Fix a base ring R.

$A \in \mathrm{Alg}_R \qquad P^\bullet_{A/R}: \underset{0}{A} \leftarrow \underset{-1}{R[A]} \leftarrow \underset{-2}{R[R[A]]} \leftarrow \cdots$

$L_{A/R} = \Omega_{P^\bullet_{A/R}}$

Thm (Flat descent) $A \to B$ faithfully flat

$$\Lambda^i L_{A/R} \simeq \varprojlim \left(\Lambda^i L_{B/R} \rightrightarrows \Lambda^i L_{B \otimes_A B/R} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

Proof $i=0$ f.f. descent

$i=1$ Write $B^\bullet$ for Čech nerve of $A \to B$

$$\dot{B} = \left(\sim B^{\otimes_A k} \rightrightarrows \right)$$

key: transitive triangle $R \to A \to 1$

$$L_{A/R} \otimes^L_A B^\bullet \to L_{B^\bullet/R} \to L_{B^\bullet/A}$$

Reduced to prove

1) $A \to B^\bullet$ induces $L_{A/R} \simeq \varprojlim (L_{A/R} \otimes^L_A B^\bullet)$

2) $\mathrm{Tot}(L_{B^\bullet/A}) \simeq 0$

For 2) by convergence of Postnikov fil sufficient to show for each i, the A-cochain complex corresponding to $\pi_i L_{B/A}$ under Dold-Kan equiv is acyclic.

$$(\pi_i L_{B^\bullet/A}) \otimes^{\mathbb{L}}_A B = \pi_i(L_{B^\bullet/A} \otimes^{\mathbb{L}}_A B)$$

$$= \pi_i(L_{C^\bullet/B})$$

$$C^\bullet = B \to B \otimes_A B$$

$$\simeq \pi_i(L_{B/B}) = 0$$

For $i > 1$. $\exists$ fil $F^\bullet$ on $\Lambda^i L_{B^\bullet/R}$

$$(F^{j+1} L_{B^\bullet/R}, F^j \Lambda^i L_{B^\bullet/R}, \Lambda^{i-j-1} L_{B^\bullet/A}) \wedge (\Lambda^{j+1} L_{A/R} \otimes^{\mathbb{L}} B^\bullet))$$

by induction $\square$

QSyn^{op} & $\mathrm{QRSPerfd}^{op}$

$\simeq$

Slogan: basis

equiv of sheaves

p-complete flat ($\hat{p}$ for p-complete)

<u>Def</u> 1) We say $M \in D(A)$ has $\hat{p}$-Tor-amplitude in $[a,b]$

if $M \otimes^{\mathbb{L}}_A A/p \in D(A/p)$ has Tor-amplitude in $[a,b]$.

2) . . . . . . – – – is $\hat{p}$'ly (faithfully) flat if

$M \otimes^{\mathbb{L}}_A A/p \in D(A/p)$ concentrated in

deg 0 as a (faithfully) flat

A/p-module.

Remark: Can replace p by p^n since

if $I \subset R \to R/I$ $I^2 = 0$

$$(M \otimes^{\mathbb{L}}_R R/I) \otimes^{\mathbb{L}}_{R/I} I \longrightarrow M \to M \otimes^{\mathbb{L}}_R R/I$$

<u>Lemma</u> Fix A, $M \in D(A)$. and $a,b \in \mathbb{Z} \cup \{\infty\}$

$$\hat{M} = R\varprojlim_n (M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}) \text{ in } D(A)$$

Then M has $\hat{p}$-Tor-amplitude in $[a,b]$ iff $\hat{M}$ is.

Sketch key $M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p \simeq \hat{M} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p$

$\otimes^{\mathbb{L}}_{\mathbb{Z}/p} A/p$

Lemma Fix a map $A \to B$ of rings,

a complex $M \in D(A)$, $a, b \in \mathbb{Z} \cup \{\infty\}$.

1) If M has $\hat{p}$-Tor-amplitude in $[a,b]$ (or $\hat{p}$ f.f.)

then so is $M \otimes^{\mathbb{L}}_A B \in D(B)$

2) if $A \to B$ is $\hat{p}$ly f.f. the converse of 1) is true.

Remark ① Example for derived p-complete

$A = \mathbb{Z}$, $I = (p)$, $M = \mathbb{Q}/\mathbb{Z}$ $\hat{M} \simeq \mathbb{Z}_p[1]$

classical. $\varprojlim (M \otimes \mathbb{Z}/p^n\mathbb{Z})$

② derived-complex forms an abelian category

derive p-complete + p-separated $\Leftrightarrow$ classical p-complete

M p-complete + p-separated $\Leftrightarrow$ derived p-complete

With bounded p^∞-torsion $A[p^\infty] = A[p^n]$

Lemma Fix a ring A with bounded p^∞-torsion and a derived $\hat{p}$- $M \in D(A)$ w/ $\hat{p}$-Tor amplitude in $[a,b]$, $a,b \in \mathbb{Z}$. Then $M \in D^{[a,b]}(M)$.

Proof. $\{A/p^n\} \simeq \{A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\}$

and $M = R\varprojlim M \otimes^{\mathbb{L}}_A A/p^n$

By assumption $M \otimes^{\mathbb{L}}_A A/p^n \in D^{[a,b]}(A/p^n)$

$H^b(M \otimes^{\mathbb{L}}_A A/p^n) \twoheadrightarrow H^b(M \otimes^{\mathbb{L}}_A A/p^{n-1})$ is surj. $\square$

<u>Lemma</u> Fix A as above.

1) If a derived $\hat{p}$-mod M is $\hat{p}$-flat then M is classically $\hat{p}$-mod in deg 0.

$M[p^\infty] = M[p^c]_*$ M/p^n flat A/p^n-mod

$M \otimes_A A[p^n] \simeq M[p^n]$

2) Conversely, if N is a classically $\hat{p}$ A-mod with bd p^∞-torsion. N/p^n is a flat A/p^n-mod then N is itself $\hat{p}$-flat

key: $A[p^i][1] \longrightarrow A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n \to A/p^n A$

$\otimes^{\mathbb{L}}_{M}$

<u>Cor</u> $A \to B$ be a map of derived $\hat{p}$-rings

1) If A has bounded p^∞-torsion, B is $\hat{p}$-flat over A then B has bounded p^∞-torsion

2) Converse is true if $A \to B$ $\hat{p}$.f.f

3) If A, B have bounded p^∞-torsion

$A/p^n \to B/p^n$ flat for all n

$\iff A \to B$ $\hat{p}$-flat

<u>Remark</u> f.f. descent of cotangent complex is also true after derived p-completion when A has bounded

p^∞-torsion.

QSyn

Def 1) A ring A is called quasi syntomic if

① A is p-complete with bounded p^∞-torsion

② $L_{A/\mathbb{Z}_p} \in D(A)$ has $\hat{p}$-Tor-amplitude in deg $[-1,0]$.

Denote by QSyn the category of quasi-syntomic rings.

2) Say $A \to B$ is a quasi-smooth map (cover) if

① B is $\hat{p}$-(f)-flat over A

② $L_{B/A} \in D(B)$ is $\hat{p}$-flat deg 0

3). $A \to B$ is quasi-syntomic map (cover) if

① ...

② $L_{B/A} \in D(B)$ has $\hat{p}$-Tor-amplitude in $[-1,0]$

<u>Remark</u> A noetherian ring A is <u>locally complete intersection</u> iff $L_{A/\mathbb{Z}}$ has Tor-amplitude in $[-1,0]$.

Remark (HKR for quasi-smooth maps)

$A \to B$ a map of $\hat{p}$-rings with bounded p^∞-torsion

Consider p-completion of HKR fil

get a $\mathbb{T}$-equiv complete descending $\mathbb{N}$-indexed

fil on $HH(B/A)$ with $gr^i_{HKR} HH(B/A, \mathbb{Z}_p) \simeq$

$(\Lambda^i L_{B/A}[i])^\wedge_p$ with trivial $\mathbb{T}$-action.

If $A \to B$ q-smooth, then $(\Lambda^i L_{B/A}[i])^\wedge_p = (\Omega^i_{B/A})^\wedge_p[i]$
in deg i

$\pi_* HH(B/A, \mathbb{Z}_p) \simeq (\Omega^*_{B/A})^\wedge_p$.