§ Flat descent for THH [BMS19 II, §3]

① Flat descent on HH

② Flat descent on THH $\in Sp_{\geq 0}$

⓪ Recall: $\mathbb{L}_{-/R}$ is an fpqc sheaf

$$⓪ \xRightarrow{HKR} ① \xRightarrow[\text{filtration}]{\text{Postnikov}} ②$$

§§ Flat descent on HH

Thm $HH(-/R)$ is an fpqc sheaf

Pf Recall the HKR filtration on HH.

$$gr^i_{HKR}(HH(-/R)) \simeq \wedge^i \mathbb{L}_{-/R}[i]$$

$$\boxed{HH(-/R)/\mathrm{Fil}^n_{HKR}} \leftarrow \text{do induction on } n$$

Similarly $(HH)_{hS^1}$ is also a fpqc sheaf.

$$HC^- = HH^{hS^1} = \varprojlim_{BS^1} HH$$

DS

HP ...

§§ t-structure on Sp

Def $E \in Sp$, $\pi_i E := [\mathbb{S}^i, E]_{Sp}$

Def (Postnikov t-structure on Sp)

$$\underbrace{\begin{cases} Sp_{\le n} := \{E \in Sp \mid \pi_i E = 0 \ \ \forall i > n\} \subseteq Sp_{\le n+1} \\ Sp_{\ge n} := \{E \in Sp \mid \pi_i E = 0 \ \ \forall i < n\} \supseteq Sp_{\ge n+1} \end{cases}}_{\text{localizing subcats}}$$

Ajoint functors

$\tau_{\le n} : Sp \rightleftarrows Sp_{\le n} : \mathrm{incl}_n$ $\quad X \to \tau_{\le n} X$ n-truncation

$\mathrm{incl}_n : Sp_{\ge n} \rightleftarrows Sp : \tau_{\ge n}$ $\quad \tau_{\ge n} X \to X$ n-connective cover

$X\langle n\rangle$

$$\left.\begin{array}{l} \operatorname{colim} \tau_{\ge -n} E \simeq E \\ \lim \tau_{\le n} E \simeq E \end{array}\right\} \text{ in } Sp$$

Remark: ① hSp admits a t-structure $\{hSp_{\ge n}, hSp_{\le n}\}$

②
$$
\begin{tikzcd}
Sp^{\heartsuit} \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Abel} \arrow[r, phantom, "\simeq"] & N(hSp)^{\heartsuit} \arrow[dll, hook] \\
\{\text{EM spectra}\} & &
\end{tikzcd}
$$

§§ Postnikov tower

$X \in Sp_{\geq 0}$

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & \tau_{\leq 2} X \arrow[d, "i_2"] & \mathrm{Fib}(i_2) \simeq \pi_2(X)[-2] \arrow[l] \\
 & \tau_{\leq 1} X \arrow[d, "i_1"] & \mathrm{Fib}(i_1) \simeq \pi_1(X)[-1] \arrow[l] \\
X \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leq 0} X &
\end{tikzcd}
$$

$$X \simeq \varprojlim \tau_{\leq n} X$$

Thm THH is an fpqc sheaf.

Pf $\mathrm{THH}(A) = \mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathrm{THH}(\mathbb{Z})$.

Lemma 3.3

$$\varprojlim \{\mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathbb{Z})} \tau_{\leq n} \mathrm{THH}(\mathbb{Z})\}_n \simeq \mathrm{THH}(A)$$

Reduced to prove that $\underbrace{THH \otimes_{THH(\mathbb{Z})} \tau_{\le n} THH(\mathbb{Z})}_{=: M_n}$ is a sheaf

Now we do induction on n.

Inductive hypothesis: $\forall j < n$, M_j is a sheaf

$$\underbrace{M_{n-1}[-1]}_{\text{sheaf}} \to \underbrace{THH(A) \otimes_{THH(\mathbb{Z})} \pi_n THH\,\mathbb{Z}[-n]}_{\substack{\uparrow \\ \text{remains to prove} \\ \text{this is a sheaf}}} \to M_n \to \underbrace{M_{n-1}}_{\text{sheaf}}$$

$$\pi_i THH\,\mathbb{Z} = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/k & i = 2k-1 \\ 0 & \text{otherwise} \end{cases}$$

we just need to prove

$THH(A) \otimes_{THH(\mathbb{Z})} \mathbb{Z}$ is a sheaf

$\mathrm{HH}(A)$.

Then by ①. we prove it.

$\square$

§ THH for perfectd rings [§6. BMS2]

<u>Thm</u> Let R be a perfectd ring

$$\boxed{\begin{array}{l} A_{inf} := A_{inf}(R) \\ \theta : A_{inf} \to R \end{array}}$$

Then $\pi_* THH(R, \mathbb{Z}_p) \cong R[u]$

where $u \in THH_2(R, \mathbb{Z}_p) \cong HH_2(R, \mathbb{Z}_p) = \ker(\theta)/\ker(\theta)^2$

is a generator.

<u>Thm</u> (Bökstedt)

$$THH(\mathbb{F}_p) = \mathbb{F}_p[u], \quad |u| = 2.$$

[Hesselholt; Nikolaus] Bökstedt SS approach

[Klause-Nikolaus] Thom approach

(Handbook) Video lectures

Pf 1. Show that $HH_i(R, \mathbb{Z}_p) \cong R$ if $i = 2n$ and vanishes o/w

▷ Recall $(\mathbb{L}_{R/\mathbb{Z}_p})^\wedge_p \cong R[1]$ $\ker\theta/(\ker\theta)^2$

Since the HKR filtration is multiplicative, we can prove it directly.

2. $R \to R'$ morphism of perfd rings

$$THH(R, \mathbb{Z}_p) \otimes^{\mathbb{L}}_R R' \to THH(R', \mathbb{Z}_p)$$

is an equivalence

▷
$$\left\{ \begin{array}{c} THH(R, \mathbb{Z}_p) \otimes_{THH(\mathbb{Z})} \tau_{\leq n} THH(\mathbb{Z}) \otimes^{\mathbb{L}}_R R' \\ \downarrow \\ THH(R, \mathbb{Z}_p) \otimes \tau \ THH(\mathbb{Z}) \end{array} \right\}$$

$\mathrm{THH}(R;\mathbb{Z}_p) \otimes_{\mathrm{THH}\,\mathbb{Z}} \tau_{\leq n}\mathrm{THH}(\mathbb{Z}) \Big)_{n\in\mathbb{N}}$

Postnikov tower

reduced to $\mathrm{THH}(R,\mathbb{Z}_p) \otimes_{\mathrm{THH}\,\mathbb{Z}} \mathbb{Z} \simeq \mathrm{HH}(R;\mathbb{Z}_p)$

and thus we just need to prove

$$\mathrm{HH}(R,\mathbb{Z}_p) \otimes_R R' \simeq \mathrm{HH}(R',\mathbb{Z}_p)$$

~~(Follows from step 1)~~ HKR.

$\mathrm{HH}_*(\mathbb{F}_p) = \mathbb{F}_p\langle x\rangle$ divided powers

$$(\wedge \mathbb{L}_{R/\mathbb{Z}_p} \otimes R')^\wedge_p \xrightarrow{\sim} (\wedge \mathbb{L}_{R'/\mathbb{Z}_p})^\wedge_p.$$

3.

<u>Fact</u> $\mathrm{THH}_i(R;\mathbb{Z}_p)$ is finitely generated. (formal)

3. Reduce general case to char p case.

▷ $R \to R/p \overset{\text{direct limit perfection}}{\rightsquigarrow} \overline{R}$

Then we have a surjective map $R \to \overline{R}$

Base change

$$\mathrm{THH}(R,\mathbb{Z}_p)\otimes_R \bar{R} \simeq \mathrm{THH}(\bar{R},\mathbb{Z}_p)$$

Tor SS.

$$\mathrm{Tor}^*_R(\mathrm{THH}_*(R),\bar{R}) \Rightarrow \pi_*(\sim\otimes_R \bar{R}).$$

$$\Rightarrow \mathrm{THH}_i(R,\mathbb{Z}_p)\otimes_R \bar{R} \simeq \underbrace{\mathrm{THH}_i(\bar{R},\mathbb{Z}_p)}_{\bar{R}[u]}.$$

$$M := \pi_i \mathrm{THH}(R,\mathbb{Z}_p)$$

$$M' := \begin{cases} R\cdot u^n & i=2n \\ 0 & \text{otherwise} \end{cases}$$

Then we consider

$$M' \to M$$

$$1 \mapsto u^n$$

$$\begin{array}{ccc} M'\otimes \bar{R} & \xrightarrow{\sim} & M\otimes \bar{R} \\ \| & & \| \\ \bar{R} & & \bar{R} \end{array}$$

By Nakayama's lemma.

$M' \longrightarrow M$ is surjective

$\ker(R \to \bar{R}) \cdot \operatorname{coker}(M' \to M) = \operatorname{coker}(M' \to M)$.

It remains to prove $M' \to M$ is injective

- special fiber $\in \operatorname{Spec} \bar{R} \subseteq \operatorname{Spec} R$. ✓
- generic fiber (rational case)

~~claim~~ Fact $THH(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$

$$THH(R, \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} = THH(R, \mathbb{Z}_p) \otimes_{THH(\mathbb{Z})} (THH(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q})$$

$$= THH(R, \mathbb{Z}_p) \otimes_{THH(\mathbb{Z})} \mathbb{Q}$$

$$\left(THH(R, \mathbb{Z}_p) \otimes_{THH(\mathbb{Z})} \mathbb{Z} \right) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$= HH(R, \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$M \otimes_{\mathbb{Z}} \mathbb{Q} \cong R \otimes \mathbb{Q}$$

reduced!

$\Rightarrow \ker(R \to M)$ is in $\mathrm{Nil}(R)$

$= 0.$ $\square$.