§ TC^- & TP for perfd rings

Thm I $\pi_* TC^-(\mathbb{F}_p) = \mathbb{Z}_p[u,v]/(uv-p)$ $|u|=2$, $|v|=-2$

$\pi_* TP(\mathbb{F}_p) = \mathbb{Z}_p[\sigma,\sigma^{-1}]$, $|\sigma|=2$

Thm II R perfd.

$$\pi_* TC^-(R;\mathbb{Z}_p) = A_{inf}[u,v]/(uv-\xi)$$

ξ generator of ker θ

Goal $\pi_* TP(R,\mathbb{Z}_p) = A_{inf}[\sigma,\sigma^{-1}]$

Assume (technical) Thm I, prove Thm II.

Recall $TC^- = (THH)^{hS^1}$, $TP = (THH)^{tS^1}$

Tools : Tate spectral sequence strongly convergent

$BS^1 = \mathbb{CP}^\infty$

$$E_2^{i,j} = \hat{H}^i(S^1, \pi_{-j} THH(R,\mathbb{Z}_p)) \Rightarrow \pi_{i+j} TP$$

Tate cohomlgy

$\hat{H}(S^1;A) = A[v,v^{-1}]$
$|v|=2$

Homotopy fixed point SS

$$E_2^{i,j} = H^i(S^1, \pi_{-j} THH(R,\mathbb{Z}_p)) \Rightarrow \pi_{i+j} TC^-$$

$E \in Sp^G$

dual tower to Postnikov

$\overleftarrow{\lim}$ $\{ \cdots \to \tau_{\geq n+1} E \to \tau_{\geq n} E \to \tau_{\geq n-1} E \to \cdots \}$ colim $= E$

= 0. Whitehead tower

Fact $\mathrm{Fib}(\tau_{\geq n} E \to \tau_{\geq n-1} E) \simeq (\pi_{n-1} E)[-n+1]$

$$\mathrm{colim}(\tau_{\geq -n} Y)^{hG} \xrightarrow{\sim} Y^{hG}$$

$$\mathrm{colim}(\tau_{\geq -n} Y)^{tG} \xrightarrow{\sim} Y^{tG}$$

[NS'18 §1.2].

$M \in \mathrm{Abel}$

$G \circlearrowright M \implies HM \in Sp^{BG}$

$$\pi_n(HM^{hG}) = H^{-n}(G, M)$$

$$\pi_n(HM_{hG}) = H_n(G, M).$$

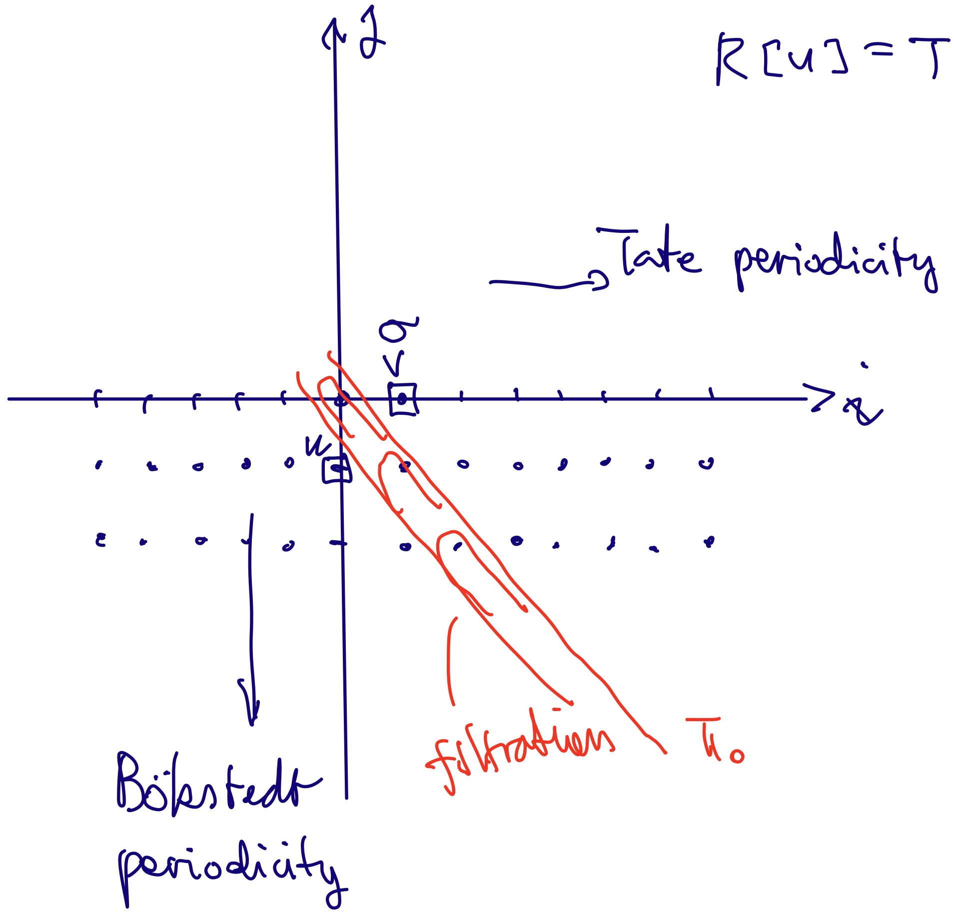
Pf of Thm II Let $F(R) = \pi_0 TP(R; \mathbb{Z}_p)$

The Whitehead tower on THH gives a filtration on $F(R)$.

$$\mathrm{Fil}^i F(R) = \mathrm{Im}\left(\pi_0(\tau_{\geq 2i} \mathrm{THH})^{tS^1} \to \pi_0 \mathrm{THH}^{tS^1}\right)$$

and the E_2-page

$R[u] = THH_*(R, \mathbb{Z}_p)$



evenness $\Rightarrow$ E_2 degenerate

$$gr^i F(R) \cong THH_{2i}(R, \mathbb{Z}_p) \cong R$$

Note that $F(R) \to \pi_0 THH(R; \mathbb{Z}_p) = R$ is a p-adically complete pro-nilpotent thickening.

$\Rightarrow$ By the universal property of A_{inf} we have $A_{inf} \longrightarrow F(R)$

$$\begin{tikzcd} & A_{inf} \arrow[r] \arrow[dl] \arrow[d, dotted] \arrow[dr] & F(R) \arrow[d] \\ {} & (\ker\theta)^2 & R \end{tikzcd}$$

Nygaard $\Big|$ $(\ker\theta)^2$ $\qquad$ R $\qquad \Big/ \; R$

filtration

$$\downarrow$$

$\ker\theta$

$$\downarrow$$

$A_{\inf}$

$\overline{R}$

$\overline{R}$

reduced to Thm I

$$\mathrm{Fil}^1(R) = \ker(F(R) \to R)$$

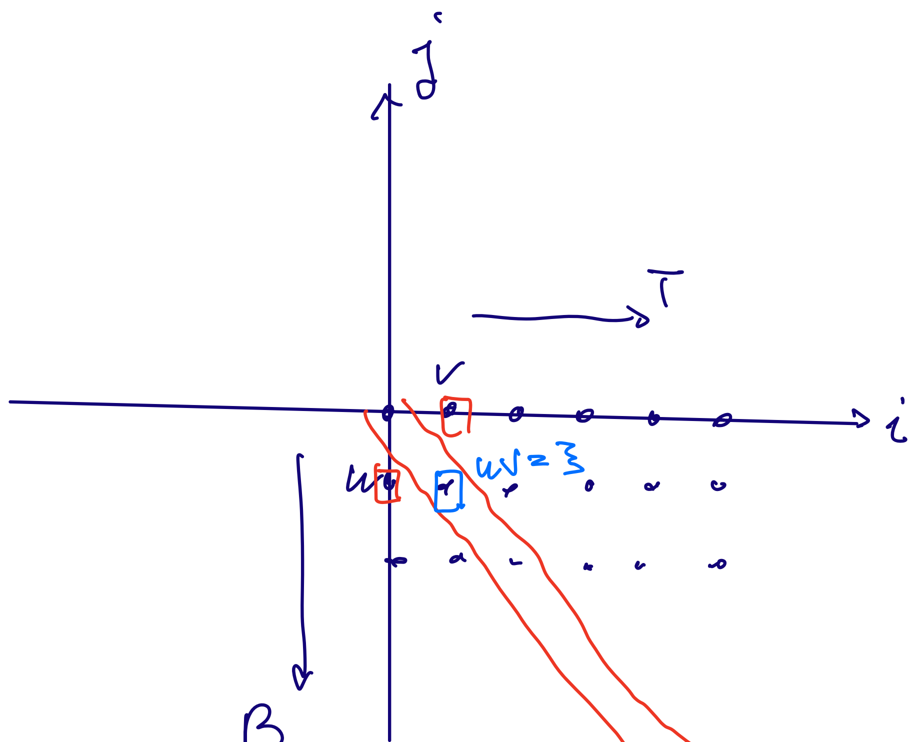
$$\ker\theta / \ker\theta^2 \simeq F(R)/\mathrm{Fil}^1(R) = \mathrm{gr}^1 F(R).$$

$$\downarrow$$

$$\simeq \pi_2 THH.$$

TC^- case

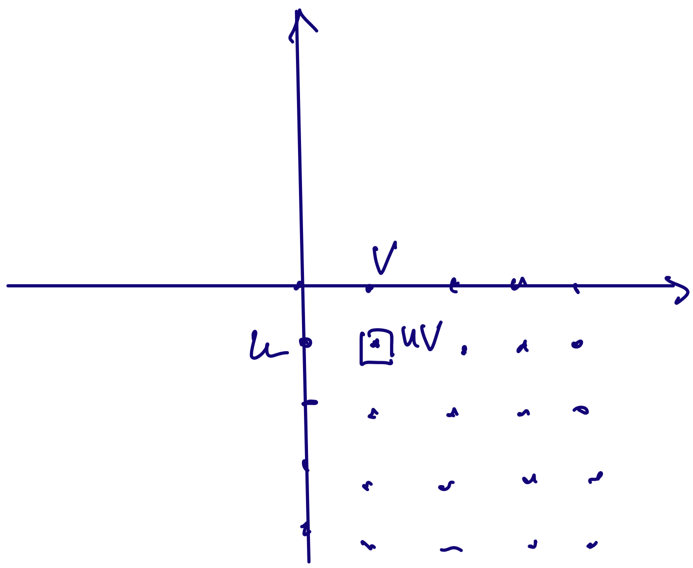
E_2



$S^1 \curvearrowright THH$ nontrivial

Consider $S^1 \curvearrowright THH$ trivially

Then the HFPSS of $THH^{hS^1_{triv}}$



$$\pi_* THH^{hS^1_{triv}}(\mathbb{F}_p)$$
$$\parallel$$
$$[\mathbb{CP}^\infty, THH]^{*}_{Sp}$$
$$\parallel$$
$$\mathbb{F}_p[[t]](u,v)/\ ?$$
$$(uv-t).$$

$\pi_0(THH^{hS^1_{triv}}) = \mathbb{F}_p[[t]]$ vs $THH_0^{hS^1}\ \mathbb{Z}_p$.

cf

($\mathbb{F}_p((t))$ vs $\mathbb{Q}_p$.

perfd ring

motivation).

Scholze ICM