Let R perfd ring.

$$\mathrm{THH}(R,\mathbb{Z}_p) \xrightarrow{\varphi_p} \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}$$

$$\mathrm{TC}^-(R;\mathbb{Z}_p) \xrightarrow{\varphi_p^{h\mathbb{T}}} \mathrm{TP}(R;\mathbb{Z}_p)$$

$$A_{\mathrm{inf}}(R)[u,v]/(uv-\xi) \simeq \mathrm{TC}^-_*(R;\mathbb{Z}_p) \xrightarrow{\mathrm{can}} \mathrm{TP}_*(R;\mathbb{Z}_p)$$

$$\simeq A_{\mathrm{inf}}[\sigma,\sigma^{-1}]$$

A_{inf}-linear

$u \longmapsto \xi\sigma$

$v \longmapsto \sigma^{-1}$

$$\begin{tikzcd}
A_{\mathrm{inf}}(R)[u,v]/(uv-\xi) \simeq \mathrm{TC}^-_*(R;\mathbb{Z}_p) \arrow[r, "\varphi_p^{h\mathbb{T}}", "{\substack{u\mapsto\sigma \\ v\mapsto\varphi(\xi)\sigma^{-1} \\ \varphi_{A_{\mathrm{inf}}}\text{-linear}}}"'] \arrow[d, "{\substack{\boxed{v\mapsto 0} \\ u\mapsto u \\ \theta\text{-linear}}}"', "\text{Fontaine}"] & \mathrm{TP}_*(R;\mathbb{Z}_p) \simeq A_{\mathrm{inf}}[\sigma,\sigma^{-1}] \arrow[d, "{\substack{\sigma\mapsto\sigma \\ \theta\circ\varphi^{-1}\text{-linear} \\ \| \\ \tilde{\theta}}}"] \\
R[u] \simeq \mathrm{THH}_*(R;\mathbb{Z}_p) \arrow[r, "\varphi_p"', "{\substack{R\text{-linear} \\ u\mapsto\sigma}}"'] & \pi_*\mathrm{THH}(R;\mathbb{Z}_p)^{tC_p} \simeq R[\sigma,\sigma^{-1}]
\end{tikzcd}$$

$h\mathbb{T}$

$$\begin{tikzcd}
A_{inf}(R)\{-1\} \oplus A_{inf}(R) \oplus \mathcal{N}^{\geq 1} A_{inf}(R)\{1\} \oplus \cdots \arrow[r, "\varphi_p^{hT}"] \arrow[d] & A_{inf}(R)\{-1\} \oplus A_{inf}(R) \oplus A_{inf}(R)\{1\} \oplus \cdots \arrow[d, "\tilde{\theta}\text{-linear}"] \\
R \oplus N^1 \oplus N^2 \oplus \cdots \arrow[r, "\varphi_p"] & R\{-1\} \oplus R \oplus R\{1\} \cdots
\end{tikzcd}$$

$$TP_2(R;\mathbb{Z}_p) \otimes TP_{-2}(R;\mathbb{Z}_p) \cong TP_0$$

$$TP_0$$

$A_{inf}(R)\{1\}$ = <u>Breuil-Kisin twist</u>

$$\pi_2 \varphi_p^{hT} : \mathcal{N}^{\geq 1} A_{inf}(R)\{1\} \longrightarrow A_{inf}(R)\{1\}$$

" ‖ " divided Frobenius

$$\varphi/\varphi(\xi) \quad \mathcal{N}^{\geq 1} A_{inf} \longrightarrow A_{inf}$$

$$TP(R;\mathbb{Z}_p) \underset{TC^-(R;\mathbb{Z}_p)}{\otimes} THH(R;\mathbb{Z}_p) \xrightarrow{\sim} THH(R;\mathbb{Z}_p)^{tC_p}$$

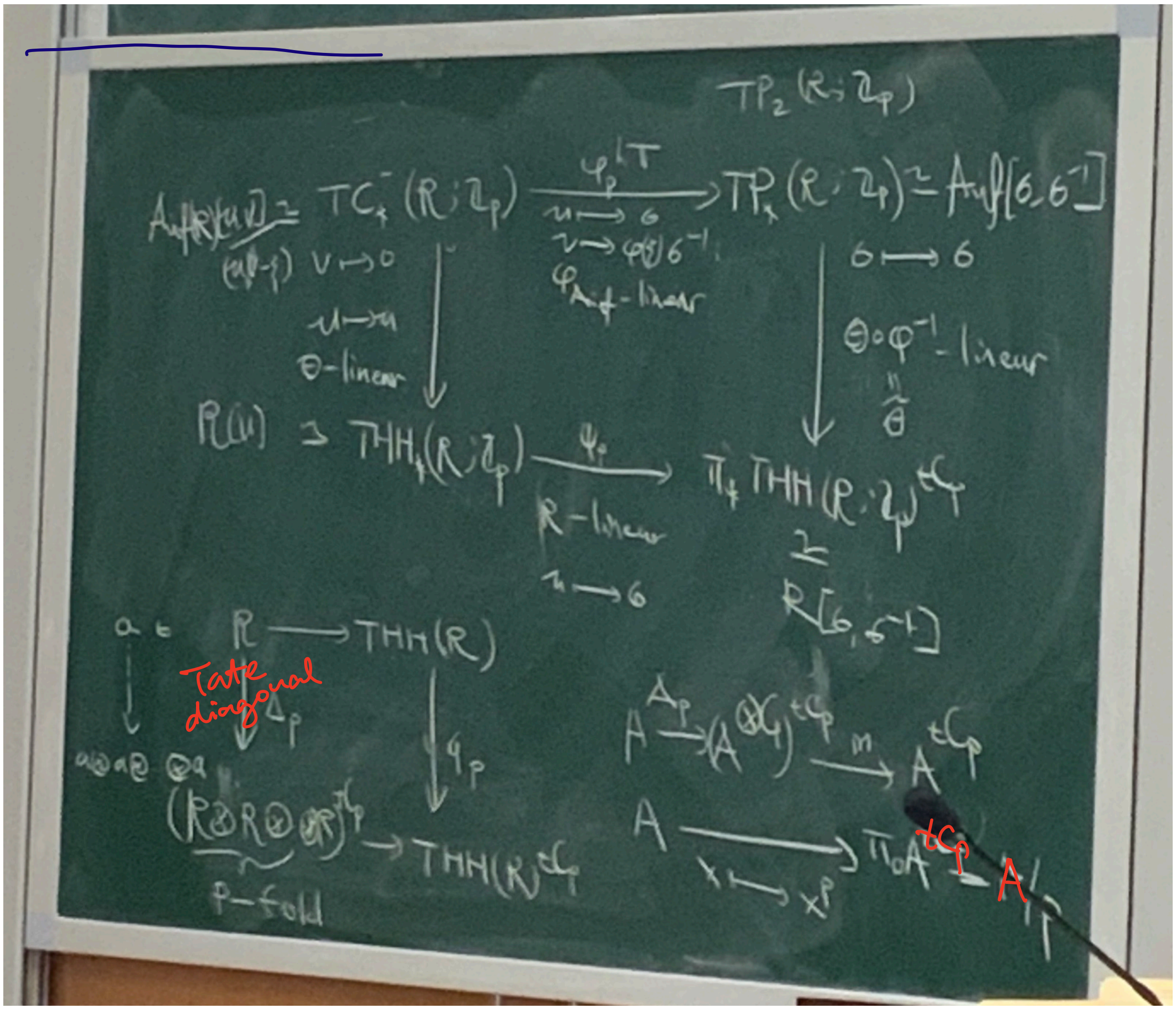
$\mathbb{Z}$ with trivial S^1-action

$$\begin{tikzcd}
\pi_* \mathbb{Z}^{hS^1} \arrow[d, phantom, "\parallel"] & = \mathbb{Z}[v], \quad |v| = -2 \\
H^{-*}(BS^1;\mathbb{Z}) &
\end{tikzcd}$$

$$BS^1 = \mathbb{CP}^\infty$$

$$\mathbb{Z}^{hS^1}[-2] \xrightarrow{v} \mathbb{Z}^{hS^1} \longrightarrow \mathbb{Z}$$

Tate periodicity.



$$TP_2(R;\mathbb{Z}_p)$$

$$\begin{tikzcd}
A_{inf}(R)[u,v]/(uv-\xi) \cong TC^-_*(R;\mathbb{Z}_p) \arrow[r, "\varphi_p^{hT}", "{u \mapsto \sigma,\ v \mapsto \varphi(\xi)\sigma^{-1},\ \varphi_{A_{inf}}\text{-linear}}"'] \arrow[d, "{v \mapsto 0,\ u \mapsto u,\ \theta\text{-linear}}"'] & TP_*(R;\mathbb{Z}_p) \cong A_{inf}[\sigma,\sigma^{-1}] \arrow[d, "{\sigma \mapsto \sigma,\ \theta \circ \varphi^{-1}\text{-linear} = \tilde{\theta}}"] \\
R[u] \cong THH_*(R;\mathbb{Z}_p) \arrow[r, "\varphi_p", "{R\text{-linear},\ u \mapsto \sigma}"'] & \pi_* THH(R;\mathbb{Z}_p)^{tC_p} \cong R[\sigma,\sigma^{-1}]
\end{tikzcd}$$

$$\begin{tikzcd}
a \in R \arrow[d, maps to] \arrow[r] \arrow[d, phantom, shift left=4, "\Delta_p"] & THH(R) \arrow[d, "\varphi_p"] \\
a\otimes a\otimes \cdots \otimes a \in \underbrace{(R\otimes R\otimes \cdots \otimes R)}_{p\text{-fold}}^{tC_p} \arrow[r] & THH(R)^{tC_p}
\end{tikzcd}$$

Tate diagonal

$$\begin{tikzcd}
A \arrow[r, "\Delta_p"] & (A^{\otimes C_p})^{tC_p} \arrow[r, "m"] & A^{tC_p}
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "x \mapsto x^p"'] & \pi_0 A^{tC_p} \cong A/p
\end{tikzcd}$$

THH for QRSPerfd

$S \in \mathrm{QRSPerfd}$

$R \to S$, $\quad R \in \mathrm{Perfd}$

$$\begin{tikzcd}
\underline{THH(S;\mathbb{Z}_p)} \arrow[d, rightsquigarrow, "\text{reduced to}"] \\
HH(S/R;\mathbb{Z}_p)
\end{tikzcd}$$

HKR-Fil

R base ring $\qquad A = R$-algebra

$\mathrm{Fil}^{*}_{HKR}\, HH(A/R)$ complete multiplicative $\mathbb{Z}_{\geq 0}$-indexed decreasing

$$\mathrm{gr}^{i}_{HKR}\, HH(A/R) = \Lambda^{i} L_{A/R}[i]$$

If $A \in \mathrm{Poly}_R$, $\quad \mathrm{Fil}^{*}_{HKR}\, HH(A/R) = \tau_{\geq *} HH(A/R)$

polynomial algebra $\qquad$ (HKR-thm)

left Kan extension

$$HH(A/R) \simeq A \otimes_{A \otimes_R A} A$$

commutes with colimits

general

HKR can be obtained by QSyn descent.

$$(\Lambda^n L_{-/R})^\wedge_p : \mathrm{Shv}(\mathrm{qrsPerfd}_R, D(R))$$

$R \in \mathrm{QSyn}$ $\qquad$ $S \in \mathrm{qrsPerfd}_R \Leftrightarrow$

$\mathrm{qrsPerfd}_R$

1) $R \to S$ p-completely flat

2) $L^1_{S/R} \otimes^L_S S/p \in D(S/p)$

is a flat S/p-module

concentrated in cohomlg'l degree -1.

i.e.,

$$(L^1_{S/R})^\wedge_p = M[1]$$

M a $^\vee$ flat S/p-module

p-completely

More generally,

$$(\Lambda^n L_{S/R})^\wedge_p = (\Gamma^n_S M)^\wedge_p [n]$$

divided
powers

p-completely flat module

$(\Lambda^n L_{-/R})^\wedge_p$ for Quasismooth R-algebras

A quasismooth over R:

1) $R \to A$ p.c. flat

2) $L'_{A/R} \otimes^L_A A/p \in D(A/p)$

flat concentrated in degree 0

$$(\Lambda L^n_{A/R})^\wedge_p = (\Omega^n_{A/R})^\wedge_p$$

Divided powers:

(A, M)

comm ring

A-module

$$M^{\otimes_A \Sigma_n}$$

Orbits $\rightsquigarrow$ Sym^n

signed orbits ⇝ Λ^n

fixed points ⇝ Γ^n

$$LSym^n_A(\Sigma M) \simeq \Sigma^n \Lambda^n_A(M), \quad \Lambda^n_A(\Sigma M) \simeq \Sigma^n \Gamma^n_A(M)$$

"Relative de Rham comparison"

$R \in \mathrm{Perfd}$ $\qquad A$ an R-algebra

$\exists$ $\mathbb{T}$-equivariant cofiber sequence $\quad Sp^{B\mathbb{T}}$

$$THH(A;\mathbb{Z}_p)[2] \xrightarrow{u} THH(A;\mathbb{Z}_p) \longrightarrow HH(A/R;\mathbb{Z}_p)$$

$u \in THH_2(R;\mathbb{Z}_p)$ a generator

Since

$$HH(A/R;\mathbb{Z}_p) \simeq THH(A;\mathbb{Z}_p) \otimes_{THH(R;\mathbb{Z}_p)} R$$

the above cofiber sequence can be obtained via base change from

$$THH(R;\mathbb{Z}_p)[2] \xrightarrow{u} THH(R;\mathbb{Z}_p) \to R.$$

$u \in TC^-_2(R;\mathbb{Z}_p)$ ⇝ $\mathbb{S}[2] \xrightarrow{u} TC^-(R;\mathbb{Z}_p)$.

$\rightsquigarrow$ $\quad\Updownarrow$ adjoint

$$\mathbb{S}[2] \xrightarrow{u} THH(R;\mathbb{Z}_p)$$

left adjoint

$$THH(R;\mathbb{Z}_p)[2] \xrightarrow{u} THH(R;\mathbb{Z}_p)$$

$\mathbb{T}$- equivariant

$\mathbb{T}$-equivariant $THH(R;\mathbb{Z}_p)$-module

$$THH(R;\mathbb{Z}_p)[2] \xrightarrow{u} THH(R;\mathbb{Z}_p) \longrightarrow R$$

$$\begin{tikzcd}
 & R \arrow[d, "u"] & Ru \arrow[d, "u"] & Ru^2 \\
R & Ru & Ru^2 &
\end{tikzcd}$$

Taking (general cofiber seq)

$$hS^1: \quad TC^-(A;\mathbb{Z}_p)[2] \xrightarrow{u} TC^-(A;\mathbb{Z}_p) \longrightarrow HC^-(A/R;\mathbb{Z}_p)$$

$$tS^1: \quad TP(A;\mathbb{Z}_p)[2] \xrightarrow{\wr\,\sigma} TP(A;\mathbb{Z}_p) \longrightarrow HP(A/R;\mathbb{Z}_p)$$

$S \in QRSPerfd_R \qquad R \in Perfd$

$$THH(S;\mathbb{Z}_p)[2] \xrightarrow{u} THH(S;\mathbb{Z}_p) \longrightarrow HH(S/R;\mathbb{Z}_p)$$

$$\begin{tikzcd}
\pi_{-2}THH \arrow[r, "u"] & \pi_0 THH \arrow[r] & \pi_0 HH \\
\arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & {} \arrow[d, dash] & & \\
 & \pi_1 HH & \pi_1 THH \arrow[l] & \pi_{-1} THH \arrow[l] \\
{} \arrow[r] & \pi_0 THH \arrow[r] & \pi_2 THH \arrow[r] & \pi_2 HH \simeq (L_{S/R})^\wedge_p[-1] \arrow[u]
\end{tikzcd}$$

HKR fil on $HH(S/R;\mathbb{Z}_p)$

$$\pi_{2n} HH(S/R;\mathbb{Z}_p) = (\Lambda^n L_{S/R})^\wedge_p [n]$$

$$\pi_{odd} = 0$$

LES ⇝ evenness of $THH_{(S)}$ hmtp gps.

$\Longrightarrow$ $THH(S;\mathbb{Z}_p)$ is even

homotopy fixed point SS $\Longrightarrow$ $TC^-(S;\mathbb{Z}_p)$ even

Tate SS $TP(S;\mathbb{Z}_p)$