Intro (Part I) Alg vs geom p-adic Hodge theory

Roughly
- (1) Complex Hodge thy
- (2) $\mathbb{Q}_p$-Hodge thy
- (3) $\mathbb{Z}_p$-Hodge thy

* $\mathbb{C}$-Hodge thy

Thm $X/\mathbb{C}$ sm proj

$$H^i_B(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^i_{dR}(X/\mathbb{C}) \simeq \bigoplus_{p+q=i} H^p(X, \Omega^q)$$

Betti

Def An $\mathbb{R}$-Hodge structure is

(1) $H_{\mathbb{R}}$: f. dim $\mathbb{R}$-v.s.

(2). $H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}$ s.t. $H^{p,q} = \overline{H^{q,p}}$

Thm (Deligne) cat of $\mathbb{R}$-H.S. $\cong$ $\{$fin. dim. $\mathbb{R}$-rep of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m\}$

Weil restriction

<u>Rmk</u> $(H^i_B \otimes_{\mathbb{Z}} \mathbb{R}$, Hodge decomp$) \in \mathbb{R}$-H.S.

$*$ p-adic Hodge thy

Notation: k char p, perfect. $W(k)$. $K_0 = W(k)[\frac{1}{p}]$
fraction field

K/K_0 fin tot ram ext.

$G_K = \mathrm{Gal}(\overline{K}/K)$.

p-adic Galois rep is $G_K \xrightarrow{\text{cont}} GL_n(\mathbb{Q}_p)$
$\circlearrowright \mathbb{Q}_p^n = V$

<u>Ex</u> χ_p: cyclotomic character $G_K \longrightarrow GL_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times$
$g \longmapsto \chi_p(g)$.

$g(\varepsilon_p) = \varepsilon_p^{a_1}$ $\quad g(\varepsilon_{p^2}) = \varepsilon_{p^2}^{a_2}, \ldots, \quad \chi_p(g) = \lim a_i \in \mathbb{Z}_p^\times$

<u>Ex</u> E/K ell curve $T_p(E) = \varprojlim_n E(\overline{K})[p^n] = \mathbb{Z}_p \oplus \mathbb{Z}_p$
$\circlearrowleft$
G_K

* Some rings in p-adic Hodge theory

$C = \widehat{\bar{K}}$ perfectoid field

$\mathcal{O}_C$ is perfectoid ring

$\mathcal{O}_C^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_C$

$= \{ (x^{(n)})_{n \geq 0} \mid (x^{(n+1)})^p = x^{(n)} \}$

Coleman norm?

= char p perfect ring

$A_{\inf} := W(\mathcal{O}_C^\flat)$

$(1, \varepsilon_p, \varepsilon_{p^2}, \ldots) = \varepsilon \in \mathcal{O}_C^\flat$

$(\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \ldots) = \varpi^\flat \in \mathcal{O}_C^\flat$

ϖ: unif of K

$\rightsquigarrow [\varepsilon], [\varpi^\flat] \in A_{\inf}$. Teichmüller lifts to Witt ring.

$$A_{\inf} = \left\{ \sum_{i=0}^{\infty} p^i [a_i],\ a_i \in \mathcal{O}_C^\flat \right\} \xrightarrow{\theta} \mathcal{O}_C$$

$$a_i = \{a_i^{(n)}\}_{n \geq 0} \qquad \longmapsto \sum_{i=0}^{\infty} p^i a_i^{(0)}$$

$\ker \theta$ is principal gen by

$$\begin{cases} \dfrac{[\varepsilon]-1}{[\varepsilon^{\frac{1}{p}}]-1} & \text{where } \varepsilon^\flat = (\varepsilon_p, \varepsilon_{p^2}, \dots) \in \mathcal{O}_C^\flat \\[2ex] E([\varpi^\flat]) & \text{where} \end{cases}$$

$$E(u) = \mathrm{Irr}(\overline{\varpi}, K_0).$$

<u>Def</u> $B_{cris} = \left(A_{inf}\left[\dfrac{(\ker\theta)^i}{i!}\right]_{i\geq 1}\right)^{\wedge p}\left[\dfrac{1}{p}\right]\left[\dfrac{1}{t}\right]$

"envelope"

$$t = \log([\varepsilon]) = \sum_{i\geq 1} \frac{([\varepsilon]-1)^i}{i}$$

(Fontaine computed the "right size" of ring)

<u>Def</u> Say a p-adic rep V is <u>crystalline</u>, if

$$\dim_{K_0}\left(V \otimes_{\mathbb{Q}_p} B_{cris}\right)^{G_K} = \dim_{\mathbb{Q}_p} V.$$

<u>Thm</u> (Colmez – Fontaine)

"Riemann-Hilbert correspondence?"

$$\mathrm{Rep}^{crys}_{\mathbb{Q}_p}(G_K) \simeq \{\text{weakly admissible filtered } \varphi\text{-mod}\}.$$

where an obj on RHS consists of

(1). D : f.d. K_0-v.s.

(2). $\varphi : D \hookrightarrow D$ semilinear $\varphi(ax) = \varphi(a)\varphi(x)$

$\forall\, a \in K_0,\ x \in D.$

(3) a decreasing filtration on $D_K = D \otimes_{K_0} K$.

<u>Ex</u> (1) χ_p is a crys rep

Say $K = \mathbb{Q}_p$. $D = \mathbb{Q}_p \cdot e$ $\quad \varphi(e) = \frac{1}{p} e$

Fil. $\quad D \supseteq D \supseteq \cdots \supseteq \underset{\mathrm{Fil}^{-1}}{\underset{\parallel}{D}} \supseteq \underset{\mathrm{Fil}^{0}.}{0} \supseteq 0 \cdots$

(2) $D_{Ex} = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2 \qquad \varphi\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$

Fil: $\quad \cdots \supseteq \mathrm{Fil}^0 \supseteq \mathrm{Fil}^1 \supseteq \mathrm{Fil}^2 \supseteq \cdots$

$\qquad\quad \mathrm{Fil}^0 = D \supseteq \mathrm{Fil}^1 = \mathbb{Q}_p e_1 \supseteq 0 \supseteq$

elliptic curve?

Thm (Faltings, Hyodo-Kato, Tsuji, Colmez-Nizioł).

$\mathfrak{X}/\mathcal{O}_K$ proper smooth formal scheme

(1) $H^i_{crys}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} K \simeq H^i_{dR}(\mathfrak{X}/\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$

(2) $H^i_{crys}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} B_{crys} \simeq H^i_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{crys}$

Rmk $\left(D = H^i_{crys}[\frac{1}{p}] \circlearrowleft \underset{\text{Frobenius}}{\varphi} \quad \middle| \quad D_K = LHS(1) \text{ has fil from } RHS(1). \right)$

$\Leftarrow$ RHS of Colmez-Fontaine thm

the corresponding Galois rep in CF thm is precisely $H^i_{et}[\frac{1}{p}]$.

* Integral p-adic Hodge theory

Fontaine-Laffaille thy treat $K=K_0$, $HT \subseteq [0, p-2]$.
($\curvearrowleft$ "naive")

Breuil-Kisin

<u>Def</u> $\mathfrak{S}$ = \mathfrak{S} $= W(k)[[u]] \circlearrowleft \varphi$

$$\varphi(u)=u^p$$

$$E(u) = \mathrm{Irr}(\varpi, K_0) \in \mathfrak{S}$$

<u>Def</u> A Breuil-Kisin module is fin. free $\mathfrak{S}$-mod $\mathfrak{M}$ together w/ $\varphi : \mathfrak{M} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}[\frac{1}{E}]$

s.t. $\varphi(ax) = \varphi(a)\varphi(x)$

and the induced map

$\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathfrak{S}[\frac{1}{E}] \xrightarrow{\varphi \otimes 1} \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}[\frac{1}{E}]$ is <u>isom</u>

of $\mathfrak{S}[\frac{1}{E}]$-modules.

Say $\mathfrak{M}$ has non-negative height if $\varphi(\mathfrak{M}) \subseteq \mathfrak{M}$.

In this case, the condition $\Leftrightarrow$ $\varphi: \mathfrak{M} \to \mathfrak{M}$ is semi-linear, and $\mathfrak{M}/\langle \varphi(\mathfrak{M})\rangle_{\mathfrak{S}}$ is killed by $E(u)^h$ for some $h \geq 0$.

Dieudonné module?

Thm (Kisin) $\exists$ fully faithful

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(G_K) \hookrightarrow \{\text{BK-mod}\}$$

essential image — Liu, Gao, BMS

Ex $\chi_p \rightsquigarrow \mathfrak{S}\{1\}$ the 1st BK-twist in some papers, but not in BMS2

$$\mathfrak{S}\{1\} = \mathfrak{S}\cdot e \quad \varphi(e) = (E)^{-1} e$$

Def A (non-φ-twisted) Breuil-Kisin-Fargues mod is a fin. free

A_{inf}-mod $\widehat{\mathfrak{M}}$, with $\varphi : \widehat{\mathfrak{M}} \to \widehat{\mathfrak{M}}[\frac{1}{E}]$.

inducing iso $1 \otimes \varphi$.

<u>Def</u> A BKF-mod is $\widehat{\mathfrak{M}} = A_{inf}^{\oplus d}$ $\varphi : \widehat{\mathfrak{M}} \to \widehat{\mathfrak{M}}[\frac{1}{\varphi(E)}]$

inducing iso $1 \otimes \varphi$.

<u>Rmk</u> Have $BK_{\mathfrak{S}} \xrightarrow[\mathfrak{S}]{\otimes A_{inf}}$ non-φ-twisted BKF-mod $\xrightarrow{\simeq} BKF$

$$\widehat{\mathfrak{M}} \mapsto \widehat{\mathfrak{M}} \otimes_{A_{inf}, \varphi} A_{inf}$$

<u>Def</u> $A_{inf}\{1\} = A_{inf}\, e$ $\quad \varphi(e) = \frac{1}{\varphi(\xi)} e$

where $\xi = \dfrac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}$ recall

$(\xi) = (E) = \ker \theta$

You can also write $\varphi(e) = \dfrac{1}{\varphi(E)} e$.

This is called BK-twist in [BMS2, §6.2].