

# Topology Course Notes (KSM1C03)

**Day 21 : 24<sup>th</sup> October, 2025**

Tychonoff corkscrew property -- completely regular space

## 21.1 Regular space and $T_3$ space (cont.)

### Proposition 21.1: (Continuous map from $S_\Omega$ is eventually constant)

Given any continuous map  $f : S_\Omega \rightarrow \mathbb{R}$ , there exists some  $\alpha \in S_\Omega$  such that  $f(x) = c$  for all  $x \geq \alpha$ . Consequently,  $f$  can only have countably many distinct values.

#### Proof

If possible, suppose there exists some  $\epsilon > 0$  such that for any  $\alpha \in S_\Omega$  there exists some  $\beta(\alpha) > \alpha$  with  $|f(\alpha) - f(\beta)| \geq \epsilon$ . Otherwise, for each  $n \geq 1$ , there exists some  $\alpha_n$  such that for all  $\beta > \alpha_n$ , we have  $|f(\beta) - f(\alpha_n)| < \frac{1}{n}$ . If the sequence  $\{\alpha_n\}$  is finite (i.e, there are finitely many points), then just take  $\theta = \max \alpha_n$ . It follows that for any  $\beta > \theta$ , we have  $|f(\beta) - f(\theta)| < \frac{1}{n}$  for all  $n$ . In particular,  $f(\beta) = f(\theta)$  for all  $\beta > \theta$ , proving the claim. If the sequence is not finite, without loss of generality, assume  $\alpha_1 < \alpha_2 < \dots$ . Now, recall that  $[0, \Omega)$  is sequentially convergent. Hence, without loss of generality, the sequence  $\{\alpha_n\}$  converges to some  $\theta \in [0, \Omega)$ , and  $\theta \geq \alpha_i$  for all  $i$ . Then, by continuity of  $f$  we have  $f(\theta) = \lim_n f(\alpha_n)$ . Now, for any  $\beta > \theta$ , we have

$$|f(\beta) - f(\theta)| \leq |f(\beta) - f(\alpha_n)| + |f(\alpha_n) - f(\theta)| \rightarrow 0, \quad n \rightarrow \infty.$$

Thus,  $f(\beta) = f(\theta)$  for any  $\beta > \theta$ , again proving the claim.

Thus, let us now assume that there exists some  $\epsilon > 0$  such that for any  $\alpha \in S_\Omega$  there exists some  $\beta(\alpha) > \alpha$  with  $|f(\alpha) - f(\beta)| \geq \epsilon$ . Starting with  $\alpha_0 = 0$ , we can construct an increasing sequence  $\alpha_0 < \alpha_1 < \dots$ , where each  $\alpha_j$  is inductively obtained as some  $\beta(\alpha_{j-1})$ . Now,  $\{\alpha_j\}$  is a countable set, and hence, upper bounded. Suppose  $\theta \in S_\Omega$  is the least upper bound of  $\{\alpha_j\}$ . Now, by continuity, we have some  $\delta < \theta$  such that

$$f((\delta, \theta]) \subset \left(f(\theta) - \frac{\epsilon}{2}, f(\theta) + \frac{\epsilon}{2}\right).$$

Since  $\theta$  is the least upper bound of the strictly increasing sequence  $\alpha_j$ , there exists some  $\delta < \alpha_{j_0} \leq \theta$ . Now, for  $\alpha_j < \alpha_{j+1} \leq \theta$ . But then,  $|f(\alpha_{j+1}) - f(\alpha_j)| < \epsilon$ , a contradiction.

Hence, we have that there is some  $\alpha \in S_\Omega$  such that  $f(x)$  is constant for all  $x \geq \alpha$ .  $\square$

**Proposition 21.2: ( $T_3 \not\Rightarrow$  Completely  $T_2$  : Tychonoff Corkscrew)**

The Tychonoff corkscrew is  $T_3$ , but not completely  $T_2$ .

*Proof*

For any point other than  $\alpha_{\pm}$ , one can easily construct a basis of open sets which are regular (i.e.,  $\text{int}(\bar{O}) = O$ ). Indeed, if the point is not on any of the “slits”, we can take product of intervals. For a point on the slit, we might need to take the intervals in two different planks, but we can still get a basis of regular open sets. For  $\alpha_+$ , the image of the basic open neighborhoods are open (Check!), and they are clearly regular open sets. Similar argument works for  $\alpha_-$ . Thus, the Tychonoff corkscrew is a regular space. In fact, it is  $T_0$  as every point is closed, and hence,  $T_3$ .

Let us now show that the space is not completely  $T_2$ . Suppose  $f$  is a real-valued continuous function. Observe that for  $n \neq 0$ , on each of the horizontal lines  $A_{\Omega} \times \{n\} \times \{k\}$ , the function  $f$  is constant on an interval of the form  $[-\alpha, \alpha]$  about  $\Omega$ . Same argument works for the  $x$ -axis as well, and we get a deleted neighborhood about  $\{(\Omega, \omega, k)\}$  where  $f$  is constant. Now, there are countable infinitely many such intervals, on each of which  $f$  is constant. Indeed, on each stage, there are countable infinitely many horizontal lines (counting two lines for the  $x$ -axis), and there are countable infinitely many stages (the positive  $x$ -axes are getting counted twice, which is not an issue). Again, using the well-ordering, we can get a common  $\alpha$  such that  $f$  is constant on each of the  $[-\alpha, \alpha] \times \{\pm n\} \times \{k\}$  and on  $([-\alpha, \alpha] \times \{\omega\} \setminus \{(\Omega, \omega)\}) \times \{k\}$ , for all  $k \in \mathbb{Z}$ .

Fix some  $-\beta \in [-\alpha, \Omega)$ , and the corresponding  $\beta \in (\Omega, \alpha]$ . Then, denote the same points (i.e., their equivalence classes) in different stages as

$$-\beta^k = (-\beta, \omega, k), \quad \beta^k = (\beta, \omega, k).$$

Next, get the sequences

$$-\beta_{\pm n}^k = (-\beta, \pm n, k), \quad \beta_{\pm n}^k = (\beta, \pm n, k).$$

Clearly, as  $\pm n \rightarrow \omega$ , we have

$$-\beta_{\pm n}^k \rightarrow -\beta^k, \quad \beta_n^k \rightarrow \beta^k, \quad \beta_{-n}^k \rightarrow \beta^{k-1},$$

where the last convergence follows since the north edge of the fourth quadrant is identified with the south edge of the first quadrant of the stage just below! Now,  $f(-\beta_{\pm n}^k) = f(\beta_{\pm n}^k)$ . Hence, by continuity,

$$f(-\beta^k) = \lim f(-\beta_n^k) = \lim f(\beta_n^k) = f(\beta^k),$$

and also,

$$f(-\beta^k) = \lim f(-\beta_{-n}^k) = \lim f(\beta_{-n}^k) = f(\beta^{k-1}).$$

But then, inductively we see that  $f(\pm\beta^k)$  are all constant. This implies that  $f$  is constant on the union of deleted intervals

$$\mathcal{I} = \bigcup_{k \in \mathbb{Z}} ([-\alpha, \alpha] \times \{\omega\} \setminus \{(\Omega, \omega)\}) \times \{k\}.$$

We can now get a sequence  $\{a_i\}_{i=-\infty}^{\infty} \in \mathcal{I}$  (in fact, taking  $a_{\pm i} = \pm\beta^i$  will do) such that  $\lim_{i \rightarrow \infty} a_i = \alpha_+$  and  $\lim_{i \rightarrow -\infty} a_i = \alpha_-$ . This follows since the basic open neighborhoods of  $\{\alpha_{pm}\}$  contains all

the stages after (resp. below) a certain 'height'. By continuity of  $f$ , we have  $f(\alpha_+) = f(\alpha_-)$ . Thus, Tychonoff corkscrew is not functionally  $T_2$ , as no continuous function is able to distinguish the points  $\alpha_{\pm}$ .  $\square$

## 21.2 Completely regular space

### Definition 21.3: (Completely regular space)

A space  $X$  is called a **completely regular space** if given any closed set  $A \subset X$  and a point  $x \in X \setminus A$ , there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$  and  $f(A) = 1$ .

### Remark 21.4

It is immediate that a completely regular space is regular.

### Definition 21.5: ( $T_{3\frac{1}{2}}$ -space)

A space  $X$  is called a  **$T_{3\frac{1}{2}}$ -space** (or a **Tychonoff space**) if it is completely regular, and  $T_0$ .

### Remark 21.6

It is immediate that a  $T_{3\frac{1}{2}}$ -space is completely  $T_2$ , and hence,  $T_{2\frac{1}{2}}$ . Also,  $T_{3\frac{1}{2}} \Rightarrow T_3$  is clear as well. Moreover, one can check that a semi-regular space is  $T_{3\frac{1}{2}}$  if and only if it is  $T_2$ . Thus, one can define  $T_{3\frac{1}{2}}$ -space as a completely regular, Hausdorff space.

### Proposition 21.7: (Metrizable $\Rightarrow$ Tychonoff)

Metrizable spaces are Tychonoff.

#### Proof

Say  $(X, d)$  is a metric space. Let  $A \subset X$  be closed, and  $p \in X \setminus A$  be a point. Consider the map

$$f(x) := \frac{d(p, x)}{d(p, x) + d(A, x)}, \quad x \in X.$$

It is easy to see that  $f : X \rightarrow \mathbb{R}$  is continuous, and  $f(p) = 0, f(A) = 1$ . Thus,  $X$  is completely regular, and hence, Tychonoff.  $\square$

### Proposition 21.8: ( $T_3 \not\Rightarrow T_{3\frac{1}{2}}$ : Tychonoff corkscrew)

The Tychonoff corkscrew  $X$  is  $T_3$  but not  $T_{3\frac{1}{2}}$ .

#### Proof

We have seen that  $X$  is  $T_3$  but not completely  $T_2$ . Since  $T_{3\frac{1}{2}}$  implies completely  $T_2$ , it follows that  $X$  is not  $T_{3\frac{1}{2}}$ .  $\square$

### Proposition 21.9: (Completely $T_2 \not\Rightarrow T_{3\frac{1}{2}}$ : Half-disc topology)

The half-disc topology  $X$  is a completely  $T_2$  space, which is not  $T_{3\frac{1}{2}}$ .

*Proof*

We have seen  $X$  is completely  $T_2$  (as it was submetrizable), but not regular (in fact not even semiregular). Hence,  $X$  cannot be  $T_{3\frac{1}{2}}$ .  $\square$