Deformable slip bubble approaching a planar shear free surface: trapezoidal rule in time, central difference in space

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Conditions

1. small ϵ expansion
2. all $\mathcal{C}$ for top surface
3. for the bubble, assume $h_2 = h_2^{static} + h_2^{dynamic} = -1 - \frac{r^2}{2} + t + h_2^{O(\mathcal{B})correction} + h_2^{dynamic}$. For now, let $h_2^{O(\mathcal{B})correction} = 0$, and $h_2^D := h_2^{dynamic}$ and $h_2^S := h_2^{static}$

Objectives

1. write the semi infinite BCs for the unknonwn variables
2. solve for top surface shape function h_1, deformation on the bubble surface h_2^D, and radial film velocity v_r.

Governing equations for $O(\epsilon^0)$

$$\frac{\partial h_1 - h_2^D}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r(h_1 - h_2^S - h_2^D)v_r) = \Theta(t - t_{stop}) \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\big[r\frac{\partial h_1}{\partial r}\big] - \mathcal{B}h_1 = -3\mathcal{C}\frac{1}{r}\frac{\partial}{\partial r}(rv_r) \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\big[r\frac{\partial h_2^D}{\partial r}\big] + \mathcal{B}h_2^D = 3\mathcal{C}\frac{1}{r}\frac{\partial}{\partial r}(rv_r) \tag{3}$$

where

$$\Theta(t - t_{stop}) = \begin{cases} 1, & t \le t_{stop} \\ 0, & t > t_{stop} \end{cases}$$

and

$$h_2^{(0)} = \begin{cases} -1 - \frac{r^2}{2} + t, & t \le t_{stop} \\ -1 - \frac{r^2}{2} + t_{stop}, & t > t_{stop} \end{cases}$$

ICBC

Initial conditions:

$$h_1(t=0) = 0 \quad h_2^D(t=0) = 0 \quad v_r(t=0) = 0 \tag{4}$$

Boundary conditions at $r = 0$:

$$\frac{\partial h_1}{\partial r}\bigg|_{r=0} = 0 \quad \frac{\partial h_2^D}{\partial r}\bigg|_{r=0} = 0 \quad v_r(r=0) = 0 \tag{5}$$

Now derive the far field boundary condition:

Plug $h_1\big|_{r\to\infty} \to 0$ into Eq. (2) to get:

$$v_r\big|_{r\to\infty} = \frac{A(t)}{r} \tag{6}$$

where $A(t)$ is an integration constant with respect to r. From simulation experience, $A(t)$ is a constant that does not depend on time. Now plug the two above conditions into Eq. (3):

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial h_2^D}{\partial r}\right] + \mathcal{B}h_2^D = 0 \tag{7}$$

Since Eq. (7) is the homogeneous Bessel equation, the solution is of the form:

$$h_2^D = \lambda Y_0(mr), \quad m = \sqrt{\mathcal{B}} \tag{8}$$

Drop the $J_0(mr)$ term, leaving the solution in the same form as the far field solution obtained for a spherical bubble to a planar free surface case ($\bar{h}_2^{(01,inner)}$ in Joe's notes). To compute λ, impose the following BC:

$$\frac{\partial h_2^D}{\partial r} = \lambda\frac{\partial Y_0(mr)}{\partial r} = -\lambda m Y_1(mr) \tag{9}$$

Combining Eq. (8) and Eq. (9) to eliminate λ:

$$\frac{\partial h_2^D}{\partial r} + \frac{mY_1(mr)}{Y_0(mr)}h_2^D = 0 \tag{10}$$

The following far field conditions are applied at $r_{max} = R1$ in the simulation:

$$h_1\big|_{r=R1} = 0 \tag{11}$$

$$\frac{\partial h_2^D\big|_{r=R1}}{\partial r} + \frac{mY_1(mr)}{Y_0(mr)}h_2^D\big|_{r=R1} = 0 \tag{12}$$

Numerical scheme: trapezoidal rule in time, central difference in space

$$
\begin{aligned}
&\frac{(h_1^{n+1}-h_1^n)-(h_2^{D,n+1}-h_2^{D,n})}{\Delta t}+\frac{1}{2}\mathbf{D_A}diag(v_{rm}^n)\Big[(h_{1m}-h_{2m}^D)^{n+1}+(h_{1m}-h_{2m}^D)^n\Big]= \\
&=\Theta(t-t_{stop})+\mathbf{D_A}diag(v_{rm}^n)h_{2m}^{S,n+\frac{1}{2}} && (13)\\
&(\mathbf{D_D}-\mathcal{B}\mathbf{I})h_1^{n+1}+3\mathcal{C}\mathbf{D_A}v_r^{n+1}=0 && (14)\\
&(\mathbf{D_D}+\mathcal{B}\mathbf{I})h_2^{D,n+1}-3\mathcal{C}\mathbf{D_A}v_r^{n+1}=0 && (15)
\end{aligned}
$$

Further rearrage Eq. (13), (14), (15) into a block matrix form:

$$
\begin{bmatrix} L_{11} & -L_{11} & \mathbf{0} \\ \mathbf{D_D}-\mathcal{B}\mathbf{I} & \mathbf{0} & 3\mathcal{C}\mathbf{D_A} \\ \mathbf{0} & \mathbf{D_D}+\mathcal{B}\mathbf{I} & -3\mathcal{C}\mathbf{D_A} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2^D \\ v_r \end{bmatrix}^{n+1} = \begin{bmatrix} b_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}^n \qquad (16)
$$

where variables with subscript m are evaluated at half grid spaces.

$$
\begin{aligned}
&L_{11}=\mathbf{I}+\frac{\Delta t}{2}\mathbf{D_A}diag(v_{rm}^n)\mathbf{D_M} && (17)\\
&\mathbf{I}=diag(1) && (18)\\
&b_1=\Delta t\Theta+h_1^n-h_2^{D,n}+\Delta t\mathbf{D_A}diag(v_{rm}^n)h_{2m}^{S,n+\frac{1}{2}}-\frac{\Delta t}{2}\mathbf{D_A}diag(v_{rm}^n)\Big[h_{1m}-h_{2m}^D\Big]^n && (19)
\end{aligned}
$$

$\mathbf{D_A}$ is the $J-1$ by $J-1$ central-difference matrix that corresponds to the operator $\frac{1}{r}\frac{\partial}{\partial r}(r\cdot)$:

$$
\mathbf{D_A} = \begin{bmatrix} 4 & & & & & \\ -\frac{1}{2} & \frac{3}{2} & & & & \\ & -\frac{3}{4} & \frac{5}{4} & & & \\ & & \ddots & \ddots & & \\ & & & & -\frac{J-\frac{3}{2}}{J-1} & \frac{J-\frac{1}{2}}{J-1} \end{bmatrix} \qquad (20)
$$

$\mathbf{D_D}$ is the central-difference matrix that corresponds to the operator $\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\cdot}{\partial r}\right]$:

$$
\mathbf{D_D} = \begin{bmatrix} -4 & 4 & & & & \\ \frac{1}{2} & -2 & \frac{3}{2} & & & \\ & \frac{3}{4} & -2 & \frac{5}{4} & & \\ & & \ddots & \ddots & & \\ & & & & -\frac{J-\frac{3}{2}}{J-1} & -2 \end{bmatrix} \qquad (21)
$$

$\mathbf{D_M}$ is the $J-1$ by $J-1$ spatial average matrix:

$$
\mathbf{D_M} = \begin{bmatrix} 0.5 & 0.5 & & & \\ & 0.5 & 0.5 & & \\ & & 0.5 & 0.5 & \\ & & & \ddots & \ddots \\ & & & & 0.5 \end{bmatrix} \qquad (22)
$$

Numerical implementation of the far field Robin boundary condition

Previously we derived the far field boundary condition on h_2^D:

$$\frac{\partial h_2^D\big|_{r=R1}}{\partial r} + \frac{mY_1(m*R1)}{Y_0(m*R1)} h_2^D\big|_{r=R1} = 0 \tag{23}$$

Introduce the following constant:

$$MYY := \frac{mY_1(m*R1)}{Y_0(m*R1)} = \sqrt{\mathcal{B}}\frac{Y_1(\sqrt{\mathcal{B}}*R1)}{Y_0(\sqrt{\mathcal{B}}*R1)} \tag{24}$$

Discretize the above equation:

$$\frac{h_{2,J}^D - h_{2,J-2}^D}{2\Delta r} + MYY * h_{2,J-1}^D = 0 \tag{25}$$

Rearrange for $h_{2,J}^D$:

$$h_{2,J}^D = h_{2,J-2}^D - 2\Delta r \cdot MYY \cdot h_{2,J-1}^D \tag{26}$$

Plug the above expression into row $J-1$ in the discretized version of Eq. (15):

$$\begin{aligned}&\frac{\mathcal{C}}{\Delta r^2}\left[\big(1 - \frac{1}{2(J-1)}\big)h_{2,J-2}^D - 2h_{2,J-1}^D + \big(1 + \frac{1}{2(J-1)}\big)h_{2,J}^D\right] = \\ &-\mathcal{C}\mathcal{B}h_{2,J-1}^D + \frac{3\mathcal{C}}{\Delta r}\left[-\big(1 - \frac{1}{2(J-1)}\big)v_{r,J-\frac{3}{2}} + \big(1 + \frac{1}{2(J-1)}\big)v_{r,J-\frac{1}{2}}\right] := RHS\end{aligned}$$

Cleaning up to get:

$$\frac{\mathcal{C}}{\Delta r^2}\left[\big(1 - \frac{1}{2(J-1)}\big)h_{2,(J-2)}^D - 2h_{2,(J-1)}^D + \big(1 + \frac{1}{2(J-1)}\big)\big(h_{2,J-2}^D - 2\Delta r \cdot MYY \cdot h_{2,J-1}^D\big)\right] = RHS \tag{27}$$

Eq. (27) is used to change the operator matrix in Eq. (16). Eq. (26) is used compute the end point value for h_2^D.

Now we examine the impact of the Robin condition on Eq. (13). For row $J-1$, the discretized verion of Eq. (13) is:

$$
\begin{aligned}
&h^{n+1}_{1,J-1} + \frac{\Delta t}{2\Delta r}\Big[-\big(1-\frac{1}{2(J-1)}\big)v^n_{rm,J-\frac{3}{2}}h^{n+1}_{1,J-\frac{3}{2}} + \big(1+\frac{1}{2(J-1)}\big)v^n_{rm,J-\frac{1}{2}}h^{n+1}_{1,J-\frac{1}{2}}\Big] \\
&- h^{D,n+1}_{2,J-1} - \frac{\Delta t}{2\Delta r}\Big[-\big(1-\frac{1}{2(J-1)}\big)v^n_{rm,J-\frac{3}{2}}h^{D,n+1}_{2,J-\frac{3}{2}} + \big(1+\frac{1}{2(J-1)}\big)v^n_{rm,J-\frac{1}{2}}h^{D,n+1}_{2,J-\frac{1}{2}}\Big] = b_{1,J-1}
\end{aligned} \tag{28}
$$

The term that is impacted by the Robin condition is:

$$
-\frac{\Delta t}{2\Delta r}\Big[\big(1+\frac{1}{2(J-1)}\big)v^n_{rm,J-\frac{1}{2}}h^{D,n+1}_{2,J-\frac{1}{2}}\Big] = -\frac{\Delta t}{2\Delta r}\Big[\big(1+\frac{1}{2(J-1)}\big)v^n_{rm,J-\frac{1}{2}}\frac{1}{2}(h^{D,n+1}_{2,J-1} + h^{D,n+1}_{2,J})\Big] \tag{29}
$$

One needs to modify the $(1,2)$ block matrix in the operator in Eq. (16) with:

$$
-\frac{\Delta t}{2\Delta r}\Big[\big(1+\frac{1}{2(J-1)}\big)v^n_{rm,J-\frac{1}{2}}\frac{1}{2}h^{D,n+1}_{2,J}\Big] = -\frac{\Delta t}{4\Delta r}\Big[\big(1+\frac{1}{2(J-1)}\big)v^n_{rm,J-\frac{1}{2}}\big(h^{D}_{2,J-2} - 2\Delta r\cdot MYY\cdot h^{D,n+1}_{2,J-1}\big)\Big] \tag{30}
$$