Joint with 胡永泉

I) Motivation

Classical local Langlands & Jacquet-Langlands (local)

for $GL_2(\mathbb{Q}_p)$ (works for $GL_n(L)$, $L/\mathbb{Q}_p$ finite)

$$\left\{\begin{array}{c}\text{2-dim'l Frobenius}\\ \text{semisimple Weil-Deligne}\\ \text{reps } /\mathbb{C}\end{array}\right\} \overset{\text{LLC}}{\hookrightarrow} \left\{\begin{array}{c}\text{irred. adm.}\\ \text{smooth } \mathbb{C}\text{-reps}\\ \text{of } GL_2(\mathbb{Q}_p)\end{array}\right\}$$

image = discrete series i.e. special series & supercuspidal.

$$\Big\uparrow \text{JL}$$

$$\left\{\begin{array}{c}\text{irred. adm smooth}\\ \mathbb{C}\text{-reps of } D^\times\end{array}\right\},$$

D = quaternion alg $/\mathbb{Q}_p$

Question. Replace $\mathbb{C}$-coeffs by $\overline{\mathbb{F}}_p$-coeffs?

$$\left\{\begin{array}{c}\bar\rho : G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{F}}_p)\\ \text{continuous}\end{array}\right\} \underset{\text{Colmez}}{\overset{\text{Breuil}}{\hookrightarrow}} \left\{\begin{array}{c}\text{(certain)}\\ \text{adm. smooth } \overline{\mathbb{F}}_p\text{-reps}\\ \text{of } GL_2(\mathbb{Q}_p)\end{array}\right\}$$

$G_{\mathbb{Q}_p} = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

(only for $GL_2(\mathbb{Q}_p)$)

$$\mathrm{Rep}^{\text{adm.sm}}_{\overline{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p))$$

$$\Big\uparrow ?$$

(certain) admissible k' :

$$\mathrm{Rep}^{\text{adm, sm}}_{\overline{\mathbb{F}}_p}(D^\times)$$

for **irred** rep'ns ;- irred rep'ns $D^\times$ are of dim ≤ 2

程创勋 : $\mathrm{Irr}_{\overline{\mathbb{F}}_p}(D^\times) \hookrightarrow \mathrm{Irr}_{\overline{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p))$

2-dim'l $\longleftrightarrow$ supersingular

uses $\{$2-dim'l irred rep'z of $G\mathbb{Q}_p\}$ $\xleftrightarrow{1-1}$ $\{$2-dim'l irred rep'z of $D^\times\}$

Problem : This is not the correspondence we want.

we want a correspondence which matches with global.

II) Two approaches. global & local.

- Global approach : motivated by the work of Buzzard-Diamond-Jarvis

| Emerton on GL_2-side

$\bar\rho$ $\quad \{\bar\rho : G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{F}}_p) \text{ cts}\}$ $\quad \bar\rho$

global method

$LL(\bar\rho)$ (certain) admissible $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p)) \longleftarrow\text{-}\text{-}\text{-}\longrightarrow$ (certain) admissible $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(D^\times)$ $\quad JL(\bar\rho)$

| Breuil, Colmez, Emerton

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
$\parallel$

- $\bar\rho$ globalize to a modular $\bar r : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$

  i.e. $\quad \bar r|_{G_{\mathbb{Q}_p}} \cong \bar\rho$

- choose B quaternion algebra $/\mathbb{Q}$,
  - ramified at p, $\quad B \otimes_{\mathbb{Q}} \mathbb{Q}_p = D$
  - split at ∞

$$G_{\mathbb{Q}} \times D^\times \times \mathbb{T}$$
$\cup$
$\mathfrak{m}_{\bar r}$ maximal ideal

$$\mathrm{Hom}_{G_{\mathbb{Q}}}\left(\bar{r}, \boxed{\varinjlim_{U_p} H^1_{et}\left(Sh_B(U^p U_p)_{\bar{\mathbb{Q}}}, \bar{\mathbb{F}}_p\right)[m_{\bar{r}}]}\right)$$

$U^p U_p \subset B^\times(\mathbb{A}_f)$ $\qquad \mathcal{O}_D^\times$

$= \underline{JL(\bar{\rho})}^{\oplus d}$. $\qquad \supset \underline{\underline{D^\times}}$

Aim: understand the structure of $\underline{JL(\bar{\rho})}$.

<u>Khare</u>: studied the quaternionic Serre wts of $\bar{\rho}$. $\left(\mathrm{soc}_{\mathcal{O}_D^\times}(JL(\bar{\rho}))\right)$

<u>Thm</u> (Breuil-Diamond, Scholze)

$JL(\bar{\rho})$ is infinite dim'l over $\bar{\mathbb{F}}_p$.

(hence is of infinite length.)

- Local <u>approach</u> $\qquad$ Scholze (ASENS 2018) $\quad \mathbb{Y}_n$

cohomological covariant δ-functor | (works for $GL_n(L)$) $\qquad$ adm. sm.

$$S^i : \mathrm{Rep}^{adm, sm}_{\bar{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p)) \longrightarrow \mathrm{Rep}_{\bar{\mathbb{F}}_p}(G_{\mathbb{Q}_p} \times D^\times)$$

$0 \le i \le 2$; $\quad \delta^i(\pi) = 0$, $i \ge 3$

JL correspondence should be given by δ^1 for interesting rep$^{\text{ns}}$.

Recall: construction of Scholze.

$\pi \in \mathrm{Rep}^{\text{adm, sm}}_{\overline{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p))$

$\mathcal{M}_{LT,\infty}$ $\quad \underline{\pi}$ constant sheaf

$$\begin{tikzcd} \mathcal{M}_{LT,\infty} \arrow[d, "\pi_{GH}"] \arrow[loop left, "GL_2(\mathbb{Q}_p)"] \\ \mathbb{P}^1 \end{tikzcd}$$

$\mathcal{F}_\pi := \underbrace{\left(\pi_{GH*}(\underline{\pi})\right)^{GL_2(\mathbb{Q}_p)}}$

$\delta^i(\pi) := H^i_{et}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_\pi)$

$\uparrow$ should be hard to compute

$G_{\mathbb{Q}_p} \times D^\times$ acts

Local-global compatibility (Scholze, Paskunas)

$\delta^1(LL(\bar\rho)) \cong \bar\rho \otimes JL(\bar\rho)$

with equality if $\bar{\rho}^{ss} \neq \omega \oplus 1$

Scholze : $\mathcal{S}^0(\pi) = \mathcal{S}^0(\pi^{SL_2(\mathbb{Q}_p)})$

$\mathcal{S}^i = 0 \quad i \geq 3$

Ludwig : $\mathcal{S}^2(\underbrace{\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \chi}) = 0$

works for $GL_2(L)$

$\mathcal{S}^2(St) = 0$

Ludwig - Johansson ($GL_n(L)$)

(Ludwig + miracle flatness)

Paskunas : $\mathcal{S}^1(LL(\bar{\rho}))$ has Gelfand-Kirillov

dim 1 over $\bar{\mathbb{F}}_p[[U_D^1]]$

U_D^1 pro-p Sylow of $\mathcal{O}_D^\times$

when $\bar{\rho}^{ss} = \chi_1 \oplus \chi_2$,

$\chi_1 \chi_2^{-1} \neq \omega^{\pm 1}$

recall $0 \to PS_1 \to LL(\bar{\rho}) \to PS_2 \to 0.$

Hansen - Mann, $\mathcal{S}^1(PS_1)$, $\mathcal{S}^1(PS_2)$

has GK-dim 1.

Gelfand-Kirillov dim $\quad \Big| \quad \dim_{\overline{\mathbb{F}}_p} \pi^{K_n}, \quad K_n \subset \mathcal{O}_D^\times$

measures the size of π adm

GKdim(π) $\quad \pi^\vee$ coherent sheaf over $\overline{\mathbb{F}}_p[[U_D^1]]$

= dim of the "support" of

III) Main results

Classification of irred. adm. sm. $\overline{\mathbb{F}}_p$-rep of $GL_2(\mathbb{Q}_p)$

(Barthel-Livné, Breuil)

	$\pi \in \mathrm{Irr}_{\overline{\mathbb{F}}_p}(GL_2(\mathbb{Q}_p))$	$S^0(\pi)$	$S^1(\pi)$	$S^2(\pi)$
Scholze	$\chi \circ \det$	1-dim	0	1-dim
Ludwig	$St \otimes \chi \circ \det$	0	?	0
Paškūnas	$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2)$ irred	0	?	0
	supersingular	0	$\bar{\rho} \otimes JL(\bar{\rho})$	0

Thm A (Hu-W.)

$\delta^2(\pi) = 0$ for (generic) supersingular rep<u>ns</u>.

i.e. $\delta^2(\mathrm{LL}(\bar{\rho})) = 0$, $\bar{\rho}$ irred. generic.

Thm B (Hu-W.)

for $\bar{\rho}$ in Thm. A.

$\mathrm{JL}(\bar{\rho})$ has GK-dim 1.

Idea of proof: Paskunas: $\delta^2 = 0 \Rightarrow \delta^1(\mathrm{LL}(\bar{\rho}))$ has GK-dim 1.

" $\Leftarrow$ " is also possible (need to use Gee-Newton)

Hence Thm. B $\Rightarrow$ Thm. A.

For Thm B, the strategy developed in

Breuil-Herzig-Hu-Morra-Schraen 2020

on $\mathrm{GL}_2(\mathbb{Q}_{p^f})$-rep<u>ns</u>

also works in our situation.

$\left(\underline{JL(\bar\rho)}\, [\mathfrak{m}^3_{u_0^1}] \right)$

on S^1, less explicit result.

Thm C

$\boxed{G_{\mathbb{Q}_p} \times D^\times}$

$\bar\rho \sim \begin{pmatrix} \omega & * \\ & 1 \end{pmatrix}$

π	$S^1(\pi)$
$\chi \circ \det$	0
$\mathrm{St} \otimes \chi \circ \det$	$1_{G_{\mathbb{Q}_p}} \otimes 1_{D^\times} - 1_{G_{\mathbb{Q}_p}} \otimes \underline{JL(\bar\rho)} - \omega^{-1} \otimes 1_{D^\times}$
$\mathrm{Ind}_B^G[\chi_1 \otimes \chi_2]$ irred.	$\chi_2 \otimes \underline{JL(\bar\rho)}$, $JL(\bar\rho) = JL(\bar\rho^{r.s.})$
supersingular	$\bar\rho \otimes \underline{JL(\bar\rho)}$

(Hansen–Mann.)