02249 - Computationally Hard Problems, Assignment II

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14 September 2020

Exercise Description

[Refutation]:Given is a disjunctive form consisting of k monomials $m_1, ..., m_k$ over n Boolean variables $x_1, ..., x_n$. The task is to decide if there is a truth assignment to the variables such that the truth value of the disjunctive form is false.

You are given a decision algorithm A_d that solves this problem, i. e., for each instance to Refutation, A_d answers YES if there is an assignment that makes the truth value of the disjunctive form false; otherwise it answers NO.

a) Describe an algorithm A_o which solves the optimization problem, i. e., which finds a truth assignment making the disjunctive form false if one exists. The running time has to be polynomial in the input size and A_o may make calls to A_d. Such calls count as one basic computational step.

b) Argue that your algorithm is correct.

c) Prove that the running time of the algorithm is bounded from above by a polynomial. Any polynomial is sufficient; you need not look for a polynomial of minimal degree. Recall that a call to Ad counts one step.

Note: The input to Refutation is a disjunctive form of monomials over n boolean variables, nothing else. In particular, a legal input to the problem cannot specify specific settings of variables. For example, you cannot simply say "set x1 to true" and then call the decision algorithm.

Exercise Resolution

Disjunctive form is a formula over n boolean variables $x_1, ..., x_n$ such that the formula is a boolean disjunction of k so-called monomials $m_1, ..., m_k$ where each monomial m_i, $1 \leq i \leq k$, is a boolean conjunction of what we refer to as **terms** of the monomial, which are variables or their negations. For example:

$$disForm = \underbrace{\left(\bar{x}_1 \wedge \overbrace{x_2}^{t} \wedge \bar{x}_3\right)}_{m_1} \vee \underbrace{\left(x_1 \wedge \overbrace{\bar{x}_2}^{t} \wedge x_3\right)}_{m_2} \vee \underbrace{(x_4 \wedge x_5 \wedge x_6)}_{m_3}$$

variable :x_2
possible terms associated with the variable: x_2, $\bar{x}_2$

a) Description of A_o: We define as "crucial" , terms without which $A_d[disForm - \{crucialterms\}] = NO$. We keep track of the crucial terms encountered in a dedicated list. This list is not necessarily unique per disjunctive form and depends on the order in which the terms are processed. We decide to process each term in each monomial, proceeding left to right, until we get to a list of k- terms, one "crucial" term per monomial, all of which need to be False in order for the disjunctive form to be False. If the term is the negation of a variable, then the variable is set as True, otherwise the variable is set as False. Once all the crucial variables have been set, the remaining non crucial variables can be arbitrarily assigned(eg all to be True).

1. We run the algorithm with the initial disjunctive form as an input.

$$A_d[disForm(x_1, ..., x_n)]$$

If "output = NO" there is no truth assignment to the variables that makes the disjunctive form False and we stop. If "output = YES", at least one truth assignment exists so that the disjunctive form is False and we continue to next point.

2. We consider the monomial m_i (eg m_1)

2.1. We consider the term t_j in monomial m_i (eg $t_1 = \bar{x}_1$), if the monomial has only one term, we append it to the list of crucial terms and continue to the next monomial. Otherwise we check if the term is in the list of crucial terms: if it is, we move to the next term, if it is not we continue with 2.3

2.2. We write a formulation of the current disjunctive form excluding that term: $disForm - \{(t_1)\}$.

2.3. We run the decision algorithm on the $A_d[disForm - \{(t_1)\}$. If "output = NO" we put the term back and append it to the crucial terms list, we then continue with the next term in the monomial . If "output = YES", we do not put the term back in (the disjForm is changed from the previous step) and continue with next term in the monomial.

2.4. once all the terms in the monomial are considered, we move to the next monomial.

3. once all the monomials are considered we will have reduced the disjunctive form to a set of terms that all need to be False. Based on that we can assign to the correspondent crucial variables truth statement.
4. Finally all the unassigned variables are then assigned randomly (eg. all True).

b) For the Disjunctive Form to be False, all the monomials must be False. For one monomial to be False, there must be at least one term that is False.

In step (i) $A_d[disForm]$ = NO implies no possible assignment and that is the end, but if $A_d[disForm]$ = YES then there is at least one configuration in which all the monomials can be False. We consider all the terms of all the monomials. If a term can be eliminated while maintaining the overall output of A_d as "YES", it means that whatever that term is assigned to does not really matter, because there is still at least one configuration of all the remaining variables that returns at least one False term per each monomial. We exclude the term from the definition of the disjunctive form and we continue. When we encounter a A_d = NO, means that that term that we just excluded, is the one that makes the monomial we are looking at False. Therefore the term can not be excluded and it is appended to the list of crucial terms. At the end we move from crucial terms to crucial variables and assign the truth value.

- *start*

$$disForm = \underbrace{(\bar{x}_1 \wedge x_2 \wedge \bar{x}_3)}_{m_1} \vee \underbrace{(x_1 \wedge \bar{x}_2 \wedge x_3)}_{m_2} \vee \underbrace{(x_4 \wedge x_5 \wedge x_6)}_{m_3}$$

$A_d = YES$
$ct = []$

- $m_1; t_1 = \bar{x}_1$

$$disForm = (x_2 \wedge \bar{x}_3) \vee (x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_4 \wedge x_5 \wedge x_6)$$

$A_d = YES$
$ct = []$

- $m_1; t_2 = x_2$

$$disForm = (\bar{x}_3) \vee (x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_4 \wedge x_5 \wedge x_6)$$

$A_d = YES$
$ct = []$

- $m_1; t_3 = \bar{x}_3$
one term left in the monomial, append $ct = [\bar{x}_3]$

- $m_2; t_1 = x_1$

$$disForm = (\bar{x}_3) \vee (\bar{x}_2 \wedge x_3) \vee (x_4 \wedge x_5 \wedge x_6)$$

$A_d = YES$
$ct = [\bar{x}_3]$

- $m_2; t_2 = \bar{x}_2$

$$disForm = (\bar{x}_3) \vee (x_3) \vee (x_4 \wedge x_5 \wedge x_6)$$

  $A_d = NO$
  $ct = [\bar{x}_3, \bar{x}_2]$

- $m_2; t_3 = x_3$

$$disForm = (\bar{x}_3) \vee (\bar{x}_2) \vee (x_4 \wedge x_5 \wedge x_6)$$

  $A_d = YES$
  $ct = [\bar{x}_3, \bar{x}_2]$

- $m_3; t_1 = x_4$

$$disForm = (\bar{x}_3) \vee (\bar{x}_2) \vee (x_5 \wedge x_6)$$

  $A_d = YES$
  $ct = [\bar{x}_3, \bar{x}_2]$

- $m_3; t_2 = x_5$

$$disForm = (\bar{x}_3) \vee (\bar{x}_2) \vee (x_6)$$

  $A_d = YES$
  $ct = [\bar{x}_3, \bar{x}_2]$

- $m_3; t_3 = x_6$
  one term left in the monomial, append $ct = [\bar{x}_3, \bar{x}_2, x_6]$

- conclusion:

$$\bar{x}_3 = False \rightarrow x_3 = True$$

$$\bar{x}_2 = False \rightarrow x_2 = True$$

$$x_6 = False \rightarrow x_6 = False$$

c) We have k monomials, each monomial can at most be of length $(2 * n)$ (this assuming in one monomial there can be both a variable and the negation of the same), so overall we can at most have $(2 * n * k)$ terms to consider. The list of crucial terms grows to be from 1 to k terms, increasing of one term after each monomial. In the first monomial we look up over an empty list, for the second monomial we look up on a list of length 1, until the last monomial in which we look up each term against terms in a list $(k-1)$ long. At the end we assign k truth statements first and then randomly $(n-k)$ for the not crucial variables. So over all the time complexity is $O((2nk) * (k-1) + n) = O(nk^2 + n)$