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Thermodynamic Uncertainty Relation in Superconducting Circuits

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Contents

1 Introduction

Goal of this chapter: short intro to superconducting circuits $\rightarrow$ importance of the study of thermal fluctuations
$\rightarrow$ TUR: statement and validity (short history, for which types of systems is TUR valid? what would be the implications?)
Goal of this thesis: what type of system do we study? when is the TUR saturated? what leads us close to saturation? ... in the end: case studies and compare to analytical predictions

2 Statistical Modeling of Superconducting Circuits

Goal of this Chapter: short intro to superconducting circuits, typical components and how to model thermal fluctuations. State that many systems can be described via tilted periodic potentials. State the general form of SDE that describe these systems (Overdamped Langevin) and equivalent forms (Fokker-Planck and Backward Kolmogorov). For applications: how can the relative accuracy σ/μ be optimized → transition: end chapter with TUR

2.1 Thermal Noise Strength of an Ohmic Resistor

Goal of this section: Present a simple classical model for thermal fluctuations of an Ohmic Resistor

- LR Circuit
- define the node flux/phase
- add the noise term (assumptions: gaussian white noise)
- Hamiltonian
- → derive noise strength so that the SDE is consistent with the Hamiltonian (FDT)

2.2 Josephson Junction

Goal of this section: Present a prime example component of a superconducting circuit →transition: JJ dynamics (like many other systems) can be described via a tilted periodic potential

- JJ Current-phase relation (no explanation)
- RSJ model → overdamped limit → dimensionless form
- Write driving force via tilted washboard potential → sketch: probability to go "right" is higher than to go "left" → running process
- General Form: ... "we find that many other systems can be described via tilted periodic potentials"

2.3 Overdamped Langevin, Fokker-Planck and Backward Kolmogorov equation

Goal of this section: State the general type of system (overdamped Langevin) and write it in dimensionless form → equivalent formulation: Fokker-Planck Equation (PDF) and Backward Kolmogorov equation (Expectations)

- Overdamped Langevin equation with tilted periodic potential
- Derive Fokker-Planck equation (FPE) from Langevin
- derive Backward Kolmogorov as the adjoint FPE

2.4 TUR

Goal of this section: State that accuracy σ/μ is desired to be minimized. Statement of the TUR and meaning for the relative accuracy σ/μ,

- state the TUR in dimensionless quantities
- relative accuracy is constrained via entropy production (to increase relative accuracy, entropy production must also increase)

3 Thermodynamic Uncertainty Relation for tilted Periodic Potentials in the Overdamped Limit

Goal of this Chapter: State the ansatz for calculating the uncertainty product (in the long-time limit). Calculate the uncertainty product. Study which conditions lead to saturation and interpret them. Derive circuit properties that lead to almost saturation

3.1 Ansatz via the Central Limit Theorem

Goal of this section: State the Ansatz for calculating the uncertainty product.

- $\varphi(\tau) = LN(\tau) + \theta(\tau)$ where θ is the process with periodic boundary conditions
- in the long-time limit, since θ is bounded, only look at the counter N
- N is the sum of first-passage times (FPT)
- FPT are iid $\rightarrow$ CLT
- from CLT for the sum of FPTs, informal derivation of the renewal theory CLT for the mean and variance of N
- express the TUR in terms of mean and variance of the FPT

3.2 Mean and Variance of the First Passage Time

Goal of this section: Calculating the mean and variance of the FPT

- Moments of the FPT follows the Backward Kolmogorov Equation (BKE)
- derive recursive relation for the moments: $\partial T_n / \partial \tau = n T_{n-1}$
- Solving the BKE
- Define $\tilde{I}_+$ and $\tilde{I}_-$
- Express the MFPT via $\tilde{I}_+$ and $\tilde{I}_-$
- Express the Variance of the FPT (cite the 2002 Seifert Paper)

3.3 Proof of the Thermodynamic Uncertainty Relation

Goal of this section: Show that the TUR is equivalent to the Cauchy-Schwartz inequality

3.4 Thermodynamic Uncertainty Relation at Saturation

Goal of this section: Study the conditions that lead to saturation and interpret them physically

- derive that the condition is $\tilde{I}_+ = \tilde{I}_-$
- Physical interpretation via the FPE
- evaluate $\tilde{I}_+ = \tilde{I}_-$ in a NESS ($\rightarrow$ constant driving)
- in an ESS $i_0 = 0$ but arbitrary periodic potential P

3.5 Thermodynamic Uncertainty Relation near Saturation

Goal of this section: Derive constraints on the current and the diffusion coefficient and the properties of the potential that lead to almost saturation.

- Derive that for fixed current i_0, the uncertainty product always increases when D decreases $\rightarrow$ increasing noise leads closer to saturation
- for $i_0 << P'(\varphi)\,\forall\varphi$, TUR is almost saturated (close to equilibrium)
- small noise approximation: Saddle point method: $i_0 > P'(\varphi)\,\forall\varphi \rightarrow$ TUR is almost saturated if $P' << i_0$ and $\overline{(P')^2}$ small
- Saturation follows for arbitrary potentials for $i_0 >> P'$ via a scaling symmetry
- $i_0 < P'(\varphi)$ for some φ: Uncertainty product is larger than when $i_0 > P'(\varphi)\,\forall\varphi$

4 Case Studies

Goal of this chapter: Examine some systems and compare case-specific results to general analytical predictions from the previous chapter.

4.1 LR Circuit

Goal of this section: Explicitly calculate mean and variance. Show that the TUR is indeed saturated for constant driving force.

4.2 Sawtooth Ratchet

Goal of this section: Parameterize the sawtooth ratchet $(a/b, V_0)$. Evaluate uncertainty product analytically. Study influence of individual parameters on uncertainty Product. Exemplary plots for different parameter configurations.

4.3 Josephson Junction

Goal of this section: Numerically evaluating the uncertainty product (MC, Crank-Nicolson, details in appendix). Exemplary plots for different parameter configurations.

5 Conclusion and Outlook

- Recap of the TURs physical meaning
- relevance for Superconducting circuit analysis in the overdamped limit (TUR is valid)
- Physical interpretation of TUR saturation
- Implications for circuit design in the overdamped limit (which regime is favorable for TUR saturation, what conditions must be met)
- Outlook to underdamped processes ($\rightarrow$ Clock circuit paper)

A References

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