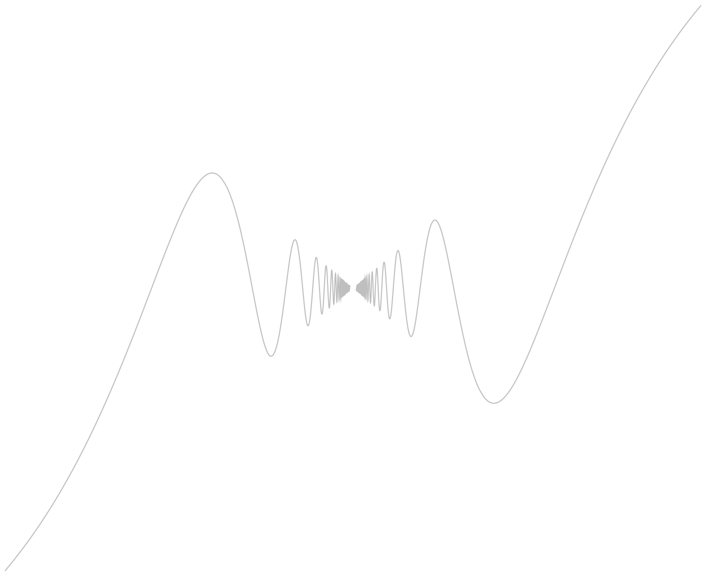
ARITHMETIC

of the

CONTINUUM



Contents

PREFACE

These notes form an introduction to doing mathematics.

Einstein once said to

> "make things as easy as possible, but no easier."

I have done my best to follow this advice.

In my quest to do so, I found certain truths:

Optimize for perspective. How one sees something begets how one solves something, and how one solves something begets how one understands something: perceive, analyze, revisit, in that order. In particular I am against exercises of the mindless sort. Not every reader is enthusiastic enough to dedicate time to the exercises in the first place, so if there is an exercise there, it better teach something at least mildly interesting.

Communicate the surprises. If a narrative is a piecewise linear path, the plot twists are the vertices, the stars from which the constellation was originally drawn. Furthermore there may be other valid paths through the vertices, and if the reader finds these, the material will be theirs forever.

Don't assume monotonicity. The reader may want to jump around. Thus, throughout the chapters there are pointers to how the material connects to sections both adjacent and distant. If you are using the digital version of this book on a tablet or phone, tapping the section heading should send you to the table of contents and vice versa. That way the contents stay just a touch away, and should serve as a functional minimap of the text as one reads.

Combine pure and applied approaches. This is self-explanatory. My criterion for including something is not whether it is pure or applied, or considered part of a certain branch of mathematics, but rather whether it is interesting or

> memorable. Mathematics ought to be a joy and inspiration, and proofs ought to showcase ingenuity wherever possible, in the hope that the reader may someday experience this form of *aristeia* for themselves, in the exercises and beyond.

Every work is a product of its time. While the core text remains mostly a classical treatment of real and functional analysis starting from logic, the endmatter of the chapters contain relevant applications and categorical insights (see Ch. 3 Remarks).

Anyone who has ever written anything might be able to relate: I started writing this book primarily as a means to catalyze my own understanding. What emerged from this premise was a story I never set out to tell.

Yet here it is. I hope you enjoy it.

Justin T. Chun
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GUIDE TO NOTATION

Throughout:

X denotes a domain space (elements: $x, y, z \in X$),

Y denotes a codomain space,

f denotes a function from X to Y

Z denotes a subspace of X,

W denotes a subspace of Y,

U denotes an open subset,

V denotes a closed subset,

S denotes an arbitrary subset,

Λ denotes an index set,

N denotes a neighborhood,

T denotes a linear map,

Ω denotes a compact operator with eigenpair (ω, e_ω),

A denotes an algebraic object (elements: $\alpha, \beta, \gamma \in A$), but

k denotes either $\mathbf{R}$ or $\mathbf{C}$,

τ denotes a topology,

$d(x, y)$ denotes a metric,

μ denotes an abstract measure,

λ denotes Lebesgue measure.

Chapter 0

ZEROTH ORDER LOGIC

Logical reasoning forms the cornerstone of mathematical thought. We thus start by summarizing the basic notions.

FUNCTIONS AND TUPLES

Definition 0.1. A **function** is a way to map elements from one set, the *domain*, to elements of another set, the *codomain*, such that every element of the domain is paired with a unique element of the codomain.

We codify logical connectives (e.g. *and*, *or*, *not*) as functions taking values in the set

$$\mathcal{V} = \{\bot, \top\}$$

where $\bot$ denotes *false* and $\top$ denotes *true*.

We collect together finitely many objects using n-tuples[1] where for example

$$(a, b, c, d, a)$$

denotes a 5-tuple containing the entries a, b, c, d, and a.

We denote by $\mathcal{V}^n$ the set of all n-tuples entries in $\mathcal{V}$, and by

$$f : \mathcal{V}^n \to \mathcal{V}$$

an n-ary truth function.

[1] Note that order matters for n-tuples but does not matter for sets. Also, n-tuples may have repeats whereas sets may not.

NOTE ON FINITENESS FOR THE CAREFUL

Observe that $\mathcal{V}$ has two elements, $\mathcal{V}^2$ has four elements, and in general $\mathcal{V}^n$ has 2^n elements. A function going from $\mathcal{V}^n$ to $\mathcal{V}$ must make 2^n binary decisions (i.e. whether to set each function value to either $\top$ or $\bot$), so there are 2^{2^n} possible functions that could go from $\mathcal{V}^n$ to $\mathcal{V}$. We point this out as a means of saying that everything we have so far defined is *finite*.

We will soon work with arbitrary (i.e. potentially infinite) sets, though. Here is why we did not just start with infinite sets: intuitively, one thinks of a set as an unordered collection of objects with no repeats. However, this naive conception can lead to logical disaster.

For instance, we have the following argument, known as *Russell's Paradox*:

> Consider the set of all sets that are not elements of themselves – call this set Ω. On the one hand, if Ω is an element of itself, then it is (by definition of Ω) not an element of itself. On the other hand, if Ω is not an element of itself, then it belongs with all the other sets that aren't elements of themselves – namely, in Ω. So Ω contains itself if and only if Ω does not contain itself – a contradiction.

Chapter 1

FIRST ORDER LOGIC

Chapter 2

THE ZFC AXIOMS

Chapter 3

RELATION AND NUMBER

Chapter 4

NEARNESS AND DISTANCE

Chapter 5

COMPLETENSS OF $\mathbf{R}$

Chapter 6

FINITE DIMENSIONAL VECTOR SPACES

Chapter 7

MEASURE AND INTEGRATION IN $\mathbf{R}^n$

Chapter 8

NORMED SPACES

Chapter 9

LINEAR AND SUBLINEAR FUNCTIONALS

Chapter 10

CONTINOUS LINEAR MAPS

Chapter 11

THE RADON-NIKODYM THEOREM

Chapter 12

PERTURBATIONAL APPROXIMATION

Chapter 13

TEST FUNCTIONS AND WEAK DERIVATIVES

Chapter 14

LOCALLY CONVEX SPACES

Chapter 15

INNER PRODUCT SPACES

Chapter 16

UNIFORM APPROXIMATION

RETROSPECT

We have just completed a gradual mathematical progression that started with propositional logic and set theory, and ended with spectral theory and distributions.

By no coincidence, the background knowledge gained through reading this book stations the reader so that they are ready to begin understanding the foundations of quantum mechanics and computation from a clean, theoretical viewpoint.

We sketch this now.

- Every quantum system can be thought of as a C^*-algebra of *observables* which we will denote $\mathcal{O}$. When $\mathcal{O}$ is commutative, we recover the setting of classical mechanics; this reflects the fact that classical observables can be measured simultaneously and in any order.

- Quantum phenomena occur when $\mathcal{O}$ is noncommutative: one usually takes $\mathcal{O}$ to be the self-adjoint (i.e. Hermitian) operators on a Hilbert space. Recall that the eigenvalues of these operators are real numbers; these correspond to physical measurements.

- A quantum state is a positive linear functional ω on $\mathcal{O}$, normalized so that $\omega(1) = 1$. Pure states arise from unit vectors $|\psi\rangle$, whereas mixed states correspond to density matrices ρ, aka positive operators with trace 1. Unitary operators evolve the state of the system through time.

- We may regard position and momentum observables as operator-valued distributions. Of course, position and momentum are Fourier conjugates; that is, each is the other's Fourier transform, up to a scaling factor.