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THE SOLUTION OF A BASE CHANGE PROBLEM FOR GL(2) (FOLLOWING LANGLANDS, SAITO, SHINTANI)

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These Notes present a survey of the results on the lifting of automorphic representations of GL(2) with respect to a cyclic extension of prime degree of the groundfield, and of some of its applications to the Artin conjecture, with some sketches of proofs. §§1–5 are devoted to the definitions and results on the lifting, §6 to the proof of the Artin conjecture in the tetrahedral case. The first part ends up with three appendices describing respectively two-dimensional representations of the Weil group (Appendix A), representations of GL(2) over a local field (Appendix B) and a global field (Appendix C). Part II gives some indications on the proof of the results on lifting. The main tools are the orbital and twisted orbital integrals, and a twisted trace formula **[Sa]**. The main references for the lifting are **[Sa]**, [S-1], [S-2], **[L]**. As a side remark, we would like to point out that the study of the example of the general linear group over a finite field [S-2] is illuminating.

I. Definitions, Theorems, Applications

1. Notation. Let F be a local or global field. Then W_F is the Weil group of F and, for F a p-field, W'_F is the Weil-Deligne group of F **[T]**. We recall that there exists a canonical surjective homomorphism $W_F \to C_F$ which identifies W_F^{ab} with C_F.

In all these notes, E is a Galois extension of F, cyclic of prime degree l, and Γ its Galois group (the "split case", where $E = F \times F \times \cdots \times F$ l-times and Γ is generated by $\sigma: (x_1, x_2, \cdots, x_l) \mapsto (x_2, \cdots, x_l, x_1)$ is handled easily and is left to the reader).

For F local, choose a nontrivial character ψ of the additive group of F, and define $\psi_{E/F} = \psi \circ \mathrm{Tr}_{E/F}$.

In the following we shall use systematically the notation given by Borel **[B]** about L-groups : $\Phi(G)$, $\cdots$, and by Tate **[T]** for Weil groups, the L- and ε-factors.

2. Base change for GL(1). From abelian class-field theory, there is a commutative diagram

$$\begin{tikzcd} & 1 \arrow[r] & W_E \arrow[r] \arrow[d] & W_F \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d, "\wr"] & 1 \\ 1 \arrow[r] & C_E^{1-\sigma} \arrow[r] & C_E \arrow[r, "N_{E/F}"] & C_F \arrow[r] & \Gamma \arrow[r] & 1 \end{tikzcd}$$

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where σ is a generator of Γ.

The one-dimensional representations of W_F are given by the quasi-characters of $W_F^{ab} = C_F$; by composing a quasi-character of C_F with the norm, a map is defined called the lifting (or more precisely, the base change lift): $\chi \mapsto \chi_{E/F} = \chi \circ N_{E/F}$ which sends the set $\mathscr{A}(F)$ of quasi-characters of C_F in the set $\mathscr{A}(E)$ of quasi-characters of C_E.

The group Γ acts on the set of quasi-characters of C_E by:

$$({}^\gamma\theta)(z) = \theta(z^\gamma), \qquad \gamma \in \Gamma, z \in C_E;$$

the group $\hat{\Gamma}$ of characters of Γ can be identified with the set of characters of C_F which are trivial on $N_{E/F}C_E$; this group $\hat{\Gamma}$ acts on the set of quasi-characters of C_F by multiplication: $\chi \mapsto \chi\zeta$, $\zeta \in \hat{\Gamma}$. Then the exactness of the second line of the above diagram implies the following result:

PROPOSITION 1. *The lifting $\chi \mapsto \chi_{E/F}$ defines a bijection from the orbits of $\hat{\Gamma}$ in $\mathscr{A}(F)$ onto the invariant elements by Γ in $\mathscr{A}(E)$; moreover, the following relations hold*:

$$\begin{aligned} L(\chi_{E/F}) &= \prod_{\zeta \in \hat{\Gamma}} L(\chi\zeta), \\ \varepsilon(\chi_{E/F}) &= \prod_{\zeta \in \hat{\Gamma}} \varepsilon(\chi\zeta) \quad \textit{for } F \textit{ global}, \\ \varepsilon(\chi_{E/F}, \psi_{E/F}) &= \lambda_{E/F}(\psi)^{-1} \prod_{\zeta \in \hat{\Gamma}} \varepsilon(\chi\zeta, \psi) \quad \textit{for } F \textit{ local}. \end{aligned}$$

3. Base change for GL(2) on the L-groups.

3.1. Let G be the group GL(2) over F, and $G_{E/F}$ be the group over F defined by restriction of scalars of the group GL(2) over E, so that $G_{E/F}(F) = \mathrm{GL}(2, E)$; the group $G_{E/F}$ is quasi-split, and there is a natural map:

$${}^LG = \mathrm{GL}(2, \boldsymbol{C}) \times \Gamma_F \longrightarrow {}^LG_{E/F} = \mathrm{GL}(2, \boldsymbol{C})^\Gamma \rtimes \Gamma_F;$$

here $\mathrm{GL}(2, \boldsymbol{C})^\Gamma$ is the set of applications from Γ in $\mathrm{GL}(2, \boldsymbol{C})$, and $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ acts by permutations of the coordinates [**B**, §5].

Given any $\rho \in \Phi(G)$, its restriction to W'_E defines an element $\rho_{E/F} \in \Phi(G_{E/F})$; the set $\Phi(G_{E/F})$ can be identified with the set $\Phi(G/E)$ [**B**], and the above map defines the application

$$\begin{aligned} \Phi(G) &\longrightarrow \Phi(G_{E/F}) = \Phi(G/E), \\ \rho &\longmapsto \rho_{E/F} \end{aligned}$$

called the base change.

For F local, let Fr_F be a Frobenius element in Γ_F; when E is unramified, then $\mathrm{Fr}_E = \mathrm{Fr}_F^l \in \Gamma_E$ is a Frobenius element for E. Moreover, if ρ is unramified and defined by $\mathrm{Fr}_F \mapsto s$, where s is a semisimple element in $\mathrm{GL}(2, \boldsymbol{C})$, then $\rho_{E/F}$ is unramified and is given by $\mathrm{Fr}_E \mapsto s^l$.

The properties of L- and ε-factors with respect to induction [T] show that:

$$\begin{aligned} L(\rho_{E/F}) &= \prod_{\zeta \in \hat{\Gamma}} L(\rho \otimes \zeta), \\ \varepsilon(\rho_{E/F}) &= \prod_{\zeta \in \Gamma} \varepsilon(\rho \otimes \zeta), \quad \text{for } F \text{ global}, \\ \varepsilon(\rho_{E/F}, \psi_{E/F}) &= \lambda_{E/F}(\psi)^{-2} \prod_{\zeta \in \Gamma} \varepsilon(\rho \otimes \zeta, \psi), \quad \text{for } F \text{ local}. \end{aligned}$$

3.2. From the classification of the two-dimensional admissible representations of W'_F (see Appendix A), one has the following result:

PROPOSITION 2. (a) *The lifting* $\rho \in \Phi(G) \mapsto \rho_{E/F} \in \Phi(G_{E/F})$ *has for image the set of* Γ*-invariants in* $\Phi(G_{E/F})$.

(b) *In the following cases, the lifting is given by*

$$(\mu \oplus \nu)_{E/F} = \mu_{E/F} \oplus \nu_{E/F},$$
$$(\operatorname{Ind}{}_{W_K}^{W_F} \theta)_{E/F} = \operatorname{Ind}{}_{W_{KE}}^{W_E} \theta_{KE/K} \quad \textit{for } E \neq K,$$
$$(\operatorname{Ind}{}_{W_K}^{W_F} \theta)_{E/F} = \theta \oplus {}^{\sigma}\theta \quad \textit{for } E = K, 1 \neq \sigma \in \Gamma,$$

$(\chi \otimes \mathrm{sp}(2))_{E/F} = \chi_{E/F} \otimes \mathrm{sp}(2)$ (*for F nonarchimedean*).

(c) *Given a nondecomposable* $\rho \in \Phi(G)$, *the representations which have the same lifting are the* $\rho \otimes \zeta$ *for all* $\zeta \in \hat{\Gamma}$; *for* $\rho = \lambda \oplus \mu$, *the representations which have the same lifting are the* $\lambda\zeta \oplus \mu\zeta'$ *for all* $\zeta, \zeta' \in \hat{\Gamma}$.

4. Base change over a local field. In this section, we denote by σ a generator of $\Gamma = \mathrm{Gal}(E/F)$.

4.1. Let $\Pi(G)$ be the set of classes of admissible irreducible representations of $G(F)$. There is a conjectural bijection $\Phi(G) \rightleftarrows \Pi(G)$ **[B]**. The base change map $\Phi(G) \to \Phi(G_{E/F})$ must reflect a map $\Pi(G) \to \Pi(G_{E/F})$. The definition of base change for representations of $G(F)$ will be given in 4.3; since the image of $\Phi(G)$ is the set of Γ-invariant elements in $\Phi(G_{E/F})$, one studies first the admissible irreducible representations of $G(E)$ equivalent to their conjugates by Γ.

4.2. Let $\tilde{\pi}$ be an admissible irreducible representation of $G(E)$ such that ${}^{\sigma}\tilde{\pi} \simeq \tilde{\pi}$; then there exists an operator C on the space of $\tilde{\pi}$ such that $C^{-1}\tilde{\pi}(z)C = \tilde{\pi}(z^{\sigma})$, $z \in G(E)$, and $C^{l} = \mathrm{Id}$. This operator is determined up to an lth root of unity. The mapping $\tilde{\pi}'\colon (\sigma^m, z) \mapsto C^m\tilde{\pi}(z)$ defines an extension $\tilde{\pi}'$ of $\tilde{\pi}$ to the semidirect product $\Gamma \ltimes G(E)$.

PROPOSITION 3. *This representation has a character given by a locally integrable function* $\mathrm{Tr}\,\tilde{\pi}'$ *on* $\Gamma \ltimes G(E)$.

On $\sigma \times G(E)$, the character $\mathrm{Tr}\,\tilde{\pi}'$ defines a σ-invariant distribution on $G(E)$, i.e., invariant under σ-conjugation: $z \mapsto y^{-\sigma}zy$, $y, z \in G(E)$.

Let us state some properties of the σ-conjugation.

For $z \in G(E)$, put

$$N_{E/F,\sigma}z = z^{\sigma^{l-1}} \cdots z^{\sigma} \cdot z;$$

or simply $N(z)$ if no confusion can arise.

PROPOSITION 4. (a) $N_{E/F,\sigma}z$ *is conjugate in* $G(E)$ *to an element of* $G(F)$;

(b) $z \mapsto N_{E/F,\sigma}z$ *defines an injection of the set of* σ*-conjugacy classes of* $G(E)$ *into the set of conjugacy classes of* $G(F)$;

(c) *the elliptic classes of* $G(F)$ *obtained by* $N_{E/F,\sigma}$ *are those with determinant in* $N_{E/F}E^{\times}$; *the hyperbolic classes of* $G(F)$ *obtained are those whose eigenvalues are norms of* $E^{\times}$; *any unipotent class of* $G(F)$ *is in the image of* N.

4.3. *Definition of the base change for* GL(2) *over a local field.* Let $\pi \in \Pi(G)$ and $\tilde{\pi}$

be an irreducible admissible representation of $G(E)$ which is equivalent to its conjugate by σ. Then $\tilde{\pi}$ is called a base change lift of π, or a lifting of π, if either

(a) $\pi = \pi(\mu, \nu)$ and $\tilde{\pi} = \pi(\mu_{E/F}, \nu_{E/F})$, or

(b) there exists an extension $\tilde{\pi}'$ of $\tilde{\pi}$ to $\Gamma \ltimes G(E)$ such that $\operatorname{Tr} \tilde{\pi}'(\sigma \times z) = \operatorname{Tr} \pi(x)$ for $z \in G(E)$ whenever $N_{E/F,\sigma}\, z$ is conjugate in $G(E)$ to a regular semisimple element $x \in G(F)$.

Some of the notation in the following theorem is explained in Appendix B.

THEOREM 1 (BASE CHANGE FOR GL(2) OVER A LOCAL FIELD). (a) *Any* $\pi \in \Pi(G)$ *has a unique lifting* $\pi_{E/F} \in \Pi(G_{E/F})$, *and any* $\pi \in \Pi(G)$ *fixed by* Γ *is a lifting*;

(b) *the lifting is independent of the choice of the generator* σ *of* Γ;

(c) $\pi_{E/F} = \pi'_{E/F} \Leftrightarrow \pi' = \pi \otimes \zeta$ *for a* $\zeta \in \hat{\Gamma}$, *or* $\pi = \pi(\mu, \nu)$, $\pi' = \pi(\mu', \nu')$ *with* $\mu^{-1}\mu'$ *and* $\nu^{-1}\nu'$ *in* $\hat{\Gamma}$;

(d) $\omega_{\pi_{E/F}} = (\omega_\pi)_{E/F}$, $(\pi \otimes \chi)_{E/F} = \pi_{E/F} \otimes \chi_{E/F}$ *for any one-dimensional representation* χ *of* $F^\times$, $(\pi_{E/F})^\vee = (\pi^\vee)_{E/F}$ *(contragredient representations)*;

(e) *for* $E \supset F \supset k$ *with* E *and* F *Galois over* k, ${}^{\tilde{\gamma}}(\pi_{E/F}) = ({}^{\gamma}\pi)_{E/F}$ *for any* $\tilde{\gamma} \in \operatorname{Gal}(E/k)$, *with image* γ *in* $\operatorname{Gal}(F/k)$;

(f) *at least for* $\rho \in \Phi(G)$ *not exceptional*, $\pi(\rho)_{E/F} = \pi(\rho_{E/F})$.

5. Global base change.

5.1. Let $\Pi(G)$ be the set of classes of irreducible admissible automorphic representations of $G(A_F) = \mathrm{GL}(2, A_F)$, where A_F is the ring of adeles of the number field F. From the principle of functoriality **[B]**, the base change on L-groups should reflect a map from $\Pi(G)$ to $\Pi(G_{E/F})$, the set of irreducible admissible automorphic representations of $G(A_E) = \mathrm{GL}(2, A_E) = G_{E/F}(A_F)$; such a map must be compatible with the local data.

For any place v of F, put $E_v = E \otimes_F F_v$; it is a cyclic Galois extension of F or a product of l copies of F; in this latter case, define the lifting π_{E_v/F_v} of $\pi \in \Pi(G)$ by

$$\pi_{E_v/F_v} = \pi \otimes \cdots \otimes \pi \quad (l \text{ times}).$$

5.2. *Definition of the global base change for* GL(2). Let $\pi \in \Pi(G)$, $\tilde{\pi} \in \Pi(G_{E/F})$; then $\tilde{\pi}$ is called a lifting of π (or more precisely a base change lift of π) if, for every place v of F, $\tilde{\pi}_v$ is the lifting of π_v.

5.3. The notations used in the following theorem are those of Appendix C.

THEOREM 2 (GLOBAL BASE CHANGE FOR GL(2)). (a) *Every* $\pi \in \Pi(G)$ *has a unique lifting* $\pi_{E/F} \in \Pi(G_{E/F})$;

(b) *a cuspidal* $\tilde{\pi} \in \Pi(G_{E/F})$ *is a lifting if and only if it is fixed by* Γ, *and then, it is a lifting of cuspidal representations*; *a cuspidal* $\pi \in \Pi(G)$ *has a lifting which is cuspidal except for* $l = 2$ *and* $\pi = \pi(\operatorname{Ind}_{W_E}^{W_F} \theta)$ *and then* $\pi_{E/F} = \pi(\theta, {}^{\sigma}\theta)$;

(c) *for a cuspidal* $\pi \in \Pi(G)$, *the representations* π', *which have* $\pi_{E/F}$ *for lifting are the* $\pi' = \pi \otimes \zeta$ *with* $\zeta \in \Gamma$;

(d)
$$\begin{aligned} \omega_{\pi_{E/F}} &= (\omega_\pi)_{E/F} && \textit{(central quasi-characters)},\\ (\pi \otimes \chi)_{E/F} &= \pi_{E/F} \otimes \chi_{E/F} && \textit{(twisting by a quasi-character)},\\ (\pi_{E/F})^\vee &= (\check{\pi})_{E/F} && \textit{(contragredient representations)}; \end{aligned}$$

(e) *for* $E \supset F \supset k$ *with* E *and* F *Galois over* k, ${}^{\tilde{\gamma}}(\pi_{E/F}) = ({}^{\gamma}\pi)_{E/F}$ *for any* $\gamma \in \operatorname{Gal}(F/k)$ *image of* $\tilde{\gamma} \in \operatorname{Gal}(E/k)$;

(f) *if* $\pi = \pi(\rho)$ *for some* $\rho \in \phi(G)$, *then* $\pi_{E/F} = \pi(\rho_{E/F})$.

REMARK. There are examples of noncuspidal $\tilde{\pi} \in \Pi(G_{E/F})$, fixed by Γ which are not liftings (cf. [L, §10]).

6. Artin conjecture for tetrahedral type. Let ρ be a two-dimensional admissible representation of the Weil group of the number field F; we assume that its image modulo the center: $W_F \to \mathrm{GL}(2, C) \to \mathrm{PGL}(2, C)$ is the tetrahedral group $\mathfrak{A}_4$. This group is solvable: $1 \to D_4 \to \mathfrak{A}_4 \to C_3 \to 1$. The action of the cyclic group C_3 on the dihedral group D_4—the so-called mattress group—is given by the cyclic permutations of its nontrivial elements. The inverse image of D_4 in W_F is a normal subgroup of index 3, hence is the Weil group W_E of a cubic Galois extension E of F:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W_E & \longrightarrow & W_F & \longrightarrow & \mathrm{Gal}(E/F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \wr & & \\ 1 & \longrightarrow & D_4 & \longrightarrow & \mathfrak{A}_4 & \longrightarrow & C_3 & \longrightarrow & 1 \end{array}$$

The restriction $\rho_{E/F}$ of ρ to W_E has for image the dihedral group D_4; it is induced from a one-dimensional representation of a subgroup of index 2 in W_E, so that there is a corresponding cuspidal automorphic representation $\pi(\rho_{E/F})$ of $\mathrm{GL}(2, A_E)$.

The inner automorphisms of $\mathfrak{A}_4$ give an action of C_3 on D_4, and the action of $\Gamma = \mathrm{Gal}(E/F)$ fixes the class of $\rho_{E/F}$, hence also the class of $\pi(\rho_{E/F})$. From Theorem 2 (5.3), this representation is the base change lift of exactly three classes of irreducible cuspidal automorphic representations of $\mathrm{GL}(2, A_F)$, and their central character has for base change lift the central character of $\pi(\rho_{E/F})$, which is equal to $\det \rho_{E/F} = (\det \rho)_{E/F}$; there is only one of them, say π, with, the central character $\det \rho$.

The Artin conjecture for the representation ρ is the holomorphy of the corresponding L-function: $s \mapsto L(s, \rho)$. According to Jacquet-Langlands [J-L, Theorem 11.1, p. 350], the L-function $s \mapsto L(s, \pi)$ corresponding to cuspidal π is holomorphic ; hence, the Artin conjecture will be proved for ρ if we show the equality of these two L-functions. From [J-L, pp. 404–407], this will be done if the following assertion is shown to be true:

(c_0) $\pi_v = \pi(\rho_v)$ for each archimedean place v of F, and for almost all v.

As E is cubic over F, each infinite place v of F splits in E, so for $w|v$, we have the equations:

$$E_w = F_v, \quad (\rho_{E/F})_w = \rho_v, \quad (\pi(\rho_{E/F}))_w = \pi(\rho_v) = \pi_v,$$

which are more generally true for any place v of F which splits in E.

Now if v does not split and is unramified in E, and if moreover ρ_v is unramified, then so is the restriction ρ_{E_v/F_v} of ρ_v to W_{E_v}, and the representation $(\pi_{E/F})_v = \pi(\rho_{E_v/F_v})$ is unramified; since E_v/F_v is unramified this representation is the base change lift of unramified representations. This shows that π_v is unramified; call ρ_{π_v} a two-dimensional representation of W_{E_v}, such that $\pi_v = \pi(\rho_{\pi_v})$. We have shown that our assertion (c_0) is equivalent to:

(c_1) if v does not split and is unramified for ρ and E, ρ_v and ρ_{π_v} are equivalent.

The adjoint representation of PGL(2) defines an injection of PGL(2) in GL(3), hence a morphism $A\colon \mathrm{GL}(2) \to \mathrm{GL}(3)$. We observe now that the condition (c_1) is equivalent to the apparently weaker condition:

(c_2) if v is unramified for ρ and E, the three-dimensional representations $A\rho_v$ and $A\rho_{\pi_v}$ are equivalent.

In fact, call a (resp. b) the image of a Frobenius in W_{F_v} through ρ_v (resp. ρ_{π_v}); if (c_2) is satisfied, $a \in \boldsymbol{C}^\times b$; but $\det a = \det b$, hence $a = \pm b$. If $a = -b$ then, since ρ_v and ρ_{π_v} have the same restriction to W_{E_v}, a^3 is conjugate to $-a^3$, that is $\mathrm{Tr}(a^3) = 0$. Hence $A(a^3)$ is of order two, and this means that $A(a)$ is of order 6; but the image of $A\rho_v$ is in the tetrahedral group which has no element of order 6; so we have $a = b$, hence $\rho_v = \rho_{\pi_v}$. The introduction of $A\rho$ is motivated by the crucial observation, due to Serre, that this three-dimensional representation is induced by a one-dimensional representation of W_E; in fact, the tetrahedral group leaves invariant the set of the three lines joining the middles of the opposite edges of the tetrahedron. This means that $A\rho$ is induced by the one-dimensional representation θ of the stabilizer of one of these lines (obtained by restriction of $A\rho$); but this stabilizer is the pull-back of the dihedral group $D_4 \subset \mathfrak{A}_4$ in W_F, which is the subgroup W_E:

$$A\rho = \mathrm{Ind}\,_{W_E}^{W_F}\, \theta.$$

From **[J-PS-S]**, to such a three-dimensional irreducible monomial representation $A\rho$ of W_F is associated an irreducible cuspidal automorphic representation $\pi(A\rho)$ of $\mathrm{GL}(3, \boldsymbol{A}_F)$. On the other hand, the morphism A reflects a lifting from irreducible cuspidal automorphic representations of $\mathrm{GL}(2, \boldsymbol{A}_F)$ to automorphic representations of $\mathrm{GL}(3, \boldsymbol{A}_F)$; and, here, the representation $A\pi$ corresponding to π is cuspidal **[G-J-2]**. To prove (c_2) it suffices to show the condition:

(c_3) the lifting $A\pi$ is equivalent to $\pi(A\rho)$.

There is a practical criterion given by **[J-S]** to prove the equivalence of such representations: π_1 and π_2 are equivalent if and only if $L(s, \pi_1 \times \check{\pi}_2)$ has a pole at $s = 1$, where L is the L-function attached to the representation of $\mathrm{GL}_3(\boldsymbol{C}) \times \mathrm{GL}_3(\boldsymbol{C})$ in $\mathrm{GL}_9(\boldsymbol{C})$ given by the tensor product.

We shall prove that almost all local factors of $L(s, A\pi \times \pi(A\rho)^\vee)$ and of $L(s, \pi(A\rho) \times \pi(A\rho)^\vee)$ are equal; by nonvanishing properties of local factors **[J-S]**, this is enough to prove that $L(s, A\pi \times \pi(A\rho)^\vee)$ has a pole at $s = 1$, and hence (c_3) will be proved. If v is split, then $\pi_v = \pi(\rho_v)$ and, at least when π_v and ρ_v are unramified, $(A\pi)_v = \pi(A\rho_v)$: the local L-factors are then equal.

If v does not split in E and is unramified for E and ρ, the two local L-functions are those associated to the nine-dimensional representation of W_F given by $A\rho_{\pi_v} \otimes ((A\rho)_v^\vee)$ and $A\rho_v \otimes ((A\rho)_v^\vee)$; but we know that $A\rho_v = \mathrm{Ind}\,_{W_{E_v}}^{W_{F_v}}\, \theta_v$.

Now if U (resp. V) are representations of a group G (resp. a subgroup H) one has $U \otimes \mathrm{Ind}_H^G\, V \cong \mathrm{Ind}\,_H^G(V \otimes \mathrm{Res}\,_H^G U)$; also recall that the two representations ρ_{π_v} and ρ_v have equivalent restrictions to W_{E_v}; hence $A\rho_{\pi_v} \otimes (A\rho)_v^\vee$ is equivalent to $(A\rho_v) \otimes (A\rho)_v^\vee$ so that they have the same L-factor.

This concludes the proof of the Artin conjecture for the tetrahedral case.

The above proof is taken from a letter of Langlands to Serre (December 1975); it also contains some indications on a method to handle the octahedral case; however the latter requires some results on group representations which are not yet available. Still, a partial result is obtained **[L, §1]**:

Assume that ρ is of octahedral type; we use the fact that $\mathfrak{S}_4$ has a normal subgroup $\mathfrak{A}_4$ of index 2; hence there is a quadratic extension E of F for which the restriction $\rho_{E/F}$ is of type $\mathfrak{A}_4$. By the above theorem, $\pi(\rho_{E/F})$ exists and, by Theorem 2, it is the base change lift of two cuspidal admissible irreducible automorphic representations π' and π'' of $G(\boldsymbol{A}_F)$. Assume now that $F = \boldsymbol{Q}$, that E is totally real and that the complex conjugations in $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ are sent by ρ into the class of $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$; in such a case the components of π' and π'' at the real place verify $\pi'_\infty = \pi''_\infty = \pi(\rho_\infty)$ with $\rho_\infty = 1 \oplus \mathrm{sign}$; then π' and π'' correspond to holomorphic automorphic forms of weight one.

This situation has been studied by Deligne-Serre **[D-S]**, and they show that $\pi' = \pi(\rho')$ and $\pi'' = \pi(\rho'')$ for some representations ρ', ρ'' of W_F in $\mathrm{GL}(2, \boldsymbol{C})$; now, by Theorem 2, $\pi(\rho')_{E/F} = \pi(\rho'_{E/F})$, and the same is true for π''; this shows that ρ' and ρ'' are the two representations which lift to $\rho_{E/F}$; hence either $\rho = \rho'$ or ρ''. Thus one concludes that either $\pi' = \pi(\rho)$ or $\pi'' = \pi(\rho)$, and this gives

THEOREM 4. *For a two-dimensional representation of $W_{\boldsymbol{Q}}$ which is of octahedral type and which sends the complex conjugation on $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, and such that the above quadratic field E is real, the Artin conjecture is satisfied.*

Appendix A. List of the two-dimensional admissible representations of W'_F [D].

Notations. F is a local (resp. global) field, C_F is $F^\times$ (resp. the group of ideles classes of F); W'_F is the Weil-Deligne group of F **[T]**.

For F global, v a place of F, there is an injection $W_{F_v} \to W_F$ which defines an application $\rho \mapsto \rho_v$ from the two-dimensional admissible representations of W_F into those of W_{F_v}; if another representation ρ' of W_F satisfies $\rho'_v \sim \rho_v$ for all but a finite number of places, then ρ' is equivalent to ρ, and $\rho'_v \sim \rho_v$ for all v. The two-dimensional admissible representations of W_F are classified by the image of the inertia group in $\mathrm{PGL}(2, \boldsymbol{C})$, called the type of the representation.

(1) Cyclic type: $\mu \oplus \nu$ is the sum of the two one-dimensional representations of W_F defined by μ and ν; $\mu \oplus \nu \simeq \nu \oplus \mu$; $\det(\mu \oplus \nu) = \mu\nu$; $(\mu \oplus \nu) \otimes \chi = (\mu\chi) \oplus (\nu\chi)$; $(\mu \oplus \nu)^\vee = \mu^{-1} \oplus \nu^{-1}$. The L- and ε-functions verify:

$$L(\mu \oplus \nu) = L(\mu)L(\nu), \qquad \varepsilon(\mu \oplus \nu) = \varepsilon(\mu)\varepsilon(\nu) \quad (\text{if } F \text{ is global}),$$
$$\varepsilon(\mu \oplus \nu, \psi) = \varepsilon(\mu, \psi)\varepsilon(\nu, \psi) \quad (\text{if } F \text{ is local}),$$
$$\mu \oplus \nu = \otimes_v (\mu_v \oplus \nu_v) \quad (\text{if } F \text{ is global}).$$

(2) Dihedral type: $\tau = \mathrm{Ind}_{W_K}^{W_F} \theta$, where θ is a quasi-character of C_K, and K a separable quadratic extension of F; τ is irreducible if and only if $\theta \neq {}^\sigma\theta$ $(1 \neq \sigma \in \mathrm{Gal}(K/F))$. For $\theta = {}^\sigma\theta$, let χ be a quasi-character of C_F such that $\theta = \chi \circ N_{K/F}$ and let $\hat{\sigma}$ be the character of C_F with Kernel $N_{K/F}C_K$. Then $\mathrm{Ind}_{W_K}^{W_F} \theta = \chi \oplus \chi\hat{\sigma}$; $\det \tau = \hat{\sigma} \cdot \theta|_{C_F}$; $\tau \otimes \chi = \mathrm{Ind}_{W_K}^{W_F} (\theta\chi \circ N_{K/F})$; $\check{\tau} = \mathrm{Ind}_{W_K}^{W_F} \theta^{-1}$; $L(\tau) = L(\theta)$, $\varepsilon(\tau) = \varepsilon(\theta)$ (F global), $\varepsilon(\tau, \psi) = \lambda_{K/F}(\psi)\varepsilon(\theta, \psi \circ \mathrm{Tr}_{K/F})$ (F local); for F global, $\tau = \otimes \tau_v$ with $\tau_v = \mathrm{Ind}_{W_{K_v}}^{W_{F_v}} \theta_v$ for K_v/F_v quadratic, and $\tau_v = \theta_v \oplus \theta_v$ for $K_v = F_v \times F_v$. The equivalences are: $\mathrm{Ind}_{W_1}^{W_1} \theta_1 \sim \mathrm{Ind}_{W_K}^{W_F} \theta_2 \Leftrightarrow$ either $K_1 = K_2$ and θ_1, θ_2 conjugate by $\mathrm{Gal}(K/F)$, or $K_1 \neq K_2$, $\theta_1^{\sigma_1}\theta_1^{-1}$ and $\theta_2^{\sigma_2}\theta_2^{-1}$ are of order 2 and $\theta_1 \circ N_{K_1K_2/K_1} = \theta_2 \circ N_{K_1K_2/K_2}$.

(3) Exceptional type: the image of the inertia group in $\mathrm{PGL}(2, \boldsymbol{C})$ is $\mathfrak{A}_4$ (tetrahedral type), $\mathfrak{S}_4$ (octahedral type), or $\mathfrak{A}_5$ (icosahedral type); they occur only for F

global or F nonarchimedean local of even residual characteristic; in this latter case, the icosahedral type does not occur.

(4) Special type (occurs only for F nonarchimedean local): $\chi \otimes \mathrm{sp}(2)$ for a quasi-character χ of $F^\times$ and sp(2) the representation of W'_F defined by

$$z \in \boldsymbol{C} \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad w \in W_F \mapsto \begin{pmatrix} |w| & 0 \\ 0 & 1 \end{pmatrix}.$$

Appendix B. List of admissible irreducible representations of GL(2, F), F local field [J-L].

Notations. $G = \mathrm{GL}(2, F)$, $|\ |$ is the absolute value defined by the dilatation of the Haar measure on F: $d(ax) = |a|\, dx$, and ψ is a nontrivial character of the additive group of F.

Representations. (1) Principal series $\rho(\mu, \nu)$, where μ, ν are quasi-characters of $F^\times$.

(a) *Definition.* Let $\rho(\mu, \nu)$ be the representation of G by right translations in the space of smooth functions for G such that

$$f(ang) = \mu(u)\nu(v)\, |uv^{-1}|^{1/2} f(g)$$

for any $g \in G$, $a = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) \in G$, $n \in \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. When this representation is irreducible, then $\pi(\mu, \nu)$ is $\rho(\mu, \nu)$. When $\rho(\mu, \nu)$ is reducible, there are exactly two irreducible subquotients; one is finite dimensional and $\pi(\mu, \nu)$ is this one. The other one is denoted $\sigma(\mu, \nu)$.

(b) *Equivalences.* $\pi(\mu, \nu) \sim \pi(\nu, \mu)$. For $F = \boldsymbol{C}$, any irreducible admissible representation of G is equivalent to a $\pi(\mu, \nu)$.

(c) *Finite dimensional representations.* F nonarchimedean: they are one-dimensional, and are the $\pi(\mu, \nu)$ for $\mu\nu^{-1} = |\ |^{\pm 1}$: $\pi(\mu, \nu)\,(x) = |\det x|^{\pm 1/2}\nu\,(\det x)$, $x \in G$: the corresponding representations $\sigma(\mu, \nu)$ are called the special representations;

$F = \boldsymbol{R}$: They are the $\pi(\mu, \nu)$ with $\mu\nu^{-1}(a) = |a|^{\pm(n+1)} \cdot \mathrm{sign}(a)^n$ for integers $n \geqq 0$;

$F = \boldsymbol{C}$: they are the $\pi(\mu, \nu)$ with $\mu\nu^{-1}(a) = [a^{n+1}(\bar{a})^{m+1}]^{\pm 1}$ for integers $n \geqq 0$, $m \geqq 0$.

(d) *Other properties.*

Restriction to the center: $\omega_{\pi(\mu,\nu)} = \mu\nu$;

twisting by a quasi-character of $F^\times$: $\pi(\mu, \nu) \otimes \chi = \pi(\mu\chi, \nu\chi)$;

contragredient representation: $\pi(\mu, \nu)^\vee \sim \pi(\mu^{-1}, \nu^{-1})$;

local factors: $L(\pi(\mu, \nu)) = L(\mu)L(\nu)$, $\varepsilon(\pi(\mu, \nu), \psi) = \varepsilon(\mu, \psi)\varepsilon(\nu, \psi)$.

(2) *Weil representations.* $\pi(\tau)$, $\tau = \mathrm{Ind}_{W_K}^{W_F} \theta$, with θ a quasi-character of $K^\times$, and K a separable quadratic extension of F.

(a) *Definition.* Let G_K be the subgroup of index two in G defined by those elements which have a norm of $K^\times$ for determinant. Fix a nontrivial character ψ of the additive group of F. Then $\pi(\tau)$ is the class of the representation of G induced by the following representation $r(\theta, \psi)$ of G_K, in the space of smooth functions f on $K^\times$ such that $f(t^{-1} x) = \theta(t)f(x)$ for $t, x \in K^\times, N_{K/F}\ t = 1$:

$$\left(r(\theta, \psi)\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f\right)(x) = \psi(uN_{K/F}x) f(x), \quad u \in F,$$

$$\left(r(\theta, \psi)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\right)(x) = \lambda_{K/F}(\psi) \int_{K^\times} f(y)\psi_{K/F}(xy^\sigma)d_\psi y,$$

$$\left(r(\theta, \psi)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} f\right)(x) = \theta(b) f(bx) \quad \text{for } a = N_{K/F} b, b \in K^\times,$$

where $\psi_{K/F} = \psi \circ \mathrm{Tr}_{K/F}$, y^σ is the conjugate of y by the nontrivial element σ of $\mathrm{Gal}(K/F)$, $d_\psi y$ is the self-dual Haar measure on K with respect to the character $\psi_{K/F}$ and $\lambda_{K/F}(\psi)$ is the unitary part of the local factor $\varepsilon(\hat{\sigma}, \psi)$, where $\hat{\sigma}$ is the nontrivial character of $F^\times$ which is trivial on $N_{K/F}K^\times$.

(b) *Equivalences.*

(1) $\pi(\mathrm{Ind}_{W_K}^{W_F} \theta) = \pi(\mathrm{Ind}_{W_K}^{W_F} {}^\sigma\theta)$;

(2) $\pi(\mathrm{Ind}_{W_K}^{W_F} \theta) \sim \pi(\chi, \chi^{\hat{\sigma}})$ if $\theta = {}^\sigma\theta$ and χ is a quasi-character of $F^\times$ such that $\theta(a) = \chi(N_{K/F}\, a)$;

(3) Other equivalences: $\pi(\mathrm{Ind}_{W_{K_1}}^{W_F} \theta_1) = \pi(\mathrm{Ind}_{W_{K_2}}^{W_F} \theta_2)$ for $K_1 \neq K_2$, if and only if $\theta_1{}^{\sigma_1}\theta_1^{-1}$ and $\theta_2{}^{\sigma_2}\theta_2^{-1}$ are of order 2 and satisfy $\theta_1 \circ N_{K_1K_2/K_1} = \theta_2 \circ N_{K_1K_2/K_2}$.

(c) *Characterization.* If a nontrivial character χ of $F^\times$ fixes a class π of irreducible admissible representations of $G: \pi \otimes \chi = \pi$, then χ is of order 2, attached to a separable quadratic extension K of F and $\pi = \pi(\tau)$ where $\tau = \mathrm{Ind}_{W_K}^{W_F} \theta$ for a suitable θ, and conversely (cf. **[L**, §5]).

(d) *Other properties.* For $F = \boldsymbol{R}$, or nonarchimedean with odd residual characteristic, any irreducible admissible representation is a $\pi(\lambda, \mu)$ of a $\pi(\mathrm{Ind}_{W_K}^{W_F} \theta)$;
restriction to the center: $\omega_{\pi(\tau)} = \hat{\sigma} \cdot \theta|_{F^\times}$;
twisting by a quasi-character of $F^\times$: $\pi(\tau) \otimes \chi = \pi(\mathrm{Ind}_{W_K}^{W_F} \theta \,.\, \chi \circ N_{K/F})$,
contragredient representation: $\pi(\check{\tau}) = \pi(\mathrm{Ind}_{W_K}^{W_F} \theta^{-1})$,
local factors: $L(\pi(\tau)) = L(\theta)$, $\varepsilon(\pi(\tau), \psi) = \lambda_{K/F}(\psi)\varepsilon(\theta, \psi_{K/F})$.

(3) *Exceptional representations.* They occur only for F nonarchimedean of residual characteristic 2, and, up to twisting by quasi-characters of $F^\times$, their number is $4(|2|^{-2} - 1)/3$ for F of characteristic 0, infinite for F of characteristic 2. They are supercuspidal (see complements below) (cf. **[Tu]**).

(4) *Special representations.* They occur only for F nonarchimedean, and are the infinite dimensional subquotient $\sigma(\mu, \nu)$ of the reducible $\rho(\mu, \nu)$, that is for $\mu\nu^{-1} = |\ |^{\pm 1}$; one has $\sigma(\mu, \nu) \sim \sigma(\nu, \mu)$, $\sigma(\mu, \nu) = \chi\sigma(|\ |^{1/2}, |\ |^{-1/2})$ for $\chi = \mu|\ |^{-1/2}$.

Complements.

(1) Representations $\sigma(\mu, \nu)$. When the induced representation $\rho(\mu, \nu)$ is not irreducible, $\sigma(\mu, \nu)$ denotes any representation equivalent to the unique infinite dimensional subquotient of $\rho(\mu, \nu)$: for $F = \boldsymbol{R}$, the representations $\sigma(\mu, \nu)$ are the representations $\pi(\mathrm{Ind}_{C^\times}^{W_F} \theta)$, $\theta \neq {}^\sigma\theta$.

(2) For a two-dimensional admissible representation ρ of W_F, there is at most one irreducible admissible representation π of G often denoted $\pi(\rho)$ when it exists such that $\omega_\pi = \det \rho$, $L(\pi \otimes \chi) = L(\rho \otimes \chi)$ and $\varepsilon(\pi \otimes \chi, \psi) = \varepsilon(\rho \otimes \chi, \psi)$ for any quasi-character χ of $F^\times$; for ρ reducible, $\rho = \mu \oplus \nu$, then $\pi = \pi(\mu, \nu)$; for $\rho = \chi \otimes \mathrm{sp}(2)$ then $\pi = \sigma(\mu, \nu)$ with $\mu = \chi|\ |^{1/2}$, $\nu = \chi|\ |^{-1/2}$; for ρ dihedral $\rho = \mathrm{Ind}_{W_K}^{W_F} \theta$, then $\pi = \pi(\mathrm{Ind}_{W_K}^{W_F} \theta)$; in the remaining cases, that is when ρ is exceptional, the existence of π is still not completly settled, but base change techniques were used to prove it in many instances. Conversely, any π should be a $\pi(\rho)$.

Appendix C. Irreducible admissible automorphic representations of GL(2) [J-L].

Notations. F is a global field, A_F its ring of adeles, $C_F = A_F^\times/F^\times$ the group of ideles classes, $|\ |$ the absolute value on A_F.

(1) *Noncuspidal representations.*

(1.1) $\pi(\lambda, \mu)$ for λ, μ Grössencharakters of F: they are the following representations: $\pi(\lambda, \mu) = \bigotimes_v \pi(\lambda_v, \mu_v)$; the one-dimensional representations are the $\pi(\lambda, \mu)$ for $\lambda\mu^{-1} = |\ |^{\pm 1}$.

(1.2) Any noncuspidal irreducible admissible automorphic representation π of $\mathrm{GL}(2, A_F)$ has the following form: there are two Grössencharakters λ and μ of F, and a finite set S of places of F, such that the components π_v of π are given by

$$\pi_v = \pi(\lambda_v, \mu_v), \quad v \notin S, \qquad \pi_v = \sigma(\lambda_v, \mu_v), \quad v \in S,$$

where $\sigma(\lambda_v, \mu_v)$ denotes the infinite dimensional subquotient of the reducible $\rho(\lambda_v, \mu_v)$ (Appendix B).

(2) *Cuspidal representations* (*examples*).

(2.1) $\pi(\tau)$ with $\tau = \mathrm{Ind}_{W_K}^{W_F} \theta$ for a separable quadratic extension K of F and a Grössencharakter θ of K, not fixed under $\mathrm{Gal}(K/F)$, is the representation

$$\pi(\tau) = \bigotimes_v \pi(\mathrm{Ind}_{W_{K_v}}^{W_{F_v}} \theta_v)$$

with $\mathrm{Ind}_{W_{K_v}}^{W_{F_v}} \theta_v = \theta_v \oplus \theta_v$ for $K_v = F_v \times F_v$.

Properties. (1) Let $\pi \in \Pi(G)$; in order that there exist a nontrivial Grössencharakter χ of F such that $\pi \otimes \chi = \pi$, it is necessary and sufficient that there exist a separable quadratic extension E of F and a Grössencharakter θ of E such that

(a) χ is the character of C_F with Kernel $N_{E/F}C_E$;

(b) $\pi = \pi(\mathrm{Ind}_{W_E}^{W_F} \theta)$ (in particular $\pi = \pi(\tau, \tau\chi)$ if $\theta = {}^\sigma\theta$, where τ is a Grössencharakter of F which has θ for lifting to E);

(2) $\pi(\mathrm{Ind}_{W_{K_1}}^{W_F} \theta_1) = \pi(\mathrm{Ind}_{W_{K_2}}^{W_F} \theta_2) \Leftrightarrow$ either $K_1 = K_2$ and $\theta_2 = \theta_1$ or ${}^\sigma\theta_1$, or $K_1 \neq K_2$ then $\theta_1 \cdot {}^{\sigma_1}\theta_1^{-1}$ and $\theta_2 \cdot {}^\sigma\theta_2^{-1}$ are of order 2, and $(\theta_1) \circ N_{K_1K_2/K_1} = (\theta_2) \circ N_{K_1K_2/K_2}$.

(3.1) More generally let ρ be a two-dimensional admissible representation of W_F; we say that $\pi = \bigotimes \pi_v$ is $\pi(\rho)$ if $\pi_v \simeq \pi(\rho_v)$ for all v. The existence of such π when ρ is irreducible is related to the Artin conjecture for the $\rho \otimes \chi$, where χ is any quasi-character of C_F [J-L, §12].

(3.2) Of course there are many other, more complicated, types of cuspidal representations: think of the classical Δ for example.

II. Base change for GL_2, a sketch of the proof.

1. The trace formula. In all the following we shall use notations close to those of **[G-J-1]** in these Proceedings.

Let F be a number field, E a cyclic extension of prime degree, put $Z_1 = N_{E/F}\, Z(A_E)$, where A_E denotes the ring of adeles of E, and $Z_1(F) = Z_1 \cap Z(F)$; as usual Z is the center of $\mathrm{GL}_2 = G$ and is identified with the multiplicative group. Since E/F is cyclic one has $Z_1(F) = N_{E/F}Z(E)$.

Choose a character ω of $Z_1/Z_1(F)$ and consider the space $L^2(Z_1 \cdot G(F)\backslash G(A), \omega) = L^2$ of functions on $G(F)\backslash G(A)$, which transform on Z_1 according to ω:

$$\varphi(z\gamma g) = \omega(z)\varphi(g), \qquad z \in Z_1,\ \gamma \in G(F),$$

and square-integrable on $Z_1 \cdot G(F)\backslash G(A)$.

In such a situation, which is slightly more general than the one studied in **[G-J-1]** (where $E = F$), one defines in an obvious way the spaces L_0^2 and L_{sp}^2. The restriction of the natural representation of $G(A)$ in L^2 to $L_0^2 \oplus L_{sp}^2$ will be denoted by r. If $f \in \mathscr{C}_c^\infty(Z_1 \backslash G(A), \omega^{-1})$, the space of smooth functions on $G(A)$ compactly supported modulo Z_1 which transform according to ω^{-1} on Z_1, the operator $r(f)$ is of trace class. The Haar measures being chosen as in **[G-J-1, §§6–7]** we assume moreover that $\mathrm{vol}(Z_1 \cdot Z(F) \backslash Z(A)) = l$. Then $\mathrm{tr}\, r(f)$ is the sum of the expressions (i)—(vii) below (we assume that f is a tensor product of local functions f_v) (cf. **[L, §8]**).

(i) $$\sum_{z \in Z_1(F) \backslash Z(F)} \mathrm{vol}(Z_1 \cdot G(F) \backslash G(A)) \cdot f(z),$$

(ii) $$\sum_{\gamma \in \mathscr{E}} \varepsilon(\gamma)\, \mathrm{vol}(Z_1 \cdot G_\gamma(F) \backslash G_\gamma(A)) \int_{G_\gamma(A) \backslash G(A)} f(g^{-1} \gamma\, g)\, dg$$

where $\mathscr{E}$ is a set of representatives of the conjugacy classes of elliptic elements (i.e., whose eigenvalues are not in F) taken modulo $Z_1(F)$, and $\varepsilon(\gamma)$ is $\frac{1}{2}$ (resp. 1) if the equation $\delta^{-1}\gamma\delta = z\gamma$ has (resp. has not) a solution in $z \in Z_1(F) - \{1\}$.

(iii) $$-\frac{1}{4} \sum_{\eta = (\mu, \mu); \eta \in D^\circ} \mathrm{tr}(M(\eta)\pi_\eta(f)),$$

where D° is the set of pairs $\eta = (\mu, \nu)$ of characters of $A^\times / F^\times$ such that $\mu\nu$ induces ω on Z_1, where $M(\eta)$ and π_η are defined in **[G-J-1, §4]**, and with $\pi_\eta(f) = \int_{Z_1 \backslash G(A)} f(g)\pi_\eta(g)\, dg$. A Haar measure $d\eta$ on D° is defined as in **[G-J-1, §7-D]** by considering D° as a union of homogeneous spaces under the group of characters of $A^\times / F^\times$ (with the dual Haar measure), acting by $\chi \cdot (\mu, \nu) = (\chi\mu, \chi^{-1}\nu)$.

This allows us to write the fourth term:

(iv) $$\frac{1}{2} \int_{D^\circ} m^{-1}(\eta) \cdot m'(\eta) \mathrm{tr}(\pi_\eta(f))\, d\eta,$$

the derivative m' being computed as in **[G-J-1, §7-D]**.

(v) $$\sum_{z \in Z_1(F) \backslash Z(F)} l \cdot \lambda_0 \prod_v \frac{\int_{Z_v N_v \backslash G_v} f_v(g^{-1} z n_0 g)\, dg}{L(1, 1_v)}, \qquad \left(n_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right),$$

where λ_0 and other notations are defined in **[G-J-I, §7-B]**. The remaining terms will not be written as in **[G-J-1]** since they are there expressed by noninvariant local distributions. For example the local distribution

$$\tfrac{1}{2} A_1(\gamma, f_v) = -\Delta(\gamma) \int_{|x_v|>1} f_v^K \begin{pmatrix} a & (a-b)x_v \\ 0 & b \end{pmatrix} \log|x_v|\, dx_v$$

where $\gamma = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, $\Delta_v(\gamma) = |(a-b)^2/ab|_v^{1/2}$ and $f_v^K(g) = \int_{K_v} f_v(k^{-1}gk)\, dk$, can be written $A_2(\gamma, f_v) + A_3(\gamma, f_v)$ where $A_2(\gamma, f_v)$ is invariant, and A_3 fits with other terms to provide an invariant expression.

More precisely one takes for a nonarchimedean place v:

$$A_2(\gamma, f_v) = \log|(a-b)/a|_v F(\gamma, f_v) + \Delta_v(\gamma) f_v^K(z) \int_{|x_v|>1} \log|x_v|\, dx_v - |a/b|^{1/2} \cdot |\bar{\omega}_v|$$
$$\cdot \log|\bar{\omega}_v| \cdot \int_{Z_v N_v \backslash G_v} f(g z n_0 g^{-1})\, dg \qquad \left(z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right),$$

where $F(\gamma, f_v) = \Delta_v(\gamma) \int_{A_v\backslash G_v} f_v(g^{-1}\gamma g)\, dg$ and $\bar{\omega}_v$ is a uniformizing parameter for F_v.

If v is archimedean one takes

$$A_2(\gamma, f_v) = \log\left|1 - \frac{b}{a}\right|_v F(\gamma, f_v) - \frac{L'(1, 1_v)}{L(1, 1_v)^2} \int_{Z_vN_v\backslash G_v} f_v(g^{-1}zn_0g)\, dg.$$

This yields the term

$$\text{(vi)} \qquad - \sum_{\gamma\in Z_1\backslash A(F);\gamma\notin Z(F)} l\lambda_{-1} \sum_v A_2(\gamma, f_v) \prod_{w\neq v} F(\gamma, f_w).$$

It can be checked that $\gamma \mapsto A_3(\gamma, f_v)$ extends to a continuous map on $A(F_v)$ and one sees that the terms 6.34 in **[G-J-1]** minus our (v) plus 6.35 minus our (vi) yield the term

$$\sum_{\gamma\in Z_1\backslash A(F)} - \lambda_{-1} l \sum_v A_3(\gamma, f_v) \prod_{w\neq v} F(\gamma, f_w).$$

This can in turn be transformed by a kind of Poisson summation formula to

$$\int_{D^0} - \sum_v B_1(\eta_v, f_v) \prod_{w\neq v} \operatorname{tr} \pi_{\eta_w}(f_w)\, d\eta.$$

One has to add the term 6.36 of **[G-J-1]** minus our (iv) to get the final term:

$$\text{(vii)} \qquad 2 \int_{D^0} \sum_v B(\eta_v, f_v) \prod_{w\neq v} \operatorname{tr} \pi_{\eta_w}(f_w)\, d\eta.$$

2. The twisted trace formula. Here we shall use definitions and results of the Kottwitz lecture (see **[K]** in these PROCEEDINGS). As above we follow closely **[L, §8]**. Let $L^2(Z(A_E)\backslash G(A_E), \bar{\omega}) = \tilde{L}^2$ where $\bar{\omega} = \omega \circ N_{E/F}$. The Galois group $\Gamma = \operatorname{Gal}(E/F)$ acts on $\tilde{L}^2$ by ${}^\sigma\rho(x) = \rho(x^\sigma)$ for $\rho \in \tilde{L}^2$, $x \in G(A_E)$ and $\sigma \in \Gamma$.

Let R_d denote the restriction of the natural representation of $G(A_E)$ in $\tilde{L}^2$ to the discrete spectrum $\tilde{L}^2_0 \oplus \tilde{L}^2_{\mathrm{sp}}$. The projection commutes with the action of the Galois group; hence R_d can be extended to a representation R'_d of $\Gamma \ltimes G(A_E)$. Let $\phi \in \mathscr{C}_c^\infty(Z(A_E)\backslash G(A_E), \bar{\omega}^{-1})$; then $R_d(\phi)$ is of trace class and $R'_d(\sigma)$ is unitary for $\sigma \in \Gamma$. The operator $R_d(\phi)$ can be represented by a kernel $K(\phi, x, y)$ and then the operator $R'_d(\sigma)R_d(\phi)$ is represented by the kernel $K(\phi, x^\sigma, y)$ and

$$\operatorname{tr}(R'_d(\sigma)R_d(\phi)) = \int_{Z(A_E)\backslash G(A_E)} K(\phi, x^\sigma, x)\, dx.$$

Assuming, as usual, that ϕ is a tensor product: $\phi = \bigotimes \phi_v$, one can proceed as in **[G-J-1]** to compute this integral; if $\sigma \neq 1$ it is the sum of the following terms (1)–(7):

$$\text{(1)} \qquad \sum_\delta \operatorname{vol}(Z(A)G_\delta^\sigma(E)\backslash G_\delta^\sigma(A_E)) \int_{Z(A_E)G_\delta^\sigma(A_E)\backslash G(A_E)} \phi(g^{-\sigma}\delta g)\, dg$$

where the sum runs over the σ-conjugacy classes of elements δ such that $N(\delta)$ is central, and G_δ^σ is the σ-centralizer of δ (cf. **[K]**).

$$\text{(2)} \qquad \sum_{\delta\in\mathscr{E}_\sigma} \varepsilon(\delta)\operatorname{vol}(Z(A)\cdot G_\delta^\sigma(E)\backslash G_\delta^\sigma(A_E)) \int_{Z(A_E)G_\delta^\sigma(A_E)\backslash G(A_E)} \phi(g^{-\sigma}\delta g)\, dg$$

where $\mathscr{E}_\sigma$ is a set of representatives of the σ-conjugacy classes that are not σ-conjugate to a triangular matrix, taken modulo $Z(E)$ in $G(E)$, and $\varepsilon(\delta)$ is $\frac{1}{2}$ or 1 according as the equation $\tau^{-\sigma}\delta\tau = z\delta$ has or not a solution in $Z(E)$ with $z \notin Z(E)^{1-\sigma}$.

$$(3) \qquad -\frac{1}{4} \sum_{{}^\sigma\eta=\eta^\vee} \operatorname{tr}(M_E({}^\sigma\eta)\pi_\eta'(\sigma)\pi_\eta(\phi))$$

where $\eta^\vee = (\nu, \mu)$ if $\eta = (\mu, \nu)$ is a pair of characters of $A_E^\times/E^\times$ with $\mu\nu = \tilde{\omega}$. The representation π_η is realized in a space of functions on $G(A_E)$, the action of $\sigma \in \Gamma$ defines an operator $\pi_\eta'(\sigma)$ from the space of π_η to the space of π_{σ_η}; then $M_E({}^\sigma\eta)$ intertwines π_{σ_η} and $\pi_{\sigma_\eta^\vee}$ but ${}^\sigma\eta^\vee = \eta$. Hence the product $M({}^\sigma\eta)\ \pi_\eta'(\sigma)\pi_\eta(\phi)$ is a well-defined operator in the space of $\pi_{\sigma_\eta^\vee}$, and the above expression is meaningful.

$$(4) \qquad \frac{1}{2l^2} \int_{\eta\in D^\circ} m_E^{-1}(\tilde{\eta})m_E'(\tilde{\eta}) \operatorname{tr}(\pi_{\tilde{\eta}}'(\sigma)\pi_{\tilde{\eta}}(\phi))\, d\eta$$

where $\tilde{\eta} = (\mu \circ N_{E/F}, \nu \circ N_{E/F})$ if $\eta = (\mu, \nu)$. There are l^2 elements $\tilde{\eta}$ giving rise to the same $\tilde{\eta}$. The reader should be aware that our notation $\tilde{\eta}$ has not the same meaning as in [L].

$$(5) \qquad \lambda_0 \prod_v \theta^\sigma(0, \phi_v),$$

where $\theta^\sigma(0, \phi_v) = L(1, 1_v)^{-1} \iiint \phi_v(k^{-\sigma}t^{-\sigma}n^{-\sigma}\tilde{n}_0 ntk)t^{-2\rho}\, dn\, dt\, dk$, with $k \in \tilde{K}_v$, the standard maximal compact subgroup of $\mathrm{GL}_2(E \otimes F_v)$,

$$t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad t^{-2\rho} = \left|\frac{a}{b}\right|^{-1} \quad \text{and} \quad \tilde{n}_0 = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

such that $\operatorname{tr}_{E/F} z = 1$. The integration is on $\tilde{K}_v \times Z(E_v)\backslash A(E_v) \times N(F_v)\backslash N(E_v)$, where $E_v = E \otimes F_v$.

$$(6) \qquad -\frac{\lambda_{-1}}{l} \sum_{\delta\in\mathscr{F}} \sum_v A_2^\sigma(\delta, \phi_v) \prod_{w\neq v} F^\sigma(\delta, \phi_w),$$

where $\mathscr{F} = \{\delta \in A^{1-\sigma}(E)Z(E)\backslash A(E) \mid N(\delta) \notin Z(F)\}$,

$$F^\sigma(\delta, \phi_w) = \Delta_w(\gamma) \int_{Z(E_v)A(F_v)\backslash G(E_v)} \phi_w(g^{-\sigma}\delta g)\, dg$$

and $\gamma = N(\delta)$. An explicit definition of $A_2^\sigma(\delta, \phi_v)$ will not be given here; we shall simply say that $A_2^\sigma(\delta, \phi_v) = lA_2(\gamma, f_v)$ if f_v is associated to ϕ_v under the base change correspondence (see [K]).

As above the remaining term can be written

$$(7) \qquad \frac{2}{l^2} \int_{\eta\in D^\circ} \sum_v B^\sigma(\tilde{\eta}_v, \phi_v) \prod_{w\neq v} \operatorname{tr} \pi_{\tilde{\eta}_w}(\phi_w).$$

For a definition and a detailed study of distributions A_2^σ and B^σ the reader is referred to [L, §7] (where the subscript σ is omitted) and to [K, Lemma 4].

3. The comparison. We assume now on that the function $f = \otimes f_v \in \mathscr{C}_c^\infty(Z_1\backslash G(A), \omega^{-1})$ and the function $\phi = \otimes\phi_v \in \mathscr{C}_c^\infty(Z(A_E)\backslash G(A_E), \tilde{\omega})$ are such that ϕ_v and f_v are "associated" in the sense defined in [K]; we consider $\mathscr{D}_1 = l \operatorname{tr}(R_d'(\sigma)R_d(\phi)) - \operatorname{tr} r(f)$.

PROPOSITION 1.

$$\mathscr{D}_1 = \frac{2}{l^2} \int_{D^\circ} \sum_v \Big(lB^\sigma(\eta_v, \phi_v) - \sum_{\eta \mapsto \tilde\eta} B(\eta_v, f_v) \Big) \prod_{w \neq v} \operatorname{tr} \pi_{\eta_w}(f_w)\, d\eta$$
$$- \frac{\delta_{l,2}}{2} \sum_{\eta = (\mu, {}^\sigma\mu);\, {}^\sigma\mu \neq \mu;\, {}^\sigma\mu\mu = \tilde\omega} \operatorname{tr}(M({}^\sigma\eta)\pi'_\eta(\sigma)\pi_\eta(\phi))$$

where

$$\delta_{l,2} = 1 \quad \text{if } l = 2, \qquad = 0 \quad \text{if } l \neq 2.$$

The proof amounts to the comparison term by term of the expressions for $\operatorname{tr}(r(f))$ and $l \cdot \operatorname{tr}(R'_d(\sigma)R'_d(\phi))$.

For example to prove that $l \cdot (1) =$ (i) note that we work there with a sum over elements $\delta \in G(E)$ such that $N(\delta)$ is central in $G(F)$; hence $G^\sigma_\delta(E)$ is the set of F-points of a twisted inner form of G. Since we use Tamagawa measures,

$$l \operatorname{vol}(Z(A)G^\sigma_\delta(E)\backslash G^\sigma_\delta(A_E)) = l \operatorname{vol}(Z(A)G(F)\backslash G(A)) = \operatorname{vol}(Z_1 G(F)\backslash G(A));$$

the expressions to be compared are products of the local analogues, and now, using properties A′ and B′ in [K] for associated functions and the fact that the number of places where a minus sign occurs is even, we obtain the desired results. To prove $l \cdot (2) =$ (ii) is even simpler since in that case $G^\sigma_\delta(E) = G_\gamma(F)$, where $\gamma = N(\delta)$: the twisting is trivial since G_γ is abelian.

To compare $l \cdot (3)$ and (iii) is slightly more complicated; we must distinguish two cases:

(a) $l \neq 2$; then ${}^\sigma\eta = \eta^\vee$ implies ${}^\sigma\eta = \eta = \eta^\vee$ and in such a case one has $M(\eta) = -1$. One the other hand one should note that $\operatorname{tr} \pi_{\eta_v}(f_v) = \int_{Z_v\backslash A_v} F(a, f_v)\eta_v(a)\, da$ and that for associated functions f_v and ϕ_v one has $F(a, f_v) = F^\sigma(b, \phi_v)$ if $a = N(b)$. Moreover

$$\operatorname{tr} \pi'_{\tilde\eta_v}(\sigma)\pi_{\tilde\eta_v}(\phi_v) = \int_{\tilde Z_v\backslash \tilde A_v} F^\sigma(b, \phi_v)\tilde\eta_v(b)\, db,$$

where $\tilde Z_v = Z(E \otimes F_v)$ and $\tilde A_v = A(E \otimes F_v)$. Then $\operatorname{tr}(\pi'_{\tilde\eta}(\sigma)\pi_{\tilde\eta}(\phi)) = \operatorname{tr}(\pi_\eta(f))$ if ϕ and f are associated and if $\tilde\eta = (\mu \circ N_{E/F}, \nu \circ N_{E/F})$ and $\eta = (\mu, \nu)$.

This yields $l \cdot (3) =$ (iii).

(b) If $l = 2$, the same arguments apply if $\eta = \eta^\vee = {}^\sigma\eta$, but there are other terms corresponding to $\eta = (\mu, \nu)$ with $\mu = {}^\sigma\nu \neq {}^\sigma\mu$ and then

$$l \cdot (3) - \text{(iii)} = -\frac{l}{4} \sum_{\mu \neq {}^\sigma\mu;\, \eta = (\mu, {}^\sigma\mu);\, {}^\sigma\mu\mu = \tilde\omega} \operatorname{tr}(M({}^\sigma\eta)\pi'_\eta(\sigma)\pi_\eta(\phi)).$$

To prove that $l \cdot (4) =$ (iv) we use the previous remarks and the fact that $m_E(\tilde\eta)^l = \prod_{\eta \mapsto \tilde\eta} m(\eta)$, where $\eta \mapsto \tilde\eta$ means that $\eta = (\mu, \nu)$ and $\tilde\eta = (\mu \circ N_{E/F}, \nu \circ N_{E/F}) = (\tilde\mu, \tilde\nu)$ and by definition:

$$m(\eta) = \frac{L(1, \mu^{-1}\nu)}{L(1, \nu^{-1}\mu)} \quad \text{and} \quad m_E(\tilde\eta) = \frac{L(1, \tilde\mu^{-1}\tilde\nu)}{L(1, \tilde\mu\tilde\nu^{-1})}.$$

Now $l \cdot (5) =$ (v) follows from the comparison of orbital and twisted orbital integrals on unipotent elements.

To conclude the proof of Proposition 1 it is enough to show that $l\cdot(6) = \text{(vi)}$, which in turn follows from the equality:

$$A_2(\gamma, f_v) = A_2^\sigma(\delta, \phi_v) \quad \text{if } \gamma = N(\delta).$$

4. The main theorem. The representation r is a discrete sum of unitary irreducible representations of $G(A)$ with multiplicity one:

$$r = \sum_{i\in I} \pi_i = \sum_{i\in I} \otimes\pi_{i,v}$$

and $\operatorname{tr} r(f) = \sum_{i\in I} \prod_v \operatorname{tr} \pi_{i,v}(f_v)$ for some set I.

Let v be a nonarchimedean place. Given f_v in the Hecke algebra of $G(F_v)$ then $\operatorname{tr} \pi_{i,v}(f_v)$ is zero unless $\pi_{i,v}$ is unramified and hence corresponds to the conjugacy class of some semisimple element $t_{i,v} \in \mathrm{GL}_2(C)$, the connected component of the L-group. Any function f_v in the Hecke algebra defines a rational class function $f_v^\vee$ on $\mathrm{GL}_2(C)$ such that $f_v^\vee(t_{i,v}) = \operatorname{tr} \pi_{i,v}(f_v)$. Choose a finite set V of places of F containing archimedean ones and assume f_v is in the Hecke algebra for $v \notin V$. Then one has

LEMMA 1. $\operatorname{tr} r(f) = \sum_i \prod_{v\in V} \operatorname{tr} \pi_{i,v}(f_v) \prod_{v\notin V} f_v^\vee(t_{i,v})$.

The representation R_d can also be written

$$R_d = \sum_{j\in J} \Pi_j = \sum_{j\in J} \bigotimes_v \Pi_{j,v}$$

where Π_j is an automorphic representation of $G(A_E)$ and $\Pi_{j,v}$ a representation of $G(F_v \otimes E)$. Since ${}^\sigma R_d \simeq R_d$ and is multiplicity free, σ permutes the Π_j. If ${}^\sigma\Pi_j \simeq \Pi_j$ one can restrict the operator $R_d'(\sigma)$ to the space of Π_j, and denote this restriction by $\Pi_j'(\sigma)$. The nonfixed Π_j do not contribute to the trace of $R_d'(\sigma)R_d(\phi)$ (a permutation matrix without fixed point has trace zero) and then

$$\operatorname{tr} R_d'(\sigma)R_d(\phi) = \sum_{{}^\sigma\Pi_j\simeq\Pi_j} \operatorname{tr} \Pi_j'(\sigma)\Pi_j(\phi).$$

Moreover $\Pi_j = \otimes\Pi_{j,v}$ and ${}^\sigma\Pi_{j,v} \simeq \Pi_{j,v}$; we can define $\Pi_{j,v}'(\sigma)$ up to lth roots of 1. If $\Pi_{j,v}$ is unramified, there is a canonical choice. If ϕ_v and f_v are associated in the Hecke algebras, we choose a semisimple element $t_{j,v} \in \mathrm{GL}_2(C)$ such that

$$\operatorname{tr} \Pi_{j,v}'(\sigma)\Pi_{j,v}(\phi_v) = \operatorname{tr} \Pi_{j,v}(\phi_v) = f_v^\vee(t_{j,v}),$$

and then for some big enough finite set V of places of F, we have:

LEMMA 2. $\operatorname{tr} R_d'(\sigma)R_d(\phi) = \sum_j \prod_{v\in V} \operatorname{tr} \Pi_{j,v}'(\sigma)\Pi_{j,v}(\phi_v) \prod_{v\notin V} f_v^\vee(t_{j,v})$.

REMARK. If v is nonarchimedean and split in E, then any f_v is associated to some ϕ_v; in fact in such a case ϕ_v may be taken to be $f_{w_1} \otimes f_{w_2} \otimes \cdots \otimes f_{w_l}$ where the w_i are the places of E above v, and $f_v = f_{w_1} * f_{w_2} * \cdots * f_{w_l}$ can be any smooth function on $G(F_v)$; the conjugacy class of $t_{j,v}$ is well defined by $\Pi_{j,v}$.

If v does not split in E, is unramified, and if ϕ_v and f_v are associated in the Hecke algebras, one can define a function $\phi_v^\vee$ on $\mathrm{GL}(2, C)$ as above; we have

$$f_v^\vee(t) = \phi_v^\vee(t^l) \quad \text{for } t \text{ semisimple in } \mathrm{GL}_2(C).$$

In such a case the conjugacy class of the $t_{j,v}$ above is not uniquely defined.

If $l \neq 2$ let $R = lR_d$.

Assume for a while $l = 2$; if μ is a character of $A_E^\times/E^\times$ such that ${}^\sigma\mu\mu = \tilde{\omega}$ and ${}^\sigma\mu \neq \mu$, we consider $\tau_\mu = \pi_\eta$, where $\eta = (\mu, {}^\sigma\mu)$. As was said above, the operator $M({}^\sigma\eta)\pi'_\eta(\sigma) = \tau'_\mu(\sigma)$ maps the space of π_η into itself. One defines in such a way a representation τ'_μ of $\Gamma \ltimes G(A_E)$. One should remark that $\tau'_\mu \simeq \tau'_{{}^\sigma\mu}$. Let us denote by $\mathscr{T}$ the set of such representations (modulo equivalence) and let

$$R' = lR'_d \oplus \sum_{\tau'_\mu \in \mathscr{T}} \tau'_\mu.$$

An analogue of Lemma 2 can be stated.

We can now state the main theorem (cf. [L, Theorem 9.1]).

THEOREM 1. *Assume f and ϕ are associated; then* $\operatorname{tr} R'(\sigma)R(\phi) = \operatorname{tr} r(f)$.

(Recall that the definition of the correspondence "f and ϕ are associated" depends on the choice of a $\sigma \in \Gamma - \{1\}$.)

The proof of the theorem can be carried out as follows: consider the expression

$$\text{(a)} \qquad \operatorname{tr} R'(\sigma)R(\phi) - \operatorname{tr} r(f).$$

Thanks to Proposition 1 above, this is equal to

$$\text{(b)} \qquad \frac{2}{l^2} \int_{D^\circ} \sum_v \Big(lB(\tilde{\eta}_v, \phi_v) - \sum_{\eta_v \mapsto \tilde{\eta}_v} B(\eta_v, f_v) \Big) \prod_{w \neq v} \operatorname{tr} \pi_{\eta_w}(f_w)\, d\eta.$$

Using properties of some weighted orbital integrals [K, Lemma 4], one can prove that

$$lB(\tilde{\eta}_v, \phi_v) - \sum_{\eta_v \mapsto \tilde{\eta}_v} B(\eta_v, f_v) = 0,$$

at least when v is nonarchimedean, unramified in E, with ϕ_v and f_v associated and in the Hecke algebras. Hence there is a finite set of places V such that (b) can be written

$$\text{(b}'\text{)} \qquad \int_{D^\circ} \beta(\eta) \prod_{v \notin V} \operatorname{tr} \pi_{\eta_v}(f_v)\, d\eta$$

with some nice function β.

Now choose a place $v_0 \notin V$ split in E; then (b′) reduces to an absolutely convergent integral:

$$\text{(b}''\text{)} \qquad \int_{-\infty}^{+\infty} \delta(s) f_{v_0}^\vee \begin{pmatrix} aq^{is} & 0 \\ 0 & bq^{-is} \end{pmatrix} ds$$

where a, b depends on the central character ω_{v_0}. All we need to know is that $\delta(s)$ is some continuous, bounded and integrable function on the real line. On the other hand (a) can be written using Lemmas 1 and 2 above

$$\text{(a}''\text{)} \qquad \sum_{k=0}^{\infty} a_k f_{v_0}^\vee(t_k)$$

with $a_k \in C$ and $t_k \in \mathrm{GL}_2(C)$ semisimple elements corresponding to inequivalent unitary representations of $\mathrm{GL}_2(F_{v_0})$. The series, as the integral, is absolutely convergent. Now f_{v_0} is arbitrary in the Hecke algebra (since v_0 is split in E) and the

Hecke algebra separates inequivalent unramified representations. The Stone-Weierstrass theorem and easy majorations prove that all a_k are zero (which is a stronger statement than Theorem 1). All the desired results can now be xtracted from Theorem 1 and from some results on the characters of representations of $\Gamma \ltimes \mathrm{GL}_2(E_v)$ (cf. [L, §5]). We shall try to explain some of the steps.

5. Existence of weak liftings. Choose a finite set V of places of F containing all archimedean places and all places ramified in E. Assume that ϕ_v and f_v are associated and in the Hecke algebras for $v \notin V$. One can, using Lemmas 1 and 2, choose element $t_{n,v} \in \mathrm{GL}_2(C)$ for $v \in V$ and $n \in N$ such that

$$\operatorname{tr} R_d'(\sigma)R_d(\phi) = \sum_n \alpha_n(\phi) \prod_{v\notin V} f_v^\vee(t_{n,v}),$$

$$\sum_{\tau_\mu'\in\mathscr{T}} \operatorname{tr} \tau_\mu'(\sigma)\tau_\mu(\phi) = \sum_n \beta_n(\phi) \prod_{v\notin V} f_v^\vee(t_{n,v}),$$

$$\operatorname{tr} r(f) = \sum_n \gamma_n(\phi) \prod_{v\notin V} f_v^\vee(t_{n,v});$$

we may assume moreover they are chosen such that the functions $T_n\colon (\phi_v)_{v\notin V} \mapsto \prod_{v\notin V} f_v^\vee(t_{n,v})$ on the product for $v \notin V$ of the Hecke algebras are distinct. (Recall the remark after Lemma 2.) Let $\delta_n = l\alpha_n + \beta_n - \gamma_n$; then the above theorem can be restated in the following form

$$\sum_{n\in N} \delta_n(\phi)T_n(\phi) = 0.$$

Another use of density arguments, the T_n being distinct, yields

PROPOSITION 2. *For all n one has $\delta_n(\phi) = 0$.*

This can be read $l\alpha_n + \beta_n = \gamma_n$.

Assume that there exist a representation $\Pi = \otimes\Pi_v$, unramified outside V, occurring in $\tilde{L}_0$ such that $\operatorname{tr} \Pi_v(\phi_v) = f_v^\vee(t_{n,v})$ for some n with α_n not zero, and any $v \notin V$; then using the strong multiplicity one theorem [C] one concludes that such a Π is unique and satisfies $\Pi \simeq {}^\sigma\Pi$. The fact that $L(s, \Pi)$ is entire allows one to conclude that no other Π occurring in $\tilde{L}^2_{\mathrm{sp}}$ or in $\mathscr{T}$ has the property that $\operatorname{tr} \Pi_v(\phi_v) = f_v^\vee(t_{n,v})$, $v \notin V$. Then $\alpha_n(\phi) = \prod_{v\in V} \operatorname{tr} \Pi_v'(\sigma)\Pi_v(\phi_v)$ and $\beta_n(\phi) = 0$; since $l\alpha_n + \beta_n = \gamma_n$ we conclude that $\gamma_n(\phi)$ is not identically zero and hence there exists (at least) one π in $L_0^2 \otimes L^2_{\mathrm{sp}}$ such that $\pi = \otimes\pi_v$ and $\operatorname{tr} \pi_v(f_v) = f_v^\vee(t_{n,v})$ for $v \notin V$ and then Π_v is the lifting of π_v for $v \notin V$. We shall say that Π is a weak lifting of π if Π_v is a lifting of π_v for almost all v. We then have proved

THEOREM 2. *If Π is a cuspidal automorphic representation of $\mathrm{GL}_2(A_E)$ such that $\Pi \simeq {}^\sigma\Pi$, then Π is the weak lifting of some automorphic representation π of $\mathrm{GL}_2(A)$.*

A direct study of L^2_{sp} allows one to prove:

PROPOSITION 3. *Any π occurring in L^2_{sp} lifts to a Π in $\tilde{L}^2_{\mathrm{sp}}$ and all σ-fixed representations in $\tilde{L}^2_{\mathrm{sp}}$ are obtained in this way.*

In the case $l = 2$, assume that $\Pi = \pi(\mu, {}^\sigma\mu)$ occurs in $\mathscr{T}$ and that μ_v is unramified for $v \notin V$; one can show using Proposition 2 and results on L-functions on $\mathrm{GL}_2 \times$

GL_2 that Π is the weak lifting of $\pi = \pi(\rho)$ where $\rho = \mathrm{Ind}_{W_E}^{W_F}\mu$ and that there exists $n \in \boldsymbol{N}$ such that

$$\prod_{v\in V} \operatorname{tr} \Pi_v'(\sigma)\Pi_v(\phi_v) = \beta_n(\phi) = \gamma_n(\phi) = \prod_{v\in V} \operatorname{tr} \pi_v(f_v).$$

This can be used to show that at all places of F: $\operatorname{tr} \Pi_v'(\sigma)\Pi_v(\phi_v) = \operatorname{tr} \pi_v(f_v)$, and hence

THEOREM 3. *Let the fields E and F be either local or global. Then* $\pi(\mu, {}^\sigma\mu)$ *is the lifting of* $\pi(\rho)$ *where* $\rho = \mathrm{Ind}_{W_E}^{W_F}\mu$.

6. Any cuspidal π has a weak lifting. Assume for a while that some π occurring in L_0^2 has no weak lifting. Let V be a finite set of places of F including archimedean places and those where E or π are ramified. Let $f = \bigotimes f_v$ be associated to some ϕ and such that f_v is in the Hecke algebra for $v \notin V$. Consider the π_k in $L_0^2 \oplus L_{\mathrm{sp}}^2$ such that $\operatorname{tr} \pi_{k,v}(f_v) = \operatorname{tr} \pi_v(f_v)$ if $v \notin V$. Since the π_k have no weak lifting, Proposition 2 shows that the sum of the π_k gives a zero contribution to $\operatorname{tr} r(f)$:

$$\sum_k \prod_{v\in V} \operatorname{tr} \pi_{k,v}(f_v) \prod_{v\notin V} \operatorname{tr} \pi_v(f_v) = 0;$$

hence there is a set $V_1 \subset V$ such that $\sum_k \prod_{v\in V_1} \operatorname{tr} \pi_{k,v}(f_v) = 0$. One has to prove that this is impossible unless the sum is empty. The idea of the proof (by induction on the cardinality of V_1) is that characters of inequivalent representations are linearly independent, but the proof is complicated here by the fact that f_v cannot assume all values, since f_v must be associated to some ϕ_v (cf. [**L**, §9, pp. 25–30]). This yields

THEOREM 4. *Any cuspidal π has a weak lifting.*

7. Local liftings. To finish the proof of the global theorem on base change for cuspidal representations, one must show that the above weak liftings are liftings at all places. Let $\Pi \simeq {}^\sigma\Pi$, occurring in L_0^2, unramified outside a finite set V chosen as before; there is an n such that, according to Proposition 2, $l\alpha_n(\phi) = \gamma_n(\phi)$, which can be written, for some $V_1 \subset V$,

$$l \prod_{v\in V_1} \operatorname{tr} \Pi_v'(\sigma)\Pi_v(\phi_v) = \sum_k \prod_{v\in V_1} \operatorname{tr} \pi_{k,v}(f_v),$$

where the $\pi_k = \bigotimes \pi_{k,v}$ have Π as weak lifting. One then proves [**L**, §9, pp. 33–34]:

LEMMA 3. *If for some v the representation* Π_v *is the lifting of some* π_v *then it is the lifting of* $\pi_{k,v}$ *for all k, that is* $\operatorname{tr} \Pi_v'(\sigma)\Pi_v(\phi_v) = \operatorname{tr} \pi_{k,v}(f_v)$.

Then one may assume that V_1 does not contain such places.

On the other hand existence and properties of local liftings are easy to prove, or are deduced from Theorem 3 above, except for some supercuspidal representations (exceptional ones). Lemma 3 and this remark show that all desired local or global results (cf. part I of this paper) can be deduced from

THEOREM 5 [**L**, §9, PROPOSITION 9.6]). (a) *Every supercuspidal* π_v *has a lifting.*
(b) *If* $\Pi_v \simeq {}^\sigma\Pi_v$ *and is supercuspidal, then* π_v *is a lifting.*

Part (a) of this theorem is proved by embedding the local situation in an ad hoc global one, where the existence of liftings is known at all places except perhaps at one place, and to use the above equation with V_1, reduced to one element.

Part (b) then follows from the orthogonality relations of [L, §5].

The last paragraph of Langlands paper [L] is devoted to the proof of the existence of lifting for noncuspidal representations, using their explicit description (cf. Appendix C).

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UNIVERSITÉ PARIS VII