Roughly BMS2 Ch. 11, but in greater generality

1. The cyclotomic bases

<u>Remark 1.1</u> $THH(A/\mathbb{S}) = A \otimes_{A \otimes_{\mathbb{S}} A} A$

cyclotomic Frobenius

<u>Remark 1.2</u> (p-typical cyclotomic structure)

$X \in Sp$ a p-typical Cyc Str on X

1) a $\mathbb{T}$-action on X $X \in Sp^{B\mathbb{T}}$

2) $\mathbb{T} \simeq \mathbb{T}/C_p$ - equivariant $\varphi_p : X \longrightarrow X^{tC_p}$

<u>Remark 1.3</u> $CAlg(CycSp_p) \simeq Cyc(CAlg)_p$

$\mathbb{E}_\infty$-cyclotomic ring

cyclotomic $\mathbb{E}_\infty$-ring

$CAlg(CycSp_p) \xrightarrow{\text{forget}} CAlg$

preserves (small) colimits

symmetric monoidal

Example 1.3.1. $\mathbb{S}$ equipped with trivial cyclotomic str.

1) trivial $\mathbb{T}$-action $\quad B\mathbb{T} \longrightarrow * \xrightarrow{\mathbb{S}} Sp$

2)
$$\mathbb{S} \longrightarrow \mathbb{S}^{h\mathbb{T}} \longrightarrow (\mathbb{S}^{hC_p})^{h(\mathbb{T}/C_p)}$$

$$\xrightarrow{can^{h(\mathbb{T}/C_p)}} (\mathbb{S}^{tC_p})^{h(\mathbb{T}/C_p)} \xrightarrow[\text{counit}]{} \mathbb{S}^{tC_p} \simeq \mathbb{S}_p^{\wedge}$$

$$\mathbb{S} \longrightarrow \mathbb{S}^{hC_p} \xrightarrow[can]{} \mathbb{S}^{tC_p}.$$

$\mathbb{T} \simeq \mathbb{T}/C_p$-equivariant

spherical Witt vectors?

Definition (The cyclotomic base) Let $B \in CAlg(CycSp_p)$.

B is called a cyclotomic base if

$$THH(B) \longrightarrow B \simeq THH(B/B)$$

can be lifted to a morphism in $CAlg(CycSp_p)$.

Remark ($\mathbb{T}$-equivariant structure of B)

B a cyclotomic base.

$$B \longrightarrow \underbrace{THH(B) \longrightarrow THH(B/B)}_{\text{equivariant}} \longrightarrow B$$

nonequivariant

the $\mathbb{T}$-action on B is trivial.

Remark (The Cyc Frob on B)

Let B be a cyclotomic base.

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\Delta_p"] \arrow[dr, phantom, "\circlearrowleft"] & THH(B) \arrow[r] \arrow[d, "\varphi_p"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\varphi_p"] \\ (B^{\otimes C_p})^{tC_p} \arrow[r] & THH(B)^{tC_p} \arrow[r] & B^{tC_p} \end{tikzcd} \Longrightarrow \begin{tikzcd} B \arrow[r, "id"] \arrow[d, "\Delta_p"] & B \arrow[d] \\ (B^{\otimes C_p})^{tC_p} \arrow[r] & B^{tC_p} \end{tikzcd}$$

The cyclotomic Frob of B is the Tate valued Frob of B

Example Consider $\mathbb{S}[z] = \Sigma^{\infty}_{+} \mathbb{N}$

$$\pi_*(\mathbb{S}[z]) = (\pi_*\mathbb{S})[z].$$

$\mathbb{S}[z]$ is a cyclotomic base $^{\deg 0}$
with trivial $\mathbb{T}$-action

Frobenius $\mathbb{S}[z] \to (\mathbb{S}[z])^{tC_p} = (\mathbb{S}[z])^{\wedge}_p$

$$z \mapsto z^p$$

$$\mathbb{N} \longrightarrow (\mathbb{N}^{\times C_p})^{hC_p}.$$

2. The Tate valued Frobenius and the Segal conjecture

$$\mathbb{S}^{tC_p} \simeq \mathbb{S}^{\wedge}_p$$

<u>Remark</u> (Tate diagonal) There is a natural transformation of functors $Sp \to Sp$

$$\Delta_p : id_{Sp} \longrightarrow T_p$$

$$X \longmapsto (X^{\otimes C_p})^{tC_p}$$

uniquely determined by $\mathbb{S} \to \mathbb{S}^{hC_p} \xrightarrow{can} \mathbb{S}^{tC_p}$

<u>Remark</u> $A \longrightarrow A/p$

$$x \mapsto x^p$$

$$\underbrace{\text{“}A\text{''} \longrightarrow \text{“}A^{\otimes C_p}}_{\text{not a group hom}} \longrightarrow A$$

replaced by
$$A \longrightarrow (A^{\otimes C_p})^{C_p} / \mathrm{Norm}$$

$$\mathrm{THH}: \quad \underbrace{A/[A,A]}_{A \otimes_{A\otimes A} A}.$$

<u>Remark</u> (Segal conjecture) The composition

$$\mathbb{S} \longrightarrow \mathbb{S}^{hC_p} \xrightarrow[\text{can}]{} \mathbb{S}^{tC_p}$$

exhibits $\mathbb{S}^{tC_p}$ as $\mathbb{S}^{\wedge}_p$

<u>Remark</u> $\pi_* \mathbb{Z}^{tC_p} = \mathbb{F}_p[\sigma, \sigma^{-1}]$, $|\sigma| = 2$

In contrast

$$\pi_* \mathbb{F}_p^{tC_p}$$

$$= (\pi_* KU)((x)) / [p](x)$$

$$\pi_* KU^{tC_p} = (\pi_* KU) \otimes \mathbb{Q}_p(\zeta_p)$$

NS.
on top cyc hom

$$[p](x) = (x+1)^p - 1$$

Tate construction $\rightsquigarrow$ chromatic blue shift

<u>Remark</u> $X \in Sp$ bounded below

$$\Delta_p : X \longrightarrow (X^{\otimes C_p})^{tC_p}$$

exhibits $(X^{\otimes C_p})^{tC_p}$ as $X_p^\wedge$

<u>Def</u> (Tate valued Frob) A $\mathbb{E}_\infty$-ring.

$$\varphi_p : A \xrightarrow{\Delta_p} (A^{\otimes C_p})^{tC_p} \xrightarrow{m} A^{tC_p}$$

<u>Remark</u> 1) $KU \longrightarrow KU[1/p] \longrightarrow KU_p^\wedge[1/p]$
pth stable
Adams operation

agrees with $KU \xrightarrow{\varphi_p} (KU^{tC_p})^{h\mathbb{F}_p^\times} \simeq KU_p^\wedge[1/p]$

$\mathbb{F}_p^\times = \mathrm{Aut}\, C_p$.

$C_p \rtimes \mathbb{F}_p^\times$-action

$\cap$

(Σ_p-action)

2). $\mathbb{F}_p \to (\mathbb{F}_p^{tC_p})^{h\mathbb{F}_p^\times}$

$p=2$. Steenrod square

p odd Steenrod power

NS Example

<u>Def</u> (δ-$\mathbb{E}_\infty$-rings) An $\mathbb{E}_\infty$-ring A is called a δ-$\mathbb{E}_\infty$-ring if the following conditions are satisfied

1) A is p-complete and connective

2) $A \to A^{hC_p} \xrightarrow{\text{can}} A^{tC_p}$ is an equivalence

(Segal conj $A^\wedge_p = A$)

3). $A \otimes_S \mathbb{Z}$ is discrete