Computationally Hard Problems

Summary

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Today

- No new material
- A very short course summary
- Some remarks on the exam
- Example of NP-completeness proof
- Results of competition
- Room for questions and discussions

Afterwards: exercise sessions where we continue to work on the old exam. You are welcome to ask further questions then.

What was in the Course

- Formal stuff: Languages, running times
- Randomized algorithms: Definition, complexity classes
- Decision vs. optimization problems
- $\mathcal{NP}$-completeness: Definition, proving a problem is in $\mathcal{NP}$, proving a problem to be $\mathcal{NP}$-complete, list of $\mathcal{NP}$-complete problems
- Using randomization to solve problems exactly, approximately or heuristically

Exam

The exam is on Monday, 14.12.2020, 9:00–13:00 in 101E/Sportshal 1 and 09:00-14:00 in 127/013 (without guarantee; check DTU Inside and watch all announcements).

The exam will consist of seven assignments.

All aids allowed (you may bring your computer with copies of lecture notes etc.)
No internet access.

The lecture are the definite source of reference and comprise the whole curriculum.

Things worth to remember

- The non-mandatory and mandatory assignments
- The curriculum of the course ($\rightarrow$ multiple-choice exercise)

Structure of exam

- Design of a language (exam exercise 1)
- Conversion decision to optimization (exam exercise 2).
- Proving a problem in $\mathcal{NP}$ (exam exercise 3).
- Proving a problem $\mathcal{NP}$-complete (exam exercise 4).
- Analysis of randomized (approximation) algorithms (exam exercises 5–6).
- Multiple choice (exam exercise 7)

Some Advice

- No need to panic.
- Read the assignment thoroughly.
- Make sure you know what you are asked to do.
- Do not answer questions that are not asked.
- Remember the assignments from the exercise sessions.
- Write your solutions directly into the answer boxes on the exam set.

Example of Exam Assignment

Exercise 1: [redacted]

a) Specify the alphabet Σ[redacted] you use.

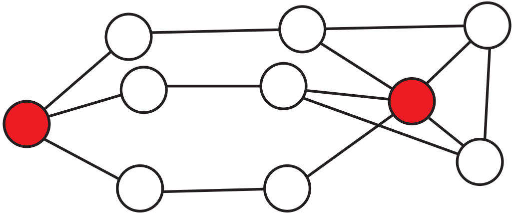
b) Specify how the language L[redacted] is defined and describe how the above example is coded in your language.

Yet Another $\mathcal{NP}$-completeness Proof

Problem [NEIGHBORWATCH]

Input: An undirected, connected graph $G = (V, E)$ and a positive integer k.

Output: YES if G has a *guard set* of size at most k, and NO otherwise. A **guard set** is a set $U \subseteq V$ such that for every $v \in V \setminus U$ there is $u \in U$ such that $\{u, v\} \in E$.



Rep.: What to do

In order to prove that a problem P is $\mathcal{NP}$-complete we have to:

1) Prove that $P \in \mathcal{NP}$.
2) Find a suitable problem P_c which is known to be $\mathcal{NP}$-complete.
3) Prove $P_c \leq_p P$, especially:
   - 3a) Desribe a transformation T which transforms every instance $\boldsymbol{X}$ of P_c into an instance $T(\boldsymbol{X})$ of P and which runs polynomial in the size $\|\boldsymbol{X}\|$ of $\boldsymbol{X}$.
   - 3b) Show that if the answer to $\boldsymbol{X}$ is YES then so is the answer to $T(\boldsymbol{X})$.
   - 3c) Show that if the answer to $T(\boldsymbol{X})$ is YES then so is the answer to $\boldsymbol{X}$.

1) NW is in $\mathcal{NP}$

As always . . .

2) NW is $\mathcal{NP}$-complete: Reference Problem

Find a problem to reduce from.

Any suggestions?

We use $\textsc{VertexCover}$.

We show

$$\textsc{NeighborWatch} \leq_p \textsc{VertexCover}.$$

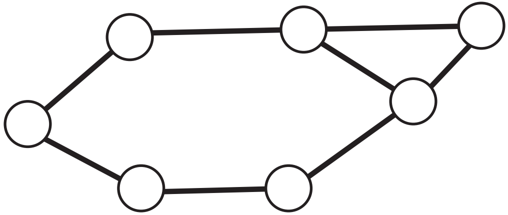
NO!!!!!!! We show

$$\textsc{VertexCover} \leq_p \textsc{NeighborWatch}.$$

3a) The Reduction

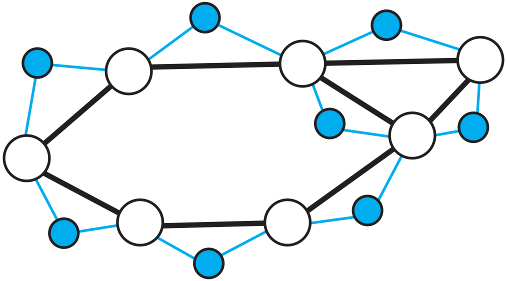
Given an instance $(G = (V, E), k)$ of VC we transform it to an instance $(G' = (V', E'), k)$ of NW as follows. G' is similar to G, with a vertex added for each edge, and connected to both ends of the original edge.

The VC instance



3a) The Reduction

The transformed NW instance



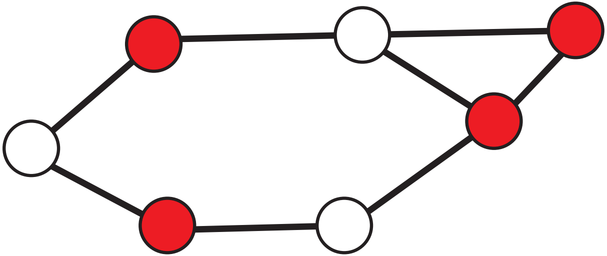
The parameter k is the same as in the VC instance.

This transformation can be done in polynomial time.

3b) $\Rightarrow$

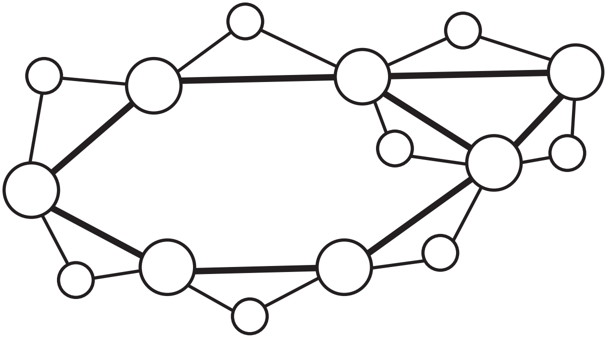
Let U be a vertex cover of G of size at most k. Then we claim that there is also a guard set of size at most k in G'.

The VC instance with vertex cover U (red)



3b) $\Rightarrow$

U is also a guard set in G'. The NW instance

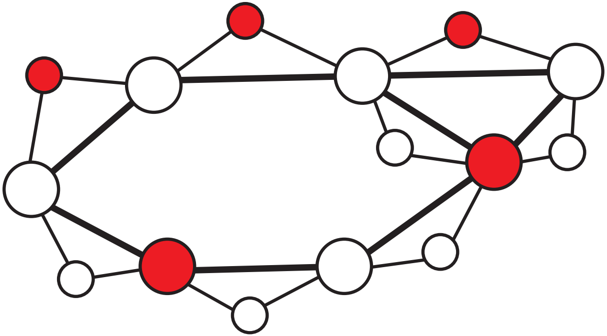


Why is that so?

3c) $\Leftarrow$

Suppose that U is a guard set in G' of size at most k. Then we claim that there is also a vertex cover of size at most k in G'.

The NW instance

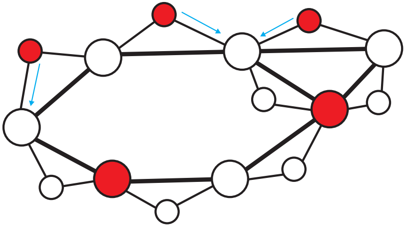


3c) $\Leftarrow$

Modify guard set for G' into another, no larger guard set for G' which is also a vertex cover for G.

Move the guards from the nodes that were introduced by the transformation to an arbitrary endpoint of the corresponding edge.

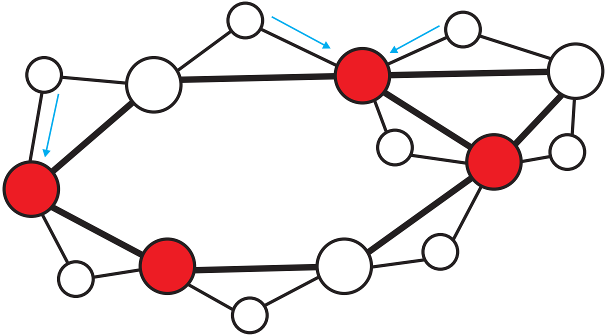
The NW instance



Note: each triangle consisting of original edge and new vertex must contain at least one guard.

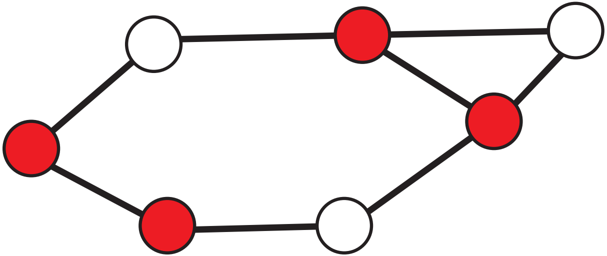
3c) $\Leftarrow$

The NW instance and the modified guard set



3c) $\Leftarrow$

The original VC instance and the vertex cover



Competition

Competition

- 25 teams participated!
- Score function to be maximized: sum of edge weights minus value of your solution
- 29 new instances:
  - Complete graphs with $n = 50, 100, \ldots, 250$ and edge weight n
  - Graphs consisting of path of length n with uniform weights in $[1, 100]$ plus an extra edge of weight $\binom{n}{2}$. Used $n = 1000, 3000, 5000, 8000$.
  - Graphs consisting of k cliques on c vertices each, joined by $k - 1$ edges; each weight uniform in $[1, 10]$. Used $(100, 2)$, $(100, 4)$, $(100, 6)$, $(100, 8)$, $(100, 10)$ and $(200, 3)$, $(200, 6)$, $(200, 9)$, $(200, 12)$, $(200, 15)$ for (c, k)
  - Erdős-Rényi random graphs with $n = 100, 150, \ldots, 550$, expected $1.1\binom{n}{2} \ln n$ edges (some of these unconnected) and uniform weights in $[1, n^2]$
- Time budget was 180 seconds in the first two settings, 60 seconds in the third and 90 seconds in the last setting.
- Showing the anonymized scoreboard now.

[redacted]	737312521 (81.43s)	58800	485100	1653900	3940200	7719000	74	95	39	50	54324	108498	163355	217155	270833	327572	655520	981873	1311937	1641011	1299518	14352862	48884702	49000362	124522328	255426937	224236476
[redacted]	718872051 (104.97s)	58800	485100	1653900	3940200	7719000	94	97	98	96	53514	106880	160947	213953	266828	324964	650298	974019			1266885	13986618	47961885	48140772	121045739	249925305	219936059
[redacted]	665174111 (83.93s)	58800	485100	1653900	3940200	7719000	74	95	39	50	54108	107990	162756	216202	269609	327383	655138	981326	1311115	1640060	1123594	12860638	44193341	43528742	112432086	232460180	198992585
[redacted]	641067855 (66.80s)	58800	485100	1653900	3940200	7719000	74	95	39	50	53455	106915	160960	214076	266740	324906	650184				1135259	12472977	42789855	41204366	108867644	227262546	191700714
[redacted]	641067787 (108.73s)	58800	485100	1653900	3940200	7719000	74	95	39	50	53516	106859	160855	213925	266743	324947	650323				1135259	12472977	42789855	41204366	108867644	227262546	191700714
[redacted]	517324871 (109.86s)	58800	485100	1653900	3940200	7719000	94	97	98	99	54312	108485	163348	217126	270802	327436		981465	1311425	1640355	1307350	14493266	49478233	50031636	125456574	257625670	
[redacted]	242859798 (324.31s)	58800	485100	1653900																	1303192	14501388	49401938	50013671	125441809		

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Rikke Bertram Rasmussen (s174493) Anton Stockmarr (s164170)	665174111 (83.93s)	58800	485100	1653900	3940200	7719000	74	95	39	50	54108	107990	162756	216202	269609	327383	655138	981326	1311115	1640060	1123594	12860638	44193341	43528742	112432086	232460180	198992585
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Sindri Pétur Ingimundarson (s192586) Laouen Pablo Killian Fernet (s192612)	718872051 (104.97s)	58800	485100	1653900	3940200	7719000	94	97	98	96	53514	106880	160947	213953	266828	324964	650298	974019			1266885	13986618	47961885	48140772	121045739	249925305	219936059
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Ask Dörge Kunding (s174266) Asbjørn Pedersen Kaad (s174282)	737312521 (81.43s)	58800	485100	1653900	3940200	7719000	74	95	39	50	54324	108498	163355	217155	270833	327572	655520	981873	1311937	1641011	1299518	14352862	48884702	49000362	124522328	255426937	224236476
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Questions

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The End

Thank you for the positive course evaluation!

Big thank you to Amir and Andrei.

Good luck with the exam!

And happy holidays!