Thermodynamic Uncertainty Relation in Superconducting Circuits

bachelor's thesis

presented by

Aristid Großmann

under the supervision of

Prof. Fabian Hassler
and
Prof. Mario Berta

Institute for Quantum Information

18.07.2025

Ich versichere, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie Zitate kenntlich gemacht habe.

Aachen, den 24. Juli 2025

Aristid Großmann

Abstract

abstract

Contents

dedication

Chapter 1

Introduction

Goal of this chapter: short intro to superconducting circuits $\rightarrow$ importance of the study of thermal fluctuations
$\rightarrow$ TUR: statement and validity (short history, for which types of systems is TUR valid? What would be the implications?)
Goal of this thesis: what type of system do we study? when is the TUR saturated? what leads us close to saturation? ... in the end: case studies and compare to analytical predictions

Chapter 2

Noise in Superconducting Circuits

Superconducting circuits are electrical networks built from materials that, when cooled below a critical temperature, exhibit zero electrical resistance and expel magnetic fields. These unique properties make them essential for advanced technologies such as quantum computing, ultra-sensitive magnetometers, amplifiers and sensors, where energy efficiency and quantum coherence are critical. Unlike conventional circuits, superconducting circuits can sustain an electrical current indefinitely without power loss and exhibit macroscopic quantum behavior. Understanding noise in these circuits is especially important, as even small thermal fluctuations can disturb quantum states and degrade performance. Precise control and mitigation of such noise sources are fundamental to maintaining coherence and achieving reliable operation.

2.1 Josephson Junction

In the context of superconducting circuits, the Josephson junction (JJ) is a fundamental component that enables a wide range of quantum and classical applications. It consists of two superconductors separated by a thin insulating barrier, allowing bound pairs of electrons (also called Cooper pairs) to tunnel through via the Josephson effect. Its current-phase relation reads

$$\begin{aligned} I &= I_{\mathrm{c}} \sin\left(\frac{\hbar}{2e}\phi\right) \\ V &= \frac{\mathrm{d}\phi}{\mathrm{d}t} \end{aligned} \tag{2.1}$$

where I and V are the current and the voltage across the JJ, as a function of its phase ϕ (also referred to as node flux) and the junction's critical current $I_{\mathrm{c}} > 0$. Its nonlinear current-phase relation makes it a crucial element in qubits and SQUIDs [2, 4]. The JJ is mathematically described by the Resistively Capacitance Shunted Junction (RCSJ) model

$$I_0 = C_{\mathrm{J}} \frac{\mathrm{d}V}{\mathrm{d}t} + I_{\mathrm{c}} \sin\left(\frac{\hbar}{2e}\phi\right) + \frac{V}{R} \tag{2.2}$$

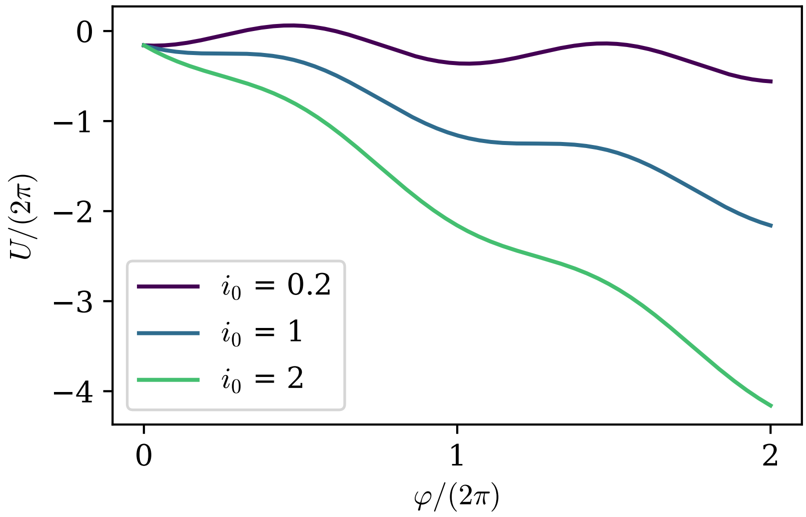

Figure 2.1: Tilted washboard potential potential $U(\varphi) = -i_0\varphi - \cos(\varphi)$ for different currents (tilts) i_0. For $i_0 < 1$ the potential exhibits local minima, in which a Brownian particle would be trapped. For $i_0 > 1$, the local minima disappear, and the Brownian particle would run towards $\phi \to \infty$ indefinitely.

where $I_0 > 0$ is an external current, C_J is the junction's capacitance, and R is the junction's resistance [1]. In the overdamped limit $C_\mathrm{J} \to 0$, the JJ's phase dynamics are described by the first-order nonlinear Ordinary Differential Equation (ODE)

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = RI_0 + RI_\mathrm{c} \sin\left(\frac{\hbar}{2e}\phi\right). \tag{2.3}$$

Introducing the variables $\varphi = \phi/\phi_0$, $\tau = t/t_0$ and $i_0 = I_0/I_\mathrm{c}$ with $\phi_0 = \hbar/(2e)$ and $t_0 = \hbar/(2eRI_\mathrm{c})$ enables us to express the phase dynamics in dimensionless quantities:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = i_0 - \sin(\varphi) = f(\varphi) := -\frac{\mathrm{d}U}{\mathrm{d}\varphi} \tag{2.4}$$

Like many other superconducting circuits, the driving force of the JJ can be expressed as minus the derivative of a potential of the form $U(\varphi) = -i_0\varphi + P(\varphi)$, where $P(\varphi)$ is an L-periodic function and i_0 is the tilt of the potential (commonly referred to as tilted periodic potential). For the JJ, the driving force $f(\varphi)$ can be written as minus the derivative $U(\varphi) = -i_0\varphi - \cos(\varphi)$, which is called the tilted washboard potential. Examining the JJ's phase dynamics in terms of the tilted washboard potential gives access to a more visual understanding: The JJ's phase dynamics are equivalent to a particle in the tilted washboard potential (note that the particle has no momentum in the overdamped limit). For $i_0 < 1$, the tilted washboard potential exhibits local minima, out of which the particle cannot escape. When $i_0 = 1 \Leftrightarrow I_0 = I_\mathrm{c}$, the local minima turn into saddle points. For $i_0 > 1$, the tilt is large enough so that all local minima

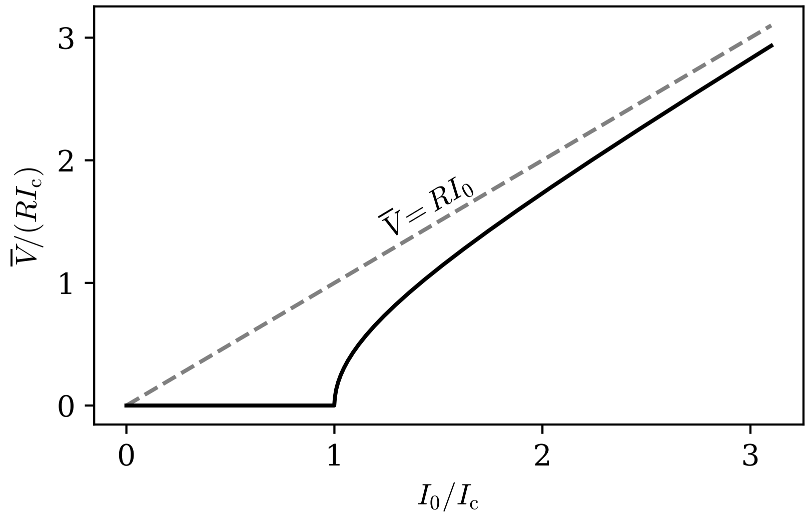

Figure 2.2: I-V-curve of the Josephson Junction. In the zero-voltage state $I_0 < I_c$, the supercurrent I_0 flows across the junction without resistance. In the finite-voltage state $I_0 > I_c$, the junction enters a resistive state. For $I_0 \gg I_c$, superconductivity breaks down and the voltage across the junction behaves linearly with $\overline{V} \approx RI_0$

disappear, and the particle runs towards $\varphi \to \infty$ indefinitely.
Indeed, when solving Equation 2.4 for $i_0 < 1$, we analytically obtain that after a short settling period due to initial conditions, the phase settles into a local minimum with $\varphi \mod 2\pi = \arcsin(i_0)$. For $i_0 > 1$, we can solve Equation 2.4 via separation of variables. The resulting integrals can be calculated via the tangent half-angle substitution. After substituting $\omega_0 = \sqrt{i_0^2 - 1}$, we obtain

$$\tan(\varphi/2) = \frac{\omega_0}{i_0} \tan\left(\frac{\tau - \tau_0}{2}\omega_0\right) + \frac{1}{i_0}.$$

Comparing the left-hand side (LHS) to the RHS, it can be seen that the phase is a running process in which φ increases by $\Delta\varphi = 2\pi$ over the period $T = 2\pi/\omega_0$. The average voltage over one period is

$$\overline{v} = \frac{\overline{V}}{RI_c} = \frac{1}{T}\int_0^T d\tau \, \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\varphi(T) - \varphi(0)}{T} = \frac{2\pi}{T} = \omega_0$$

for $i_0 > 1$ and $\overline{v} = 0$ for $i_0 < 1$. Plotting the average dimensionless voltage $\overline{v}$ with respect to the dimensionless current i_0, we obtain the I-V-curve: Below I_c, it is possible to transmit the current I_0 without resistance.

2.2 Thermal Noise Strength of an Ohmic Resistor

In practice, the voltage over a circuit with resistance R is subject to noise $\delta V_R(t)$ due to thermal fluctuations. To account for this, we consider the circuit to be in thermal equilibrium with an idealized infinite heat bath at temperature T. Heat exchange between the circuit and the bath occurs through the circuit's resistance R, which is assumed to be constant. A simple yet effective way to model thermal fluctuations is to represent them as Gaussian white noise $\delta V_{\mathrm{R}}(t) = B\xi(\tau)$ where $\xi(\tau)$ is a stochastic process with mean $\langle \xi(\tau) \rangle = 0$ and correlation $\langle \xi(\tau)\xi(\tau') \rangle = \delta(\tau - \tau')$. The parameter B characterizes the strength of the noise and must be consistent with the equipartition theorem from statistical physics, which states that in thermal equilibrium, the expected energy associated with each quadratic degree of freedom equals $k_{\mathrm{B}}T$, where k_{B} is the Boltzmann constant.

To derive the thermal noise strength of an ohmic resistor consistent with the equipartition theorem, we analyze a simple LR circuit, consisting of an ohmic resistor R and an inductance L. First, we aim to express the circuit dynamics in terms of the phase ϕ. The current-phase relations of the respective components are obtained from their current-voltage relation:

$$
\begin{aligned}
V_{\mathrm{R}} &= \frac{\mathrm{d}\phi_{\mathrm{R}}}{\mathrm{d}t} = RI_{\mathrm{R}} \\
V_{\mathrm{L}} &= \frac{\mathrm{d}\phi_{\mathrm{L}}}{\mathrm{d}t} = L\frac{\mathrm{d}I_{\mathrm{L}}}{\mathrm{d}t} \qquad \to \phi_{\mathrm{L}} = LI_{\mathrm{L}}
\end{aligned}
$$

where V_{R} and V_{L} are the voltages over the resistor and the inductance, respectively. According to Kirchhoff's rules, without thermal noise $I_{\mathrm{L}} + I_{\mathrm{R}} = 0$ and $V_{\mathrm{L}} + V_{\mathrm{R}} = 0$. Expressed in terms of phase $\phi_{\mathrm{R}} = \phi_{\mathrm{L}} = \phi$, this is equivalent to

$$
\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{R}{L}\phi = 0.
$$

When considering voltage fluctuations due to thermal noise, the RHS is no longer equal to zero. Instead,

$$
\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{R}{L}\phi = B\xi(t).
$$

Solving for the node flux yields the general solution

$$
\begin{aligned}
\phi(t) &= \phi(0)\exp\left(-\frac{R}{L}t\right) + \int_0^t dt' \; RB\xi(t')\exp\left(-\frac{R}{L}(t-t')\right) \\
&= \int_0^\infty dt' \; RB\xi(t')\exp\left(-\frac{R}{L}(t-t')\right) \quad \text{in the long-time limit } \; t \to \infty
\end{aligned}
$$

The expectation of the circuit's Hamiltonian is equal to the energy stored inside a coil, which is given by

$$
\langle H \rangle = \left\langle \frac{1}{2} L I_{\mathrm{L}}^2 \right\rangle = \frac{\langle \phi_{\mathrm{L}}^2 \rangle}{2L}
$$

and must be equal to $k_{\mathrm{B}}T$ according to the equipartition theorem. In the long time limit, the Hamiltonian can be calculated by taking the ensemble average of the integrand and substituting the correlation:

$$\begin{aligned}
\frac{\langle \phi_{\mathrm{L}}^2 \rangle}{2L} &= \frac{1}{2L} \lim_{t\to\infty} \int_0^t dt' \int_0^t dt''\ B^2 \exp\left(-\frac{R}{L}(2t - t' - t'')\right) \underbrace{\langle \xi(t')\xi(t'') \rangle}_{=\delta(\tau' - \tau'')} \\
&= \frac{1}{2L} \lim_{t\to\infty} \int_0^t dt'\ B^2 \exp\left(-2\frac{R}{L}(t - t')\right) \\
&= \lim_{t\to\infty} \frac{B^2}{2R}\left[1 - \exp\left(-\frac{2R}{L}t\right)\right] \\
&= \frac{B^2}{2R} = k_{\mathrm{B}}T \quad \to B = \sqrt{2Rk_{\mathrm{B}}T}
\end{aligned}$$

Using the equipartition theorem in the last line and rearranging, we obtain the voltage fluctuations from an ohmic resistor $\delta V_{\mathrm{R}}(t) = \sqrt{2Rk_{\mathrm{B}}T}\xi(t)$. This expression is also commonly referred to as Nyquist–Johnson noise.

Combining this result for the thermal noise strength of an ohmic resistor with the insights gained from analyzing the JJ, we conclude that in the overdamped limit and including thermal noise, the phase dynamics of any circuit whose phase ϕ follows a tilted periodic potential $U(\phi) = -I_0\phi + P(\phi)$ [1] can be expressed via an overdamped Langevin equation with shape

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\mathrm{d}U(\phi)}{\mathrm{d}\phi} + \sqrt{2Rk_{\mathrm{B}}T}\xi(t) \tag{2.5}$$

where R is the resistance of the circuit, T is the temperature of the surrounding heat bath, k_{B} is the Boltzmann constant, and $\xi(t)$ is Gaussian white noise with $\langle \xi(t')\xi(t) \rangle = \delta(t' - t)$.

2.3 Constraints on Accuracy

Many applications of superconducting circuits for high-precision technologies, most famously for quantum computing, but also ultra-sensitive magnetometers, amplifiers and sensors, pose tight constraints on precision and noise. Even small fluctuations in the system's degrees of freedom can lead to loss of coherence, readout errors, or degraded signal resolution. A simple measure of precision is the ratio of the phase's variance and its mean $\langle\langle \phi^2 \rangle\rangle / \langle \phi \rangle^2$. To ensure accurate operation, this ratio must be kept as small as possible. Reducing fluctuations, however, generally requires dissipation in the form of energy exchange with the environment, which produces entropy. While dissipation suppresses noise, it also leads to heat generation, which negatively affects performance and efficiency. On the other hand, minimizing dissipation tends to increase fluctuations. This tradeoff is described by the Thermodynamic Uncertainty Relation (TUR), which places a theoretical bound on the precision of any fluctuating current. For the phase ϕ, it states:

$$\frac{\langle\langle \phi^2 \rangle\rangle}{\langle \phi \rangle^2} \Sigma_\phi \geq 2k_{\mathrm{B}} \tag{2.6}$$

[1] Note that here, $U(\phi)$ and $P(\phi)$ have the dimensions of voltage

where Σ_ϕ is the total entropy production [3]. The TUR therefore sets a fundamental limit on precision: lowering noise in superconducting circuits requires a minimal thermodynamic cost in the form of a higher entropy production rate.

Acknowledgements

acknowledge

Bibliography

[1] James A. Blackburn, Matteo Cirillo, and Niels Grønbech-Jensen. A survey of classical and quantum interpretations of experiments on josephson junctions at very low temperatures. *Physics Reports*, 611:1–33, 2016. A survey of classical and quantum interpretations of experiments on Josephson junctions at very low temperatures.

[2] Boris Chesca, Reinhold Kleiner, and Dieter Koelle. *SQUID Theory*, chapter 2, pages 29–92. John Wiley Sons, Ltd, 2004.

[3] Todd R. Gingrich, Jordan M. Horowitz, Nikolay Perunov, and Jeremy L. England. Dissipation bounds all steady-state current fluctuations. *Phys. Rev. Lett.*, 116:120601, Mar 2016.

[4] Jens Koch, Terri M. Yu, Jay Gambetta, A. A. Houck, D. I. Schuster, J. Majer, Alexandre Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf. Charge-insensitive qubit design derived from the cooper pair box. *Phys. Rev. A*, 76:042319, Oct 2007.