Breuil-Schraen $\mathcal{L}$-invariants for GL_n $\quad [E:\mathbb{Q}_p]<+\infty$

$$\left\{ \rho: Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_n(E) \right\} \xleftrightarrow{\text{bijection}} \star \left\{ \text{"certain" adm locally analytic reps of } GL_n(\mathbb{Q}_p) \right\}$$

$$\left\{ \text{"certain" adm unitary Banach reps of } GL_n(\mathbb{Q}_p) \right\} \xrightarrow{\text{loc an vector}} \left\{ \text{"certain" adm locally analytic reps of } GL_n(\mathbb{Q}_p) \right\}$$

$1_T : T(\mathbb{Q}_p) \to E^\times$

$$\left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_n(\mathbb{Q}_p)} 1_T\right)^{cont} := C^{cont}(B(\mathbb{Q}_p)\backslash GL_n(\mathbb{Q}_p), E)$$

$$\uparrow$$

$$\left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_n(\mathbb{Q}_p)} 1_T\right)^{an} := C^{an}(B(\mathbb{Q}_p)\backslash GL_n(\mathbb{Q}_p), E)$$

has one-parameter

$n=2$ $\quad \rho = \begin{pmatrix} \varepsilon & * \\ & 1 \end{pmatrix}$ $\quad \varepsilon \hookrightarrow \rho \twoheadrightarrow 1$ non-split

$G_{\mathbb{Q}_p} := Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ $\quad \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(1, \varepsilon)$ $\quad$ 2-dim

$$\begin{aligned} &\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(1, \varepsilon) \\ &\quad \cong \mathrm{Hom}_{cont}(G_{\mathbb{Q}_p}, E)^\vee \\ &\quad \cong \mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E)^\vee \end{aligned}$$

$\log_p, \ \mathrm{val}_p$

$\mathrm{Ext}^1_{\mathbb{Q}_p^\times}(1_{\mathbb{Q}_p^\times}, 1_{\mathbb{Q}_p^\times}) \cong \mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E)$

$T(\mathbb{Q}_p) \cong (\mathbb{Q}_p^\times)^2 \quad T/Z(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times$

$T := T(\mathbb{Q}_p)$ (maximal torus in $GL_2(\mathbb{Q}_p)$)

$\mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E) \rightsquigarrow 1_T \hookrightarrow V_T \twoheadrightarrow 1_T$

$$(*) \quad \left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} 1_T\right)^{an} \hookrightarrow \left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} V_T\right)^{an} \twoheadrightarrow \left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} 1_T\right)^{an}$$

$$(**) \quad 1_{GL_2} \hookrightarrow \left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} 1_T\right)^{an} \twoheadrightarrow St_2^{an}$$

locally analytic (Steinberg rep for $GL_2(\mathbb{Q}_p)$)

subquotient of $(*)$ V. st

$$St_2^{an} \hookrightarrow V \twoheadrightarrow 1_{GL_2}$$

Thm (Breuil) $\mathrm{Ext}^1_{GL_2(\mathbb{Q}_p)}(1_{GL_2}, St_2^{an}) \cong \mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E)$

$\rho = \begin{pmatrix} \varepsilon & * \\ & 1 \end{pmatrix} \rightsquigarrow$ E-line in $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(1, \varepsilon) \cong \mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E)^\vee$

E-line in $\mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E) \rightsquigarrow St_2^{an} \hookrightarrow V \twoheadrightarrow 1_{GL_2}$

$n=3$ Breuil, Ding, Schraen. $GL_3(\mathbb{Q}_p)$

- Breuil-Ding explicit reps ℓ-inv Ext^1 classes
- Schraen $\mathrm{Ext}^1, \mathrm{Ext}^2$ between generalized Steinberg.

$G = GL_n$ $B = \Delta$ $T = \diagdown$ $B^+ = \nabla$ $\Delta =$ positive simple roots for B^+

$I \subseteq \Delta \rightsquigarrow P_I \supseteq B$ $P_I \twoheadrightarrow L_I$

Example. $n=3$ $B = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix}$ $\Delta = \{\alpha_1, \alpha_2\} \rightsquigarrow \{1,2\}$

$P_\emptyset = B$ $P_{\{1\}} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$ $P_{\{2\}} = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$ $P_\Delta = G$

$i_I^{an} := \left(\mathrm{Ind}_{P_I}^G 1_{L_I}\right)^{an} := C^{an}(P_I \backslash G, E)$ $i_\Delta^{an} = 1_G$

$i_I^{an} \hookrightarrow i_{I'}^{an}$ $I \supseteq I'$

$v_I^{an} := i_I^{an} \Big/ \sum_{I' \supsetneq I} i_{I'}^{an}$

Example : $n=2$ $v_\Delta^{an} = i_\Delta^{an} = 1_G$ $v_\emptyset^{an} = i_\emptyset^{an} / i_\Delta^{an} = St_2^{an}$

$n=3$ $\Delta = \{1,2\}$ $v_\Delta^{an} = 1_G$ $v_\emptyset^{an} =: St_3^{an}$

$v_{\{1\}}^{an}, v_{\{2\}}^{an}$ $i_{\{1\}}^{an} = 1_G — v_{\{1\}}^{an}$

$i_{\{2\}}^{an} = 1_G — v_{\{2\}}^{an}$ $i_\emptyset^{an} =$
$$\begin{tikzcd}[arrows=dash] & v_{\{1\}}^{an} \arrow[dr] & \\ 1_G \arrow[ur] \arrow[dr] & & St_3^{an} \\ & v_{\{2\}}^{an} \arrow[ur] & \end{tikzcd}$$

§ $n=3$ Schraen's thesis

$P_{\{1\}} \twoheadrightarrow GL_2$

$P_{\{2\}} \twoheadrightarrow GL_2$

$$\left(\mathrm{Ind}_{P_{\{1\}}}^{GL_3} V\right)^{an} = \begin{tikzcd} & 1 \arrow[dr, dash] & \\ v^{an}_{\{2\}} \arrow[ur, dash] \arrow[dr, dash] & & v^{an}_{\{1\}} \\ & St_3^{an} \arrow[ur, dash] & \end{tikzcd}$$

$n=2$ Breuil $V = St_2^{an} - 1$

$$n=3 \quad V_\# = St_3^{an} \begin{matrix} \diagup \, v^{an}_{\{1\}} \diagdown \\ \diagdown \, v^{an}_{\{2\}} \diagup \end{matrix} 1_{GL_3}$$

$\mathrm{Ext}^1_{GL_3}(v^{an}_{\{k\}}, St_3^{an}) \xrightarrow{\sim} \mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E)$

$k=1,2$

$$\rho = \begin{pmatrix} \varepsilon^2 & * & * \\ & \varepsilon & * \\ & & 1 \end{pmatrix}$$

simple $\mathcal{L}$-inv

$\mathrm{Dim}_E \mathrm{Ext}^1_{GL_3}(1_{GL_3}, V\#) = 1 \quad \mathrm{Dim}_E \mathrm{Ext}^2_{GL_3}(1_{GL_3}, V\#) = 2$

Schraen use this Ext^2 to define an object in the derived category, which has three parameters

n general

$$St_n^{an} \begin{cases} v^{an}_{\{1\}} \\ v^{an}_{\{2\}} \\ \vdots \\ v^{an}_{\{n-1\}} \end{cases}$$

simple $\mathcal{L}$-inv for GL_n

$$\rho = \begin{pmatrix} \varepsilon^{n-1} & * & \cdots & * \\ & \varepsilon^{n-2} & & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix}$$

$\frac{n(n-1)}{2}$ $\mathcal{L}$-inv

$St_3^{an} \hookrightarrow V\#$

$k: \mathrm{Ext}^2_{GL_3}(1_{GL_3}, St_3^{an}) \twoheadrightarrow \mathrm{Ext}^2_{GL_3}(1_{GL_3}, V\#)$

5-dim 2-dim

cup_k $\mathrm{Ext}^1_{GL_3}(1_{GL_3}, v^{an}_{\{k\}}) \otimes \mathrm{Ext}^1_{GL_3}(v^{an}_{\{k\}}, St_3^{an}) \xrightarrow{\cup} \mathrm{Ext}^2_{GL_3}(1_{GL_3}, St_3^{an})$

$k=1,2$ 2-dim 2-dim 5-dim

$\mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E)$ $\mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E)$ 4 dim image

$\ker(k) = \sum_{k=1}^{2} \mathrm{Ext}^1_{GL_3}(1_{GL_3}, v^{an}_{\{k\}}) \cup$ "$St_3^{an} - v^{an}_{\{k\}}$" 3-dim $\subseteq \mathrm{Ext}^2_{GL_3}(1_{GL_3}, St_3^{an})$

Schraen's choice $W_k \subseteq$ E-line in $\mathrm{Ext}^1_{GL_3}(v^{an}_{\{k\}}, St_3^{an})$ for each $k=1,2$

E-line $W_3 \subseteq \mathrm{Ext}^2_{GL_3}(1_{GL_3}, V\#)$

$$W := \kappa^{-1}(W_3) \subseteq \mathrm{Ext}^2_{GL_3}(1_{GL_3}, St_3^{an})$$

4-dim 5-dim

Claim: $(W_1, W_2, W_3) \longleftrightarrow W$

$cup_k^\infty : \mathrm{Ext}^1_{GL_3}(1, v^{an}_{\{k\}})^\infty \times \mathrm{Ext}^1_{GL_3}(v^{an}_{\{k\}}, St_3^{an}) \to \mathrm{Ext}^2_{GL_3}(1, St_3^{an})$

made a stupid mistake in the talk

1-dim smooth Ext^1 $\mathrm{Ext}^1_{GL_3}(1, v_{\{k\}}^{\cancel{an}\,\boxed{\infty}}) \cup W_k = W \cap \mathrm{Im}(cup_k^\infty)$

W_1, W_2 determines κ, $W_3 = \kappa(W)$ $\dim \mathrm{Im}(cup_k^\infty) = 2$

Reformulation $W \subseteq \mathrm{Ext}^2_{GL_3}(1, St_3^{an})$

General n: $\mathrm{Ext}^{n-1}_{GL_n}(1_{GL_n}, St_n^{an})$ $(St_n^{an} = v_\emptyset^{an})$

define B-S $\mathcal{L}$inv hyperplane $\overset{\cup}{W}$ (with further conditions

compatibility with

$$cup_{I,I',I''} : \mathrm{Ext}^{\#I\setminus I'}_{GL_n}(v_I^{an}, v_{I'}^{an}) \times \mathrm{Ext}^{\#I'\setminus I''}_{GL_n}(v_{I'}^{an}, v_{I''}^{an}) \xrightarrow{\cup} \mathrm{Ext}^{\#I\setminus I''}_{GL_n}(v_I^{an}, v_{I''}^{an})$$

$I \supseteq I' \supseteq I''$

* $cup_{I,I',I''}$ is always injective
* $\#I\setminus I' = 1$ $\mathrm{Ext}^1(\cdot\,\cdot) \cong \mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, E)$
* when is $cup_{I,I',I''}$ an isom
* cokernel of $\bigoplus_{I'' \subsetneq I' \subsetneq I} cup_{I,I',I''}$