Crystalline representations, filtered (φ, N)-modules and Breuil-Kisin modules

K CDVF $\mathcal{O}_K \ni \varpi$

$$\mathcal{O}_K \twoheadrightarrow k \quad \text{perfect char } p \qquad \begin{array}{c} K \\ | \\ K_0 = W(k)[\tfrac{1}{p}] \\ | \\ \mathbb{Q}_p \end{array}$$

$$\begin{array}{c} \overline{K} \\ | \\ K \end{array} \Big) G_K \qquad \hat{\overline{K}} = C \quad \mathcal{O}_C \qquad \mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_K/p$$

$$\mathcal{O}_{C^\flat} \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_C \ni (x^{(n)})_{n \in \mathbb{N}}$$

$$A_{\mathrm{inf}}(\mathcal{O}_C) = W(\mathcal{O}_{C^\flat}) = A_{\mathrm{inf}}$$

$$\theta : A_{\mathrm{inf}} \longrightarrow \mathcal{O}_C$$

$$(x_n) \longmapsto \sum p^i x_i^{(i)}$$

$$x_n = (x_n^{(i)})_{i \in \mathbb{N}}$$

Rmk ① $\ker\theta$ principal $\quad \xi := [\pi] + p \in W(\mathcal{O}_{C^\flat}) \ni \pi$

$$\pi^{(0)} = -p \qquad \pi = (-p, \sqrt[p]{-p}, \dots)$$

$$\theta(\xi) = \theta[\pi] + p = -p + p = 0.$$

② $[\varepsilon]-1$ $\quad \theta([\varepsilon])-1 = 1-1=0$

$\varepsilon^{(0)}=1, \quad \varepsilon=(1, \varepsilon_p, \varepsilon_{p^2}, \text{---})$
($p=2$ is a generator)

③ $\theta_{\mathbb{Q}} : A_{inf}[\frac{1}{p}] \longrightarrow C$

$\ker\theta_{\mathbb{Q}} \cap A_{inf} = \ker\theta$

Def $\quad B_{dR}^{+} = \varprojlim_{n} A_{inf}[\frac{1}{p}]/(\xi)^n \ni t := \log[\varepsilon] = \log([\varepsilon]-1+1)$

$A_{cris} = \widehat{A_{inf}[\frac{\xi^n}{n!}]}^{p}$ $\qquad \ni \qquad = \sum_{n\geq 1} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n}$

$([\varepsilon]-1)^{(n)}(n-1)!$

Rmk: ① $t \in A_{cris}$

② $\theta_{dR} : B_{dR}^{+} \twoheadrightarrow A_{inf}[\frac{1}{p}]/\xi \simeq C$

③ B_{dR}^{+} : CDVR $\quad t$ is a uniformizer

④ $B_{dR} := \operatorname{Frac} B_{dR}^{+}$

⑤ $B_{cris} = A_{cris}[\frac{1}{t}]$ $\quad B_{cris} \hookrightarrow K \otimes_{k} B_{cp} \supset B_{dR}$

Def $\quad \operatorname{Fil}^{i} B_{dR} := B_{dR}^{+} \cdot t^{i}$

Rmk ① $g(t) = \chi(g) \cdot t$

② $$\begin{array}{c} \mathbb{Z}_p \cdot t \subseteq B_{dR}^{+} \\ \wr\!\wr \\ \mathbb{Z}_p(1) \end{array}$$

§ Galois repn

G top gp. B top ring with continuous G-action

Def $(\mathrm{Rep}_B(G))$ finite free B-module + continuous semi-linear G-action

Rmk ① $\mathrm{Rep}_B^d(G) \overset{(\text{exe})}{\simeq} H^1_{\mathrm{cont}}(G, GL_d(B))$

② $G = G_K$, $B = \mathbb{Q}_p, \mathbb{Z}_p$

$\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ $\quad \rho : G_K \xrightarrow{\text{continuous}} GL_d(\mathbb{Q}_p)$

$\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$

Def $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(G_K)$. $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ if $\dim_{\mathbb{Q}_p} V = \dim_{K_0} (V \otimes B_{\mathrm{cris}})^{G_K}$

Rmk $B_{\mathrm{st}}^{G_K} = B_{\mathrm{cris}}^{G_K} = K_0$, $B_{dR}^{G_K} = K$

semistable? $D_{cris}(V)$

$\mathrm{Rep}^{st}_{\mathbb{Q}_p}(G_K)$ —— $\dim_{\mathbb{Q}_p} V = \dim (V \otimes B_{st})^{G_K}$

$\mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K)$ $(\sim B_{dR})$

§. Filtered (φ, N)-module

<u>Def</u> (Fil_K) a Filtered K-module is a K-vector space D

with $\{\mathrm{Fil}^i D\}$ s.t.
① decreasing $\mathrm{Fil}^{i+1} D \subseteq \mathrm{Fil}^i D$
② separated $\bigcap \mathrm{Fil}^i D = 0$
③ exhaustive $\bigcup \sim = D$.

($\Longleftrightarrow$ $i \ll 0$, $\mathrm{Fil}^i = D$
$i \gg 0$, $\mathrm{Fil}^i = 0$

<u>Def</u> ($\mathrm{Mod}_{K_0}(\varphi, N)$) a (φ, N)-module over K_0 is

a K_0-vector space + a semi-linear inj

($\overset{\text{exe}}{\Longleftrightarrow}$ iso)

Frobenius action + K_0-linear N s.t.

$p\varphi N = N\varphi$.

filtered (φ,N)-module

Def ($MF_K(\varphi,N)$) a filtered (φ,N)-module is a

(φ,N)-module over K_0, (D,φ,N)

$+\ \mathrm{Fil}^i D_K \in \mathrm{Fil}_K$

$D_K := K \bigotimes_{K_0} D$

Def (1) $\forall D \in \mathrm{Fil}_K$ $\quad t_H(D) \overset{\dim=1}{:=} \min\{i \mid \mathrm{Fil}^i D = D\}$

$= \max\{i \mid \mathrm{gr}^i D \neq 0\}$

$t_H(D) \overset{\dim=d}{:=} t_H(\Lambda^d D)$

$\uparrow$ 1-dim K_0

(2) $\forall D \in MF(\varphi,N)$, $\quad t_N(D) \overset{\dim 1}{:=} v_p(\lambda)$

$\varphi(e) = \lambda \cdot e$

$\in K_0^\times$

$t_N(D) \overset{\dim d}{=} t_N(\Lambda^d D)$

$(\Leftrightarrow v_p(\det(A_\varphi)))$

(3) $D \in MF_K(\varphi,N)$ is admissible if $\quad MF^{ad}_{(\varphi,N)}$

① $t_N(D) = t_H(D)$

② $D' \subseteq D$ subobj $\quad t_N(D') \geq t_H(D')$

(Fontaine $\iff$ $P_N(D')$ over $P_H(D')$)

(4) ______ <u>effective</u> if $\mathrm{Fil}^0 D = D$.

<u>Rmk/Ex</u> dim 1 admissible (φ, N)-module

$$D_\lambda \quad \boxed{\mathrm{Fil}^i D = \begin{cases} D & i \leq v_p(\lambda) \\ 0 & i > v_p(\lambda) \end{cases}} \quad t_H(D) = r$$

$v_p(\lambda) \quad \varphi e = \lambda e$

$$D_\lambda \simeq D_{\lambda'} \iff \lambda' = \lambda \frac{\sigma(a)}{a} \qquad a \in W(k)^\times$$

$$\mathbb{Q}_p(i) = \mathbb{Q}_p \cdot t^i \qquad \varphi(t) = pt$$

$$g(t) = \chi(g) \cdot t$$

§ equivalence of cats

$$D_{dR} \qquad \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{Fil}_K$$

$$V \longmapsto (V \otimes_{\mathbb{Q}_p} B_{dR})^G$$

$$\underbrace{\qquad\qquad}_{D_{dR}(V)}$$

$$\mathrm{Fil}^i D_{dR}(V) := (\mathrm{Fil}^i B_{dR} \otimes V)^G$$

Thm (Colmez-Fontaine)

$\underline{D_{st}}$: $\quad \mathrm{Rep}^{st}_{\mathbb{Q}_p}(G_K) \xhookrightarrow{\sim} MF^{ad}(\varphi, N)$

(Dcrys)

$$V \longmapsto (V \otimes B_{st})^{G_K}$$

$$\ldots \longleftarrow D$$

$\underline{\text{Def}}$ (Hodge-Tate weights) $V \in \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_K)$

$$\mathrm{Fil}^i D_{dR}(V) \qquad \{i,\ gr^i D_{dR}(V) \neq 0\}$$

§ Breuil-Kisin module

$$MF^{eff}_K(\varphi, N) \underset{\underline{D}}{\overset{\underline{\mathcal{M}}}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}}(\varphi, N_\nabla)$$

$$\mathcal{O} \subseteq K_0[[u]]$$

$$\cup\!|\qquad\qquad\qquad \cup\!|$$

$N := N_\nabla \bmod \varphi$

$$MF^{eff,ad}_K(\varphi, N) \xrightarrow{\sim} \mathrm{Mod}^{o}_{\mathcal{O}}(\varphi, N_\nabla) \xhookrightarrow{\Psi} \mathrm{Mod}_{\mathcal{O}}(\varphi, N)$$

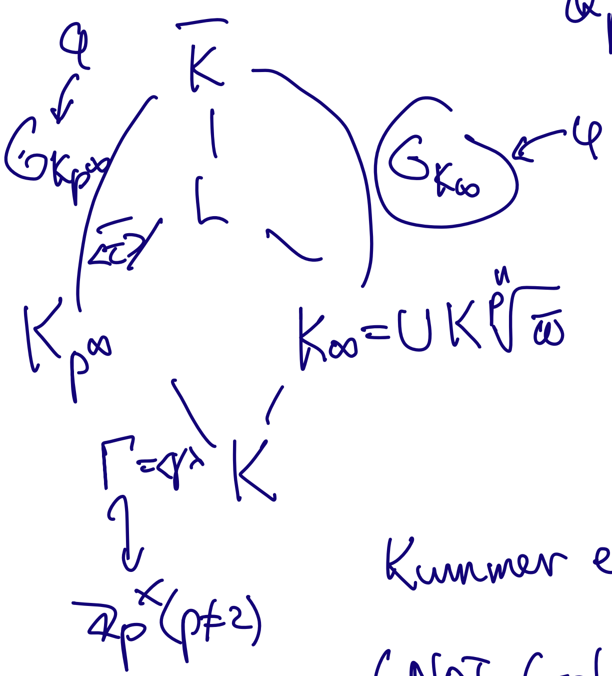
$$\Big\downarrow \text{(H)}$$

$$\mathrm{Mod}_{\mathfrak{S}}(\varphi, N) \otimes \mathbb{Q}$$

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_{K_\infty}) \underset{\text{Fontaine}}{\overset{N=0}{\simeq}}$$

<u>Prop</u> (Kisin)

$$\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \hookrightarrow \mathrm{Rep}_{\mathbb{Q}_p}(G_{K_\infty})$$



Kummer exten

(NOT Galois)

$\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ $\qquad$ (φ, Γ)

$\mathrm{Rep}^{\mathrm{st}}(G_K)$ $\qquad$ $\mathrm{MF}^{\mathrm{ad}}(\varphi, N)$

$\mathrm{Rep}^{\mathrm{crys}}(G_K)$ $\qquad$ $\mathrm{MF}^{\mathrm{ad}}(\varphi)$

Fontaine–Laffaille $\qquad$ (φ, τ)-module

Wach

$(\varphi, \hat{a})$ Liu, Gao

BK-module

Rmk ① finite flat gp scheme $/\mathcal{O}_K$

② p-div $/\mathcal{O}_K$

③ many applications