Pascal's Tetrahedron, Combinatorics and the Higher-Dimensional Complex Forms

Viator Lapis

Department of Applied Mathematics

University of Washington

Abstract

In this paper, we explore the relationship between higher dimensional complex forms, combinatorics and multinomial theorem. We look in particular at the applications of the trinomial theorem and the relationship between complex trigonometric infinite series expansions, combinatorics and geometrics insights in complex analysis and the higher dimensions.

Introduction

Pascal's triangle is named after Blaise Pascal, who in 1653 published his Traité du Triangle Arithmétique [1], which explored many of the uses of the triangle's numbers. Although it bears his name, discussion of the form can be traced back many centuries before. The form is a triangular representation of the expansion of binomial coefficients. There exist higher dimensional representations that are referred to a Pascal's simplices. The trinomial expansion of coefficients can be represented by a three-dimensional version called Pascal's tetrahedron. This form has multiple triangular layers that form a tetrahedron when taken as an ensemble. Normally the form is considered for real numbers that are commutative.

In this paper, we consider the trinomial expansion of complex forms with emphasis on the relationship with matrix groups. A matrix group is a group of invertible matrices and their study and applications spans many subsets of mathematics and physics. Matrix groups of higher dimensional representation tend to be non-commutative in nature. We consider this fact and apply this to multinomial theorem, with an emphasis on combinatorics. We use the SO(3,C) and U(3) group that we derived in a previous work [2], and discover a unique symmetry with trinomial theory that does not exist in a simple representation for quaternions or octonions. We explore this deep link between these different fields of mathematics and find they are different representations of the same form.

Trinomial Expansion

A trinomial expansion is the power expansion of the sum of three terms into monomials and is typically defined as

$$(x+y+z)^n = \sum_{\substack{i,j,k \\ i+j+k=n}} \binom{n}{i,j,k} x^i y^j z^k \,, \tag{1}$$

Where we define further for the trinomial expansion that

$$\text{"n choose i,j,k"} = \binom{n}{i,j,k} = \frac{n!}{i!\,j!\,k!}. \tag{2}$$

Similar to the binomial expansion and Pascal's triangle, we can arrange the expansion of the trinomial terms into a three-dimensional form called Pascal's tetrahedron. In the above formula, 'n' represents the respective layers of the tetrahedron and the indices '*i,j,k*' are the number of ways to choose $i!\,j!\,k!$ objects from the set of 'n', where $i!\,j!\,k! = n$. When it comes to how many terms we will have in each layer of our Pascal's tetrahedron we will find that it can be determined by

$$terms = \frac{(n+2)(n+1)}{2}, \tag{3}$$

where once again *'n'* is the layer of the tetrahedron. As an example, to find the number of terms in the tetrahedron layer 4

$$\frac{(4+2)(4+1)}{2} = \frac{30}{2} = 15. \tag{4}$$

When it comes to applications of the trinomial expansion with regard to complex-valued objects, we find deep layers of connection with multinomial theory, combinatorics, geometry, group and Lie theory. It is the main goal of this paper to explore these links with what we mathematically derived as Lie groups ,that represent hyper-dimensional complex forms. We start by applying objects of the imaginary form in *ix, iy and iz* where

$$(ix + iy + iz)^n = \sum_{\substack{i,j,k \\ i+j+k=n}} \binom{n}{i,j,k} x^i y^j z^k \,. \tag{5}$$

We give the example of creating layer 2 in the tetrahedron form where we see there are a total of 6 terms as

$$\frac{(2+2)(2+1)}{2} = \frac{12}{2} = 6. \tag{6}$$

The second layer will include all of the different arrangements in which we can choose 2 objects from the set *{ix,iy,iz}*. As noted above with objects that commute, there are six ways to do so as follows

$$\{(ix)^2, (iy)^2, (iz)^2, (ix \cdot iy), (ix \cdot iz), (iy \cdot iz)\}.$$

In the same order, we then have

$$(ix + iy + iz)^2 = \left\{\binom{2}{i,j,k}\right\} = \left\{\frac{2!}{2!\,0!\,0!}, \frac{2!}{0!\,2!\,0!}, \frac{2!}{0!\,0!\,2!}, \frac{2!}{1!\,1!\,0!} \frac{2!}{1!\,0!\,1!} \frac{2!}{0!\,1!\,1!}\right\}.$$

Which yields our set

$$\{-x^2, -y^2, -z^2, -2xy, -2xz, -2yz\}.$$

We proceed to determine layers 0 thru 5 below where we have turned the layers of the tetrahedron upside down to distinguish them from the Pascal's triangle. We can take note of the similarities of the tetrahedron with the cosine and sine values that we obtain with the infinite expansion of a complex multivariate exponential series, minus the factorial denominators. It turns out that when we do calculate a trinomial expansion of the form $e^{i(x+y+z)}$, there are very beautiful simplifications that result, yields geometric insights and leads us to conclude that radial measurements between complex planes are non-Euclidian, which is in contrast to what we have been lead to believe in mathematics and physics.

Layer 0

$$e^0$$

Layer 1

$$iz + iy$$
$$ix$$

Layer 2

$$-z^2 - 2yz - y^2$$
$$-2xz - 2xy$$
$$-x^2$$

Layer 3

$$-iz^3 \ - 3iyz^2 \ - 3iy^2z \ - \ iy^3$$
$$-3ixz^2 - 6ixyz - 3ixy^2$$
$$-3ix^2 - 3ix^2y$$
$$-ix^3$$

Layer 4

$$z^4 \ + \ 4yz^3 \ + \ 6y^2z^2 \ + \ 4y^3z \ + \ y^4$$
$$4xz^3 + 12xyz^2 + 12xy^2z + 4xy^3$$
$$6x^2z^2 + 12x^2yz + 6x^2y^2$$
$$4x^3z + 4x^3y$$
$$x^4$$

Layer 5

$$iz^5 + 5iyz^4 + 10iy^2z^3 + 10iy^3z^2 + 5iy^4z + iy^5$$
$$5ixz^4 + 20ixyz^3 + 30ixy^2z^2 + 20ixy^3z + 5ixy^4$$
$$10ixz^3 + 20ixyz^2 + 20ixy^2z + 10ixy^3$$
$$6x^2z^2 + 12x^2yz + 6x^2y^2$$
$$4x^3z + 4x^3y$$
$$x^4$$

Fig. 1 - Pascal's Tetrahedron for $(ix + iy + iz)^n \ for \ n = 0 \ to \ 5$.

Further consideration reveals that the total number of ways to combine our imaginary objects in non-commutative fashion is 3^n. We once again use our tetrahedron layer 2 set, and find nine total non-commutative values in the layer as such

$$\{-x^2, -y^2, -z^2, -xy, -yx, -xz, -zx, -yz, -zy\}.$$

Non-commutative objects play a very important role in Lie theory and linear algebra. We keep this consideration in mind as we proceed.

Lie Representation of the SO(3,C) and U(3) Group

In previous work [2], we derived the Lie group that represented the orthogonal grouping of $3\times\mathbb{C}_2$cyclic groups by exponentiation of the three imaginary axes of this group into a six-form. These imaginary permutations are the group generators of the whole six-dimensional group, where combinations of these groups yield all of the six permutations. We show below these 3 permutation matrices and with squaring them we arrive at the negative identity of the matrix.

$$b = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \quad (7)$$

$$b^2, d^2, f^2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{8}$$

We also, for completeness, show the real axes in the way of the permutations *'a', 'c' and 'e'* below. We note that *'a'* is the identity permutation and *'c'* and *'e'* run the other two off-diagonals of the group.

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, c = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, e = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{9}$$

We form the representation of the group by combining the three imaginary axes below

$$\mathfrak{g} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}. \tag{10}$$

We proceed and take varying powers of the $\mathfrak{g}$ matrix and find a very unique output that is consistent with the trigonometric functions of the complex exponential series. Even powers of $\mathfrak{g}$ are consistent with real values and represent the higher-dimensional cosine series, while odd powers of $\mathfrak{g}$ represent the imaginary values of the sine series. We see that we have appropriate progress of the even and odd signs in the power series and note how real values occupy real permutations *('a','c','e')* and that imaginary values occupy our imaginary permutations *('b','d','f')*. For our negative imaginary values, we simply see a reversal of the position of the negative sign as seen in $\mathfrak{g}^1$ vs g^3.

$$g^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, g^1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} g^2 = \begin{bmatrix} -3 & 0 & -3 & 0 & -3 & 0 \\ 0 & -3 & 0 & -3 & 0 & -3 \\ -3 & 0 & -3 & 0 & -3 & 0 \\ 0 & -3 & 0 & -3 & 0 & -3 \\ -3 & 0 & -3 & 0 & -3 & 0 \\ 0 & -3 & 0 & -3 & 0 & -3 \end{bmatrix} \tag{11}$$

$$g^3 = \begin{bmatrix} 0 & -9 & 0 & -9 & 0 & -9 \\ 9 & 0 & 9 & 0 & 9 & 0 \\ 0 & -9 & 0 & -9 & 0 & -9 \\ 9 & 0 & 9 & 0 & 9 & 0 \\ 0 & -9 & 0 & -9 & 0 & -9 \\ 9 & 0 & 9 & 0 & 9 & 0 \end{bmatrix}, g^4 = \begin{bmatrix} 27 & 0 & 27 & 0 & 27 & 0 \\ 0 & 27 & 0 & 27 & 0 & 27 \\ 27 & 0 & 27 & 0 & 27 & 0 \\ 0 & 27 & 0 & 27 & 0 & 27 \\ 27 & 0 & 27 & 0 & 27 & 0 \\ 0 & 27 & 0 & 27 & 0 & 27 \end{bmatrix}, g^5 = \begin{bmatrix} 0 & 81 & 0 & 81 & 0 & 81 \\ -81 & 0 & -81 & 0 & -81 & 0 \\ 0 & 81 & 0 & 81 & 0 & 81 \\ -81 & 0 & -81 & 0 & -81 & 0 \\ 0 & 81 & 0 & 81 & 0 & 81 \\ -81 & 0 & -81 & 0 & -81 & 0 \end{bmatrix}.$$

It may not be obvious to the reader right now, but the values of the entries in each matrix when summed along a row or a column, represents the total number of non-commutative objects in each layer of the Pascal's tetrahedron. For example, we recall that we would have with *'n'*= 4, 81 terms $(3^4 = 81)$ and we have that the sum of row one of $g^4 = (27 + 27 + 27) = 81$.

We define $X = x \cdot b, Y = y \cdot d \text{ and } Z = z \cdot f$ and redefine $\mathfrak{g} = \{X + Y + Z \colon \in \mathbb{R}\}$

$$\mathfrak{g} = \begin{bmatrix} 0 & x & 0 & z & 0 & y \\ -x & 0 & -z & 0 & -y & 0 \\ 0 & z & 0 & y & 0 & x \\ -z & 0 & -y & 0 & -x & 0 \\ 0 & y & 0 & x & 0 & z \\ -y & 0 & -x & 0 & -z & 0 \end{bmatrix}. \tag{12}$$

We now have our permutations in the matrix form that we can exponentiate, to arrive at our Lie group for our hyper-dimensional complex form

$$e^{i(x+y+z)} = e^{\mathfrak{g}} = \frac{\mathfrak{g}^0}{0!} + \frac{\mathfrak{g}^1}{1!} + \frac{\mathfrak{g}^2}{2!} + \frac{\mathfrak{g}^3}{3!} + \frac{\mathfrak{g}^4}{4!} + \frac{\mathfrak{g}^5}{5!} + \cdots, + \frac{\mathfrak{g}^\infty}{\infty!}. \tag{13}$$

Where we show the first 4 examples

$$\frac{\mathfrak{g}^0}{0!} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \frac{\mathfrak{g}^1}{1!} = \begin{bmatrix} 0 & x & 0 & z & 0 & y \\ -x & 0 & -z & 0 & -y & 0 \\ 0 & z & 0 & y & 0 & x \\ -z & 0 & -y & 0 & -x & 0 \\ 0 & y & 0 & x & 0 & z \\ -y & 0 & -x & 0 & -z & 0 \end{bmatrix}, \therefore$$

$$\frac{\mathfrak{g}^2}{2!} = \begin{bmatrix} -a_2 & 0 & -c_2 & 0 & -e_2 & 0 \\ 0 & -a_2 & 0 & -c_2 & 0 & -e_2 \\ -e_2 & 0 & -a_2 & 0 & -c_2 & 0 \\ 0 & -e_2 & 0 & -a_2 & 0 & -c_2 \\ -c_2 & 0 & -e_2 & 0 & -a_2 & 0 \\ 0 & -c_2 & 0 & -e_2 & 0 & -a_2 \end{bmatrix}, \frac{\mathfrak{g}^3}{3!} = \begin{bmatrix} 0 & -b_3 & 0 & -f_3 & 0 & -d_3 \\ b_3 & 0 & f_3 & 0 & d_3 & 0 \\ 0 & -f_3 & 0 & -d_3 & 0 & -b_3 \\ f_3 & 0 & d_3 & 0 & b_3 & 0 \\ 0 & -d_3 & 0 & -b_3 & 0 & -f_3 \\ d_3 & 0 & b_3 & 0 & f_3 & 0 \end{bmatrix}. \tag{14}$$

Where for $\frac{\mathfrak{g}^2}{2!}, a_2 = \left(\frac{x^2+y^2+z^2}{2}\right), c_2 = \left(\frac{xy+xz+yz}{2}\right), e_2 = \left(\frac{xy+xz+yz}{2}\right) \text{ and } \frac{\mathfrak{g}^3}{3!} b_3 = \frac{x^3}{6} + \frac{xy^2}{3} + \frac{xyz}{3} + \frac{xz^2}{3} + \frac{y^2z}{6} + \frac{yz^2}{6},\ d_3 = \frac{y^3}{6} + \frac{x^2y}{3} + \frac{x^2z}{3} + \frac{xyz}{3} + \frac{xz^2}{6} + \frac{yz^2}{3}, z_3 = \frac{z^3}{6} + \frac{x^2y}{3} + \frac{x^2z}{3} + \frac{xy^2}{6} + \frac{xyz}{3} + \frac{y^2z}{3}$.

We arrive at the following form for the six-dimensional SO(3,C) representation below, where we have all real values for the matrix as we went from $\mathbb{C}^3 \rightarrow \mathbb{R}^6$, and we can recall that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \therefore \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i. \tag{15}$$

$$\begin{bmatrix} \frac{2\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\cdot\frac{2\sin(\boldsymbol{r})}{3}+\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}}\cdot\frac{2\sin(\boldsymbol{r})}{3}+\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}}\cdot\frac{2\sin(\boldsymbol{r})}{3}+\frac{\sin(\boldsymbol{\gamma})}{3} \\ -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\cdot\frac{2\sin(\mathbf{r})}{3}-\frac{\sin(\boldsymbol{\gamma})}{3} & \frac{2\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}}\cdot\frac{2\sin(\boldsymbol{r})}{3}-\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}}\cdot\frac{2\sin(\boldsymbol{r})}{3}-\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} \\ -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}}\cdot\frac{2\sin(\boldsymbol{r})}{3}+\frac{\sin(\boldsymbol{\gamma})}{3} & \frac{2\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}}\cdot\frac{2\sin(\boldsymbol{r})}{3}+\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\cdot\frac{2\sin(\boldsymbol{r})}{3}+\frac{\sin(\boldsymbol{\gamma})}{3} \\ -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}}\cdot\frac{2\sin(\boldsymbol{r})}{3}-\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}}\cdot\frac{2\sin(\boldsymbol{r})}{3}-\frac{\sin(\boldsymbol{\gamma})}{3} & \frac{2\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\cdot\frac{2\sin(\boldsymbol{r})}{3}-\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} \\ -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}}\cdot\frac{2\sin(\boldsymbol{r})}{3}+\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\cdot\frac{2\sin(\boldsymbol{r})}{3}+\frac{\sin(\boldsymbol{\gamma})}{3} & \frac{2\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}}\cdot\frac{2\sin(\boldsymbol{r})}{3}-\frac{\sin(\boldsymbol{\gamma})}{3} \\ -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}}\cdot\frac{2\sin(\boldsymbol{r})}{3}-\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\cdot\frac{2\sin(\boldsymbol{r})}{3}-\frac{\sin(\boldsymbol{\gamma})}{3} & -\frac{\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} & -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}}\cdot\frac{2\sin(\boldsymbol{r})}{3}+\frac{\sin(\boldsymbol{\gamma})}{3} & \frac{2\cos(\boldsymbol{r})}{3}+\frac{\cos(\boldsymbol{\gamma})}{3} \end{bmatrix}$$

Fig. 2 – Lie Group Matrix $\boldsymbol{G}$. $\boldsymbol{r} = \sqrt{x^2 - xy - xz + y^2 - yz + z^2}$, $\boldsymbol{\gamma} = (x + y + z)$, $\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} = \frac{(x - \frac{y}{2} - \frac{z}{2})}{\boldsymbol{r}}$, $\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}} = \frac{(-\frac{x}{2} + y - \frac{z}{2})}{\boldsymbol{r}}$, $\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}} = \frac{(-\frac{x}{2} - \frac{y}{2} + z)}{\boldsymbol{r}}$

In regard to the quaternion form, we have that $r = \sqrt{(x^2 + y^2 + z^2)}$, whereas with the orthogonal representation of $3\times\mathbb{C}_2$ cyclic groups we have that $r = \sqrt{(x^2 - xy - xz + y^2 - yz + z^2)}$. This is an interesting result! In the two-dimensional complex form we are accustomed to taking the modulus to arrive at our radius where by the modulus we mean

$$(a + ib)(a - ib) = |a^2 + b^2|, \tag{16}$$

we can now argue the validity of this assumption for higher dimensional complex groups. The use of the complex conjugate seems to work by sweeping imaginary numbers under the rug. We will show that this radius is imaginary for angle representations that are real in the Lie group.

We end with also showing the reader the ***U(3)*** representation that we arrive at when going back from $\mathbb{R}^6 \rightarrow \mathbb{C}^3$

$$\begin{bmatrix} \frac{e^{i\gamma}}{3}+\frac{2\cos(\boldsymbol{r})}{3}+\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\cdot\frac{2i\sin(\boldsymbol{r})}{3} & \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{r})}{3}+\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}}\cdot\frac{2i\sin(\boldsymbol{r})}{3} & \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{r})}{3}+\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}}\cdot\frac{2i\sin(\boldsymbol{r})}{3} \\ \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{r})}{3}+\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}}\cdot\frac{2i\sin(\boldsymbol{r})}{3} & \frac{e^{i\gamma}}{3}+\frac{2\cos(\boldsymbol{r})}{3}+\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}}\cdot\frac{2i\sin(\boldsymbol{r})}{3} & \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{r})}{3}+\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\cdot\frac{2i\sin(\boldsymbol{r})}{3} \\ \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{r})}{3}+\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}}\cdot\frac{2i\sin(\boldsymbol{r})}{3} & \frac{e^{i\gamma}}{3}-\frac{\cos(\boldsymbol{r})}{3}+\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\cdot\frac{2i\sin(\boldsymbol{r})}{3} & \frac{e^{i\gamma}}{3}+\frac{2\cos(\boldsymbol{r})}{3}+\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}}\cdot\frac{2i\sin(\boldsymbol{r})}{3} \end{bmatrix}.$$

Fig. 3 – $G_{U(3)}$ Lie group. $\boldsymbol{r} = \sqrt{x^2 - xy - xz + y^2 - yz + z^2}$, $\boldsymbol{\gamma} = (x + y + z)$, $\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} = \frac{(x - \frac{y}{2} - \frac{z}{2})}{\boldsymbol{r}}$, $\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}} = \frac{(-\frac{x}{2} + y - \frac{z}{2})}{\boldsymbol{r}}$, $\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{z}} = \frac{(-\frac{x}{2} - \frac{y}{2} + z)}{\boldsymbol{r}}$.

The Trinomial Expansion of the $3\times\mathbb{C}_2$ Exponential Series

We can define the exponential series as a trinomial expansion for positive integers of *n*, from $0 \rightarrow \infty$

$$e^{i(x+y+z)} = \frac{(ix + iy + iz)^n}{n!} = \sum_{\substack{i,j,k \\ i+j+k=n}} \binom{n}{i,j,k} \frac{x^i y^j z^k}{n!}. \tag{17}$$

We proceed with the above definition to expand into our Pascal's tetrahedron and find

Layer 0 (**positive cosine function**)

$$\frac{e^0}{0!\cdot 0!\cdot 0!}\left(\frac{1}{1}\right)$$

Layer 1 (**positive sine function**)

$$\frac{iz}{0!\cdot 0!\cdot 1!}\left(\frac{1}{3}\right)+\frac{iy}{0!\cdot 1!\cdot 0!}\left(\frac{1}{3}\right)$$
$$\frac{ix}{1!\cdot 0!\cdot 0!}\left(\frac{1}{3}\right)$$

Layer 2 (**negative cosine function**)

$$-\frac{z^2}{0!\cdot 0!\cdot 2!}\left(\frac{1}{9}\right)-\frac{yz}{0!\cdot 1!\cdot 1!}\left(\frac{2}{9}\right)-\frac{y^2}{0!\cdot 2!\cdot 0!}\left(\frac{1}{9}\right)$$
$$-\frac{xz}{1!\cdot 0!\cdot 1!}\left(\frac{2}{9}\right)-\frac{xy}{1!\cdot 1!\cdot 0!}\left(\frac{2}{9}\right)$$
$$-\frac{x^2}{2!\cdot 0!\cdot 0!}\left(\frac{1}{9}\right)$$

Layer 3 (**negative sine function**)

$$-\frac{iz^3}{0!\cdot 0!\cdot 3!}\left(\frac{1}{27}\right)-\frac{iyz^2}{0!\cdot 1!\cdot 2!}\left(\frac{3}{27}\right)-\frac{iy^2z}{0!\cdot 2!\cdot 1!}\left(\frac{3}{27}\right)-\frac{iy^3}{0!\cdot 0!\cdot 3!}\left(\frac{1}{27}\right)$$
$$-\frac{ixz^2}{1!\cdot 0!\cdot 2!}\left(\frac{3}{27}\right)-\frac{ixyz}{1!\cdot 1!\cdot 1!}\left(\frac{6}{27}\right)-\frac{ixy^2}{1!\cdot 2!\cdot 0!}\left(\frac{3}{27}\right)$$
$$-\frac{ix^2z}{2!\cdot 0!\cdot 1!}\left(\frac{3}{27}\right)-\frac{ix^2y}{2!\cdot 1!\cdot 0!}\left(\frac{3}{27}\right)$$
$$-\frac{ix^3}{0!\cdot 0!\cdot 3!}\left(\frac{1}{27}\right)$$

Layer 4 (**positive cosine function**)

$$\frac{z^4}{0!\cdot 0!\cdot 4!}\left(\frac{1}{81}\right)+\frac{yz^3}{0!\cdot 1!\cdot 3!}\left(\frac{4}{81}\right)+\frac{y^2z^2}{0!\cdot 2!\cdot 2!}\left(\frac{6}{81}\right)+\frac{y^3z}{0!\cdot 3!\cdot 1!}\left(\frac{4}{81}\right)+\frac{y^4}{0!\cdot 4!\cdot 0!}\left(\frac{1}{81}\right)$$
$$\frac{xz^3}{1!\cdot 0!\cdot 3!}\left(\frac{4}{81}\right)+\frac{xyz^2}{1!\cdot 1!\cdot 2!}\left(\frac{12}{81}\right)+\frac{xy^2z}{1!\cdot 2!\cdot 1!}\left(\frac{12}{81}\right)+\frac{xy^3}{1!\cdot 3!\cdot 0!}\left(\frac{4}{81}\right)$$
$$\frac{x^2z^2}{2!\cdot 0!\cdot 2!}\left(\frac{6}{81}\right)+\frac{x^2yz}{2!\cdot 1!\cdot 1!}\left(\frac{12}{81}\right)+\frac{x^2y^2}{2!\cdot 2!\cdot 0!}\left(\frac{6}{81}\right)$$
$$\frac{x^3z}{3!\cdot 0!\cdot 1!}\left(\frac{4}{81}\right)+\frac{x^3y}{3!\cdot 1!\cdot 0!}\left(\frac{4}{81}\right)$$
$$\frac{x^4}{4!\cdot 0!\cdot 0!}\left(\frac{1}{81}\right)$$

Layer 5 (**positive sine function**)

$$\frac{iz^5}{0!\cdot 0!\cdot 5!}\left(\frac{1}{243}\right)+\frac{iyz^4}{0!\cdot 1!\cdot 4!}\left(\frac{5}{243}\right)+\frac{iy^2z^3}{0!\cdot 2!\cdot 3!}\left(\frac{10}{243}\right)+\frac{iy^3z^2}{0!\cdot 3!\cdot 2!}\left(\frac{10}{243}\right)+\frac{iy^4z}{0!\cdot 4!\cdot 1!}\left(\frac{5}{243}\right)+\frac{iy^5}{0!\cdot 5!\cdot 0!}\left(\frac{1}{243}\right)$$
$$\frac{ixz^4}{1!\cdot 0!\cdot 4!}\left(\frac{5}{243}\right)+\frac{ixyz^3}{1!\cdot 1!\cdot 3!}\left(\frac{20}{243}\right)+\frac{ixy^2z^2}{1!\cdot 2!\cdot 2!}\left(\frac{30}{243}\right)+\frac{ixy^3z}{1!\cdot 3!\cdot 1!}\left(\frac{20}{243}\right)+\frac{ixy^4}{1!\cdot 4!\cdot 0!}\left(\frac{5}{243}\right)$$
$$\frac{ix^2z^3}{2!\cdot 0!\cdot 3!}\left(\frac{10}{243}\right)+\frac{ix^2yz^2}{2!\cdot 1!\cdot 2!}\left(\frac{30}{243}\right)+\frac{ix^2y^2z}{2!\cdot 2!\cdot 1!}\left(\frac{30}{243}\right)+\frac{ix^2y^3}{2!\cdot 3!\cdot 0!}\left(\frac{10}{243}\right)$$
$$\frac{x^3z^2}{3!\cdot 0!\cdot 2!}\left(\frac{10}{243}\right)+\frac{x^3yz}{3!\cdot 1!\cdot 1!}\left(\frac{20}{243}\right)+\frac{x^3y^2}{3!\cdot 2!\cdot 0!}\left(\frac{10}{243}\right)$$
$$\frac{x^4z}{4!\cdot 0!\cdot 1!}\left(\frac{5}{243}\right)+\frac{x^4y}{4!\cdot 1!\cdot 0!}\left(\frac{5}{243}\right)$$
$$\frac{x^5}{5!\cdot 0!\cdot 0!}\left(\frac{1}{243}\right)$$

Fig. 4 - Pascal's Tetrahedron for $\sum_{n=0}^{\infty}\frac{g^n}{n!}$

We see above in ***Fig. 4,*** that each layer represents either the cosine of imaginary sine series values. There is a very nice simplification of the tetrahedron for our values in each layer. We see that with the addition of the factorial in the denominator of the series cancels exactly with the $n!$ in the numerator

$$\frac{(ix+iy+iz)^n}{n!} = \sum_{\substack{i,j,k \\ i+j+k=n}} \binom{n}{i,j,k} \frac{x^i y^j z^k}{n!} = \frac{n!}{n!\, i!\, j!\, k!} = \frac{1}{i!\, j!\, k!}. \tag{18}$$

So, we see that our entries in each layer can be represented with each respective *"choose i,j,k"* in each denominator. For example, in Layer 5, we choose the second entry from the left from row 3 which is

$$\frac{ix^2yz^2}{2! \cdot 1! \cdot 2!}\left(\frac{30}{243}\right). \tag{19}$$

We have found that with the factorial in the denominator we would have

$$\frac{5!}{2!\,1!\,2!} = \frac{120}{6} = 30ix^2yz^2, \tag{20}$$

but with cancellation we have

$$\frac{5!}{5!\,2!\,1!\,2!} = \frac{1}{2!\,1!\,2!} = \frac{1}{6}. \tag{21}$$

This result can be interpreted as a ratio to the other entries in the layer. So, we compare with entry in the sixth row/first entry from left, with the forth row/second entry from left, to see

$$\frac{iz^5}{0! \cdot 0! \cdot 5!} : \frac{ix^2yz^2}{2! \cdot 1! \cdot 2!} \rightarrow \frac{iz^5}{120} : \frac{ix^2yz^2}{6} \rightarrow iz^5 : 30ix^2yz^2. \tag{22}$$

So, there are 30 more entries of the form ix^2yz^2 than iz^5's. We know there are 243 total entries in the layer and that there is only one occurrence of iz^5 in layer 5. So, there are 30 entries in layer 5 of the monomial form ix^2yz^2. We have put brackets next to each entry, indicating to indicate this ratio as

$$\left(\frac{number\ of\ monomials\ of\ particular\ form\ \left(\frac{n!}{i!\,j!\,k!}\right)}{total\ number\ of\ non-commutative\ entries\ in\ layer\ (3^n)}\right). \tag{23}$$

We can also see that the exponential series for $e^{ix}, e^{iy}\ and\ e^{iz}$ run along the corners of each layer. This gives us geometric clues about the monomials in the tetrahedron. We see that this is an analogue to a six-dimensional form, as the apical edges (the edges that are each singular exponential series) are two-dimensional in their own right in being $\mathbb{C}_2$ cyclic groups. So, we see that each layer is orthogonal to the layer both above and below itself. All of the monomials outside of the apical edges represent the geometric links between the six axes of the Lie group and can be represented as directed monomials when the non-commutative algebraic nature is considered.

	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	-a	f	-e	d	-c
c	c	d	e	f	a	b
d	d	-c	b	-a	f	-e
e	e	f	a	b	c	d
f	f	-e	d	-c	b	-a

Fig. 5 – Standard Form Cayley Table of Permutation matrix ***(8)***.
The identity element is noted as "a" above.

Above we include the Standard Form Cayley Table of the non-commutative relations of our group's permutations. Recall that we have defined $X = b, Y = d \; and \; Z = f$. We examine Layer 4 in our tetrahedron form, which represents the positive cosine function for the group and as such we will find that the monomials will yield positive real values of the group *'a','c', and 'e'*. $\{a, c, e{:} +\mathbb{R} \in G\}$

Layer 4 $(\boldsymbol{positive\ cosine\ function})$

$$\frac{z^4}{0!\cdot 0!\cdot 4!}\left(\frac{1}{81}\right) + \frac{yz^3}{0!\cdot 1!\cdot 3!}\left(\frac{4}{81}\right) + \frac{y^2z^2}{0!\cdot 2!\cdot 2!}\left(\frac{6}{81}\right) + \frac{y^3z}{0!\cdot 3!\cdot 1!}\left(\frac{4}{81}\right) + \frac{y^4}{0!\cdot 4!\cdot 0!}\left(\frac{1}{81}\right)$$
$$\frac{xz^3}{1!\cdot 0!\cdot 3!}\left(\frac{4}{81}\right) + \frac{xyz^2}{1!\cdot 1!\cdot 2!}\left(\frac{12}{81}\right) + \frac{xy^2z}{1!\cdot 2!\cdot 1!}\left(\frac{12}{81}\right) + \frac{xy^3}{1!\cdot 3!\cdot 0!}\left(\frac{4}{81}\right)$$
$$\frac{x^2z^2}{2!\cdot 0!\cdot 2!}\left(\frac{6}{81}\right) + \frac{x^2yz}{2!\cdot 1!\cdot 1!}\left(\frac{12}{81}\right) + \frac{x^2y^2}{2!\cdot 2!\cdot 0!}\left(\frac{6}{81}\right)$$
$$\frac{x^3z}{3!\cdot 0!\cdot 1!}\left(\frac{4}{81}\right) + \frac{x^3y}{3!\cdot 1!\cdot 0!}\left(\frac{4}{81}\right)$$
$$\frac{x^4}{4!\cdot 0!\cdot 0!}\left(\frac{1}{81}\right)$$

We first start by expanding the non-singular monomials and determining the permutation results of each with the help of our Standard Form Cayley Table

$$\frac{z^4}{0!\cdot 0!\cdot 4!}\binom{4}{0,0,4}, \frac{y^4}{0!\cdot 4!\cdot 0!}\binom{4}{0,4,0}, \frac{x^4}{0!\cdot 0!\cdot 4!}\binom{4}{0,0,4} \to \{a, a, a\} \tag{24.1}$$

$$\frac{yz^3}{0!\cdot 1!\cdot 3!}\binom{4}{0,1,3} \to \{yz^3, z^2yz, z^3y, yz^3\} \to \{e, e, c, e\}, \tag{24.2}$$

$$\frac{y^3z}{0!\cdot 3!\cdot 1!}\binom{4}{0,3,1} \to \{zy^3, y^2zy, y^3z, zy^3\} \to \{c, c, e, c\}, \tag{24.3}$$

$$\frac{xz^3}{1!\cdot 0!\cdot 3!}\binom{4}{1,0,3} \to \{xz^3, z^2xz, z^3x, xz^3\} \to \{c, c, e, c\}, \tag{24.4}$$

$$\frac{xy^3}{1!\cdot 3!\cdot 0!}\binom{4}{1,3,0} \to \{xy^3, y^2xy, y^3x, xy^3\} \to \{e, e, c, e\}, \tag{24.5}$$

$$\frac{x^3z}{3!\cdot 0!\cdot 1!}\binom{4}{3,0,1} \to \{zx^3, x^2zx, x^3z, zx^3\} \to \{e, e, c, e\}, \tag{24.6}$$

$$\frac{x^3y}{3!\cdot 1!\cdot 0!}\binom{4}{3,1,0} \to \{yx^3, x^2yx, x^3y, yx^3\} \to \{c, c, e, c\}, \tag{24.7}$$

$$\frac{y^2z^2}{0!\cdot 2!\cdot 2!}\binom{4}{0,2,2} \to \{y^2z^2, yz^2y, yzyz, zyzy, z^2y^2, zy^2z\} \to \{a, a, c, e, a, a\}, \tag{24.8}$$

$$\frac{x^2z^2}{2!\cdot 0!\cdot 2!}\binom{4}{2,0,2} \to \{x^2z^2, xz^2x, xzxz, zxzx, z^2x^2, zx^2z\} \to \{a, a, e, c, a, a\}, \tag{24.9}$$

$$\frac{x^2y^2}{2!\cdot 2!\cdot 0!}\binom{4}{2,2,0} \to \{x^2y^2, xy^2x, xyxy, yxyx, y^2x^2, yx^2y\} \to \{a, a, c, e, a, a\}, \tag{24.10}$$

$$\begin{aligned}\frac{xyz^2}{1!\cdot 1!\cdot 2!}\binom{4}{1,1,2} &\to \{xyz^2, xz^2y, xzyz, yxz^2, yz^2x, yzxz, z^2yx, z^2xy, zxyz, zyxz, zxzy, zyzx\}\\ &\to \{e, e, a, c, c, a, c, e, c, e, a, a\},\end{aligned} \tag{24.11}$$

$$\begin{aligned}\frac{xy^2z}{1!\cdot 2!\cdot 1!}\binom{4}{1,2,1} &\to \{xzy^2, xy^2z, xyzy, zxy^2, zy^2x, zyxy, y^2zx, y^2xz, yxzy, yzxy, yxyz, yzyx\}\\ &\to \{c, c, a, e, e, a, e, c, e, c, a, a\},\end{aligned} \tag{24.12}$$

$$\begin{aligned}\frac{x^2yz}{2!\cdot 1!\cdot 1!}\binom{4}{2,1,1} &\to \{zyx^2, zx^2y, zxyx, yzx^2, yx^2z, yxzx, x^2yz, x^2zy, xzyx, xyzx, xzxy, xyxz\}\\ &\to \{c, c, a, e, e, a, e, c, e, c, a, a\}.\end{aligned} \tag{24.13}$$

We find with counting that there are twenty-seven permutations for each *'a'*,*'c'* and *'e'* group. We take our $\mathfrak{g}$ matrix and raise it to the 4th power

$$\frac{\mathfrak{g}^3}{4!} = \frac{1}{4!}\begin{bmatrix} 0 & x & 0 & z & 0 & y \\ -x & 0 & -z & 0 & -y & 0 \\ 0 & z & 0 & y & 0 & x \\ -z & 0 & -y & 0 & -x & 0 \\ 0 & y & 0 & x & 0 & z \\ -y & 0 & -x & 0 & -z & 0 \end{bmatrix}^4, \tag{25}$$

and find for the real values in row 1 and columns 1,3,5 of $\frac{\mathfrak{g}^3}{4!}$ to be

$$\frac{\mathfrak{g}^3}{4!}\,\mathbb{R}_{1,1} = \frac{x^4}{24} + \frac{x^2y^2}{6} + \frac{x^2yz}{6} + \frac{x^2z^2}{6} + \frac{xy^2z}{6} + \frac{xyz^2}{6} + \frac{y^4}{24} + \frac{y^2z^2}{6} + \frac{z^4}{24}, \tag{26.1}$$

$$\frac{\mathfrak{g}^3}{4!}\,\mathbb{R}_{1,3} = \frac{x^3y}{12} + \frac{x^3z}{12} + \frac{x^2y^2}{24} + \frac{x^2yz}{6} + \frac{x^2z^2}{24} + \frac{xy^3}{12} + \frac{xy^2z}{6} + \frac{xyz^2}{6} + \frac{xz^3}{12} + \frac{y^3z}{12} + \frac{y^2z^2}{24} + \frac{yz^3}{12}, \tag{26.2}$$

$$\frac{\mathfrak{g}^3}{4!}\,\mathbb{R}_{1,5} = \frac{x^3y}{12} + \frac{x^3z}{12} + \frac{x^2y^2}{24} + \frac{x^2yz}{6} + \frac{x^2z^2}{24} + \frac{xy^3}{12} + \frac{xy^2z}{6} + \frac{xyz^2}{6} + \frac{xz^3}{12} + \frac{y^3z}{12} + \frac{y^2z^2}{24} + \frac{yz^3}{12}. \tag{26.3}$$

We find with looking at $\mathbb{R}_{1,1,}$ $\mathbb{R}_{1,3,}$ *and* $\mathbb{R}_{1,5}$ that they have respective permutations of *'a','c' and 'e'.* We demonstrate this with entry $\mathbb{R}_{1,1}$ below where we find twenty-seven *'a'* permutations as expected

$$\frac{z^4}{0!\cdot 0!\cdot 4!}\binom{4}{0,0,4}, \frac{y^4}{0!\cdot 4!\cdot 0!}\binom{4}{0,4,0}, \frac{x^4}{0!\cdot 0!\cdot 4!}\binom{4}{0,0,4} \to \{a,a,a\}, \tag{27.1}$$

$$\frac{x^2y^2}{2!\cdot 2!\cdot 0!}\binom{4}{2,2,0} \to \{x^2y^2, xy^2x, y^2x^2, yx^2y\} \to \{a,a,a,a\}, \tag{27.2}$$

$$\frac{xy^2z}{1!\cdot 2!\cdot 1!}\binom{4}{1,2,1} \to \{zy^2x, zyxy, yxyz, yzyx\} \to \{a,a,a,a\}, \tag{27.3}$$

$$\frac{x^2z^2}{2!\cdot 0!\cdot 2!}\binom{4}{2,0,2} \to \{x^2z^2, xz^2x, z^2x^2, zx^2z\} \to \{a,a,a,a\}, \tag{27.4}$$

$$\frac{xy^2z}{1!\cdot 2!\cdot 1!}\binom{4}{1,2,1} \to \{xyzy, zyxy, yxyz, yzyx\} \to \{a,a,a,a\}, \tag{27.5}$$

$$\frac{xyz^2}{1!\cdot 1!\cdot 2!}\binom{4}{1,1,2} \to \{xzyz, yzxz, zxzy, zyzx\} \to \{a,a,a,a\}, \tag{27.6}$$

$$\frac{y^2z^2}{0!\cdot 2!\cdot 2!}\binom{4}{0,2,2} \to \{y^2z^2, yz^2y, z^2y^2, zy^2z\} \to \{a,a,a,a\}. \tag{27.7}$$

There are general pattern tendencies on Layer 4, we see that for *'a'* permutations we have all of these coming from monomials that are with at least one object raised to an even exponent as noted above, whereas with the *'c' and 'e'* permutations there is a tendency to appear in the oddly raised exponents with the monomials with the cubic terms coming entirely from the subset of *'c' and 'e'*. With closer inspection, these tendencies are due to the nature of the non-commutative relationships, as any imaginary axis squared or raised to the 4th power, is equal to the negative and positive identities (*'a'*) and odd powers tend to involve the product of adjacent permutation matrices, which will equal the off-diagonal permutations (*'c' and 'e'*).

Layer 4 Permutation Distribution

$$\binom{4}{0,0,4}\{a\} \quad \binom{4}{0,1,3}\{e,e,c,e\} \quad \binom{4}{0,2,2}\begin{matrix}\{a,a,e,\\ c,a,a\}\end{matrix} \quad \binom{4}{0,3,1}\{c,c,e,c\} \quad \binom{4}{0,4,0}\{a\}$$

$$\binom{4}{1,0,3}\{c,c,e,c\} \quad \binom{4}{1,1,2}\begin{matrix}\{e,e,a,c,\\ c,a,c,e,\\ c,e,a,a\}\end{matrix} \quad \binom{4}{1,2,1}\begin{matrix}\{c,c,a,e,\\ e,a,e,c,\\ e,c,a,a\}\end{matrix} \quad \binom{4}{1,3,0}\{e,e,c,e\}$$

$$\binom{4}{2,0,2}\begin{matrix}\{a,a,e,\\ c,a,a\}\end{matrix} \quad \binom{4}{2,1,1}\begin{matrix}\{c,c,a,e,\\ e,a,e,c,\\ e,c,a,a\}\end{matrix} \quad \binom{4}{2,2,0}\begin{matrix}\{a,a,c,\\ e,a,a\}\end{matrix}$$

$$\binom{4}{3,0,1}\{e,e,c,e\} \quad \binom{4}{3,1,0}\{c,c,e,c\}$$

$$\binom{4}{4,0,0}\{a\}$$

Fig. 6 – Permutations - Layer 4 of the tetrahedron representative of the cosine values of $\frac{9^4}{4!}$.

Above we restate the Layer 4 in terms of the distribution of the permutations. We see a very nice pattern of symmetry that emerges that is akin to skew symmetry between the '*e*' and '*a*' permutations, if we were to run an axis down the middle of the distribution. This symmetry plays out in the rotation group when we rotate in clockwise vs counter clockwise fashion.

Layer 3 Permutation Distribution

$$\binom{3}{0,0,3}\{-f\} \quad \binom{3}{0,1,2}\{-d,-b,-d\} \quad \binom{3}{0,2,1}\{-f,-b,-f\} \quad \binom{3}{0,3,0}\{-d\}$$

$$\binom{3}{1,0,2}\{-b,-d,-b\} \quad \binom{3}{1,1,1}\begin{matrix}\{-d,-f,-b,\\ -d,-f,-b\}\end{matrix} \quad \binom{3}{1,2,0}\{-b,-f,-b\}$$

$$\binom{3}{2,0,1}\{-f,-d,-f\} \quad \binom{3}{2,1,0}\{-d,-f,-d\}$$

$$\binom{3}{3,0,0}\{-b\}$$

Fig. 7 – Permutations of Layer 3 of the tetrahedron representative of the imaginary sine values of $\frac{g^3}{3!}$.

We have been working with the real values so far, so we show the reader a similar relationship above for the sine function that is Layer 3. These values in the trinomial expansion are valued as negative imaginary cubic monomials. We once again see symmetry patterns amongst the monomial values when non-commutativity is accounted for. We see that the corners of the tetrahedral layer represent the respective imaginary axes and the other monomials the directed rotational relations amongst all of the imaginary axes.

More on the Non-Euclidian Nature of the Higher Dimensional Complex Forms

In complex analysis, we have all been taught the Argand Plane, which is a nice way to visualize complex numbers in relationship to one another in two-dimensions. The study of this SO(3,C)/*U(3)* group does raise the questions as to how distance between $\mathbb{C}_2$ planes should be represented. In our two-dimensional complex cases, the formula

$$r = \sqrt{x^2 - xy - xz + y^2 - xz + z^2} \tag{28}$$

reduces simply to

$$r = \sqrt{x^2} \tag{29}$$

So, in the case of rotating by a unique axis of rotation consistent with the form e^{ix}, we find the unit circle/Argand plane relation that we are accustomed to. When it comes to splitting the exponential function across multiple $\mathbb{C}_2$ planes, this does not seem to hold. We have found with looking at other similar groups of the form $2\times\mathbb{C}_2 and\ 4\times\mathbb{C}_2$, that these also have the similar radial representation, albeit with different splitting of the exponential across the matrix group. We have then that the radial formulae are

$$2\times\mathbb{C}_2 r = \sqrt{x^2 - xy + y^2}. \tag{30}$$

$$4\times\mathbb{C}_2\, r = \sqrt{x^2 - xy - xz - xw + y^2 - yz - yw + z^2 - zw + w^2}. \tag{31}$$

With this notion in mind, perhaps we can redefine arc length between complex planes in a new light. We are accustomed thru calculus, that the form for differential arc length is

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(1 + y'^2)}\, dx. \tag{32}$$

Then one can measure the arc length between two points by taking the integral of the above as

$$s = \int_a^b \sqrt{(1 + y'^2)}\, dx\,. \tag{33}$$

We can reformulate the differential arc length between our $2\times\mathbb{C}_2$ case with respect to the $\angle x$ as

$$ds_{\,2\times\mathbb{C}_2} = \sqrt{(dx)^2 - (dx\cdot dy) + (dy)^2} = \sqrt{\left(\frac{dx}{dx}\right)^2 - \frac{dx\cdot dy}{dx^2} + \left(\frac{dy}{dx}\right)^2} \tag{34}$$

which will then yield

$$ds_{\,2\times\mathbb{C}_2} = \sqrt{1 - y' + y'^2}\, dx. \tag{35}$$

This is a very interesting aspect of these groups ***SO(3,C)U/(3)*** and represents an exciting potential field of further study!

Conclusion

In this paper, we have explored the connection between matrix groups of the symmetric complex forms, with equal real axes to imaginary axes, to the trinomial theorem and combinatorics. When taking into consideration the non-commutative nature of the rotation group, we are able to demonstrate the one-to-one relationship between the two, view the inter-relationships between the complex planes, and gain a better appreciation of the geometry that is $e^{i(x+y+z)}$. Along the way we have come across a potentially profound understanding that sheds light on non-Euclidian distance metric between complex cyclic $\mathbb{C}_2$ planes.

References

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