Remark ($\delta$-$\mathbb{E}_\infty$-rings are cyclotomic bases)

$A$, trivial $\mathbb{T}$-action

equivalence

$$A \longrightarrow A^{hC_p} \longrightarrow A^{tC_p} \implies \begin{matrix} A \\ \wr\| \\ A^{tC_p} \end{matrix} \quad \text{residual}$$

$\mathbb{T}$-equivariant

$\wr\!\wr$

$\mathbb{T}/C_p$

$\mathbb{T}/C_p$-action is trivial.

Then Tate valued Frobenius

$$\varphi_p : A \longrightarrow A$$

Remark ($\delta$-structure) $A$ a $\delta$-$\mathbb{E}_\infty$-ring

$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d] & A \arrow[d, "\varphi_p"] \\ A^{hC_p} \arrow[r, "can"'] & A^{tC_p} \simeq A \end{tikzcd}$$

Tate valued Frob

$A \longrightarrow A$ $\mathbb{T}$-equivariant
$C_p$-equivariant

$\wr$

$A \longrightarrow A^{hC_p}$ via adjunction

$$
\begin{tikzcd}
A \arrow[r] \arrow[drr, bend right, "\mathrm{id}"'] & A^{hC_p} \times_{A^{tC_p}} A \simeq TR^2(A) \text{ "} \simeq W_2(A) \text{"} \arrow[dr] & \\
& & A
\end{tikzcd}
$$

Take $\pi_0$:

$$
\begin{tikzcd}
\pi_0 A \arrow[r] \arrow[dr, "\mathrm{id}"'] & W_2(\pi_0 A) \arrow[d, "R"] \\
& \pi_0 A
\end{tikzcd}
$$

$\rightsquigarrow$ $\delta$-structure ring

$W_2(-)$ if $A$ $p$-torsion free

$$
\begin{tikzcd}
A \arrow[r, "{(a,\delta(a))}"'] \arrow[dr, bend right, "a^p + p\delta(a)"'] & W_2(A) \arrow[r, "R"] \arrow[d, "F"] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "\mathrm{Frob}"] \\
& A \arrow[r, "\mathrm{mod}\ p"'] & A/p
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\phi"'] & A \arrow[d, "\mathrm{Frob}"] \\
A \arrow[r, "\mathrm{mod}\ p"'] & A/p
\end{tikzcd}
$$

**Example** $\mathbb{S}_p^\wedge$ $\pi_0 \mathbb{S}_p^\wedge = \mathbb{Z}_p$

$\mathbb{S}_{W(k)}$ $\pi_0 \mathbb{S}_{W(k)} = W(k)$

$\mathbb{S}_{W(k)}$ flat over $\mathbb{S}$

$$\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{Z} = W(k)$$

Lurie Elliptic Coh II

$(\mathbb{S}[z])_p^\wedge$

**Remark** (The Segal conjecture for THH)

$A$ $E_\infty$-ring

$$\mathrm{THH}(A;\mathbb{Z}_p) \xrightarrow{\varphi_p} \mathrm{THH}(A;\mathbb{Z}_p)^{tC_p}$$

sufficiently large degree equivalence.

$\mathbb{F}_p$, $\mathbb{Z}_p$.

$MU$, $BP\langle n\rangle$.

<u>Notation</u> (Prism) $\underline{\mathbb{S}_A}$ $\qquad \mathbb{S}_A \otimes_{\mathbb{S}} \mathbb{Z} = A$

$\delta$-$\mathbb{E}_\infty$-ring

Let $I$ be derived? an ideal of $A$.

$(\mathbb{S}_A, I)$ is called a <u>prism</u>

if $(A, I)$ is a prism.

<u>Example</u> $(\hat{\mathbb{S}}_p, p)$

$(\mathbb{Z}_p, p) \qquad (\mathcal{O}_K, \mathbb{S}_{W(k)}[[z]])$

<u>Thm</u> (Nikolaus, Krause, ---) Bökstedt periodicity

Let $(\mathbb{S}_A, I)$ be a prism. Then there is a canonical identification

(relative $THH_*(\bar{A}/\mathbb{S}_A) = \bar{A} \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$

$\qquad\qquad 0 \qquad 2 \qquad 4$

absolute $WCart^{HT}/\mathbb{S}_{WCart}$)

"这个定理有一定局限性，我觉得 in general 是错的
我们也可以考虑使之成立的那些 objects"

**Remark** (Perfect $\delta$-$\mathbb{E}_\infty$-ring)

$$\mathbb{S}_A \xrightarrow{\varphi_p} \mathbb{S}_A$$

$$\mathbb{S}_{W(k)}$$

**Remark** $THH(-/\mathbb{S}) \longrightarrow THH(-/\mathbb{S}_{W(k)})$
is an equivalence after $p$-completion.

$$THH(\mathbb{S}_{W(k)}) \to \mathbb{S}_{W(k)}$$
is $p$-completion

**Remark** (Perfection)

$$(\mathbb{S}_A)_{perf} = (\operatorname*{colim}_{\varphi_p} \mathbb{S}_A)^{\wedge}_p$$

Example $\quad R \quad (\mathbb{S}_{W(R^\flat)}, \ker\theta)$

$(\mathfrak{S}_{W(k)}[\![z]\!], E_K)_{perf} = (\mathfrak{S}_{W(\mathcal{O}^\flat_{K_\infty})}, \ker\theta)$

$$W(k)[\![z]\!] \xrightarrow{\varphi} W(k)[\![z]\!]$$

faithfully flat

Theorem (Krause, Nikolaus)

$(\mathfrak{S}_A, I)$ be a prism (transversal, orientable)

$$TC^-_*(\bar{A}/\mathfrak{S}_A) \xrightarrow{can} TP_*(\bar{A}/\mathfrak{S}_A)$$

$$\bigoplus \varphi_A^*(A) \oplus I\varphi_A^*(A\{1\}) \oplus I^2\varphi_A^*(A\{2\})$$

0 2

$$\downarrow$$

$$\varphi_A^*(A\{-1\}) \oplus \varphi_A^*(A) \oplus \varphi_A^*(A\{1\}) \oplus \varphi_A^*(A\{2\})$$

-2 0 2

$$\begin{tikzcd}
TC^- \arrow[r, "\varphi_p^{hT}"] \arrow[d] & TP \arrow[d] \\
THH \arrow[r, "\varphi_p"] & THH^{tC_p}
\end{tikzcd}$$

$$I\varphi_A^*(A\{1\}) \xrightarrow[\varphi]{\sim} A\{1\}$$

Frob

$$\begin{tikzcd}
\varphi_A^*(A) \oplus I\varphi_A^*(A\{1\}) \oplus \cdots \arrow[r] \arrow[d] \arrow[dr] & \oplus\, \varphi_A^*(A) \oplus \varphi_A^*(A\{1\}) \arrow[d] \\
& A\{-1\} \oplus A \oplus A\{1\} \oplus \cdots \arrow[u] \\
A/I \oplus I/I^2 \oplus I^2/I^3 \arrow[r] \arrow[dr] & ? \\
& \arrow[u]
\end{tikzcd}$$

cf yesterday first talk Guo

**Theorem** (cont'd). $(S_A, I)$ transversal / orientable

$A/I$ p-tors free

$I$ principal $I=(d)$

$(W(k)[[z]], E_k)$

$$\begin{tikzcd}
A[u,v]/(uv-d) \arrow[r, "u\mapsto\sigma", "{v\mapsto\varphi(d)\sigma^{-1}}"' , "\varphi_A\text{-linear}"' near end] \arrow[d, "u\mapsto u", "v\mapsto 0"'] & A[\sigma,\sigma^{-1}] \arrow[d] \\
A/d[u] \arrow[r, "u\mapsto\sigma"] & A/\varphi(d)[\sigma,\sigma^{-1}]
\end{tikzcd}$$

$$\varphi: A/d \to A/\varphi(d)$$

$$\begin{tikzcd} TC^- \arrow[r] \arrow[d] & TP \arrow[d] \\ THH \arrow[r] & THH^{tC_p} \end{tikzcd} \qquad \begin{tikzcd} TC^-(\bar{A}/\mathbb{S}_A) \arrow[r] \arrow[d] & TC^-(\bar{A}/\mathbb{S}_A)[u^{-1}] \arrow[r] \arrow[d] & TP(\bar{A}/\mathbb{S}_A) \arrow[d] \\ THH(\bar{A}/\mathbb{S}_A) \arrow[r] & THH(\bar{A}/\mathbb{S}_A)[u^{-1}] \arrow[r] & THH^{tC_p} \end{tikzcd}$$

<u>Example</u> $(\mathbb{S}_{A_{inf}(R)}, \ker\theta)$, $(\mathbb{S}_{W(k)}[z], (E_k(z)))$

Breuil-Kisin prism

Perfd

$\mathbb{S}_{W(R^\flat)}$ perfect prism

$\left(\mathbb{S}_p^\wedge[[q-1]], \left([p]_q = \frac{q^p-1}{q-1}\right)\right)$

q-de Rham prism

$$\left(\mathbb{S}_p^\wedge[[q-1]], ([p]_q)\right)^{h\mathbb{F}_p^\times} = \left(\mathbb{S}_p^\wedge[[\tilde{p}]], (\tilde{p})\right)$$

$\tilde{p}$-de Rham

## <u>Relative de Rham comparison</u>

BK

$\mathbb{S}_{W(k)}[z]$ $\quad$ $\mathbb{S}[z] \to \mathcal{O}_K$, $z \mapsto \varpi$

evenness

$$THH(-/\mathbb{S}_A; \mathbb{Z}_p)[2] \xrightarrow{u} THH(-/\mathbb{S}_A; \mathbb{Z}_p) \to HH(-/A; \mathbb{Z}_p)$$

$$TC^-[2] \xrightarrow{u} TC^- \to HC^-$$

$$TP[2] \xrightarrow{d\sigma} TP \longrightarrow HP$$

Hodge-Tate :

$$THH \xrightarrow{u} THH[-2] \xrightarrow{u} THH[-4] \xrightarrow{u} \cdots \longrightarrow THH[u^{-1}]$$

$$(S_{W(k)[[z]]}, (E_K(z))) \longmapsto (S_{A_{inf}(\mathcal{O}_{K_\infty})}, \ker\theta).$$

perfection
base change
THH multiplicativity

descriptions from perfectoid descent

$$THH(\mathcal{O}_K/S[z]; \mathbb{Z}_p) \hat{\otimes}^p_{THH(\mathcal{O}_K;\mathbb{Z}_p)} THH(\mathcal{O}_K/S[z]; \mathbb{Z}_p).$$

$$= THH(\mathcal{O}_K/S[z_0,z_1]; \mathbb{Z}_p).$$

$$\pi_0 TP(-) = W(k)[[z_0,z_1]] \left\{ \frac{z_0^p - z_1^p}{\varphi(E_K(z_0))} \right\}^\wedge_{(p,N)}$$

$(E_K(z_0))$

prismatic envelope

$N^{\geq 1}$ $E_K(z_0)$, $(z_0 - z_1)$