Theorem (Breuil-Kisin Cohomology) There is a $\mathfrak{S}$-linear cohomology theory $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ equipped with a φ-linear Frobenius $\varphi : R\Gamma(\mathfrak{X}) \longrightarrow R\Gamma(\mathfrak{X})$, s.t.

1) $$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S}} A_{inf} \simeq R\Gamma_{A_{inf}}(\mathfrak{X}_{O_C}),$$

and φ induces

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S},\varphi} \mathfrak{S}[\tfrac{1}{E}] \xrightarrow{\sim} R\Gamma_{\mathfrak{S}}(\mathfrak{X})[\tfrac{1}{E}]$$

moreover, after scalar extension to $A_{inf}[\frac{1}{\mu}]$ one recovers étale cohomology.

2) After scalar extension along $\tilde{\theta} := \hat{\theta} \circ \varphi : \mathfrak{S} \to O_K$

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{L}_{\mathfrak{S}} O_K \simeq R\Gamma_{dR}(\mathfrak{X}/O_K)$$

3) After scalar extension along $\mathfrak{S} \xrightarrow[\varphi_{W(k)}\text{-linear}]{u \mapsto 0} W(k)$

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{L}_{\mathfrak{S}} W(k) \simeq R\Gamma_{crys}(\mathfrak{X}_k / W(k))$$

4) If $\mathfrak{X}$ affine, $\mathfrak{X} = Spf(R)$

$$\Omega^{i}_{R/O_K}\{-i\} \cong H^{i}(R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{L}_{\mathfrak{S},\tilde{\theta}} O_K)$$

1. Topological Hochschild homology

$P \in K_0(R)$, $\mathrm{Hom}_R(P, P) \simeq P \underset{R}{\otimes} \mathrm{Hom}_R(P, R)$

$\xrightarrow{ev} R/[R,R]$

$id_P \longmapsto tr(P) \in R/[R,R]$

Hattori-Stallings trace

$R \underset{\mathbb{Z}}{\otimes} R$

$R \underset{\mathbb{Z}}{\otimes} R^{op}$

$\rightsquigarrow \quad K(R) \to HH(R)$.

$\underline{THH(-)}$

Let R be a perfectoid ring.

$\mathcal{O}_C \qquad C = \widehat{\overline{\mathbb{Q}}}_p$

$THH(R, \mathbb{Z}_p) \simeq R \otimes \Sigma^\infty_+ \Omega S^3$

$\Longleftrightarrow$

$THH_*(R, \mathbb{Z}_p) \simeq R[u] \qquad |u| = 2.$

$THH(R;\mathbb{Z}_p)$ $\circlearrowleft$

$\mathbb{T} = S^1$ -action

$\varphi_p : THH(R;\mathbb{Z}_p) \longrightarrow THH(R,\mathbb{Z}_p)^{tC_p}$ cyclotomic Frobenius

taking fixed points $\rightsquigarrow$ deformations

$\pi_0(THH(R;\mathbb{Z}_p)^{h\mathbb{T}}) \simeq A_{inf}(R)$

THH :

<u>QSyn</u>

$\overset{\cup}{R}$ $(L\Omega^1_{R/\mathbb{Z}_p}) \overset{L}{\underset{R}{\otimes}} R/pR \in D(R/pR)$

has Tor amplitude in $[-1,0]$.

$THH \in Shv(QSy, Sp)$

and is <u>even</u> with respect to the canonical t-structure on $Shv(QSyn, Sp)$

Def Motivic filtration

$$\mathrm{Fil}^i_M \, THH(-;\mathbb{Z}_p) := \tau_{\geq 2i} THH(-;\mathbb{Z}_p)$$

$$\mathrm{Fil}^i_M \, TC^-(-;\mathbb{Z}_p) := \tau_{\geq 2i} TC^-(-;\mathbb{Z}_p)$$

$$\mathrm{Fil}^i_M \, TP(-;\mathbb{Z}_p) = \cdots$$

$$\mathrm{Fil}^i_M \, TC(-;\mathbb{Z}_p) = \cdots$$

compatible w/ fil on A_{inf}

HFPSS ker θ

2. Nygaard-completed absolute prismatic cohomology

$$\hat{A}\_\{i\} := \underbrace{gr^i_M \, TP(-;\mathbb{Z}_p)[-2i]}_{\text{BK-twist}}$$

w/ $\frac{\varphi}{p^n}$ char p

$\frac{\varphi}{\varphi(\xi)^n}$ char 0.

$$N^{\geq i} \hat{A}\_\{i\} = gr^i_M \, TC^-(-;\mathbb{Z}_p)[-2i]$$

Nygaard filtration

$$\mathbb{Z}_p(i)(-) := gr^i_M \, TC(-;\mathbb{Z}_p)[-2i]$$

divided Frobenius fixed points

"eigenspace of Frobenius"

<u>HH</u> $\in Shv(Qsyn, Sp)$ w/o cyclotomic Frobenius

$\mathbb{T}$-action

$$gr^i_M HH(-/R;\mathbb{Z}_p) = (L\Omega^i_{-/R})^\wedge_p[-i]$$

$$gr^i_M HC^-(-/R;\mathbb{Z}_p) = Fil^i_{Hod}\widehat{dR}_{(p,Hod)}$$

$$gr^i_M HP(-/R;\mathbb{Z}_p) = \widehat{dR}_{(p,Hod)}$$

$HH(A) = HH(A/\mathbb{Z})$

$A\otimes^L A.$

$A\otimes^L_R A$

<u>$\mathbb{S}$ - theory</u>

$THH(-/\mathbb{S}[z];\mathbb{Z}_p) \rightsquigarrow Fil_M$

$\mathbb{S}[z] \longrightarrow \mathcal{O}_K$

$$z \mapsto \bar{\omega}$$

$$\mathfrak{S} = W(k)[[z]]$$

$$\tilde{\theta} : \mathfrak{S} \longrightarrow \mathcal{O}_K$$

$$z \mapsto \bar{\omega}$$

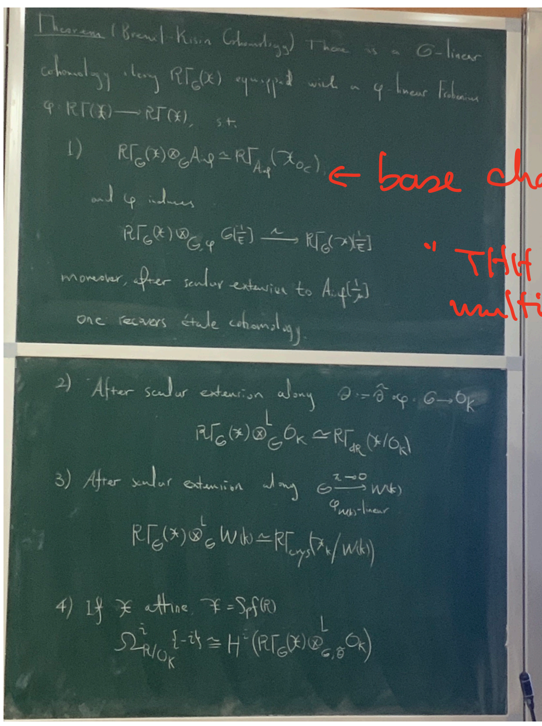
$$\theta : \mathfrak{S} \xrightarrow{\varphi} \mathfrak{S} \xrightarrow{\tilde{\theta}} \mathcal{O}_K$$

$$\mathfrak{S} \longrightarrow A_{inf}(\mathcal{O}_{K_\infty}) \longrightarrow A_{inf}(\mathcal{O}_C)$$

$$A_{inf}(\mathcal{O}_{K_\infty}) \cong \left(\operatorname{colim}_{\varphi} \mathfrak{S}\right)^{\wedge}_{(p, E_K(z))}$$

$\mathfrak{X}$ smooth formal scheme $/ \mathcal{O}_K$

$$\boxed{\begin{gathered} A_{inf} = W(\mathcal{O}_C^{\flat}) \\ \mathbb{S}_{A_{inf}} \otimes_{\mathbb{S}} \mathbb{Z} \simeq A_{inf} \end{gathered}}$$



← base change

"THH multiplicativity":

$$THH(\mathfrak{X}/\mathbb{S}[z])^{\wedge}_{p} \otimes_{THH(\mathcal{O}_K/\mathbb{S}[z])} THH(\mathcal{O}_C/\mathbb{S}_{A_{inf}})$$

$$\simeq THH(\mathfrak{X}_{\mathcal{O}_C}/\mathbb{S}_{A_{inf}})^{\wedge}_{p}$$

(Recall

$THH(A/R) = A \otimes A$).

$$\mu = [\varepsilon] - 1, \quad \varepsilon = (1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \ldots)$$

$$\xi = \frac{[\varepsilon] - 1}{[\varepsilon]^{1/p} - 1} \qquad \mu = \frac{[\varepsilon] - 1}{\cdots} \cdot \frac{[\varepsilon]^{1/p} - 1}{\cdots} \cdots$$

$[\varepsilon]^{1/p}-1 \quad [\varepsilon]^{1/p^2}-1$

$= \xi \cdot \varphi^{-1}(\xi) \cdot \varphi^{-2}(\xi) \cdots$

(A, I) be a perfect prism

$(A_{inf}(\mathcal{O}_C), \ker\theta)$

Let R be a p-complete A/I-algebra.
(derived)

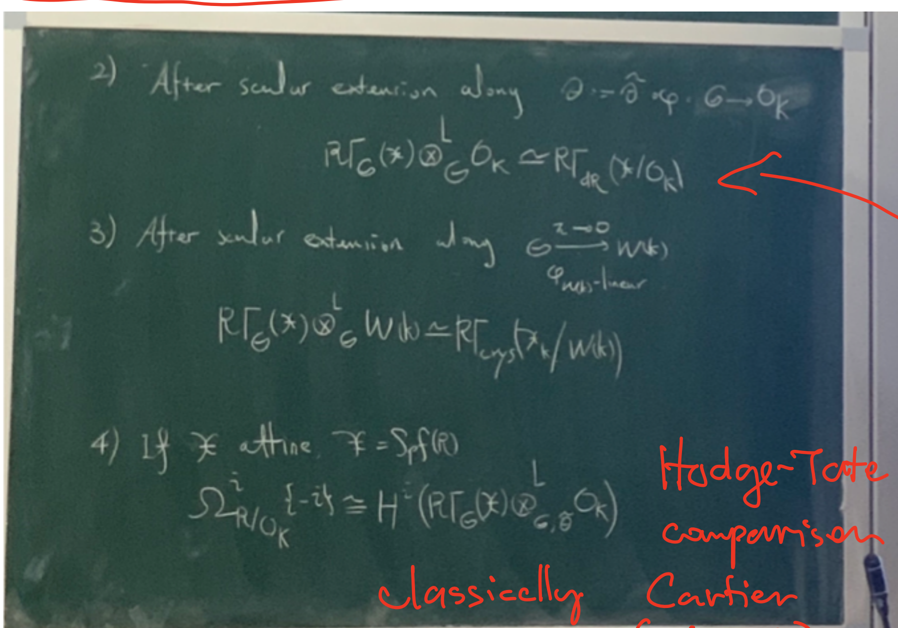
$$R\Gamma_{et}(\mathrm{Spec}\, R[p^{-1}], \underline{\mathbb{Z}/p^n}) \cong (\Delta_{R/A}[I^{-1}]/p^n)^{\phi=1}$$

take Frobenius fixed points

motivic filtration

prismatic cohomolgy

$K \xrightarrow{tr} TC$



Hodge-Tate comparison

classically Cartier (char p)

Bökstedt periodicity

$$TC^-(\mathcal{X}/\mathbb{S}[z])^\wedge_p[z] \xrightarrow{u} TC^-(\mathcal{X}/\mathbb{S}[z])^\wedge_p \rightsquigarrow HC^-(\mathcal{X}/\mathcal{O}_K)^\wedge_p$$

cofiber sequence

Syntomic First Chern Class

$$c_1^{syn} : R\Gamma_{et}(\mathrm{Spec}(R), \mathbb{G}_m)[-1] \longrightarrow R\Gamma_{syn}(\mathrm{Spf}(R), \mathbb{Z}_p(1))$$

Prop. 7.5.2 of Absolute Prismatic Cohomology
Bhatt-Lurie