space $\Omega \subset \mathbb{R}^2 = [-1,1] \times [-1,1]$ as

$$A = |\Pi_{i=1}^{n} \arg\max_{x \in \Omega}(x[i]) - \arg\min x \in \Omega(x[i])|$$

$$= (1 - (-1)) \times (1 - (-1)) = 4$$

Thus, the probabilistic problem we must solve is $4 \frac{1}{N} \sum_{i=1}^{N} f(\vec{x}_i)$ for N samples of $\vec{x}$. With a pseudo-random number generator and a given N, this is no more than a problem of summing a map of N generated numbers.

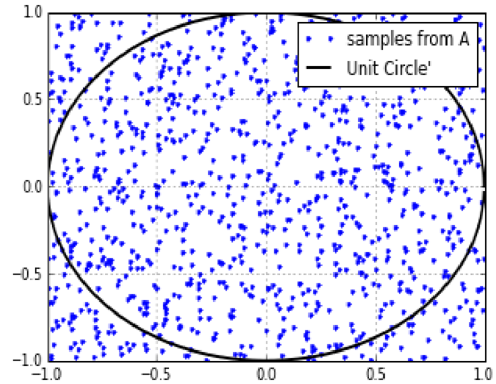
We calculate the error in our approximation using the Central Limit Theorem [4]. Let I denote the true integral I and $S_n = A \sum_{i=1}^{N} f(\vec{x}_i) = \sum_{i=1}^{N} A f(\vec{x}_i)$. Note that σ, the standard deviation of the integral, is a constant whose value won't vary. With respect to how many standard deviations it is from the true value [3, p.77], $\sigma \times \epsilon, \epsilon \in \mathbb{Q}$ and positive, the error for the approximation I_n can be calculated probabilistically:

$$P(|\frac{S_n - N\mu}{\sqrt{N\sigma^2}}| \leq \epsilon)$$

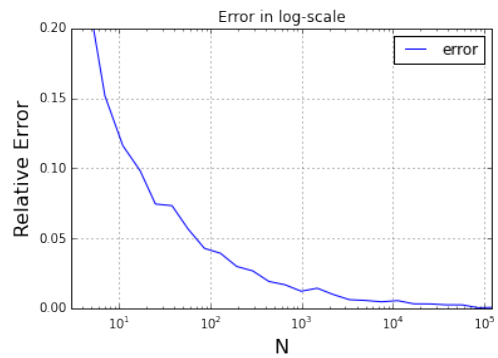
$$= P(|\frac{(\frac{N}{A} \times I_n) - N(I)}{N}| \leq \frac{\epsilon \times \sigma}{\sqrt{N}})$$

$$= P(|I_n - I| \leq \frac{\epsilon \times \sigma}{\sqrt{N}})$$

We can thus say that, for a certain number of standard deviations from the true integral, the estimate I_n converges within it with a probability of $P(\chi \leq \epsilon)$. Typically, in applications, σ is taken to just be 1 [5, p.398], but should still be considered an important factor due to the probablistic nature of the Monte Carlo Method. Similarly, for error-analysis, we say that the Monte Carlo Integration method converges to an error inversely-proportional to the square-root of N, such that the error$\approx \frac{1}{\sqrt{N}}$, which lacks the σ to signify how the error is assumed to scale with the variability of the problem.



(A) A graphical representation of the integration process for a sample of 100 points. The ratio of points inside the points and points overall is approximately π



(B) How the error of the approximation decreases as N increases

FIGURE 1. Monte Carlo Integration for a pseudo-random number generator, integrating the unit circle