Perceptron

Instructor: Alan Ritter

Many Slides from Luke Zettemoyer and Ian Murray

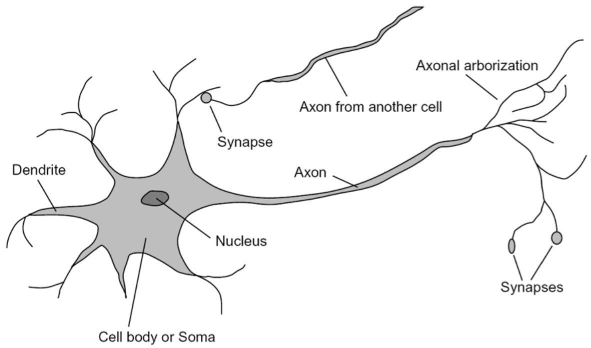
Who needs probabilities?

- Previously: model data with distributions
- Joint: P(X,Y)
  - e.g. Naïve Bayes
- Conditional: P(Y|X)
  - e.g. Logistic Regression
- But wait, why probabilities?
- Lets try to be error-driven!

mpg	cylinders	displacemen	horsepower	weight	acceleration	modelyear	make
good	4	97	75	2265	18.2	77	asia
bad	6	199	90	2648	15	70	ameri
bad	4	121	110	2600	12.8	77	europ
bad	8	350	175	4100	13	73	ameri
bad	6	198	95	3102	16.5	74	ameri
bad	4	108	94	2379	16.5	73	asia
bad	4	113	95	2228	14	71	asia
bad	8	302	139	3570	12.8	78	ameri
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
good	4	120	79	2625	18.6	82	ameri
bad	8	455	225	4425	10	70	ameri
good	4	107	86	2464	15.5	76	europ
bad	5	131	103	2830	15.9	78	europ

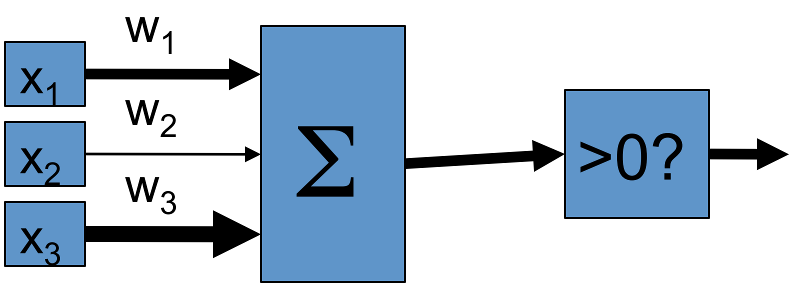
Linear Classifiers

- Inputs are feature values
- Each feature has a weight
- Sum is the activation

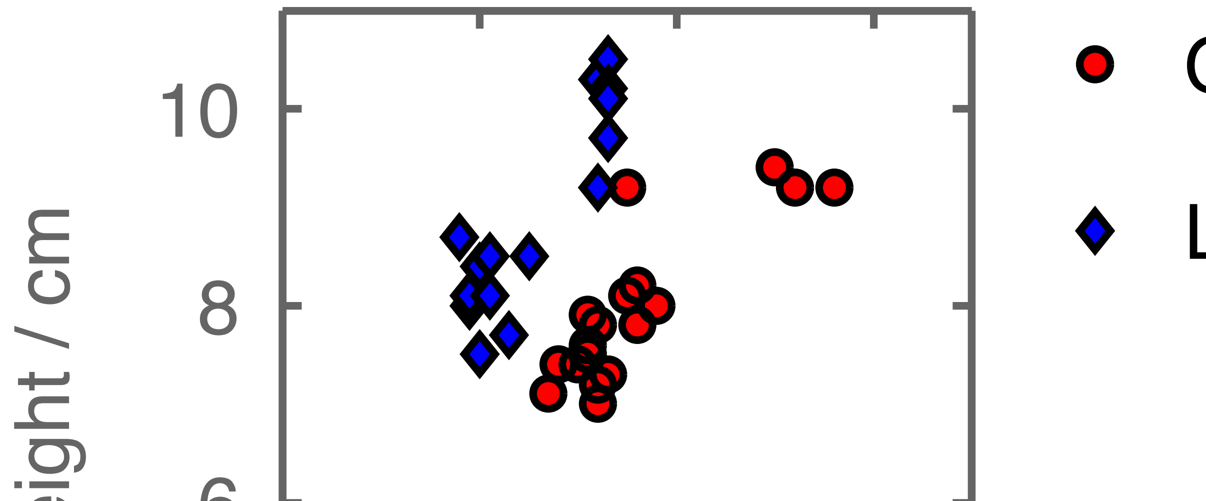


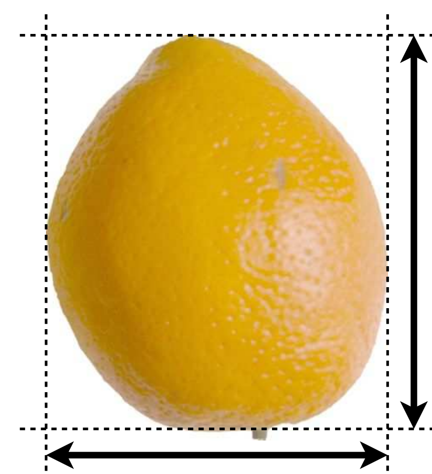
$$\text{activation}_w(x) = \sum_i w_i x_i = w \cdot x$$

- If the activation is:
  - Positive, output *class 1*
  - Negative, output *class 2*



A two-dimensional space



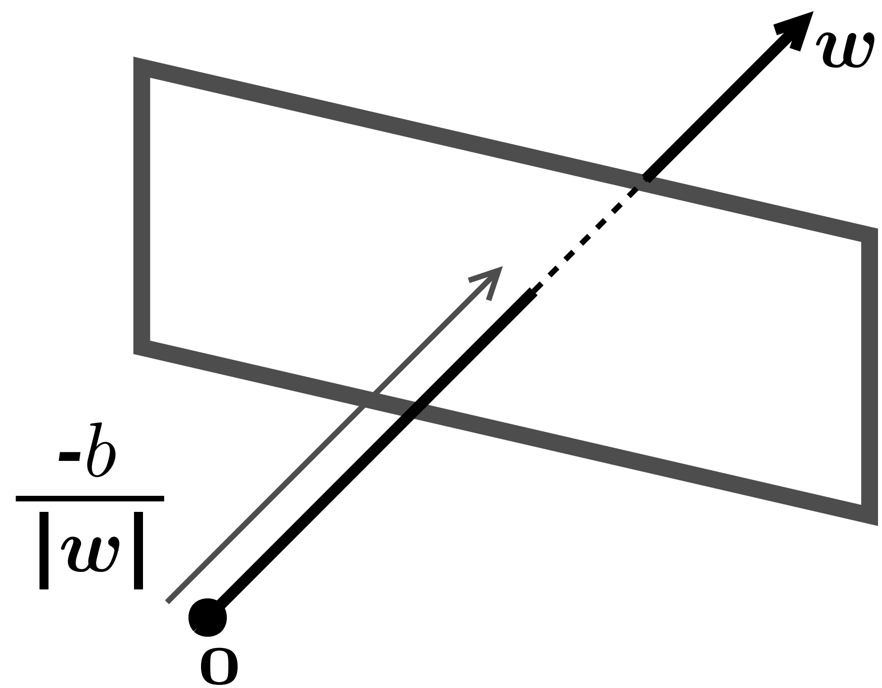


The weight vector

Define positive class region: $\mathbf{w}^\top \mathbf{x} + b > 0$

$w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots b > 0$

$\sum_d w_d x_d + b > 0$



We will set $b = 0$:

$$\mathbf{w}^\top \mathbf{x} > 0$$

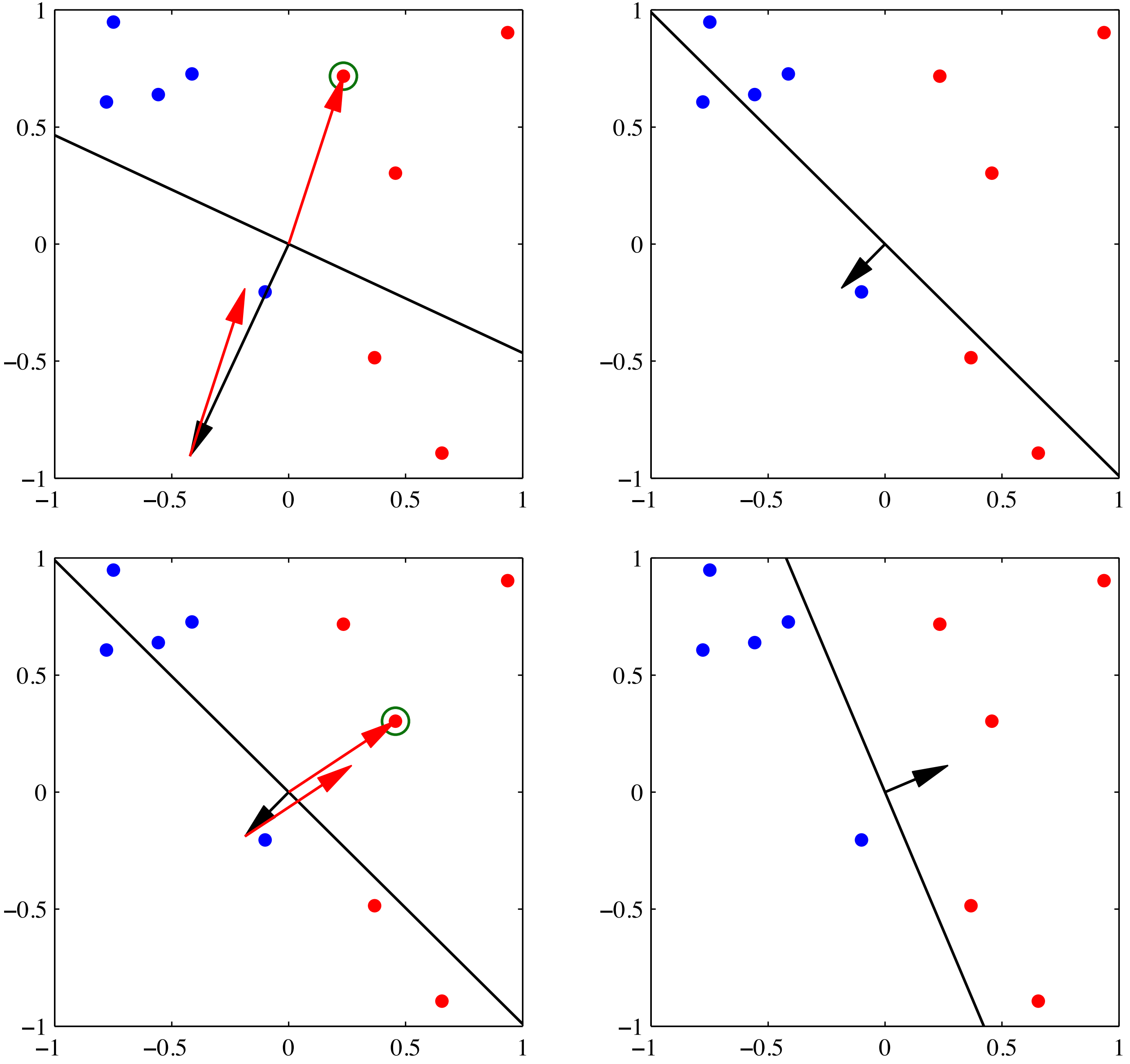
Learning the weights

A ‘perceptron’ learning rule:

$$\hat{y} \leftarrow \mathrm{sgn}(\mathbf{w}^\top \mathbf{x})$$

$$\mathbf{w} \leftarrow \mathbf{w} + (y - \hat{y})\mathbf{x}$$

```
% Matlab/Octave code
old_ww = []; ww = zeros(size(xx,2), 1);
while ~isequal(ww, old_ww)
    old_ww = ww;
    for ii = 1:N
       pred = sign(xx(ii, :)*ww);
       ww = ww + (yy(ii) - pred)*xx(ii, :)';
    end
end
```



Pattern Recognition and Machine Learning C. M. Bishop (2006)

Example: Spam

- Imagine 3 features (spam is "positive" class):
  - free (number of occurrences of "free")
  - money (occurrences of "money")
  - BIAS (intercept, always has value 1)

"free money"

x

```
BIAS  :  1
free  :  1
money :  1
...
```

w

```
BIAS  : -3
free  :  4
money :  2
...
```

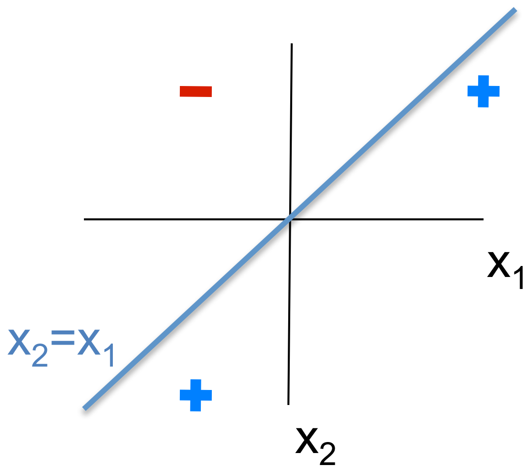
$$
\begin{aligned}
&(1)(-3) \quad + \\
&(1)(4) \quad + \\
&(1)(2) \quad + \\
&\ldots \\
&= 3
\end{aligned}
$$

w•x > 0 ➔ SPAM!!!

- For t=1..T, i=1..n:
  - $y = sign(w \cdot x^i)$
  - if $y \neq y^i$

    $$w = w + y^i x^i$$

x_1	x_2	y
3	2	1
-2	2	-1
-2	-3	1



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Converged!!!

- $y=w_1x_1+w_2x_2 \rightarrow y=x_1+-x_2$
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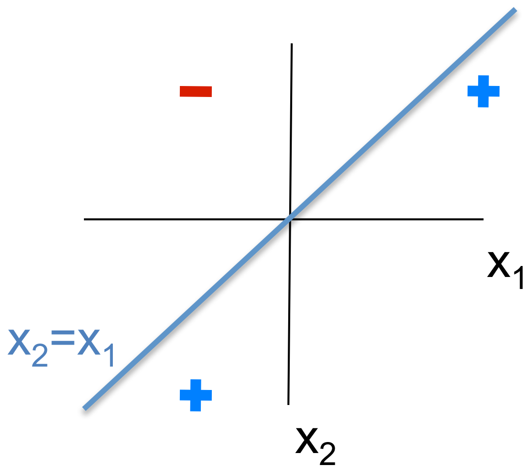

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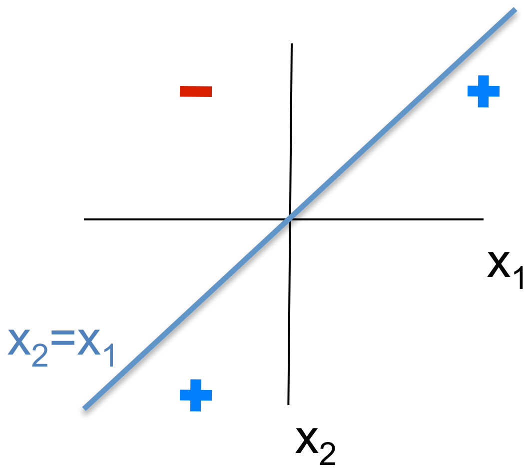

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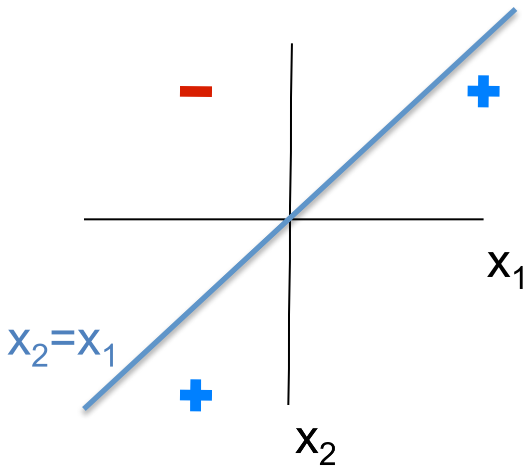

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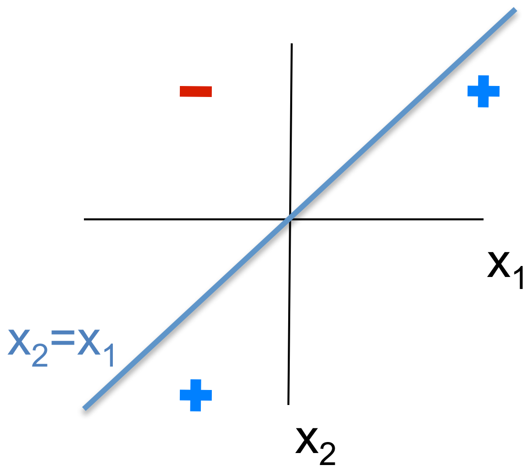

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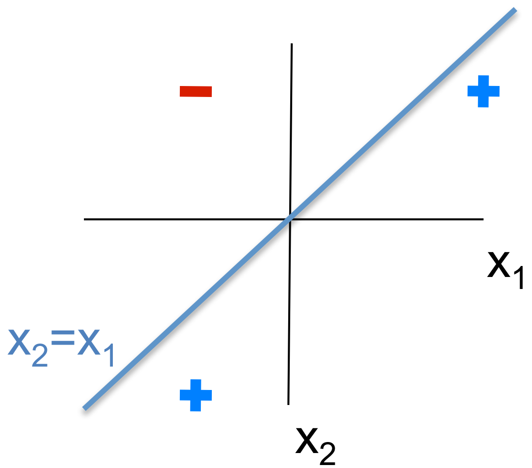

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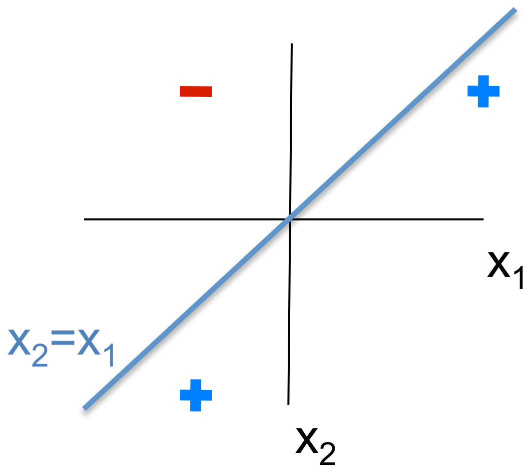

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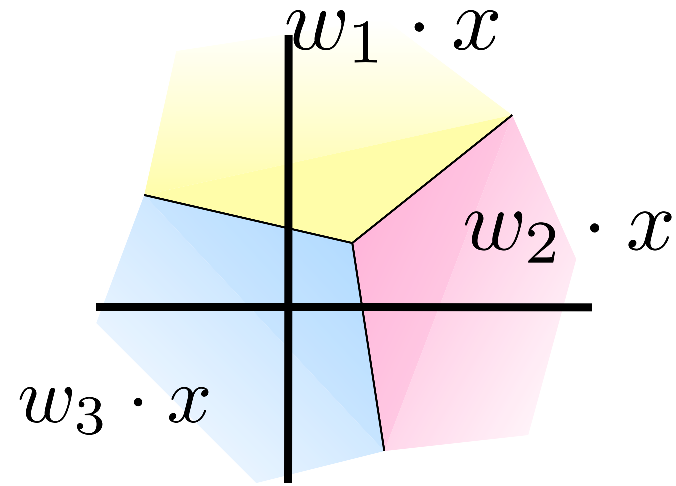
Multiclass Decision Rule

Example: y is {1,2,3}

- We are fitting three planes: w_1, w_2, w_3
- Predict i when $w_i \cdot x$ is highest



- If we have more than two classes:
  - Have a weight vector for each class: w_y
  - Calculate an activation for each class

$$\text{activation}_w(x, y) = w_y \cdot x$$

  - Highest activation wins

$$y^* = \arg\max_y(\text{activation}_w(x, y))$$

Example

“win the vote”

x

```
BIAS  :  1
win   :  1
game  :  0
vote  :  1
the   :  1
...
```

w_{SPORTS}

```
BIAS  : -2
win   :  4
game  :  4
vote  :  0
the   :  0
...
```

$w_{POLITICS}$

```
BIAS  :  1
win   :  2
game  :  0
vote  :  4
the   :  0
...
```

w_{TECH}

```
BIAS  :  2
win   :  0
game  :  2
vote  :  0
the   :  0
...
```

$x \cdot w_{SPORTS} = 2$

$x \cdot w_{POLITICS} = 7$

$x \cdot w_{TECH} = 2$

POLITICS wins!!!

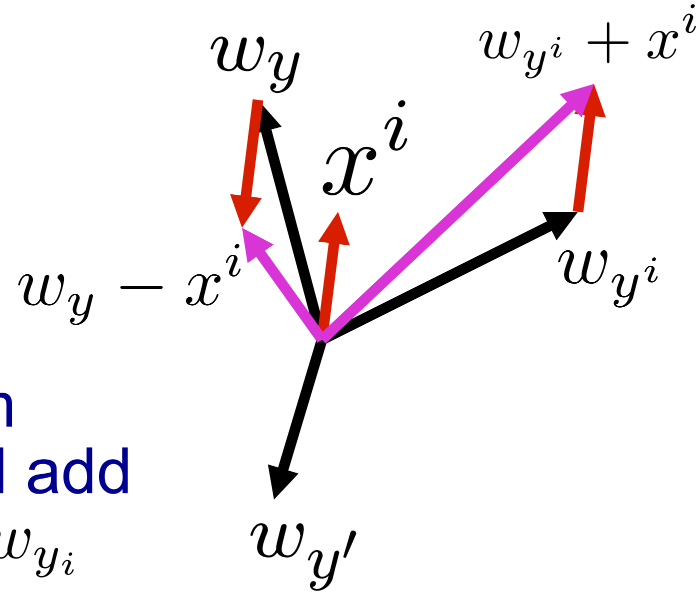
The Multi-class Perceptron Alg.

- Start with zero weights
- For t=1..T, i=1..n (T times over data)
  - Classify with current weights

$$y = \arg\max_y w_y \cdot x^i$$

  - If correct ($y=y_i$), no change!
  - If wrong: subtract features x^i from weights for predicted class w_y and add them to weights for correct class w_{y_i}

$$w_y = w_y - x^i$$
$$w_{y^i} = w_{y^i} + x^i$$



Interpreting Perceptron Weights

$$f(x) = \text{sign}(\sum_d w_d x_d + b)$$

$$\frac{\partial}{\partial x_7}(\sum_d w_d x_d + b) =?$$

Interpreting Perceptron Weights

$$f(x) = \operatorname{sign}\left(\sum_d w_d x_d + b\right)$$

$$\frac{\partial}{\partial x_7}\left(\sum_d w_d x_d + b\right) = w_7$$

Questions

» Is the perceptron guaranteed to converge?
» Under what circumstances?
» What does it converge to?
» How long does it take?

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How about a dataset with zero features?

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- Is the perceptron guaranteed to converge?
- Under what circumstances?
- What does it converge to?
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How about a dataset with zero features?

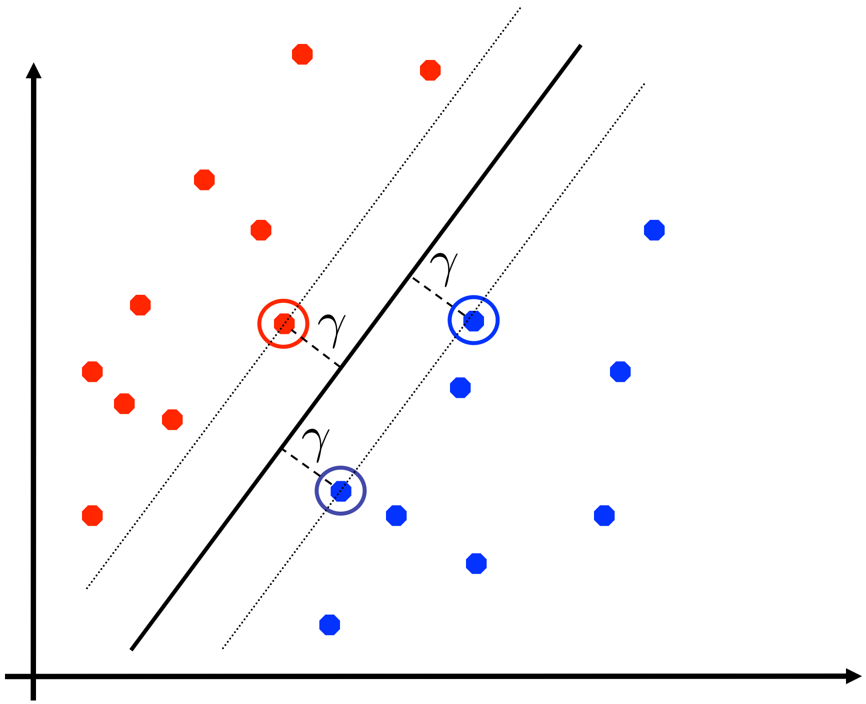
Linearly Separable -> Perceptron Guaranteed to Converge

Not Linearly Separable -> Never Converges

Linearly Separable (binary case)

- The data is linearly separable with *margin γ*, if:

$$\exists w.\forall t.y^t(w \cdot x^t) \geq \gamma > 0$$



- For $y^t=1$

$$w \cdot x^t \geq \gamma$$

- For $y^t=-1$

$$w \cdot x^t \leq -\gamma$$

Linearly Separable (binary case)

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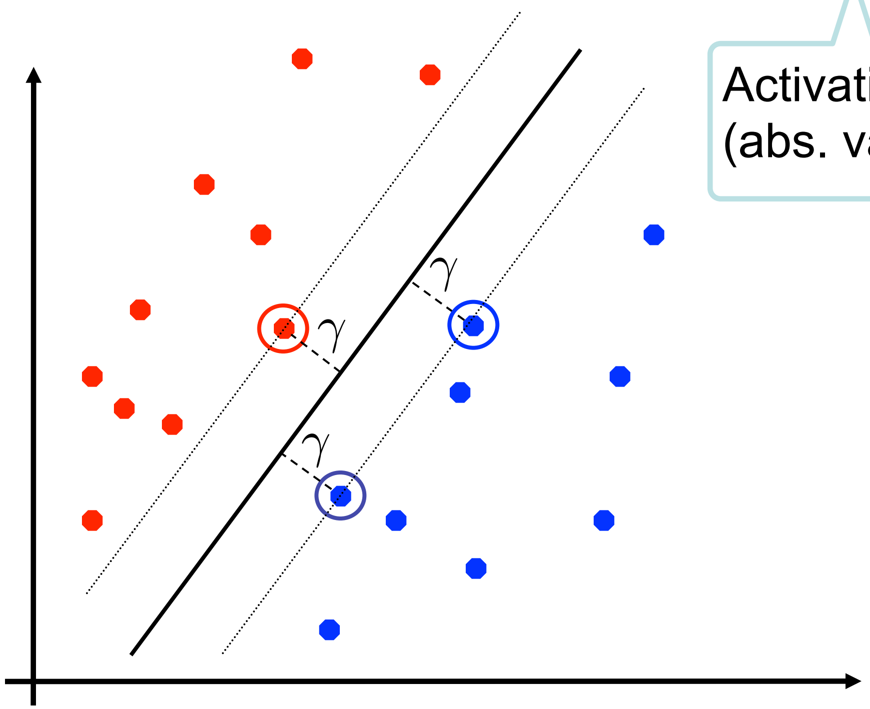
Activation (abs. val.)



- For $y^t=1$
  $$w \cdot x^t \geq \gamma$$
- For $y^t=-1$
  $$w \cdot x^t \leq -\gamma$$

Mistake Bound for Perceptron

$$\|x\|_2 = \sqrt{\sum_i x_i^2}$$

- Assume data is separable with margin γ:

$$\exists w^* \text{ s.t. } \|w^*\|_2 = 1 \text{ and } \forall t. y^t (w^* \cdot x^t) \geq \gamma$$

- Also assume there is a number R such that:

$$\forall t. \|x^t\|_2 \leq R$$

- Theorem: The number of mistakes (parameter updates) made by the perceptron is bounded:

$$mistakes \leq \frac{R^2}{\gamma^2}$$

Perceptron Convergence (by Induction)

- Let w^k be the weights after the k-th update (mistake), we will show that:

$$k^2\gamma^2 \leq \|w^k\|_2^2 \leq kR^2$$

- Therefore:

$$k \leq \frac{R^2}{\gamma^2}$$

- Because R and γ are fixed constants that do not change as you learn, there are a finite number of updates!
- Proof does each bound separately (next two slides)

Lower bound

Perceptron update:

$$w = w + y^t x^t$$

- Remember our margin assumption:

$$\exists w^* \text{ s.t. } \|w^*\|_2 = 1 \text{ and } \forall t. y^t(w^* \cdot x^t) \geq \gamma$$

- Now, by the definition of the perceptron update, for k-th mistake on t-th training example:

$$\begin{aligned} w^{k+1} \cdot w^* &= (w^k + y^t x^t) \cdot w^* \\ &= w^k \cdot w^* + y^t (w^* \cdot x^t) \\ &\geq w^k \cdot w^* + \gamma \end{aligned}$$

- So, by induction with $w^0=0$, for all k:

$$\begin{aligned} k\gamma &\leq w^k \cdot w^* \\ &\leq \|w^k\|_2 \end{aligned}$$

$$k^2\gamma^2 \leq \|w^k\|_2^2$$

Because:

$$w^k \cdot w^* \leq \|w^k\|_2 \times \|w^*\|_2$$

and $\|w^*\|_2 = 1$

Upper Bound

Perceptron update:
$$w = w + y^t x^t$$

Data Assumption:
$$\forall t. \|x^t\|_2 \leq R$$

- By the definition of the Perceptron update, for k-th mistake on t-th training example:

$$\|w^{k+1}\|_2^2 = \|w^k + y^t x^t\|_2^2$$
$$= \|w^k\|_2^2 + (y^t)^2\|x^t\|_2^2 + 2y^t x^t \cdot w^k$$
$$\leq \|w^k\|_2^2 + R^2$$

$\leq R^2$ because $(y^t)^2 = 1$ and $\|x^t\|_2 \leq R$

< 0 because Perceptron made error (y^t has different sign than x^t•w^t)

- So, by induction with w_0=0 have, for all k:

$$\|w_k\|_2^2 \leq kR^2$$

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$$k \leq \frac{R^2}{\gamma^2}$$

- Because R and γ are fixed constants that do not change as you learn, there are a finite number of updates!1
- If there is a linear separator, Perceptron will find it!!!
- Proof does each bound separately (last two slides)

From Logistic Regression to the Perceptron: 2 easy steps!

Perceptron update when y is {-1,1}:

$$w = w + y^j x^j$$

- Logistic Regression: (in vector notation): y is {0,1}

$$w = w + \eta \sum_j [y^j - P(y^j|x^j, w)]x^j$$

- Perceptron: when y is {0,1}:

$$w = w + [y^j - sign^0(w \cdot x^j)]x^j$$

  - $sign^0(x)$ = +1 if x>0 and 0 otherwise

Differences?

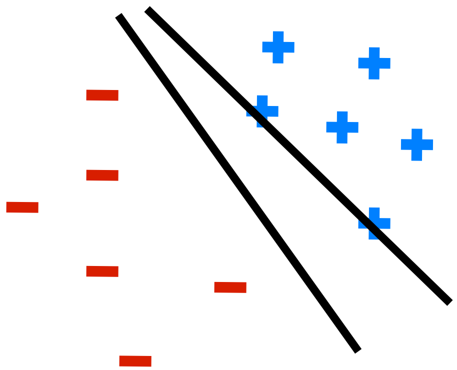
- Drop the Σ_j over training examples: online vs. batch learning
- Drop the dist'n: probabilistic vs. error driven learning

Properties of Perceptrons

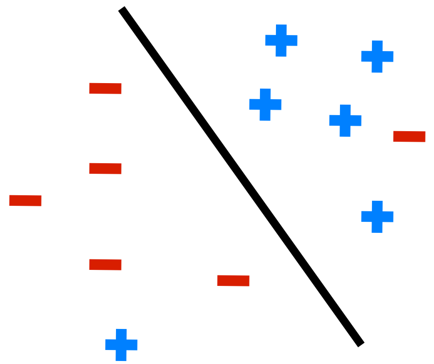
- Separability: some parameters get the training set perfectly correct
- Convergence: if the training is separable, perceptron will eventually converge (binary case)
- Mistake Bound: the maximum number of mistakes (binary case) related to the margin or degree of separability

$$mistakes \leq \frac{R^2}{\gamma^2}$$

Separable

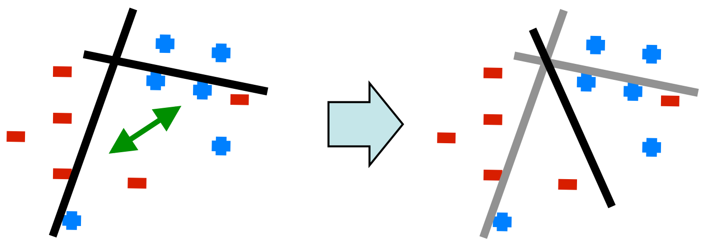


Non-Separable

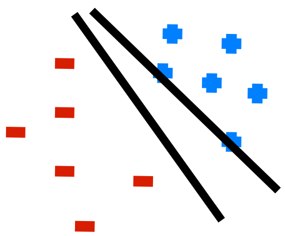


Problems with the Perceptron

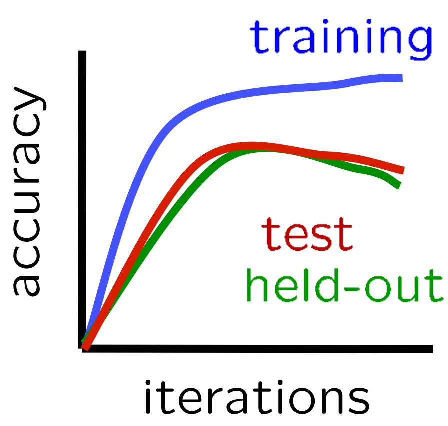
- Noise: if the data isn't separable, weights might thrash
  - Averaging weight vectors over time can help (averaged perceptron)



- Mediocre generalization: finds a "barely" separating solution



- Overtraining: test / held-out accuracy usually rises, then falls
  - Overtraining is a kind of overfitting



Voted Perceptron

- **Problem:**
  - Later training examples are counted more than earlier ones
- **Example:**
  - 10,000 training examples
  - After first 100, no no more updates until the 10,000th example
  - (could ruin the weight vector that did well on most of the data)

Voted Perceptron

» Would like weight vectors that "survive" to get more say

» **Voting!**

» As the perceptron runs, remember how long each hyperplane survives

» Prediction:

$$\hat{y} = \text{sign}\left(\sum_{k=1}^{K} c^{(k)} \text{sign}(w^{(k)} \cdot \hat{x} + b^{(k)})\right)$$

Voted Perceptron

» Would like weight vectors that "survive" to get more say

» **Voting!**

» As the perceptron runs, remember how long each hyperplane survives

Too slow / too much memory!

» Prediction:

$$\hat{y} = \text{sign}\left(\sum_{k=1}^{K} c^{(k)} \text{sign}(w^{(k)} \cdot \hat{x} + b^{(k)})\right)$$

Averaged Perceptron (much more practical)

$$\hat{y} = \operatorname{sign}\left(\sum_{k=1}^{K} c^{(k)}(w^{(k)} \cdot \hat{x} + b^{(k)})\right)$$

$$\hat{y} = \operatorname{sign}\left(\left(\sum_{k=1}^{K} c^{(k)} w^{(k)}\right) \cdot \hat{x} + \sum_{k=1}^{K} c^{(k)} b^{(k)})\right)$$

Averaged Perceptron (much more practical)

Algorithm 7 AveragedPerceptronTrain($\mathbf{D}$, *MaxIter*)

$\boldsymbol{w} \leftarrow \langle 0, 0, \ldots 0 \rangle \quad , \quad b \leftarrow 0$ // initialize weights and bias
$\boldsymbol{u} \leftarrow \langle 0, 0, \ldots 0 \rangle \quad , \quad \beta \leftarrow 0$ // initialize cached weights and bias
$c \leftarrow 1$ // initialize example counter to one
for $iter = 1 \ldots MaxIter$ **do**
    for all $(x,y) \in \mathbf{D}$ **do**
        if $y(\boldsymbol{w} \cdot \boldsymbol{x} + b) \leq 0$ **then**
            $\boldsymbol{w} \leftarrow \boldsymbol{w} + y\,\boldsymbol{x}$ // update weights
            $b \leftarrow b + y$ // update bias
            $\boldsymbol{u} \leftarrow \boldsymbol{u} + y\,c\,\boldsymbol{x}$ // update cached weights
            $\beta \leftarrow \beta + y\,c$ // update cached bias
        end if
        $c \leftarrow c + 1$ // increment counter regardless of update
    end for
end for
return $\boldsymbol{w} - \frac{1}{c}\,\boldsymbol{u},\ b - \frac{1}{c}\,\beta$ // return averaged weights and bias

Averaged Perceptron (much more practical)

Algorithm 7 AveragedPerceptronTrain($\mathbf{D}$, *MaxIter*)

$w \leftarrow \langle 0, 0, \ldots 0 \rangle \quad , \quad b \leftarrow 0$ // initialize weights and bias
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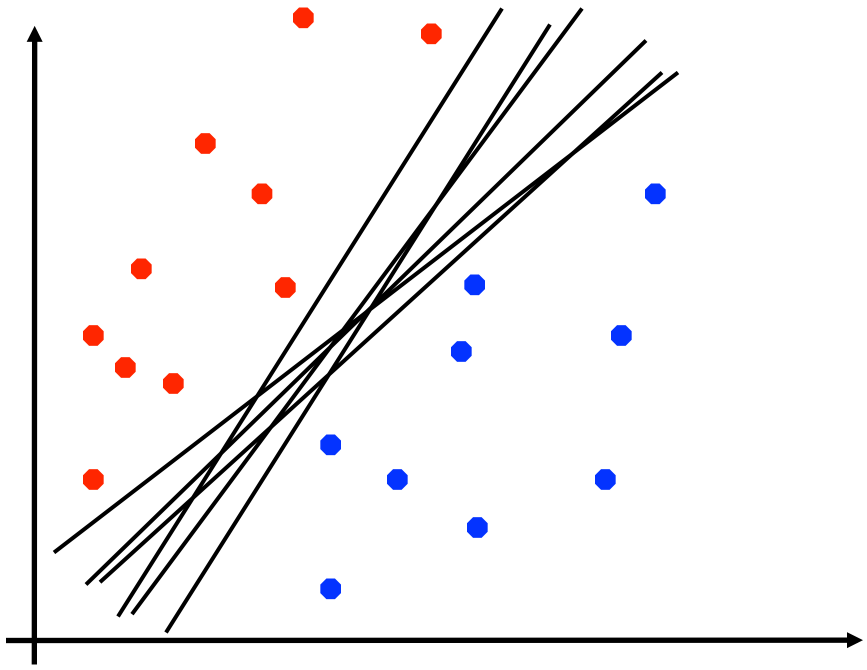
for $iter = 1 \ldots MaxIter$ **do**
    for all $(x,y) \in \mathbf{D}$ **do**
        if $y(w \cdot x + b) \leq 0$ **then**
            $w \leftarrow w + y\, x$ // update weights
            $b \leftarrow b + y$ // update bias
            $u \leftarrow u + y\, c\, x$ // update cached weights
            $\beta \leftarrow \beta + y\, c$ // update cached bias
        end if
        $c \leftarrow c + 1$ // increment counter regardless of update
    end for
end for
return $w - \frac{1}{c}\, u,\ b - \frac{1}{c}\, \beta$ // return averaged weights and bias

Efficiency Trick

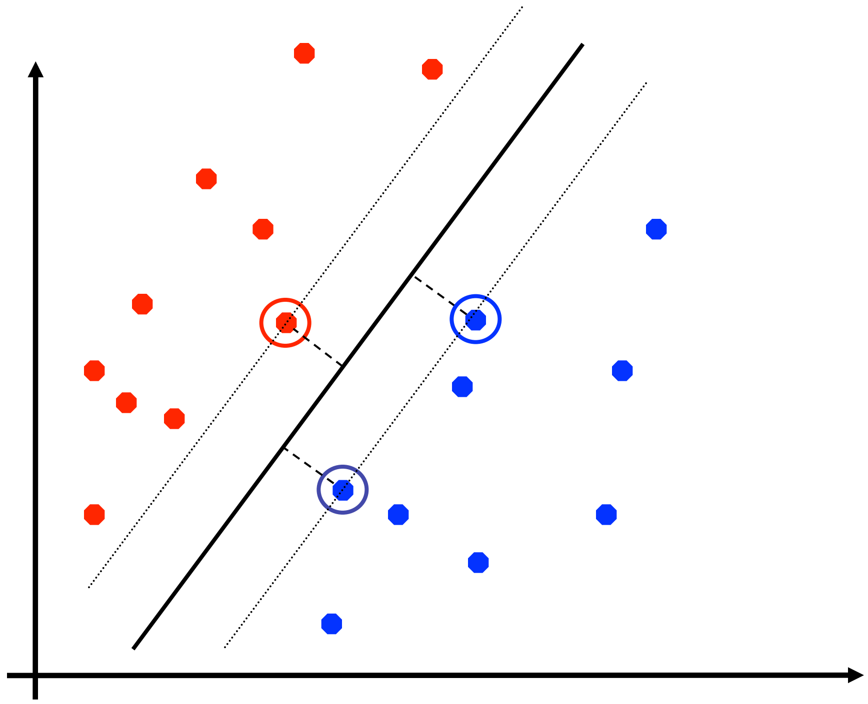
Linear Separators

- Which of these linear separators is optimal?



Support Vector Machines

- Maximizing the margin: good according to intuition, theory, practice
- Support vector machines (SVMs) find the separator with max margin



SVM

$$\min_w \frac{1}{2}||w||^2$$

$$\forall i, y \quad w_{y^*} \cdot x^i \geq w_y \cdot x^i + 1$$