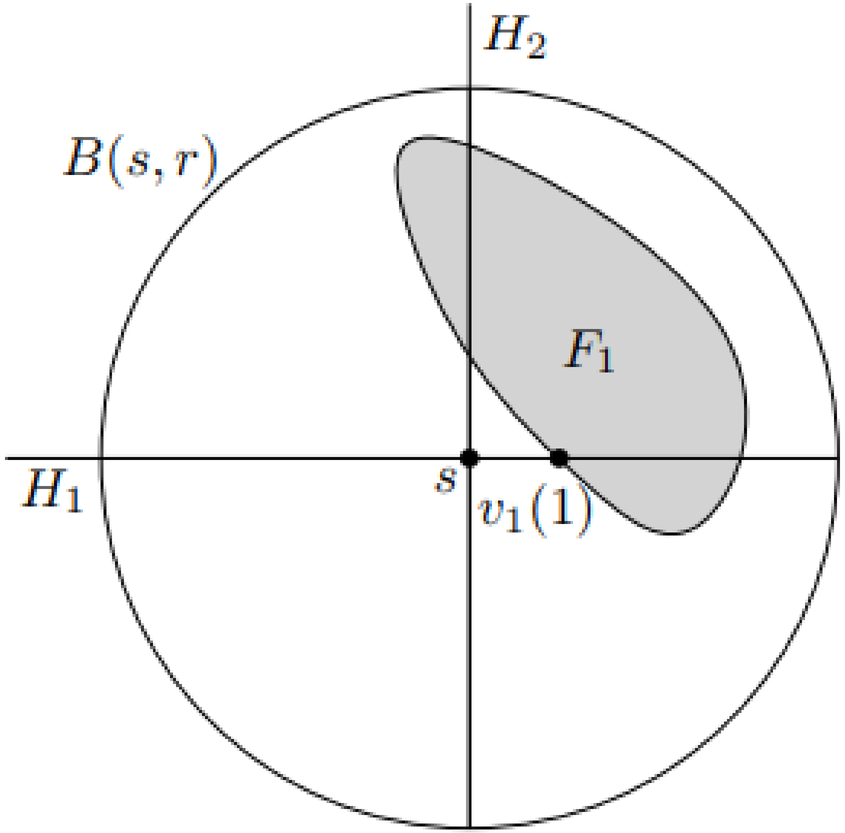
4. Recursive Greedy Method

- The first $f(d)$-competitive algorithm for Chasing Nested Convex Bodies
- Competitive ratio: $O(6^d(d!)^2)$
- Tries to tackle the problem of the “naïve centroid” approach in which the algorithm was moving back and forth
- It is called recursive because it solves “subproblems” a lower dimension

4. Recursive Greedy Method

- We will call the algorithm $Chase_d$.
- The algorithm runs in phases.
- Consider a phase that starts at center s and radius r.
- Let $H_1, \ldots, H_d$ denote the axis-aligned hyperplanes passing through s.
- **Hyperplane step**: When a request K_i arrives find the smallest index k such that H_k intersects K_i and run $Chase_{d-1}$ at H_k, until a request K_j arrives that cannot be served by H_k. Then try the hyperplanes with index $k' > k$ to serve the request K_j.
- **Recentering step:** If a request K_i arrives and none of the remaining hyperplanes intersects it, compute the smallest ball $B(s', r')$ that contains K_i, move to s' and start a new phase.

4. Recursive Greedy Method



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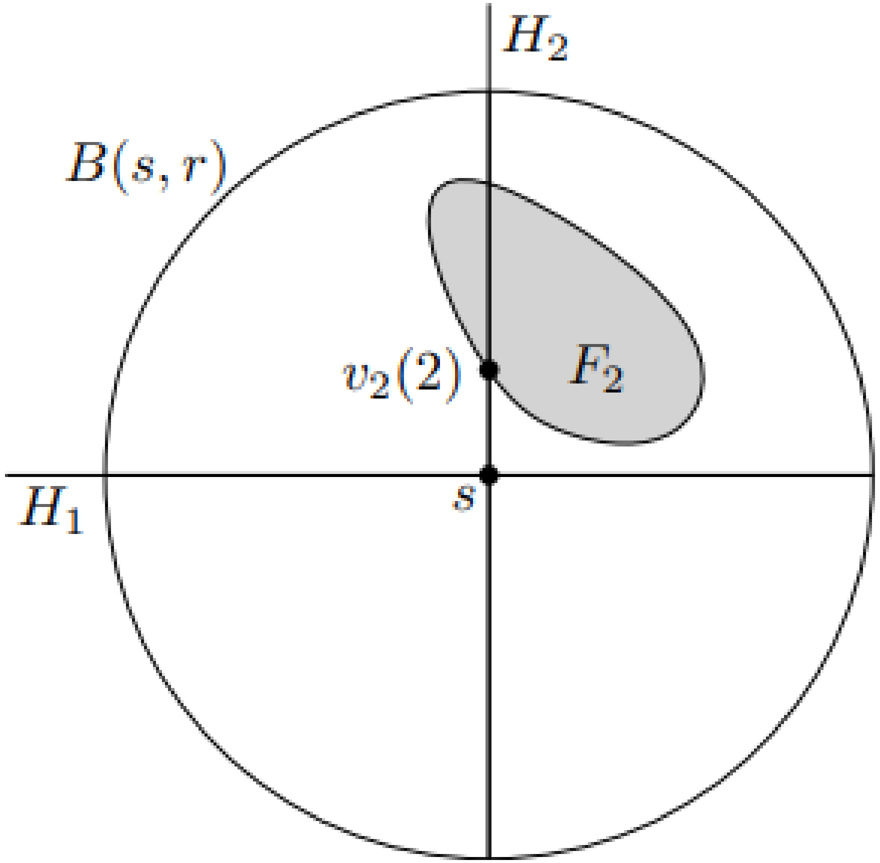

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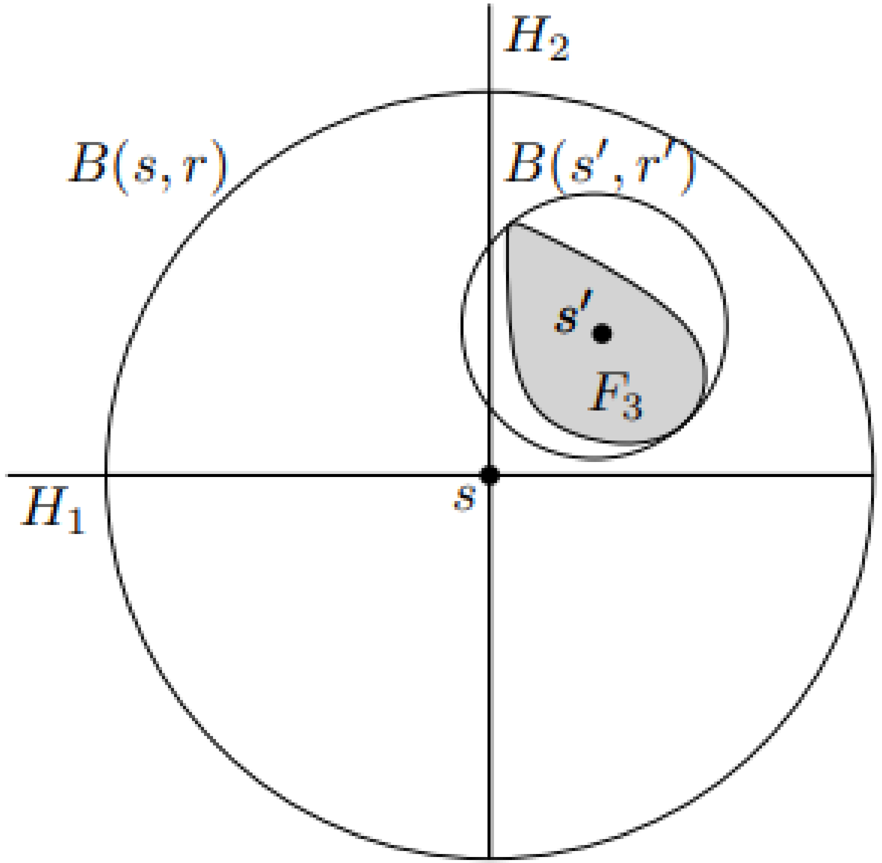

4. Recursive Greedy Method

- Let the algorithm has cost $g(d) \cdot r$, and that we are at phase j
- Movement due to hyperplane steps is $g(d-1) \cdot r_j$, because on each of the d hyperplanes H_k, we move at most $g(d-1) \cdot r_j$ by following $Chase_{d-1}$.
- Movement due to switching hyperplanes. We switch hyperplanes at most $d-1$ times, so this is at most $(d-1) \cdot 2r_j$.
- Movement due to recentering is at most $2r_j$.
- Thus, the total distance traveled in phase j is at most:

$$d(g(d-1)+2) \cdot r_j \leq 3d \cdot g(d-1) \cdot r_j, \text{ because } g(d-1) \geq 1$$

4. Recursive Greedy Method

- It is proven that the radii of the enclosing balls decrease geometrically across phases: $\mathrm{r_j} \leq r_{j-1}\left(1-\frac{1}{d}\right)^{1/2}$ and we have $r_1 = r$.
- $3d \cdot g(d-1) \cdot \frac{r}{\left(1-\frac{1}{d}\right)^{\frac{1}{2}}} \leq 3d \cdot g(d-1) \cdot 2dr = 6d^2 \cdot g(d-1) \cdot r$, by using Taylor series
- By solving $g(d) \cdot r = 6d^2 \cdot g(d-1) \cdot r$ with base case $g(1) = 1$ we get $g(d) = 6^d (d!)^2$

5. Steiner Point Method

- Imagine you are standing at the origin of a coordinate system and looking in various directions $\theta \in \mathbb{S}^{d-1}$
- For each direction θ, find the point on the boundary of K that is the furthest in that direction.
- Now, imagine you could average out all these furthest points by considering all possible directions.
- The result of this averaging gives you the Steiner point $s(K)$

5. Steiner Point Method

For a convex body $K \subset \mathbb{R}^d$, its Steiner Point $s(K)$ is defined in these two equivalent ways:

1. For any direction $\theta \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$, let $f_K(\theta) = argmax_{x \in K}(\theta \cdot x)$ be the extremal point in K in direction θ. Then compute the average of this extremal point for a random direction: $s(K) = \int_{\theta \in \mathbb{S}^{d-1}} f_K(\theta) d\theta$
2. For any direction $\theta \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$, let $h_K(\theta) = max_{x \in K}(\theta \cdot x)$ be the support function for K in direction θ, and compute: $s(K) = d \int_{\theta \in \mathbb{S}^{d-1}} h_K(\theta) \theta d\theta$

5. Steiner Point Method

- Algorithm: At each timestep t move to the $s(K_t)$
- It is a memoryless Algorithm
- It works only for Euclidean Spaces
- Competitive ratio: $O(\min(d, \sqrt{d\ logT}))$, where T is the number of timesteps
- We will prove the $O(d)$ part of the competitive ratio

5. Steiner Point Method

Proof: Let $B_1 \supseteq K_1 \supseteq \cdots \supseteq K_T \in \mathbb{R}^d$ (by scaling) a sequence of convex bodies

Total cost of the movement is:

$$\sum_{t=1}^{T-1} ||s(K_t) - s(K_{t+1})||_2 = \sum_{t=1}^{T-1} d| \int_{\theta \in \mathbb{S}^{d-1}} \big(h_{K_t}(\theta) - h_{K_{t+1}}(\theta)\big)\theta \, d\theta| \text{ (by definition)}$$

$$\leq d \sum_{t=1}^{T-1} | \int_{\theta \in \mathbb{S}^{d-1}} \left(h_{K_t}(\theta) - h_{K_{t+1}}(\theta)\right) \, d\theta| \text{ (}\theta\text{ is unit vector)}$$

$$= d \int_{\theta \in \mathbb{S}^{d-1}} \sum_{t=1}^{T-1} |h_{K_t}(\theta) - h_{K_{t+1}}(\theta)| d\theta$$

5. Steiner Point Method

Now, as $K_t \supseteq K_{t+1}$, we have that $h_{K_t}(\theta) \geq h_{K_{t+1}}(\theta)$ and so:

$$d \int_{\theta \in \mathbb{S}^{d-1}} \sum_{t=1}^{T-1} \left| h_{K_t}(\theta) - h_{K_{t+1}}(\theta) \right| = d \int_{\theta \in \mathbb{S}^{d-1}} \sum_{t=1}^{T-1} (h_{K_t}(\theta) - h_{K_{t+1}}(\theta)) d\theta$$

$$= d(\int_{\theta \in \mathbb{S}^{d-1}} h_{K_1}(\theta) d\theta - \int_{\theta \in \mathbb{S}^{d-1}} h_{K_T}(\theta) d\theta) \text{ (telescoping sum)}$$

The first integral is at most one because $K_1 \subseteq B_1$. As $h_K(\theta) + h_K(-\theta) \geq 0$ for any convex body K, the second integral is non-negative. So we conclude that

$$\sum_{t=1}^{T-1} ||s(K_t) - s(K_{t+1})||_2 \leq d$$

References

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- Sébastien Bubeck, Bo'az Klartag, Yin Tat Lee, Yuanzhi Li, Mark Sellke (2018). *Chasing Nested Convex Bodies Nearly Optimally*. https://arxiv.org/abs/1811.00999