A Novel *U(3)* Group and Applications to Quantum Chromodynamics

Jason Glowney
University of Washington
Applied Mathematics

Abstract

We review a finite group-derived *U(3)* and analyze the structure of its matrix group. This *U(3)* contains its primary generators thru which all group permutations can be obtained, as well as a Cartan subalgebra that has a profound correlation with the strong force in Quantum Chromodynamics. The *SU(3)* group and its Gell-Mann matrices are explored with its short-comings and contrasted to this mathematically complete *U(3)* group.

Introduction

Quantum Chromodynamics is the theory that represents the strong interaction between quarks and gluons that make up hadrons such as the proton and neutron. It is a quantum field theory that is normally represented by the *SU(3)* group, which is a non-Abelian gauge theory. In this theory, gluons are the force carrier of color charge and these colors are typically referred to as red, green and blue. The force in QCD is analogous to the electro-magnetic force of quantum electro-dynamics.

With years of experimental data, QCD seems to exhibit two main properties that are color confinement and asymptotic freedom. Color confinement refers to the phenomenon that color charged particles like quarks and gluons are not found in isolation and will not be observed in normal circumstances. Gluons and quarks, by way of this color confinement property, bind together to form hadrons. Asymptotic freedom refers to the property that interactions between particles will become asymptotically weaker as the scale of energy increases and the corresponding length scale decreases.

The Gell-Mann Matrices of *SU(3)*

SU(3) is the group of all 3×3 matrices with unit determinants denoted as

$$\det(U) = 1, \qquad UU^\dagger = U^\dagger U = 1. \tag{1}$$

We present the eight Gell-Mann[1] matrices below, where all of these matrices are traceless and Hermitian. The generators of the *SU(3)* are connected to these matrices where $T_A = \frac{1}{2}\lambda_A$.

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda_5 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3}$$

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[1] Gell-Mann, M. (1961). "***The Eightfold Way: A Theory of strong interaction symmetry***" (No. TID-12608; CTSL-20). California Inst. of Tech., Pasadena. Synchrotron Lab

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{4}$$

As this represents a Lie group, we can write these as an exponential function

$$U = e^{iT_A \theta_A}. \tag{5}$$

As the generators are traceless and Hermitian, the can generate unitary matrix group elements by way of exponentiation. The trace of a pairwise product of $\lambda_i\ and\ \lambda_j$ will obey the relation

$$trace(\lambda_i \lambda_j) = 2\delta_{ij}\ and\ \delta_{ij} = \begin{cases} 0\ if\ i \neq j \\ 1\ if\ i = j \end{cases}. \tag{6}$$

The λ_8 and λ_3 matrices represent the Cartan generators of the group where $\lambda_3 \lambda_8 = \lambda_8 \lambda_3$ and the eigenvalues of T_3 and T_8 are

$$T_3 = -\frac{1}{2}, \frac{1}{2}\ and\ T_8 = \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{sqrt(3)}, \tag{7}$$

and the eigenvectors of T_3 and T_8 are both respectively

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{8}$$

One way of representing color charge for strong force interacting fermions, would be to arrange them into a triplet in the basis spanned by the Cartan generators and their eigenvectors. One assignment of labelling could be [2]

$$\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)\ for\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \psi \rightarrow color\ red, \tag{9}$$

$$\left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)\ for\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \psi \rightarrow color\ blue, \tag{10}$$

$$\left(0, -\frac{1}{\sqrt{3}}\right)\ for\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \psi \rightarrow color\ green. \tag{11}$$

The idea for a colorless state would come from adding the three colors with the net result of zero where we find

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \psi \rightarrow \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) + \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) + \left(0, -\frac{1}{\sqrt{3}}\right) = 0. \tag{12}$$

From the standpoint of the definition of a group, the Gell-Mann generators might give the mathematician chest pain, as the mathematical definition of a group includes

$$Closure \rightarrow\ \forall\ \lambda_i, \lambda_j \in G, \lambda_i\ \cdot \lambda_j \in G, \tag{13}$$

$$Associativity \rightarrow\ \forall\ \lambda_i, \lambda_j, \lambda_k \in G, (\ \lambda_i\ \cdot \lambda_j) \cdot \lambda_k = \lambda_i \cdot (\lambda_j \cdot \lambda_k), \tag{14}$$

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[2] Schwichtenberg, Jakob - ***Physics from Symmetry***, Springer 2015

$$Identity \rightarrow \; \exists \;\; e \in G: \forall \; a \in G, a \cdot e = e \cdot a, \tag{15}$$

$$Inverse\ Element \rightarrow \forall \; a \in G \; \exists \; b \in G: a \cdot b = b \cdot a = e. \tag{16}$$

When we apply some of these definitions to the generators of the Gell-Mann matrices, we find many do not standup to the test. Some examples would be

$$Inverse\text{:}\ \lambda_6^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{0}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \infty, \tag{17}$$

$$Closure\text{:}\ \lambda_6 \cdot \lambda_3 \cdot \lambda_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin G, \tag{18}$$

$$Identity\text{:}\ \lambda_6^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = e \rightarrow \; e \cdot \lambda_4 = \lambda_4\ but\ e \cdot \lambda_4 \neq \lambda_4. \tag{19}$$

So, above we see that not all of the generators of *SU(3)* have inverses, the group is not closed under multiplication of its generators, and there is the lack of a true multiplicative identity that holds for all elements in the group. These issues with *SU(3)* give rise to debate as to whether or not this group actually exists. This issue is the impetus behind this paper and the application of a mathematically consistent *U(3)* group, that meets the group definition criteria, and exploring its potential role in quantum chromodynamics.

The Unitary Representation *U(3)* of the Complex Unit Sphere

In a previous paper, we detailed the derivation of the $\mathbb{R}^6$ matrix group that represented *SO(3,3)* [3]. This group represents a mathematical fashion of symmetrically joining complex two-dimensional subgroups into a higher-dimensional form. This matrix group when represented in real form, has three imaginary permutations and three real permutations. This is in contrast to the quaternionic three imaginary permutations and one real. We can represent the $\mathbb{R}^2$ form for $\mathbb{C}$ simply as $\sqrt{-1}$ seen below

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = i. \tag{20}$$

For the *U(3)* representation, we are transforming $\mathbb{R}^{2n} \rightarrow \mathbb{C}^n$, or more specifically, $\mathbb{R}^6 \rightarrow \mathbb{C}^3$. We setup our permutations as such

$$X_1 = i\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, X_2 = i\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, X_3 = i\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$Y_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Y_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, Y_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{21}$$

It is simple to note that $X_1^2 = X_2^2 = X_3^2 = -\mathbb{I}$. From a mathematical standpoint, when we multiply an positive imaginary number by another positive imaginary number, the result will be a negative real number. We see this with our X_i permutations as

[3] Glowney, Jason "***The Lie Representation of the Complex Unit Sphere***", https://arxiv.org/abs/1707.03283, 2018

$$X_1X_2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad X_2X_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix},$$

$$X_3X_1 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad X_1X_3 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \tag{22}$$

$$X_2X_3 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad X_3X_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}.$$

We will have the reader note, that for any $-X_i \times X_i$, the result is a positive real permutation. The above representations contrast to the quaternion, where there can be three real permutations, instead of one, that result from multiplication of the X_i group generators. In essence, we are placing together three *U(1)* subgroups that are orthogonal to one another and in $\mathbb{C}^3$.

We distinguish the different X_i permutations by labelling them, and then adding them together in a linear combination to arrive at

$$\mathfrak{g} = \begin{bmatrix} X_1 & X_3 & X_2 \\ X_3 & X_2 & X_1 \\ X_2 & X_1 & X_3 \end{bmatrix}. \tag{23}$$

We will use only the X_i generators, which is similar to the matrix representation of the complex unit circle where we exponentiate only ix without numbers in $\mathbb{R}$. This will result in the trigonometric form that represent *U(1) as cos(x)+isin(x).* So, to find our *U(3)* matrix group we would exponentiate the matrix $\mathfrak{g}$

$$\mathfrak{G} = e^{\mathfrak{g}}. \tag{24}$$

Our matrix $\mathfrak{g}$ turns out to be diagonalizable, and we are able to take advantage of the fact that $\mathfrak{g}$ can be represented as

$$\mathfrak{g} = VDV^{-1}, \tag{25}$$

Where D is the diagonal representations of the eigenvalues of $\mathfrak{g}$ and V is the eigenvectors of $\mathfrak{g}$ as column vectors. This simplifies the derivation of the matrix group *U(3)* as

$$\mathfrak{G} = Ve^{D}V^{-1}. \tag{26}$$

Using this approach to diagonalize $\mathfrak{g}$, we find the follow representations for D and U as

$$D = \begin{bmatrix} i\boldsymbol{\lambda} & 0 & 0 \\ 0 & -i\boldsymbol{\lambda} & 0 \\ 0 & 0 & i\boldsymbol{\gamma} \end{bmatrix} \tag{27}$$

where $\boldsymbol{\lambda} = \sqrt{X_1^2 - X_1X_2 - X_1X_3 + X_2^2 - X_2X_3 + X_3^2}$ and $\boldsymbol{\gamma} = X_1 + X_2 + X_3$. We proceed to find the eigenvectors to be

$$U = \begin{bmatrix} \frac{(X_3 - X_1 - \boldsymbol{\lambda})}{(X_1 - X_2)} & -\frac{(X_1 - X_3 - \boldsymbol{\lambda})}{(X_1 - X_2)} & 1 \\ -\frac{(X_3 - X_2 - \boldsymbol{\lambda})}{(X_1 - X_2)} & \frac{(X_2 - X_3 - \boldsymbol{\lambda})}{(X_1 - X_2)} & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{28}$$

We continue and calculate the matrix form of *U(3)* using ***Eq. (26)*** and find

$$\mathfrak{G} = \begin{bmatrix} \frac{e^{i\gamma}}{3}+\frac{2\cos(\boldsymbol{\lambda})}{3}+\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_1}}\cdot\frac{isin(\boldsymbol{\lambda})}{3} & \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{\lambda})}{3}+\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_3}}\cdot\frac{isin(\boldsymbol{\lambda})}{3} & \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{\lambda})}{3}+\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_2}}\cdot\frac{isin(\boldsymbol{\lambda})}{3} \\ \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{\lambda})}{3}+\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_3}}\cdot\frac{isin(\boldsymbol{\lambda})}{3} & \frac{e^{i\gamma}}{3}+\frac{2\cos(\boldsymbol{\lambda})}{3}+\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_2}}\cdot\frac{isin(\boldsymbol{\lambda})}{3} & \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{\lambda})}{3}+\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_1}}\cdot\frac{isin(\boldsymbol{\lambda})}{3} \\ \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{\lambda})}{3}+\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_2}}\cdot\frac{isin(\boldsymbol{\lambda})}{3} & \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{\lambda})}{3}+\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_1}}\cdot\frac{isin(\boldsymbol{\lambda})}{3} & \frac{e^{i\gamma}}{3}+\frac{2\cos(\boldsymbol{\lambda})}{3}+\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_3}}\cdot\frac{isin(\boldsymbol{\lambda})}{3} \end{bmatrix}, \tag{29}$$

where $\boldsymbol{\lambda} = \sqrt{X_1^2 - X_1X_2 - X_1X_3 + X_2^2 - X_2X_3 + X_3^2}$, $\boldsymbol{\gamma} = (X_1 + X_2 + X_3)$, $\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_1}} = \frac{(2X_1 - X_2 - X_3)}{\lambda}$, $\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_2}} = \frac{(-X_1 + 2X_2 - X_3)}{\lambda}$, and $\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{X_3}} = \frac{(-X_1 - X_2 + 2X_3)}{\lambda}$.

The matrix group above, represents the linear combination of X_i generators $\in \mathfrak{g}$ that are exponentiated as

$$\mathfrak{G} = \exp\,(x_1X_1 + x_2X_2 + x_3X_3) \in U(3) \textit{ and } x_iX_i \in \mathbb{C}^3. \tag{30}$$

We can calculate the determinant of $\mathfrak{G}$ and find

$$\det(\mathfrak{G}) = e^{i\gamma}\cos^2(\boldsymbol{\lambda}) + e^{i\gamma}\sin^2(\boldsymbol{\lambda}) = e^{i\gamma}\,(\cos^2(\boldsymbol{\lambda}) + \sin^2(\boldsymbol{\lambda})) = e^{i\boldsymbol{\gamma}}\,. \tag{31}$$

The result above in ***Eq. (31)*** coincides with the fact that for any complex square matrix, we have that

$$\det(e^{\mathfrak{g}}) = e^{Trace(\mathfrak{g})}. \tag{32}$$

Recall that $\gamma = (x + y + z)$, and we have that the modulus of the determinate will always be equal to one. This is a notable result, as we normally require a trace equal to zero in our group matrix to ensure it has a determinant equal to unity in the Lie group form. It is also easy to appreciate that the $\mathfrak{G}$ Lie group is equal to its own transpose as $\mathfrak{G} - \mathfrak{G}^T = 0$ and as such is a symmetric matrix. We can also appreciate that it is possible to rotate by three angles simultaneously, while remaining invariant.

Finally we can further can see that our *U(3)* matrix group $\mathfrak{G}$ is unitary as

$$\mathfrak{G} \cdot \mathfrak{G}^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{33}$$

To give the reader a better intuition of the $\mathfrak{G}$ matrix group, we can set two of X_i values to zero and leave one X_i non-zero. In doing so, we find the *U(1)* subgroup forms as

$$e^{X_1} = \begin{bmatrix} \boldsymbol{e^{iX_1}} & 0 & 0 \\ 0 & \cos(X_1) & isin(X_1) \\ 0 & isin(X_1) & \cos(X_1) \end{bmatrix}, \tag{34}$$

$$e^{X_2} = \begin{bmatrix} \cos(X_2) & 0 & isin(X_2) \\ 0 & \boldsymbol{e^{iX_2}} & 0 \\ isin(X_2) & 0 & \cos(X_2) \end{bmatrix}, \tag{35}$$

$$e^{X_3} = \begin{bmatrix} \cos(X_3) & isin(X_3) & 0 \\ isin(X_3) & \cos(X_3) & 0 \\ 0 & 0 & \boldsymbol{e^{iX_3}} \end{bmatrix}. \tag{36}$$

In the above, we have matrix representation of *U(1)* subgroups that span a $\mathbb{C}^3$ space, with the *U(1)* forms in the respective entries along the main diagonals, where we note these in bold-type in the above matrices. For each of the above *U(1)* forms the determinants are e^{iX_i}, which is the same determinant we find for any e^{iX_i} in a 1x1 representation. We point out to the reader that for our generators above, when $X_i = 0$, we have an identity matrix as a result.

We can see a remarkable resemblance to the matrices R_x, R_y, R_z of the *SO(3)* group, where we have the generators as

$$G_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, G_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, G_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

An element of *SO(3)* would be a linear combination of the above generators where

$$\alpha_1 G_x + \alpha_2 G_y + \alpha_3 G_z \in SO(3) \; and \; \alpha_i \in \mathbb{R}^3. \tag{38}$$

Exponentiation of the G_i generators yields

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}, R_y = \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0) \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix}, R_z = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{39}$$

With our *U(3)* group, instead of rotations in $\mathbb{R}^3$, we have rotations that are a linear combination of 3 complex planes in $\mathbb{C}^3$. We use the example of our X_1 matrix below and compare with the *SO(3)* R_x matrix above, showing the complex axis of rotation as $\boldsymbol{e^{iX_1}}$, with other entries representing rotation in the complex $\mathbb{C}^2$ planes of X_2X_3

$$e^{X_1} = \begin{bmatrix} \boldsymbol{e^{iX_1}} & 0 & 0 \\ 0 & \cos(X_1) & isin(X_1) \\ 0 & isin(X_1) & \cos(X_1) \end{bmatrix}. \tag{40}$$

By taking the complex modulus of $\boldsymbol{e^{iX_1}}$, and keeping in mind that $i \cong \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix}$ to project $\mathbb{C}^3$ down to $\mathbb{R}^3$, we have

$$e^{X_1} = \begin{bmatrix} |\boldsymbol{e^{iX_1}}| & 0 & 0 \\ 0 & \cos(X_1) & isin(X_1) \\ 0 & isin(X_1) & \cos(X_1) \end{bmatrix} \mathbb{C}^3 \to \mathbb{R}^3 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(X_1) & \mp\sin(X_1) \\ 0 & \pm\sin(X_1) & \cos(X_1) \end{bmatrix}. \tag{41}$$

The similarities between our exponentiated X_i generators and that of *SO(3)* begets the question, is the three-dimensional space we are all familiar with actually $\mathbb{C}^3$? We would argue that it is indeed. The notion of the need for a real axis to rotate about in a space is not necessary. To show this, we call attention to *U(1)* and its $\mathbb{R}$ form in *SO(2)*, have no fixed real axes, but rotate in two-dimensional space seamlessly. In the *U(3)* group, rotations in $\mathbb{C}^3$ by one X_i generator, leaves the argument for X_i proper, rotating in its respective complex two-dimensional plane. This occurs while rotation occurs with the argument for X_i in the X_jX_k subspace. Looking at ***Eq. (40)*** above, $\boldsymbol{e^{iX_1}}$ is the fixed two-dimensional "axis" in $\mathbb{C}^1$, and the primary rotation is occurring in the $\mathbb{C}^2$ plane that is X_2X_3.

Complete *U(3)* Matrix Group with Real-Valued Permutations

With regard to ***Eq. (22)***, we see that multiplication of our X_i generator permutations yield real-valued permutations that we designate as Y_i. We define the order of progression as $mod\; X_1 \to X_2 \to X_3 \to X_1 \dots$. The result for multiplication of X_i in this order is

$$X_n X_{n+1} = \begin{bmatrix} 0 & 0 & -Y_3 \\ -Y_3 & 0 & 0 \\ 0 & -Y_3 & 0 \end{bmatrix}. \tag{42}$$

For the reverse order we would have

$$X_n X_{n-1} = \begin{bmatrix} 0 & -Y_2 & 0 \\ 0 & 0 & -Y_2 \\ -Y_2 & 0 & 0 \end{bmatrix}, \tag{43}$$

and for any X_n^2 we have

$$X_n^2 = \begin{bmatrix} -Y_1 & 0 & 0 \\ 0 & -Y_1 & 0 \\ 0 & 0 & -Y_1 \end{bmatrix}. \tag{44}$$

For our Y_i permutations we have the following relationships

$$Y_2^2 = Y_3, \qquad Y_3^2 = Y_2, \qquad Y_2 Y_3 = Y_1, \qquad Y_3 Y_2 = Y_1. \tag{45}$$

Rotations to orthogonal axes in the *U(3)* group are the result of taking the product of $X_i Y_2, X_i Y_3, Y_2 X_i, Y_3 X_i$. We demonstrate for $X_1 Y_2, Y_2 X_1$ and see

$$X_1 Y_2 = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & i \end{bmatrix} = X_3, \tag{46}$$

$$Y_2 X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{bmatrix} = X_2. \tag{47}$$

These relations are similar for Y_3, but in the reverse order of rotation as $Y_2' = Y_3$.

We can combine our X_i and Y_i permutations into this $\mathfrak{g}$ group as

$$\mathfrak{g} = \begin{bmatrix} Y_1 & 0 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & Y_1 \end{bmatrix} + \begin{bmatrix} iX_1 & Y_2 + iX_3 & Y_3 + iX_2 \\ Y_3 + iX_3 & iX_2 & Y_2 + iX_1 \\ Y_2 + iX_2 & Y_3 + iX_1 & iX_3 \end{bmatrix}. \tag{48}$$

We can collect the Y_i permutations together and realize that this is the Abelian cyclic order group Z_3 that is a subgroup of $\mathfrak{g}$ and we denote the subgroup $\mathfrak{h}$ as

$$\mathfrak{h} = \begin{bmatrix} Y_1 & Y_2 & Y_3 \\ Y_3 & Y_1 & Y_2 \\ Y_2 & Y_3 & Y_1 \end{bmatrix}. \tag{49}$$

We purposely call the Z_3 group above $\mathfrak{h}$ as this represents the Cartan Subalgebra[4] of *U(3)* and is the maximal commuting subalgebra of our Lie Algebra $\mathfrak{g}$. Where

$$\forall\, Y_i, Y_j \in \mathfrak{h}, \left[Y_i, Y_j\right] = 0, \tag{50}$$

$$if\ X \in \mathfrak{g}\ and\ [X, Y] = 0\ for\ Y \in \mathfrak{h}, then \in \mathfrak{h}, \tag{51}$$

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[4] Hall, Brian C. – ***Lie Groups, Lie Algebras, and Representations***, Graduate Texts in Mathematics, Springer, 2015

$$\forall\, Y \in \mathfrak{h}, ad_Y \; is \; diagonalizable. \tag{52}$$

As an example, noting as stated in ***Eq. (45)*** that $Y_2 Y_3 = Y_1$ and $Y_3 Y_2 = Y_1$, we have that

$$[Y2, Y3] = \left(\begin{bmatrix} 0 & Y_2 & 0 \\ 0 & 0 & Y_2 \\ Y_2 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & Y_3 \\ Y_3 & 0 & 0 \\ 0 & Y_3 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & Y_3 \\ Y_3 & 0 & 0 \\ 0 & Y_3 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & Y_2 & 0 \\ 0 & 0 & Y_2 \\ Y_2 & 0 & 0 \end{bmatrix} \right) =$$

$$\left(\begin{bmatrix} Y_2 Y_3 & 0 & 0 \\ 0 & Y_2 Y_3 & 0 \\ 0 & 0 & Y_2 Y_3 \end{bmatrix} - \begin{bmatrix} Y_3 Y_2 & 0 & 0 \\ 0 & Y_3 Y_2 & 0 \\ 0 & 0 & Y_3 Y_2 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{53}$$

The Lie Algebra $\mathfrak{g}$ is also diagonalizable and we can proceed as above in ***Eq. (48)*** to calculate this matrix group as $\mathfrak{G} = V e^D V$ and find

$$\mathfrak{G} = \begin{bmatrix} e^{Y_1} & 0 & 0 \\ 0 & e^{Y_1} & 0 \\ 0 & 0 & e^{Y_1} \end{bmatrix} \times \tag{54}$$

$$\begin{bmatrix} \frac{e^{(\gamma+\mathcal{R})}}{3} + e^{-\frac{\mathcal{R}}{2}}\left(\frac{2\cosh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3} - \frac{i\frac{\partial\boldsymbol{\lambda}}{\partial X_1}\sinh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3}\right) & \frac{e^{(\gamma+\mathcal{R})}}{3} - e^{-\frac{\mathcal{R}}{2}}\left(\frac{\cosh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3} + \frac{\frac{\partial\boldsymbol{\lambda}}{\partial Y_2} + i\frac{\partial\boldsymbol{\lambda}}{\partial X_3}\sinh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3}\right) & \frac{e^{(\gamma+\mathcal{R})}}{3} - e^{-\frac{\mathcal{R}}{2}}\left(\frac{\cosh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3} + \frac{-\frac{\partial\boldsymbol{\lambda}}{\partial Y_3} - i\frac{\partial\boldsymbol{\lambda}}{\partial X_2}\sinh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3}\right) \\ \frac{e^{(\gamma+\mathcal{R})}}{3} - e^{-\frac{\mathcal{R}}{2}}\left(\frac{\cosh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3} + \frac{-\frac{\partial\boldsymbol{\lambda}}{\partial Y_3} - i\frac{\partial\boldsymbol{\lambda}}{\partial X_3}\sinh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3}\right) & \frac{e^{(\gamma+\mathcal{R})}}{3} + e^{-\frac{\mathcal{R}}{2}}\left(\frac{2\cosh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3} - \frac{i\frac{\partial\boldsymbol{\lambda}}{\partial X_2}\sinh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3}\right) & \frac{e^{(\gamma+\mathcal{R})}}{3} - e^{-\frac{\mathcal{R}}{2}}\left(\frac{\cosh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3} + \frac{\frac{\partial\boldsymbol{\lambda}}{\partial Y_2} + i\frac{\partial\boldsymbol{\lambda}}{\partial X_1}\sinh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3}\right) \\ \frac{e^{(\gamma+\mathcal{R})}}{3} - e^{-\frac{\mathcal{R}}{2}}\left(\frac{\cosh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3} + \frac{\frac{\partial\boldsymbol{\lambda}}{\partial Y_2} + i\frac{\partial\boldsymbol{\lambda}}{\partial X_2}\sinh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3}\right) & \frac{e^{(\gamma+\mathcal{R})}}{3} - e^{-\frac{\mathcal{R}}{2}}\left(\frac{\cosh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3} + \frac{-\frac{\partial\boldsymbol{\lambda}}{\partial Y_3} - i\frac{\partial\boldsymbol{\lambda}}{\partial X_1}\sinh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3}\right) & \frac{e^{(\gamma+\mathcal{R})}}{3} + e^{-\frac{\mathcal{R}}{2}}\left(\frac{2\cosh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3} - \frac{i\frac{\partial\boldsymbol{\lambda}}{\partial X_3}\sinh\left(\frac{\boldsymbol{\lambda}}{2}\right)}{3}\right) \end{bmatrix}.$$

Where $\mathcal{R} = (Y_2 + Y_3), \gamma = (X_1 + X_2 + X_3), \boldsymbol{\lambda} = \sqrt{-3Y_2^2 + 6Y_2Y_3 - 3Y_3^2 - 4X_1^2 + 4X_1X_2 + 4X_1X_3 - 4X_2^2 + 4X_2X_3 - 4X_3^2}.$

The matrix group above, represents the linear combination of X_i, Y_i generators that are exponentiated as

$$\mathfrak{G} = \exp\,(y_1 Y_1 + x_1 X_1 + y_2 Y_2 + x X_2 + y_3 Y_3 + x_3 X_3) \in U(3) \; and \; x_i X_i \in \mathbb{C}^3, y_i Y_i \in \mathbb{R}^3. \tag{55}$$

When we calculate the determinate of the matrix group $\mathfrak{G}$, we find that it is equal to $e^{3Y_1 + iX_1 + iX_2 + iX3}$ and we find that Y_1 is a scaling factor for the matrix group. Leaving the Y_1 permutation out of the matrix group, we would find the determinate is equal to $e^{iX_1 + iX_2 + iX3}$, which is the same determinate value we obtained in ***Eq. (29)*** for our other *U(3)* group representation. So, the addition of our Y_2 and Y_3 permutations, do not affect the volume of the group, while the Y_1 permutation acts as a scaling factor.

The Quantum Chromodynamics (QCD) Representation of the *U(3)* Group

When compared to the Gell-Mann matrices, we find that the *U(3)* group also contains a Cartan Subalgebra where we utilize the Y_2, Y_3 generators in a similar way to make color assignments for fermion triplets. In *U(3)* we have a legitimate group as it is closed under multiplication and represents what we believe to be the mathematically correct way of arranging three $\mathbb{C}$ groups together. The strong force would appear to be automatically generated by the product of $X_i X_j$ where for $i \neq j$ we find color representations and for $i = j$, we have our identity element Y_i representing color singlets or color neutral elements. These will only occur for fermions with a single phase and be bound on respective subalgebra $\{X_j, Y_1\}$. An example could be for single phase representation where we set $\alpha_1 = 0 \; as \; well \; as \; \alpha_3 = 0$ for our linear combination

$$x_1 X_1 + x_2 X_2 + x_3 X_3 \in U(3) \; and \; x_i X_i \in \mathbb{C}^3. \tag{56}$$

We can expand this out as

$$(0 \times X_1) + x_2X_2 + (0 \times X_3) \rightarrow \begin{bmatrix} 0 & 0 & ix_2 \\ 0 & ix_2 & 0 \\ ix_2 & 0 & 0 \end{bmatrix}. \tag{57}$$

With this as the only the only non-Cartan permutation in the group, we would only have $\pm Y_1$ $and \pm iX_2$ as the products on this subspace. For the above example the product of X_2 with itself would be

$$X_2X_2 = \begin{bmatrix} 0 & 0 & ix_2 \\ 0 & ix_2 & 0 \\ ix_2 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} -x_2^2 & 0 & 0 \\ 0 & -x_2^2 & 0 \\ 0 & 0 & -x_2^2 \end{bmatrix} = -x_2^2Y_1. \tag{58}$$

We can now state the trace of products of X_i for *U(3)* formally as

$$trace(X_iX_j) = -3\delta_{ij} \; and \; \delta_{ij} = \begin{cases} 0 \; if \; i \neq j \\ 1 \; if \; i = j \end{cases}. \tag{59}$$

We note above that for our *U(3)* group, we have integer values of negative three for the trace of products where $i = j$, compared to a value of two for the Gell-Mann matrices.

We continue in a similar fashion to what we did above with the Gell-Mann matrices and calculate the eigenvalues and eigenvectors of our Y_2, Y_3 Cartan generators for both positive and negative values. We denote $-Y_2 \; as \; \overline{Y}_2$ similar to the anti-color bar representation. We arrange our eigenvectors as a matrix *V* with each eigenvector representing a column and we arrange our eigenvalues in a diagonal matrix form.

$$Y_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} + \frac{i}{2} & -\frac{1}{2\sqrt{3}} - \frac{i}{2} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} - \frac{i}{2} & -\frac{1}{2\sqrt{3}} + \frac{i}{2} & -\frac{1}{\sqrt{3}} \end{bmatrix} D = \begin{bmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{60}$$

$$\overline{Y}_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} - \frac{i}{2} & -\frac{1}{2\sqrt{3}} + \frac{i}{2} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} + \frac{i}{2} & -\frac{1}{2\sqrt{3}} - \frac{i}{2} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix}. \tag{61}$$

$$Y_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, V = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} - \frac{i}{2} & -\frac{1}{2\sqrt{3}} + \frac{i}{2} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} + \frac{i}{2} & -\frac{1}{2\sqrt{3}} - \frac{i}{2} & -\frac{1}{\sqrt{3}} \end{bmatrix} D = \begin{bmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{62}$$

$$\overline{Y}_3 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, V = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} - \frac{i}{2} & -\frac{1}{2\sqrt{3}} + \frac{i}{2} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} + \frac{i}{2} & -\frac{1}{2\sqrt{3}} - \frac{i}{2} \end{bmatrix} D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix}. \tag{63}$$

For completeness we also include eigenvalues and eigenvectors for our commutator $[X_i, X_j]$ $where\ i \neq j$

$$[X_i, X_j]\ where\ i \neq j = (Y_2 - Y_3) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, V = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} - \frac{i}{2} & \frac{1}{2\sqrt{3}} + \frac{i}{2} & \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} + \frac{i}{2} & \frac{1}{2\sqrt{3}} - \frac{i}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} D = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (64)$$

Contrary to the representation of the eigenvalues in the Gell-Mann *SU(3)* group, the symmetry of the *U(3)* Cartan generators stands out in the fact the eigenvalues are evenly spaced by $\frac{2\pi}{3}$ about the unit circle. We have for $\overline{Y}_2, \overline{Y}_3$ a very nice conjugate relationship to their Y_2, Y_3 counterparts. We demonstrate on unit circles below this very nice symmetry in ***Fig 1.***

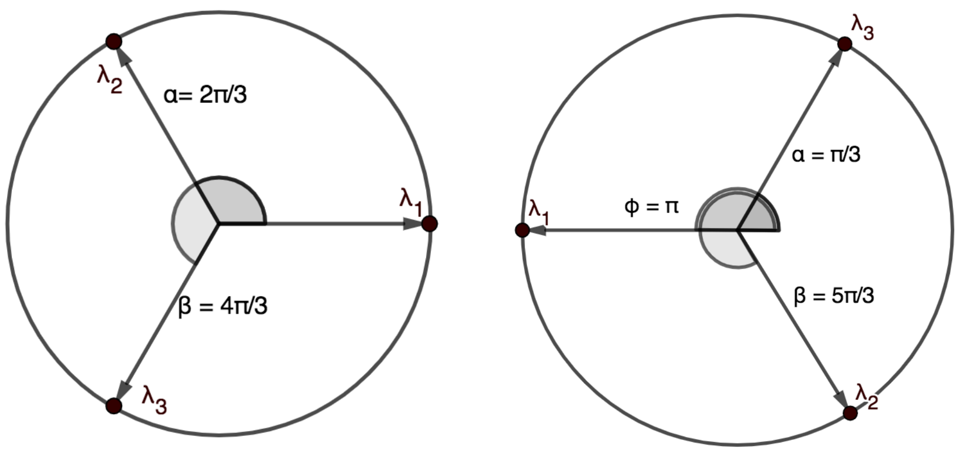

Figure 1. – Plotted Eigenvalues (λ_i) on the Unit Circle for Y_2 (***left***) and $\overline{Y}_2$ (***right***).

We overlay the $\overline{Y}_{2,3}, Y_{2,3}$ eigenvalues together to demonstrate again the symmetry of the group *U(3).*

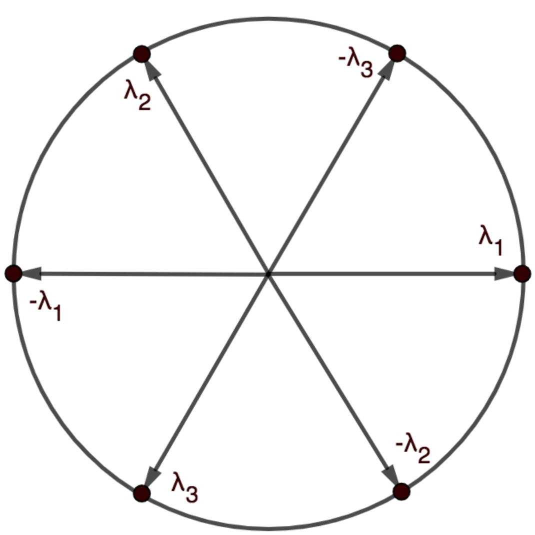

Figure 2. – Eigenvalue (λ_i) symmetry of $Y_{2,3}, \overline{Y}_{2,3}$ Cartan Subalgebra generators.

We take a similar approach to forming fermion triplets as we did in *SU(3)*, ***Eqs. (9-11)*** but this results in legitimate vectors in $\mathbb{C}^3$ space. We list all of the combinations for values attained with eigenvalues and eigenvectors from ***Eqs. (60-63)*** and find

$$Y_2(\lambda_i V_j) = \begin{pmatrix} -\frac{1}{2\sqrt{3}}+\frac{i}{2} \\ -\frac{1}{2\sqrt{3}}-\frac{i}{2} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}}+\frac{i}{2} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}}-\frac{i}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{3}}-\frac{i}{2} \\ \frac{1}{2\sqrt{3}}-\frac{i}{2} \\ \frac{1}{2\sqrt{3}}-\frac{i}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}}-\frac{i}{2} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}}+\frac{i}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}}-\frac{i}{2} \\ -\frac{1}{2\sqrt{3}}+\frac{i}{2} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{3}}+\frac{i}{2} \\ \frac{1}{2\sqrt{3}}+\frac{i}{2} \\ \frac{1}{2\sqrt{3}}+\frac{i}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}}+\frac{i}{2} \\ -\frac{1}{2\sqrt{3}}-\frac{i}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}}-\frac{i}{2} \\ -\frac{1}{2\sqrt{3}}+\frac{i}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \tag{65}$$

$$\overline{Y_2}(\lambda_i V_j) = \begin{pmatrix} \frac{1}{2\sqrt{3}}-\frac{i}{2} \\ \frac{1}{2\sqrt{3}}+\frac{i}{2} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{3}}-\frac{i}{2} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}}+\frac{i}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}}+\frac{i}{2} \\ -\frac{1}{2\sqrt{3}}+\frac{i}{2} \\ -\frac{1}{2\sqrt{3}}+\frac{i}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{3}}+\frac{i}{2} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}}-\frac{i}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{3}}+\frac{i}{2} \\ \frac{1}{2\sqrt{3}}-\frac{i}{2} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}}-\frac{i}{2} \\ -\frac{1}{2\sqrt{3}}-\frac{i}{2} \\ -\frac{1}{2\sqrt{3}}-\frac{i}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}}-\frac{i}{2} \\ \frac{1}{2\sqrt{3}}+\frac{i}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}}+\frac{i}{2} \\ \frac{1}{2\sqrt{3}}-\frac{i}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{66}$$

We notice above that all of the fermion triplets in Y_2 and $\overline{Y_2}$ are related in the fact that $Y_2 = -\overline{Y_2}$. Before moving on to $Y_3 . \overline{Y_3}$, we will assign color values based on the orientation of the value on the unit circle, with red, blue and green progressing counter clockwise and anti-red, anti-blue and anti-green progressing clockwise.

$$-\frac{1}{2\sqrt{3}}+\frac{i}{2} = red, \qquad \frac{1}{2\sqrt{3}}-\frac{i}{2} = \overline{red} \tag{67}$$

$$-\frac{1}{2\sqrt{3}}-\frac{i}{2} = blue, \qquad \frac{1}{2\sqrt{3}}+\frac{i}{2} = \overline{blue} \tag{68}$$

$$\frac{1}{\sqrt{3}} = green, \qquad -\frac{1}{\sqrt{3}} = \overline{green}. \tag{69}$$

We also point out that adding the 3 colors results in zero, as well as adding two distinct colors yield the anti-color of that which is not being added.

$$red + blue + green = \left(-\frac{1}{2\sqrt{3}}+\frac{i}{2}\right) + \left(-\frac{1}{2\sqrt{3}}-\frac{i}{2}\right) + \left(\frac{1}{\sqrt{3}}\right) = 0, \tag{70}$$

$$Red + Blue = \left(-\frac{1}{2\sqrt{3}}+\frac{i}{2}\right) + \left(-\frac{1}{2\sqrt{3}}-\frac{i}{2}\right) = -\frac{1}{\sqrt{3}} = \overline{Green}, \tag{71}$$

$$Red + Green = \left(-\frac{1}{2\sqrt{3}}+\frac{i}{2}\right) + \left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{2\sqrt{3}}+\frac{i}{2}\right) = \overline{Blue}, \tag{72}$$

$$Green + Blue = \left(\frac{1}{\sqrt{3}}\right) + \left(-\frac{1}{2\sqrt{3}}-\frac{i}{2}\right) = \frac{1}{2\sqrt{3}}-\frac{i}{2} = \overline{Red}. \tag{73}$$

We use our new notation above to state our eigenvalues and eigenvectors of $Y_3, \overline{Y_3}$ as

$$Y_3(\lambda_i V_j) = \begin{pmatrix} r \\ g \\ b \end{pmatrix}, \begin{pmatrix} r \\ b \\ g \end{pmatrix}, \begin{pmatrix} \bar{r} \\ \bar{r} \\ \bar{r} \end{pmatrix}, \begin{pmatrix} b \\ r \\ g \end{pmatrix}, \begin{pmatrix} b \\ g \\ r \end{pmatrix}, \begin{pmatrix} \bar{b} \\ \bar{b} \\ \bar{b} \end{pmatrix}, \begin{pmatrix} g \\ b \\ r \end{pmatrix}, \begin{pmatrix} g \\ r \\ b \end{pmatrix}, \begin{pmatrix} \bar{g} \\ \bar{g} \\ \bar{g} \end{pmatrix} \tag{74}$$

$$\bar{Y}_3(\lambda_i V_j) = \begin{pmatrix} g \\ g \\ g \end{pmatrix}, \begin{pmatrix} \bar{b} \\ \bar{g} \\ \bar{r} \end{pmatrix}, \begin{pmatrix} \bar{r} \\ \bar{g} \\ \bar{b} \end{pmatrix}, \begin{pmatrix} b \\ b \\ b \end{pmatrix}, \begin{pmatrix} \bar{r} \\ \bar{b} \\ \bar{g} \end{pmatrix}, \begin{pmatrix} \bar{g} \\ \bar{b} \\ \bar{r} \end{pmatrix}, \begin{pmatrix} r \\ r \\ r \end{pmatrix}, \begin{pmatrix} \bar{g} \\ \bar{r} \\ \bar{b} \end{pmatrix}, \begin{pmatrix} \bar{b} \\ \bar{r} \\ \bar{g} \end{pmatrix}. \tag{75}$$

We see above that we have 9 distinct representations for gluons as well as the anti-gluon representations. There is a very nice geometric interpretation for color representations within our *U(3)* group, where each entry in the vector represents a coordinate in the $\mathbb{C}^3$ space and the vectors all norm to one by default. Color singlets are not the result of adding $red + blue + green$ or the respective anti-colors as surmised in the Gell-Mann chromodynamics interpretation. We opine that lone phase representations of *U(3)* are at the heart of color singlets and mesons in quantum chromodynamics. Using our $\mathbb{C}^3$ coordinate space interpretation, we see that color neutrality is the result of a sole *U(3)* generator (X_i), being isolated in a $\mathbb{C}^1$ subspace and this is a fermion singlet instead of the triplets that we have mentioned above. Below we show the trivial eigenvalues and eigenvectors for Y_1

$$Y_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{76}$$

We can normalize our eigenvectors above and consider both clockwise and counter clockwise rotations in one complex plane. We can We would have 3 trivial fermion singlets that are the product of the eigenvalue with the eigenvectors that we note as

$$Y_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, V = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{77}$$

$$\bar{Y}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, V = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{78}$$

We see that our Cartan generators $Y_1, \bar{Y}_1$ yield color rotation in only one plane, and this results in no change of color charge. We have for every vector in $Y_1, \bar{Y}_1$

$$Y_1(\lambda_i q^j) = \lambda_i \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}, \tag{79}$$

$$\bar{Y}_1(\bar{\lambda}_i q^j) = \bar{\lambda}_i \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \bar{g} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \bar{g} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \bar{g} \end{pmatrix}. \tag{80}$$

With our geometric interpretation, we have an intuition for why color neutral fermions are represented this way, where the norm of any three numbers of colors would be equal to one. Put plainly, the tri-color triplet is compressed from $\mathbb{C}^3 \rightarrow \mathbb{C}^1$. This fits very well with the meson contents of one quark and its color anti-quark ($q\bar{q}$) and their lack of color exchange between quarks. We can see that color is locked up in the single gluon by remembering for the example above that

$$Red + Blue = \left(-\frac{1}{2\sqrt{3}} + \frac{i}{2}\right) + \left(-\frac{1}{2\sqrt{3}} - \frac{i}{2}\right) = -\frac{1}{\sqrt{3}} = \overline{Green}, \tag{81}$$

$$\overline{Red} + \overline{Blue} = \left(\frac{1}{2\sqrt{3}} - \frac{i}{2}\right) + \left(\frac{1}{2\sqrt{3}} + \frac{i}{2}\right) = \frac{1}{\sqrt{3}} = Green. \tag{82}$$

Pair	$g^{\overline{rrr}}$	g^{rbg}	g^{rgb}	$g^{\overline{bbb}}$	g^{bgr}	g^{brg}	$g^{\overline{ggg}}$	g^{gbr}	g^{grb}
$g^{\overline{rrr}}$	λ^b	**0**	**0**	λ^g	**0**	**0**	λ^r	**0**	**0**
g^{rbg}	**0**	**0**	λ^b	**0**	**0**	λ^g	**0**	λ^r	**0**
g^{rgb}	**0**	λ^b	**0**	**0**	λ^g	**0**	**0**	**0**	λ^r
$g^{\overline{bbb}}$	λ^g	**0**	**0**	λ^r	**0**	**0**	λ^b	**0**	**0**
g^{bgr}	**0**	**0**	λ^g	**0**	**0**	λ^r	**0**	λ^b	**0**
g^{brg}	**0**	λ^g	**0**	**0**	λ^r	**0**	**0**	**0**	λ^b
$g^{\overline{ggg}}$	λ^r	**0**	**0**	λ^b	**0**	**0**	λ^g	**0**	**0**
g^{gbr}	**0**	λ^r	**0**	**0**	λ^b	**0**	**0**	**0**	λ^g
g^{grb}	**0**	**0**	λ^r	**0**	**0**	λ^b	**0**	λ^g	**0**

Table 1. *– Gluon pairing values -* $\lambda^r = -\frac{1}{2} + \frac{i\sqrt{3}}{2},\ \lambda^b = -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \lambda^g = 1.$

Gluons that are adjacent to one another, as well as opposite (see ***Figs. 3,4***), will interact by way of a complex inner-product to form a strong force that equals to one of the eigenvalues of the *U(3)* group. What is meant by a complex inner-product is $< z, w > z, w \in \mathbb{C}^3$ and does not involve a complex conjugate like $< z, \overline{w} >$. We take the example for the two gluons g^{rbg}, g^{rgb} and define the complex inner product as

$$< g^{rbg}, g^{rgb} >_{\mathbb{C}} \ \{r, b, g \in \mathbb{C}\} = rr + bg + gb = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = \lambda^b. \tag{83}$$

Where λ^b above is the one of our Cartan generator eigenvalues from ***Eqs. (60-63)***.

Above in ***Table 1.*** we show all the pairing values between gluon vector triplets and after accounting for redundancy, there are 15 distinct pairs that result in non-zero outcomes that are the eigenvalues

$$\lambda^r = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \qquad \lambda^b = -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \qquad \lambda^g = 1. \tag{84}$$

The product of a colored quark with one of the colored gluon pair products is shown below and represents gluon field interactions and their effects on quark color charges.

$$q^r\lambda^r \to q^b, q^r\lambda^b \to q^g, q^r\lambda^g \to q^r \tag{85}$$

$$q^b\lambda^r \to q^g, q^b\lambda^b \to q^r, q^b\lambda^g \to q^b. \tag{86}$$

$$q^g\lambda^r \to q^r, q^g\lambda^b \to q^b, q^g\lambda^g \to q^g \tag{87}$$

We have the following orthogonal gluon relationships where $< g^{ijk} \perp g^{jki} >_{\mathbb{C}}= 0$.

$$< g^{rbg} \perp g^{bgr} \perp g^{grb} >_{\mathbb{C}} \tag{88}$$

$$< g^{rgb} \perp g^{brg} \perp g^{gbr} >_{\mathbb{C}} \tag{89}$$

$$< g^{\overline{c_i c_i c_i}} \perp g^{ijk} >_{\mathbb{C}} \tag{90}$$

We note above that for cyclic permutations where for example $rbg \to bgr \to grb\ or\ rgb \to brg \to gbr$, we have orthogonal relationships between these gluons. There is no color rotation in these pairings, and we are just in essence viewing the color relationship form a different perspective. For $< g^{\overline{c_i c_i c_i}} \perp g^{ijk} >_{\mathbb{C}}$, we are only multiplying each component of the gluon that sums to zero as this is $\overline{c_1}(red + blue + green) = \overline{c_1}(0) = 0$.

In terms of the additive relationships between quark triplets, we can easily calculate them and find

$$q^r + q^g \to q^{\bar{b}}, q^b + q^g \to q^{\bar{r}}, q^b + q^r \to q^{\bar{q}}. \tag{91}$$

Pair	g^{rrr}	$g^{\overline{rbg}}$	$g^{\overline{rgb}}$	g^{bbb}	$g^{\overline{bgr}}$	$g^{\overline{brg}}$	g^{ggg}	$g^{\overline{gbr}}$	$g^{\overline{grb}}$
$g^{\overline{rrr}}$	$\overline{\lambda^b}$	**0**	**0**	$\overline{\lambda^g}$	**0**	**0**	$\overline{\lambda^r}$	**0**	**0**
g^{rbg}	**0**	**0**	$\overline{\lambda^b}$	**0**	**0**	$\overline{\lambda^g}$	**0**	$\overline{\lambda^r}$	**0**
g^{rgb}	**0**	$\overline{\lambda^b}$	**0**	**0**	$\overline{\lambda^g}$	**0**	**0**	**0**	$\overline{\lambda^r}$
$g^{\overline{bbb}}$	$\overline{\lambda^g}$	**0**	**0**	$\overline{\lambda^r}$	**0**	**0**	$\overline{\lambda^b}$	**0**	**0**
g^{bgr}	**0**	**0**	$\overline{\lambda^g}$	**0**	**0**	$\overline{\lambda^r}$	**0**	$\overline{\lambda^b}$	**0**
g^{brg}	**0**	$\overline{\lambda^g}$	**0**	**0**	$\overline{\lambda^r}$	**0**	**0**	**0**	$\overline{\lambda^b}$
$g^{\overline{ggg}}$	$\overline{\lambda^r}$	**0**	**0**	$\overline{\lambda^b}$	**0**	**0**	$\overline{\lambda^g}$	**0**	**0**
g^{gbr}	**0**	$\overline{\lambda^r}$	**0**	**0**	$\overline{\lambda^b}$	**0**	**0**	**0**	$\overline{\lambda^g}$
g^{grb}	**0**	**0**	$\overline{\lambda^r}$	**0**	**0**	$\overline{\lambda^b}$	**0**	$\overline{\lambda^g}$	**0**

Table 2. *– Gluon Anti-Gluon pairing values -* $\overline{\lambda^r} = \frac{1}{2} - \frac{i\sqrt{3}}{2},\ \overline{\lambda^b} = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \overline{\lambda^g} = -1.$

Gluons that are adjacent to one another, as well as opposite (see ***Figs. 3&4***), will interact by way of a complex inner-product to form a strong force that equals to one of the eigenvalues of the *U(3)* group.

Summing our results from ***Table 1*** above, we have the following eigenvalue results for gluon pairings

$$< g^{\overline{rrr}}, g^{\overline{ggg}} >_{\mathbb{C}}, < g^{\overline{bbb}}, g^{\overline{bbb}} >_{\mathbb{C}}, < g^{rbg}, g^{gbr} >_{\mathbb{C}}, < g^{rgb}, g^{grb} >_{\mathbb{C}}, < g^{bgr}, g^{brg} >_{\mathbb{C}} = \lambda^{r}, \tag{92}$$

$$< g^{\overline{bbb}}, g^{\overline{ggg}} >_{\mathbb{C}}, < g^{\overline{rrr}}, g^{\overline{rrr}} >_{\mathbb{C}}, < g^{rbg}, g^{rgb} >_{\mathbb{C}}, < g^{bgr}, g^{gbr} >_{\mathbb{C}}, < g^{brg}, g^{grb} >_{\mathbb{C}} = \lambda^{b}, \tag{93}$$

$$< g^{\overline{rrr}}, g^{\overline{bbb}} >_{\mathbb{C}}, < g^{\overline{ggg}}, g^{\overline{ggg}} >_{\mathbb{C}}, < g^{rbg}, g^{brg} >_{\mathbb{C}}, < g^{rgb}, g^{bgr} >_{\mathbb{C}}, < g^{gbr}, g^{grb} >_{\mathbb{C}} = \lambda^{g}. \tag{94}$$

For all gluons pairings that result in a non-zero λ^c value, we see that one color is kept static, while the two other colors are exchanged. It is the color that remains static, that represents the complex axis (really a complex two-dimensional plane) that exchanging colors charges rotate around. We take the gluon pairing example from ***Eq.(83)*** where $< g^{brg}, g^{grb} >_{\mathbb{C}} = \lambda^b$. We can see that the color *r* stays fixed while *g* and *b* are exchanged when comparing the gluons. We can take this gluon pairing to represent a strong charge and take the product of it with a quark coordinate triplet where $X_1 = b, X_2 = r, X_3 = g$ and view the color change that occurs.

$$\lambda^b q \begin{pmatrix} b \\ r \\ g \end{pmatrix} = \begin{pmatrix} \lambda^b q^b \\ \lambda^b q^r \\ \lambda^b q^g \end{pmatrix} = \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \begin{pmatrix} -\frac{1}{2\sqrt{3}} - \frac{i}{2} \\ -\frac{1}{2\sqrt{3}} + \frac{i}{2} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2\sqrt{3}} + \frac{i}{2} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} - \frac{i}{2} \end{pmatrix} = q \begin{pmatrix} r \\ g \\ b \end{pmatrix}. \tag{95}$$

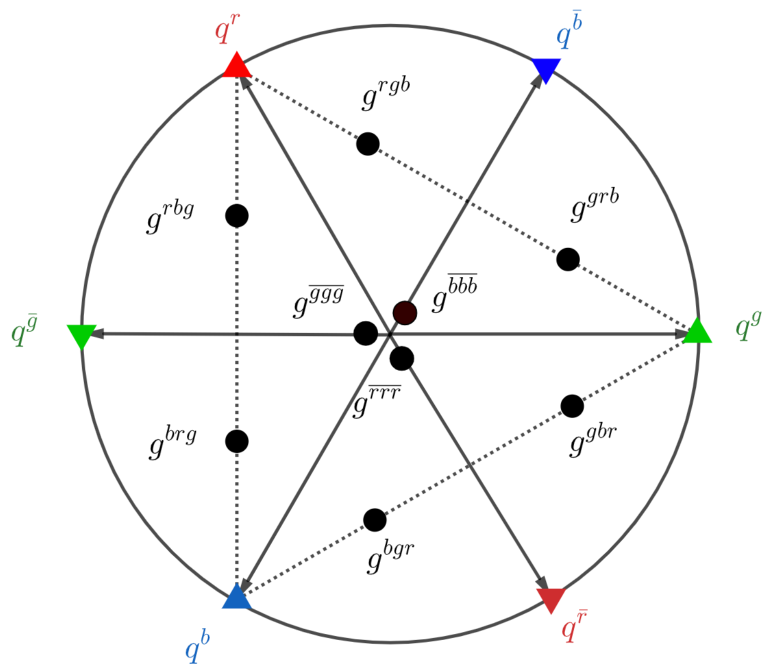

Figure 3. – *U(3)* Strong charge pattern for 3 quark colors, 3 anti-quark colors, and 9 gluons for Y_2, Y_3 Cartan generators. Quarks and anti-quarks are colored triangles and gluons as block circles.

We show in ***Figure 3*** above, the symmetric arrangement of quarks and gluons from the *U(3)* group. We can see that there are six mixed color gluons as well as three single anti-colored gluons. We can see the three same anit-colored triplets in the center, that do not interact with the other gluons, but do interact amongst themselves to initiation color charge exchanges as well. In ***Figure 4*** below we show the non-zero λ^c results that represent the strong force interactions from the gluons pairings. We can see a nice linear relationship with how λ^gs line up between q^g *and* $q^{\bar{q}}$, λ^rs line up between q^b *and* $q^{\bar{b}}$, and λ^bs line up between q^r *and* $q^{\bar{r}}$ representing the strong force in $\mathbb{C}^3$.

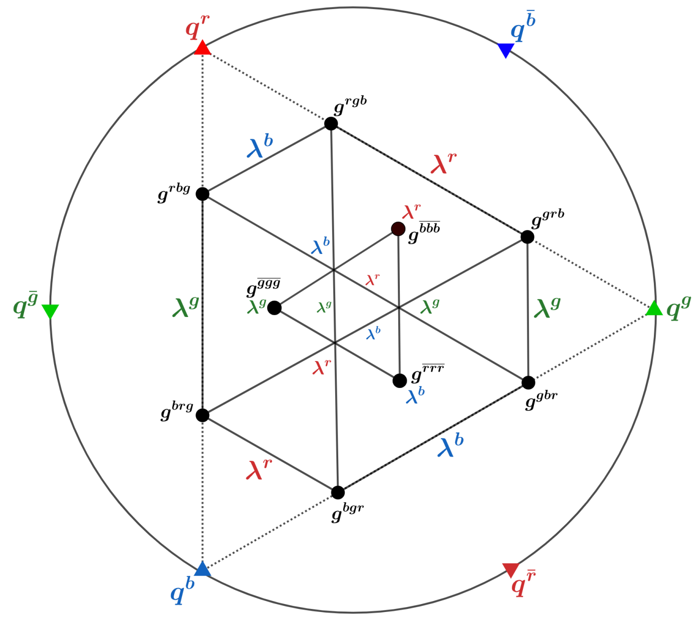

Figure 4. – *U(3)* Strong charge pattern for gluon pairings for Y_2, Y_3 Cartan generators. λs are the gluon products between interacting pairs. Black lines connect gluon-gluon pair products, where as $< g^{\overline{ccc}}, g^{\overline{ccc}} >_{\mathbb{C}}$ self-pairs are represented by black circles.

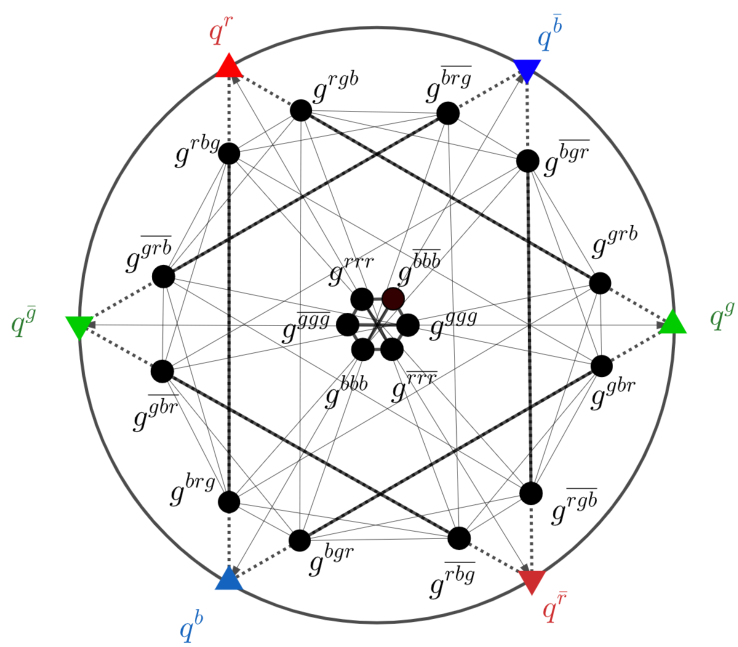

Figure 5. – *U(3)* Strong charge pattern for 3 quark colors, 3 anti-quark colors, and 9 gluons/9 anti-gluons for Y_2, Y_3 Cartan generators. Quarks and anti-quarks are colored triangles and gluons as block circles. Solid lines connect non-zero gluon pairs.

Taking the gluon pairing products between $\overline{Y}_2, \overline{Y}_3$ Cartan generators will result in the same $\lambda^c s$ as the Y_2, Y_3 Cartan generators. We will find $\overline{\lambda}^c s$ when we pair $< \overline{Y}_\iota, Y_j >_{\mathbb{C}}$ though. We show in ***Figure 5*** above all of the gluon pairing interactions as solid line connections between gluons. In essence, each gluon can pair with six of the eighteen total gluons represented in the figure.

QCD Matrix Representations

We can use what we know of the *U(3)* group, to form matrix representations of color charges and color changes. We start by using vectors in the form tri-colored quark triplets and we multiply these by our Cartan generators with our colored eigenvalues as entries. For these quark triplet vectors, the three entries correspond to the color charges for all three quarks. We denote the vector triplet as q^{rbg} for a row vector, q_{rbg} as our column vector and normalize them.

$$q_{rbg} = \left(\frac{r}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{g}{\sqrt{3}}\right), q^{rbg} = \left(\frac{r}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{g}{\sqrt{3}}\right)'. \tag{96}$$

We assign our Cartan generators with eigenvalues that represent the product of some $X_i \cdot X_j$. We will use the normalized example of $\sqrt{3}(X_2 \cdot X_3)$, where this results in a Y_3 Cartan generator with λ^b eigenvalues.

$$Y_{3(X_2X_3)} = \sqrt{3}(X_2 \cdot X_3) = \sqrt{3}\begin{bmatrix} 0 & 0 & b \\ 0 & b & 0 \\ b & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & g & 0 \\ g & 0 & 0 \\ 0 & 0 & g \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda^b \\ \lambda^b & 0 & 0 \\ 0 & \lambda^b & 0 \end{bmatrix}. \tag{97}$$

We can now use our q^{rbg}, q_{rbg} vectors and express the color changes that would occur with the gluon interaction represented by $Y_{3(X_2X_3)}$ above. We calculate both products to find

$$q_{rbg} \cdot Y_{3(X_2X_3)} = \left(\frac{r}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{g}{\sqrt{3}}\right)\begin{bmatrix} 0 & 0 & \lambda^b \\ \lambda^b & 0 & 0 \\ 0 & \lambda^b & 0 \end{bmatrix} = \left(\frac{g}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = q_{gbr}, \tag{98}$$

$$Y_{3(X_2X_3)} \cdot q^{rbg} = \begin{bmatrix} 0 & 0 & \lambda^b \\ \lambda^b & 0 & 0 \\ 0 & \lambda^b & 0 \end{bmatrix}\left(\frac{r}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{g}{\sqrt{3}}\right)' = \left(\frac{b}{\sqrt{3}}, \frac{g}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = q^{bgr}. \tag{99}$$

We see that each of the above differ in the direction of the color charge rotation in the *bg* color plane.

Row/Column	λ^r	λ^b	λ^g
λ^r	λ^b	λ^g	λ^r
λ^b	λ^g	λ^r	λ^b
λ^g	λ^r	λ^b	λ^g

Table 3. – Eigenvalue products for λ^c pairings.

Above in ***Table 3*** we show the reader the eigenvalue products of the λ^cs, and below the eigenvalue products of $\lambda^c, \overline{\lambda^c}$s.

Row/Column	$\overline{\lambda^r}$	$\overline{\lambda^b}$	$\overline{\lambda^g}$
λ^r	$\overline{\lambda^b}$	$\overline{\lambda^g}$	$\overline{\lambda^r}$
λ^b	$\overline{\lambda^g}$	$\overline{\lambda^r}$	$\overline{\lambda^b}$
λ^g	$\overline{\lambda^r}$	$\overline{\lambda^b}$	$\overline{\lambda^g}$

Table 4. – Eigenvalue products for $\overline{\lambda^c}, \lambda^c$ pairings.

The Cartan generator that is representative of the single colored gluons ($g^{\overline{rrr}}, g^{\overline{bbb}}, g^{\overline{ggg}}$) would be of a similar form as above. We will use the example where using once again $Y_{3(X_2X_3)} = \sqrt{3}(X_2 \cdot X_3)$, with $q^{\overline{bbb}}, q_{\overline{bbb}}$ as our vectors.

$$q^{\overline{bbb}} \cdot Y_{3(X_2X_3)} = \left(\frac{\bar{b}}{\sqrt{3}}, \frac{\bar{b}}{\sqrt{3}}, \frac{\bar{b}}{\sqrt{3}}\right) \begin{bmatrix} 0 & 0 & \lambda^b \\ \lambda^b & 0 & 0 \\ 0 & \lambda^b & 0 \end{bmatrix} = \left(\frac{\bar{g}}{\sqrt{3}}, \frac{\bar{g}}{\sqrt{3}}, \frac{\bar{g}}{\sqrt{3}}\right) = q^{\overline{ggg}}, \tag{100}$$

$$Y_{3(X_2X_3)} \cdot q_{\overline{bbb}} = \begin{bmatrix} 0 & 0 & \lambda^b \\ \lambda^b & 0 & 0 \\ 0 & \lambda^b & 0 \end{bmatrix} \left(\frac{\bar{b}}{\sqrt{3}}, \frac{\bar{b}}{\sqrt{3}}, \frac{\bar{b}}{\sqrt{3}}\right)' = \left(\frac{\bar{r}}{\sqrt{3}}, \frac{\bar{r}}{\sqrt{3}}, \frac{\bar{r}}{\sqrt{3}}\right) = q_{\overline{rrr}}. \tag{101}$$

Beyond the *U(3)* Group

We are able to define matrix groups in a similar fashion to how we derived our *U(3)* in higher-dimensional forms that are unitary and of the form *U(**n**)*, where ***n*** is an odd integer value. We show the complete matrix of generators $\mathfrak{g}$ for *U(5)* as

$$\mathfrak{g} = (x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_5X_5) \in U(5)\ and\ x_iX_i \in \mathbb{C}^5 = \begin{bmatrix} ix_1 & ix_4 & ix_2 & ix_5 & ix_3 \\ ix_4 & ix_2 & ix_5 & ix_3 & ix_1 \\ ix_2 & ix_5 & ix_3 & ix_1 & ix_4 \\ ix_5 & ix_3 & ix_1 & ix_4 & ix_2 \\ ix_3 & ix_1 & ix_4 & ix_2 & ix_5 \end{bmatrix}. \tag{102}$$

This represents a symmetric and complex matrix and as such we can show for *U(5),* as we did for *U(3)* that

$$\det(e^{\mathfrak{g}}) = e^{Trace(\mathfrak{g})} = e^{i(x_1+x_2+x_3+x_4+x_5)}. \tag{103}$$

Conclusion

We have demonstrated that a finite-derived *U(3)* group, where three *U(1)* subgroups are interconnected together, can form a framework to describe quantum chromodynamics. The product of imaginary axes X_i, form the elements of a Cartan subalgebra in *U(3),* that describes the strong force, color exchange and the intuition as to why quarks and gluons exist in tandem. Fermions that do not exhibit color charge are isolated on a two-dimensional subgroup of this *U(3)* represented by X_i, Y_1, while fermions demonstrative of color charge exchange, are rotating on the Cartan subalgebra Y_2, Y_3. Within the *U(3)* group we are able to appreciate a much clearer geometric interpretation of gluons, quarks and the strong force.