1. Motivation: Homotopy coherence
2. ∞-categories (Joyal, Lurie)
3. Stable ∞-categories (higher algebra, Lurie)
4. Equivariant homotopy theory
5. Towers and spectral sequences (SS)

<u>Thm</u> X space

G cpt Lie gp

$$[X, BG] \cong \mathrm{Prin}_G(X)$$

<u>Thm</u> $A \in Ab$ $\quad X$ space

$$[X, K(A,n)] \cong H^n(X; A)$$

HA spectrum

$$[X, K_{mot}(A(p), q)] \cong H^{p,q}_{mot}(X, A)$$

$B_0 G$: $\{*\}$, $\mathrm{Aut}\{*\} = G$

$\mathrm{Fun}(B_0 G, \mathcal{C})$

$$\mathcal{C} = \mathrm{Top} \xrightarrow{\pi} h\mathrm{Top}$$

Def homotopy group action

$$\mathrm{Fun}(B_\circ G, h\mathrm{Top})$$

Thm (Cooke) $X \in \mathrm{Fun}(B_\circ G, h\mathrm{Top})$

$$\left(G \leftleftarrows G\times G \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; G\times G\times G \cdots\right)$$

$$BG \longrightarrow B\mathrm{Aut}_\circ(X)$$

X is a strictly G-equivariant space iff there is a lifting

$$\begin{tikzcd} & B\mathrm{Aut}(X) \arrow[d] & \simeq \mathrm{Aut}_{\mathrm{Top}}(X) \\ BG \arrow[r] \arrow[ur, dashed] & B\mathrm{Aut}_\circ(X) & = \mathrm{Aut}_{h\mathrm{Top}}(X) \end{tikzcd}$$

Thm (Dwyer-Kan-Smith)

A homotopy commutative diagram can be lifted to a comm diagram iff it is homotopy coherent.

§ Homotopy coherence

$$I = [0,1] = |\Delta^1|$$

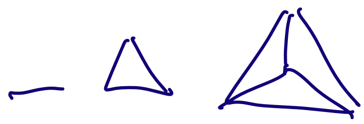
$$\mathrm{Map}(X\times I, Y) \cong \mathrm{Map}(I, \mathrm{Map}(X,Y))$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \qquad$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Y^X \qquad$$

$$\Downarrow$$

$\{$ [illegible] $\} \Longrightarrow \{$ [illegible] Y^{X} 2

$\{1\text{-homotopy}\} \Longleftrightarrow \{\text{a } 1\text{-simplex in } Y^X\}$

Slogan: $\{n\text{-homotopy}\} \Longleftrightarrow \{n\text{-simplex in } Y^X\}$.



slogan Homotopy coherence is parametrized by simplicial sets.

$$\mathrm{sSet} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathrm{Top} \qquad \mathrm{Sing}_n(X) = \mathrm{Map}(|\Delta^n|, X).$$

§ Nerve of a category

$N: \mathrm{Cat} \longrightarrow \mathrm{sSet}$

$$N(\mathcal{C})_n = \mathrm{Hom}(\underset{(0\to 1\to 2\to 3\to \cdots \to n)}{[n]}, \mathcal{C}).$$

$$\tau_0 : \mathrm{sSet} \rightleftarrows \mathrm{Cat} : N$$

$\uparrow$

homotopy category

Def A ∞-cat X is an ∞-groupoid / Anima

iff $\tau_0(X)$ is a groupoid

Kan complex

$$X_0 \rightleftarrows X_1 \;\; X_2 \cdots$$

objects, morphisms

Def (homotopy)

$$\begin{tikzcd} & B \arrow[d, equal] \\ A \arrow[ur, "f"] \arrow[r, "g"'] & B \end{tikzcd}$$

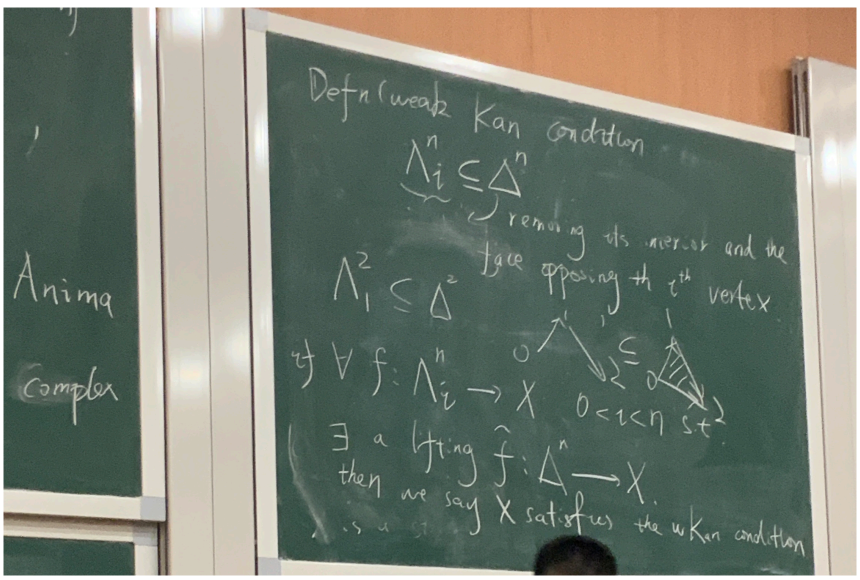
$f, g : A \to B$

a homotopy $f \simeq g$ is a 2-simplex filling ✡

$$A \xrightarrow{f_1} B \xrightarrow{f_2} C$$

composition?

weak Kan condition



Def An ∞-cat is a simplicial set satisfying the weak Kan condition.

$f \nearrow \searrow g$ $\xrightarrow[\text{condition}]{\text{weak Kan}}$ $f \nearrow \searrow g$ $g \circ f \simeq h$

Def ∞-cat of spaces

homotopy coherent nerve $N_\Delta(\mathrm{Kan})$. ?

$$
\begin{tikzcd}
\mathrm{Cat}^{\mathrm{Kan}} \arrow[rr, hook] \arrow[d, shift right, "N_\Delta"'] & & s\mathrm{Cat} \arrow[dl, shift left, "N_\Delta"] \\
\boxed{\mathrm{Cat}_\infty} \arrow[u, shift right, "|\cdot|"'] \arrow[r, phantom, "\subseteq"] \arrow[rrd, bend right] & s\mathrm{Set} \arrow[ur, shift left, "|\cdot|_C"] \arrow[r, shift left, "\tau_0"] & \mathrm{Cat} \arrow[l, shift left, "N"] \arrow[u, "\text{simplicial enhancement}"'] \\
\mathrm{Kan} \arrow[u, phantom, "\cup"] & & \infty\text{-cat of } \infty\text{-cats}
\end{tikzcd}
$$

§ Dold-Kan correspondence

$$
\begin{tikzcd}
\mathrm{Top} \arrow[r, shift right, "\mathrm{Sing}"'] & s\mathrm{Set} \arrow[l, shift right, "|\cdot|"'] \arrow[r, shift left, "\mathbb{Z}[\cdot]"] & s\mathrm{Abel} \arrow[l, shift left, "\mathrm{forget}"] \arrow[r, shift left, "|\cdot|_C"] & \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \arrow[l, shift left, "\mathrm{DK}", "\mathrm{Sing}_C"']
\end{tikzcd}
$$

π_i H_i

Prop: $s\mathrm{Abel} \subseteq \mathrm{Kan}$

"12"

$$
\begin{tikzcd}
\mathbb{E}_\infty\text{-group} \arrow[r, "\text{delooping}"] & \boxed{\text{connective spectra}}
\end{tikzcd}
$$

$B\mathbb{Z}$ machine

$\rightsquigarrow$ connective $H\mathbb{Z}$-module

"naive spectra" May spectra

$$\{X_i\}_{i=0}^{\infty} \quad \text{with } X_i \simeq \Omega X_{i+1} \quad i \geq 0$$

Ex Eilenberg-Mac Lane spectrum

Given $A \in \mathrm{Abel}$

$$(HA)_n = K(A,n).$$

Roughly speaking

$$\{HR\text{-module}\} \simeq D(R)$$

§ Stable ∞-categories

Spc: ∞-category of spaces

$$\mathrm{Sp} := \underbrace{\mathrm{holim}\left(\mathrm{Spc} \xleftarrow[\Omega]{\mathrm{Map}(S^1,-)} \mathrm{Spc} \xleftarrow{\Omega} \mathrm{Spc} \xleftarrow{\Omega} \right)}_{\text{stabilization}}$$

FACT: Sp admits a symmetric monoidal structure

$$\Sigma^{\infty} : Spc_* \rightleftarrows Sp : \Omega^{\infty}$$

$$Sp = Spc[S^{-1}]$$

$\mathbb{P}^1$-stable motivic category.

Huang