Geometric Thoughts with Guest Algebra i^i ☺

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When considering the complex number/algebra i^i, online mathematics resources typical show a solution by the following derivation

$$i^i = e^{\ln(i^i)} \rightarrow e^{i\ln(i)} \rightarrow e^{i\left(\frac{i\pi}{2}\right)} \rightarrow e^{-\frac{\pi}{2}} \in \mathbb{R}.$$

The solution is an oversimplification of what the question revolves around. It poses problems with vector-based physics, geometric spaces, unit singularities, and it can break a pre-defined operation in a chosen group space.

From the standpoint of Lie theory, the tangent space represents the Lie algebra space that, once exponentiated, becomes the Lie group. So, the Lie group $e^{i\theta}$ has a Lie algebra which is simply i. You add the variable with exponentiation to gain infinite possible orientations on the complex unit circle's boundary, returning you to $e^{i\theta}$ with θ as our said variable.

Lie matrix groups are smooth manifolds that tend to be curved objects, but it turns out that the characteristic mathematics of a matrix group can be encapsulated in the flat space, that is the tangent space $\mathfrak{T}_\epsilon(G)$. The tangent space of a group G consists of smooth paths, $A(t)$, moving through the identity, ϵ. Smooth defines that derivatives of these paths exist, $A'(t)$, and if $A(0) = \epsilon$, then $A'(0)$ is called the tangent vector of $A(t)$ at ϵ. Determining the matrix derivative of A centered at the identity, $A'(0)$, to evaluate the smooth paths is straightforward for the general and special linear groups $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$. The tangent vectors of these groups will take the form of a $n \times n$ skew-symmetric matrix X, defined for $X \in \mathbb{C}$ as $X + X^* = 0$, where X^* defines the complex conjugate of X.

So, the tangent space of a group G is a beautiful thing. Put simply, it uses a flat tangential function passing in the neighborhood of the groups' identity ϵ, to capture the algebra of a Lie group that is often spherical. The tangent space is thus defined as

$$\mathfrak{T}_\epsilon\big(G(\theta)\big) = \left(\frac{d}{d\theta}[G(\theta)]\right)\bigg|_{\theta=0}.$$

Let's use the simple example for $e^{i\theta}$, where the tangent space algebra is found as follows

$$\mathfrak{T}_\epsilon\big(G(\theta)\big) = \left(\frac{d}{d\theta}\left[e^{i\theta}\right]\right)\bigg|_{\theta=0} = i.$$

This is the expected result given that $\ln\left(e^{i\theta}\right) = i\theta$, where we're assuming a principal valued logarithm and that θ is our variable. Taking i^i is an entirely different thing in comparison to the simple group example above. It is a problem that involves the interaction of two groups, one extended by the exponential mapping, aka local, so to say, the other in the phase space/tangent space. This dual-algebraic group will have the form

$$G(\theta, \phi) = \left(e^{i\theta}\right)^{e^{i\phi}},$$

for $\theta, \phi \in \mathbb{R}$. Geometrically the problem i^i is asking what's happening when the phase/algebra variables for the Lie group $G(\theta, \phi)$ are synchronized? To find out, we first consider the tangent space $\mathfrak{T}_{\epsilon,\theta}\big(G(\theta, \phi)\big)$

$$\mathfrak{T}_{\epsilon,\theta}\big(G(\theta, \phi)\big) = \left(\frac{d}{d\theta}[G(\theta, \phi)]\right)\bigg|_{\theta=0} = \left(\frac{d}{d\theta}\left[\left(e^{i\theta}\right)^{e^{i\phi}}\right]\right)\bigg|_{\theta=0} = \left(ie^{i\phi}\left(e^{i\theta}\right)\right)\bigg|_{\theta=0} = ie^{i\phi},$$

which by reassigning the θ phase variable, will result in our original dual Lie group with the exponential mapping $\left(e^{i\theta}\right)^{e^{i\phi}}$. So, the synchronized variable, denoted as τ, for this first step occurs when ϕ is equal to $\tau = \frac{\pi}{2}$, given that

$$\left(e^{i\phi}\right)\Big|_{\phi=\frac{\pi}{2}} = i.$$

This result equals the algebra of the tangent space found regarding the neighborhood of the $e^{i\theta}$ portion of the dual-algebraic group. Next, we travel to the next "dimension" down in the iterative logarithmic function process, otherwise stated as one layer down in phase space, by evaluating what we define as the *subtangent* space, finding the algebra is equal to -1

$$\mathfrak{T}_{\epsilon,\phi}\left(\mathfrak{T}_{\epsilon,\theta}\big(G(\theta,\phi)\big)\right) = \left(\frac{d}{d\phi}\left[\mathfrak{T}_{\epsilon,\theta}\big(G(\theta,\phi)\big)\right]\right)\Big|_{\phi=0} = \left(\frac{d}{d\theta}\left[ie^{i\phi}\right]\right)\Big|_{\phi=0} = -1.$$

Note, this *subtangent* space is the phase space of the tangent space, which is not considered in Lie theory, but it exists assuming we're not taking the logarithm of a zero-based algebra space.

Returning to our thoughts, the dual-algebraic interaction of the phase spaces will have the following algebra

$$T_\phi \tau = -\tau,$$

and recalling that for the i^i problem we're considering, the synchronized variable is $\tau = \frac{\pi}{2}$, which implies that

$$T_\phi\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}.$$

We're still in phase space, but the exponential mapping brings us out to the vanilla derivation that we began with

$$e^{T_\phi\left(\frac{\pi}{2}\right)} = e^{-\frac{\pi}{2}}.$$

To consider the problem further, in a like-phased dimensioned problem like

$$e^{i\theta}e^{i\phi} = e^{i\theta+i\phi},$$

the phase algebras add when commutativity is assumed, which they do in the stated problem case for $\theta,\phi \in \mathbb{R}$, since it is implied, we're working $\in \mathbb{C}^1$. This adding and subtraction is consistent with the assigned adding and subtraction for a group's Lie algebra/phase space. When they don't commute, we're into Baker-Campbell-Hausdorff territory and all the glorious commutators that come with it!

For the group $G(\theta,\phi) = \left(e^{i\theta}\right)^{e^{i\phi}}$, we can take advantage of its exponential form to write it as

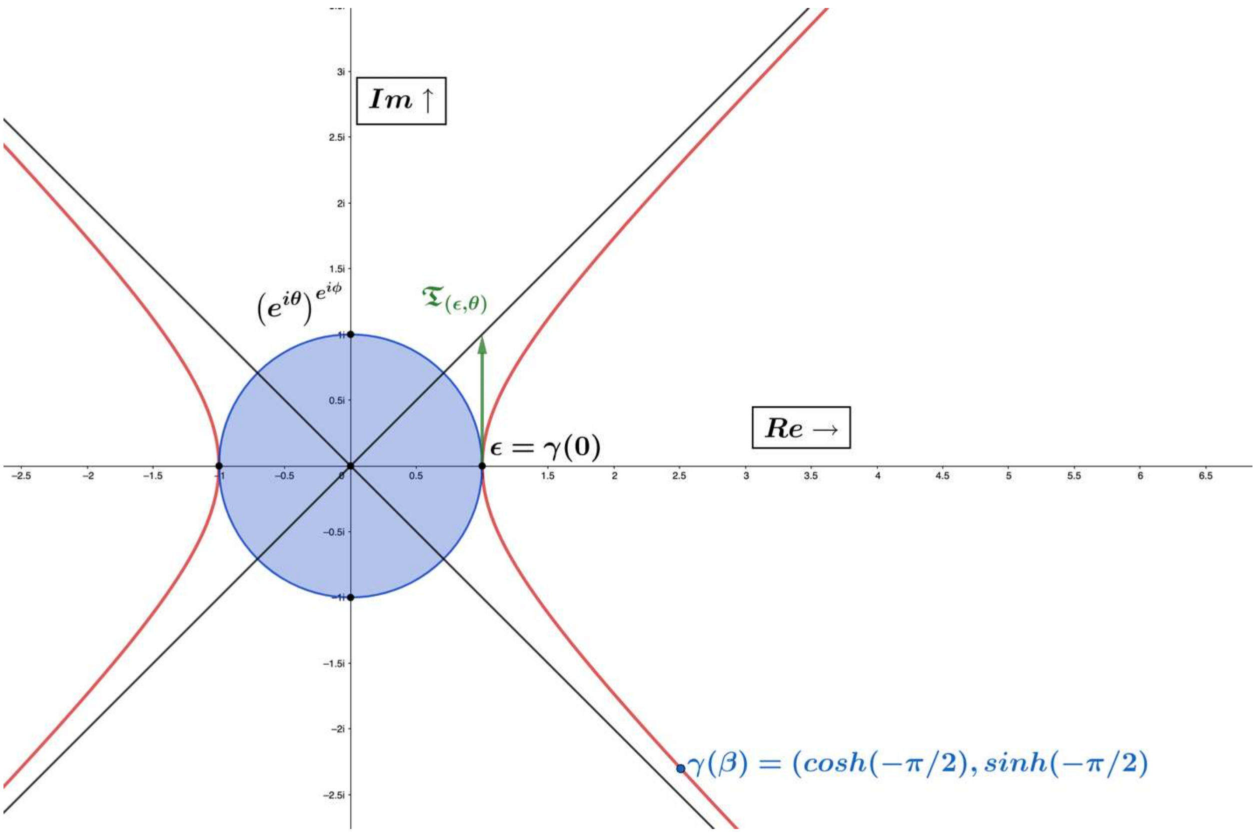
$$G(\theta,\phi) = \left(e^{i\theta}\right)^{e^{i\phi}} = e^{i\theta e^{i\phi}},$$

where we see the presence of an exponent product, showing it's not a like-phased dimensional problem. We can synchronize the variables in the above form to $\theta,\phi = \frac{\pi}{2}$, to also find that

$$G\left(\frac{\pi}{2},\frac{\pi}{2}\right) = e^{\frac{i\pi}{2}\left(e^{\frac{i\pi}{2}}\right)} = e^{-\frac{\pi}{2}},$$

but there's way too much information lost in this translation as well.

It becomes a more enlightening endeavor by taking the initiative to consider the problem using the Lie theory, the tangent space, and the subtangent space. This is because it shows us the problem is asking what happens when a unit singularity's waveform interacts with a waveform in spacetime with relative phase variables that are synchronized?



The answer is that it's transformed into a real exponential function oriented in anti-spacetime. The reason is that $e^{-\frac{\pi}{2}}$ is smaller than the one-dimensional plank length set at $p_l = 1 = e^0$, and when the parametric form for $e^{-\frac{\pi}{2}} = \gamma(\beta) = \cosh(\beta)\,, \sinh(\beta)$ is plotted in the figure above, you can that the function has crossed the horizontal axis of the right-oriented unit hyperbola. Also in the figure is the $G(\theta,\phi) = \left(e^{i\theta}\right)^{e^{i\phi}}$ group oriented along the vertical axis, the identity at $\epsilon = \gamma(0)$, and the tangent vector $\mathfrak{T}_{\epsilon,\theta}\big(G(\theta,\phi)\big) = \left(\frac{d}{d\theta}[G(\theta,\phi)]\right)\Big|_{\theta=0}$ passing in the neighborhood of ϵ.

So, there are many realized insights by taking a more consistent geometric approach to this seemingly straightforward problem.