Bell's Theorem and the Mathematical Universe

Arithmoi Foundation

June 6, 2025

Introduction

Bell's Theorem is a landmark achievement in physics, demonstrating a fundamental conflict between the principles of local realism and the predictions of quantum mechanics. This guide will walk through the essential mathematical concepts, the historical development of the ideas, the theorem itself, its experimental verification, and its profound philosophical implications, including its alignment with the concept of a mathematical universe.

Foundational Concepts: Probability and Statistics

To grasp Bell's Theorem, a basic understanding of probability and statistics is crucial.

Probability Basics

Probability quantifies the likelihood of an event occurring. For an event A, its probability $P(A)$ ranges from 0 (impossible) to 1 (certain).

- **Mutually Exclusive Events**: If events A and B cannot occur simultaneously, the probability of either A or B occurring is $P(A \cup B) = P(A) + P(B)$.
- **Independent Events**: If the occurrence of event A does not affect the probability of event B, then $P(A \cap B) = P(A)P(B)$.

Expectation Value

The expectation value (or expected value) of a random variable X, denoted $E[X]$ or $\langle X \rangle$, is the weighted average of all possible values that X can take, with weights being their respective probabilities.

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

If X can take values $x_1, x_2, \ldots, x_n$ with probabilities $P(x_1), P(x_2), \ldots, P(x_n)$, then:

$$E[X] = \sum_{i=1}^{n} x_i P(x_i)$$

For example, if a measurement can yield outcomes $+1$ or -1, each with probability $1/2$, the expectation value is $E[X] = (+1) \cdot (1/2) + (-1) \cdot (1/2) = 0$.

Correlation Coefficient

A correlation coefficient measures the statistical relationship, or association, between two random variables. For two variables A and B, the correlation coefficient $C(A, B)$ (often denoted ρ_{AB}) typically ranges from -1 to $+1$.

- $C(A, B) = +1$: Perfect positive correlation (A and B always agree).
- $C(A, B) = -1$: Perfect negative correlation (A and B always disagree).
- $C(A, B) = 0$: No linear correlation.

The expectation value of the product of two variables is often used to define correlation in the context of Bell's theorem. If A and B are variables that can take values ± 1, their correlation is given by:

$$E(A, B) = P(+, +|A, B) - P(+, -|A, B) - P(-, +|A, B) + P(-, -|A, B)$$

Where $P(i, j|A, B)$ is the joint probability of observing outcome i for A and outcome j for B.

Quantum Mechanical Background

Quantum mechanics describes the physical properties of nature at the scale of atoms and subatomic particles.

Spin

Spin is an intrinsic form of angular momentum carried by elementary particles. It's a quantum mechanical property without a classical analogue. For example, an electron has spin 1/2. When measured along a specific axis (e.g., z-axis), its spin component can be found to be either "up" ($+\hbar/2$) or "down" ($-\hbar/2$), where $\hbar$ (h-bar) is the reduced Planck constant. We often simplify this to outcomes $+1$ and -1.

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

Entanglement

Quantum entanglement is a phenomenon where two or more quantum particles become linked in such a way that their fates are intertwined, regardless of the distance separating them. Measuring a property of one particle instantaneously influences the properties of the other(s).

Consider a pair of entangled spin-1/2 particles, like electrons, created in a singlet state. The total spin of the pair is zero. If one particle is measured to have spin-up along a certain axis, the other particle, when measured along the same axis, will instantly be found to have spin-down, and vice-versa. This correlation is perfect, no matter how far apart the particles are.

The quantum state of such a pair (e.g., an electron-positron pair from a pion decay) can be written as:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\downarrow_B\rangle - |\downarrow_A\uparrow_B\rangle)$$

Where:

- $|\psi\rangle$ (psi) represents the quantum state of the entangled pair.
- $|\uparrow_A\downarrow_B\rangle$ means particle A has spin-up and particle B has spin-down along a chosen axis.
- $|\downarrow_A\uparrow_B\rangle$ means particle A has spin-down and particle B has spin-up along that axis.

Measurement in Quantum Mechanics

In quantum mechanics, measurement is an active process that generally disturbs the system being measured. The outcome of a measurement is probabilistic. For a particle in a superposition of states (like one of the entangled particles before measurement), the act of measurement "collapses" the wavefunction into a definite state.

The EPR Paradox and the Quest for Completeness

In 1935, Albert Einstein, Boris Podolsky, and Nathan Rosen (EPR) published a paper titled "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?". They argued that quantum mechanics was incomplete.

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

Local Realism

The EPR argument was based on two fundamental assumptions, collectively known as **local realism**:

- **Realism**: Physical properties of objects exist independently of whether they are observed. An electron has a definite spin in a given direction even if we don't measure it.
- **Locality**: The result of a measurement on one particle cannot instantaneously influence the result of a measurement on another particle that is spatially separated from it. Information cannot travel faster than the speed of light.

EPR considered an entangled pair. If we measure the spin of particle A along the z-axis and find it to be spin-up, locality implies this measurement cannot affect particle B. Realism implies that particle B must have had a definite spin-down property along the z-axis all along, even before A was measured. Since quantum mechanics doesn't assign such a definite property before measurement, EPR concluded it was incomplete.

Hidden Variables

To complete quantum mechanics, EPR and others suggested the existence of **hidden variables**. These would be additional parameters, not yet known or described by quantum theory, that determine the outcome of individual measurements. If we knew these hidden variables, the probabilistic nature of quantum mechanics would disappear, and outcomes would be deterministic.

For example, for an entangled pair, a hidden variable λ (lambda) might predetermine the outcome of any spin measurement on either particle. The apparent randomness would be due to our ignorance of λ.

John Bell and His Theorem

In 1964, John Stewart Bell, a physicist at CERN, took the EPR argument further. He showed that the assumptions of local realism (or, more precisely, local hidden variable theories) lead to testable predictions that differ from those of quantum mechanics.

Bell's Assumptions

Bell considered an experiment with a source emitting pairs of entangled particles. Two observers, Alice and Bob, are spatially separated. Alice measures a property of her

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

particle (e.g., spin along axis $\mathbf{a}$), and Bob measures a property of his particle (e.g., spin along axis $\mathbf{b}$). Alice can choose her measurement setting $\mathbf{a}$ from a set of possibilities, and Bob can choose his setting $\mathbf{b}$.

Bell's theorem relies on the following assumptions for a local hidden variable theory:

1. **Realism**: The measurement outcomes $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$ are determined by the settings $\mathbf{a}$, $\mathbf{b}$ and the hidden variable(s) λ. The outcomes are definite values (e.g., $+1$ or -1).
2. **Locality**: Alice's outcome A cannot depend on Bob's setting $\mathbf{b}$, and Bob's outcome B cannot depend on Alice's setting $\mathbf{a}$. So, $A = A(\mathbf{a}, \lambda)$ and $B = B(\mathbf{b}, \lambda)$.
3. **Statistical Independence (or Free Will/Measurement Independence)**: The choice of measurement settings $(\mathbf{a}, \mathbf{b})$ is independent of the hidden variables λ. The distribution of hidden variables $\rho(\lambda)$ is fixed and does not depend on the settings chosen by the experimenters.

Bell's Inequality

Bell derived an inequality that must be satisfied by any local hidden variable theory. There are several forms of Bell's inequality. One of the most famous and experimentally convenient is the **CHSH inequality**, named after John Clauser, Michael Horne, Abner Shimony, and Richard Holt, who derived it in 1969.

Let Alice choose between two settings $\mathbf{a}$ and $\mathbf{a}'$, and Bob choose between $\mathbf{b}$ and $\mathbf{b}'$. The outcomes are $A = \pm 1$ and $B = \pm 1$.

The expectation value of the product of their outcomes for settings $\mathbf{a}$ and $\mathbf{b}$ is:

$$E(\mathbf{a}, \mathbf{b}) = \int A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \rho(\lambda) d\lambda$$

Where:

- $A(\mathbf{a}, \lambda)$ is Alice's outcome for setting $\mathbf{a}$ and hidden variable λ.
- $B(\mathbf{b}, \lambda)$ is Bob's outcome for setting $\mathbf{b}$ and hidden variable λ.
- $\rho(\lambda)$ (rho) is the probability distribution of the hidden variables λ.

The CHSH inequality states:

$$|E(\mathbf{a}, \mathbf{b}) - E(\mathbf{a}, \mathbf{b}')| + |E(\mathbf{a}', \mathbf{b}) + E(\mathbf{a}', \mathbf{b}')| \leq 2$$

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

Let S be the CHSH quantity:

$$S = E(\mathbf{a}, \mathbf{b}) - E(\mathbf{a}, \mathbf{b}') + E(\mathbf{a}', \mathbf{b}) + E(\mathbf{a}', \mathbf{b}')$$

Then the CHSH inequality is $|S| \leq 2$.

Derivation Sketch of CHSH Inequality (Simplified):

Assume local hidden variables. For each particle pair (characterized by λ), the outcomes $A_1 = A(\mathbf{a}, \lambda)$, $A_2 = A(\mathbf{a}', \lambda)$, $B_1 = B(\mathbf{b}, \lambda)$, $B_2 = B(\mathbf{b}', \lambda)$ are all ± 1. Consider the quantity $A_1 B_1 - A_1 B_2 + A_2 B_1 + A_2 B_2 = A_1(B_1 - B_2) + A_2(B_1 + B_2)$. Since $B_1, B_2 = \pm 1$, either $B_1 - B_2 = 0$ (if $B_1 = B_2$) or $B_1 + B_2 = 0$ (if $B_1 \neq B_2$).

- If $B_1 = B_2$, then $A_1(B_1 - B_2) + A_2(B_1 + B_2) = A_2(2B_1) = \pm 2$.
- If $B_1 = -B_2$, then $A_1(B_1 - B_2) + A_2(B_1 + B_2) = A_1(2B_1) = \pm 2$.

So, for any specific λ, the expression $A(\mathbf{a}, \lambda)B(\mathbf{b}, \lambda) - A(\mathbf{a}, \lambda)B(\mathbf{b}', \lambda) + A(\mathbf{a}', \lambda)B(\mathbf{b}, \lambda) + A(\mathbf{a}', \lambda)B(\mathbf{b}', \lambda)$ must be equal to ± 2.

Taking the average over all λ (i.e., the expectation value):

$$|\langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle| \leq \langle |A_1(B_1 - B_2) + A_2(B_1 + B_2)| \rangle = \langle 2 \rangle = 2$$

This gives $|S| \leq 2$.

Quantum Mechanical Prediction

Quantum mechanics predicts correlations that can violate this inequality. For entangled spin-1/2 particles in the singlet state, the expectation value of the product of spin measurements along directions $\mathbf{a}$ and $\mathbf{b}$ is given by:

$$E_{QM}(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} = -\cos(\theta_{ab})$$

Where θ_{ab} (theta-sub-ab) is the angle between the measurement directions $\mathbf{a}$ and $\mathbf{b}$.

Let's choose specific angles for the CHSH inequality:

- Alice's setting $\mathbf{a}$ (angle $0°$)

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

- Alice's setting $\mathbf{a}'$ (angle 90°)
- Bob's setting $\mathbf{b}$ (angle 45°)
- Bob's setting $\mathbf{b}'$ (angle 135°)

Then:

- $E(\mathbf{a}, \mathbf{b}) = -\cos(45°) = -1/\sqrt{2}$
- $E(\mathbf{a}, \mathbf{b}') = -\cos(135°) = -(-\cos(45°)) = +1/\sqrt{2}$
- $E(\mathbf{a}', \mathbf{b}) = -\cos(45°) = -1/\sqrt{2}$ (angle between 90° and 45° is 45°)
- $E(\mathbf{a}', \mathbf{b}') = -\cos(45°) = -1/\sqrt{2}$ (angle between 90° and 135° is 45°)

Plugging these into the CHSH quantity S:

$$S_{QM} = (-1/\sqrt{2}) - (1/\sqrt{2}) + (-1/\sqrt{2}) + (-1/\sqrt{2}) = -4/\sqrt{2} = -2\sqrt{2}$$

So, $|S_{QM}| = 2\sqrt{2} \approx 2.828$.

Since $2\sqrt{2} > 2$, quantum mechanics predicts a violation of the CHSH inequality. This means that quantum mechanics is incompatible with local hidden variable theories.

Experimental Tests of Bell's Theorem

Numerous experiments have been conducted to test Bell's inequality. The overwhelming majority have confirmed the predictions of quantum mechanics and violated Bell's inequalities.

Typical Experimental Setup

- **Source**: Produces pairs of entangled particles (e.g., photons from parametric down-conversion, or atoms).
- **Detectors**: Alice and Bob each have a detector that can measure a property of their particle along a chosen axis. The settings (axes) can be changed.
- **Random Setting Choice**: Ideally, the settings **a** and **b** are chosen randomly and fast enough to prevent any light-speed communication between the choice of setting at one station and the measurement at the other (closing the "locality loophole").
- **High Detection Efficiency**: Detectors should be efficient enough to ensure that the detected pairs are a fair sample of all emitted pairs (closing the "detection loophole" or "fair sampling loophole").

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

Key Experiments

- **Freedman and Clauser (1972)**: One of the first experiments, using entangled photons from calcium atom decay. Their results were consistent with quantum mechanics and violated Bell's inequality.
- **Aspect, Grangier, and Roger (1981, 1982)**: Alain Aspect and his team performed a series of increasingly sophisticated experiments. Their 1982 experiment used time-varying polarizers, changing settings while photons were in flight, addressing a potential loophole related to the settings being fixed.
- **Weihs et al. (1998)**: Gregor Weihs and Anton Zeilinger's group performed an experiment with random setting choices and spacelike separation of measurements, significantly strengthening the closure of the locality loophole.
- **Hensen et al. (2015)**: Bas Hensen and Ronald Hanson's group at Delft University performed the first "loophole-free" Bell test, simultaneously closing both the locality and detection loopholes using entangled electron spins in diamond nitrogen-vacancy centers separated by 1.3 kilometers.

These experiments, and many others, have consistently shown violations of Bell's inequalities, supporting the predictions of quantum mechanics over local realism.

Historical Timeline and Key Figures

- **1935**: Einstein, Podolsky, and Rosen publish the EPR paradox paper.
- **1951**: David Bohm reformulates the EPR paradox in terms of spin, making it conceptually simpler.
- **1964**: John Stewart Bell publishes his theorem and inequality.
- **1969**: Clauser, Horne, Shimony, and Holt (CHSH) derive a more experimentally practical version of Bell's inequality.
- **1972**: Stuart Freedman and John Clauser perform the first significant experimental test.
- **1981-1982**: Alain Aspect, Philippe Grangier, and Gérard Roger conduct experiments with time-varying settings.
- **1990s-2000s**: Numerous experiments refine techniques, closing more loopholes (e.g., Zeilinger's group).
- **2015**: Hensen et al. (Delft) perform the first widely accepted loophole-free Bell test.
- **2022**: Alain Aspect, John F. Clauser, and Anton Zeilinger are awarded the Nobel Prize in Physics "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science."

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

Philosophical and Metaphysical Implications

Bell's theorem and its experimental verification have profound implications for our understanding of reality.

Rejection of Local Realism

The most direct consequence is that the worldview of local realism, which Einstein cherished, is untenable. At least one of its core assumptions (locality or realism) must be abandoned.

- **If Locality is Abandoned (Non-locality)**: This implies that there are influences that can act instantaneously over arbitrary distances. This doesn't mean we can send information faster than light (as that would violate causality in relativity), but it suggests a deep interconnectedness in the universe. This is the more common interpretation among physicists.
- **If Realism is Abandoned**: This means that physical properties do not have definite values until they are measured. The act of measurement plays a crucial role in bringing reality into being. This aligns with some interpretations of quantum mechanics like the Copenhagen interpretation.
- **Both could be false, or our understanding of them is flawed.**

Determinism and Free Will

Bell's theorem also touches upon determinism. If hidden variables determined outcomes, the universe would be deterministic at a deeper level. The failure of local hidden variable theories challenges this simple picture.

- **Superdeterminism**: A radical way to save local realism is to abandon the assumption of "free will" or "measurement independence." This means the choice of measurement settings is not truly free but is itself correlated with the hidden variables in a way that conspires to produce the quantum mechanical results. Most physicists find this explanation highly contrived.

Impact on Interpretations of Quantum Mechanics

Bell's theorem has shaped the debate around different interpretations of quantum mechanics:

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

- **Copenhagen Interpretation**: Generally accepts non-locality or a non-realist stance on unmeasured properties.
- **Many-Worlds Interpretation (MWI)**: Avoids the measurement problem by positing that all possible outcomes of a measurement occur, each in a separate, parallel universe. Locality is preserved within each branch, but the global structure is non-local in a sense.
- **Bohmian Mechanics (Pilot-Wave Theory)**: This is a non-local hidden variable theory. It is explicitly non-local and deterministic, consistent with Bell's theorem (as Bell's theorem rules out *local* hidden variables).

Nature of Reality

Bell's theorem forces us to confront that the universe is stranger than classical intuition suggests. It points towards a reality where interconnectedness and the role of observation are fundamental.

Bell's Theorem and the Mathematical Universe Hypothesis

The Mathematical Universe Hypothesis (MUH), proposed by Max Tegmark, posits that our external physical reality is a mathematical structure. In this view, the universe *is* mathematics, not just described by it.

Bell's theorem aligns with and can be seen as supporting aspects of the MUH in several ways:

- **Abstract Nature of Quantum Reality**: Quantum mechanics, particularly phenomena like entanglement and the violation of Bell inequalities, describes a reality that is far removed from everyday, tangible experience. The properties and behaviors it describes (wavefunctions, superposition, non-local correlations) are inherently abstract and mathematical. Bell's theorem underscores that this mathematical description is not just a tool but reflects a deep, counter-intuitive structure of reality itself.
- **Primacy of Mathematical Consistency**: Bell's theorem is a purely mathematical derivation based on logical assumptions (local realism). Its conflict with the mathematical structure of quantum mechanics, and the subsequent experimental verification favoring quantum mechanics, highlights that the universe adheres to a specific, and perhaps surprising, mathematical framework. If the universe is a mathematical structure, then its properties are dictated by mathematical consistency and theorems like Bell's.
- **Information-Theoretic Aspects**: Quantum mechanics, and particularly entanglement, has strong connections to information theory. The correlations violating

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.

Bell's inequalities can be viewed as a fundamental property of how information is structured and correlated in the universe. If the universe is mathematical, then information (which can be mathematically defined) could be a primary constituent. Bell's theorem reveals constraints on how this information can be locally encoded and correlated.

- **Non-Intuitive Structures**: The non-locality or non-realism implied by Bell's theorem suggests that the underlying mathematical structure of our universe is not necessarily one that aligns with our evolved intuitions based on macroscopic, classical physics. The MUH allows for any self-consistent mathematical structure to constitute a universe, and Bell's theorem points to the specific, non-classical type of structure our universe embodies.
- **Unreasonable Effectiveness of Mathematics**: The fact that a purely mathematical argument (Bell's theorem) can lead to such profound and testable insights about the fundamental nature of reality reinforces Eugene Wigner's observation about "the unreasonable effectiveness of mathematics in the natural sciences." The MUH offers an explanation for this: if reality *is* mathematics, then its effectiveness is not unreasonable but definitional.

In essence, Bell's theorem shows that certain intuitive mathematical assumptions about reality (local realism) are incompatible with the more complex mathematical structure of quantum mechanics that experiments confirm. This supports the idea that reality is governed by, or even *is*, a specific mathematical structure, one that is inherently quantum and non-local.

Conclusion

Bell's Theorem is more than just a piece of theoretical physics; it is a profound statement about the nature of reality. It emerged from a deep philosophical debate initiated by Einstein and his colleagues and, through Bell's mathematical ingenuity, transformed this debate into an experimentally testable question. The results have consistently favored quantum mechanics, forcing us to abandon the intuitive picture of a world governed by local realism.

The implications stretch far beyond physics, touching upon philosophy, metaphysics, and our understanding of information and determinism. It underscores that the universe at its most fundamental level operates in ways that defy classical intuition, hinting at a deeply interconnected and abstract reality, potentially aligning with the idea of a universe that is, at its core, a mathematical structure.

John Bell's work has not only solidified our understanding of quantum mechanics but has also opened new frontiers in quantum information science, including quantum computing and quantum cryptography, which harness the very non-local correlations that Bell's theorem brought to light.

 © 2025 Arithmoi Foundation | arithmoi.org. All rights reserved.