(Relative) Prismatic cohomology

Notation k perfect field in char p

$\rightsquigarrow W(k) \rightsquigarrow \mathfrak{S} = W(k)[[u]]$

$K/W(k)[\frac{1}{p}]$ fin. tot. ram. ext. $\varpi \in \mathcal{O}_K$ uniformizer

$E \in \mathfrak{S}$ minimal poly of ϖ

Recall $\mathfrak{X}/\mathcal{O}_K$ smooth formal scheme

$X = \mathfrak{X}_\eta^{ad}$ adic generic fiber

$\exists$ p-adic coh thy

(1) crystalline $R\Gamma_{crys}(\mathfrak{X}_k/W)$.

(2) de Rham $R\Gamma_{dR}(\mathfrak{X}/\mathcal{O}_K)$ + comparison thm

(3). étale $R\Gamma_{et}(X, \mathbb{Z}_p)$

In BMS1 & BMS2

$\exists$ coho thy $\quad R\Gamma_{\mathbb{G}}(\mathfrak{X})$ which universal

in the sense that it recovers (1)~(3)

after base change

<u>Thm</u> Fix (A, I) (bounded) prism $\quad \overline{A} := A/I$ ($\rightsquigarrow \phi_A : A \to A$)

$\mathfrak{X}/\overline{A}$ smooth formal scheme, $X := \mathfrak{X}^{ad}_{\eta}$

$\exists$ coho thy $R\Gamma_{\Delta}(\mathfrak{X}/A) = R\Gamma((\mathfrak{X}/A)_{\Delta}, \mathcal{O}_{\Delta})$

which is a comm alg in $D(A)$ & is equipped

with a ϕ_A-linear endo φ.

(1) (crystalline). $I = (p)$.

$$\phi_A^* R\Gamma_{\Delta}(\mathfrak{X}/A) \simeq R\Gamma_{crys}(\mathfrak{X}/A).$$

(2) (de Rham)

$$R\Gamma_{\Delta}(\mathfrak{X}/A) \underset{A, \phi_A}{\hat{\otimes}^{L}} A/I \simeq R\Gamma_{dR}(\mathfrak{X}/\overline{A})$$

(3) (étale) A perfect ($\Leftrightarrow \bar{A}$ perfd)

$$\left(R\Gamma_{\Delta}(\mathfrak{X}/A)/p^n \left[\tfrac{1}{I}\right]\right)^{\phi=1} = R\Gamma_{et}(X, \mathbb{Z}_p/p^n)$$

(4) (Hodge-Tate) $\mathfrak{X} = \mathrm{Spf}\, R$ affine

$$H^i\left(R\Gamma_{\Delta}(\mathfrak{X}/A) \hat{\otimes}^{\mathbb{L}}_A \bar{A}\right) \simeq \Omega^i_{\mathfrak{X}/\bar{A}}\{-i\}$$

(5) (Base change) $(A, I) \to (B, J)$

$$\mathfrak{X}_{\bar{B}} := \mathfrak{X} \times_{\mathrm{Spf}\,\bar{A}} \mathrm{Spf}\,\bar{B}$$

$$R\Gamma_{\Delta}(\mathfrak{X}/A) \hat{\otimes}^{\mathbb{L}}_A B \simeq R\Gamma_{\Delta}(\mathfrak{X}_{\bar{B}}/B)$$

(6) $\phi_A^* R\Gamma_{\Delta}(\mathfrak{X}/A)\left[\tfrac{1}{I}\right] \simeq R\Gamma_{\Delta}(\mathfrak{X}/A)\left[\tfrac{1}{I}\right]$

<u>Rmk</u> if $(A, I) = (\mathfrak{S}, (E))$.

$\Rightarrow R\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S}) \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X})$

BMS2 Breuil-Kisin cohomlgy

§ Prism & prismatic coho

Def A δ-ring is a comm ring R & $\delta: R \to R$

Joyal!

s.t. $\delta(0) = \delta(1) = 0$

& if we put $\varphi(x) := x^p + p\,\delta(x)$

(1) $\delta(xy) = x^p \delta(y) + \varphi(y)\delta(x)$

(2) $\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x^i y^{p-i}$

$\Rightarrow$ $\phi: R \to R$ is a lifting of $F: R/p \to R/p$
$x \mapsto x^p$

Rmk $\{\delta\text{-structure on } R\} \xleftrightarrow{\sim} \{\text{sections of } W_2(R) \to R\}$, $s: r \mapsto (r, \delta(r))$

Def (1) A prism (A, I) is a pair of

(i) A: δ-ring.

(ii). $I \subseteq A$ invertible ideal

s.t. / A is (derived) (p, I)-complete

$\Big\{ \; p \in I + \varphi(I)A.$

(2). $f:(A,I) \to (B,J)$ is a morphism

if $f: A \to B$ is a map of δ-rings

& $f(I) \cong J$

<u>Rmk</u> if $I=(d)$, then $p \in (d, \varphi(d))$ iff $\delta(d) \in A^\times$.

<u>Ex</u> (1). $\left(\mathfrak{S} = W(k)[\![u]\!], (E)\right)$ is a prism if we set $\delta(u)=0$.

(2) $\left(A_{\inf} = W(\mathcal{O}^\flat_{\hat{K}}), \ker(\theta) = (\xi)\right)$, is a prism

(3) k perfect ring in char p

$(W(k), (p))$ is a prism

<u>Def</u> Say a prism (A, I) is

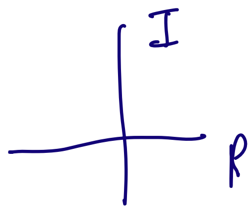
① <u>bounded</u> if $\bar{A} := A/I$ s.t.

$\bar{A}[p^\infty] = \bar{A}[p^N]$

$A[p] = A[p]$

$\rightsquigarrow$ A is derived (p,I)-complete $\Leftrightarrow$ classical (p,I)-complt

② <u>transversal</u> if A/I p-torsion free



③ <u>crystalline</u> if $I=(p)$

④ <u>perfect</u> if $\phi_A : A \longrightarrow A$ is an isom

<u>Prop</u> $\{\text{perfect prisms}\} \simeq \{\text{perfectd rings}\}$

$$(A, I) \longmapsto \overline{A} := A/I.$$

$$(W(R^\flat), \ker\theta) \longleftarrow\!\shortmid\ R$$

<u>Rmk</u> $(A, I) \xrightarrow{f} (B, J) \Longrightarrow J = IB.$

Def $f: (A, I) \to (B, J)$ is a mor of prisms

Say flat if $A \to B$ is (p, I)-completely flat.

faithfully flat (cover) ———— f. flat

Def $\mathfrak{X} / \bar{A} = A/I$ smooth formal scheme

Define $(\mathfrak{X}/A)_\Delta$ obj $(B, IB)/(A, I)$ & $\begin{array}{c} \mathrm{Spf}\, \bar{B} = B/IB \\ \downarrow \\ \mathfrak{X} \end{array}$

Morphism mor of prisms satisfy some obvious compatibilities

Cover flat cover

Def-Prop $\mathcal{O}_\Delta : (\mathfrak{X}/A)_\Delta \longrightarrow \mathrm{Alg}_A$

$(B, IB) \longmapsto B$

$\overline{\mathcal{O}}_\Delta : (\mathfrak{X}/A)_\Delta \longrightarrow \mathrm{Alg}_{\bar{A}}$

$$(B, IB) \longmapsto \overline{B} = B/IB$$

are sheaves on $(\mathfrak{X}/A)_{\Delta}$.

$\left(\Rightarrow I_{\Delta} : (B, IB) \longmapsto IB \text{ is a sheaf} \right)$

Moreover, $\forall (B, IB) \in (\mathfrak{X}/_A)_{\Delta}$

For $i \geq 1$. $H^i((B, IB), \mathcal{O}_{\Delta} \text{ or } \overline{\mathcal{O}}_{\Delta}) = 0$.

<u>Def</u> $R\Gamma_{\Delta}(\mathfrak{X}/A) := R\Gamma((\mathfrak{X}/_A)_{\Delta}, \mathcal{O}_{\Delta})$

$$R\Gamma_{\Delta}(\mathfrak{X}/_A) \hat{\otimes}_A^{\mathbb{L}} A/I = R\Gamma((\mathfrak{X}/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta})$$

"Hodge-Tate structure sheaf"

§ Hodge-Tate comparison

<u>Fact</u> $\exists\ \nu : \mathrm{Shv}((\mathfrak{X}/A)_{\Delta}) \longrightarrow \mathrm{Shv}(\mathfrak{X}_{\text{ét}})$

s.t. $\nu_* \mathcal{F}(U \xrightarrow{\text{ét}} \mathfrak{X}) = H^0((U/A)_{\Delta}, \mathcal{F})$.

Def $\Delta_{X/A} := Rv_* \mathcal{O}_{\Delta}$, $\overline{\Delta}_{X/A} := Rv_* \overline{\mathcal{O}}_{\Delta} \in D(X_{et}, \mathcal{O}_X)$

Thm (HT) $H^i(\overline{\Delta}_{X/A}) \simeq \Omega^i_{X/\overline{A}}\{-i\}$

Pf (sketch) $\mathcal{O}_X \longrightarrow H^0(\overline{\Delta}_{X/A})$

$$\Delta_{X/A} \hat{\otimes}^{\mathbb{L}}_A I^{n+1}/I^{n+2} \longrightarrow \Delta_{X/A} \hat{\otimes}^{\mathbb{L}}_A I^n/I^{n+2} \longrightarrow \Delta_{X/A} \hat{\otimes}^{\mathbb{L}}_A I^{n+1}/I^{n+2}$$

$\rightsquigarrow$ long exact sequence

$\rightsquigarrow$ $\beta : H^n(\overline{\Delta}_{X/A}\{n\}) \longrightarrow H^{n+1}(\overline{\Delta}_{X/A}\{n+1\})$

Bockstein differential

$(H^*(\overline{\Delta}_{X/A}\{*\}), \beta)$ is cdga $\in D(X_{et})$.

$\Rightarrow$ $\eta_X : (\Omega^*_{X/\overline{A}}, d) \to (H^*(\overline{\Delta}_{X/A}\{*\}, \beta)$

Check: $\eta_{\mathfrak{X}}$ is an isom (reduced to dim p)

$\square$.

<u>Rmk</u> $\exists$ $\hat{\mathbb{L}}_{\mathfrak{X}/\bar{A}} \xrightarrow{\sim} \tau^{\leq 1} \bar{\Delta}_{\mathfrak{X}/A}$

Recall (BMS1) $\mathfrak{X}/\mathcal{O}_C$ $A = A_{\inf}$

$$\mu : X_{\text{proét}} \longrightarrow \mathfrak{X}_{\text{ét}}$$

$$\tilde{\Omega}_{\mathfrak{X}} := L\eta_{\zeta_p - 1} R\mu_* \hat{\mathcal{O}}^+_X$$

décalage

$$\Rightarrow \hat{L}_{\mathfrak{X}/\bar{A}} \xrightarrow{\sim} \tau^{\leq 1} \tilde{\Omega}_{\mathfrak{X}}$$

$$\rightsquigarrow \tau^{\leq 1} \bar{\Delta}_{\mathfrak{X}/A} \simeq \tau^{\leq 1} \tilde{\Omega}_{\mathfrak{X}}$$

$$\rightsquigarrow \bar{\Delta}_{\mathfrak{X}/A} \simeq \tilde{\Omega}_{\mathfrak{X}}$$

$$\varphi^* R\Gamma_{A_{\inf}}(\mathfrak{X})/\xi = R\Gamma(\mathfrak{X}_{\text{ét}}, \tilde{\Omega}_{\mathfrak{X}})$$

$$\rotatebox{90}{=}$$

$$R\Gamma_{\Delta}(\mathfrak{X}/A_{\inf})/\xi = R\Gamma(\mathfrak{X}_{et}, \overline{\Delta}_{\mathfrak{X}/A})$$

§ filtrations on prismatic coho

$\mathfrak{X} = \mathrm{Spf}\, R$

$$\mathrm{Fil}^{conj}_n \overline{\Delta}_{R/\overline{A}} = \tau^{\leq n} \overline{\Delta}_{R/\overline{A}}$$

$$\Rightarrow \mathrm{gr}^{conj}_n \overline{\Delta}_{R/\overline{A}} = \Omega^i_R\{-i\}$$

$\Rightarrow$ Left Kan ext ...

<u>Thm</u> S: qrsp. Then

$$(S)_{\Delta} = \{(B, J)\ \&\ S \to B/J\}$$

has an initial obj. (Δ_S, I) ($\simeq$ cd).

(1). Δ_S/I is p-completely flat.

& $\forall\ R \twoheadrightarrow S$ with R perfectd

$\exists$ increasing fil (= conjugate)

$\mathrm{Fil}_n^{\mathrm{conj}}\ \overline{\Delta}_S$ s.t.

$$\mathrm{gr}_n^{\mathrm{conj}}\ \overline{\Delta}_S = \left(\wedge^n \left(L_{S/R}\{-1\}\right)\right)_p^{\wedge}[-n].$$

(2) Δ_S admits a decreasing fil (= Nygaard)

$$\mathrm{Fil}_N^n\ \Delta_S = \{x \in \Delta_S \mid \varphi(x) \in d^n \Delta_S\}$$

$$f: \mathrm{Fil}_N^n \Delta_S \xrightarrow{\frac{\phi}{d^n}} \Delta_S \longrightarrow \overline{\Delta}_S$$

$$\mathrm{Im}(f) = \mathrm{Fil}_n^{\mathrm{conj}}\ \overline{\Delta}_S$$

$$\ker(f) = \mathrm{Fil}_N^{n+1}\ \Delta_S.$$

$$\Rightarrow\ \mathrm{gr}_N^n \Delta_S = \mathrm{Fil}_n^{\mathrm{conj}}\ \overline{\Delta}_S$$

independent depending on choices (R, d.)

?

Thm $\mathfrak{X} = \mathrm{Spf}\, R$

(1) $\Delta_{-/A}$ is a sheaf of (p, I)-compl'f'ly flat δ-A-alg

+ $\exists$ Nygaard filtration Fil_N^i on

$$\Delta^{(1)}_{-/A} := \Delta_{-/A} \hat{\otimes}^{I}_{A, \phi_A} A$$

s.t. $\mathrm{Fil}^n_N \Delta^{(1)}_{-/A} := \{ x \in \Delta^{(1)}_{-/A} \mid \phi(x) \in I^n \Delta_{-/A} \}$

(2). (qsyn descent)

$$R\Gamma_{\Delta}(\mathfrak{X}/A) = R\Gamma_{qsyn}(\mathfrak{X}, \Delta_{-/A})$$

$\Rightarrow \exists$ Nygaard filt on $R\Gamma_{\Delta}(\mathfrak{X}/A)^{(1)} = R\Gamma_{\Delta}(\mathfrak{X}/A) \hat{\otimes}^{L}_{A, \phi_A} A$

$= R\Gamma_{qsyn}(\mathfrak{X}, \Delta^{(1)}_{-/A})$

by

$$\mathrm{Fil}^n_N R\Gamma_{\Delta}(\mathfrak{X}/A)^{(1)} = R\Gamma_{qsyn}(\mathfrak{X}, \mathrm{Fil}^n_N \Delta^{(1)}_{-/A}).$$

(3) $\mathrm{gr}^q_N R\Gamma_\Delta(X/A)^{(1)} = \mathrm{Fil}^{\mathrm{conj}}_q \overline{\Delta}_{R/A}$.

<u>Rmk</u> S qrsp $\qquad \hat{\Delta}_S = \pi_0 TP(S, \mathbb{Z}_p)$.