Derivation of the Complex Unit Sphere and its Hyperbolic Analogue

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Abstract

We present the derivation of the 6-dimensional Eulerian Lie group of the form SO(3,C). We describe our derivation process, which involves creation of finite group by using permutation matrices, and the exponentiation of the adjoint representation of the subset representing the generators of the finite group. We take clues from the 2-dimensional complex rotation matrix to present, what we believe, is a true representation of the Lie group for the six-dimensional complex unit sphere and proceed to study its dynamics. We also derive the 6-dimensional form of the hyperbolic Lie group representing the higher dimensional exponential, apply this to special relativity considerations, and show its relation to its Eulerian counterpart. With this approach, we discover a profound link with SO(3,C) and SO(3,3) and proceed to show the isomorphism to the Lie group SU(3). The following findings will likely prove useful in mathematical physics, complex analysis and applications in deriving higher dimensional forms of similar division algebras.

Introduction

We start with introducing permutations that represent the two-dimensional complex form of e^{ix}. This is an order 2 finite group and the following permutation matrices are the elements of the group

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; and \; b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

We can create the adjoint representation then as follows

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{2}$$

The permutation "a", represents the identity of the group, as it runs the diagonal. The "b" permutation represents the imaginary axis of the rotation matrix, as evidenced by when taking its exponent, we find it equal to the negative identity

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3}$$

The above is equivalent to $i^2 = -1$ and represents an important way to determine the nature of permutations when looking at the derivation of division algebras. We can go on to set up our permutations matrices and exponentiate separately and find

$$\exp\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\right) = \begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix} \begin{bmatrix} \cos b & -sinb \\ sinb & cosb \end{bmatrix}. \tag{4}$$

From the above we have the rotation matrix for our two-dimensional complex equation $e^{ix} = 1$ in the form of

$$\begin{bmatrix} \cos b & -sinb \\ sinb & cosb \end{bmatrix}. \tag{5}$$

We define the distance function of ***(5)*** as the determinant of the matrix. In this two-dimensional complex form (C_2), we find this distance function to be equal to unity, as we are describing the complex unit circle

$$\det\left(\begin{bmatrix} cosb & -sinb \\ sinb & cosb \end{bmatrix}\right) = \cos^2 b + \sin^2 b = 1. \tag{6}$$

Derivation of the Complex Unit Sphere

With the aforementioned permutations of C_2 in mind, our goal is to create a group of permutation matrices, that once exponentiated with yield a 6x6 rotation matrix, with three imaginary permutations, spaced symmetrically, and with a determinate equal to unity. Typically, in an order 6 finite group, there are a total of 6! permutation matrices to choose from, resulting in 720 different possibilities.

The permutations we present below, are consistent with being representative of imaginary axes, as when squared they are equal to the negative identity

$$
b = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} and\ b^2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},
$$

$$
d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} and\ d^2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{7}
$$

$$
f = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} and\ f^2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.
$$

We continue by setting the "a" permutation to be our identity and "c" and "e" to represent the last 2 permutation matrices to arrive at our adjoint representation $\mathfrak{G}$.

$$
\mathfrak{G} = \begin{bmatrix} a & b & c & f & e & d \\ -b & a & -f & c & -d & e \\ e & f & a & d & c & b \\ -f & e & -d & a & -b & c \\ c & d & e & b & a & f \\ -d & c & -b & e & -f & a \end{bmatrix}. \tag{8}
$$

We continue by looking at the inter-relations of our permutations from above. We find that the "c" and "e" permutations represent real-valued elements by way of the fact that taking the product of any two of our imaginary permutations with one another, will yield either "-c" or "-e". This is akin to the imaginary values yielding "-a" when squared. It will turn out that the permutations we consider in "-c" and "-e", are important in that they will play the role of scaling factors, that can be set to create a gradient for progressing rotations. It is easy verify that $\mathfrak{G}$ is closed under multiplication that we will prove shortly with a Standard Form Cayley Table.

When our adjoint matrix $\mathfrak{G}$ is represented as positive and negative ones, its determinate is valued at zero. This plays an important role with ensuring that after exponentiation, the determinant of our rotation matrix will equate to unity, as $e^0 = 1$.

	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	-a	f	-e	d	-c
c	c	d	e	f	a	b
d	d	-c	b	-a	f	-e
e	e	f	a	b	c	d
f	f	-e	d	-c	b	-a

Fig. 1 – Standard Form Cayley Table of Permutation matrix ***(8)***.
The identity element is noted as “a” above.

We can further establish the relationships between the elements of the permutation matrix by way of a Standard Cayley table [1] seen in ***Fig. 1***. The table is read by taking the respective row and multiplying by the column. So, as an example taking *db* will yield *-c* as shown above. The reader will notice the symmetry between the alternating signs of the values for “a”, “b” and “c” as one moves along the columns from left to right. We note that by way of our Standard Form Cayley table, that this is a non-abelian form, as it is lacking symmetry along the diagonal and that some products do not commute.

With quaternions and octonion forms, the general approach has been to assume there is one scalar value and three imaginary values for quaternions, and one scalar and seven imaginary values for octonions. To have the symmetry that we desire for the complex unit sphere form, there has to be real values associated to each imaginary axis. A central tenet in arriving at this form, is the realization that permutations “a”, ”c” and “e” represent real values that are oriented plane(s). This is akin to a Clifford algebra form where $e_1e_2 = -e_2e_1$. Instead we have the following relations

$$
\begin{gathered}
b^2 = d^2 = f^2 = -1 \\
bd = -e, df = -e, fb = -e \\
bf = -c, fd = -c, db = -c \\
bf = fd, df = fb, bd = df \\
e^3 = -1, c^3 = -1, ce = 1, (-c)(-e) = 1\,.
\end{gathered} \tag{9}
$$

We can use the relations above and simplify our algebras. For example

$$
bfddbdfb \to\ -bfbdfb \to -bfbddf \to bfbf \to bffd \to -bd = e. \tag{10}
$$

So, the “c” and “e”, are these oriented plane(s). In our permutation matrix, a key realization is the fact the imaginary permutations “b”,”d” and “f” comprise the basis of the group $\mathfrak{G}$, by way of the fact that with the products of just these three permutations, we can arrive at all of the permutations in our finite group. We cannot do the same with the “a”, “c” and “e” permutations. This is similar to using only the products between 2 imaginary values, you can arrive at any real and or imaginary number, but we cannot take real numbers and arrive at imaginary ones with products of integers only. For example, the real number “4” can be obtained by taking the product of -2i and 2i. With these ideas in mind, we can set to zero the permutations “a”, “c” and “e” from the matrix ***(8)*** above and the following we denote as $\mathfrak{g}$.

$$
\mathfrak{g} = \begin{matrix} 0 & b & 0 & f & 0 & d \\ -b & 0 & -f & 0 & -d & 0 \\ 0 & f & 0 & d & 0 & b \\ -f & 0 & -d & 0 & -b & 0 \\ 0 & d & 0 & b & 0 & f \\ -d & 0 & -b & 0 & -f & 0 \end{matrix}. \tag{11}
$$

In the parlance of group theory, we state that $\mathfrak{g}$ is a subset of $\mathfrak{G}$.

To aid in getting a familiar feel, we substituted x=b, y=d and z=f into the above to arrive at

$$\mathfrak{g} = \begin{matrix} 0 & x & 0 & z & 0 & y \\ -x & 0 & -z & 0 & -y & 0 \\ 0 & z & 0 & y & 0 & x \\ -z & 0 & -y & 0 & -x & 0 \\ 0 & y & 0 & x & 0 & z \\ -y & 0 & -x & 0 & -z & 0 \end{matrix}. \tag{12}$$

SO(3,C) Lie Group and Algebras

Our commutator relationships as Lie brackets are as follows

$$\begin{gathered} [b,d] = c - e, [d,b] = e - c, [d,f] = c - e, [f,d] = e - c, [f,b] = c - e, [b,f] = e - c, \\ [b,c] = f - d, [c,b] = d - f, [d,c] = b - f, [c,d] = f - b, [f,c] = d - b, [c,f] = b - d, \\ [b,e] = d - f, [e,b] = f - d, [d,e] = f - b, [e,d] = b - f, [f,e] = b - d, [e,f] = d - b, \\ [c,e] = 0, [e,c] = 0. \end{gathered} \tag{13}$$

We can verify that the permutations that form our adjoint matrix $\mathfrak{G}$, form a Lie algebra by way of [2]:

- Bilinearity: $[ax + by, z] = a[x,z] + b[y,z]$ and $[z, ax + by] = a[z,x] + b[z,y]$ $for\ \forall\ x,y,z\ \in$ $\mathfrak{g}$ $and\ arbitrary\ numbers\ a\ \&\ b,$

- Anticommutatvity: $[x,y] = -[y,x], [x,z] = -[z,x], [y,x] = -[x,y], [y,z] = -[z,y], [z,x] = -[x,z], [z,y] = -[y,z]\ \forall\ x,y,z\ \in\ \mathfrak{g},$

- Jacobi Identity: $\big[x,[y,z]\big] + \big[z,[x,y]\big] + \big[y,[z,x]\big] = 0\ \forall\ x,y,z\ \in\ \mathfrak{g}.$

We can now proceed to exponentiate the matrix $\mathfrak{g}$ ***(12)*** above to arrive at our Lie Group that represents $e^{ix+iy+iz} = 1$. We utilized MATLAB and the *expm(*$\mathfrak{g}$*)* function to determine the result and after by hand simplification, a consistent pattern emerged in the form for the final result seen in ***Fig. 2.***

$$\begin{bmatrix} 2A_c + B_c & (2x - y - z)A_s + B_s & -A_c + B_c & (-x - y + 2z)A_s + B_s & -A_c + B_c & (-x + 2y - z)A_s + B_s \\ (-2x + y + z)A_s - B_s & 2A_c + B_c & (x + y - 2z)A_s - B_s & -A_c + B_c & (x - 2y + z)A_s - B_s & -A_c + B_c \\ -A_c + B_c & (-x - y + 2z)A_s + B_s & 2A_c + B_c & (-x + 2y - z)A_s + B_s & -A_c + B_c & (2x - y - z)A_s + B_s \\ (x + y - 2z)A_s - B_s & -A_c + B_c & (x - 2y + z)A_s - B_s & 2A_c + B_c & (-2x + y + z)A_s - B_s & -A_c + B_c \\ -A_c + B_c & (-x + 2y - z)A_s + B_s & -A_c + B_c & (2x - y - z)A_s + B_s & 2A_c + B_c & (-x - y + 2z)A_s + B_s \\ (x - 2y + z)A_s - B_s & -A_c + B_c & (-2x + y + z)A_s - B_s & -A_c + B_c & (x + y - 2z)A_s - B_s & 2A_c + B_c \end{bmatrix}$$

Fig. 2 – Lie Group Matrix $- M$. $A_c = \frac{cosh\lambda_e}{3}, B_c = \frac{cos\gamma}{3}, A_s = \frac{\sinh\lambda_e}{3\lambda_e}, B_s = \frac{sin\gamma}{3}, \lambda_e = \sqrt{-x^2 + xy + xz - y^2 + yz - z^2}, \gamma = (x + y + z)$

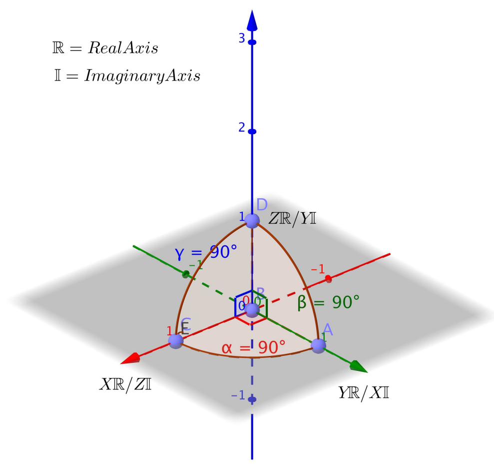
We see our rotation matrix is a mixture of both hyperbolic and Euclidean trigonometric functions. One should note the value for λ_e in the matrix above. We see that the presence of negative values can cause changes from hyperbolic to Euclidean trigonometric forms when imaginary values arise by way of taking the square roots of negative values. This will occur with the use of real-valued angles to rotate with. On the contrary, when we take imaginary-valued angles, we find the hyperbolic functions tend to remain in their original form and change the cosine and sines to their hyperbolic counterparts (cosh, sinh). We see that λ_e is a six-valued term representing the relation amongst the angles of rotation and γ representing the individual 3 angles $\angle X, \angle Y, and\ \angle Z$.

We can verify our Lie group matrix M is "special" (S), as

$$\det(M) = 1 \tag{14}$$

and is orthogonal (O) as

$$M^T M = 1. \tag{15}$$



Arrowheads represent positive axes and X,Y,Z are Complex.

Fig. 3 – Graphic Representation of Axes of 6D Complex Unit Sphere

In ***Fig. 3,*** we attempt to create a sense of the relationships between the angles on the complex unit sphere. We see it is an amalgam of the three C_2 cyclic groups that combined are $e^{ix+iy+iz}$. The three C_2 groups are arranged in orthogonal relation to one another and form a closed loop.

Calculating the eigenvalues of the matrix from ***Fig. 2***, where we ask the reader to not confuse the eigenvalues λ_i with the $\lambda_e = \sqrt{-x^2 + xy + xz - y^2 + yz - z^2}$ *and* $\lambda_h = \sqrt{x^2 - xy - xz + y^2 - yz + z^2}$ of the rotation matrices, we find

$$\lambda_1 = e^{\lambda_h}, \lambda_2 = e^{\lambda_h}, \lambda_3 = e^{\lambda_e}, \lambda_4 = e^{\lambda_e}, \lambda_5 = e^{-i\gamma}, \lambda_6 = e^{i\gamma}. \tag{16}$$

We see that the axes of rotation would be constantly changing based on the complex angle values we are calculating. This is a similar when compared to the eigenvalues one finds by a similar calculation for the complex two-dimensional rotation matrix ***(5)***. This equates to an infinite number of rotation axes, and is much different from the eigenvalues the reader might be accustomed to in the form of fixed rotation axes.

Taking a step further, we can calculate our Lie group with the "c" and "e" permutations and we leave "a" out as this is identity permutation along the main diagonal that acts to uniformly scale the outputs. We fill in our $\mathfrak{g}$ matrix from ***(12)*** to find

$$\mathfrak{g}_{(c,e)} = \begin{matrix} 0 & b & c & f & e & d \\ -b & 0 & -f & c & -d & e \\ e & f & 0 & d & c & b \\ -f & e & -d & 0 & -b & c \\ c & d & e & b & 0 & f \\ -d & c & -b & e & -f & 0 \end{matrix}. \tag{17}$$

We exponentiate the $\mathfrak{g}_{(c,e)}$ matrix above to find our Lie group as seen in ***Fig. 4*** below.

$$\begin{bmatrix} 2\mathrm{A}_c + \mathrm{B}_c & (2x-y-z)\mathrm{A}_s + \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c & (-x-y+2z)\mathrm{A}_s + \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c & (-x+2y-z)\mathrm{A}_s + \mathrm{B}_s \\ (-2x+y+z)\mathrm{A}_s - \mathrm{B}_s & 2\mathrm{A}_c + \mathrm{B}_c & (x+y-2z)\mathrm{A}_s - \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c & (x-2y+z)\mathrm{A}_s - \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c \\ -\mathrm{A}_c + \mathrm{B}_c & (-x-y+2z)\mathrm{A}_s + \mathrm{B}_s & 2\mathrm{A}_c + \mathrm{B}_c & (-x+2y-z)\mathrm{A}_s + \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c & (2x-y-z)\mathrm{A}_s + \mathrm{B}_s \\ (x+y-2z)\mathrm{A}_s - \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c & (x-2y+z)\mathrm{A}_s - \mathrm{B}_s & 2\mathrm{A}_c + \mathrm{B}_c & (-2x+y+z)\mathrm{A}_s - \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c \\ -\mathrm{A}_c + \mathrm{B}_c & (-x+2y-z)\mathrm{A}_s + \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c & (2x-y-z)\mathrm{A}_s + \mathrm{B}_s & 2\mathrm{A}_c + \mathrm{B}_c & (-x-y+2z)\mathrm{A}_s + \mathrm{B}_s \\ (x-2y+z)\mathrm{A}_s - \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c & (-2x+y+z)\mathrm{A}_s - \mathrm{B}_s & -\mathrm{A}_c + \mathrm{B}_c & (x+y-2z)\mathrm{A}_s - \mathrm{B}_s & 2\mathrm{A}_c + \mathrm{B}_c \end{bmatrix}$$

Fig. 4 – Lie Group Matrix – *N*. $\mathrm{A}_c = \frac{e^{\left(-\frac{c}{2}-\frac{e}{2}\right)} cosh\lambda_e}{3}, \mathrm{B}_c = \frac{e^{(c+e)} cos\gamma}{3}, \mathrm{A}_s = \frac{e^{\left(-\frac{c}{2}-\frac{e}{2}\right)} \sinh\lambda_e}{3\lambda_e}, \mathrm{B}_s = \frac{e^{(c+e)} sin\gamma}{3}, \lambda_e = \sqrt{-x^2 + xy + xz - y^2 + yz - z^2}, \gamma = (x+y+z)$

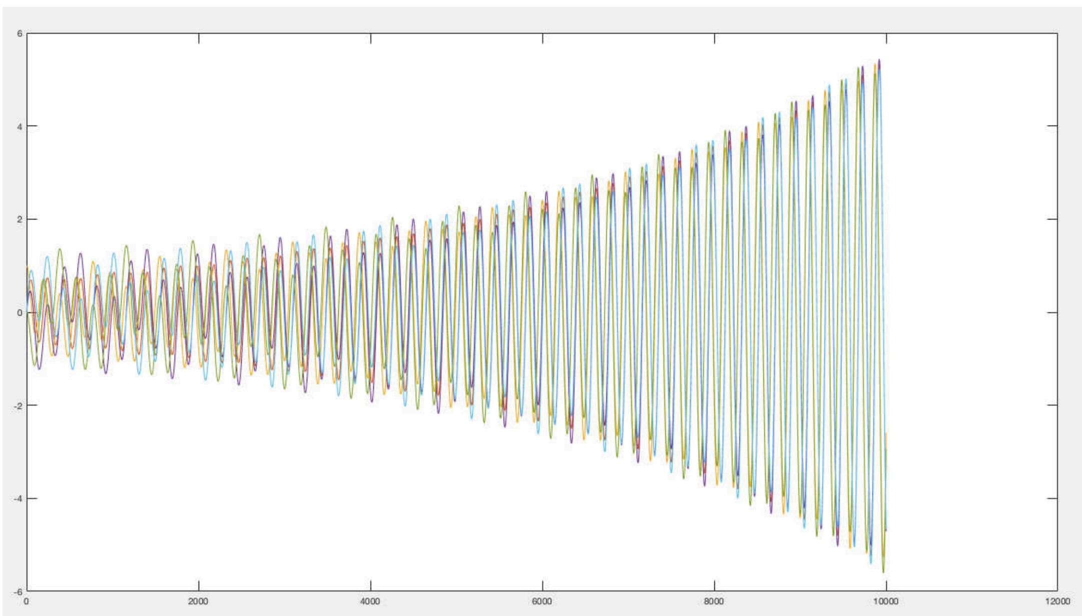


Fig. 5 **–** Waveform plot with scaling factors $c = \frac{-\pi}{30000}$ and $e = \frac{\pi}{10000}$

In ***Fig. 5*** above we show an example of a waveform plot where we have set scaling factors for “c” and “e”. We can see as the waveforms move from left to right, they are increasing in magnitude and becoming more symmetric. If we were to switch the signs of “c” and “e”, we would see waveforms shrinking in magnitude as we progress from left to right. If “c” and “e” are equal in absolute value but of opposite sign, we would see no net change in the waveforms. Alternatively, if “c” and “e” are of the same sign their effects would add, and increase or decrease the waveform magnitude for positive or negative values respectively.

Distance Functions

In regard to the Euclidean plane and that of space-time, physicists and mathematicians speak of distance functions that should equate to unity. In the Euclidean case, we have become accustomed to taking the determinate of the two-dimensional complex rotation matrix and finding

$$\cos(x)^2 + \sin(x)^2 = 1. \tag{18}$$

In the hyperbolic rotation matrix, we find the determinant represents rotations in space-time otherwise known as space-time boost, and takes the form of [3]

$$\cosh(x)^2 - \sinh(x)^2 = 1. \tag{19}$$

There seems to be quite the incongruence as to how you get from one to the other and the use of the complex conjugate. With the Euclidean two-dimensional form, the complex conjugate seems to have been pulled out of nowhere, which leaves us with questions as to its mathematical origins. By way of taking the determinate of the rotation matrices, we are able to find our distance functions and do away with the need for the complex conjugate. Taking the distance function of the complex unit sphere's Lie group matrix *M*, we find that both ***Eq.(18)*** and ***Eq.(19)*** merge together and form the correct mathematically derived distance function for Euclidean space. With taking the determinant we first find

$$\begin{gathered}\cos^2\gamma\cosh^4\lambda_e + \cos^2\gamma\sinh^4\lambda_e + \sin^2\gamma\cosh^4\lambda_e + \sin^2\gamma\sinh^4\lambda_e \\ -2\cos^2\gamma\cosh^2\lambda_e\sinh^2\lambda_e - 2\sin^2\gamma\cosh^2\lambda_e\sinh^2\lambda_e\,.\end{gathered} \tag{20}$$

Simplifies to

$$(\cos^2\gamma + \sin^2\gamma)(cosh\lambda_e - sinh\,\lambda_e)^2(cosh\lambda_e + sinh\,\lambda_e)^2\,. \tag{21}$$

Which ultimately yields

$$\frac{(\cos^2\gamma + \sin^2\gamma)(cosh\lambda_e + sinh\,\lambda_e)^2}{(cosh\,\lambda_e + sinh\,\lambda_e)^2} = 1^2 \rightarrow (\cos^2\gamma + \sin^2\gamma) = 1\,. \tag{22}$$

Remarkably, by taking the determinant for the Lie group matrix *N,* we find that it results in the same values as the above, where the "c" and "e" permutations cancel out in the calculation.

Hyperbolic Six-Dimensional Rotation Matrix

We can define the hyperbolic functions in the 6-dimensional sense by a similar fashion to how we solved for the Eulerian form. We set the basis rows to -1 with the Lorentz group in mind. We can setup our matrix permutations as follows

$$b = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}. \tag{23}$$

We can, once again, set x=b, y=d and f=z, add the permutations together and exponentiate the resultant matrix (M_h) as follows

$$M_h = exp\begin{bmatrix} 0 & -x & 0 & -z & 0 & -y \\ -x & 0 & -z & 0 & -y & 0 \\ 0 & -z & 0 & -y & 0 & -x \\ -z & 0 & -y & 0 & -x & 0 \\ 0 & -y & 0 & -x & 0 & -z \\ -y & 0 & -x & 0 & -z & 0 \end{bmatrix}. \tag{24}$$

We can find below, in ***Fig. 5***, the expanded hyperbolic rotation matrix we derived above, where we have set $\lambda_h = \sqrt{x^2 - xy - xz + y^2 - yz + z^2}$ and $\gamma = (x + y + z)$. We also can take note that the λ_h differs from λ_e being opposite in sign.

$$\begin{bmatrix} 2\mathrm{A}_{hc} + \mathrm{B}_{hc} & (-2x + y + z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (x + y - 2z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (x - 2y + z)\mathrm{A}_{hs} - \mathrm{B}_{hs} \\ (2x - y - z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & 2\mathrm{A}_{hc} + \mathrm{B}_{hc} & (-x - y + 2z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (-x + 2y - z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} \\ -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (x + y - 2z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & 2\mathrm{A}_{hc} + \mathrm{B}_{hc} & (x - 2y + z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (-2x + y + z)\mathrm{A}_{hs} - \mathrm{B}_{hs} \\ (-x - y + 2z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (-x + 2y - z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & 2\mathrm{A}_{hc} + \mathrm{B}_{hc} & (2x - y - z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} \\ -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (x - 2y + z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (-2x + y + z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & 2\mathrm{A}_{hc} + \mathrm{B}_{hc} & (x + y - 2z)\mathrm{A}_{hs} - \mathrm{B}_{hs} \\ (-x + 2y - z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (2x - y - z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & -\mathrm{A}_{hc} + \mathrm{B}_{hc} & (-x - y + 2z)\mathrm{A}_{hs} - \mathrm{B}_{hs} & 2\mathrm{A}_{hc} + \mathrm{B}_{hc} \end{bmatrix}$$

Fig. 5 – M_h rotation matrix. $\mathrm{A}_{hc} = \frac{cosh\lambda_h}{3}, \mathrm{B}_{hc} = \frac{cosh\gamma}{3}, \mathrm{A}_{hs} = \frac{\mathrm{sinh}\lambda_\mathrm{h}}{3\lambda_h}, \mathrm{B}_{hs} = \frac{sinh\gamma}{3}, \lambda_h = \sqrt{x^2 - xy - xz + y^2 - yz + z^2}, \gamma = (x + y + z).$

For the hyperbolic rotation matrix, the use of real angles will result in hyperbolic forms and the use of imaginary angles will result in Eulerian rotational forms, once again, by way of the value for λ_h in the above. We have left M_h in its original form purposely. We look and see that our A_{hs} value is $\frac{\mathrm{sinh}\lambda_\mathrm{h}}{3\lambda_h}$. Any λ_h value equal to zero, would result in a singular rotation matrix, which is possibly reminiscent of the fact that all matter in space-time must remain in motion.

We find the eigenvalues for our 6-dimensional hyperbolic Lie group and note the similarities to those of our Eulerian form from ***Eq. (16)***

$$\lambda_1 = e^{\lambda_h}, \lambda_2 = e^{\lambda_h}, \lambda_3 = e^{\gamma}, \lambda_4 = e^{\lambda_e}, \lambda_5 = e^{\lambda_e}, \lambda_6 = e^{\gamma}. \tag{25}$$

We calculate the determinant for M_h to yield our distance function equating to the speed of light squared (c^2), where $\gamma = (x + y + z)$ and $\lambda_h = \sqrt{x^2 - xy - xz + y^2 - yz + z^2}$ to find

$$\frac{1}{(\cosh^2\gamma - \sinh^2\gamma)\,(\cosh^2\lambda_h - \sinh^2\lambda_h)^2} = c^2 \tag{26}$$

With our calculated determinant, we can proceed to find the familiar space-time distance function by further simplifying and setting c=1 and find

$$\frac{1}{(\cosh^2\gamma - \sin^2\gamma)(1)} = 1^2 \rightarrow cosh^2\,\gamma - sinh^2\,\gamma = 1\,. \tag{27}$$

We rearrange, divide thru by $\cosh^2\gamma$, and solve for $cosh\gamma$ to find the familiar Lorentz transform

$$\cosh\gamma = \frac{c}{\sqrt{1 - \tanh^2\gamma}}. \tag{28}$$

We can also solve by first setting $(\cosh^2\gamma - \sin^2\gamma) = 1$, and find

$$\cosh\lambda_h = \sqrt{\frac{c}{1 - \tanh^2\lambda_h}}. \tag{29}$$

So, in the end we have 2 forms of the Lorentz transform when we set c=1 and find

$$\gamma_{SR1} = \frac{1}{\sqrt{1-\tanh^2\gamma}} \; and \; \gamma_{SR2} = \frac{1}{\sqrt{1-\tanh^2\lambda_h}} \,. \tag{30}$$

We note that γ_{SR} is the gamma from special relativity not to be confused with our γ from our Lie groups.

Plotting our distance functions would be equivalent to plotting on a unit hyperbola that has been transformed from 6-dimensions down to the 2-dimensional unit hyperbola as seen in ***Fig. (6).***

Special relativity, in its original form, is actually a two-dimensional theory. Minkowski invented 4-vectors in 1907, that when applied to special relativity, give correct answers most of the time. They do not represent a true division algebra, as there is no multiplicative closure, nor a multiplicative inverse. [4]

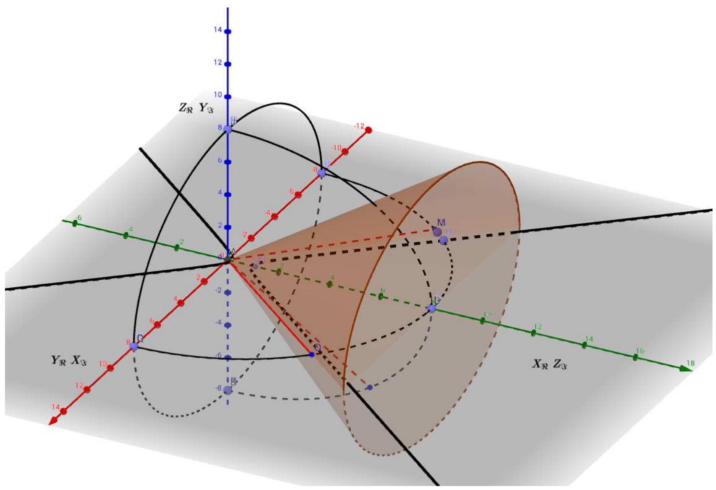


Fig. 6 – 2D Unit Hyperbola in multidimensional space

Space-time considerations with SO(3,3)

The Lorentz transformation matrix $B(v)$ represents the means to do boosts in 4D space-time. We can represent $B(v)$ [2] as

$$B(v) = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\eta_x^2 & (\gamma-1)\eta_x\eta_y & (\gamma-1)\eta_x\eta_z \\ -\gamma\beta_y & (\gamma-1)\eta_y\eta_x & 1+(\gamma-1)\eta_y^2 & (\gamma-1)\eta_y\eta_z \\ -\gamma\beta_z & (\gamma-1)\eta_z\eta_x & (\gamma-1)\eta_z\eta_y & 1+(\gamma-1)\eta_z^2 \end{bmatrix}, \tag{31}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v_x^2+v_y^2+v_z^2}{c^2}}}$, $\beta_x = \frac{v_x}{c}$, $\beta_y = \frac{v_y}{c}$, $\beta_z = \frac{v_z}{c}, \eta_i = 1 \; for \; \beta_i \neq 0, and \; \eta_i = 0 \; for \; \beta_i = 0.$

Normally, for ease in calculations, we set *c=1* in the above. The v_i's are then set as some value from $0 \; to < 1$. Using either our unit hyperbola matrix M_h with real valued angles, or using our complex unit sphere matrix M with negative imaginary numbers, we are able to reproduce similar results for pure boosts along collinear worldlines as do the Lorentz transforms. For example, setting $v_x = 0.5, \beta_x = \frac{0.5}{1},\ \beta_y = 0,\ and\ \beta_z = 0$, our Lorentz transform matrix reduces to

$$B(v) = \begin{bmatrix} \gamma & -\gamma\beta_x & 0 & 0 \\ -\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1.1547 & -0.5774 & 0 & 0 \\ -0.5774 & 1.1547 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{32}$$

We take a starting vector $\hat{v}_l = (1\ 0\ 0\ 0)$ and boost along the x-axis to find

$$\hat{v}_l' = \hat{v}_l B(v) \approx (1.1547, -0.5774, 0,0). \tag{33}$$

We can do a similar boost in SO(3,3) by taking a starting vector $\hat{v}_h = (1\ 0\ 0\ 0\ 0\ 0)$ and multiplying it by our Lie group matrix M_h, where we take the angle of rotation to be $acosh\gamma$ from $B(v)$ above and find

$$\hat{v}_h' = \hat{v}_h M_h \approx \hat{v}_h \begin{bmatrix} 1.1547 & -0.5774 & 0 & 0 & 0 & 0 \\ -0.5774 & 1.1547 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.1547 & 0 & 0 & -0.5774 \\ 0 & 0 & 0 & 1.1547 & -0.5774 & 0 \\ 0 & 0 & 0 & -0.5774 & 1.1547 & 0 \\ 0 & 0 & -0.5774 & 0 & 0 & 1.1547 \end{bmatrix} \approx (1.1547, -0.5774\ ,0,0,0,0). \tag{34}$$

Once we take into account that the convention between the 6D form and the Lorentz one has the $\angle Y\ and\ \angle Z$ switched, we can easily create similar pure boosts along the y and z axes.

To proceed, we take the example where we use again our vectors $\hat{v}_h\ and\ \hat{v}_l$, are rotate thru the compound angle, where, $\beta_x = \frac{0.5}{1},\ \beta_y = \frac{0.3}{1},\ and\ \beta_z = \frac{0.4}{1}$ and we take our angles for the SO(3,3) form to be $\angle X = \text{acosh}\left(\frac{1}{\sqrt{1-(0.5)^2}}\right), \angle Y = \text{acosh}\left(\frac{1}{\sqrt{1-(0.3)^2}}\right), and\ \angle Z = \text{acosh}\left(\frac{1}{\sqrt{1-(0.4)^2}}\right)$.

We have for this arbitrary boost its Lorentz form as

$$\hat{v}_l' \approx \hat{v}_l \begin{bmatrix} 1.4142 & -0.7071 & -0.4243 & -0.5657 \\ -0.7071 & 1.2071 & 0.1243 & 0.1657 \\ -0.4243 & 0.1243 & 1.0746 & 0.0994 \\ -0.5657 & 0.1657 & 0.0994 & 1.1325 \end{bmatrix} \approx (1.4142, -0.7071, -0.4243, -0.5657), \tag{35}$$

and for the SO(3,3) form

$$\hat{v}_h' = \hat{v}_h M_h \approx \hat{v}_h \begin{bmatrix} 1.3283 & -0.6774 & 0.3066 & -0.4359 & 0.3066 & -0.5508 \\ -0.6774 & 1.3283 & -0.4359 & 0.3066 & -0.5508 & 0.3066 \\ 0.3066 & -0.4359 & 1.3283 & -0.5508 & 0.3066 & -0.6774 \\ -0.4359 & 0.3066 & -0.5508 & 1.3283 & -0.6774 & 0.3066 \\ 0.3066 & -0.5508 & 0.3066 & -0.6774 & 1.3283 & -0.4359 \\ -0.5508 & 0.3066 & -0.6774 & 0.3066 & -0.4359 & 1.3283 \end{bmatrix}$$

$$\approx (1.3283, -0.6774, 0.3066, -0.4359\ , 0.3066, -0.5508). \tag{36}$$

When we compare the values for our resultant vectors, we note the space-time values are not the same. The result from our SO(3,3) Lie group, represents a rotation of the exponential form along 3 separate orthogonal hyperbolae centered on the X,Y and Z axes as seen in ***Fig. 7*** below. We will show that the result in ***Eq. (36)***, is rotated boost of the unit hyperbola.

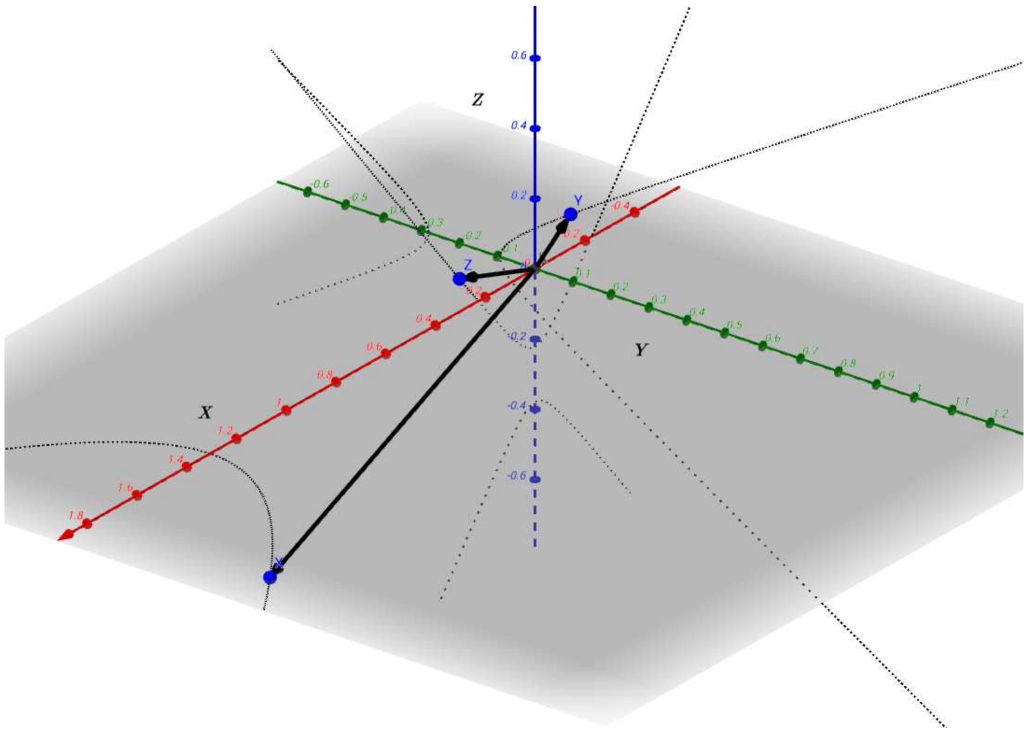


Fig. 7 – Rotation of unit hyperbolic boost within 6D hyperbolic Lie group. We see respective hyperbolae centered along X,Y and Z axes.

To reorient ourselves along a single axis, we can take advantage of the fact that we are working with a rotated exponential, and simply add the 3 hyperbolae together, where we remind the reader

$$\hat{v}_h' \approx (1.3283, -0.6774, 0.3066, -0.4359\;, 0.3066, -0.5508).$$

Each respective hyperbola is represented as

$$\hat{v}_x'^2 = (1.3283^2 - 0.6774^2), \hat{v}_y'^2 = (0.3066^2 - 0.4359^2) \; and \; \hat{v}_z'^2 = (0.3066^2 - 0.5508^2). \tag{37}$$

We add the above hyperbolae together and find

$$\hat{v}_x'^2 + \hat{v}_y'^2 + \hat{v}_z'^2 = 1. \tag{38}$$

We can also represent our initial boost by grouping and adding the time and space components to find

$$\left(\hat{v}'_{h(1,1)} + \hat{v}'_{h(1,3)} + \hat{v}'_{h(1,5)}\right)^2 - \left(\hat{v}'_{h(1,2)} + \hat{v}'_{h(1,4)} + \hat{v}'_{h(1,6)}\right)^2 = ((1.9415)^2 - (1.6641)^2) = 1. \tag{39}$$

The reader can note that above, we have our distance function we calculated in ***Eq. (27)***. We can further state that an arbitrary rotation on the six-dimensional hyperbolic Lie group, is the rotation of a boost on the unit hyperbola. We can show this with our prior example by adding our initial angles

$$\angle X + \angle Y + \angle Z \rightarrow \; \text{acosh}\left(\frac{1}{\sqrt{1-(0.5)^2}}\right) + \text{acosh}\left(\frac{1}{\sqrt{1-(0.3)^2}}\right) + \text{acosh}\left(\frac{1}{\sqrt{1-(0.4)^2}}\right) \approx 1.2825. \tag{40}$$

Setting our $\angle X$ to 1.2825, we calculate our rotation matrix M_h and rotate our vector $\hat{v}_h = (1\ 0\ 0\ 0\ 0\ 0)$ and similarly find

$$\hat{v}_h' = \hat{v}_h M_h \approx \hat{v}_h \begin{bmatrix} 1.9415 & -1.6641 & 0 & 0 & 0 & 0 \\ -1.6641 & 1.9415 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.9415 & 0 & 0 & -1.6641 \\ 0 & 0 & 0 & 1.9415 & -1.6641 & 0 \\ 0 & 0 & 0 & -1.6641 & 1.9415 & 0 \\ 0 & 0 & -1.6641 & 0 & 0 & 1.9415 \end{bmatrix} \approx (1.9415, -1.6641, 0,0,0,0). \quad (41)$$

We can see the relationship with the Lorentz boost matrix setting

$$B(v) = \begin{bmatrix} \gamma & -\gamma\beta_x & 0 & 0 \\ -\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} \cosh(1.2825) & -\sinh(1.2825) & 0 & 0 \\ -\sinh(1.2825) & \cosh(1.2825) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.9415 & -1.6641 & 0 & 0 \\ -1.6641 & 1.9415 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (42)$$

We take once again, our starting vector $\hat{v}_l = (1\ 0\ 0\ 0)$ and boost along the x-axis to find our same result from ***Eq. (41)***

$$\hat{v}_l' = \hat{v}_l B(v) \approx (1.9415, -1.6641, 0,0). \quad (43)$$

We can return to our Lorentz boost matrix from ***Eq. (35-shown below)*** and realize that it is actually a two-dimensional hyperbolic rotation that has been manipulated by adding 2 additional real dimensions to it. This represents the key problem with the Minkowski four-vectors and the lack of mathematical rigor in terms of the Lorentz group.

$$\hat{v}_l' \approx \hat{v}_l \begin{bmatrix} 1.4142 & -0.7071 & -0.4243 & -0.5657 \\ -0.7071 & 1.2071 & 0.1243 & 0.1657 \\ -0.4243 & 0.1243 & 1.0746 & 0.0994 \\ -0.5657 & 0.1657 & 0.0994 & 1.1325 \end{bmatrix} \approx (1.4142, -0.7071, -0.4243, -0.5657).$$

When we find the rotation angle by taking acosh $(\hat{v}'_{l_{1,1}})$ we see that we are rotating by 0.8814 radians. We plug this angle into a two-dimensional form and find

$$[1 \quad 0] \begin{bmatrix} \cosh(0.8814) & -\sinh(0.8814) \\ -\sinh(0.8814) & \cosh(0.8814) \end{bmatrix} = (1.14142, -1). \quad (44)$$

We can take our prior four-vector and compare to our result

$$(1.4142)^2 - ((0.7071)^2 + (-0.4243)^2 + (-0.5657)^2) = (2-1) = 1, \quad (45)$$

and from the result of ***Eq. (44)***

$$(1.4142)^2 - (1)^2 = (2-1) = 1. \quad (46)$$

We can see that the results equate to the same values and that the mathematics of the Lorentz form usually yields the correct results, but is ultimately not a proper derivation. We believe in the six-dimensional form we have the correct derivation by way of 3 time dimensions and 3 space dimensions. We proceed to do the similar boost/transformation from ***Eq. (35)*** with our proposed Lie group. We first calculate our two-dimensional angle, representing our unit hyperbola, by the following

$$\operatorname{acosh}(\gamma) = \operatorname{acosh}\left(\frac{1}{\sqrt{1-(0.5^2+0.3^2+0.4^2)}}\right) \approx 0.8814\, rad. \quad (47)$$

We can then split our angle into $\angle X, \angle Y$ *and* $\angle Z$ components with

$$\angle X = \frac{\beta_x}{(\beta_x+\beta_y+\beta_z)}(0.8814) = \frac{0.5}{1.2}(0.8814) \approx 0.3673\, rad$$

$$\angle Y = \frac{\beta_x}{(\beta_x+\beta_y+\beta_z)}(0.8814) = \frac{0.4}{1.2}(0.8814) \approx 0.2938\, rad \tag{48}$$

$$\angle Z = \frac{\beta_x}{(\beta_x+\beta_y+\beta_z)}(0.8814) = \frac{0.3}{1.2}(0.8814) \approx 0.2203\, rad$$

We input our calculated angles into our Lie group and multiply by $\hat{v}_h = (1\ 0\ 0\ 0\ 0\ 0)$ and find

$$\hat{v}'_h = \hat{v}_h M_h \approx \hat{v}_h \begin{bmatrix} 1.1435 & -0.4070 & 0.1354 & -0.2596 & 0.1354 & -0.3333 \\ -0.4070 & 1.1435 & -0.2596 & 0.1354 & -0.3333 & 0.1354 \\ 0.1354 & -0.2596 & 1.1435 & -0.3333 & 0.1354 & -0.4070 \\ -0.2596 & 0.1354 & -0.3333 & 1.1435 & -0.4070 & 0.1354 \\ 0.1354 & -0.3333 & 0.1354 & -0.4070 & 1.1435 & -0.2596 \\ -0.3333 & 0.1354 & -0.4070 & 0.1354 & -0.2596 & 1.1435 \end{bmatrix}$$

$$\approx (1.1435, -0.4070, 0.1354, -0.2596\,, 0.1354, -0.3333). \tag{49}$$

We add the like time and space components to align our orientation with our rotated unit hyperbola and find

$$((1.1435 + 0.1354 + 0.1354), -(0.4070 + 0.2596 + 0.3333)) = (1.4142, -1). \tag{50}$$

We apply our result to the unit hyperbola and find

$$\cosh^2(0.8814) - \sinh^2(0.8814) = (1.4142)^2 - (1)^2 = 1\,. \tag{51}$$

So, we see we have the same result of our Lorentz form. We also take into account that the results of the time and space components from ***Eq. (49)*** lay on respective hyperbolae as the proper form should, and this is why they differ from the result we initially found in ***Eq. (35)***.

We can proceed and take our six-vector result $\hat{v}'_h$ from Eq. (49) and rotation again by our rotation matrix M_h and find

$$\hat{v}''_h = \hat{v}'_h M_h \approx \hat{v}'_h \begin{bmatrix} 1.1435 & -0.4070 & 0.1354 & -0.2596 & 0.1354 & -0.3333 \\ -0.4070 & 1.1435 & -0.2596 & 0.1354 & -0.3333 & 0.1354 \\ 0.1354 & -0.2596 & 1.1435 & -0.3333 & 0.1354 & -0.4070 \\ -0.2596 & 0.1354 & -0.3333 & 1.1435 & -0.4070 & 0.1354 \\ 0.1354 & -0.3333 & 0.1354 & -0.4070 & 1.1435 & -0.2596 \\ -0.3333 & 0.1354 & -0.4070 & 0.1354 & -0.2596 & 1.1435 \end{bmatrix}$$

$$\approx (1.6884, -1.0913, 0.6558\,, -0.7943, 0.6558, -0.9428) \tag{52}$$

We again add the like time and space components to align our orientation with our rotated unit hyperbola and find

$$((1.6884 + 0.6558 + 0.6558), -(1.0913 + 0.7943 + 0.9428)) \approx (3.0000, -2.8284)$$

We again find this result is on the unit hyperbola and we remind the reader that the angles simply add being of the exponential form

$$\cosh^2(0.8814 + 0.8814) - \sinh^2(0.8814 + 0.8814) = (3.0000)^2 - (2.8284)^2 = 1\,. \tag{53}$$

We perform a similar repeated arbitrary boost with our Lorentz matrix with our resultant four-vector $\hat{v}_l'$ from ***Eq. (35)***

$$\hat{v}_l'' \approx \hat{v}'_l \begin{bmatrix} 1.4142 & -0.7071 & -0.4243 & -0.5657 \\ -0.7071 & 1.2071 & 0.1243 & 0.1657 \\ -0.4243 & 0.1243 & 1.0746 & 0.0994 \\ -0.5657 & 0.1657 & 0.0994 & 1.1325 \end{bmatrix} \approx (3.0000, -2.0000, -1.2000, -1.6000)$$

$$(3.0000)^2 - ((2.0000)^2 + (1.2000)^2 + (1.6000)^2) = (9 - 8) = 1, \tag{54}$$

The invariance of the space-time interval in the Lorentz group is demonstrated by

$$B(v)^T g\, B(v) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = g, \tag{55}$$

where *g* is the Minkowski metric.

In our SO(3,3) form we are able to show that our space-time interval is invariant as well by first denoted a six-dimensional metric g_h, of the form

$$g_h = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{56}$$

We once again take our matrix

$$M_h = \begin{bmatrix} 1.1435 & -0.4070 & 0.1354 & -0.2596 & 0.1354 & -0.3333 \\ -0.4070 & 1.1435 & -0.2596 & 0.1354 & -0.3333 & 0.1354 \\ 0.1354 & -0.2596 & 1.1435 & -0.3333 & 0.1354 & -0.4070 \\ -0.2596 & 0.1354 & -0.3333 & 1.1435 & -0.4070 & 0.1354 \\ 0.1354 & -0.3333 & 0.1354 & -0.4070 & 1.1435 & -0.2596 \\ -0.3333 & 0.1354 & -0.4070 & 0.1354 & -0.2596 & 1.1435 \end{bmatrix}$$

and can show a similar space-time invariance in our SO(3,3) form

$$M_h^T g_h M_h = g_h. \tag{57}$$

Space-time considerations with SO(3,C) and the isomorphism with SU(3)

There would appear to be a very deep and profound link between our Lie groups SO(3,3) and SO(3,C). We start by doing a similar transformation as we will for our space-time interval in the last section. We take the inverse of our SO(3,C) matrix M, calling this M_i, using imaginary values for our same space-time rotation angles to find

$$M_i = \begin{bmatrix} 1.1435 & 0.4070i & 0.1354 & 0.2596i & 0.1354 & 0.3333i \\ 0.4070i & 1.1435 & 0.2596i & 0.1354 & 0.3333i & 0.1354 \\ 0.1354 & 0.2596i & 1.1435 & 0.3333i & 0.1354 & 0.4070i \\ 0.2596i & 0.1354 & 0.3333i & 1.1435 & 0.4070i & 0.1354 \\ 0.1354 & 0.3333i & 0.1354 & 0.4070i & 1.1435 & 0.2596i \\ 0.3333i & 0.1354 & 0.4070i & 0.1354 & 0.2596i & 1.1435 \end{bmatrix}. \tag{58}$$

We once again perform a similar double boost, where we have a six-vector $v_i = (1, 0, 0,0,0,0)$ that we boost by way of the matrix M_i and boost the resultant vector $\hat{v}_i'$ by the matrix M_i again as seen below

$$\hat{v}_i'' = \hat{v}_i' M_i \approx \hat{v}'_i \begin{bmatrix} 1.1435 & 0.4070i & 0.1354 & 0.2596i & 0.1354 & 0.3333i \\ 0.4070i & 1.1435 & 0.2596i & 0.1354 & 0.3333i & 0.1354 \\ 0.1354 & 0.2596i & 1.1435 & 0.3333i & 0.1354 & 0.4070i \\ 0.2596i & 0.1354 & 0.3333i & 1.1435 & 0.4070i & 0.1354 \\ 0.1354 & 0.3333i & 0.1354 & 0.4070i & 1.1435 & 0.2596i \\ 0.3333i & 0.1354 & 0.4070i & 0.1354 & 0.2596i & 1.1435 \end{bmatrix}$$

$$\approx (1.6884, -1.0913i, 0.6558\,, -0.7943i, 0.6558, -0.9428i). \tag{59}$$

We can now simply add the respective space and time components of our resultant six-vector above and find similar results to our SO(3,3) Lie group after squaring

$$((1.6884 + 0.6558 + 0.6558), +(-1.0913i - 0.7943i - 0.9428i)) \approx (3.0000, -2.8284i)$$

$$(3.0000)^2 + (-2.8284i)^2 = 1. \tag{60}$$

We push further and remarkably find that our SO(3,C) Lie group with imaginary angle values is actually unitary as we can see

$$M_i^\dagger M_i =$$

$$\begin{bmatrix} 1.1435 & 0.4070i & 0.1354 & 0.2596i & 0.1354 & 0.3333i \\ -0.4070i & 1.1435 & -0.2596i & 0.1354 & -0.3333i & 0.1354 \\ 0.1354 & 0.2596i & 1.1435 & 0.3333i & 0.1354 & 0.4070i \\ -0.2596i & 0.1354 & -0.3333i & 1.1435 & -0.4070i & 0.1354 \\ 0.1354 & 0.3333i & 0.1354 & 0.4070i & 1.1435 & 0.2596i \\ -0.3333i & 0.1354 & -0.4070i & 0.1354 & -0.2596i & 1.1435 \end{bmatrix} \begin{bmatrix} 1.1435 & -0.4070i & 0.1354 & -0.2596i & 0.1354 & -0.3333i \\ 0.4070i & 1.1435 & 0.2596i & 0.1354 & 0.3333i & 0.1354 \\ 0.1354 & -0.2596i & 1.1435 & -0.3333i & 0.1354 & -0.4070i \\ 0.2596i & 0.1354 & 0.3333i & 1.1435 & 0.4070i & 0.1354 \\ 0.1354 & -0.3333i & 0.1354 & -0.4070i & 1.1435 & -0.2596i \\ 0.3333i & 0.1354 & 0.4070i & 0.1354 & 0.2596i & 1.1435 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I. \tag{61}$$

The dagger above signifies the transpose of the complex conjugate. This unitary relationship exists for real, imaginary and complex angles for the transforms.

We still have a special Lie group as

$$\det(M_i) = 1. \tag{62}$$

So, we can see that we are dealing with a Lie group of the form SU(3) with its 3 complex rotation angles. We proceed and derive the SU(3) form with imaginary permutations that act as group generators for the Lie algebra where

$$b = \begin{bmatrix} 0 & i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \end{bmatrix}, d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 \end{bmatrix}. \tag{63}$$

Once again, we substitute x=b, y=d and z=f into the above, add the permutation together, and arrive at

$$
\mathfrak{g}_{su(3)} = \begin{matrix} 0 & ix & 0 & iz & 0 & iy \\ ix & 0 & iz & 0 & iy & 0 \\ 0 & iz & 0 & iy & 0 & ix \\ iz & 0 & iy & 0 & ix & 0 \\ 0 & iy & 0 & ix & 0 & iz \\ iy & 0 & ix & 0 & iz & 0 \end{matrix}. \tag{64}
$$

To find our Lie group we can now simply matrix exponentiate $\mathfrak{g}_{su(3)}$

$$
\mathfrak{G}_{SU(3)} = \exp\left(\mathfrak{g}_{su(3)}\right), \tag{65}
$$

which yields our SU(3) Lie group

$$
\begin{bmatrix}
2\mathrm{A}_{uc} + \mathrm{B}_{uc} & (-2x+y+z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} & (x+y-2z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} & (x-2y+z)\mathrm{A}_{us} - \mathrm{B}_{us} \\
(2x-y-z)\mathrm{A}_{us} - \mathrm{B}_{us} & 2\mathrm{A}_{uc} + \mathrm{B}_{uc} & (-x-y+2z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} & (-x+2y-z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} \\
-\mathrm{A}_{uc} + \mathrm{B}_{uc} & (x+y-2z)\mathrm{A}_{us} - \mathrm{B}_{us} & 2\mathrm{A}_{uc} + \mathrm{B}_{uc} & (x-2y+z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} & (-2x+y+z)\mathrm{A}_{us} - \mathrm{B}_{us} \\
(-x-y+2z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} & (-x+2y-z)\mathrm{A}_{us} - \mathrm{B}_{us} & 2\mathrm{A}_{uc} + \mathrm{B}_{uc} & (2x-y-z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} \\
-\mathrm{A}_{uc} + \mathrm{B}_{uc} & (x-2y+z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} & (-2x+y+z)\mathrm{A}_{us} - \mathrm{B}_{us} & 2\mathrm{A}_{uc} + \mathrm{B}_{uc} & (x+y-2z)\mathrm{A}_{us} - \mathrm{B}_{us} \\
(-x+2y-z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} & (2x-y-z)\mathrm{A}_{us} - \mathrm{B}_{us} & -\mathrm{A}_{uc} + \mathrm{B}_{uc} & (-x-y+2z)\mathrm{A}_{us} - \mathrm{B}_{us} & 2\mathrm{A}_{uc} + \mathrm{B}_{uc}
\end{bmatrix}.
$$

Fig. 8 – $\mathfrak{G}_{SU(3)}$ Lie group. $\mathrm{A}_{uc} = \frac{cosh\lambda_u}{3}, \mathrm{B}_{uc} = \frac{cosh\gamma}{3}, \mathrm{A}_{us} = \frac{\mathrm{isinh}\lambda_\mathrm{u}}{3\lambda_u}, \mathrm{B}_{us} = \frac{isinh\gamma}{3}, \lambda_u = \sqrt{x^2 - xy - xz + y^2 - yz + z^2}, \gamma = (x + y + z).$

Many might decry that the existing SU(3) group and the Gell-Mann matrices are rather contrived by the lack of division algebra derivation. We believe the above SU(3) Lie group fills this gap and goes a long way to prove the existence of the SU(3) Lie group mathematically and physically.

Rotations of SO(3,C)

With regard to the dynamics of rotations of the complex unit sphere, we find the angles are coupled when doing spin rotations (complex Euclidean rotations). When you rotate by imaginary angles, the elements of the Lie group become hyperbolics. Rotations are oriented and as shown with the aforementioned verification of distance functions, we always arrive at unity. The rotation matrix is robust and we find we can rotate by single points, complex six-vectors and also complex six-dimensional volumes. Rotation angles can be real, imaginary as well as complex (both real and imaginary). Being non-abelian we find the arrangement of rotations does make a difference with the results we find.

We find there are 2 main ways to rotate on the complex unit sphere. The first, is by rotation by a single angle $\angle X, \angle Y \ and \ \angle Z$, or by way of rotation by an oriented plane that is represented by "-c" and "-e" from our permutations.

We present the dynamics of rotations on the complex unit sphere, by first showing rotation by a single angle. We use the convention that a rotation by a positive real angle yields a real counter clock-wise rotation. For these rotations about a single angle, we find the following relationships for a real unit volume being rotated about the respective angles $\angle X, \angle Y \ and \ \angle Z$.

$$
\begin{gathered}
\angle X \ Progression \\
X_r \rightarrow X_i \rightarrow -X_r \rightarrow -X_i \rightarrow X_r \\
Y_r \rightarrow X_i \rightarrow -Y_r \rightarrow -X_i \rightarrow Y_r \\
Z_r \rightarrow X_i \rightarrow -Z_r \rightarrow -X_i \rightarrow Z_r
\end{gathered} \tag{66}
$$

$$
\begin{gathered}
\angle Y \ Progression \\
X_r \rightarrow Y_i \rightarrow -X_r \rightarrow -Y_i \rightarrow X_r
\end{gathered}
$$

$$\begin{gathered} Y_r \rightarrow\ Y_i \rightarrow\ -Y_r \rightarrow -Y_i \rightarrow Y_r \\ Z_r \rightarrow\ Y_i \rightarrow\ -Z_r \rightarrow -Y_i \rightarrow Z_r \end{gathered} \tag{67}$$

$$\begin{gathered} \angle Z\ Progression \\ X_r \rightarrow\ +Z_i \rightarrow\ -X_r \rightarrow -Z_i \rightarrow X_r \\ Y_r \rightarrow\ +Z_i \rightarrow\ -Y_r \rightarrow -Z_i \rightarrow Y_r \\ Z_r \rightarrow\ +Z_i \rightarrow\ -Z_r \rightarrow -Z_i \rightarrow Z_r \end{gathered} \tag{68}$$

Next, we present rotations of the oriented volumes, by the oriented planes of *–c* and *–e,* will yield the following relations when an oriented volume $\hat{V}\ rotates\ by\ \angle\frac{\pi}{2}$ show in ***Tables 1.*** and ***2.***

$\underline{\boldsymbol{-c\ oriented\ plane\ rotation\ progression}}$

Axis	$\hat{V}(-c^1)$	$\hat{V}(-c^2)$	$\hat{V}(-c^3)$	$\hat{V}(-c^4)$	$\hat{V}(-c^5)$	$\hat{V}(-c^6)$
$X \rightarrow$	$-X_y$	X_z	$-X_x$	X_y	$-X_z$	X_x
$Y \rightarrow$	$-Y_z$	Y_x	$-Y_y$	Y_z	$-Y_x$	Y_y
$Z \rightarrow$	$-Z_x$	Z_y	$-Z_z$	Z_x	$-Z_y$	Z_z

Table 1.

$\underline{\boldsymbol{-e\ oriented\ plane\ rotation\ progression}}$

Axis	$\hat{V}(-e^1)$	$\hat{V}(-e^2)$	$\hat{V}(-e^3)$	$\hat{V}(-e^4)$	$\hat{V}(-e^5)$	$\hat{V}(-e^6)$
$X \rightarrow$	$-X_z$	X_y	$-X_x$	X_z	$-X_y$	X_x
$Y \rightarrow$	$-Y_x$	Y_z	$-Y_y$	Y_x	$-Y_z$	Y_y
$Z \rightarrow$	$-Z_y$	Z_x	$-Z_z$	Z_y	$-Z_x$	Z_z

Table 2.

We can notice that the above progressions will pass thru 6 different relationships before arriving back at the starting point when rotating by "*–c"* and "*–e"*. The "*–c"* rotation will proceed in an overall clockwise fashion, but it is important to note there is alternation between positive and negative axes. So, for "*–c"*, we see that X,Y,Z represents the order or successive rotations, whereas with "*–e"*, we find the order opposite and ordered as Z,Y,X.

One very profound relationship that deeper inspection reveals, is that when we are rotating on the complex unit sphere from $0\ to\ 2\pi$, we have the same resultant of net-zero distance when a complete revolution of 2π is made. This is related to the same phenomena we see on the complex unit circle where

$$\int_0^{2\pi} e^{ix} dx = 0. \tag{69}$$

The reader can see for both "-c" and "-e" plane rotations, starting with a unit volume that is all real, it will become negative and opposite to the initial starting for $\hat{V}(-e^3)$ and $\hat{V}(-c^3)$ rotations. The coresponding integral for ***Eqn. (48)*** from above in terms of the complex unit sphere is

$$\iiint_0^{2\pi} e^{ix+iy+iz}\, dx\, dy\, dz = 0. \tag{70}$$

Rotations of SO(3,C) – Examples

We demonstrate the power of our complex unit sphere rotation matrix in the following examples. As mentioned prior, the matrix can handle complex rotation angles, but it can also rotate by compound angles . We used vectors and volumes that represented unit elements to be rotated. For example, to represent a real six-vector that is positive we have a vector $\hat{v}$ represented as

$$\hat{v} = \left(\sqrt{\frac{1}{3}}, 0, \sqrt{\frac{1}{3}}, 0, \sqrt{\frac{1}{3}}, 0\right). \tag{71}$$

To represent a real unit volume, we can define $\hat{V}$ as

$$\hat{V} = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} \end{bmatrix}. \tag{72}$$

We can also define a single point $\hat{p}$ located at $x_{real} = 1$ as

$$\hat{p} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]. \tag{73}$$

In the proceeding examples, we set the rotation angle to $\frac{\pi}{2}$ for each $\angle X, \angle Y\ and\ \angle Z$, then use the fact that our matrix is of the exponential form and exponentiate to different degrees to elicit our chosen rotations. We define our rotation matrix as

$$M_x^n. \tag{74}$$

Where above M is the Lie group, the subscript represents which angle we are rotating, and *n* is the exponent we raise our basis $\frac{\pi}{2}$ angle by.

Example 1. – Rotating a single point on complex unit sphere

We rotate a single point $\hat{p} = [1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$ by $\angle X = -\frac{3\pi}{2}$ and $\angle Y = -\frac{\pi}{4}$ as follows

$$\hat{p}' = \hat{p} M_{(\angle X)}^{(-3)} M_{(\angle Y)}^{(-0.5)} = [0, \quad 0.7071, \quad 0, \quad 0, \quad 0.7071, \quad 0]. \tag{75}$$

It is very convenient to mix rotations as shown above. There is no need to re-orientate the plane after a rotation and there is no worry about Gimbal lock on the complex unit sphere.

Example 2. – Rotation of a single point by a compound angle

We rotate our singular point $\hat{p} = [1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$ by the compound angle $\angle X = \frac{\pi}{2}\ and\ \angle Y = \frac{\pi}{4}$ as follows

$$\hat{p}' = \hat{p} M_{(\angle X, \angle Y)}^{(1,0.5)} = [-0.0964, \quad 0.8003, \quad -0.3053, \quad -0.3289, \quad -0.3053, \quad 0.2357]. \tag{76}$$

Example 3. – Rotation of our single point $\hat{p}$ by $\angle X = \pi\ and\ \angle Y = \pi$

$$\hat{p}^{'} = \hat{p}M^{(2)}_{(\angle X)}M^{(2)}_{(\angle Y)} = [1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]. \tag{77}$$

So, as expected after a full rotation about the x-axis, and then a full rotation about the y-axis we find ourselves back at the starting point.

Example 4. – Rotating a six-vector and verifying the Euclidean distance is unity
We take a six-vector $\hat{v} = \left[\frac{1}{\sqrt{6}}, \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}}\right]$ and rotate by $\angle Z = \frac{3\pi}{2} and\ \angle X = \frac{i\pi}{4}$

$$\hat{v}' = \hat{p}M^{(3)}_{(\angle Z)}M^{(0.5)}_{(\angle iX)} = \begin{pmatrix} 0.5408 + \ 0.3546\mathrm{i}, -0.5408 + \ 0.3546\mathrm{i}, 0.5408 + \ 0.3546\mathrm{i}\ , \\ -0.5408 + \ 0.3546\mathrm{i}, \ \ 0.5408 \ + \ 0.3546\mathrm{i}, -0.5408 \ + \ 0.3546\mathrm{i} \end{pmatrix} \tag{78}$$

We can now calculate our distance function as

$$distance = \left(\hat{v}'^2_{1,1} + \hat{v}'^2_{1,2} + \hat{v}'^2_{1,3} + \hat{v}'^2_{1,4} + \hat{v}'^2_{1,5} + \hat{v}'^2_{1,6}\right) \rightarrow$$

$$\begin{pmatrix} (0.5408 + \ 0.3546\mathrm{i})^2 + (-0.5408 + \ 0.3546\mathrm{i})^2 + \ (0.5408 + \ 0.3546\mathrm{i})^2 \\ +(-0.5408 + \ 0.3546\mathrm{i})^2 \ + (0.5408 + \ 0.3546\mathrm{i})^2 \ + \ (-0.5408 + \ 0.3546\mathrm{i})^2 \end{pmatrix} = 1. \tag{79}$$

Next, we will rotate a unit volume on the complex unit sphere.

Example 5. – We rotate our unit volume by oriented plane rotations $(-c^2)$ $and\ then$ $(-e^1)$. We define our complex unit volume as $\hat{V}$ as described in ***(51)***. We once again define "*-c" as "db" and "-e" as "bd",* where the choice of imaginary permutations "b" and "d" is arbitrary, as the orientation plays the role in which direction the plane rotates.

$$\hat{V}' = \hat{V}M^{(2)}_{(-c)}M^{(1)}_{(-e)} = \begin{bmatrix} 0 & 0 & -0.4082 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4082 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.4082 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.4082 \\ -0.4082 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.4082 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{80}$$

Our results above coincide with the expected transform of our unit volume. If we look at our tables of plane rotation in Tables 1 and 2, we would expect to be at $\hat{V}(-e^4)$ after our $\hat{V}(-c^2)$ rotation. We look up our $\hat{V}(-c^2)$ orientation result in our "–e" table and progress to the right by one for our $\hat{V}(-e^1)$ rotation, seeing the result $-X_y, -Y_z, -Z_x$ perfectly matches our resulting unit volume orientation.

As our last example, we will demonstrate that a compound imaginary rotation yields the equivalent of a rotation on a unit hyperbola .

Example 6. – We rotate our vector $\hat{v}$ ***(50)*** with a 3-angle compound space-time transform with $\angle X = \frac{i\pi}{4}, \angle Y = \frac{i\pi}{10}, and\ \angle Z = \frac{i\pi}{5}$.

$$\hat{v}' = \hat{v}M^{(0.5,0.2,0.4)}_{(\angle iX,\angle iY,\angle iZ)} = \begin{pmatrix} 0.0000 \ + \ 1.6761\mathrm{i}, -1.5736 \ + \ 0.0000\mathrm{i}, -0.0000 \ + \ 1.6761\mathrm{i}, \\ -1.5736 \ + \ 0.0000\mathrm{i}, 0.0000 \ + \ 1.6761\mathrm{i}, -1.5736 \ + \ 0.0000\mathrm{i} \end{pmatrix} \tag{81}$$

We notice the alternating real and imaginary values in the above result for ***(60)***. We proceed to square as follows

$$time = \hat{v}'^2_{1,1} + \hat{v}'^2_{1,3} + \hat{v}'^2_{1,5} = -8.4284, \tag{82}$$

and

$$space = \hat{v}'^2_{1,2} + \hat{v}'^2_{1,4} + \hat{v}'^2_{1,6} = 7.4284. \tag{83}$$

Where we see we lie on the negative portion of the six-dimensional unit hyperbola as

$$time + space = -1. \quad (84)$$

We could also take the inverse of the rotation matrix M, and arrive at the more familiar *time-space=1*.

Compound Angles

In regard to rotations using compound angles, we can see that the angles add together as we would expect in an exponential form. For example, if we take the vector $\hat{v}_1 = (1,0,0,0,0,0)$, and rotate by the compound angle $\angle X = \frac{\pi}{4}, \angle Y = \frac{\pi}{6}, and\ \angle Z = \frac{\pi}{8}$, we find that

$$\hat{v}_1{}' = \hat{v}_1 M_{1_{(\angle X, \angle Y, \angle Z)}} = (0.5836 \quad 0.5443 \quad -0.3571 \quad 0.1594 \quad -0.3571 \quad 0.2877) \quad (85)$$

If we were to add the angles together from above we find that

$$\frac{\pi}{4} + \frac{\pi}{6} + \frac{\pi}{8} = \frac{13\pi}{24}. \quad (86)$$

We can see now that if we rotate our vector $\hat{v}_2 = (1,0,0,0,0,0)$ about $\angle X = \frac{13\pi}{24}$ we find

$$\hat{v}_2{}' = \hat{v}_2 M_{2_{(\angle X)}} = (-0.1305 \quad 0.9914 \quad 0 \quad 0 \quad 0 \quad 0). \quad (87)$$

At first glance the vectors $\hat{v}_1' and\ \hat{v}_2{}'$ look quite dissimilar. We can find the relationship between the two results by adding

$$\hat{v}_1'(1,1) + \hat{v}_1'(1,3) + \hat{v}_1'(1,5) \to (0.5836 - 0.3571 - 0.3571) = -0.1305 = \hat{v}_{2_{(1,1)}}', \quad (88)$$

and

$$\hat{v}_1'(1,2) + \hat{v}_1'(1,4) + \hat{v}_1'(1,6) \to (0.5433 + .1594 + 0.2877) = 0.9914 = \hat{v}_{2_{(1,2)}}'. \quad (89)$$

When we rotate six-vectors of higher dimensionality to start, we see that we are rotating a rigid body and as such, this is an affine transformation. For instance, if we set our initial vector $\hat{v}_3 = \left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right)$, and rotate once again by the compound angle $\angle X = \frac{\pi}{4}, \angle Y = \frac{\pi}{6}, and\ \angle Z = \frac{\pi}{8}$, we find

$$\hat{v}_3' = \hat{v}_3 M_{1_{(\angle X, \angle Y, \angle Z)}} = (-0.0754 \quad 0.5724 \quad -0.0754 \quad 0.5724 \quad -0.0754 \quad 0.5724). \quad (90)$$

We also see the same results now that if we rotate our vector $\hat{v}_4 = \left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right)$ about $\angle X = \frac{13\pi}{24}$ we find

$$\hat{v}_4' = \hat{v}_4 M_{2_{(\angle X)}} = (-0.0754 \quad 0.5724 \quad -0.0754 \quad 0.5724 \quad -0.0754 \quad 0.5724). \quad (91)$$

The above result holds even if we rotate by $\angle Y = \frac{13\pi}{24}\ or\ \angle Z = \frac{13\pi}{24}$.

Plotting on the Complex Unit Sphere

We continue by looking into some basic plots on the complex unit sphere. We define our six-vector $\hat{v}$ as

$$\hat{v} = (1,0,0,0,0,0). \tag{92}$$

We first confirm that the SO(3,C) form reduces to U(1) by one rotating $\hat{v}$ by the $\angle X = \frac{\pi}{1000}$ and iterate five thousand times. We see in ***Fig. 9***, a perfect circular form and the familiar cosine and sine waveforms.

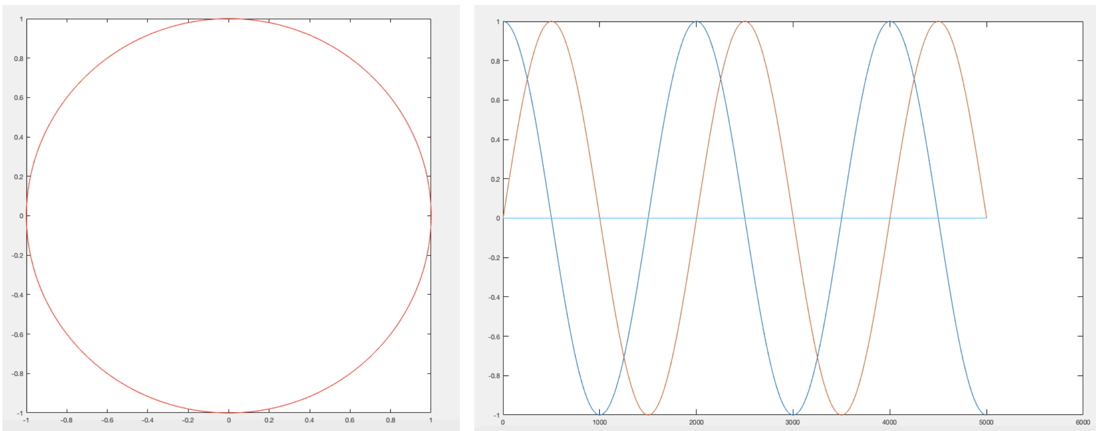


Fig. 9 **–** Geodesic on the complex unit sphere – plot of rotation by angle $\angle X$ which reduces to e^{ix} and its waveforms, cos(x) and isin(x).

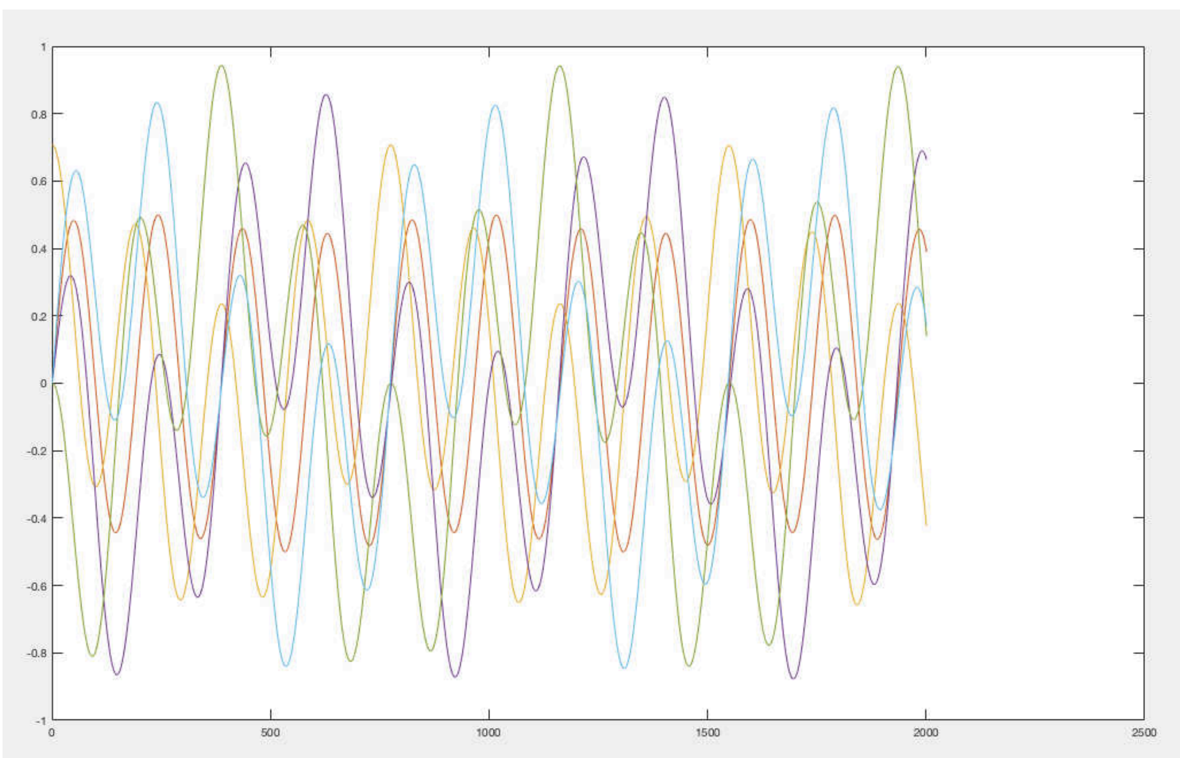


Fig. 10 **–** Compound rotation waveforms – rotation angles $\angle X = \frac{\pi}{200}, \angle Y = \frac{\pi}{300}, \angle Z = \frac{\pi}{500}$

In ***Fig. 10*** above we rotate our vector $\hat{v}$ by $\angle X = \frac{\pi}{200}, \angle Y = \frac{\pi}{300}, \angle Z = \frac{\pi}{500}$, and iterate two thousand times. We can see the inter-relation amongst all six waveforms over time.

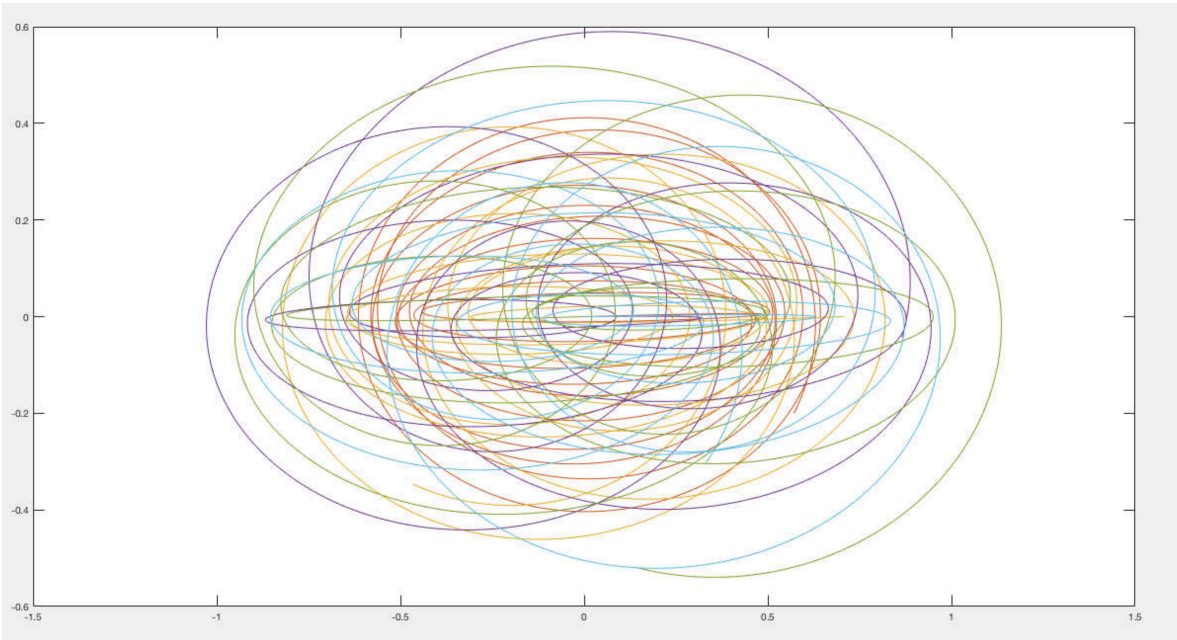


Fig. 11 **–** Complex Compound rotation waveforms– rotation angles $\angle X = \frac{\pi}{200} + \frac{i\pi}{10000}, \angle Y = \frac{\pi}{300} + \frac{i\pi}{20000}, \angle Z = \frac{\pi}{500} + \frac{i\pi}{30000}$.

In ***Fig. 11*** we rotate our vector $\hat{v}$ by the complex and compound angles $\angle X = \frac{\pi}{200} + \frac{i\pi}{10000}, \angle Y = \frac{\pi}{300} + \frac{i\pi}{20000}, \angle Z = \frac{\pi}{500} + \frac{i\pi}{30000}$ thru two thousand iterations. We can see the waveforms that we noted before start to become elliptical from the effects of the imaginary angles that we added.

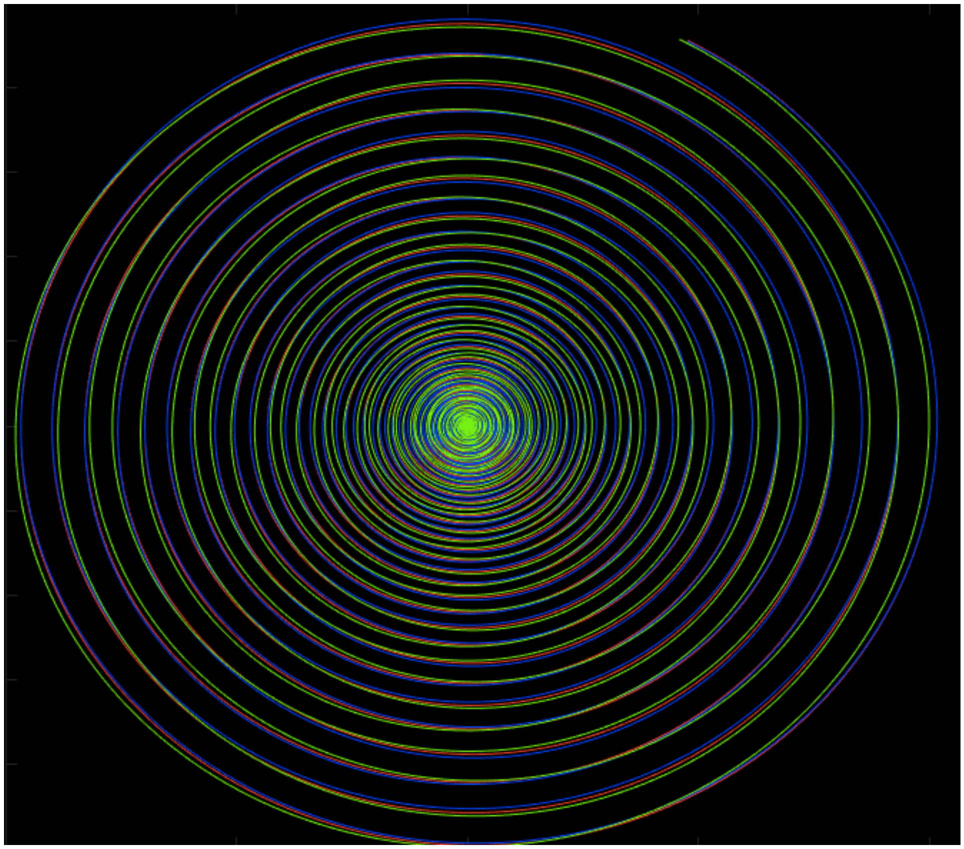


***Fig. 12* –** Complex Compound rotation waveforms– rotation angles $\angle X = \frac{\pi}{200} + \frac{i\pi}{10000}, \angle Y = \frac{\pi}{300} + \frac{i\pi}{20000}, \angle Z = \frac{\pi}{500} + \frac{i\pi}{30000}$ Twenty-thousand iterations.

In ***Fig. 12*** we continue the same rotation angles from ***Fig. 10*** and iterate for twenty-thousand times to see that the waveforms have become more elliptical as we move away from the origin.

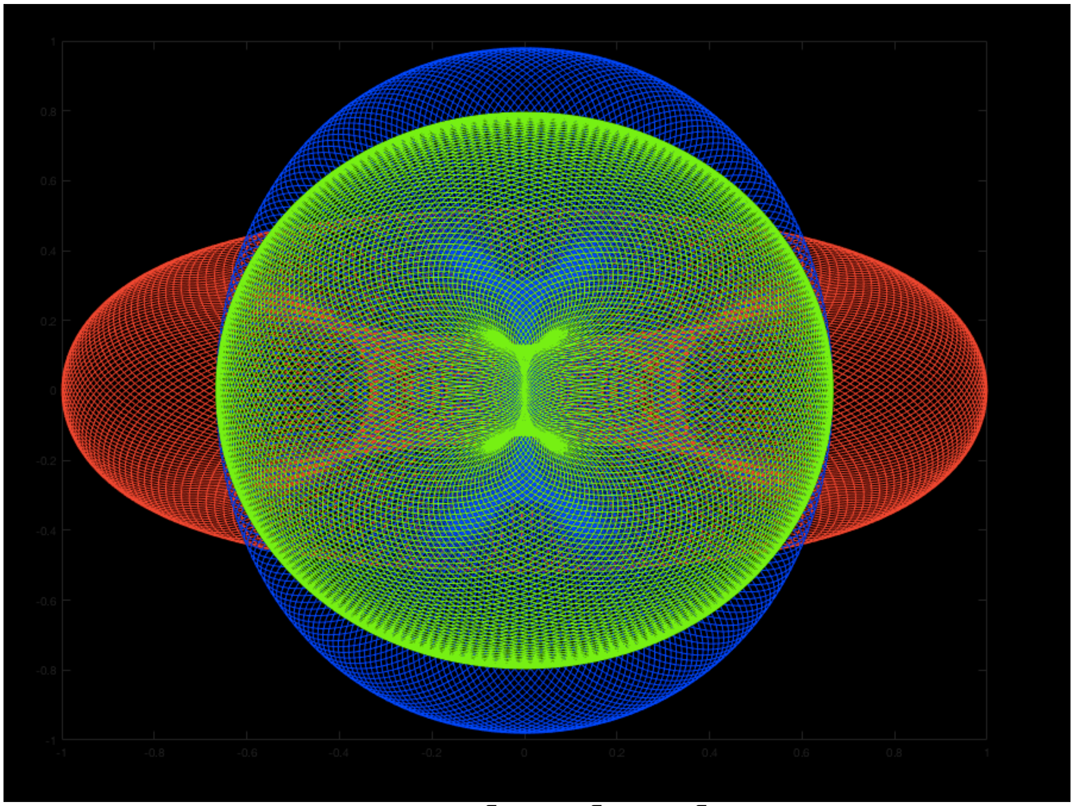


Fig. 13 **–** Compound rotation plot – Toroidal forms $\angle X = \frac{\pi}{200}, \angle Y = \frac{\pi}{300}, \angle Z = \frac{\pi}{500}$.

In ***Fig. 13***, we rotate our vector $\hat{v}$ by the angles $\angle X = \frac{\pi}{200}, \angle Y = \frac{\pi}{300}, \angle Z = \frac{\pi}{500}$ over twenty-five thousand iterations. We plot each respective 2-D form that is X,Y and Z. We notice beautiful forms that take shape in the form of toroidal shapes for each.

Conclusion

By using a very simple approach that started with using selective permutation matrices of a finite group and by way of exponentiation, we are able to form the Lie groups of the complex unit sphere SO(3,C) and its hyperbolic analogue SO(3,3). We are able appreciate the deep isomorphism between SO(3,C) and SO(3,3) and derive a true mathematical form of SU(3). Rotating with these Lie groups is straightforward, robust and takes advantage of exponential functions. With these higher dimensional Lie groups, we are able to find our familiar distance functions, demonstrate complex angular rotations, apply to space-time rotations all while being reassured that the derivations represent division algebras. We propose that these Lie groups represent how rotations occur in our empty space, that each plane of rotation requires a respective C_2 algebra and to make rotations in what we perceive as three-dimensions, requires the orthogonal arrangement of three C_2 algebras to do so. We believe the aforementioned derivations have the potential to advance our approaches to relativity, quantum field theory, complex analysis, deriving higher dimensional forms of similar division algebras and technologies requiring orientated rotations.

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