Course 02249, DTU Compute
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Computationally Hard Problems – Fall 2020
Assignment 4

Date: 22.09.2020, **Due date**: 28.09.2020, 21:00

The following exercises are **not** mandatory:

Exercise 4.1: Consider the following problem:

Problem: [TRIANGLE] The input is an undirected graph $G = (V, E)$. The objective is to decide if the graph contains a *triangle* in G, i. e., three distinct vertices $v_1, v_2, v_3 \in V$ such that all three edges are present: $\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\} \in E$.

We propose the following randomized algorithm for TRIANGLE. Let $n = |V|$ and let p denote a polynomial.
The algorithm sets a counter c to 0. It then repeatedly performs the following steps.

1. Increment the counter c by 1.
2. The algorithm picks three (not necessarily distinct) vertices from V at random according to uniform distribution. It then checks whether they from a triangle. If this is the case, the algorithm answers YES and stops.
3. If $c > p(n)$ then the algorithm answers NO and stops.

Show that there is a choice for the polynomial $p(n)$ such that the algorithm is an $\mathcal{RP}$-algorithm.
Hint: To estimate success probabilities, the inequality $(1 - 1/x)^x \leq 1/2$ for $x \geq 1$ may be useful.

End of Exercise 1

Exercise 4.2: Show that the problem TRIANGLE defined in Exercise 4.1 is in the class $\mathcal{NP}$. Try to suggest two alternative proofs of this fact.

End of Exercise 2

Continued on next page.

Exercise 4.3: Prove for each of the following problems that they are in the class $\mathcal{NP}$:

a) Road Maintenance,

b) Glasses in a cupboard,

c) Satisfiability.

You need not (and should not try to) prove that the problems are $\mathcal{NP}$-complete.

End of Exercise 3

Exercise 4.4: Consider the following problem:

Problem: [MinimumCliqueCover]
Input: An undirected graph $G = (V, E)$ and a natural number k.
Output: YES if there is clique cover for G of size at most k; that is, a collection $V_1, V_2, \ldots, V_k$ of not necessarily disjoint subsets of V such that each V_i induces a complete subgraph of G and such that for each edge $\{u, v\} \in E$ there is some V_i that contains both u and v. NO otherwise.
For a subset $V' \subseteq V$ of the nodes of an undirected graph $G = (V, E)$, the subgraph *induced by* V' has node set V' and edge set $E' \subseteq E$, where $e = \{v, w\} \in E'$ if and only if both $e \in E$ and $v, w \in V'$.

Show that MinimumCliqueCover is in the class $\mathcal{NP}$.
You need not show that the problem is $\mathcal{NP}$-complete.

End of Exercise 4

The following exercise is **mandatory**:

Exercise 4.5: Consider the following problem:

Problem: [Simple-3-Partition]
Input: Given are n natural numbers $s_1, s_2, \ldots, s_n$.
Output: YES if the numbers can be partitioned into three sets with equal sums, i. e., if there are disjoint sets $A_1, A_2, A_3 \subseteq \{1, \ldots, n\}$ such that $A_1 \cup A_2 \cup A_3 = \{1, \ldots, n\}$ and

$$\sum_{i \in A_1} s_i = \sum_{j \in A_2} s_j = \sum_{k \in A_3} s_k \ ,$$

and NO otherwise.

Prove that Simple-3-Partition is in the class $\mathcal{NP}$.

End of Exercise 5