An Imaginary Derivation of e and the Complex Exponential Function

Viator Lapis

Abstract

Using new insights into extending Euler's Pi function to encompass negative real numbers and complex numbers we take an imaginary series expansion approach to derive the constant *e* and the complex exponential form.

A brief history of *e and the complex exponential form*

In 1618 John Napier was the first to indirectly reference the number *e* in the appendix of a book of his work on logarithms where he published an assortment of logarithm values [1].

Many years later in 1683 Jacob Bernoulli attempted to find the result of the limit, which is the value of *e*:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \quad (1)$$

The first mathematician to use the value of the constant *e* was Gottfried Leibniz and this was noted in letters written between himself and Christiaan Huygens in and around 1690. At this time Leibniz referred to the constant as "b".

It was not until the time frame around 1728 that the constant was given the name as we know today as *e* by Leonard Euler in work that he was doing around that time. Euler would go on to publish is 1748 the expression know well as:

$$e^{ix} = cosx + isinx \quad (2)$$

Euler did so by way of series expansions of the trigonometric functions and the exponential expressions.

Factorials and new insights into negative and complex representations

Factorials are ubiquitous in mathematical equations and notably in infinite series representations. We are well aware of the common notation as follows [2]:

$$n! = 1 \times 2 \times 3 \times 4 \ldots (n-1)(n) = \prod_{k=1}^{n} k \quad (3)$$

Where we also define:

$$0! = 1$$

And the recurrence relation is:

$$n! = n(n-1)! \quad (4)$$

In 1730 Euler proved that the following integral was equivalent to the factorial expression *x!*:

$$x! = \prod(x) = \int_0^1 (-lnt)^x dt = \int_0^\infty t^x e^{-t}\, dt, x > -1 \quad (5)$$

This became known as Euler's factorial function and also as the Pi function. Nearly 40 years later Euler went on to define the gamma function $\Gamma(z)$ which extended factorials to all real negative numbers except zero and negative integers. The gamma function is defined as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \quad (6)$$

The gamma functions is related to the Pi function as:

$$\prod(x) = \Gamma(x+1) = x! \quad (7)$$

Complex numbers may be represented in the gamma function with setting the dependent variable to *z=x+iy*, where *x* is the real component of z and *iy* the imaginary component of z. Where we can see:

$$i! = \Gamma(1+i) = \Gamma(z) = i\Gamma(i) = \int_0^\infty t^{z-1} e^{-t}\, dt = \int_0^\infty t^i e^{-t} dt \approx 0.498016 - 0.15495i$$

In 2014 Thukral proposed an extension of Euler's Pi function to relate to factorials of all real negative numbers. In the following *z* represents a real positive number and *c* is a factorial constant not equal to zero:

$$(cz)! = (c^z)z! = \prod(c,z) = c^z \int_0^\infty t^z e^{-t}\, dt, z > 0 \quad (8)$$

In the above if we choose *c=1* this would be equivalent to Euler's Pi function for real positive numbers. If we choose *c=-1*, our resulting function would be as follows:

$$\prod(-1,z) = (-1)^z \int_0^\infty t^z e^{-t}\, dt, z > 0 \quad (9)$$

For example:

$$\prod(-1,4) = (-1)^4 \int_0^\infty t^4 e^{-t}\, dt = (1) * 24 = 24$$

We can once again set our *z* values equal to complex numbers as *x+iy:*

$$\prod(-1,z) = (-z)! = x + iy \quad (10)$$

We would find an interesting pattern in the above where for real negative values at half fractions the values are imaginary only, for quarter fractions we find real and imaginary contributions are equal and for three quarter fractions we will have real and imaginary contributions that are equal but differ in sign.

For example we will use a half fraction for *z* and we will verify that the result is imaginary only. As such for *z =-1.5:*

$$\prod(-1,z) = (-1)^z \prod(z) = (-1)^{1.5} \prod(z) = (-1)^{\frac{1}{2}}(-1)^1 \int_0^\infty t^{1.5} e^{-t}\, dt$$

$$= i(-1)^1 \int_0^\infty t^{1.5} e^{-t} dt = -1.32934i$$

Imaginary values and the exponential series in a new light

We can apply what we have learned in the aforementioned functions and expand its application to the exponential series as well.

To remind the reader:

$$e^1 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \dots = \sum_{n=0}^{\infty} \frac{1}{n!} \quad (11)$$

We can extend our evaluation of factorials of complex numbers to represent imaginary numbers. We will define imaginary numbers of positive integers as follows:

$$(in)! = (i)^n n! = i(i2)(i3) \dots (in) \quad (12)$$

Putting this in Euler's Pi function form we would have:

$$(iz)! = i^z z! = \prod(i, z) = i^z \int_0^\infty t^z e^{-t}\, dt, z > 0 \quad (13)$$

For example to find *(i6)!* :

$$(i6)! = i^6 6! = \prod(i, 6) = i^6 \int_0^\infty t^6 e^t\, dt = -720$$

Another example is for *(i3)!:*

$$(i3)! = i^3 3! = \prod(i, 3) = i^3 \int_0^\infty t^3 e^t\, dt = -i6$$

Let us revisit the exponential series and restate it as such:

$$e = \frac{1^0}{0!} + \frac{1^1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} \dots = \sum_{n=0}^{\infty} \frac{1^n}{n!} \quad (14)$$

We will try a similar process for the exponent but now with imaginary components instead while applying what we have learned from the material we have covered thus far. I will first calculate the *(iz)!* for *z=0* as follows to verify:

$$\prod(i, 0) = i^0 \int_0^\infty t^0 e^{-t}\, dt = 1 \;\therefore\; i^0 0! = 1$$

With this is mind we can extend the concept of the exponential series to one for an imaginary series.

$$\sum_{z=0}^{\infty} \frac{i^z}{(iz)!} = \frac{i^0}{0!} + \frac{i^1}{i!} + \frac{i^2}{(i2)!} + \frac{i^3}{(i3)!} + \frac{i^4}{(i4)!} + \frac{i^5}{(i5)!} \dots$$

Which yields:

$$\frac{1}{1} + \frac{i}{i} + \frac{-1}{-2} + \frac{-i}{-i6} + \frac{1}{24} + \frac{i}{i120} \dots$$

We further simplify and find that:

$$\sum_{z=0}^{\infty} \frac{i^z}{(iz)!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \dots = e$$

We can also extend the imaginary exponential series as such:

$$\sum_{z=0}^{\infty} \frac{(ix)^z}{(iz)!} = \frac{(ix)^0}{0!} + \frac{(ix)^1}{i!} + \frac{(ix)^2}{(i2)!} + \frac{(ix)^3}{(i3)!} + \frac{(ix)^4}{(i4)!} + \frac{(ix)^5}{(i5)!} \dots$$

Which yields:

$$\frac{1}{1} + \frac{ix}{i} + \frac{-x^2}{-2} + \frac{-ix^3}{-i6} + \frac{x^4}{24} + \frac{ix^5}{i120} \dots$$

We simplify and find:

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \dots = e^x$$

Finally we can also verify for Euler's formula:

$$\sum_{z=0}^{\infty} \frac{(i^2x)^z}{(iz)!} = \frac{(i^2x)^0}{0!} + \frac{(i^2x)^1}{i!} + \frac{(i^2x)^2}{(i2)!} + \frac{(i^2x)^3}{(i3)!} + \frac{(i^2x)^4}{(i4)!} + \frac{(i^2x)^5}{(i5)!} + \frac{(i^2x)^6}{(i6)!} + \frac{(i^2x^7)}{(i7)!} \dots$$

Which yields:

$$\frac{1}{1} + \frac{-x}{i} + \frac{x^2}{-2} + \frac{-x^3}{-i6} + \frac{x^4}{24} + \frac{-x^5}{i120} + \frac{x^6}{-720} + \frac{-x^7}{-i5040} \dots$$

We can now simplify:

$$1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \frac{x^4}{24} + \frac{ix^5}{120} - \frac{x^6}{720} - \frac{ix^7}{5040} \dots$$

We can separate cosine and sine values to find our exponential form:

$$\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}\ldots\right)+i\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}\ldots\right)=\cos(x)+isin(x)=\ e^{ix}$$

To sum up we can relate the real to imaginary forms of *e* and the exponential series as follows:

$$\sum_{k=0}^{\infty}\frac{(1)^k}{k!}=\sum_{z=0}^{\infty}\frac{(i)^z}{(iz)!}=e \quad (15)$$

$$\sum_{k=0}^{\infty}\frac{(x)^k}{k!}=\sum_{z=0}^{\infty}\frac{(ix)^z}{(iz)!}=e^x \quad (16)$$

$$\sum_{k=0}^{\infty}\frac{(ix)^k}{k!}=\sum_{z=0}^{\infty}\frac{(i^2x)^z}{(iz)!}=e^{ix} \quad (17)$$

What we have found is that the constant *e* is also an infinite series of the imaginary kind. This makes the number *e* and its exponential form all the more profound. We know that *e* when raised to the power of *x* is both its own derivative and its own integral. We can now see that *e* is also an amalgam of both a real number series expansion and at the same an imaginary number series expansion.

The constant *e* and its exponential form are so influential in mathematics and represent a crossroads of sorts by acting as a meeting point for the real with the imaginary. With this in mind we can better appreciate and intuit the profound role they play in complex analysis.

References

[1] Maor, Eli, "*e* – The Story of a Number ", *Princeton University Press*, Princeton, 1994.

[2] Thukral, Ashwani, K, "Factorials of real negative and imaginary numbers – A new perspective".*SpringerPlus*. **3**:658, 2014.