Asymptotic Light Geometry and its Relativistic Implications

Jason Glowney, MD, MSc

In this paper, we present a brief portion of material in the upcoming book ***Arithmoi – Mathematical Foundations and the Singularity***. This material focuses on exploring the complex plane of the areas bound between adjacent arms of horizontally and vertically oriented unit hyperbolae. We use geometric algebra to navigate the plane and arrive at an insightful and beautiful interpretation of special relativity based on its inherent geometric form.

We a rotation to arrive at the right portion of the unit hyperbola, $(cosh\beta e_2 + sinh\beta e_1)$ oriented in the positive vertical axis. We choose to use geometric algebra to represent the lower arm of the right horizontally oriented arm of the hyperbola as

$$\gamma^{-1}(\beta) = cosh\beta e_1 - sinh\beta e_2,$$

and we rotate this counterclockwise by $\frac{i\pi}{2}$ radians by taking the product with the bivector $e_1 \wedge e_2$

$$\gamma^{-1}(\beta) e_1 \wedge e_2 = (cosh\beta e_1 - sinh\beta e_2) e_1 \wedge e_2 = cosh\beta e_2 + sinh\beta e_1,$$

which brings us to the desired location in the complex hyperbolic plane. This rotation becomes clear when we view the figure below, which shows the orientation of the unit hyperbola arms with unit basis vectors e_1 and e_2 in the orientation.

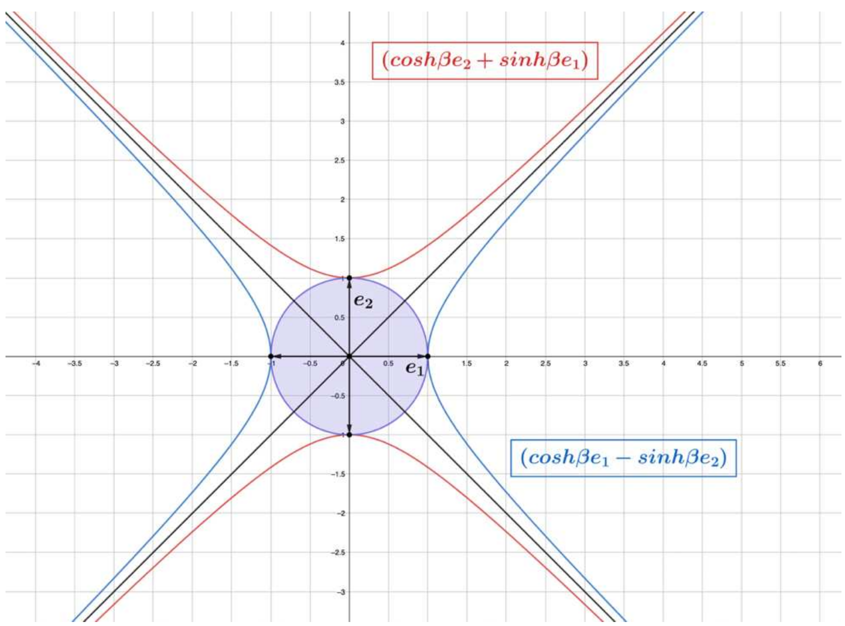


Figure 1 *– Oriented unit hyperbolae and their light asymptotes.*

Next, we denote $\gamma_s(\beta)$ and $\gamma_t(\beta)$ as the following parametric vector functions

$$\gamma_s(\beta) = cosh\beta e_1 + sinh\beta e_2,\ \gamma_t(\beta) = sinh\beta e_1 + cosh\beta e_2,$$

and adding the two vectors will give us

$$\gamma_s(\beta) + \gamma_t(\beta) = e^{\beta} e_1 + e^{\beta} e_2,$$

which represents the parametric form of the light asymptote in that quadrant, and we define the light asymptote vector to represent this sum

$$\hat{\rho}_c = \gamma_s(\beta) + \gamma_t(\beta) = \left(e^{\beta} e_1 + e^{\beta} e_2\right).$$

We plot $\hat{\rho}_c$ along with $\gamma_s(\beta)$ and $\gamma_t(\beta)$ in the following figure to visualize the exact orientation the vector functions have to one another.

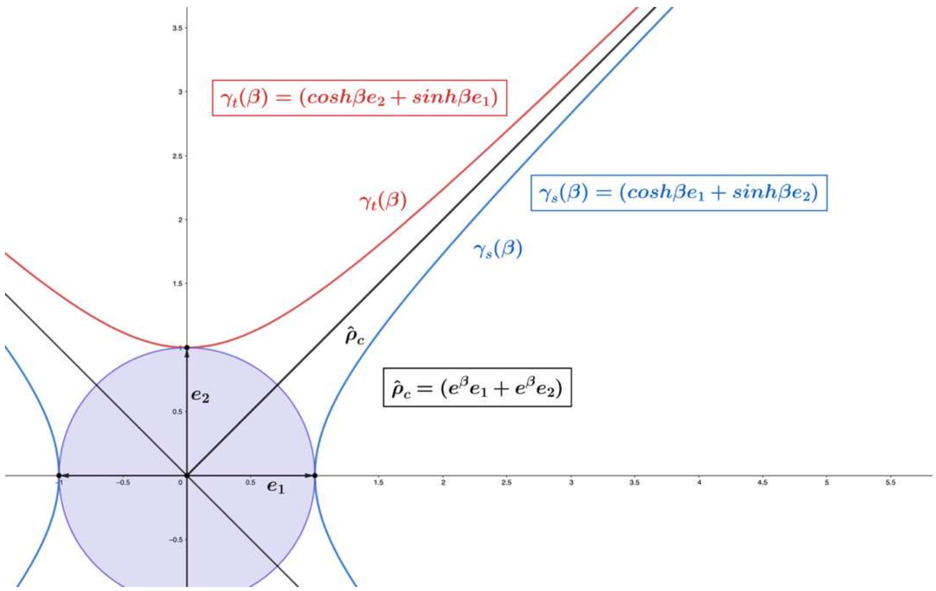


Figure 2 *– The parametric vector functions of the positively-oriented light asymptote.*

So, in the figure we definitively see the flat asymptotic light function $\hat{\rho}_c$, and we recall that the hyperbolic vector functions $\gamma_s(\beta)$ and $\gamma_t(\beta)$ are orthogonal to one another by the hyperbolic standard, differing from the Euclidean one. The key insight into the relationship between the light asymptote and the orthogonal unit hyperbolae is that they act together to form what we will define as a complex unit parallelogram, noting that a square is a parallelogram. This is akin to the complex unit circle, but instead of rotations about the fixed origin, the unit square experiences a shearing transform with its bottom left vertex remaining fixed at the origin. The complex unit square is the form noted when $\beta = 0$ for the $\gamma_s(\beta), \gamma_t(\beta)$ vectors, which is pictured in the following figure as the orange square with vertices at the origin, $\gamma_s(0), \gamma_t(0)$, and $\hat{\rho}_c(0)$.

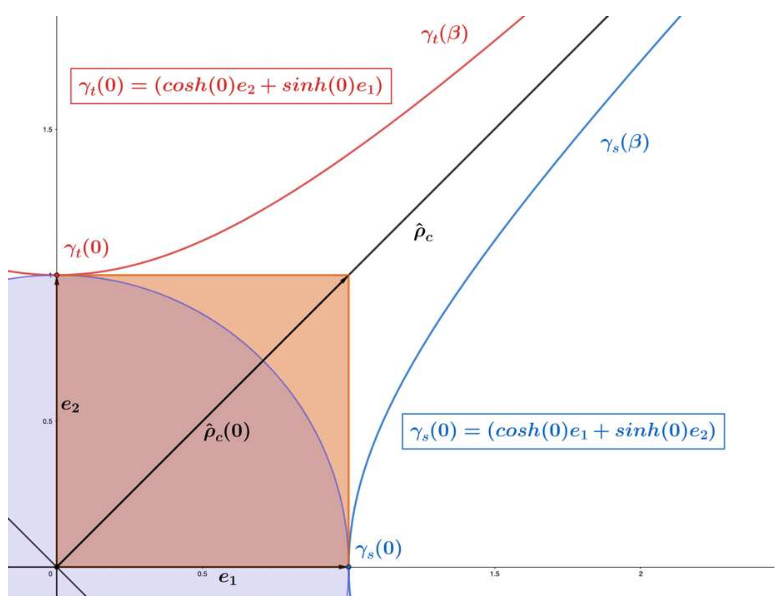


Figure 3 *– The complex unit parallelogram as the complex unit square for $\beta = 0$.*

We can find the area of this square as the wedge product $\gamma_s(\beta) \wedge \gamma_t(\beta)$

$$\gamma_s(\beta) \wedge \gamma_t(\beta) = (cosh\beta e_1 + sinh\beta e_2) \wedge (sinh\beta e_1 + cosh\beta e_2) = cosh^2 \beta\, e_1 \wedge e_2 + sinh^2 \beta\, e_2 \wedge e_1,$$

which we find simplifies to the bivector form

$$\gamma_s(\beta) \wedge \gamma_t(\beta) = I(cosh^2 \beta - sinh^2 \beta) = I,$$

and this holds for all values of $\beta \in \mathbb{R}$ including $\beta = 0$.

This is an invariant oriented area that defines light, and we note that all values for the rapidity β will have the same value of unit imaginary. The diagonal vector denoted in the complex unit parallelogram will take the value

$$\hat{\rho}_c = \gamma_s(0) + \gamma_t(0) = (e_1 + e_2),$$

which we can translate into a complex polar form as

$$(e_1 + e_2) = \sqrt{e_1^2 + e_2^2}\, e^{itan^{-1}\left(\frac{e_2}{e_1}\right)} = \sqrt{2} e^{\frac{i\pi}{4}},$$

and this is consistent with what is pictured in the figure. Next, we look at the shear transform of the complex unit square to the complex unit parallelogram by setting $\beta = 1$ for our boost, where the values for the non-origin vertices become

$$\gamma_s(1) = (cosh(1)\, e_1 + sinh(1)\, e_2),$$

$$\gamma_t(1) = (cosh(1)\, e_2 + sinh(1)\, e_1),$$

$$\hat{\rho}_c = \gamma_s(1) + \gamma_t(1) = e^1 e_1 + e^1 e_2.$$

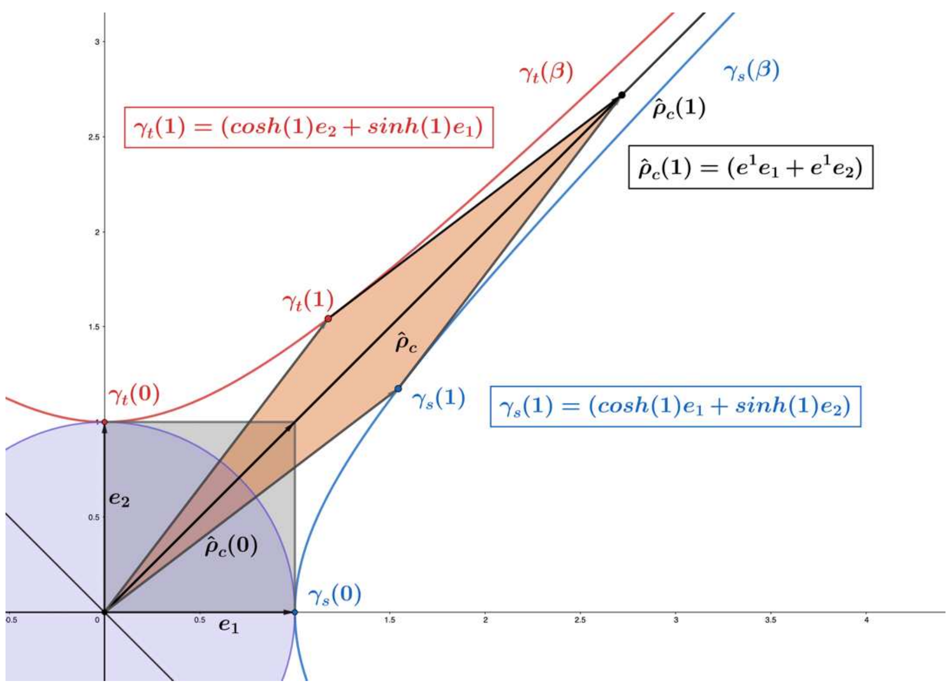


Figure 4 *– The complex unit parallelogram boost transform for* $\beta = 1$.

In ***Figure 4*** above, we plot the shear transform to the parallelogram with the above vertices, which is denoted in orange with its volume maintained in this asymptotic light region of the complex plane. If we were to draw a segment connecting the points $\gamma_s(1)$ and $\gamma_t(1)$, this would transect the $\hat{\rho}_c(1)$ vector at its center point, and we can also denote the square that is formed around this center point of $\hat{\rho}_c(1)$ with vertices defined as

$$\gamma_s(1), \gamma_t(1), (sinh_t(1)\, e_1 + sinh_s(1)\, e_2), (cosh_s(1)\, e_1 + cosh_t(1)\, e_2),$$

where we've combined the $cosh$ and $sinh$ component functions into proper vectors as

$$cosh_{s,t}(1) = (cosh_s(1)\, e_1 + cosh_t(1)\, e_2),$$

$$sinh_{t,s}(1) = (sinh_t(1)\, e_1 + sinh_s(1)\, e_2),$$

along with $\gamma_s(1), \gamma_t(1)$ vector functions. In the figure below, we zoom into the square formed by these vertices, where $\left(\frac{\hat{\rho}_c(1)}{2}\right)$ is the center point of the $\hat{\rho}_c(1)$ vector and we have denoted two vectors, one pointing from $\gamma_t(1)$ to $\gamma_s(1)$ and the other from $sinh_{t,s}(1)$ to $cosh_{s,t}(1)$.

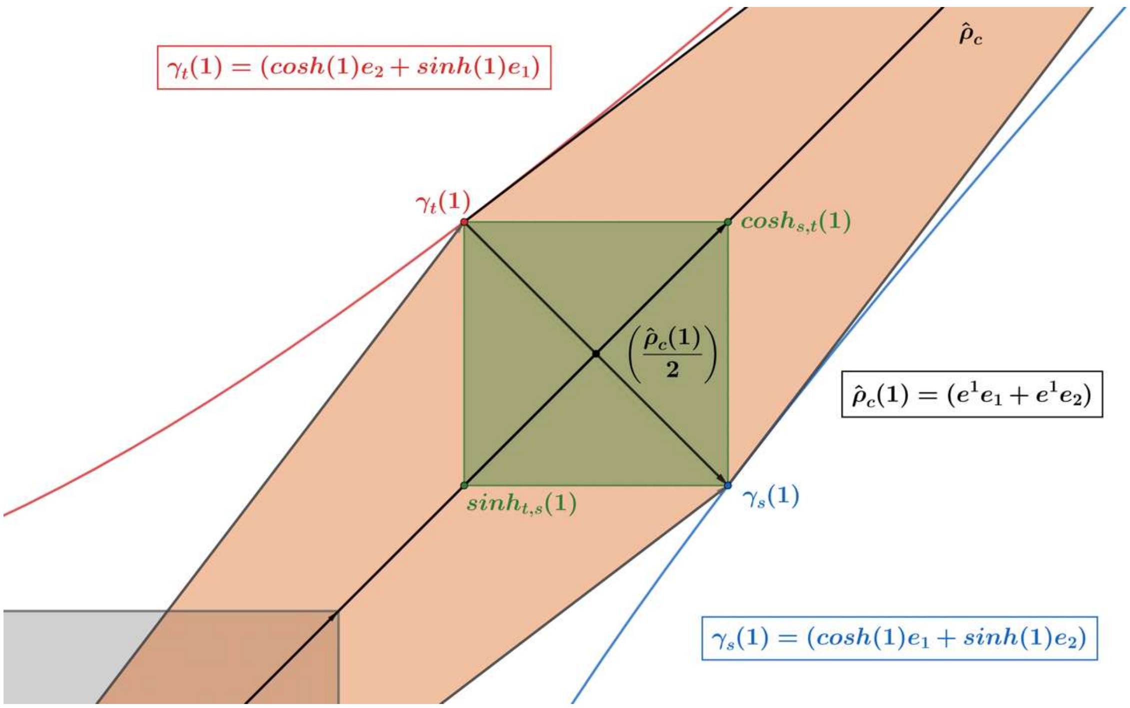


Figure 4 *– The inverse square oriented along the minor axis of the parallelogram.*

The vector running from $\gamma_t(1)$ to $\gamma_s(1)$ has the following representation

$$\gamma_s(1) - \gamma_t(1) = (cosh(1)\, e_1 + sinh(1)\, e_2) - (cosh(1)\, e_2 + sinh(1)\, e_1) = e^{-1}e_1 - e^{-1}e_2,$$

and takes the following form for any β value

$$\gamma_s(\beta) - \gamma_t(\beta) = e^{-\beta}e_1 - e^{-\beta}e_2.$$

And the vector running from $sinh_{t,s}(1)$ to $cosh_{s,t}(1)$ is equal to

$$cosh_{s,t}(1) - sinh_{t,s}(1) = (cosh_s(1)\, e_1 + cosh_t(1)\, e_2) - (sinh_t(1)\, e_1 + sinh_s(1)\, e_2) = e^{-1}e_1 + e^{-1}e_2,$$

taking the following form for any β value

$$cosh_{s,t}(\beta) - sinh_{t,s}(\beta) = e^{-\beta}e_1 + e^{-\beta}e_2.$$

Both vectors have components with the inverse exponential function $e^{-\beta}$, and the geometric product of $(e^{-\beta}e_1 + e^{-\beta}e_2)$ with the center point vector $\left(\frac{\hat{\rho}_c(\beta)}{2}\right)$, is equal to the identity

$$\left(e^{-\beta}e_1 + e^{-\beta}e_2\right)\left(\frac{\hat{\rho}_c(\beta)}{2}\right) = \left(e^{-\beta}e_1 + e^{-\beta}e_2\right)\left(\frac{e^{\beta}e_1 + e^{\beta}e_2}{2}\right) = \frac{e_1^2 + e_1e_2 + e_2e_1 + e_2^2}{2} = 1,$$

and a similar calculation with the $(e^{-\beta}e_1 - e^{-\beta}e_2)$ vector, we find yields the unit imaginary

$$\left(e^{-\beta}e_1 - e^{-\beta}e_2\right)\left(\frac{e^{\beta}e_1 + e^{\beta}e_2}{2}\right) = \frac{e_1^2 + e_1e_2 - e_2e_1 - e_2^2}{2} = I.$$

These vector product results underscore the fact that light is the perfect balance between the real and imaginary numbers. Taking with the fact that the area of the complex unit parallelogram is invariant, we have geometric rational as to why we always perceive the speed of light as invariant from the point of view of spacetime observers. If we consider ourselves the spacetime observer at the point $\gamma_s(1)$, we find the vector that points in the orthogonal direction to our location is equal to the unit imaginary, I, just as it was under initial conditions of $\beta = 0$, which reduces to the invariant expression of the form

$$\left(e^{0}e_1 - e^{0}e_2\right)\left(\frac{e^{0}e_1 + e^{0}e_2}{2}\right) = (e_1 - e_2)\left(\frac{e_1 + e_2}{2}\right) = \frac{e_1^2 + e_1e_2 - e_2e_1 - e_2^2}{2} = I.$$

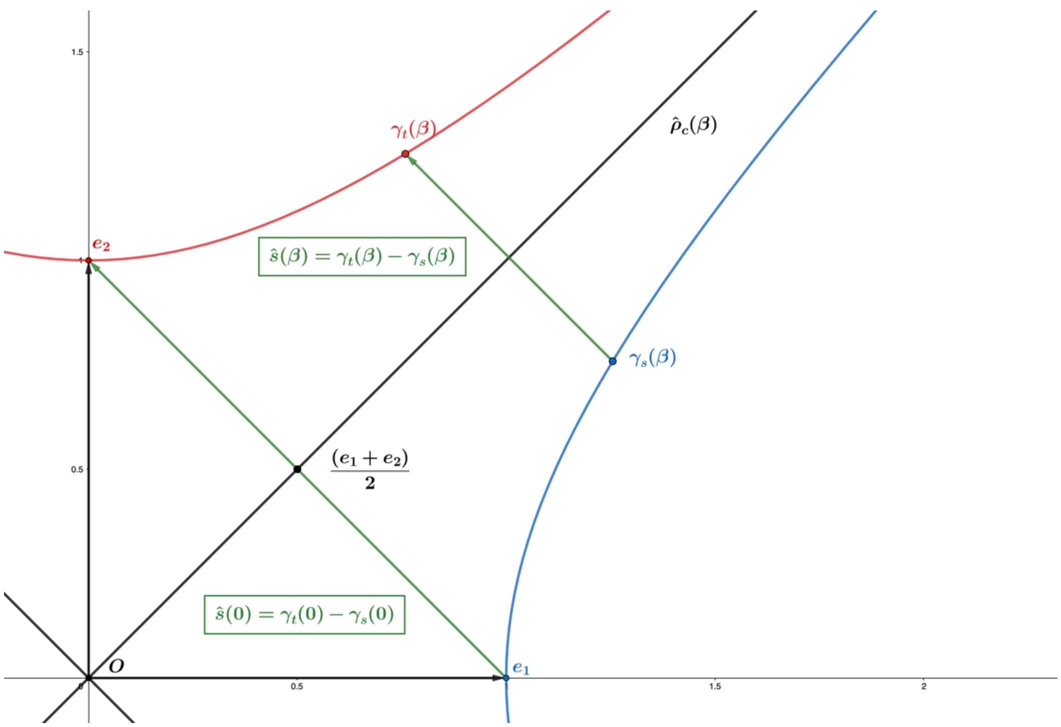


Figure 5 *– Oriented area of the hyperbolic asymptotic region.*

Next, we look to define the area contained between the arms of the hyperbolae by first considering the vector quantities that span the distance between the positive-valued unit hyperbolic curves from $\gamma_s(\beta)$ to $\gamma_t(\beta)$, as $\hat{s}(\beta)$

$$\hat{s}(\beta) = \gamma_t(\beta) - \gamma_s(\beta) = (cosh(\beta)\, e_2 + sinh(\beta)\, e_1) - (cosh(\beta)\, e_1 + sinh(\beta)\, e_2) = -e^{-\beta}e_1 + e^{-\beta}e_2.$$

The base of the asymptotic area running from the vector function $\gamma_s(0)$ to $\gamma_t(0)$, will then represents the starting span that we will be using.

Shown in green in ***Figure 5*** is the $\hat{s}(\beta)$ vector starting at $\beta = 0$, which is oriented from $\gamma_s(\beta)$ to $\gamma_t(\beta)$. The sum of all the $\hat{s}(\beta)$ vectors will yield the oriented area of the asymptotic section. We put $\hat{s}(\beta)$ into a polar coordinate form as

$$\hat{s}(\beta) = (e_2 - e_1)e^{-\beta} = \sqrt{(-e^{-\beta}e_1)^2 + (e^{-\beta}e_2)^2}e^{-\beta}e^{\frac{3i\pi}{4}} = \sqrt{2}e^{\left(-\beta+\frac{3i\pi}{4}\right)},$$

and consider the integral with integration bounds of β going from $0 \to \infty$

$$\hat{A} = \int_0^\infty \sqrt{2}e^{\left(-\beta+\frac{3i\pi}{4}\right)}\,d\beta = \sqrt{2}\,e^{-\beta-\frac{i\pi}{4}}\Big|_{\beta=0}^{\beta=\infty} = \sqrt{2}e^{-\infty-\frac{i\pi}{4}} - \left(\sqrt{2}e^{-0-\frac{i\pi}{4}}\right) = \sqrt{2}e^{\frac{3i\pi}{4}} = (i-1),$$

where we find an oriented area as a result. This represents a beautiful result of asymptotic geometry, given that this converges to an oriented area which is the same value as the vector of integration. In terms of geometric algebra, we could have simply represented the oriented area integral as

$$\hat{A} = \int_0^\infty (e_2 - e_1)e^{-\beta}\,d\beta = (e_1 - e_2)e^{-\beta}\Big|_{\beta=0}^{\beta=\infty} = (e_2 - e_1).$$

Given that this is an oriented area, we can choose to instead integrate the oppositely oriented vector, where the distance vector segments would be

$$\hat{s}' = \gamma_s(\beta) - \gamma_t(\beta) = (cosh(\beta)\,e_1 + sinh(\beta)\,e_2) - (cosh(\beta)\,e_2 + sinh(\beta)\,e_1) = e^{-\beta}e_1 - e^{-\beta}e_2,$$

and the oriented area integral would be the negative-valued $\hat{A}$

$$\hat{A}' = \int_0^\infty (e_1 - e_2)e^{-\beta}\,d\beta = (e_2 - e_1)e^{-\beta}\Big|_{\beta=0}^{\beta=\infty} = (e_1 - e_2).$$

We can take the oriented area results and find the square root of their geometric product,

$$A = \sqrt{\hat{A}\hat{A}'} = \sqrt{(e_2 - e_1)(e_1 - e_2)} = \sqrt{e_2 \wedge e_1 - e_2^2 - e_1^2 + e_1 \wedge e_2} = I\sqrt{2},$$

to be the unit imaginary bivector I scaled by $\sqrt{2}$. Note that this result is not dependent on the order of $\sqrt{\hat{A}\hat{A}'}$, evidenced by the fact that $\hat{A}$ and $\hat{A}'$ commute

$$\hat{A}\hat{A}' - \hat{A}'\hat{A} = 0.$$

The asymptotic light algebra we have been working with can be expressed in the matrix form as

$$G = \begin{pmatrix} (1+i)cosh\beta & (1+i)sinh\beta \\ (1+i)sinh\beta & (1+i)cosh\beta \end{pmatrix},$$

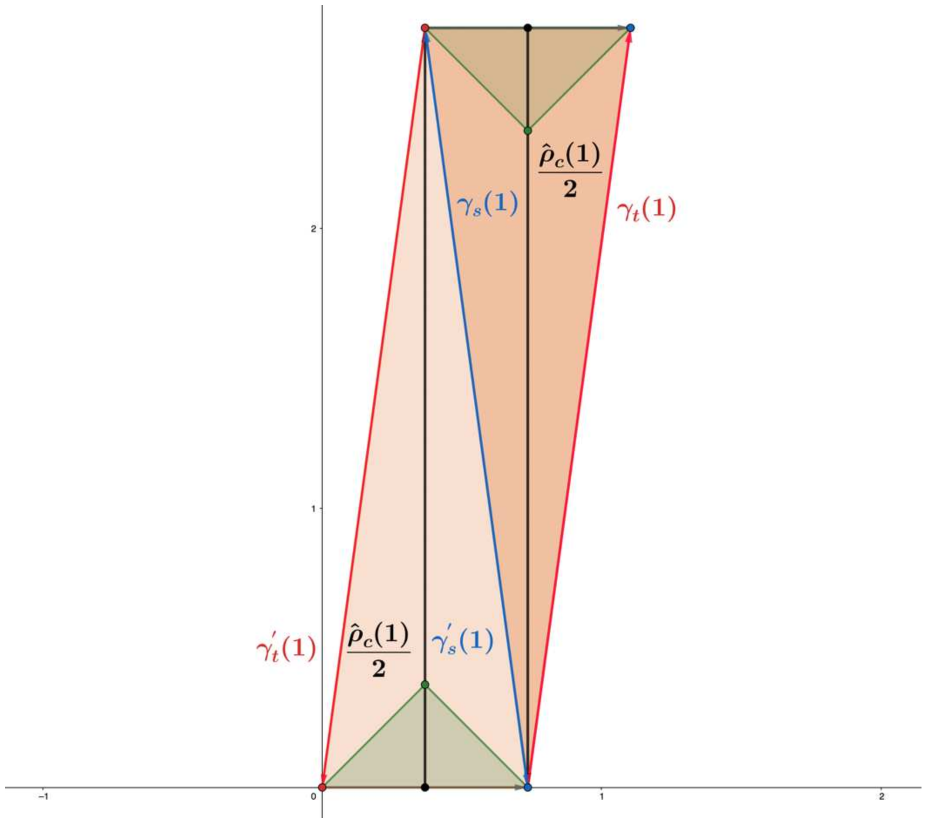
which is the result of the matrix exponential of the Lie algebra $\mathfrak{g}$

$$\mathfrak{g} = \begin{pmatrix} \frac{ln(2)}{2} + \frac{i\pi}{4} & b \\ b & \frac{ln(2)}{2} + \frac{i\pi}{4} \end{pmatrix}.$$

The square root of the determinant of G is isomorphic to the result found for $A = \sqrt{\hat{A}\hat{A}'} = I\sqrt{2}$

$$\sqrt{det(G)} = \left(det\begin{pmatrix}(1+i)cosh\beta & (1+i)sinh\beta \\ (1+i)sinh\beta & (1+i)cosh\beta\end{pmatrix}\right)^{\frac{1}{2}} = \sqrt{2icosh^2\beta - 2isinh^2\beta} = i\sqrt{2}.$$

Now, we turn to look at the area of the parallelogram which is equal to the product of the base and its height. By cutting the elongated parallelogram at the center point of the $\hat{\rho}_c$ vector, we arrange the portions as seen in the figure below, where the vectors $\gamma_t'(1), \gamma_s'(1)$ are the opposing parallelogram edges to the vectors $\gamma_t(t), \gamma_s(t)$.



The value of the base of the parallelogram in the figure is $\gamma_t(1) - \gamma_s(1) = -e^{-1}e_1 + e^{-1}e_2$, and the height of the vector is equal to the vector $\left(\frac{\hat{\rho}_c(1)}{2}\right) = \left(\frac{e_1+e_2}{2}\right)e^1$. The geometric product of these vectors equals the following area

$$\hat{A} = (-e^{-1}e_1 + e^{-1}e_2)\left(\frac{e^1e_1 + e^1e_2}{2}\right) = \frac{-1 - e_1 \wedge e_2 + e_2 \wedge e_1 + 1}{2} = -I,$$

which we find is an oriented unit imaginary bivector. Taking the oriented area of the parallelogram using the base vector $\gamma_s(1) - \gamma_t(1) = e^{-1}e_1 - e^{-1}e_2$, we find

$$\hat{A}' = (e^{-1}e_1 - e^{-1}e_2)\left(\frac{e^1e_1 + e^1e_2}{2}\right) = \frac{1 + e_1 \wedge e_2 - e_2 \wedge e_1 - 1}{2} = I,$$

which is the contrary oriented area of the parallelogram. We can define the square root of the product $\hat{A}\hat{A}'$ as the scalar value of the area

$$\sqrt{\hat{A}\hat{A}'} = \sqrt{\hat{A}'\hat{A}} = 1.$$

This relationship holds for all the parallelogram forms and takes the generalized expression

$$\hat{A}(\beta) = \left(-e^{-\beta}e_1 + e^{-\beta}e_2\right)\left(\frac{e^{\beta}e_1 + e^{\beta}e_2}{2}\right) = -I,$$

$$\hat{A}'(\beta) = \left(e^{-\beta}e_1 - e^{-\beta}e_2\right)\left(\frac{e^{\beta}e_1 + e^{\beta}e_2}{2}\right) = I,$$

where the invariant result does not depend on the rapidity β. This means that for the following limit

$$\lim_{\beta\to\infty}\sqrt{\hat{A}(\beta)\hat{A}'(\beta)} = \lim_{\beta\to\infty}\sqrt{\left((-e_1 + e_2)e^{-\beta}\left(\frac{e_1 + e_2}{2}\right)e^{\beta}\right)\left((e_1 - e_2)e^{-\beta}\left(\frac{e_1 + e_2}{2}\right)e^{\beta}\right)}$$

$$= \lim_{\beta\to\infty}\sqrt{\left((-e_1 + e_2)\left(\frac{e_1 + e_2}{2}\right)\right)\left((e_1 - e_2)\left(\frac{e_1 + e_2}{2}\right)\right)} = \sqrt{(e_2 \wedge e_1)(e_1 \wedge e_2)} = 1,$$

the result also holds and is equal to one. What this means geometrically is that the vector infinite in length, resulting when $\beta \to \infty$, will have an area equal to one. This means that we have found a Dirac delta function residing in the geometry of the asymptotic light region. This comes to us in a most intuitive form, stemming from the inverse arrangement of oppositely oriented unit imaginary bivectors and inversely related exponential functions.