Imaginary Angles and the Connection between Euclidean and Space-time Rotations

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Abstract
Rotations thru imaginary angles represent a very clean and concise means to link Euclidean rotations with that of space-time. Starting with Euler's formula we are easily able to rotate thru an imaginary angle and see that the Lorentz boosts seamlessly appear in the form of the hyperbolic functions.

Rotation by an Imaginary Angle
Many a mathematician and physicist seem to struggle with the concept of imaginary values. They will too frequently square and take their absolute value to make their preference for real numbers be satisfied. It is the goal of the proceeding writings to unify Euclidean rotations with that of space-time. We will start with Euler's Formula as

$$e^{i\theta} = \cos(\theta) + isin(\theta). \quad (1)$$

We will assume the θ value in the above is that of a phasor and represents the speed of change of the angle in terms of the real vs. the imaginary components. When its value is real we have rotation in a Euclidean sense. If we were to assume a θ value that is imaginary with rotation in the clockwise direction we would find Eq. 1 turns into the hyperbolic form, where we have pulled out the imaginary value of *i* from the rotation angle and represent it as $-i\theta$

$$e^{-i^2\theta} = \cos(-i\theta) + isin(-i\theta) \rightarrow e^{\theta} = \cosh(\theta) + \sinh(\theta). \quad (2)$$

Eq. 2 above shows the intimate link between the Euler's formula and the hyperbolic functions. Many are familiar with the complex unit circle representation of Eq. 1, but fewer may be familiar with the unit hyperbola representation. We will see later the role these play with the Lorentz boosts.

In Eq. 2 we will assume that $\cosh(\theta)$ represent speed thru space and that $\sinh(\theta)$ represents a speed thru time. To find the distance equivalents we will integrate and find that

$$\int \cosh(\theta) + \sinh(\theta) = \sinh(\theta) + \cosh(\theta). \quad (3)$$

It is important to note that in Eq. 3 $\sinh(\theta)$ is now the distance thru time and that $\cosh(\theta)$ represents the distance thru space.

There is no sufficient reason to not believe that there exists rotation thru imaginary angles. Once we are open to it and incorporate it into our mathematical mindset it makes the link between the quantum and space-time all the more intuitive and tenable.

The Lorentz Boost and Hyperbolic Geometry

In 1905 Einstein introduced a unification of time and space into what is called space-time in his seminal work "On the Electrodynamics of Moving Bodies". Today this unification is known as special relativity. For many this unification is very counterintuitive, as we perceive space and time very differently. The core of special relativity is based on the Lorentz transformations and these are actually just hyperbolic rotations in disguise. With this fact in mind, we can describe Special Relativity as an extension of hyperbolic geometry in a very concise and beautiful way. We will replace θ with the more commonly used β for our angles.

To start we will remind the reader of the hyperbolic trigonometric functions

$$cosh\beta = \frac{e^{\beta} + e^{-\beta}}{2}, \qquad sinh\beta = \frac{e^{\beta} - e^{-\beta}}{2}, \qquad and\ tanh\beta = \frac{sinh\beta}{cosh\beta}. \quad (4)$$

Where as Euclidean distance is based on the unit circle distance, in space-time it is based on the unit hyperbola and denoted as

$$x^2 - y^2 = 1\,. \quad (5)$$

We can create a hyperbolic rotation matrix (H) such that

$$H = \begin{pmatrix} cosh\beta & sinh\beta \\ sinh\beta & cosh\beta \end{pmatrix}. \quad (6)$$

Taking the determinate yields

$$det \begin{pmatrix} cosh\beta & sinh\beta \\ sinh\beta & cosh\beta \end{pmatrix} = \cosh^2\beta - \sinh^2\beta. \quad (7)$$

Where we know that our hyperbolic distance formula is

$$\cosh^2\beta - \sinh^2\beta = 1. \quad (8)$$

The Lorentz transformation γ is know as

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (9)$$

Let us start with eq. 8 and solve for $cosh\beta$

$$\cosh^2\beta - \sinh^2\beta = 1 \rightarrow \frac{\cosh^2\beta}{\cosh^2\beta} - \frac{\sinh^2\beta}{\cosh^2\beta} = \frac{1}{\cosh^2\beta}$$
$$\rightarrow 1 - \tanh^2\beta = \frac{1}{\cosh^2\beta} \rightarrow \cosh^2\beta = \frac{1}{1-\tanh^2\beta}$$
$$\rightarrow cosh\beta = \frac{1}{\sqrt{1-\tanh^2\beta}}. \quad (10)$$

Eq. 10 looks very much like the Lorentz transformation for γ with $\tanh^2\beta = \left(\frac{v}{c}\right)^2$. Let us consider the Lorentz transformations between the rest frame (x, t) and the frame in relative motion along the positive x direction at speed v (x', t') [1]

$$x = \gamma(x' + vt') \quad (11)$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right). \quad (12)$$

The key to converting to hyperbolic geometry is to measure space and time in the same units and to replace t by ct:

$$x = \gamma\left(x' + \frac{v}{c}ct'\right) \quad (13)$$

$$ct = \gamma\left(ct' + \frac{v}{c}x'\right) \quad (14)$$

Now we can solve for $sinh\beta$ as

$$\frac{v}{c}\gamma = tanh\beta * cosh\beta = sinh\beta. \quad (15)$$

We can now insert our hyperbolic identities into the Lorentz transformations and find that

$$x = x'cosh\beta + ct'sinh\beta \quad (16)$$
$$ct = x'sinh\beta + ct'cosh\beta. \quad (17)$$

Putting this into matrix form we now have

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} cosh\beta & sinh\beta \\ sinh\beta & cosh\beta \end{pmatrix}\begin{pmatrix} x' \\ ct' \end{pmatrix}. \quad (18)$$

We can also multiply thru by the inverse matrix H^{-1} on each left side of Eq. 34 with

$$H^{-1} = \frac{1}{detH}\begin{pmatrix} cosh\beta & -sinh\beta \\ -sinh\beta & cosh\beta \end{pmatrix}. \quad (19)$$

Calculating the determinate of our original Hyperbolic Matrix H

$$det\ H = (\cosh^2\beta - \sinh^2\beta) = 1\,. \quad (20)$$

So we arrive at

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} cosh\beta & -sinh\beta \\ -sinh\beta & cosh\beta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}. \quad (21)$$

If needed we can calculate our value of β by taking the inverse hyperbolic tangent of velocity (v) divided by the speed of light (c)

$$\beta = \tanh^{-1}\left(\frac{v}{c}\right) \quad (22)$$

Addition of Velocities

If we were to consider an observer moving at a velocity v along the positive x-axis, his or her word line would intersect the unit hyperbola at a point $(cosh\beta, sinh\beta)$

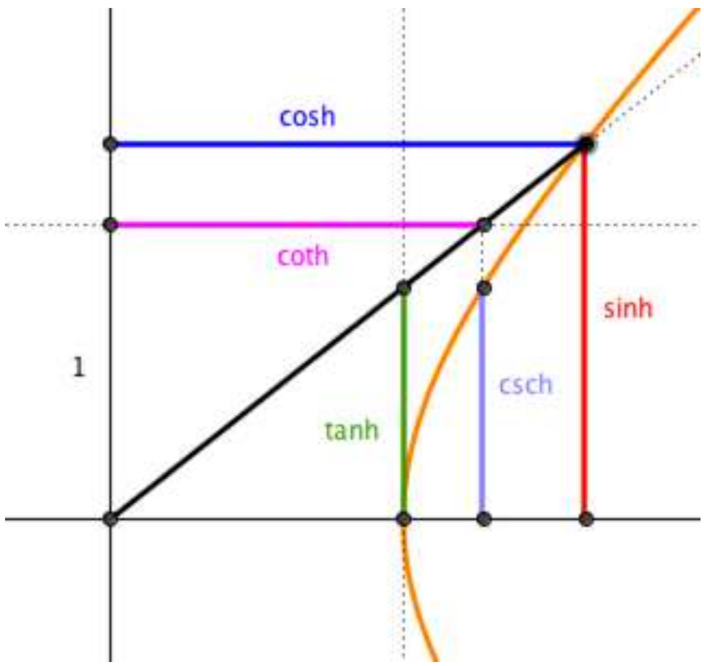

Figure 1. The value of β is from the point $(\cosh 0\,, \sinh 0) = (1,0)$ on the x-axis along the hyperbola (orange curve) to $(sinh\beta, cosh\beta)$.

with the slope of this line being equal to:

$$\frac{v}{c} = tanh\beta \quad (23)$$

We can think of β as the hyperbolic angle between the ct-axis and a moving object *x's* word line. The value of β is the actual distance from the point where the unit hyperbola intersects the x-axis, to the point on the unit hyperbola at $(cosh\beta, sinh\beta)$. See Fig. 1.

If we were to consider a person in a spaceship moving at a speed of *u* relative to another in motion observer moving at a speed of *v.* The rapidities that would apply to our *u* and *v* would be the following:

$$\frac{u}{c} = tanh\alpha \quad (24)$$

$$\frac{v}{c} = tanh\beta. \quad (25)$$

Now to determine the additive speed a stationary observer would perceive for these moving individuals we would add the rapidities. This is where the ease and the inherent beauty of hyperbolic geometry in relation to special relativity comes into play. We will call the addition of the rapidities *w* and we can denote and calculate as

$$\frac{w}{c} = \tanh(\alpha + \beta) = \frac{tanh\alpha + tanh\beta}{1 + tanh\alpha * tanh\beta} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}}. \quad (26)$$

We can now see that this relation from our hyperbolic geometry is exactly the Einstein addition formula.

A quick example would be to say we have a stationary observer relative to *u* and *v,* with *u* travelling at 0.5c and with *v* travelling at 0.6c relative to *u*. Normally in Galilean space we would expect the stationary observer to perceive *v* travelling at faster than the speed of light, but in terms of hyperbolic geometry we will find that the stationary observer will see traveller *v* as moving at

$$w = \tanh(0.5 + 0.6) = \frac{\tanh(0.5) + \tanh(0.6)}{1 + \tanh(0.5) * \tanh(0.6)} = 0.8005c. \quad (27)$$

So we see that the stationary observer will see space traveller *v* as moving at approximately 80% the speed of light! Note we have set c=1 in the previous calculation.

We can also take a transformation where we solve for sinhβ, which would represent the distance thru time function.

Let us start with eq. 8 and solve for $sinh\beta$

$$\cosh^2\beta - \sinh^2\beta = 1 \to \frac{\cosh^2\beta}{\sinh^2\beta} - \frac{\sinh^2\beta}{\sinh^2\beta} = \frac{1}{\sinh^2\beta}$$
$$\to \coth^2\beta - 1 = \frac{1}{\sinh^2\beta} \to \sinh^2\beta = \frac{1}{\coth^2\beta - 1}$$
$$\to sinh\beta = \frac{1}{\sqrt{\coth^2\beta - 1}}. \quad (28)$$

Conclusion

By starting with the mathematical framework of Euler's formula and by not giving real angles precedence over those imaginary, we arrive at the core tenets of special relativity in a much more mathematically sound and intuitive manner.

[1] Dray, Tevian, "The Geometry of Special Relativity", A K Peters/CRC Press, Boca Raton, July 2012.