Foundations for Mathematical Reality

Arithmoi Foundation

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Reality as a Mathematical Structure

"*Mathematics is the easiest of art and science to teach a mind.*"

The quote above has certainly never been said with any seriousness…

As we know, often by first-hand experience, or inadvertently in our every day modern life encounterings, mathematics, while governing the laws of matter around us, is seldom understood and appreciated for its beauty and elegance.

Fool No More

We drudge ourselves with the basics such as arithmetic and geometry in school, enough to get out of those uncomfortable chair of the classroom and into the real world of *the market and demand for your time and attention*. For some of us, we pursue the hard science and learn the mathematics that goes along the field we chose. Most often than not, we use mathematics as we use a hammer, to get the job done and move on with our lives.

But what if this way of teaching, this way of knowing and pondering about reality was just a mere reflection, a poor simulacrum of what mathematics truly is.

We invite the reader to think about and explore the realm of numbers as one would play around and explore the realm of notes on a grand piano in a dark and cozy room: **you** are the master of your thoughts, may you find gracious mathematics within the paths of your mind…

Strike a chord, be bold: Do the math.

In the following text, we briefly explore the powerful mathematical object known as *group theory*—a foundation that underlies much of number theory, symmetry, and the mathematical structure of reality.

The goal of the present text is to open the mind to a fruitful, energizing and healthy relationship with mathematics.

Contextualizing the Math with Mind

We invite the reader to ponder and wonder about the mathematics presented here, with in mind this thought:

You are a mathematical being, becoming entirely knowledgeable by incarnating within a human form in the spacetime physical domain to experience, grow, learn and finally emanate mathematics, i.e. total mastery of the mind, and its emanation, matter.

By mastering the mathematics of existence, you can navigate the spacetime domain with ease and firmness while maintaining great focus and clarity in the frequency domain, where the mind lives. Spacetime is within light, as light, the frequency domain (and the elusive ether for the scientists *out there*), is the mathematical singularity encompassing the totality of the mathmos (i.e. the cosmos as a mathematical reality) energy, which has a net sum of 0, but is eternally in movement.

A Grand Truth

Group theory is a branch of mathematics that studies symmetry and structure through sets equipped with a special operation. It is essential not only in pure mathematics such as number theory, but also in physics, chemistry, and computer science.

A **group** is a set G together with a binary operation $\star$, which combines any two elements to form another element of the set, and satisfies four fundamental properties:

- **Closure:** For all $a, b \in G$, the result $a \star b$ is also in G.
- **Associativity:** For all $a, b, c \in G$, the grouping of operations does not matter:

  $$(a \star b) \star c = a \star (b \star c)$$

  Here, a, b, and c are elements of G.
- **Identity Element:** There is an element $e \in G$, called the **identity** (pronounced "ee-dent-uh-tee"), such that for every $a \in G$:

  $$a \star e = e \star a = a$$

  The identity element leaves any element unchanged under the group operation.

- **Inverse Element:** For each element $a \in G$, there exists an inverse element $a^{-1} \in G$ (read as "a inverse"), such that:

$$a \star a^{-1} = a^{-1} \star a = e$$

  The inverse element "undoes" the effect of a under the group operation.

The Integers as a Group

An important and familiar example of a group is the set of integers, denoted by $\mathbb{Z}$ (pronounced "zee" or "zed"), under the operation of addition.

- The set $\mathbb{Z}$ consists of all whole numbers: $\ldots, -2, -1, 0, 1, 2, \ldots$.
- The operation is standard addition, denoted by $+$.

This group satisfies all group properties:

- **Closure:** The sum of any two integers is an integer.
- **Associativity:** Addition of integers is associative.
- **Identity:** The number 0 acts as the identity element, since $a + 0 = 0 + a = a$ for any integer a.
- **Inverse:** The inverse of a is $-a$, since $a + (-a) = 0$.

Euler Diagrams: Visualizing Subsets and Structure

To better understand group structure and relationships within sets, mathematicians use diagrams known as Euler diagrams. These are visual tools for illustrating how different subsets relate within a universal set.

For example, consider the set $\mathbb{Z}_n$ (pronounced "zee sub n"), which consists of integers modulo n:

$$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$$

Here, n is a positive integer called the **modulus** (pronounced "mod-yoo-lus").

Within the circle for $\mathbb{Z}_n$, we can draw smaller circles for interesting subsets, such as:

- The set of even numbers modulo n.
- The set of numbers that are generators (defined below).

Each intersection or inclusion can be clearly visualized, revealing the relationships among subsets and the structure of the group.

Prime Numbers and Generators

Prime numbers, denoted by p (pronounced "pee"), are integers greater than 1 whose only positive divisors are 1 and p itself. Primes are fundamental in group theory, particularly when studying the group $\mathbb{Z}_n$ under addition modulo n.

A group is **cyclic** (pronounced "sigh-click") if it can be generated by a single element, called a **generator**. In other words, every element of the group can be obtained by repeatedly applying the group operation to the generator.

For $\mathbb{Z}_n$ under addition, an element k is a generator if the set:

$$\{0, k, 2k, 3k, \ldots, (n-1)k\} \mod n$$

contains every element of $\mathbb{Z}_n$. Here, $\mod n$ means we take the remainder after division by n.

The criterion for k to be a generator is that it is **coprime** (shares no common factors except 1) with n. This is expressed using the greatest common divisor function, $\gcd(k, n)$ (pronounced "gee-cee-dee of k and n"). Thus,

$$k \text{ is a generator of } \mathbb{Z}_n \iff \gcd(k, n) = 1$$

When n is a prime number p, every nonzero element k in $\mathbb{Z}_p$ is a generator, since $\gcd(k, p) = 1$ for all $1 \leq k < p$.

Example: $\mathbb{Z}_5$ (Integers Modulo 5)

Let $n = 5$, which is prime.

- The elements are $\{0, 1, 2, 3, 4\}$.
- Each k in $\{1, 2, 3, 4\}$ is coprime to 5.

Thus, every nonzero element is a generator.

Example: $\mathbb{Z}_6$ (Integers Modulo 6)

Let $n = 6$, which is not prime.

- The elements are $\{0, 1, 2, 3, 4, 5\}$.
- Only $k = 1$ and $k = 5$ are coprime to 6.

Thus, only 1 and 5 are generators in $\mathbb{Z}_6$.

Mathematical Table of Generators

Let us summarize the above results in a clear table.

Group	Generators Exist?	Explanation
$\mathbb{Z}_p$	Yes, all k with $1 \leq k < p$	p is prime, so $\gcd(k, p) = 1$ for all nonzero k
$\mathbb{Z}_n$	Yes, for k with $\gcd(k, n) = 1$	n not prime, only some k are coprime to n

Here, $\gcd(k, n)$ stands for the greatest common divisor of k and n.

Visualizing Generators in Euler Circles

Imagine a large circle representing all elements of $\mathbb{Z}_n$. Inside, shade the subset corresponding to the generators:

- For $n = 5$, the shaded subset includes all $1, 2, 3, 4$.
- For $n = 6$, only 1 and 5 are shaded as generators.

This visualization helps to see which elements have the power to generate the entire group through repeated application of the group operation.

Key Mathematical Notation Explained

- $\mathbb{Z}$: The set of all integers. Pronounced "zee" or "zed".
- $\mathbb{Z}_n$: The set of integers modulo n, with elements $\{0, 1, \ldots, n-1\}$.

- $\gcd(a, b)$: Greatest common divisor of a and b. Pronounced "gee-cee-dee".
- k: An integer, often used as a candidate for a generator.
- p: A prime number.
- $\phi(n)$: Euler's totient function (pronounced "fee"), which counts how many numbers less than n are coprime to n.

$$\phi(n) = |\{1 \leq k < n \mid \gcd(k, n) = 1\}|$$

  Here, the vertical bars $|\cdot|$ denote the size of the set, that is, the number of elements it contains.
- $\exp(x)$: The exponential function, e^x.
- i: The imaginary unit, satisfying $i^2 = -1$.
- θ: An angle in radians.

Understanding Prime Numbers

A **prime number**, p, is a natural number greater than 1 that has exactly two distinct positive divisors: 1 and itself. In other words, p is a prime number if it cannot be written as a product of two smaller positive integers. Formally, p is prime if:

$$\forall\, a,\, b \in \mathbb{N}, \quad 1 < a < p,\ 1 < b < p \implies a \cdot b \neq p$$

Here, the symbol $\forall$ (pronounced "for all") means that the statement applies to all possible values of a and b in the set of natural numbers $\mathbb{N}$ (pronounced "en"), and $\cdot$ denotes multiplication.

For example: - 2 is a prime number because its only divisors are 1 and 2. - 3 is prime (divisors: 1, 3). - 4 is not prime, since $2 \cdot 2 = 4$.

Prime numbers are the building blocks of the integers, since every integer greater than 1 can be uniquely written as a product of primes (this is known as the Fundamental Theorem of Arithmetic).

A Journey Forward

Group theory is a profound tool for understanding the structure of numbers and the symmetries of nature. Its elegant rules give rise to astonishing results and connections, some of which you have glimpsed here.

Music and Group Theory

Mathematics and music are deeply intertwined, as both are structured by patterns, repetition, and symmetry. Group theory provides the mathematical language to describe musical transformations—such as transpositions, inversions, and rhythms.

By exploring how groups act on sets of musical notes or rhythms, one can see the harmony between mathematical structure and auditory beauty. The study of musical scales, chord progressions, and permutations of themes all reveal the presence of group-theoretic concepts, offering a new perspective on the universality of symmetry and pattern.

Prime Numbers

Prime numbers are not merely numerical curiosities; they are the fundamental bricks of mathematics and, by extension, existence itself. Each prime p stands as a source of mathematical purity that cannot be factored into a product of two smaller natural numbers.

On the Euler circle, primes are perfectly intuited as singular, irreducible geometric points or shapes in the 2D landscape of numbers (real and imaginary numbers). Their distribution encodes mysteries at the heart of reality, and their role in the construction of all numbers echoes the process by which complexity arises from simplicity—a perennial theme in both mathematics and the metaphysical traditions.

Geometry, Intuition, and the Prime Mandala

Visualizing primes on the Euler circle reveals a hidden order and aesthetic in their arrangement. Each prime, when mapped as a node or region on the circle, forms an eternal sacred geometry: a mandala of number that points to deep symmetries and archetypes.

The study of these patterns not only refines our mathematical intuition but also attunes the mind to the ethereal music of the cosmos, where each prime resonates as a unique frequency in the tapestry of existence.

Certainly! Here's a clear, didactic explanation of how to visualize primes on the Euler circle (complex unit circle), using the formula $\exp(i2\pi n/p)$ and related concepts. This will help readers see the connection between group theory, Euler's formula, and the distribution of primes on the circle.

Visualizing Prime Numbers on the Euler Circle

To gain deeper intuition into the nature of prime numbers, we can use the **Euler circle**, also known as the **complex unit circle** in the complex plane.

The **unit circle** consists of all complex numbers z of the form:

$$z = \exp(i\theta)$$

where θ is a real number between 0 and 2π, i is the imaginary unit ($i^2 = -1$), and exp denotes the exponential function. The unit circle has radius 1 and is centered at the origin.

Roots of Unity and Primes

An important concept in both group theory and number theory is the n**th roots of unity**. These are the complex solutions to the equation:

$$z^n = 1$$

All nth roots of unity are equally spaced points on the unit circle, given by:

$$z_k = \exp\left(\frac{2\pi i k}{n}\right)$$

where k is an integer with $0 \leq k < n$.

For example, if $n = 5$, the five 5th roots of unity are:

$$\exp(0),\ \exp\left(\frac{2\pi i}{5}\right),\ \exp\left(\frac{4\pi i}{5}\right),\ \exp\left(\frac{6\pi i}{5}\right),\ \exp\left(\frac{8\pi i}{5}\right)$$

These points are vertices of a regular pentagon inscribed in the unit circle.

Primes and the Unit Circle

If p is a prime number, the pth roots of unity divide the circle into p equally spaced, irreducible points. These points correspond to the elements of the cyclic group $\mathbb{Z}_p$ under addition modulo p.

In this context, each generator k of the group $\mathbb{Z}_p$ corresponds to a rotation by an angle of $2\pi k/p$. Since p is prime, every nonzero k is coprime to p and acts as a generator, so iterating this rotation will visit all nonzero points on the circle before returning to the starting point.

Seeing Primes on the Euler Circle

- To visualize a prime p on the Euler circle, plot the pth roots of unity:

$$z_k = \exp\left(\frac{2\pi i k}{p}\right), \quad k = 0, 1, 2, \ldots, p-1$$

- Each z_k represents a unique element of the group $\mathbb{Z}_p$.
- These points cannot be constructed by combining smaller subdivisions of the circle (that is, the pth roots of unity are not also nth roots of unity for any $n < p$), reflecting the primality of p.

Example: Prime $p = 7$ Plotting

$$z_k = \exp\left(\frac{2\pi i k}{7}\right), \quad k = 0, 1, \ldots, 6$$

yields seven points, each spaced by an angle of $2\pi/7$ radians, forming a regular heptagon on the unit circle. Each point is irreducible in the sense that it cannot be reached by repeating a smaller step that divides 2π evenly (other than the trivial one).

Indivisible Points

Prime numbers, when visualized on the Euler circle as roots of unity, appear as sets of indivisible, equally spaced points. This geometric perspective reflects their role as the fundamental "building blocks" of number theory: they create the simplest, most symmetrical divisions of the circle, and thus of the mathematical universe.

Where to Go Next

For those seeking to understand the unified mathematical foundations of reality—with a focus on equations and rigorous reasoning—the **Arithmoí** series by Jason Glowney is an excellent path. This six-part series investigates mathematics, physics, and metaphysics through a cohesive framework rooted in reason and the ontology of mathematics.

- **Mathematics Book I: Arithmoí – Mathematical Foundations and the Singularity**
  The opening volume explores existence as a mathematical construct, presenting mathematics not merely as a tool, but as the very language in which reality is written.
- **Physics Book I: Arithmoí – Phenomenology and the Geometry of Light**
  The second volume will examine how phenomena arise from mathematical light and geometric synthesis, laying the groundwork for a unified understanding of physical reality.

As stated in the Emerald Tablet:

"That which is above is from that which is below, and that which is below is from that which is above, working the miracles of one."

Ultimately, mathematics is not something we discover afresh—it is a memory we awaken at the deepest level of our soul.

Learn more at arithmoi.org

