Recall: A ring A is called quasi-syntomic if

1) A is p-complete with bounded p^∞-torsion
2) $L_{A/\mathbb{Z}_p} \in D(A)$ has $\hat{p}$-Tor-amplitude in $[-1,0]$

A map $A \to B$ of $\hat{p}$-rings is called q-syn (cover) if

1) B is $\hat{p}$-(faithfully)-flat over A
2) $L_{B/A} \in D(B)$ has $\hat{p}$-Tor-amplitude in $[-1,0]$

Lemma: Let $A \to B$ be quasi-syntomic cover

Then $A \in QSyn \Leftrightarrow B \in QSyn$

Proof:
$$L_{A/\mathbb{Z}_p} \hat{\otimes}^{\mathbb{L}}_A B \longrightarrow L_{B/\mathbb{Z}_p} \longrightarrow L_{B/A}$$

$[-1,0]_{\mathrm{I}}$ $\qquad$ $[-1,0]_{\mathrm{II}}$ $\qquad$ $\overline{[-1,0]}$

$\Rightarrow [-1,0]_{\mathrm{II}}$ $\qquad$ $\Rightarrow [-1,1]_{\mathrm{I}}$

$\Downarrow$

$[-1,0]$

Lemma: q-smooth & q-syntomic map are stable under composition and arbitrary $\hat{p}$-base change

$\Rightarrow$ Lemma: $QSyn^{op}$ forms a site, with q-syn cover
(Cor)

Perfectoid rings

Def A ring R is perfectoid if

1) it is p-adically complete
2) $\exists \pi \in R$ s.t. $\pi^p = p \cdot u$ $u \in R^\times$
3) R/p is semi-perfect

   i.e., $x \mapsto x^p$ is surjection
4) $\ker(A_{inf}(R) \to R)$ is principal (cf. $\mathcal{O}_{\mathbb{C}}$ from Gao)

Remark: $A_{inf}(R) = W(R^\flat) \xrightarrow[\text{Fontaine map}]{\theta_R} R$ $\quad [(x_i)_{i=0}^{+\infty}] \mapsto x_0$

$$R^\flat = \varprojlim_{x \mapsto x^p} R/p = \{ (x_i)_{i=0}^{+\infty} \mid \underset{\substack{\uparrow \\ R}}{x_{i+1}^p} = x_i \} \overset{\text{monoid}}{=} \varprojlim_{x \mapsto x^p} R$$

$[-] : R^\flat \to W(R^\flat)$ Teichmüller lift

All $a \in W(R^\flat)$ has unique expansion

$$\sum_{n=0}^{+\infty} [a_n] p^n \quad a_n \in R^\flat$$

For perfect ring A of char p, B p-complete

$$\mathrm{Hom}(A, B/p) = \mathrm{Hom}(W(A), B)$$

θ is constructed this way:

$$\left(R^\flat \xrightarrow{pr_0} R/p\right) \mapsto \theta_R$$

Prop R be a perfectoid ring

1) The ideal $\ker \theta_R = (p + [\pi^\flat]^p \alpha)$, $\alpha \in A_{inf}(R)^\times$

$$\pi^\flat = (\pi, \pi^{\frac{1}{p}}, \pi^{\frac{1}{p^2}}, \dots)$$

2) $L_{R/\mathbb{Z}_p}$ has $\hat{p}$-Tor-amplitude concentrated in degree -1.

$$\hat{L}_{R/\mathbb{Z}_p} \simeq R[1]$$

3) R has bounded p^∞-torsion. In fact,

$R[p^\infty] = R[p]$.

4). R is reduced

5). Let $A \to B$ be of perfectoid rings, then

$\hat{L}_{B/A} = 0$.

Proof 1) BMS1 (Integral p-adic Hodge theory)

$$\theta[\pi^b] = \pi \qquad pu - \pi^p = 0$$

$$\exists\, \theta(x) = u \Rightarrow [\pi^b]^p + px \in \ker\theta_R \quad = (\pi^{bp} x_0, \underline{1 + (\pi^b)^{p^2} x_1}, \dots)$$

$$\begin{tikzcd} & A_{inf}/\xi \arrow[r, "\theta_R"] \arrow[d] & R \arrow[d] \\ R^b/[\pi^b]^p \simeq & A_{inf}/(\xi p) \arrow[r] & R/\pi^p \end{tikzcd}$$

$$\ker\theta_R = (\xi')$$

$$\xi = \xi' a$$

$$a = (a_0, \dots, a_n, \dots)$$

$$\xi' = \sum p^n [\xi_n]^{\frac{1}{p^n}} = (\xi'_0, \xi'_1, \dots, \xi'_n, \dots)$$

$$\xi' a = (a_0 \xi'_0, \underline{a_0^p \xi'_1 + \xi_0'^p a_1}, \dots)$$

$$\xi'_1 a_0^p = 1 + \underline{(\pi^b)^{p^2} x_1} - \xi_0'^p a_1 \mapsto 1$$

$$\xi'_1 a_0^p \in R^b \xrightarrow{pr_0} R/\pi^p, \qquad (\pi^b)^{p^2} x_1 \mapsto 0, \quad \xi_0'^p a_1 \mapsto 0$$

$$\xi'_1 a_0^p \in (R^b)^\times \qquad a_0 \in (R^b)^\times \Rightarrow a \in A_{inf}^\times$$

2) $\underline{\mathbb{Z}_p \xrightarrow{\text{is perfect mod } p} A_{inf}(R) \to R}$

$$\underline{\hat{L}_{A_{\inf}(R)/\mathbb{Z}_p}} \otimes^{\mathbb{L}} R \longrightarrow \hat{L}_{R/\mathbb{Z}_p} \longrightarrow \underline{\hat{L}_{R/A_{\inf}(R)}}$$

$$\cong$$

$$\ker\theta/(\ker\theta)^2[1] \simeq R[1]$$

3) Sufficient to show if $p^2 f \in (\xi) \Rightarrow pf \in (\xi)$

$$p^2 f = \xi \cdot g \qquad \xi = \sum [a_i] p^i \quad g = \sum [g_i] p^i$$

$$g\xi = \underbrace{[a_0 g_0]}_{0} + ([a_0 g_1] + [a_1 g_0])p + hp^2$$

$$a_0 g_0 = 0 \qquad\qquad a_0 g_1 + a_1 g_0 = 0 \text{ in } R^\flat$$

$$\times g_1 \qquad\qquad \times g_0$$

$$a_1 g_0^2 = 0$$

$$\Rightarrow g_0^2 = 0 \xRightarrow{R^\flat \text{ perfect}} g_0 = 0$$

4)
$$\begin{array}{ccc} R & \longrightarrow & R/R[p] \\ \downarrow & & \downarrow \\ R/(\sqrt{p}) & \longrightarrow & R/(R[p], \sqrt{p}) \end{array}$$

$R/R[p]$ ~ p-torsion free

$\pi^p = pu$

inductively.

$a \in \pi^n R$, $n \geq 0$

$R/(\sqrt{p})$ perfect (reduced)

$a \in \pi^n R \not\Rightarrow a \in \pi^{n+1} R$

$a = \pi^n b \qquad a^p = \pi^{pn} b^p = 0 \Rightarrow b^p = 0$ in R/p

$R/p \to R/p$

$x \mapsto x^p$

$\ker \mathrm{Frob} = (\pi) \qquad \Rightarrow \pi \mid b$

5) $A \to B \qquad B = A_{\inf}(B) \otimes^{\mathbb{L}}_{A_{\inf}(A)} A$

$\Rightarrow L_{A_{\inf}(B)/A_{\inf}(A)} \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{F}_p = L_{B^\flat/A^\flat}$; $L_{A_{\inf}(B)/A_{\inf}(A)} \otimes^{\mathbb{L}} B \simeq L_{B/A}$

$\mathbb{F}_p \to A^\flat \to B^\flat$

$\underset{0}{\underset{\|}{L_{A^\flat/\mathbb{F}_p}}} \otimes^{\mathbb{L}} B^\flat \to \underset{0}{\underset{\|}{L_{B^\flat/\mathbb{F}_p}}} \to \underset{0}{\underset{\|}{L_{B^\flat/A^\flat}}}$

$\Rightarrow \hat{L}_{B/A} = 0$

Def A ring S is called quasi-regular semi-perfectoid if

1) $S \in QSyn$

2) $\exists R \to S$ from a perfd

3) S/p semi-perfect

$\Bigg\} \iff \exists R \twoheadrightarrow S$ from perfd

<u>Lemma</u> Fix a $\hat{p}$-ring with bounded p^∞-torsion S/p semi-perf.

Then S is qrsp iff $\exists R \to S$ from a perf'd $L_{S/R} \in D(S)$ has $\hat{p}$-Tor-amplitude in degree -1.

<u>Lemma</u> $\mathrm{QRSPerfd}^{op}$ is a site with q-syn cover

<u>Proof</u>

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B \hat{\otimes}_A C = D \end{tikzcd}$$

$$R_B \hat{\otimes} R_C \longrightarrow B \hat{\otimes} C$$

$$B/p \otimes C/p$$

<u>Lemma</u> A $\hat{p}$-ring A is q-syn iff $\exists$ q-syn cover in QRSPerfd

<u>Pf</u> "$\Leftarrow$" ✓

"$\Rightarrow$" Let A q-syn

$$\overbrace{\mathbb{Z}_p[X_i]_{i\in I}} \longrightarrow \overbrace{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}, X_i^{\frac{1}{p^\infty}}]_{i\in I}}$$

$$\begin{tikzcd} \cdot \arrow[d] \arrow[r, phantom, "\square"] & \cdot \arrow[d] \\ A \arrow[r] & S \end{tikzcd}$$

<u>Cor</u> $A \to S$ is a q-syn cover in QSyn.

$S \in$ Perfd

Then $[S_A^{\otimes k}] = S^\bullet$ is in QRSPerfd

Čech nerve

<u>Prop</u> $\mathrm{QRSPerfd}^{op} \subset \mathrm{QSyn}^{op}$ $\rightsquigarrow$ $\mathrm{Sh}_e(\mathrm{QSyn}^{op}) \xrightarrow{\sim} \mathrm{Sh}_e(\mathrm{QRSPerfd}^{op})$

easier

<u>Proof</u> $\mathrm{Sh}_e(\mathrm{QRSPerfd}^{op}) \xrightarrow{e} \mathrm{Sh}_e(\mathrm{QSyn}^{op})$ presentable ∞-cate

$$\mathcal{F} \longmapsto \mathcal{F}^J = (A \mapsto \mathrm{Hom}_{/\mathrm{QRSP}^{op}}(h_A, \mathcal{F}))$$

"unfolding"

$\mathcal{F}$ on QRSP^{op} $\quad u(\mathcal{F}^J)(A) = \mathrm{Hom}_{\mathrm{QRSP}}(h_A, \mathcal{F}) = \mathcal{F}(A)$

$\mathcal{G}$ on $QSyn^{op}$ $\quad (u\mathcal{G})^{\mathcal{F}}(B) = Hom_{/QSyn^{op}}(h_B, \mathcal{F}|_{QRSp^{op}})$

$$= Hom_{/QRSp^{op}}(\lim h_{S^\bullet}, \mathcal{F}|_{QRSp^{op}})$$

$$= \lim Hom_{QRSp^{op}}(h_{S^\bullet}, \mathcal{F})$$

$$= \lim \mathcal{F}(S^\bullet)$$

$$= \mathcal{F}(A)$$