Course 02249, DTU Compute
Carsten Witt

Computationally Hard Problems – Fall 2020
Assignment 3

Date: 15.09.2020, **Due date**: 21.09.2020, 21:00

The following exercises are **not** mandatory:

Exercise 3.1: Consider two random variables X and Y. Variable X assumes values $0, 2, 3$ with probabilities $1/3$, $1/2$, $1/6$, respectively. Variable Y assumes values $0, 5, 6$ with probabilities $1/4$, $1/4$, $1/2$, respectively. Let Z be the random variable $X \cdot Y$, i. e., if X and Y assume values x and y then Z is assigned $x \cdot y$.

a) Compute the expected values of X and Y.

b) Which values can Z assume?

c) What are the probabilities for these values?

d) Compute the expected values of Z, $2 + Z$, $2 \cdot Z$ and 2^Z.

End of Exercise 1

Exercise 3.2: Let the random variable X be uniformly distributed on $\{0, 1\}$. Compute the three expected values $\boldsymbol{E}\left[\frac{1}{2} + X\right]$, $\boldsymbol{E}\left[\frac{1}{2} \cdot X\right]$, and $\boldsymbol{E}\left[\left(\frac{1}{2}\right)^X\right]$.

End of Exercise 2

Exercise 3.3: Consider the following program:

$i \leftarrow -9$
while $(\text{rand}(1, 10) \neq \text{rand}(1, 10))$ **do**
  $i \leftarrow i + 1$
end while
print(i)

What is the probability that the value 3 is printed? What is the expected value of i that is printed?
Hint: Appendix B.3 in the lecture notes might be useful.

End of Exercise 3

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Exercise 3.4: Suppose you have access to a fair coin. The aim is to produce "sufficiently uniform" random numbers out of the following two sets: $A := \{1, 2, 3, 4\}$ and $B := \{1, 2, 3\}$.

1. Describe a randomized algorithm with **worst-case** running time $O(1)$ (i. e., bounded by a constant) such that the probability of every number does not differ by more than 0.001 from that of the uniform distribution. Describe such an algorithm first with respect to A and then with respect to B.

2. Describe a randomized algorithm with **expected** running time $O(1)$ such that the probability of every number equals that of the uniform distribution. Do this again for both A and B.

End of Exercise 4

Exercise 3.5: Consider the following complexity class.
Definition: A yes-no–problem is in $\mathcal{BPP}^*$ if there is a polynomial p and a randomized p-bounded algorithm A such that for every input $\boldsymbol{X}$ the following holds:

$$\begin{array}{lllll} \text{True answer for } \boldsymbol{X} \text{ is} & \text{YES} & \text{then} & \boldsymbol{P}_R\left[\, A(\boldsymbol{X}, R) = \text{YES}\,\right] & \geq & 1/2, \\ \text{True answer for } \boldsymbol{X} \text{ is} & \text{NO} & \text{then} & \boldsymbol{P}_R\left[\, A(\boldsymbol{X}, R) = \text{NO}\,\right] & \geq & 1/2. \end{array}$$

State an algorithm with running time $O(1)$ to prove that every yes-no–problem is in $\mathcal{BPP}^*$.

End of Exercise 5

The following exercise is **mandatory**:

Exercise 3.6: Consider the following complexity class.
Definition: A yes-no–problem is in $\mathcal{RP}_{1/10}$ if there is a polynomial p and a randomized p-bounded algorithm A such that for every input $\boldsymbol{X}$ the following holds:

$$\begin{array}{lllll} \text{True answer for } \boldsymbol{X} \text{ is} & \text{YES} & \text{then} & \boldsymbol{P}_R\left[\, A(\boldsymbol{X}, R) = \text{YES}\,\right] & \geq & 1/10, \\ \text{True answer for } \boldsymbol{X} \text{ is} & \text{NO} & \text{then} & \boldsymbol{P}_R\left[\, A(\boldsymbol{X}, R) = \text{NO}\,\right] & = & 1. \end{array}$$

Prove that $\mathcal{RP}_{1/10} = \mathcal{RP}$ and $\mathcal{RP} \subseteq \mathcal{BPP}$.

Hint: You may consider invoking the algorithm A from the definition of $\mathcal{RP}_{1/10}$ several times. Note, however, that the classes $\mathcal{RP}_{1/10}$ and $\mathcal{RP}$ themselves do not contain algorithms.

End of Exercise 6
