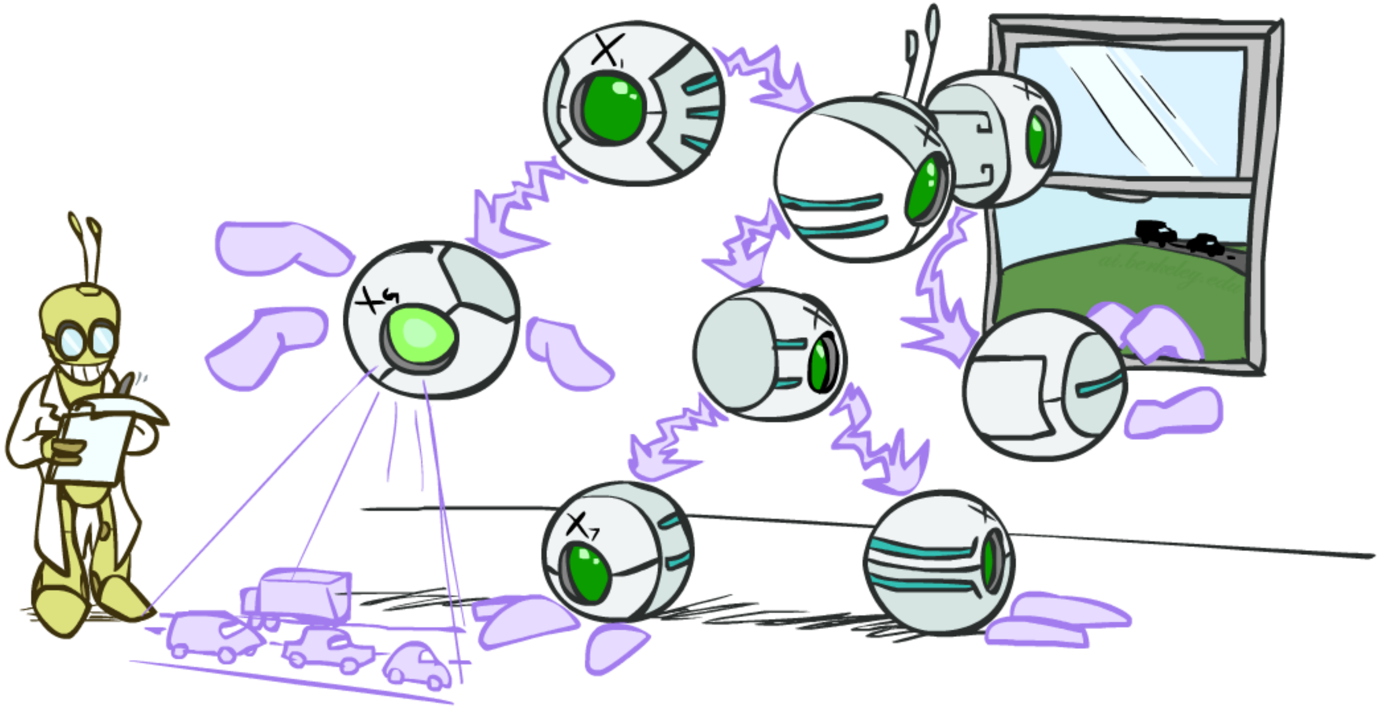
CS 188: Artificial Intelligence

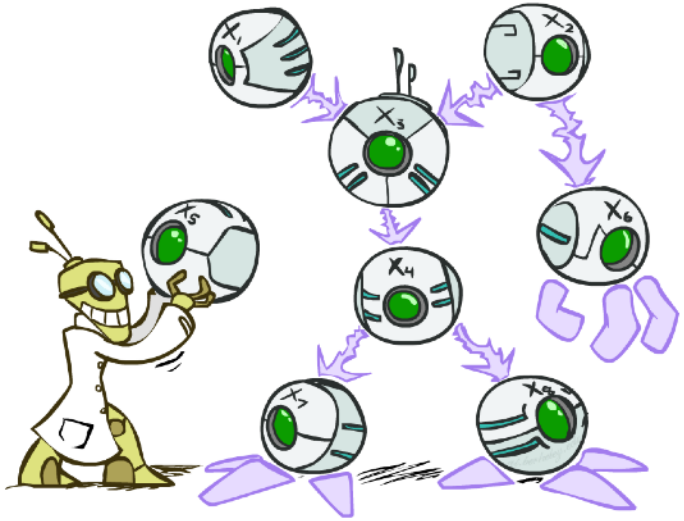
Bayes' Nets: Inference



Instructors: Dan Klein and Pieter Abbeel --- University of California, Berkeley

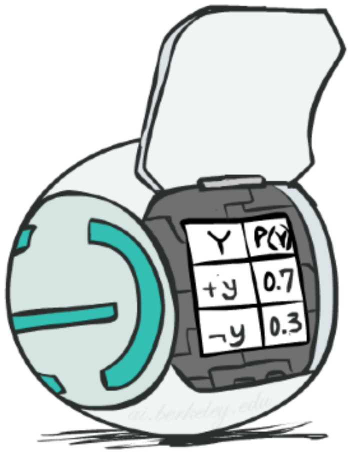
Bayes' Net Representation

- A directed, acyclic graph, one node per random variable
- A conditional probability table (CPT) for each node
  - A collection of distributions over X, one for each combination of parents' values $P(X|a_1 \ldots a_n)$



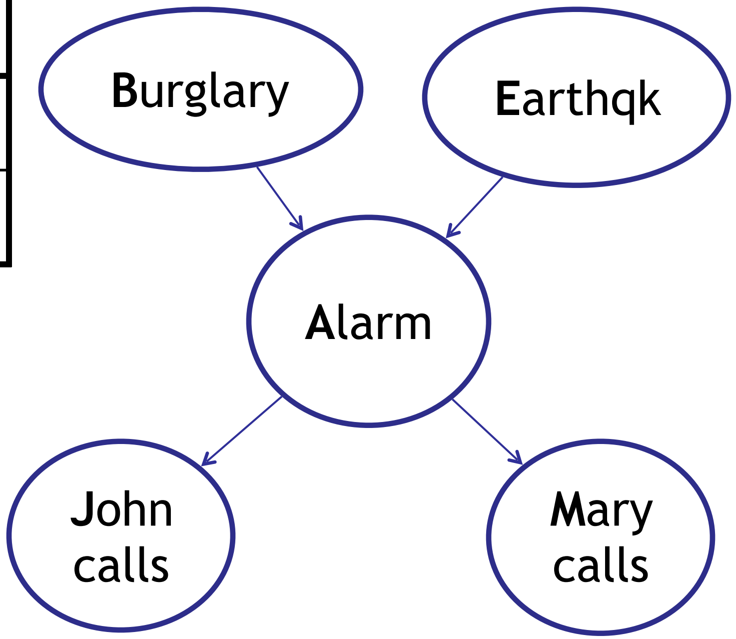
- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

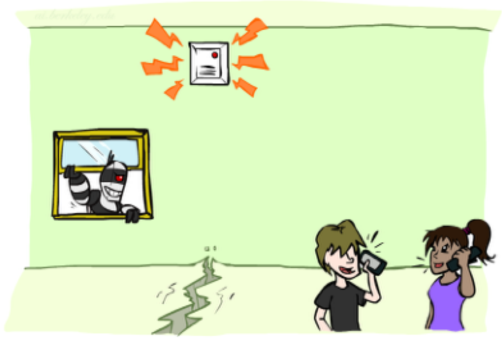


Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998



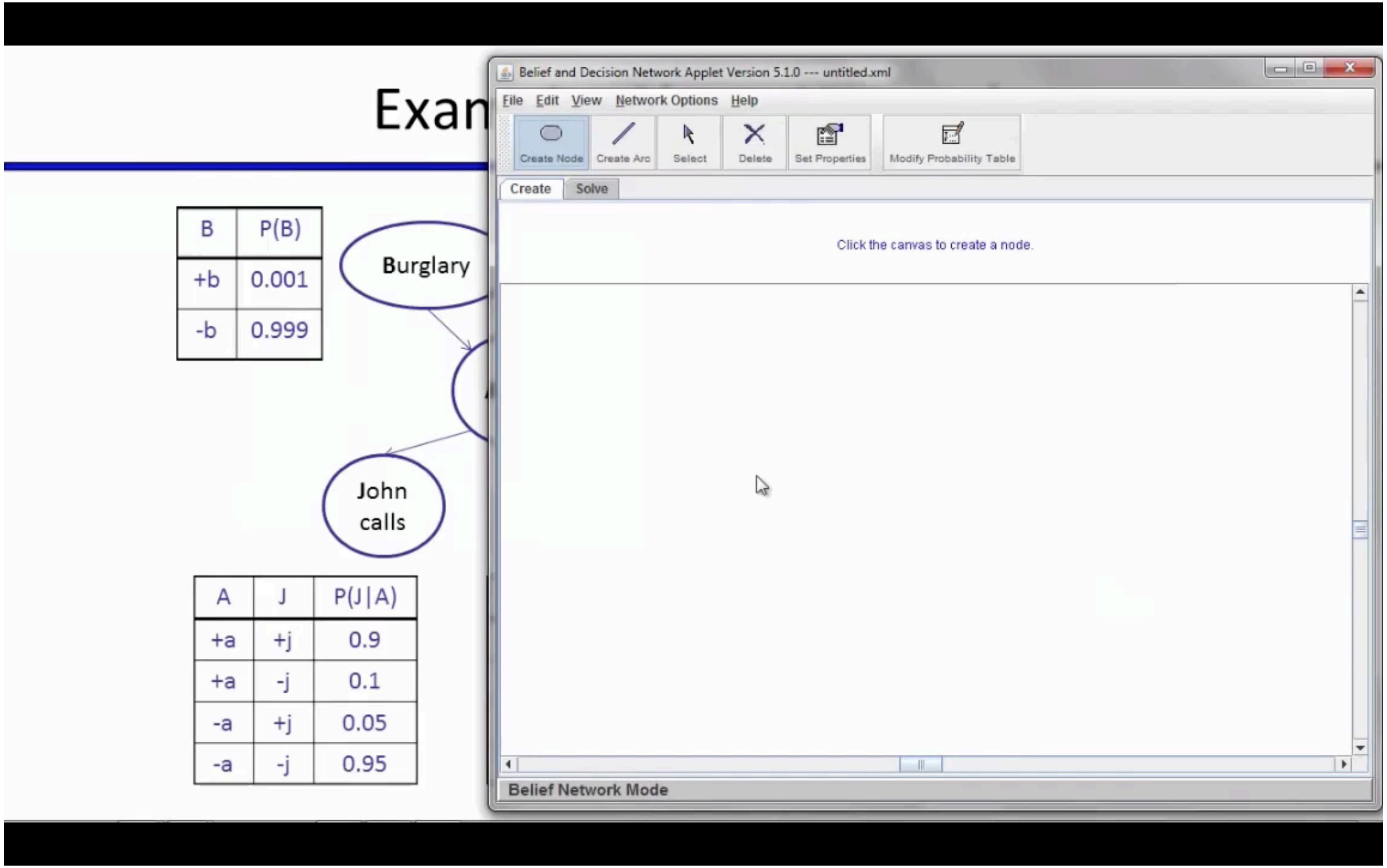
A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

[Demo: BN Applet]

Video of Demo BN Applet



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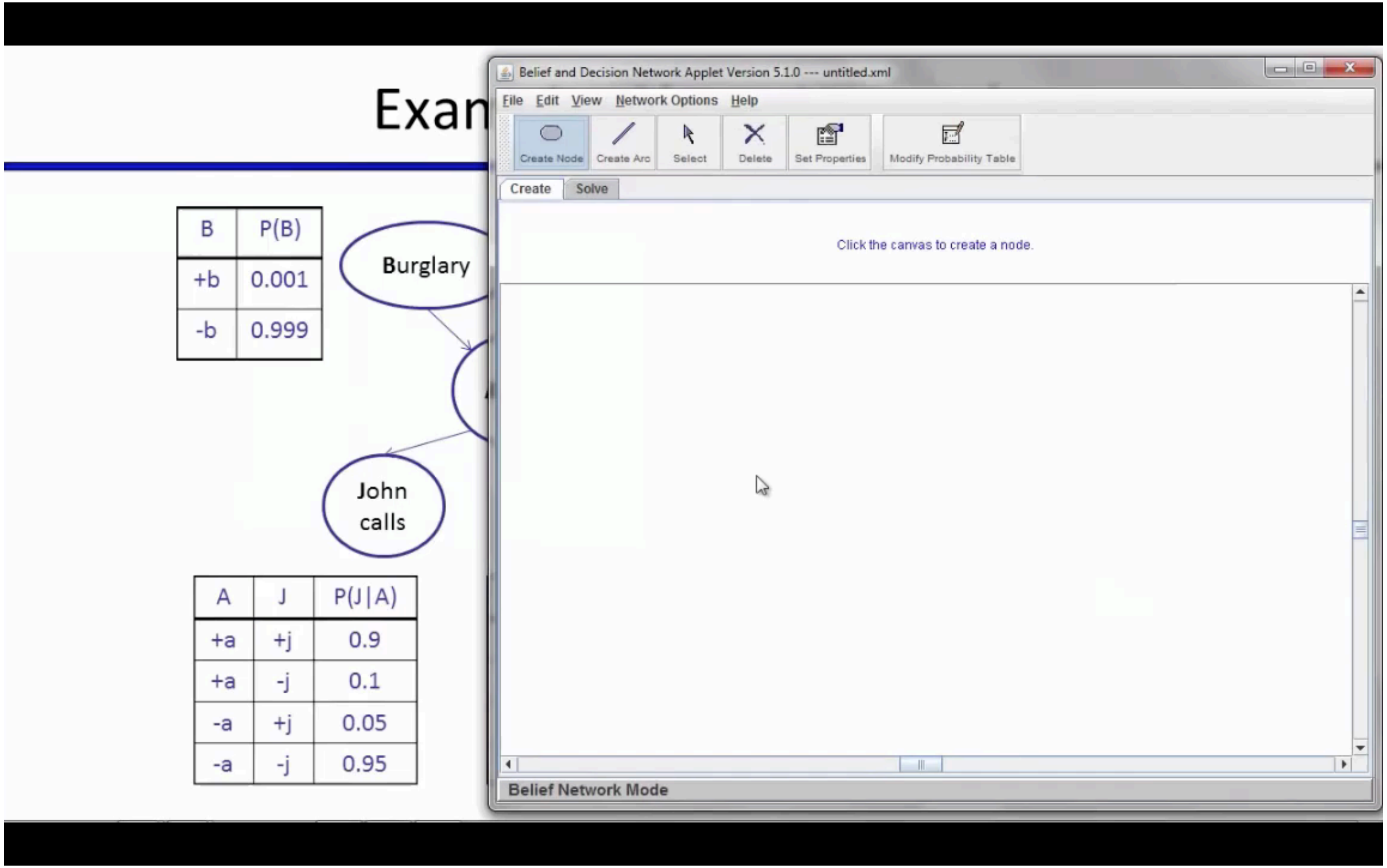

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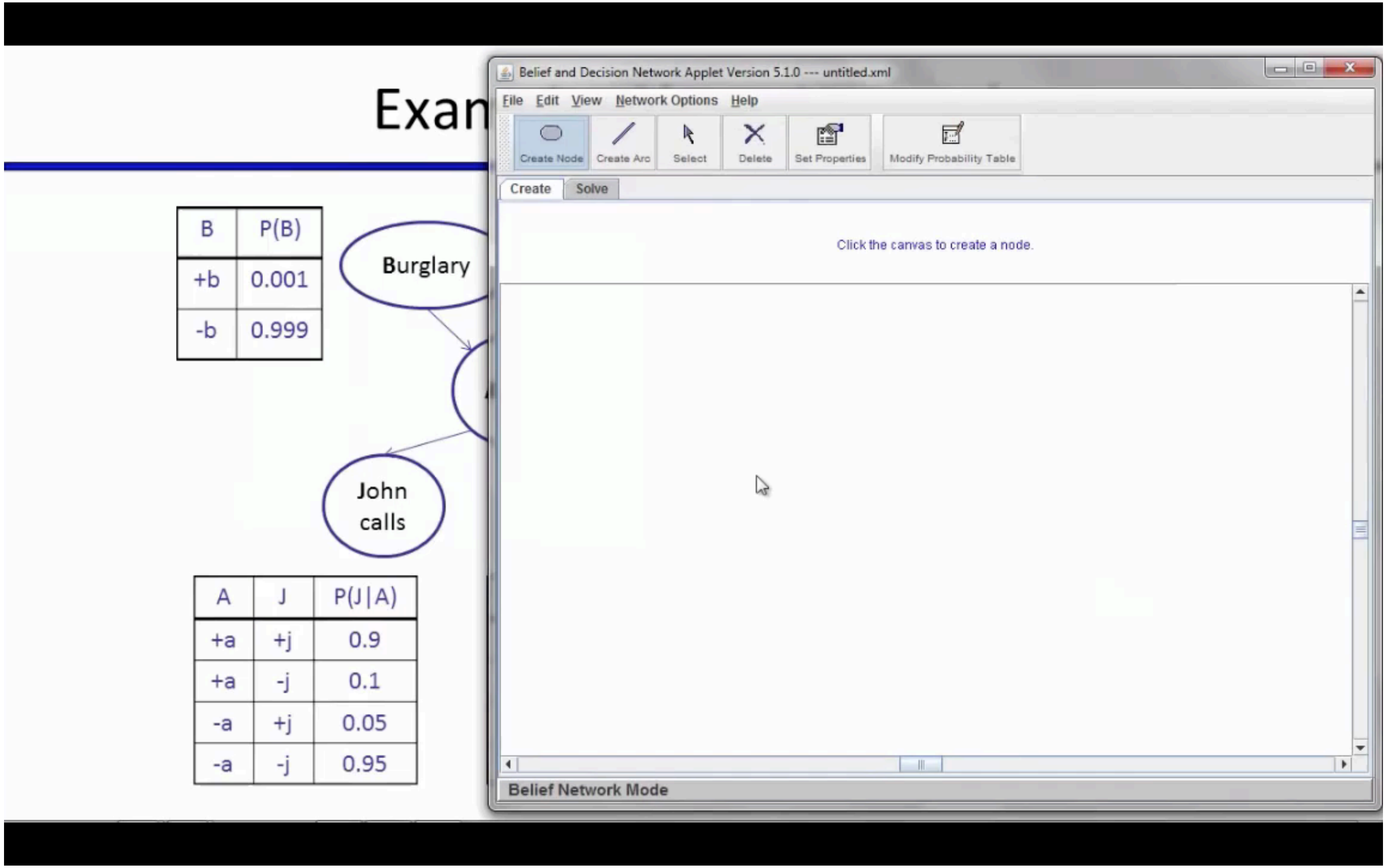

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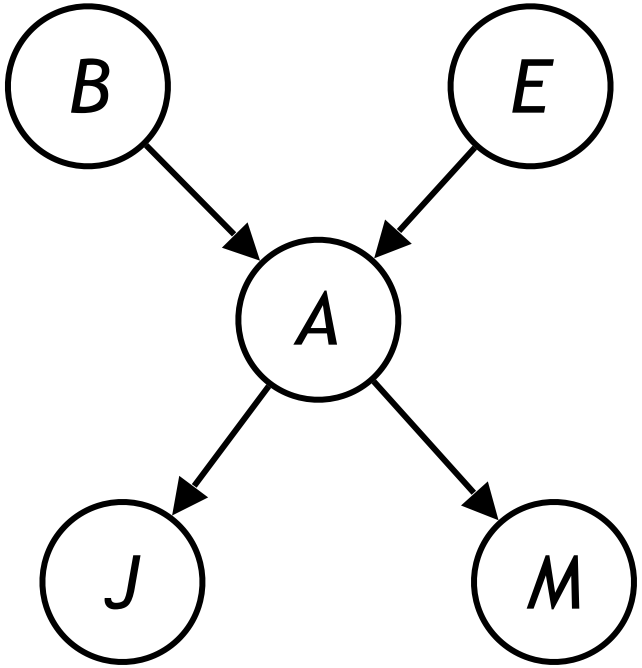

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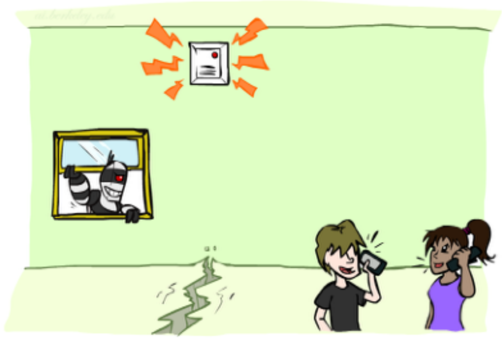

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$$P(+b, -e, +a, -j, +m) =$$

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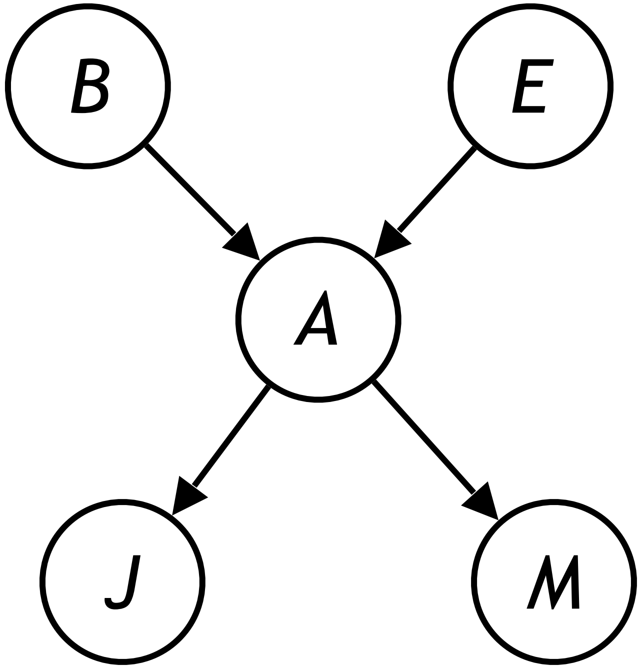

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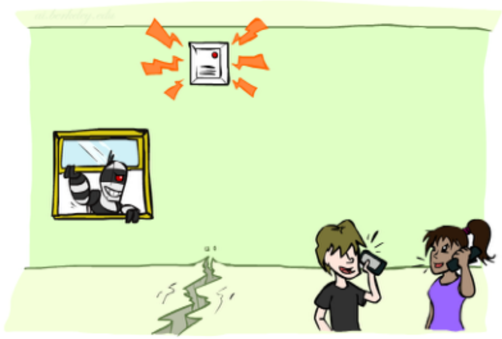

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$$P(+b, -e, +a, -j, +m) =$$
$$P(+b)P(-e)P(+a| + b, -e)P(-j| + a)P(+m| + a) =$$

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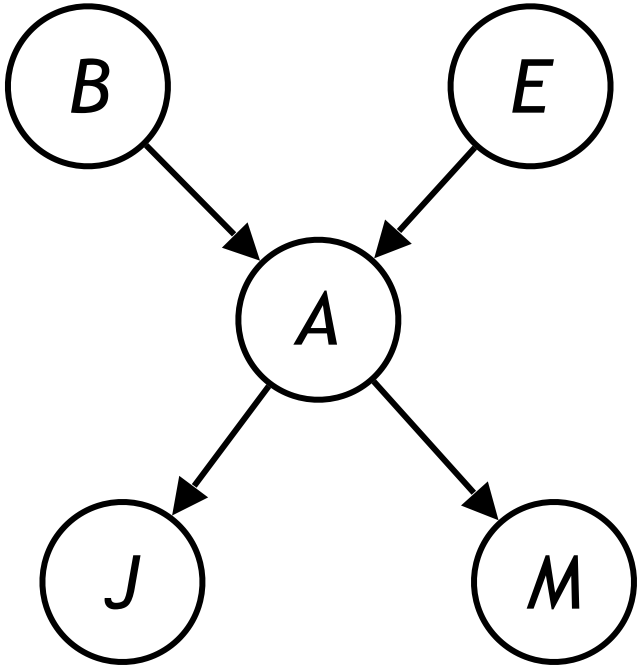

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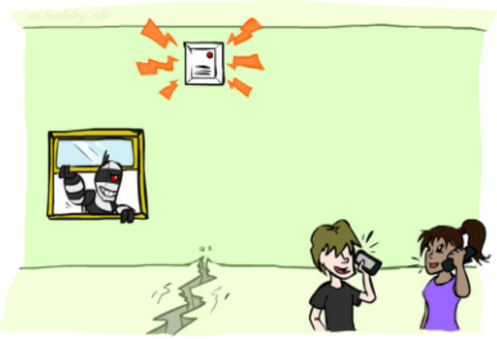

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$$P(+b,-e,+a,-j,+m) =$$

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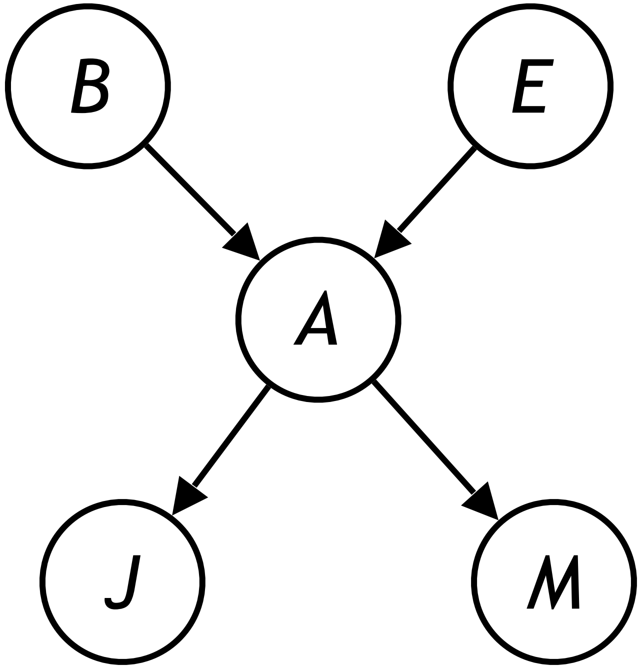

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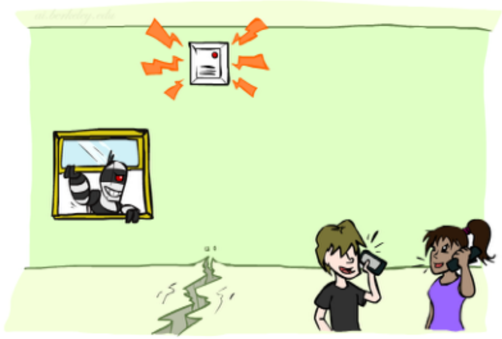

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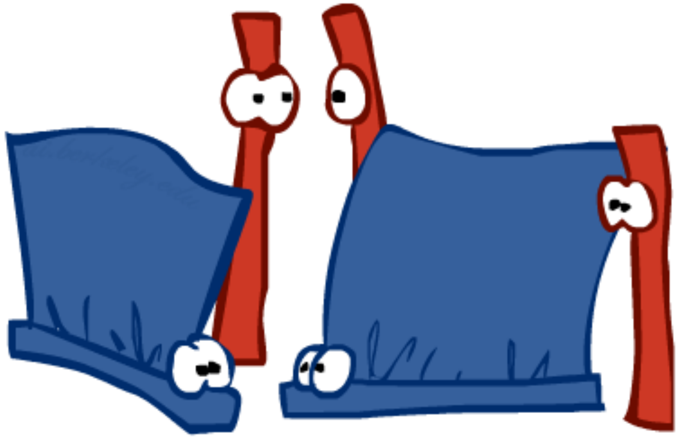
$$0.001 \times 0.998 \times 0.94 \times 0.1 \times 0.7$$

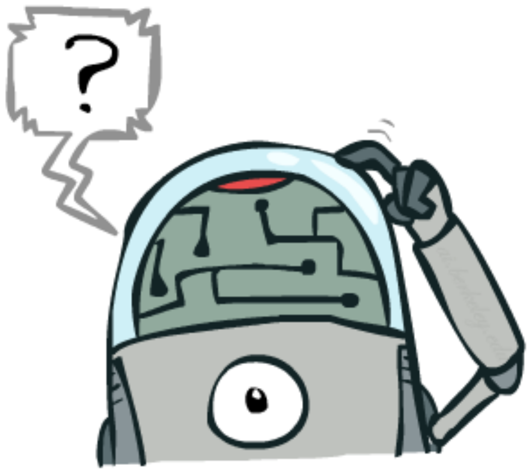
Bayes' Nets

- ✓ Representation
- ✓ Conditional Independences
- Probabilistic Inference
  - Enumeration (exact, exponential complexity)
  - Variable elimination (exact, worst-case exponential complexity, often better)
  - Inference is NP-complete
  - Sampling (approximate)
- Learning Bayes' Nets from Data

Inference

- Inference: calculating some useful quantity from a joint probability distribution





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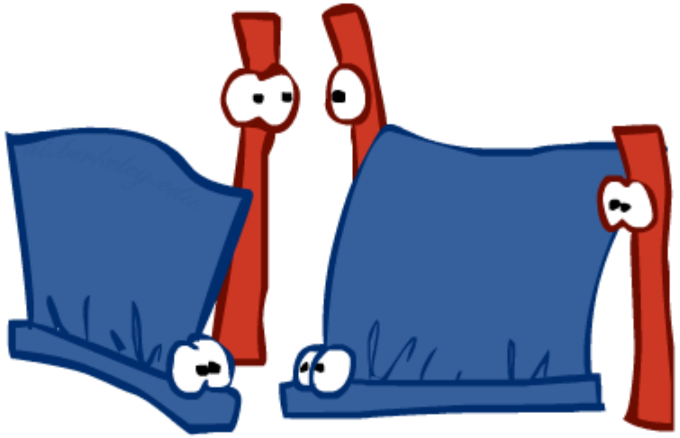

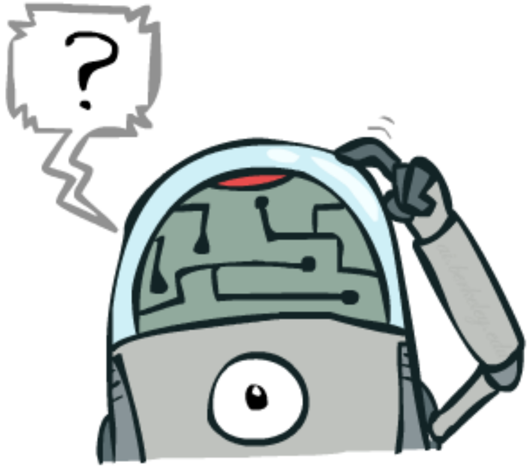

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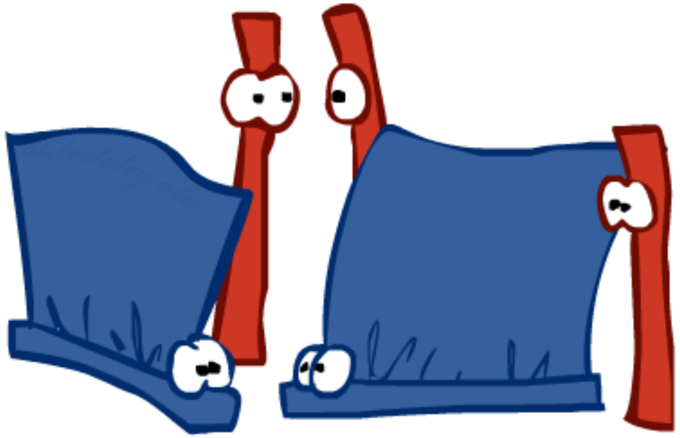
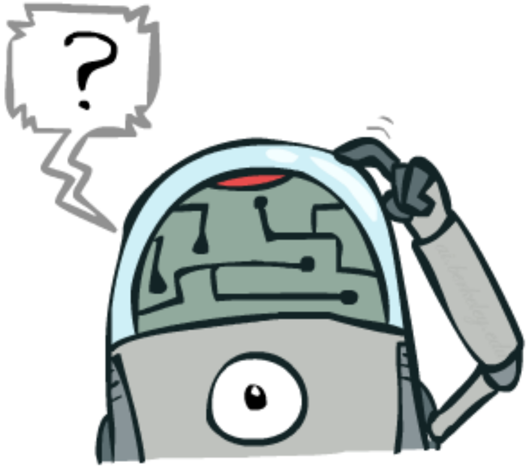
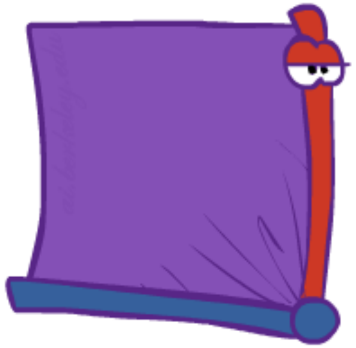
$$P(Q|E_1 = e_1, \dots E_k = e_k)$$







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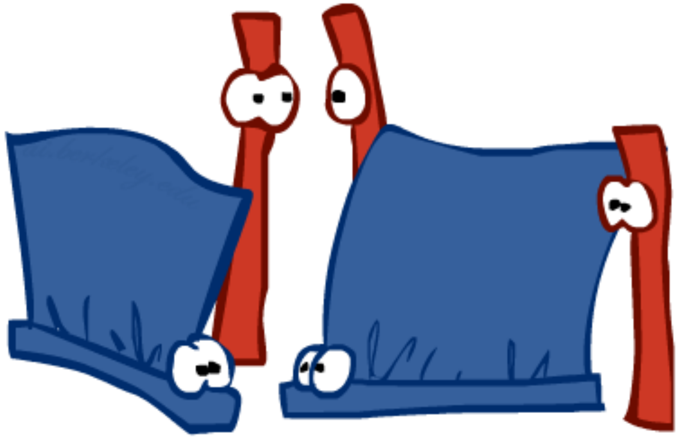
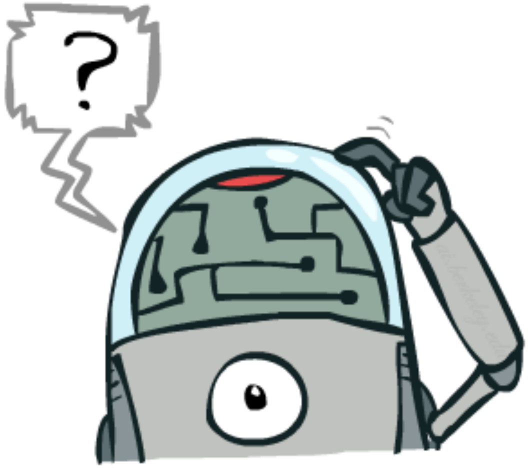
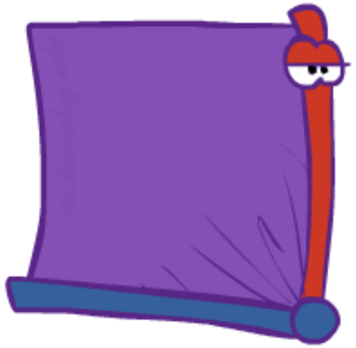
    $$\text{argmax}_q \; P(Q = q|E_1 = e_1 \dots)$$







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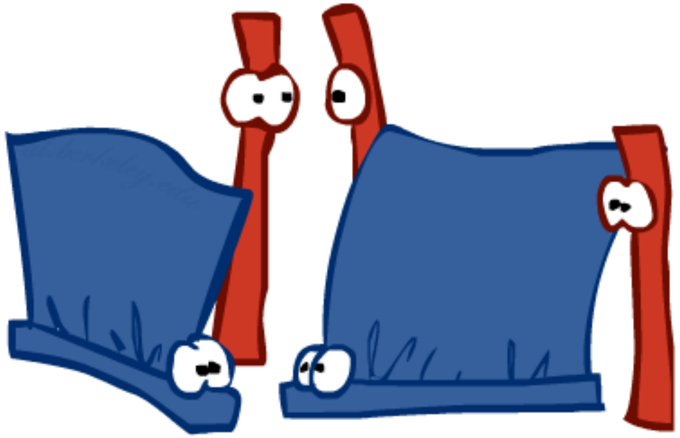

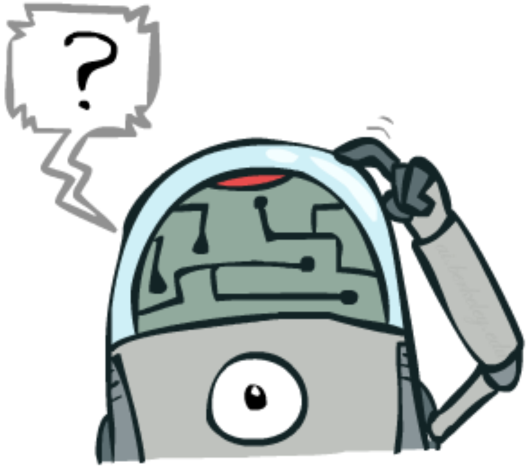

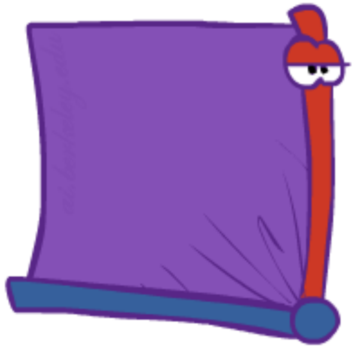

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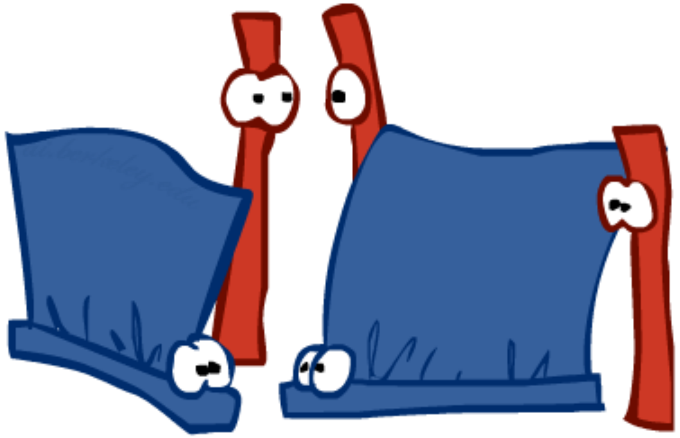

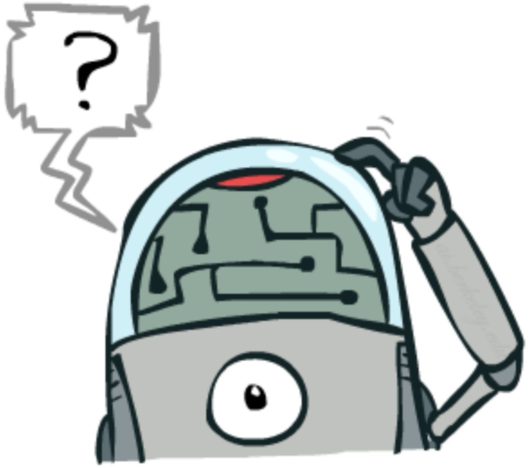

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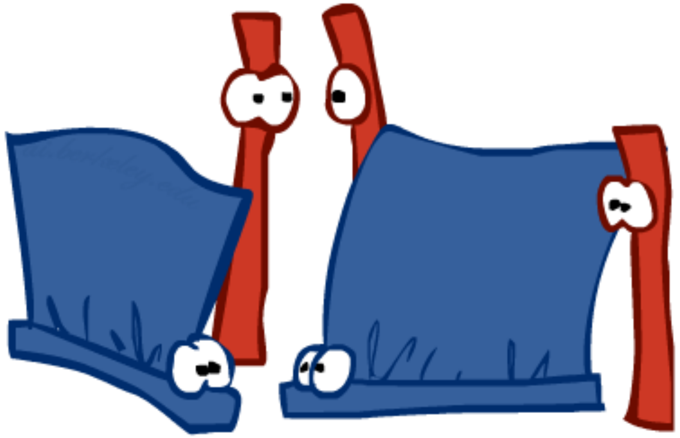
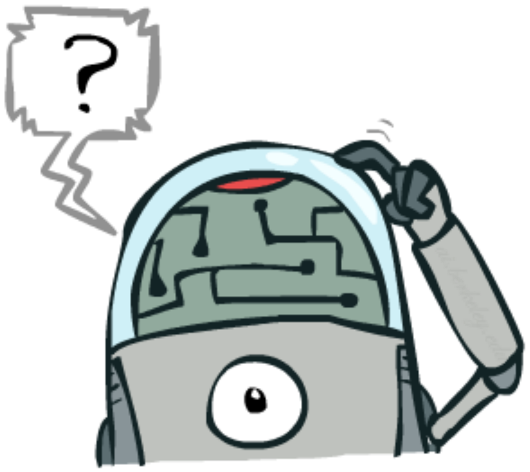


Inference by Enumeration

- General case:
  - Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
  - Query* variable: Q
  - Hidden variables: $H_1 \dots H_r$

$\left.\begin{array}{l} E_1 \dots E_k = e_1 \dots e_k \\ Q \\ H_1 \dots H_r \end{array}\right\} X_1, X_2, \dots X_n$ *All variables*

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- We want:

** Works fine with multiple query variables, too*

$$P(Q|e_1 \dots e_k)$$

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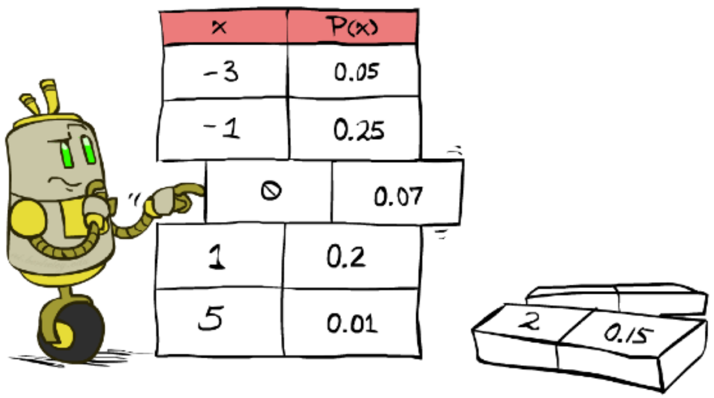

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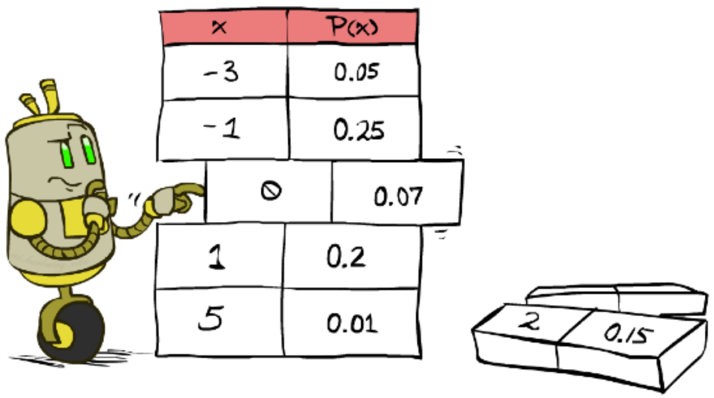
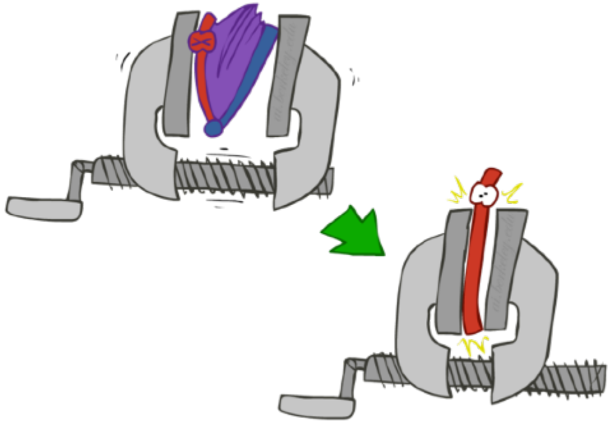


- Step 2: Sum out H to get joint of Query and evidence



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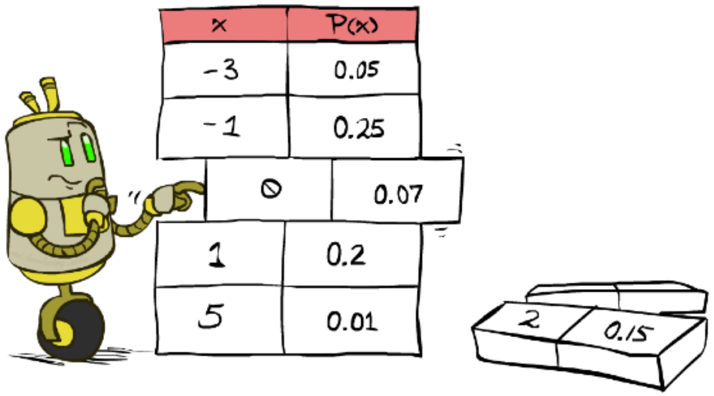

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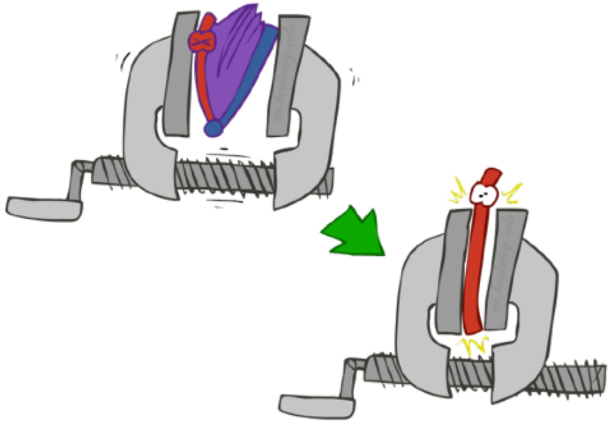

$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots X_n}$$

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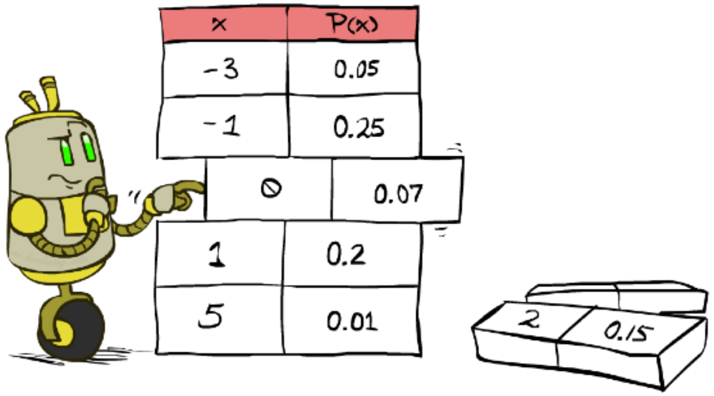

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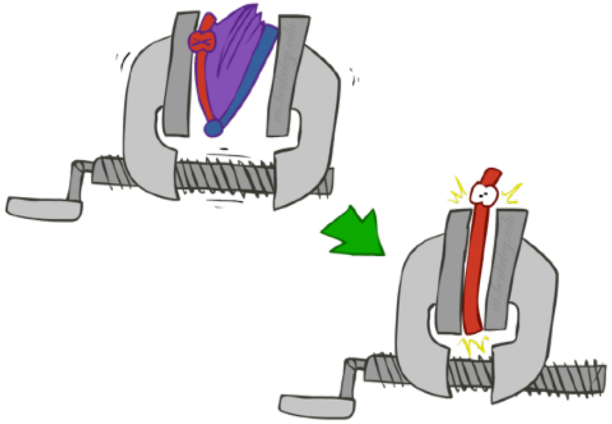

- Step 3: Normalize

$$\times \frac{1}{Z}$$

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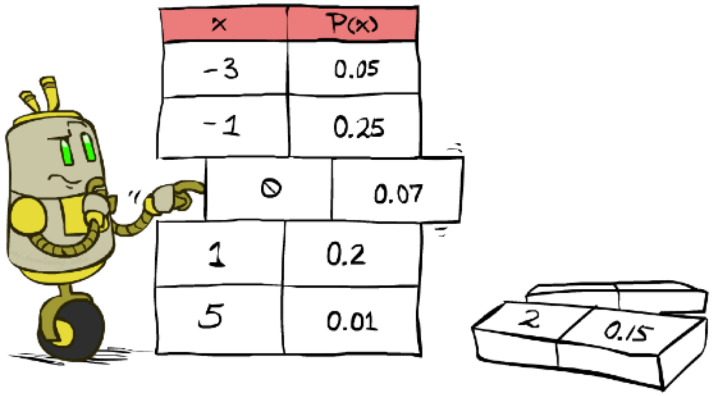
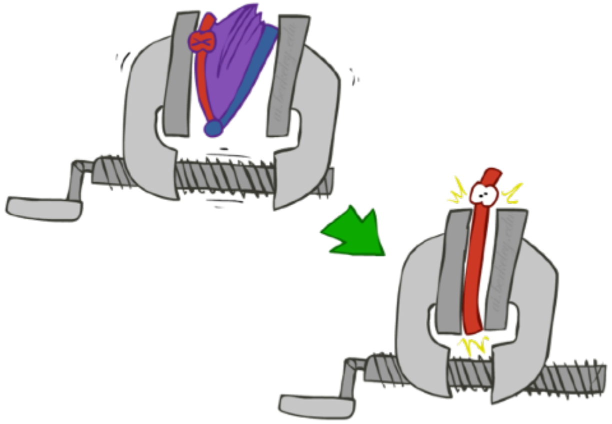


- Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots X_n}$$

- Step 3: Normalize

$$\times \frac{1}{Z}$$

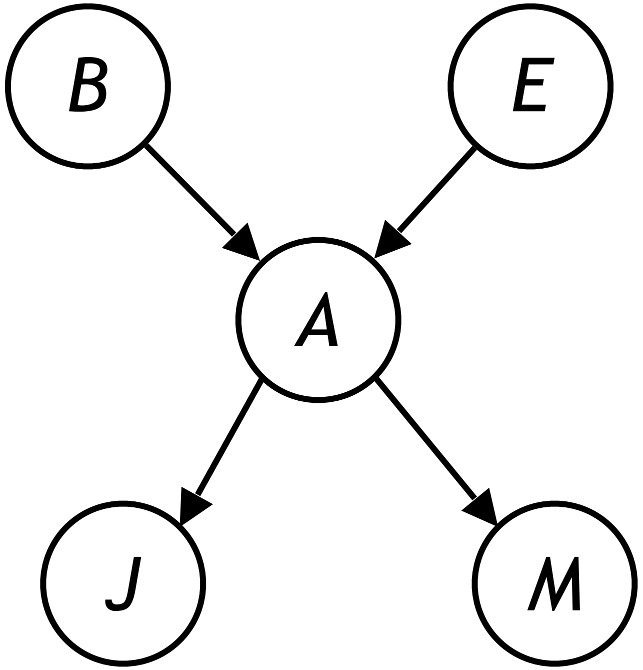
$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

Inference by Enumeration in Bayes' Net

- Given unlimited time, inference in BNs is easy
- Reminder of inference by enumeration by example:

$$P(B \mid +j, +m)$$



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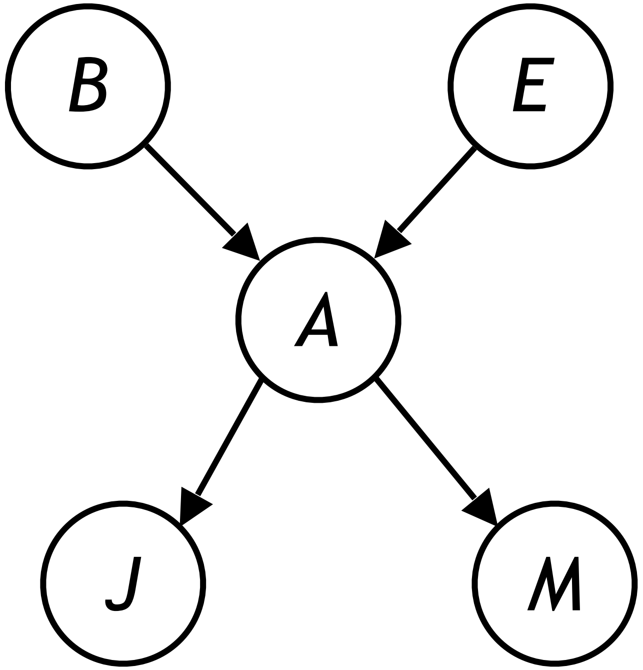
$$P(B \mid +j, +m) \propto_B P(B, +j, +m)$$



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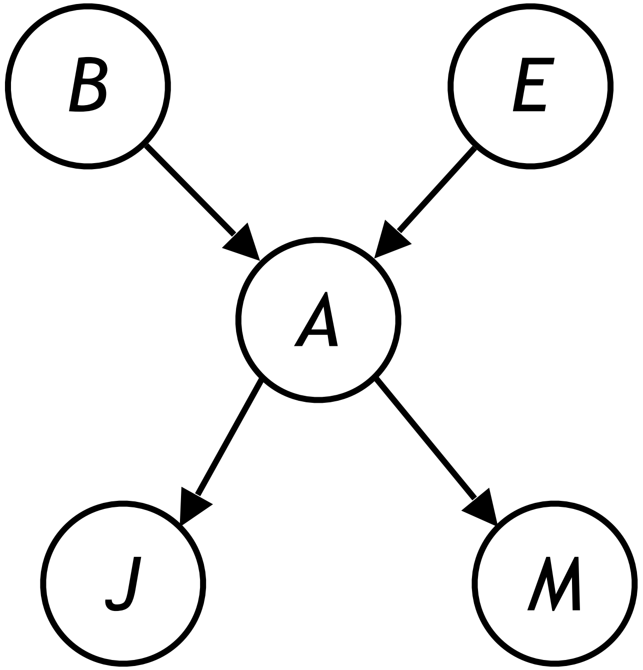
$$= \sum_{e,a} P(B, e, a, +j, +m)$$



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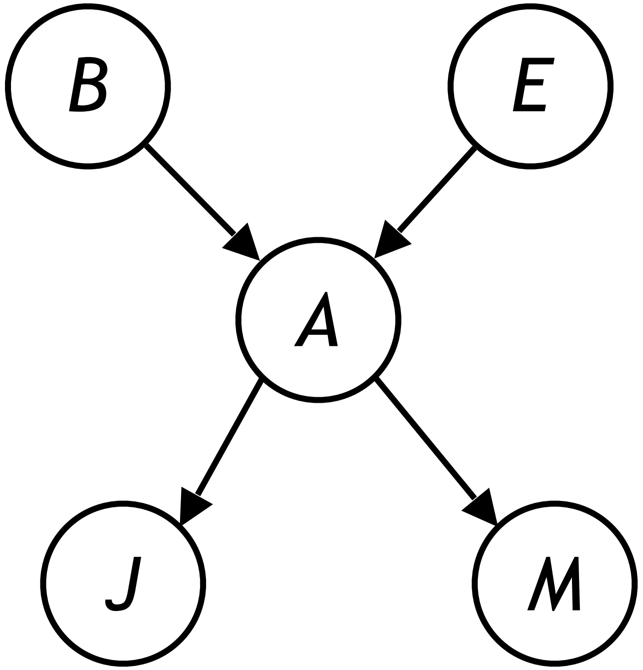

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$$= \sum_{e,a} P(B, e, a, +j, +m)$$

$$= \sum_{e,a} P(B)P(e)P(a|B, e)P(+j|a)P(+m|a)$$

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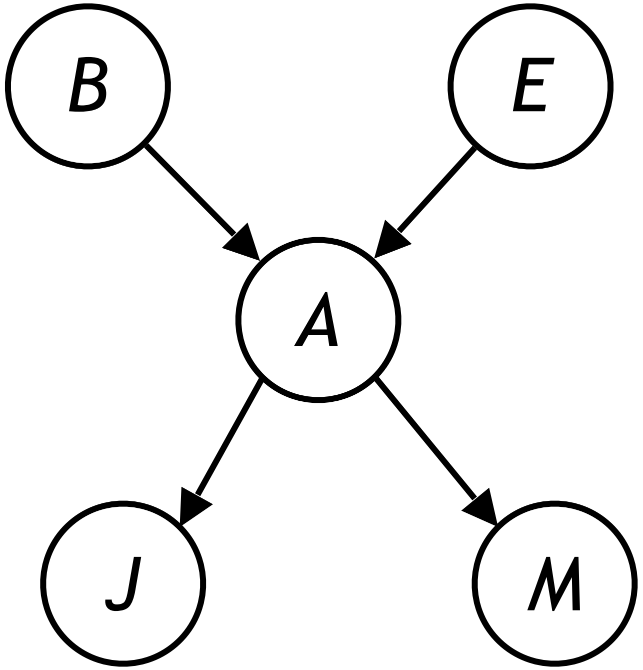

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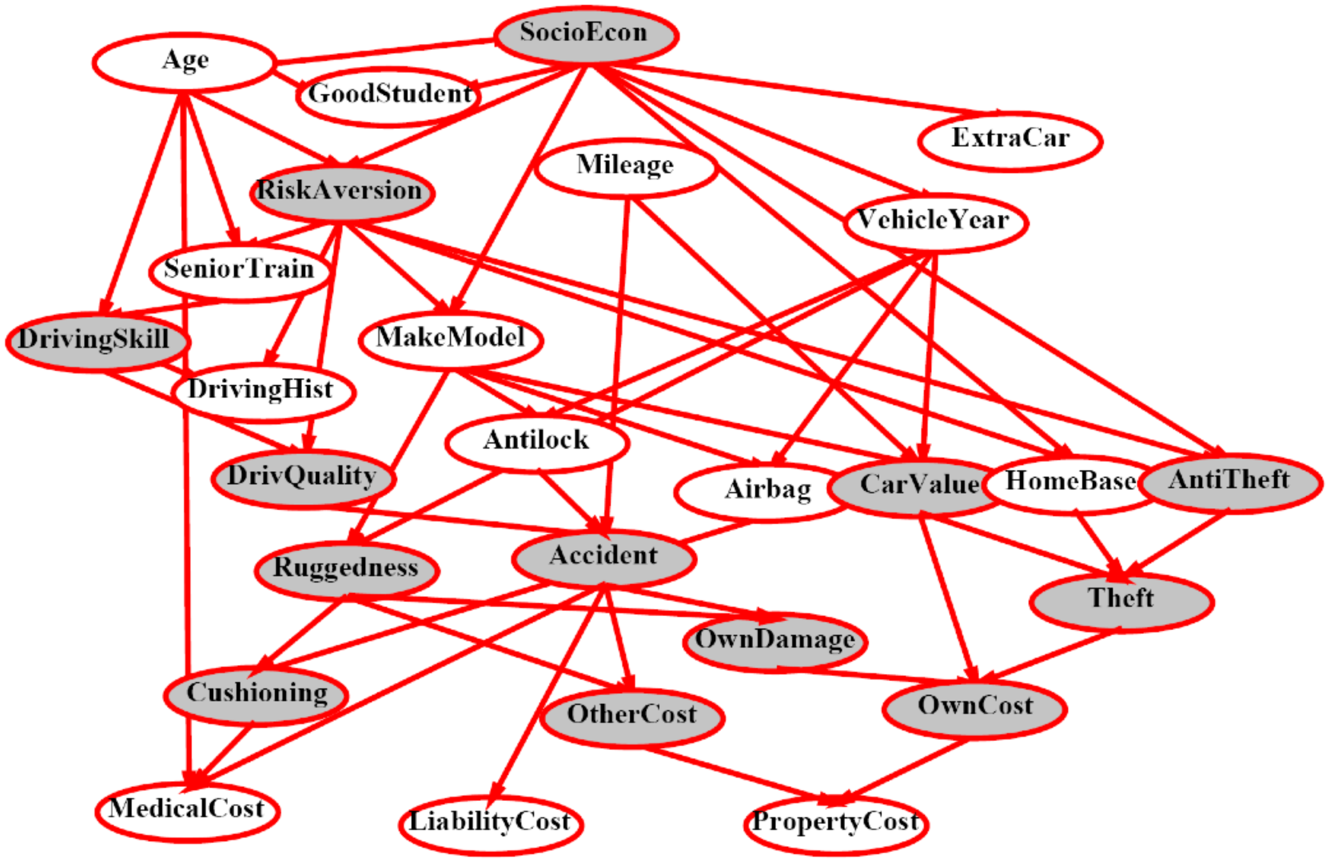
$$= P(B)P(+e)P(+a|B, +e)P(+j| + a)P(+m| + a) + P(B)P(+e)P(-a|B, +e)P(+j| - a)P(+m| - a)$$
$$P(B)P(-e)P(+a|B, -e)P(+j| + a)P(+m| + a) + P(B)P(-e)P(-a|B, -e)P(+j| - a)P(+m| - a)$$

Inference by Enumeration?

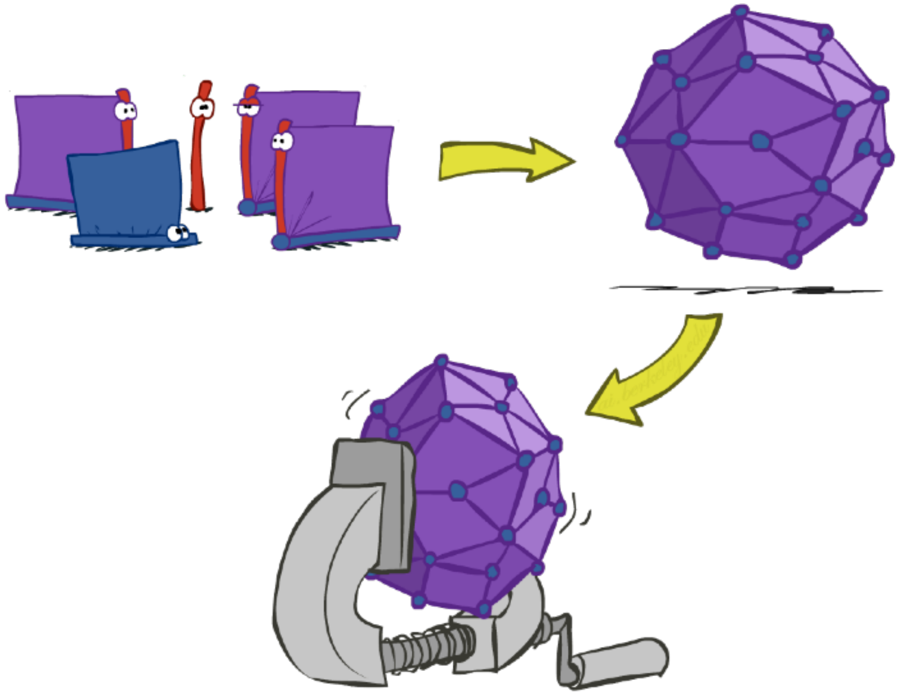


$$P(Antilock|observed\ variables) = ?$$

Inference by Enumeration vs. Variable Elimination

Inference by Enumeration vs. Variable Elimination

- Why is inference by enumeration so slow?
  - You join up the whole joint distribution before you sum out the hidden variables



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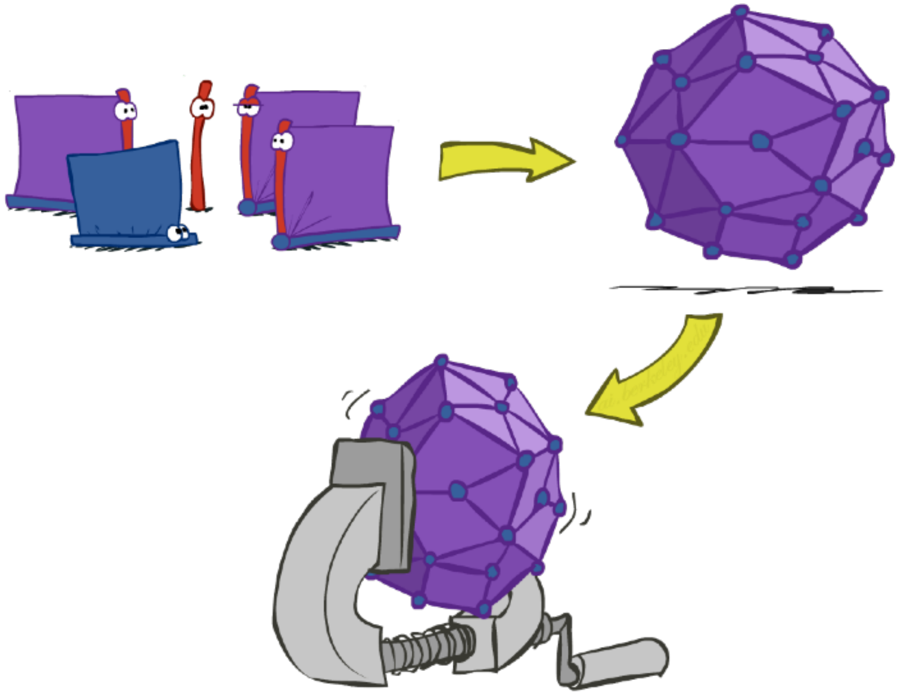
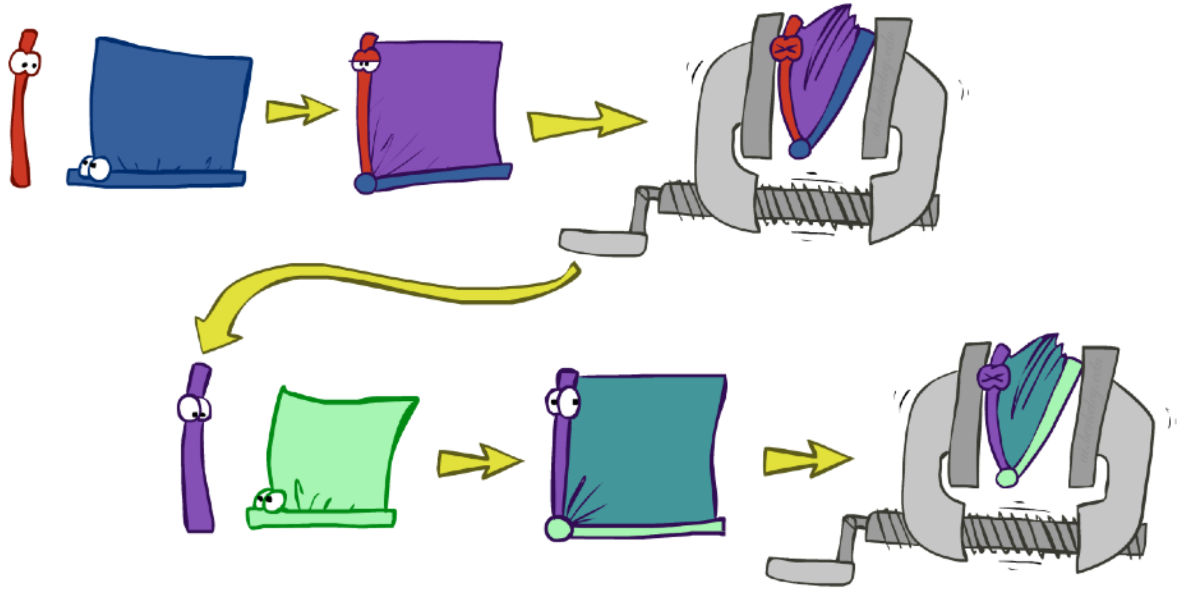


- Idea: interleave joining and marginalizing!



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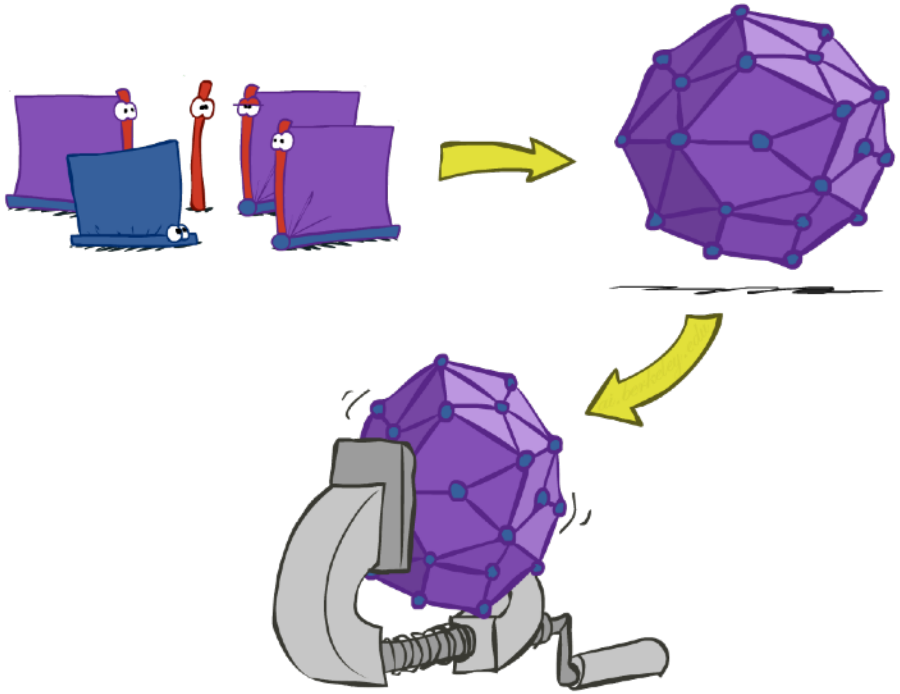

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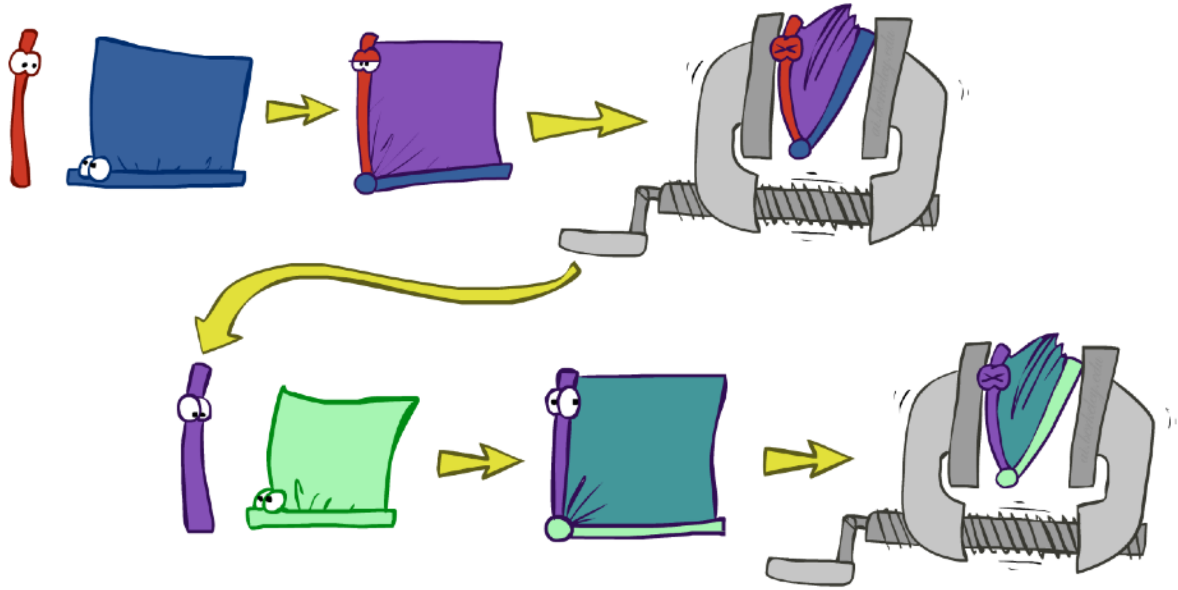

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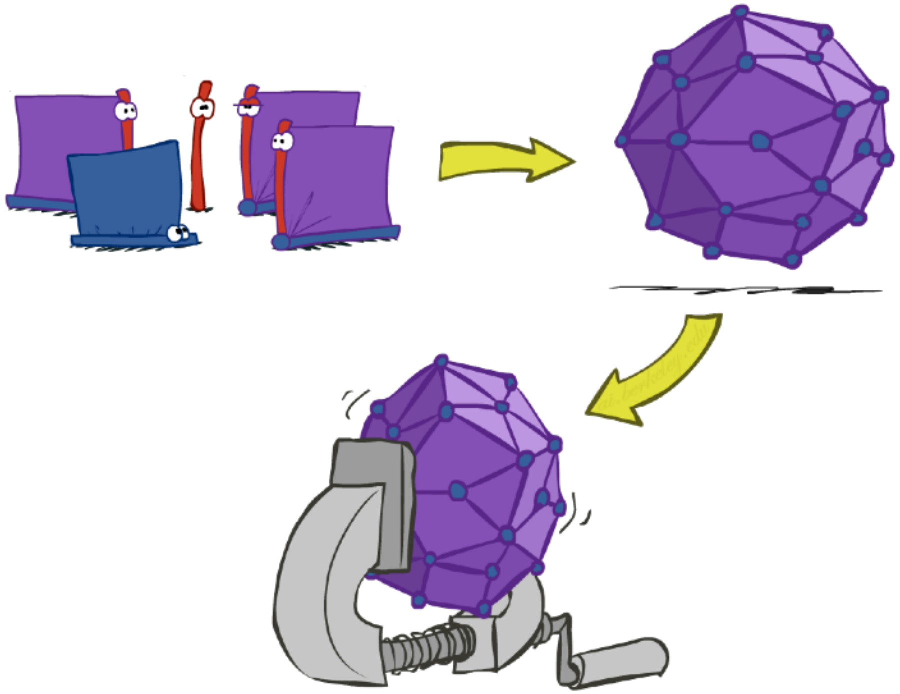

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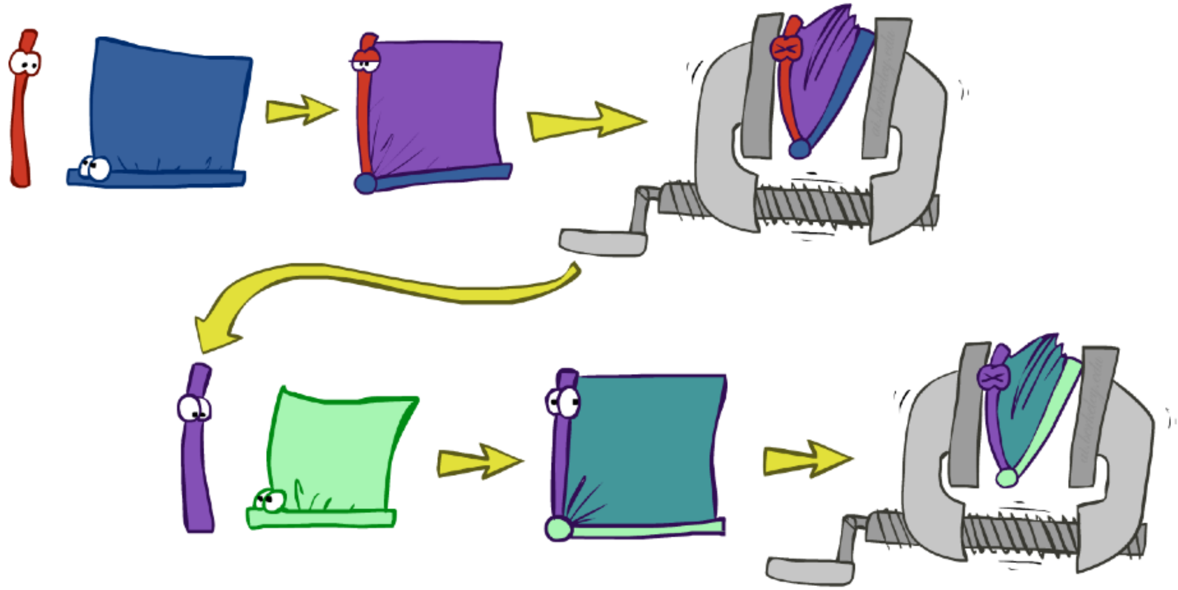

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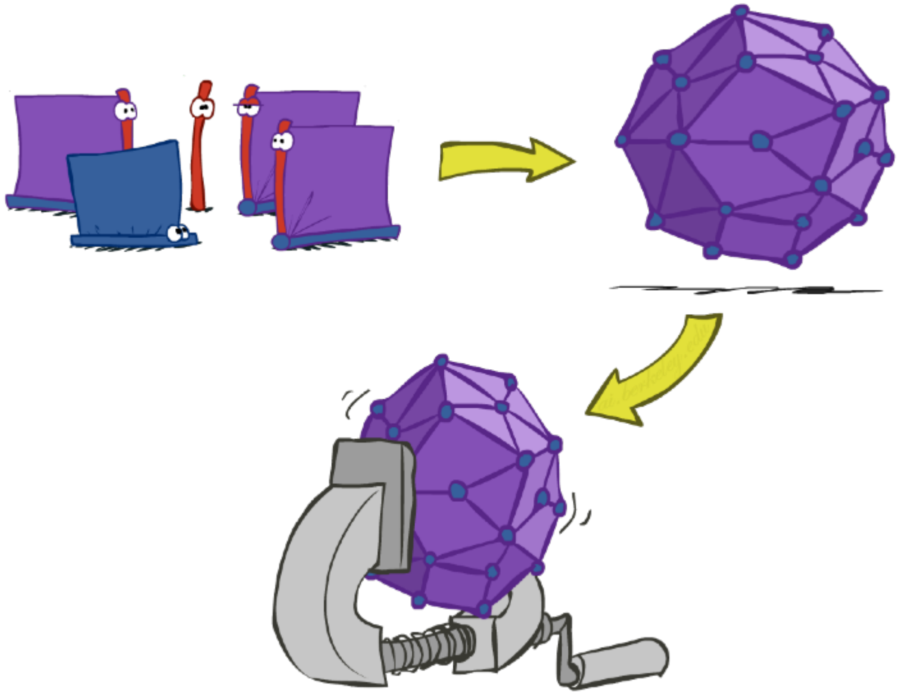

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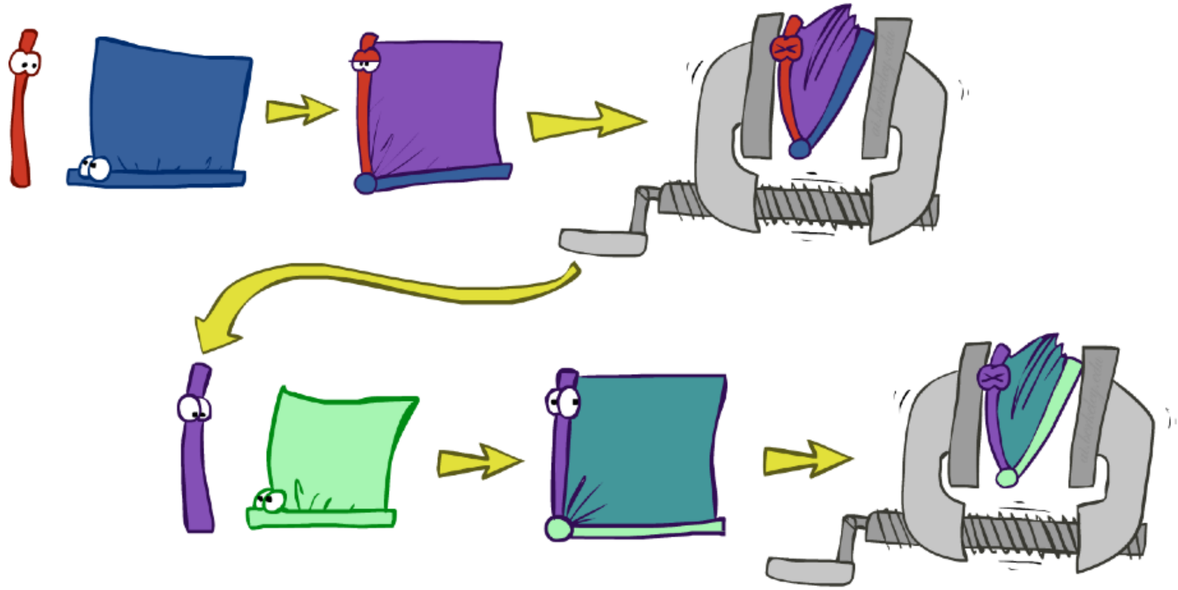

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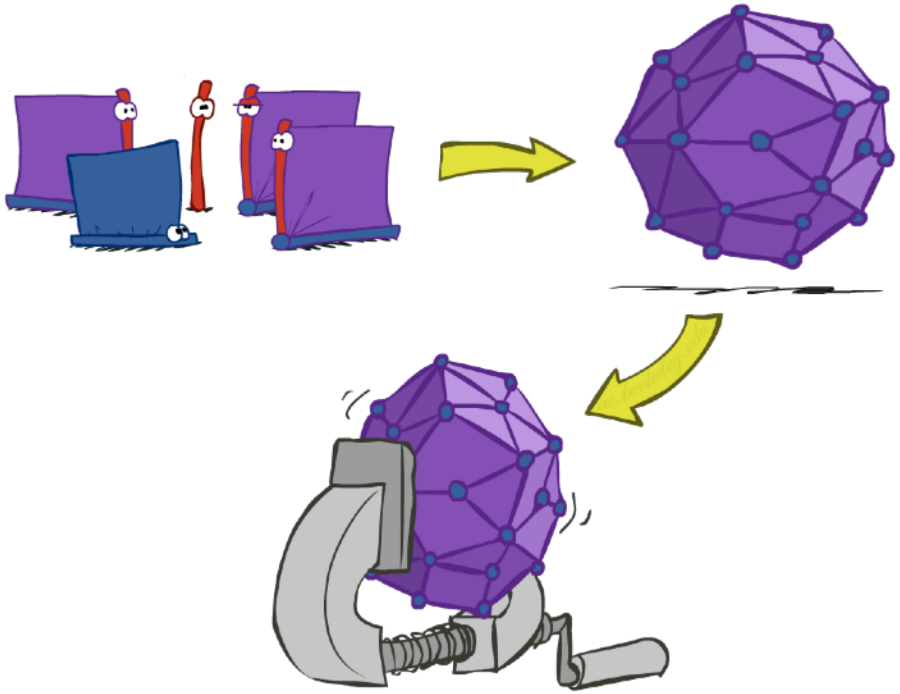

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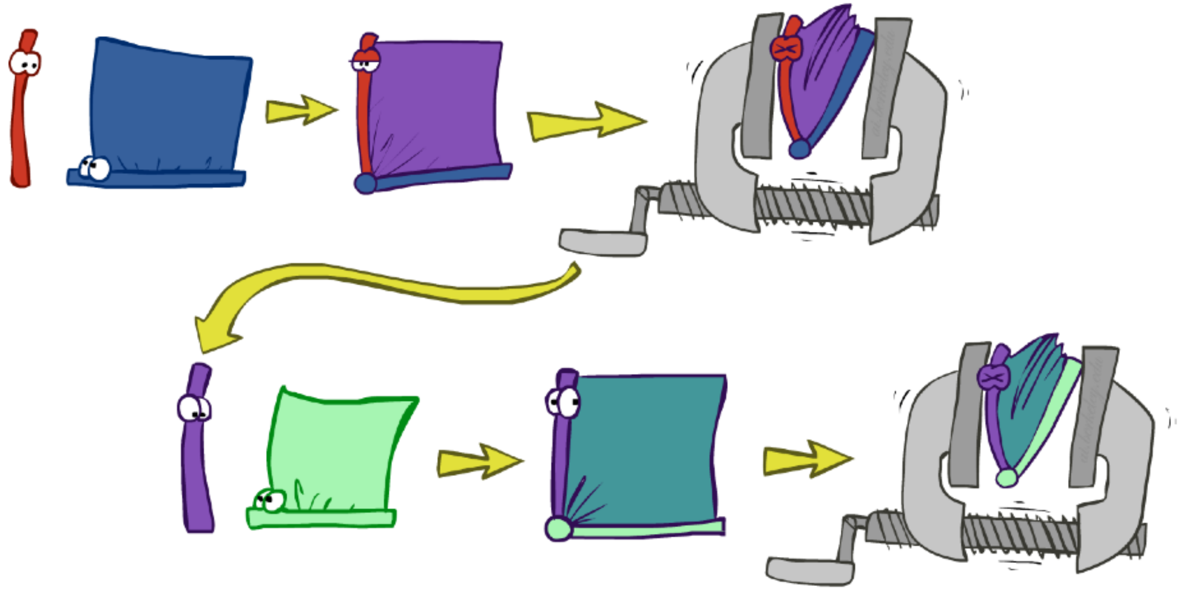
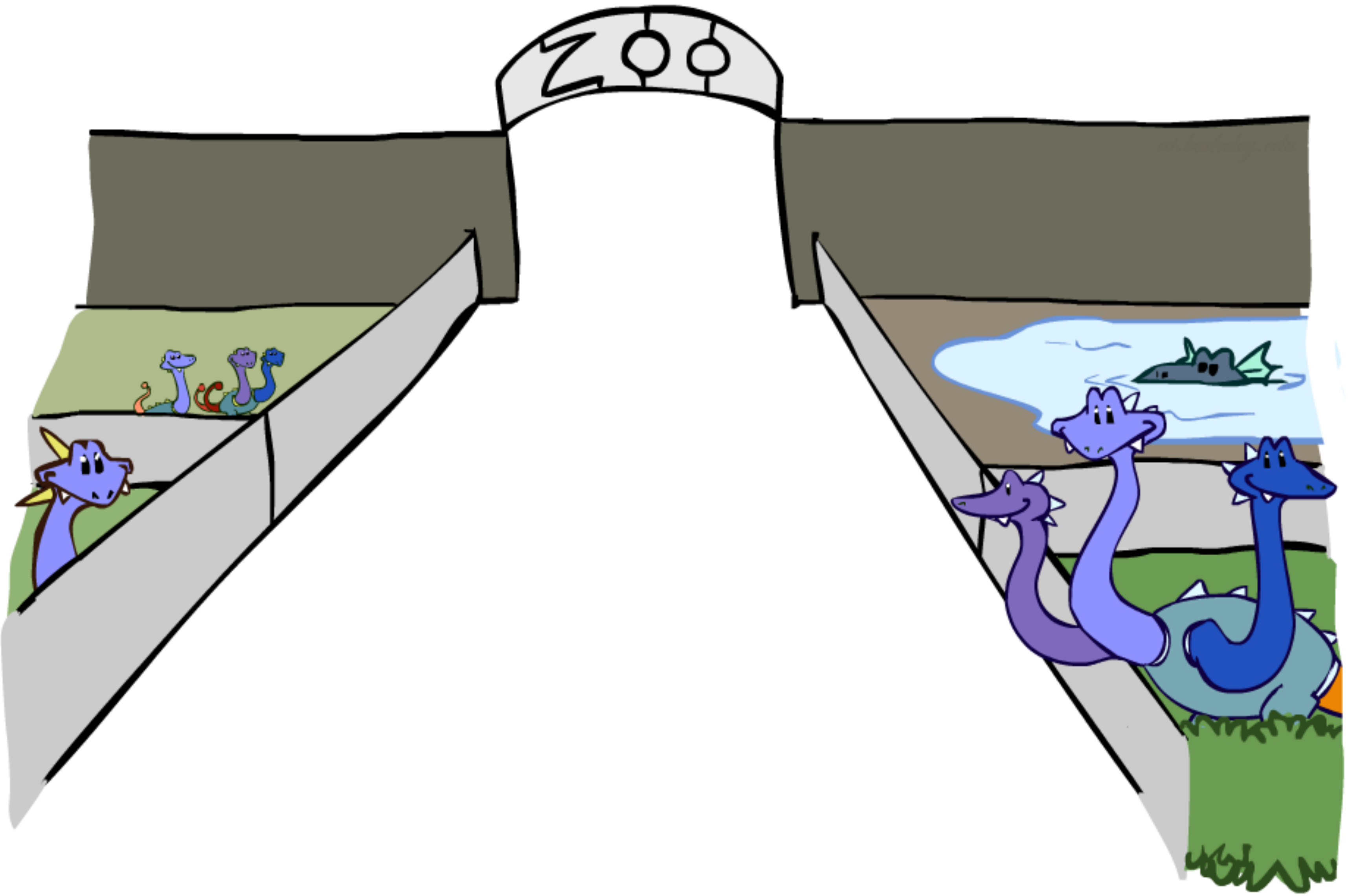
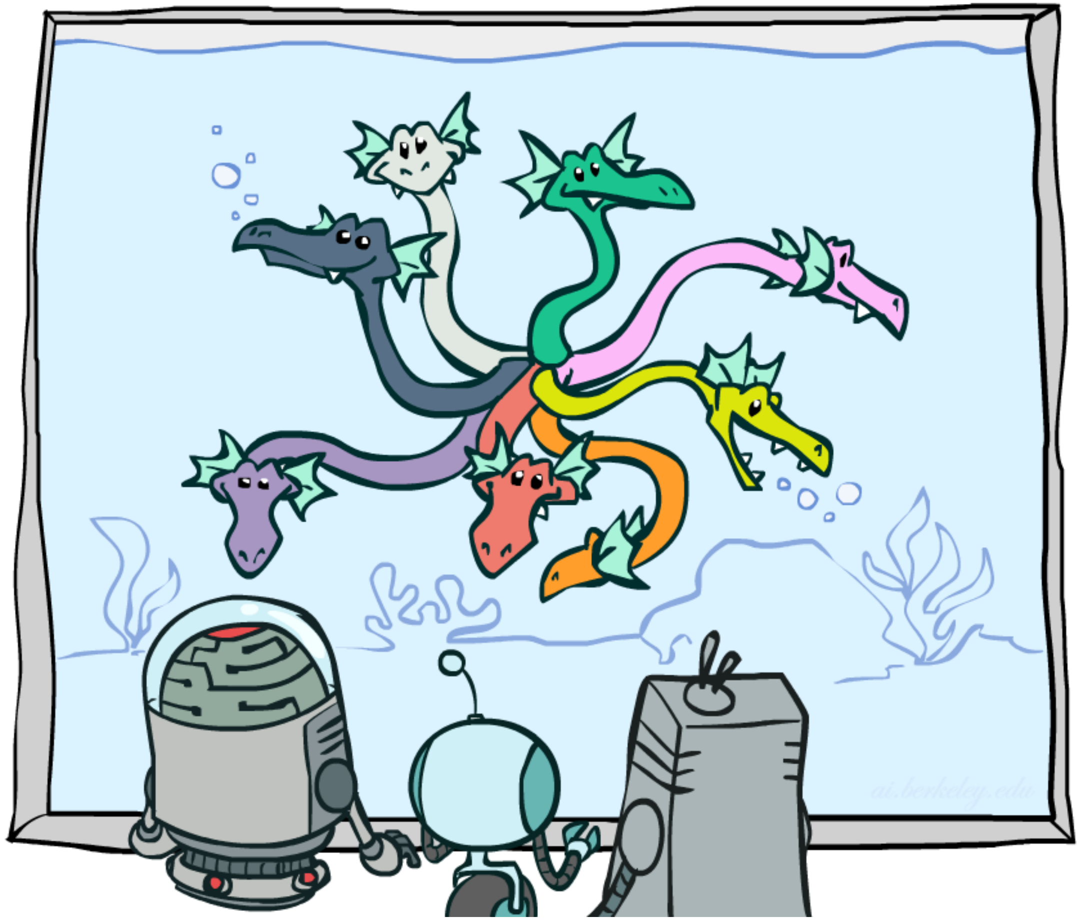


Factor Zoo



Factor Zoo I

- Joint distribution: P(X,Y)
  - Entries P(x,y) for all x, y
  - Sums to 1



Factor Zoo I

- Joint distribution: P(X,Y)
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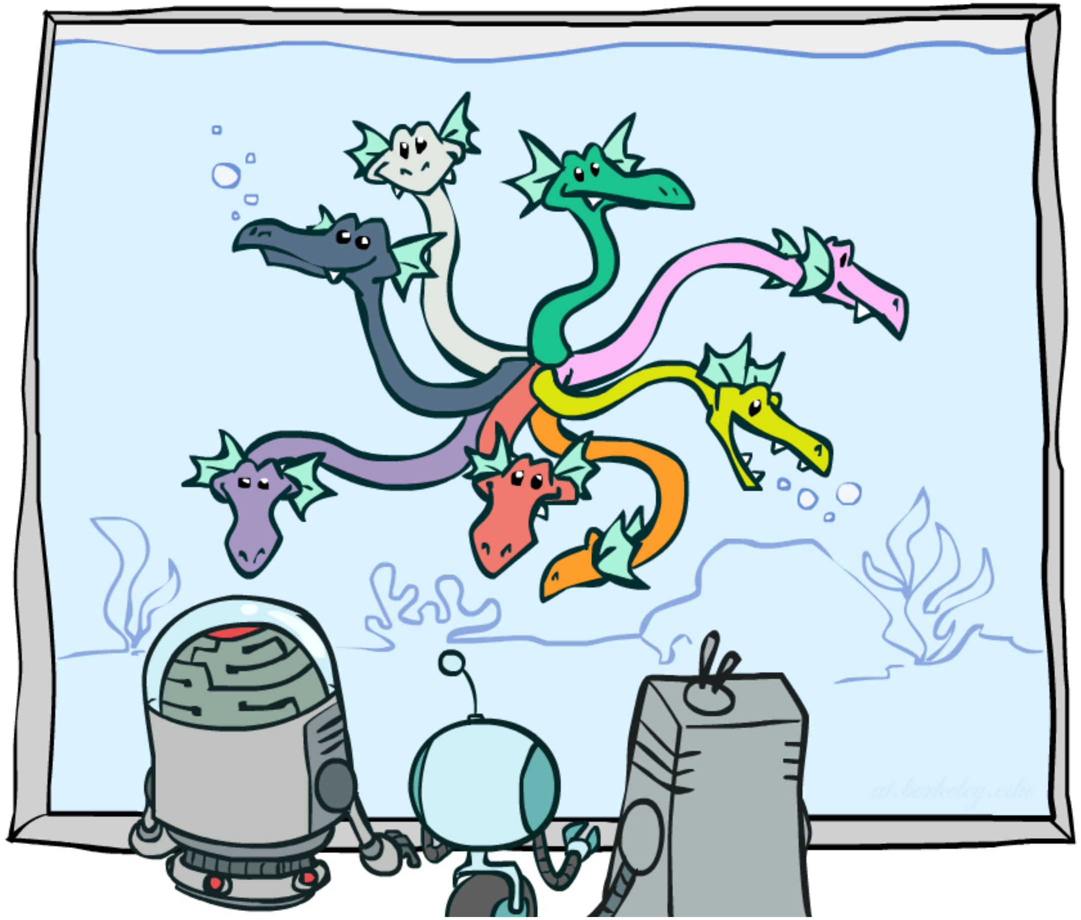
$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3



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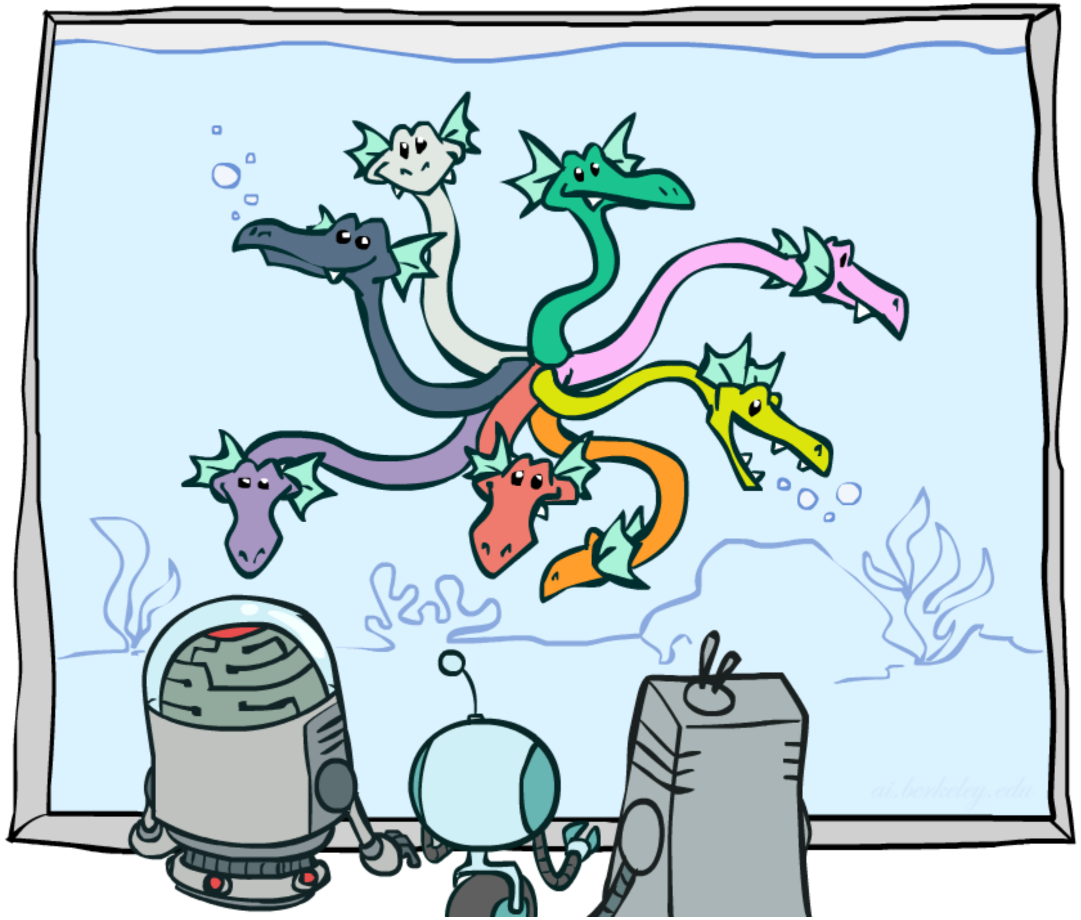

- Selected joint: P(x,Y)
  - A slice of the joint distribution
  - Entries P(x,y) for fixed x, all y
  - Sums to P(x)

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$P(T, W)$

T	W	P
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- Selected joint: P(x,Y)
  - A slice of the joint distribution
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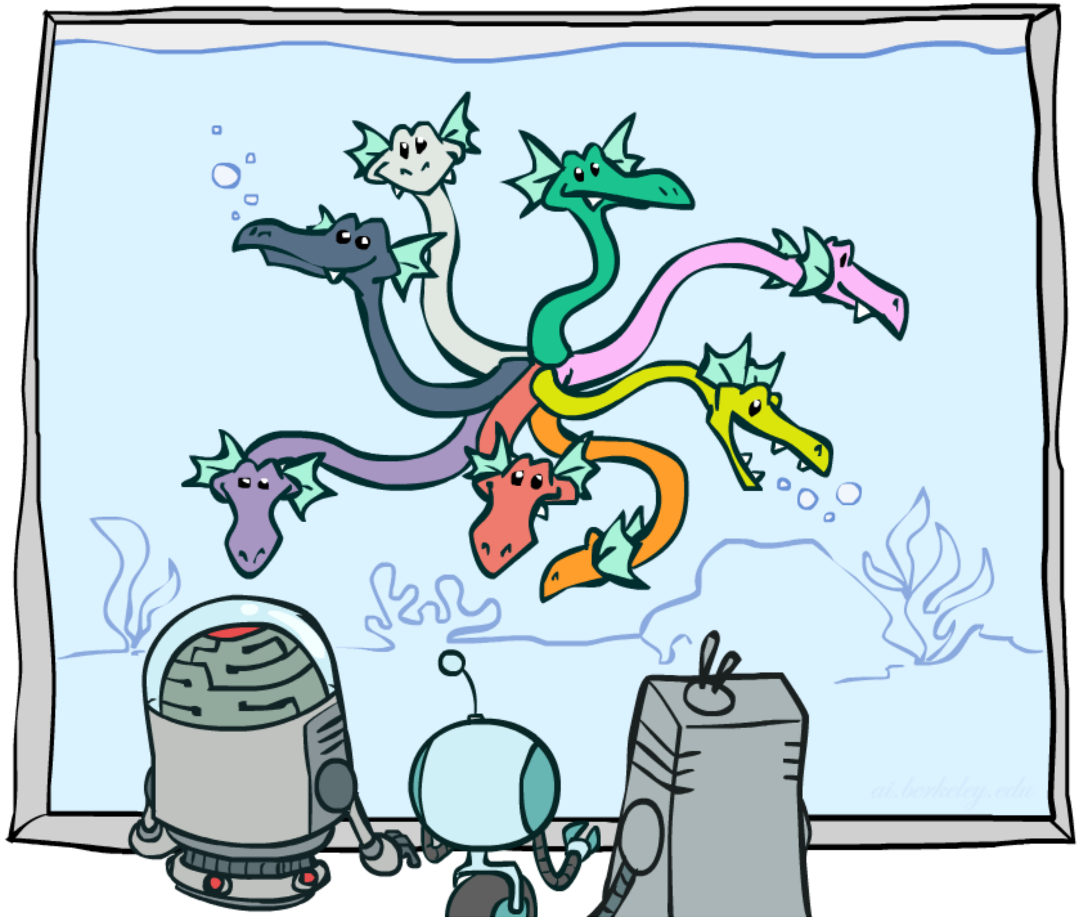
  - Sums to P(x)

$P(cold, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3



Factor Zoo I

- Joint distribution: $P(X,Y)$
  - Entries $P(x,y)$ for all x, y
  - Sums to 1
- Selected joint: $P(x,Y)$
  - A slice of the joint distribution
  - Entries $P(x,y)$ for fixed x, all y
  - Sums to $P(x)$
- Number of capitals = dimensionality of the table

$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(cold,W)$$

T	W	P
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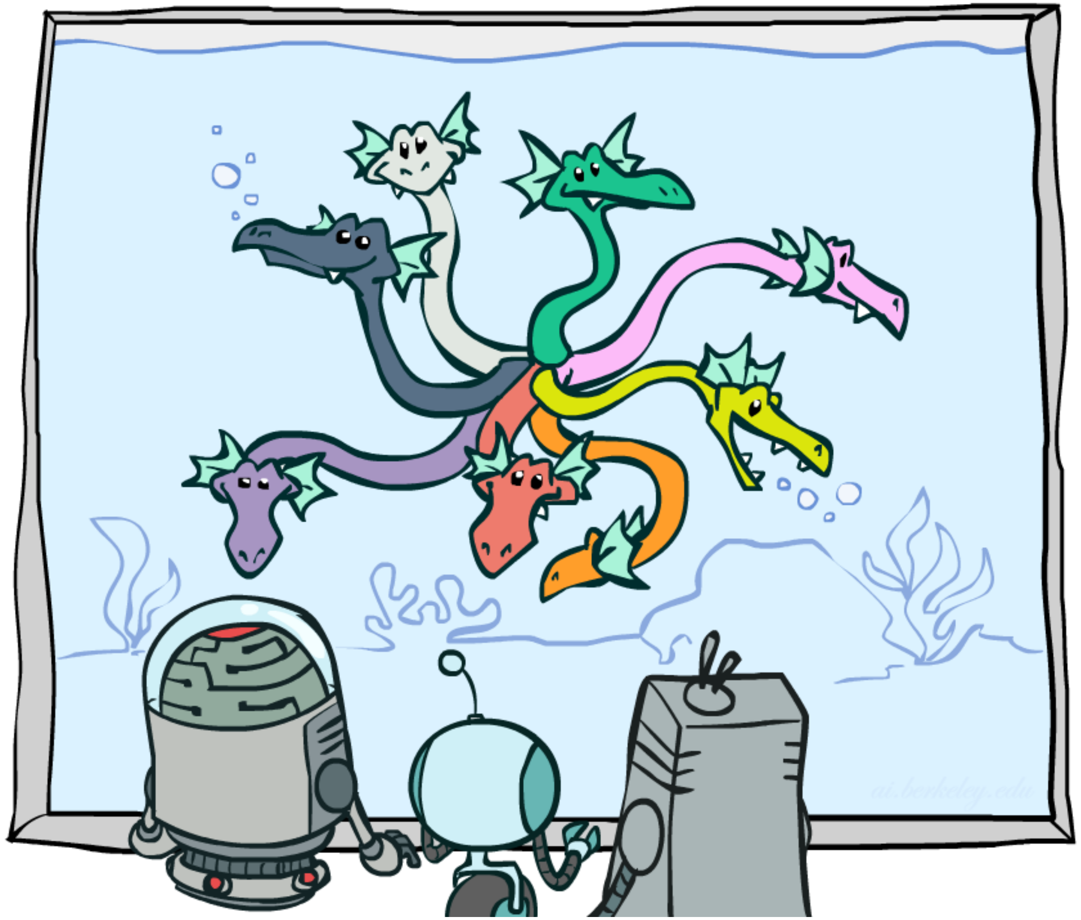
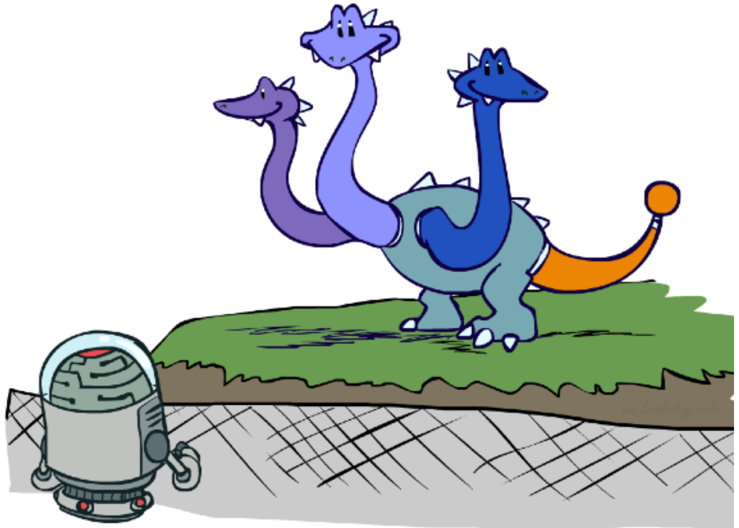
cold	rain	0.3



Factor Zoo II

- Single conditional: P(Y | x)
  - Entries P(y | x) for fixed x, a
  - Sums to 1



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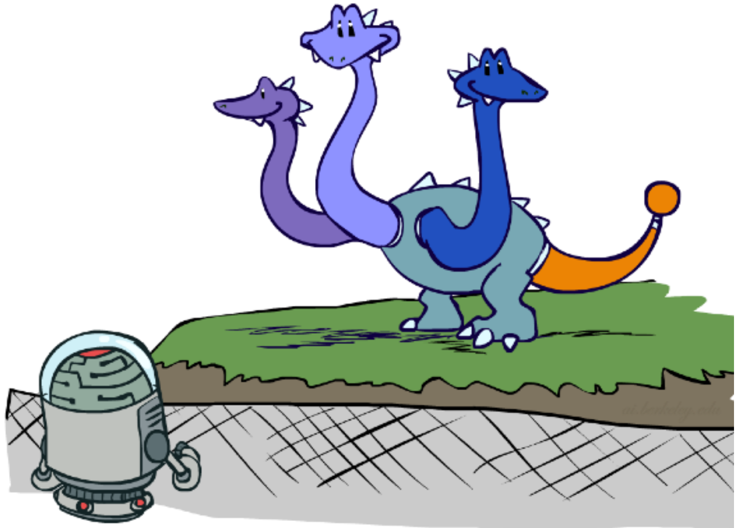

$$P(W|cold)$$

T	W	P
cold	sun	0.4
cold	rain	0.6

Factor Zoo II

- Single conditional: P(Y | x)
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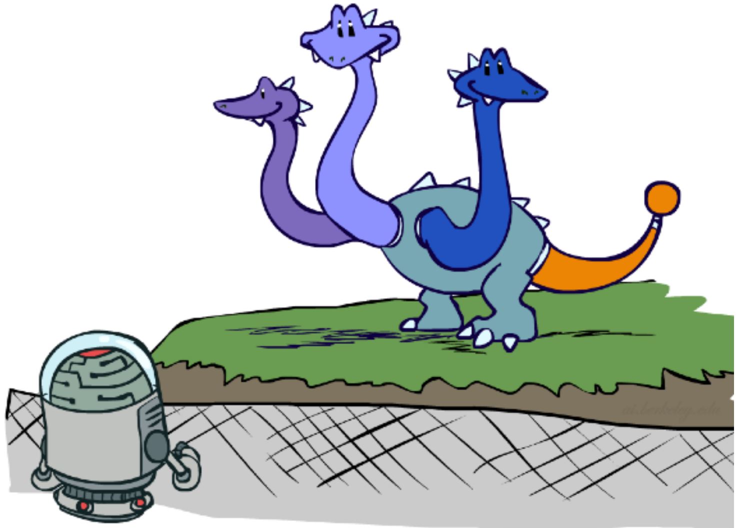

$$P(W|cold)$$

T	W	P
cold	sun	0.4
cold	rain	0.6

- Family of conditionals:
  P(X |Y)
  - Multiple conditionals
  - Entries P(x | y) for all x, y
  - Sums to |Y|

Factor Zoo II

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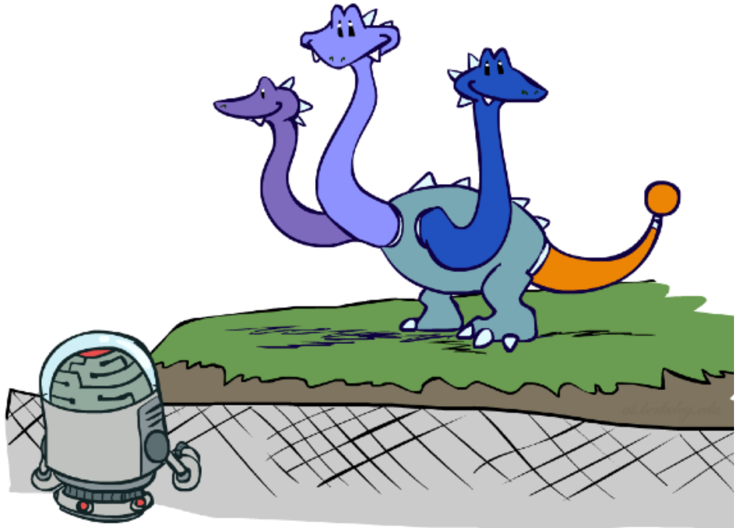

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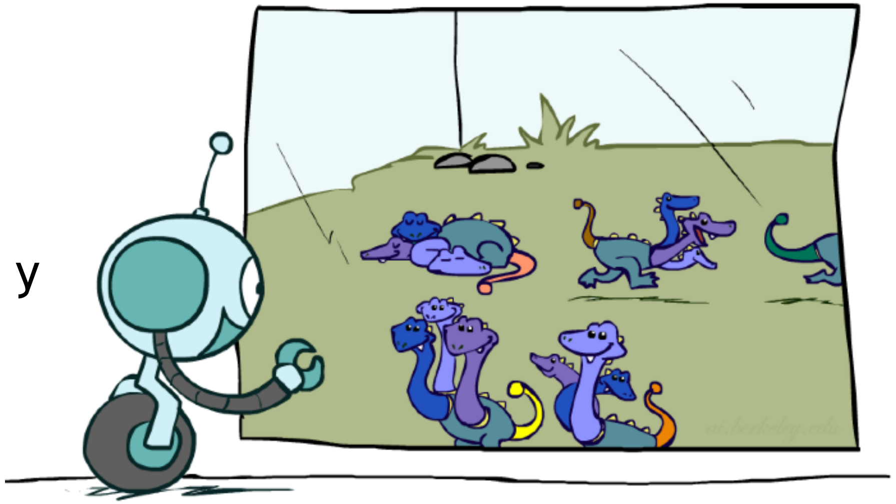

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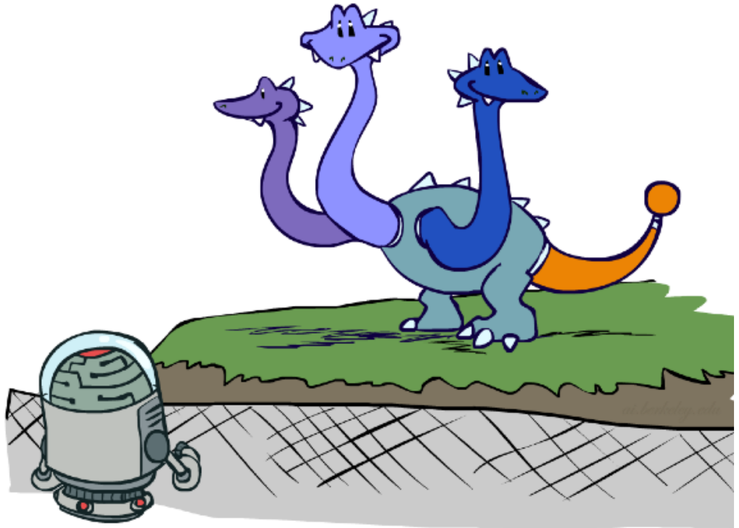

$P(W|cold)$

T	W	P
cold	sun	0.4
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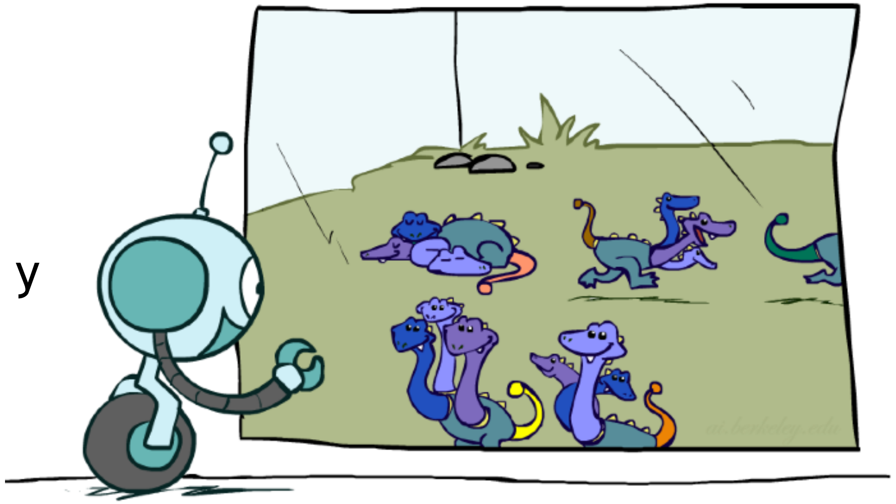


$P(W|T)$

T	W	P
hot	sun	0.8
hot	rain	0.2
cold	sun	0.4
cold	rain	0.6

$\left.\begin{array}{l}\text{hot, sun, 0.8}\\ \text{hot, rain, 0.2}\end{array}\right\} P(W|hot)$

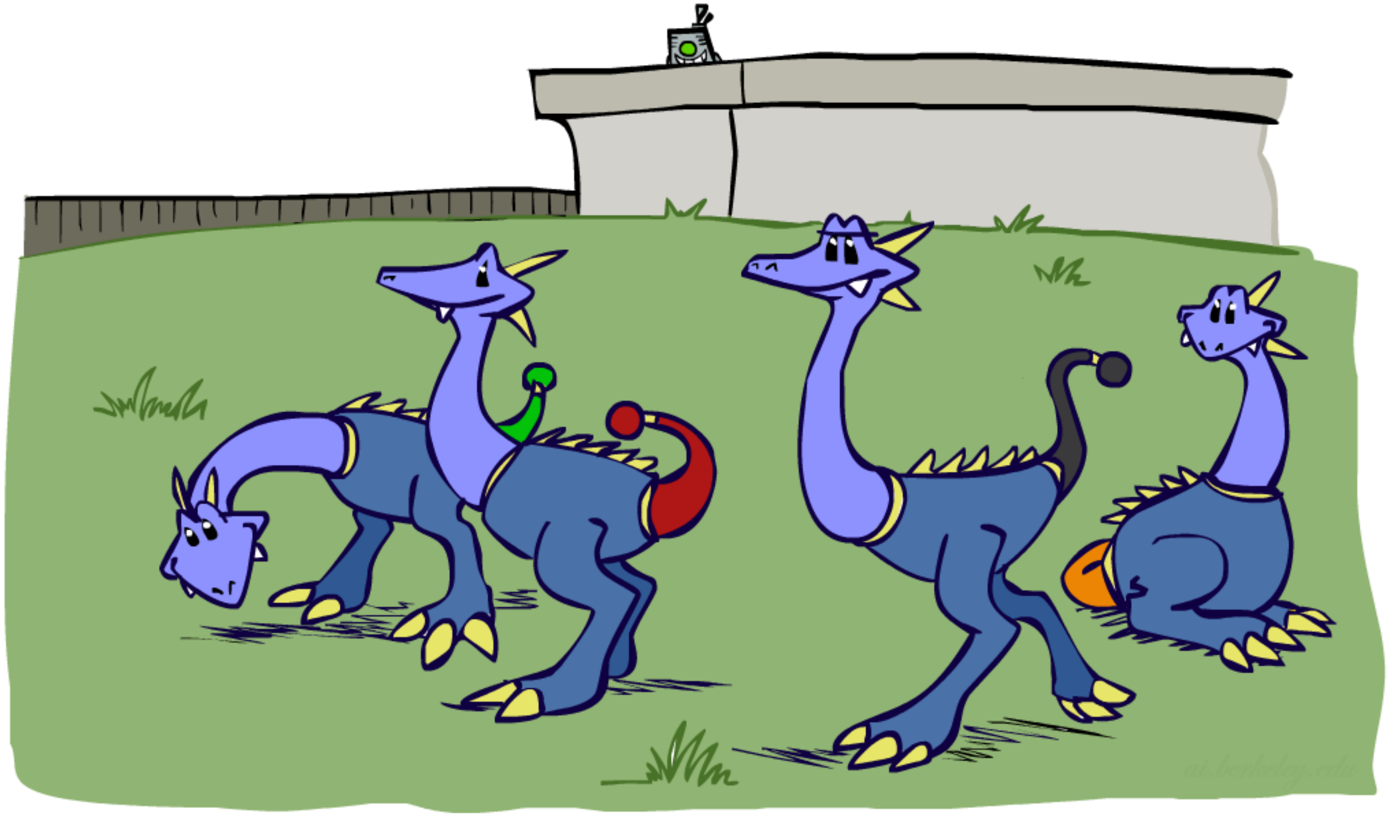
$\left.\begin{array}{l}\text{cold, sun, 0.4}\\ \text{cold, rain, 0.6}\end{array}\right\} P(W|cold)$

Factor Zoo III

- Specified family: P(y | X)
  - Entries P(y | x) for fixed y, but for all x
  - Sums to … who knows!

$$P(rain|T)$$

T	W	P
hot	rain	0.2
cold	rain	0.6



Factor Zoo III

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  - Entries P(y | x) for fixed y, but for all x
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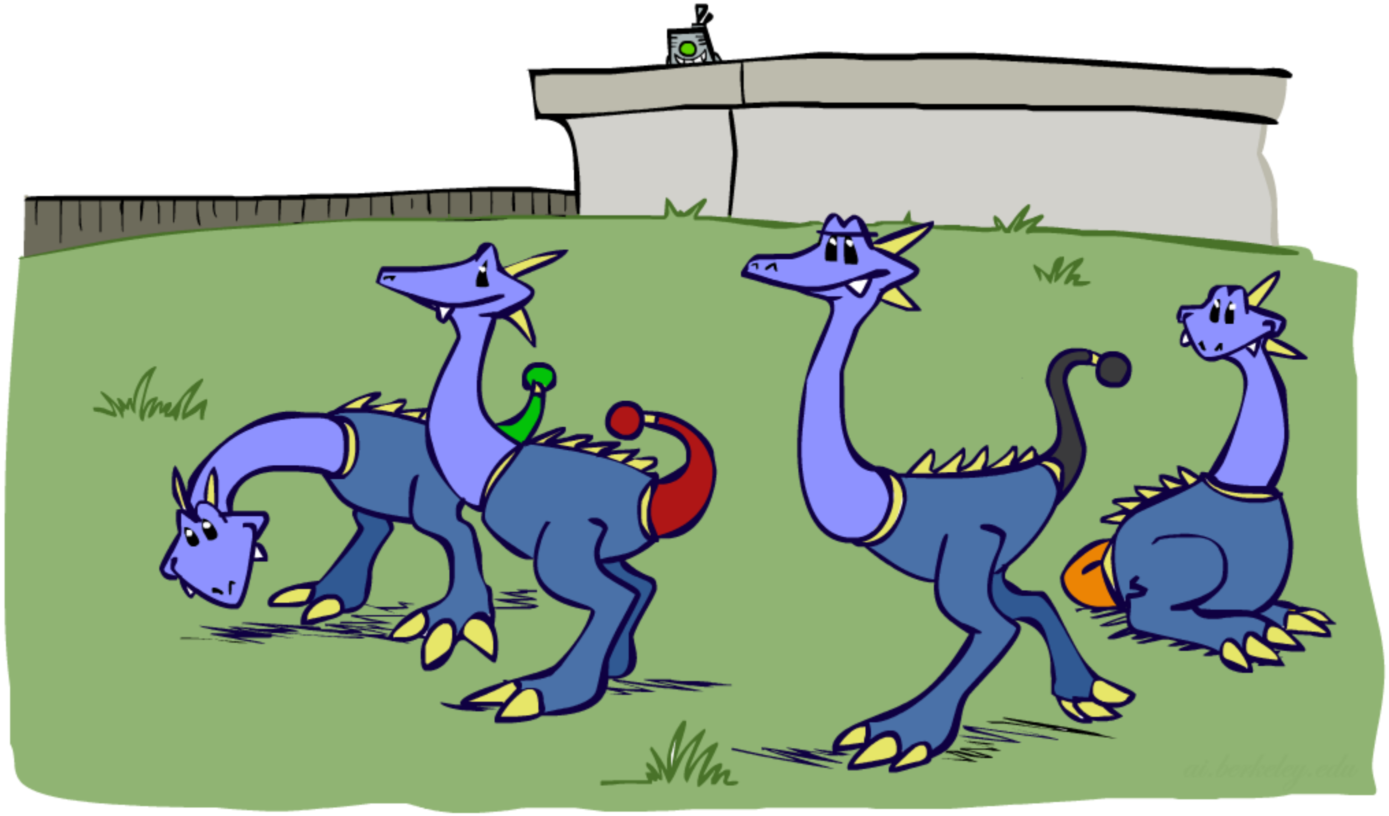
$P(rain|T)$

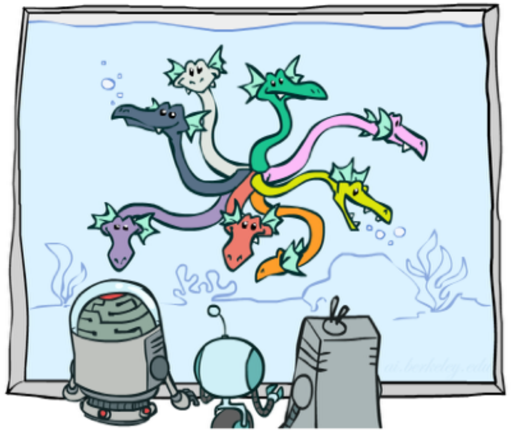
T	W	P
hot	rain	0.2
cold	rain	0.6

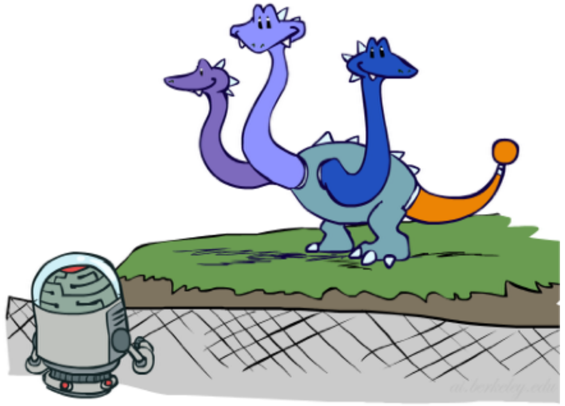
$\} P(rain|hot)$

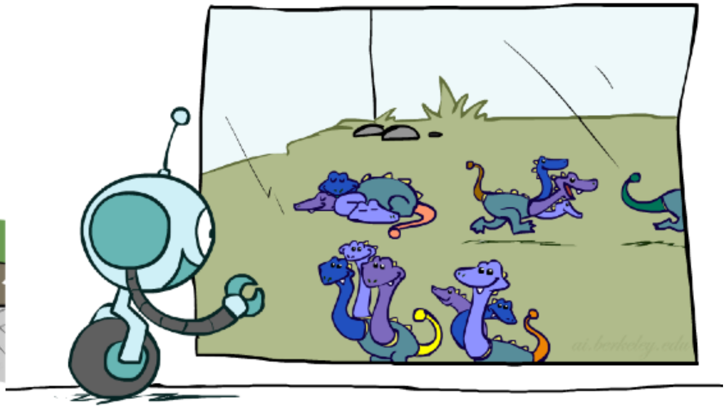
$\} P(rain|cold)$

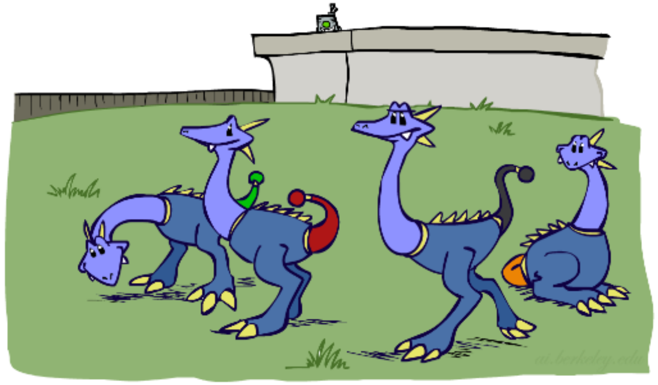


Factor Zoo Summary



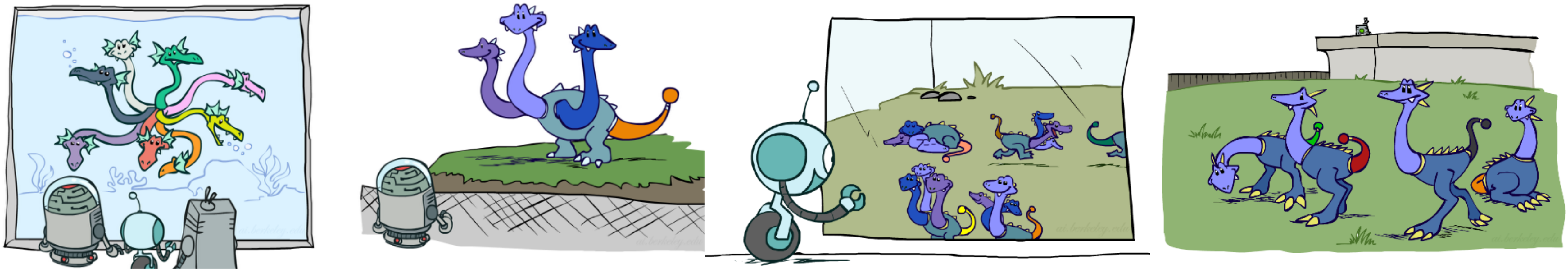






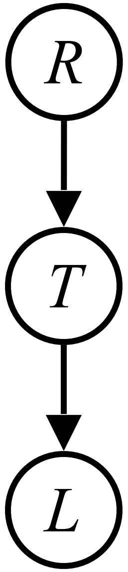
Factor Zoo Summary

- In general, when we write $P(Y_1 \dots Y_N \mid X_1 \dots X_M)$
  - It is a "factor," a multi-dimensional array
  - Its values are $P(y_1 \dots y_N \mid x_1 \dots x_M)$
  - Any assigned (=lower-case) X or Y is a dimension missing (selected) from the array



Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!



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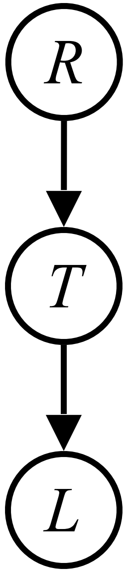

$P(R)$

+r	0.1
-r	0.9

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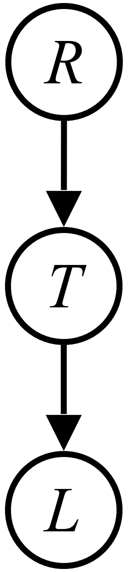

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

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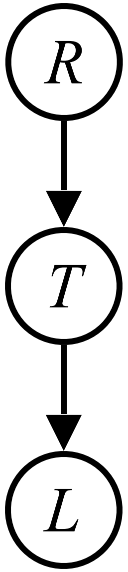

$P(R)$

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-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

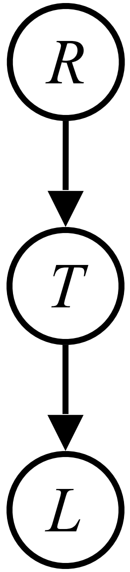
$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!

$$P(L) = ?$$



$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

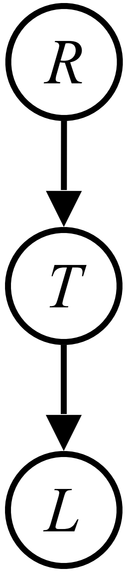
+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!

$$P(L) = ?$$

$$= \sum_{r,t} P(r,t,L)$$



$P(R)$

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$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
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$P(L|T)$

+t	+l	0.3
+t	-l	0.7
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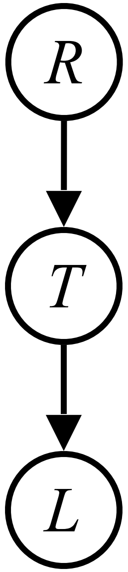
Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!



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+r	-t	0.2
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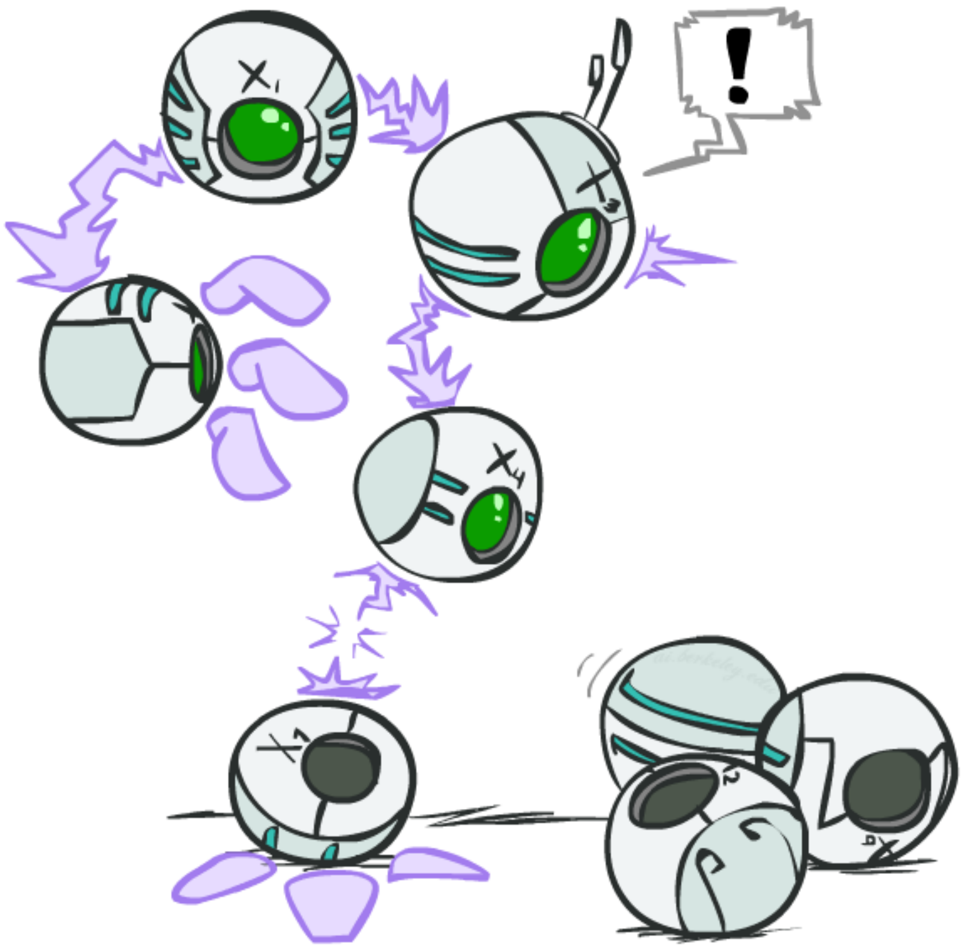
$$P(L) = ?$$

$$= \sum_{r,t} P(r, t, L)$$

$$= \sum_{r,t} P(r)P(t|r)P(L|t)$$

Inference by Enumeration: Procedural Outline

- Track objects called factors
- Initial factors are local CPTs (one per node)



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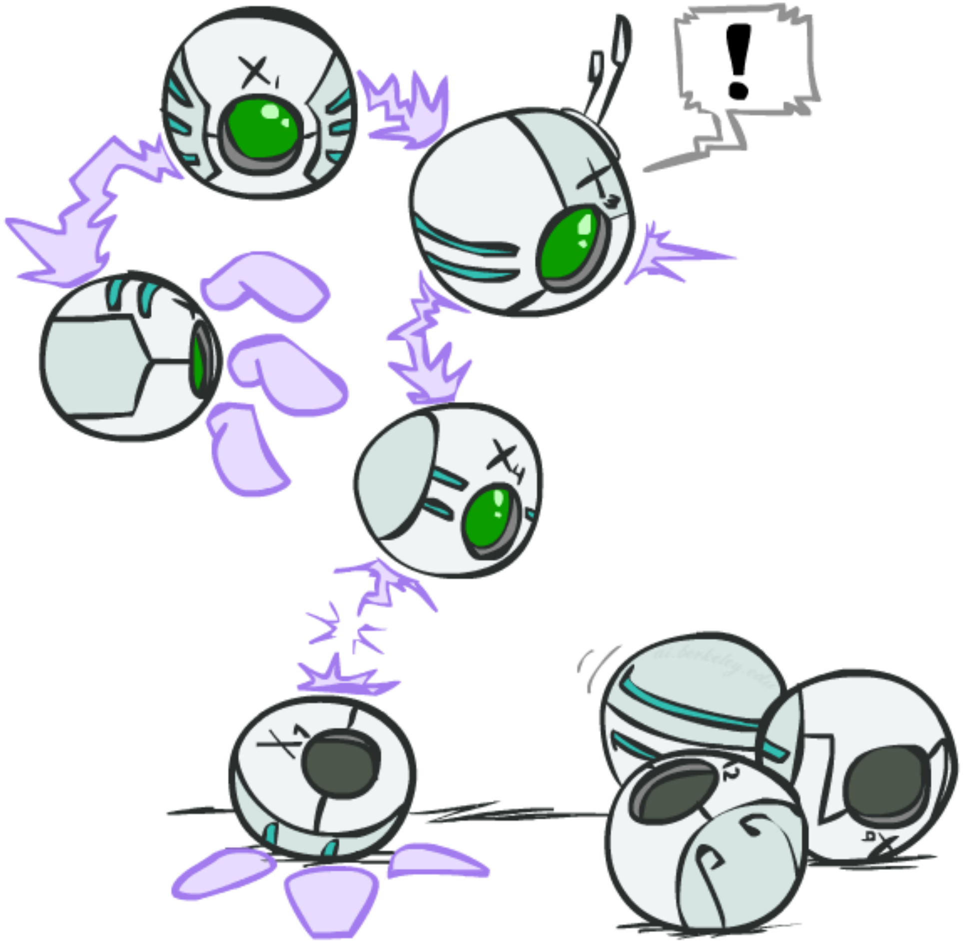

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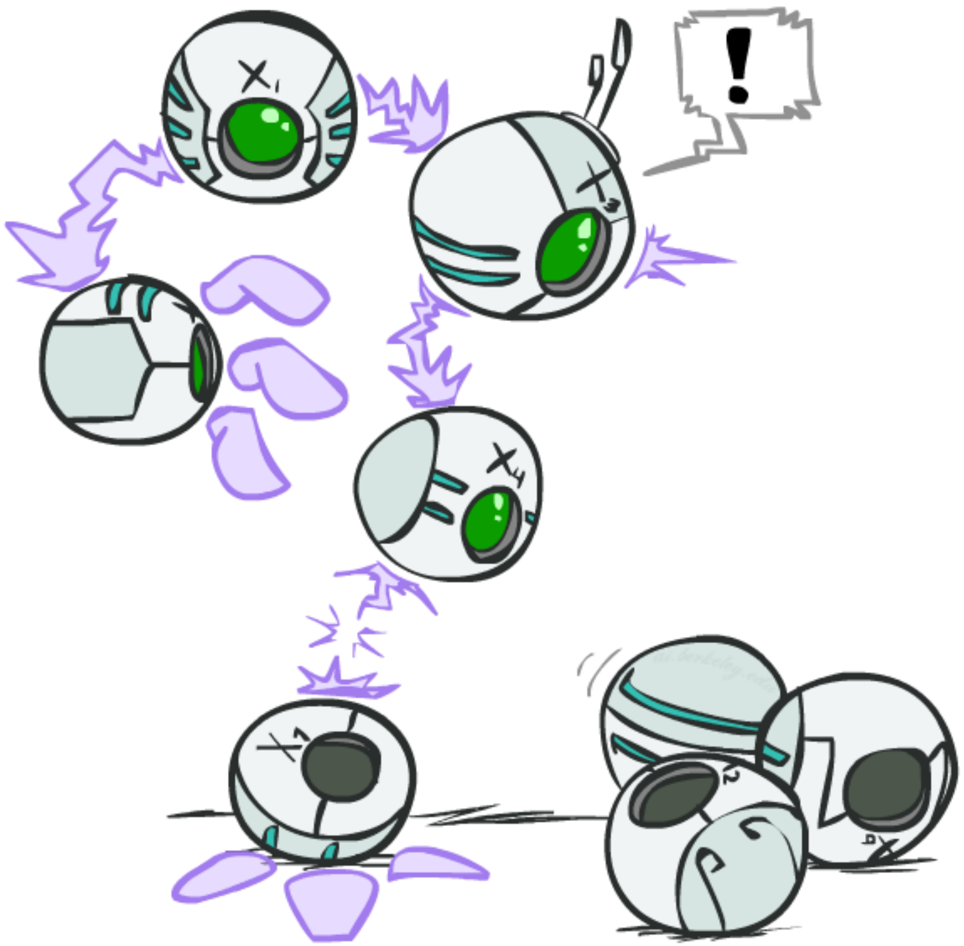
- Any known values are selected
  - E.g. if we know $L = +\ell$, the initial factors are



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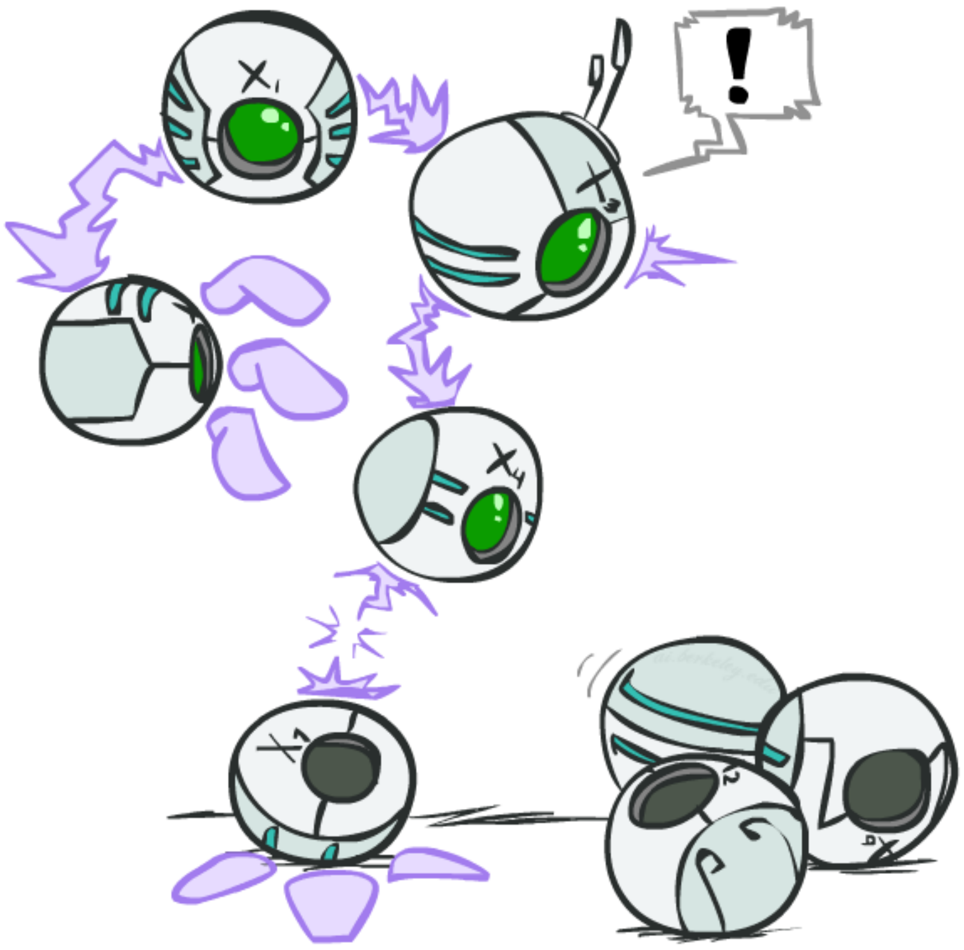

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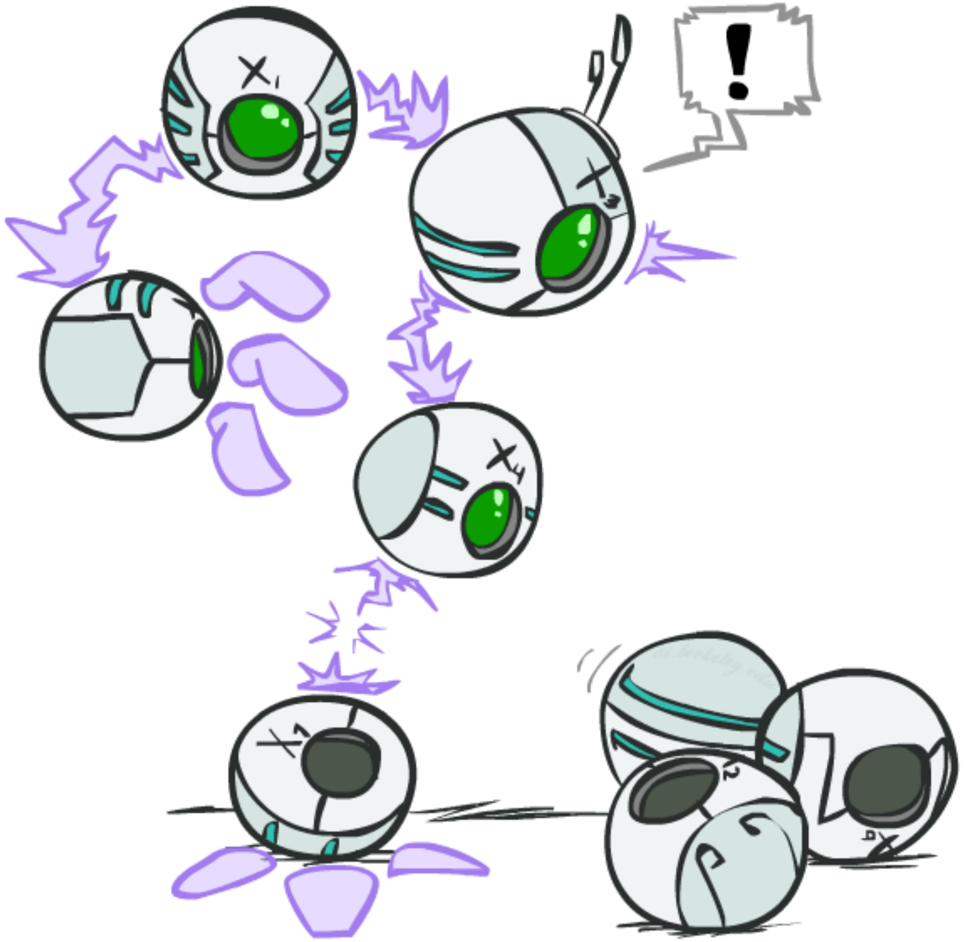
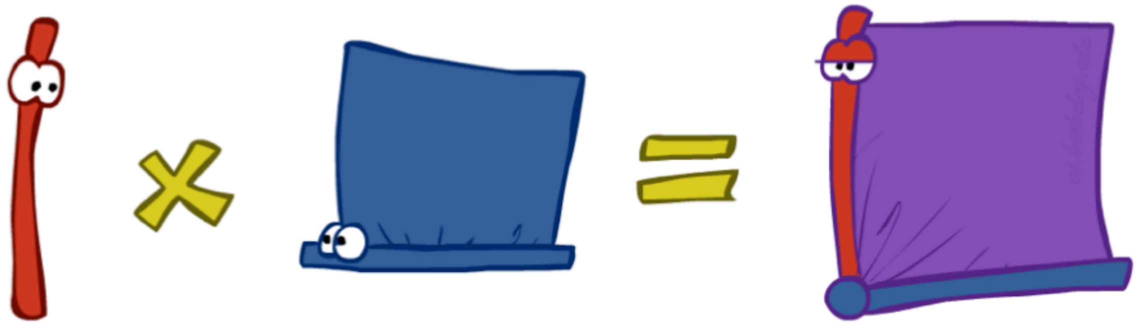


- Procedure: Join all factors, then eliminate all hidden variables

Operation 1: Join Factors

- First basic operation: joining factors
- Combining factors:
  - Just like a database join
  - Get all factors over the joining variable
  - Build a new factor over the union of the variables involved



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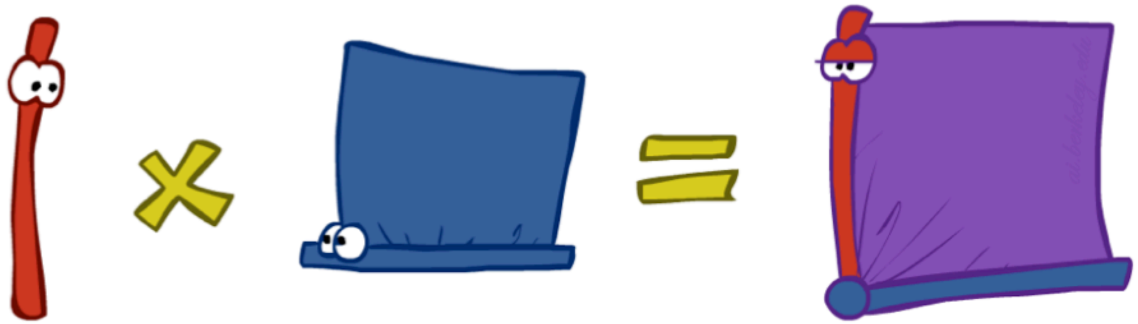
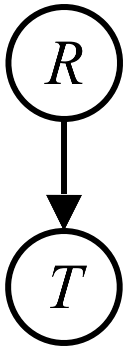


- Example: Join on R



$P(R)$

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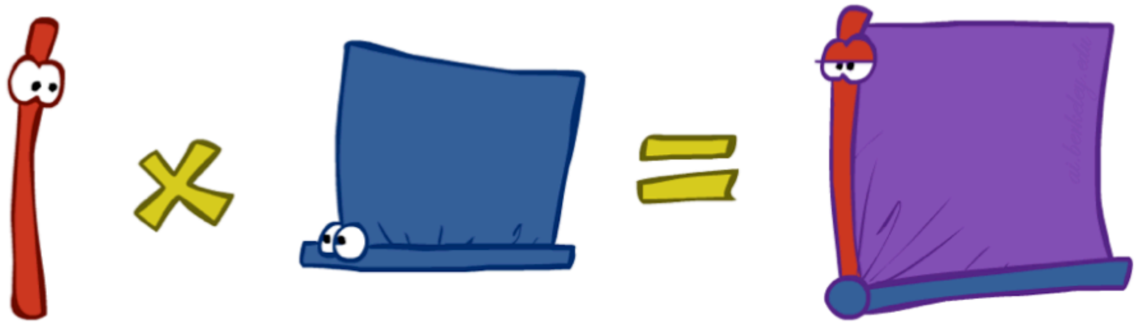

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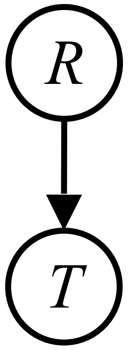

$$P(R) \quad \times \quad P(T|R)$$

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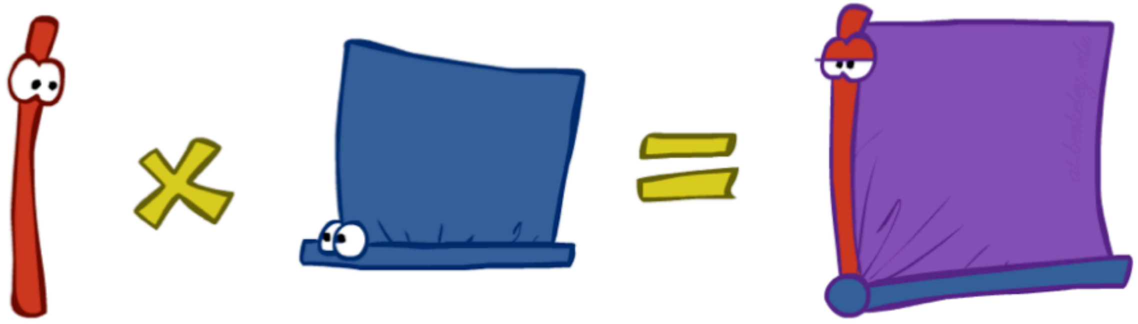

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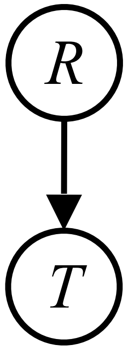

$$P(R) \times P(T|R) \rightarrow P(R,T)$$

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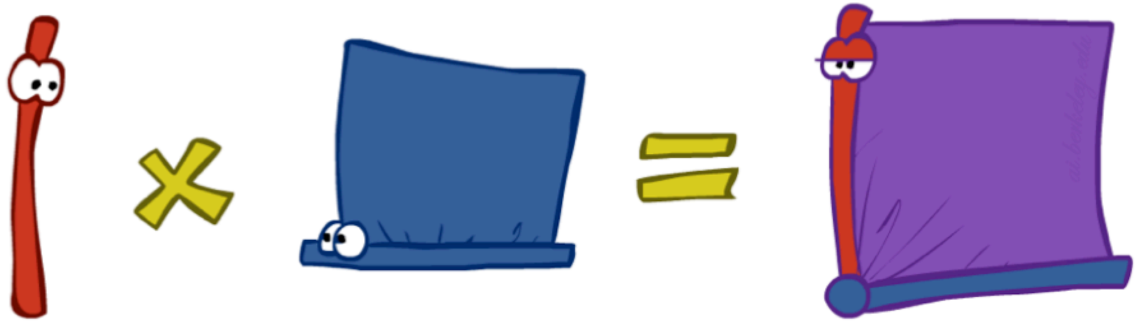
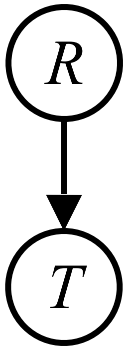


- Example: Join on R



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  - Computation for each entry: pointwise products

$$\forall r, t: \quad P(r,t) = P(r) \cdot P(t|r)$$

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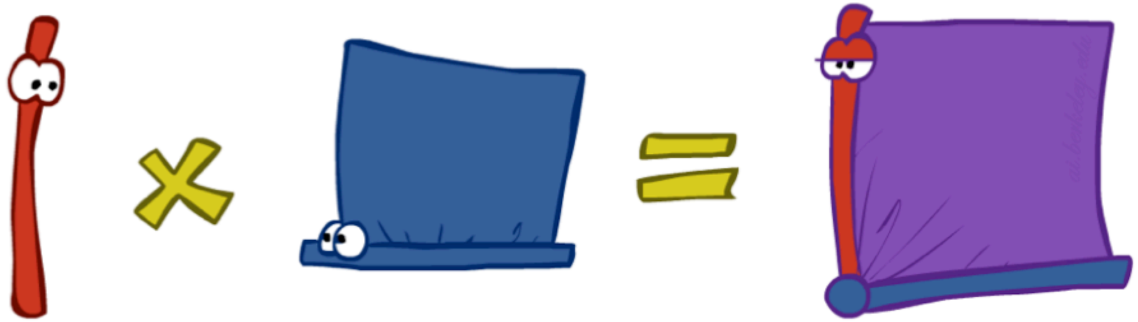

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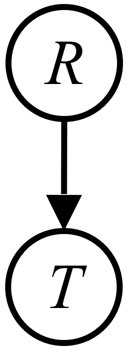

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$\times$ $P(T|R)$

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$\rightarrow$ $P(R,T)$

+r	+t	0.08
+r	-t	0.02
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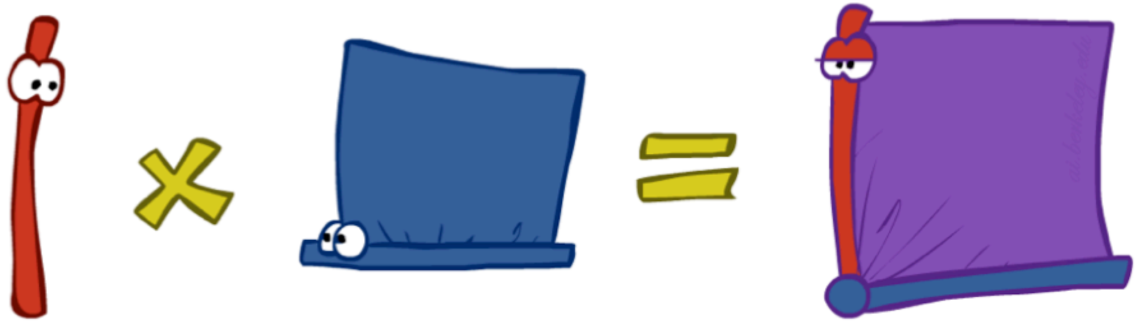

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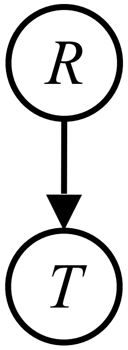

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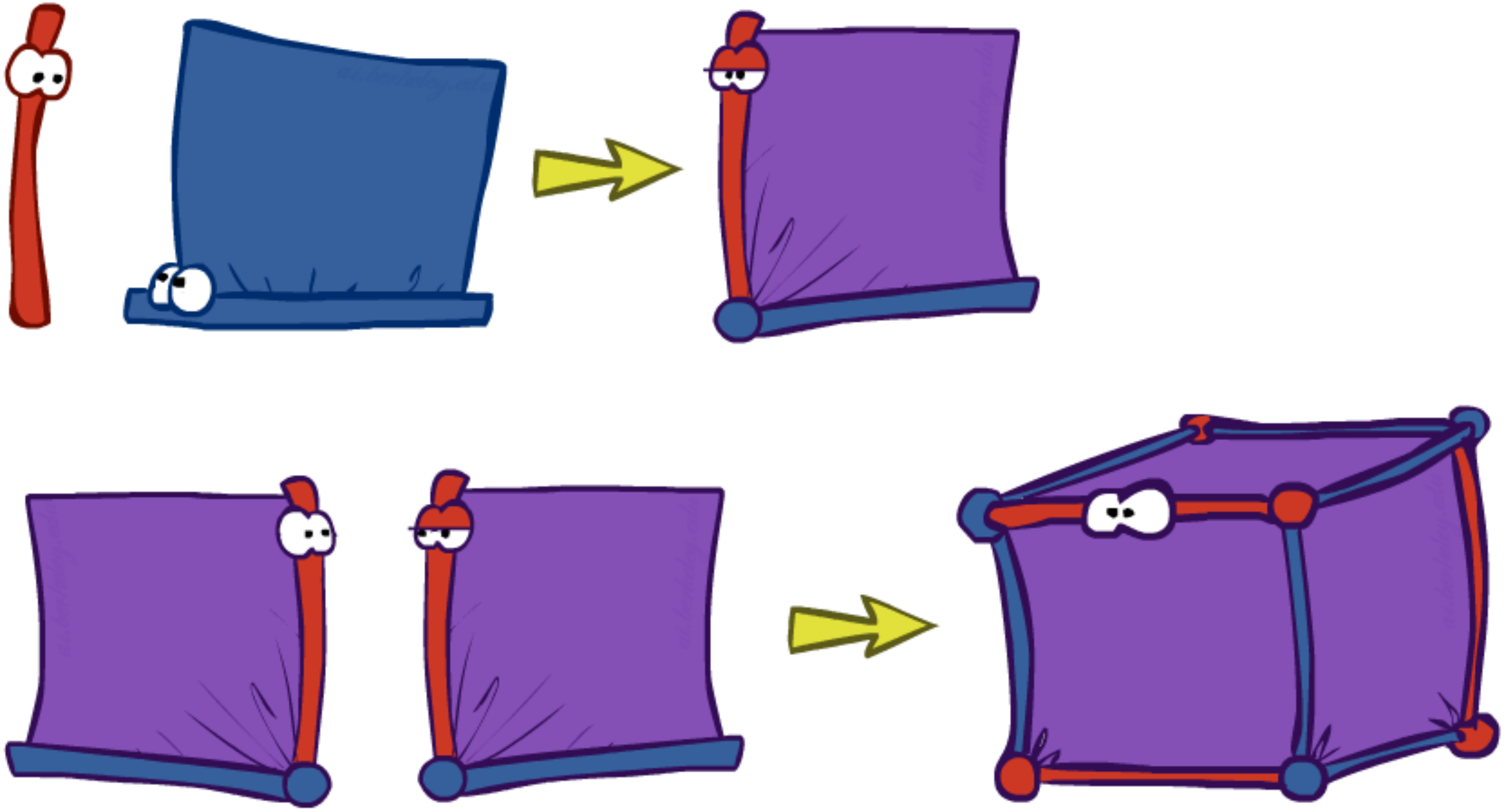
R,T

  - Computation for each entry: pointwise products

$$\forall r,t : \quad P(r,t) = P(r) \cdot P(t|r)$$

Example: Multiple Joins



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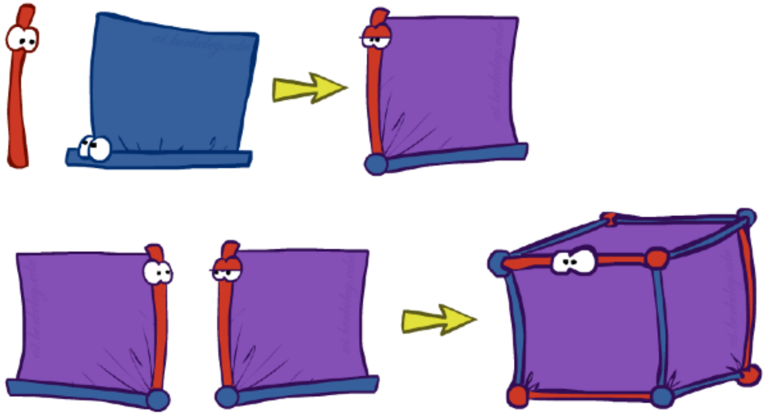

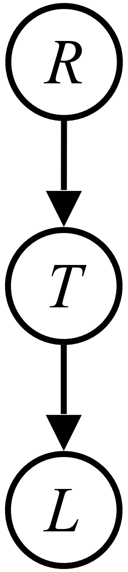

$$P(R)$$

+r	0.1
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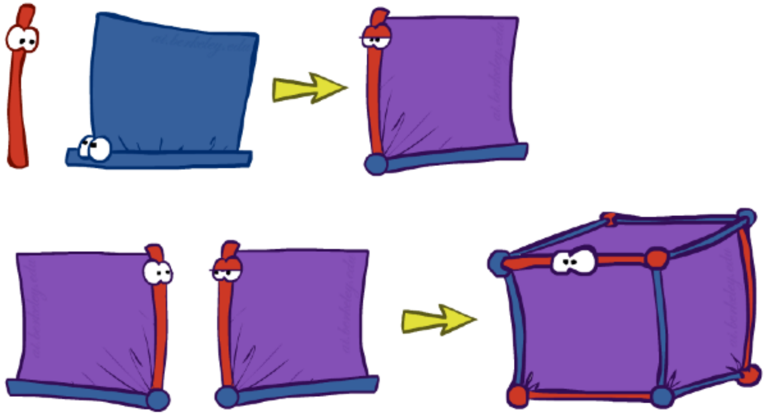

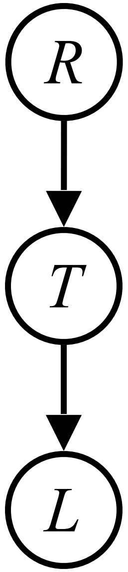

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Join R

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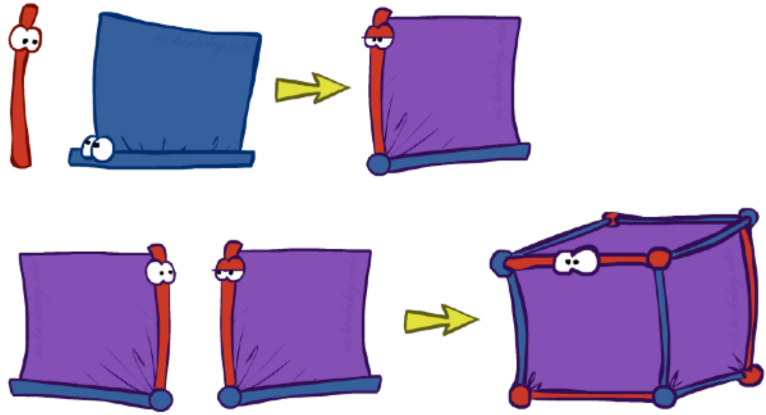

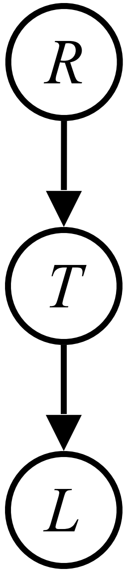

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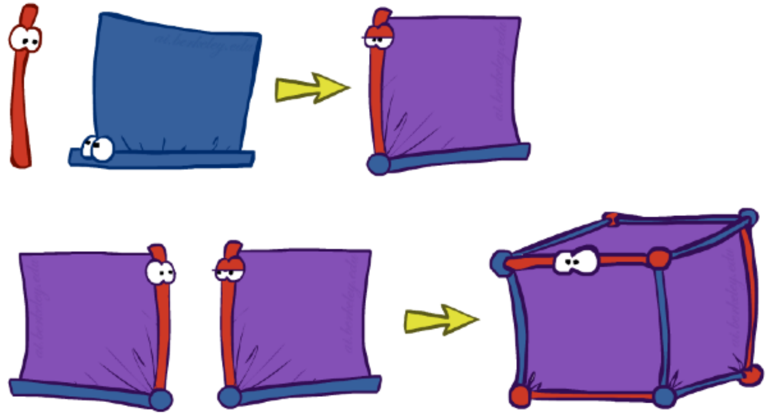

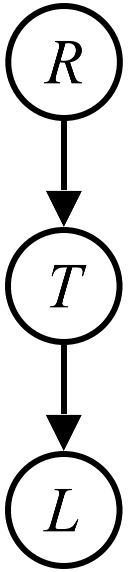

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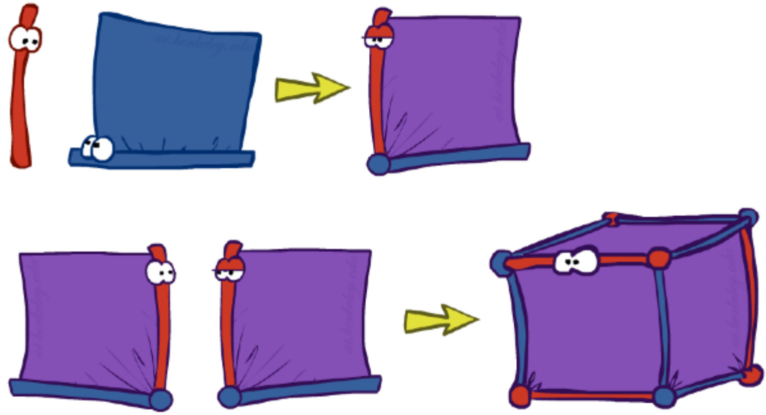

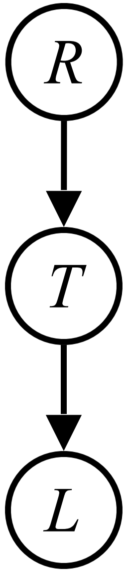

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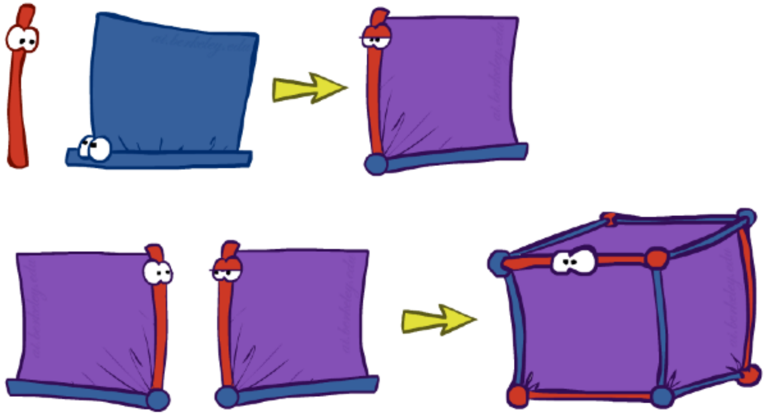

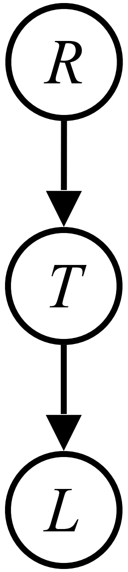

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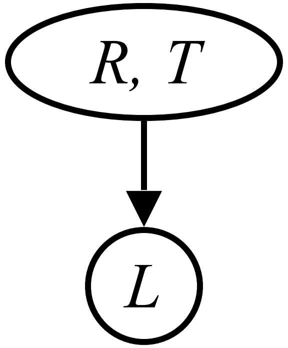

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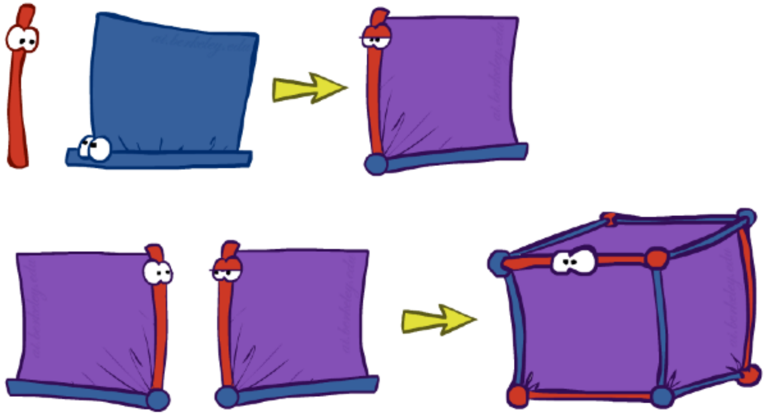

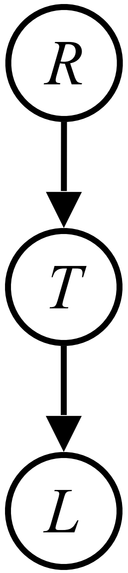

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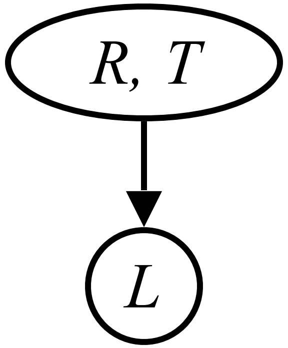
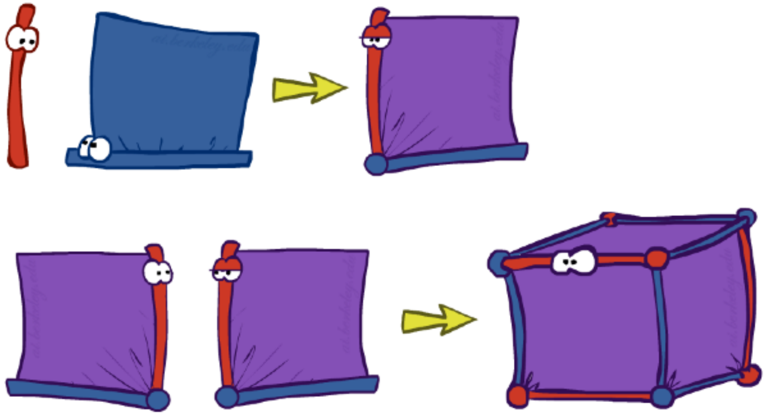
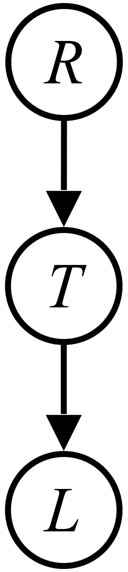


Join T

Example: Multiple Joins





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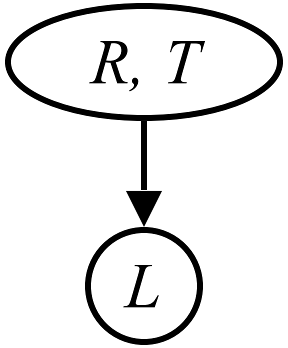
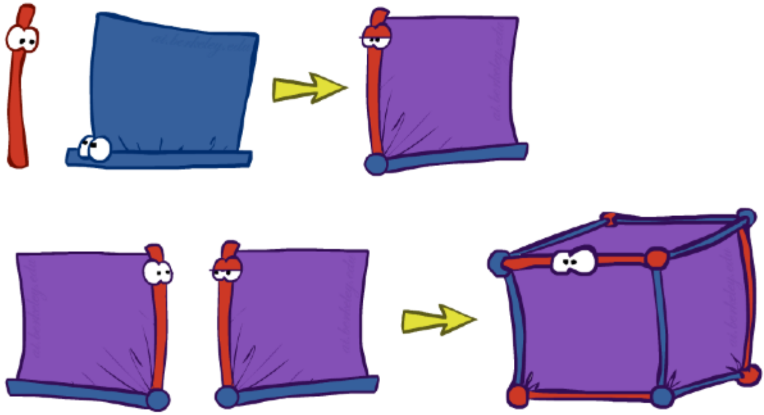
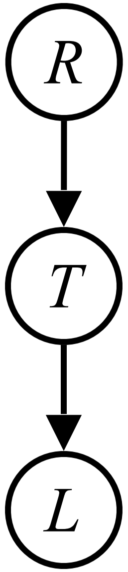


Join T

$P(R,T,L)$

Example: Multiple Joins





$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
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Join R

$P(R, T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
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+t	+l	0.3
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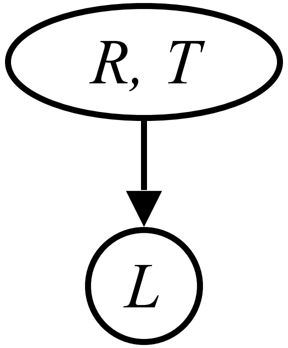
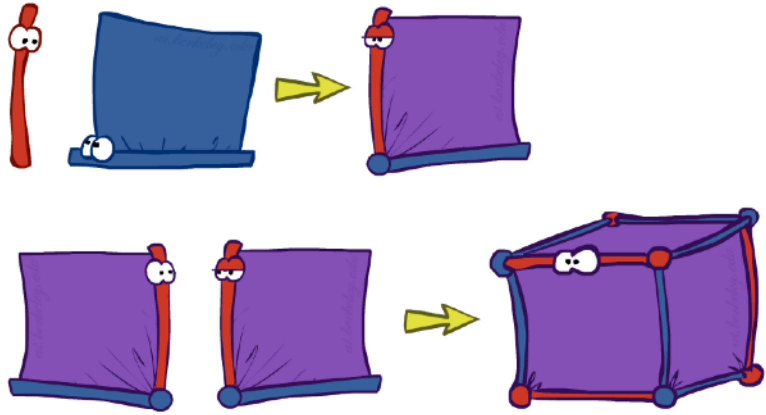


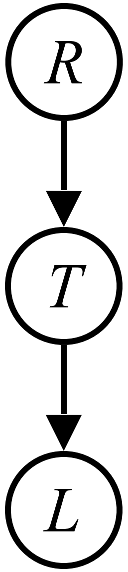
Join T

$P(R, T, L)$

+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

Example: Multiple Joins





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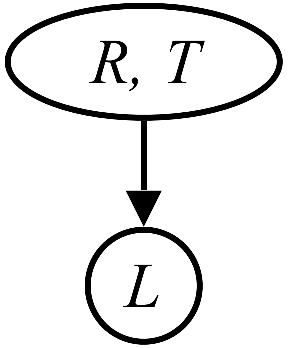
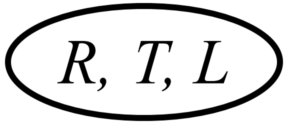


Join T



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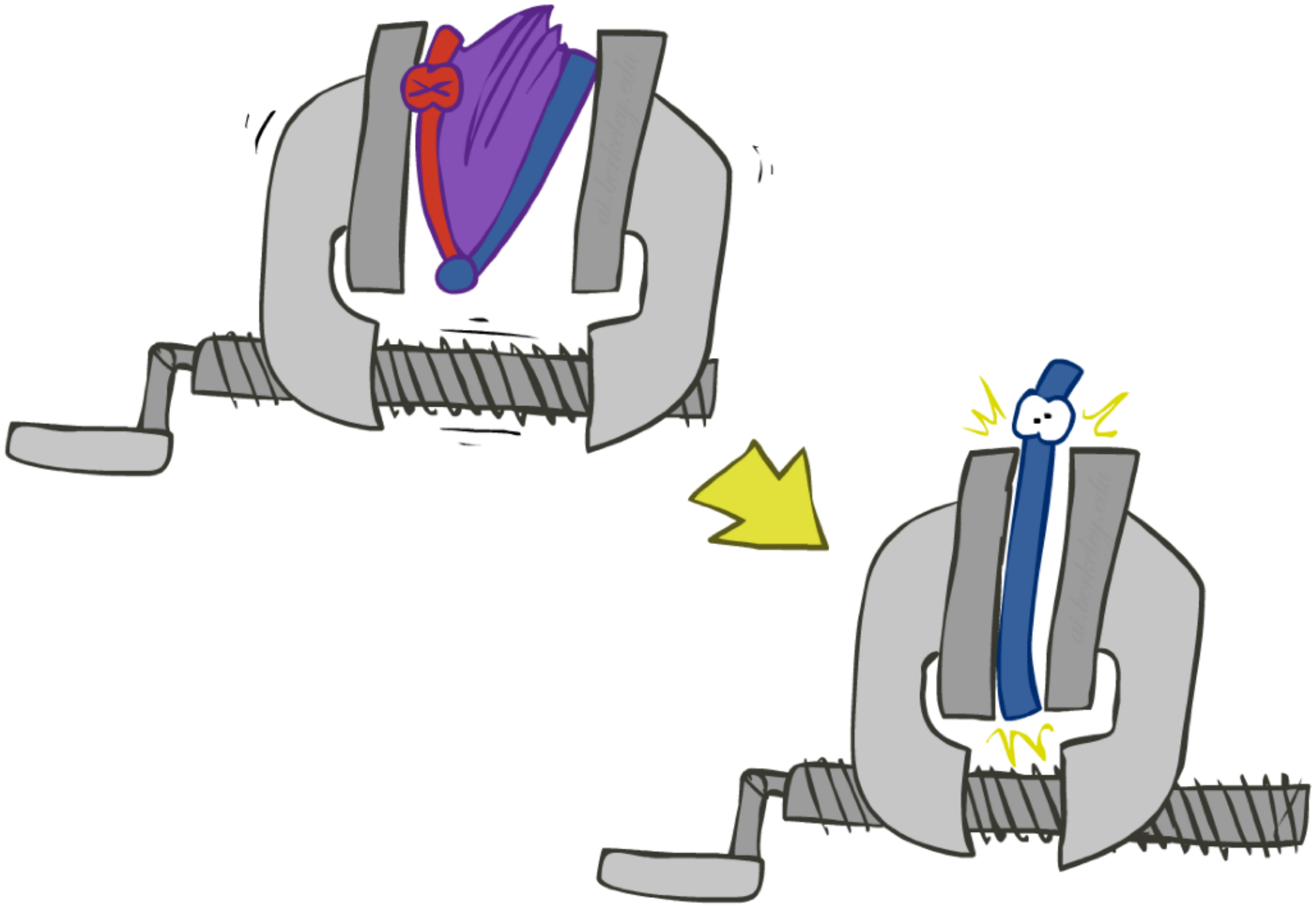
Operation 2: Eliminate



Operation 2: Eliminate

- Second basic operation: marginalization
- Take a factor and sum out a variable
  - Shrinks a factor to a smaller one
  - A projection operation
- Example:

$$P(R,T)$$

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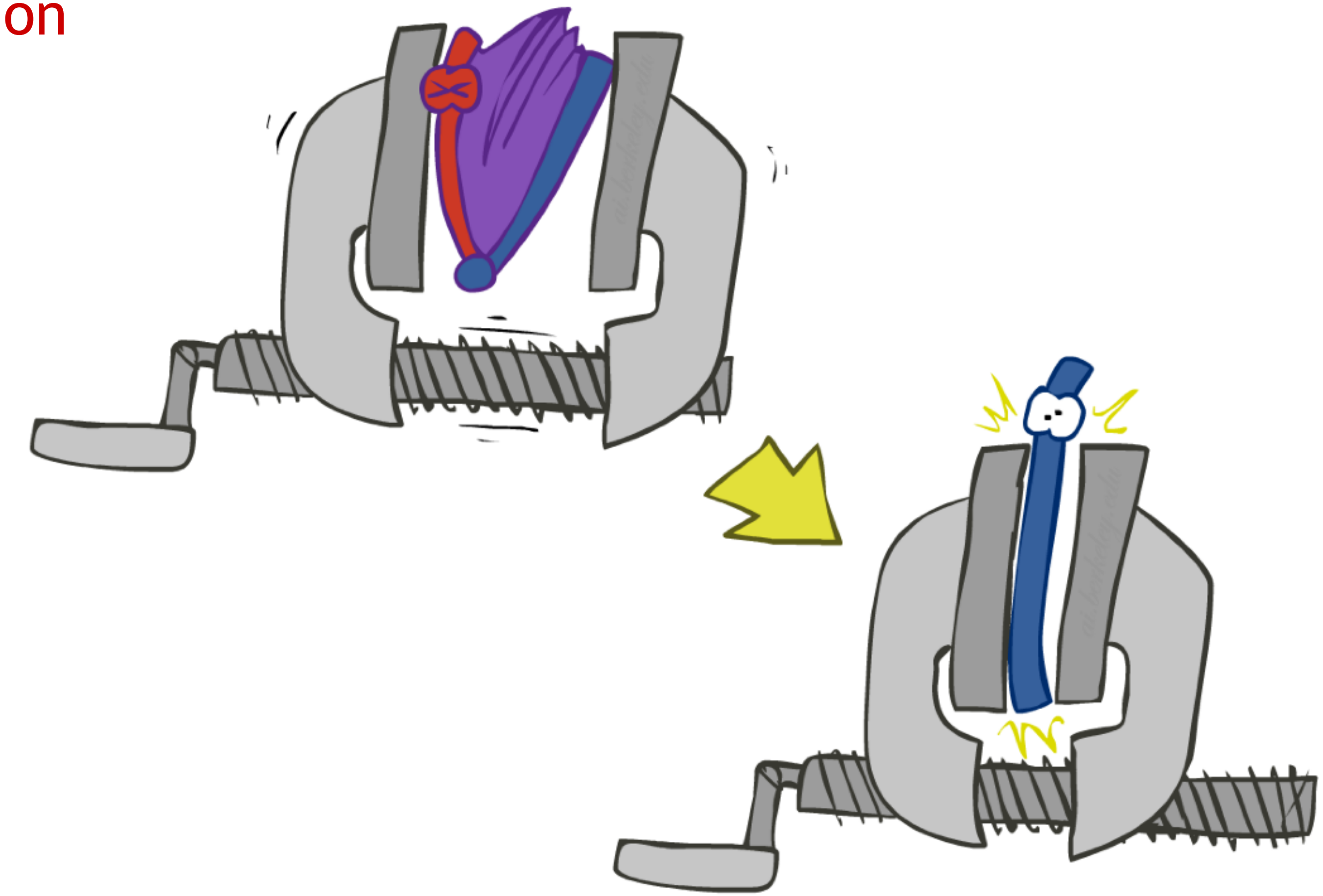

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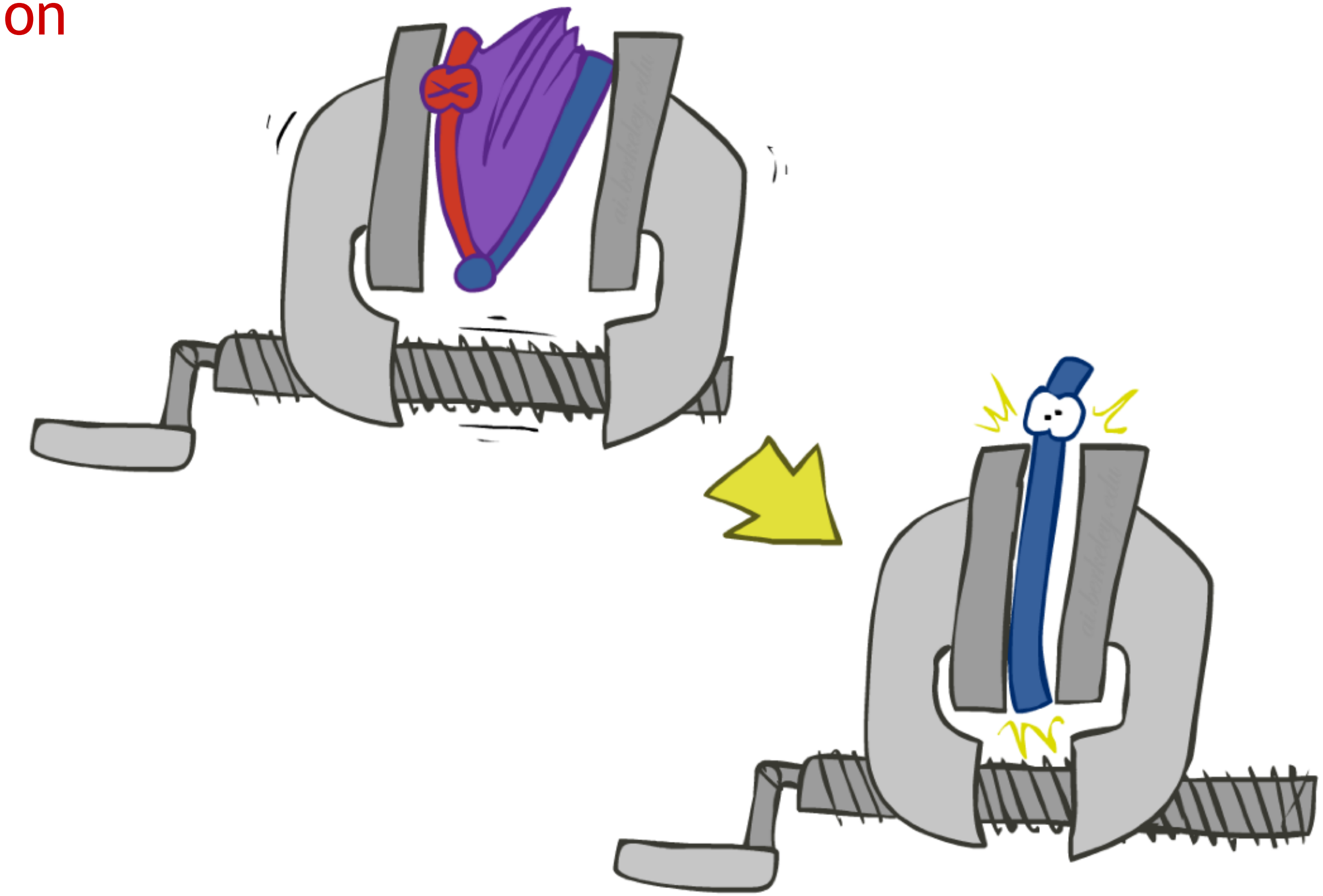
$\xrightarrow{\text{sum } R}$



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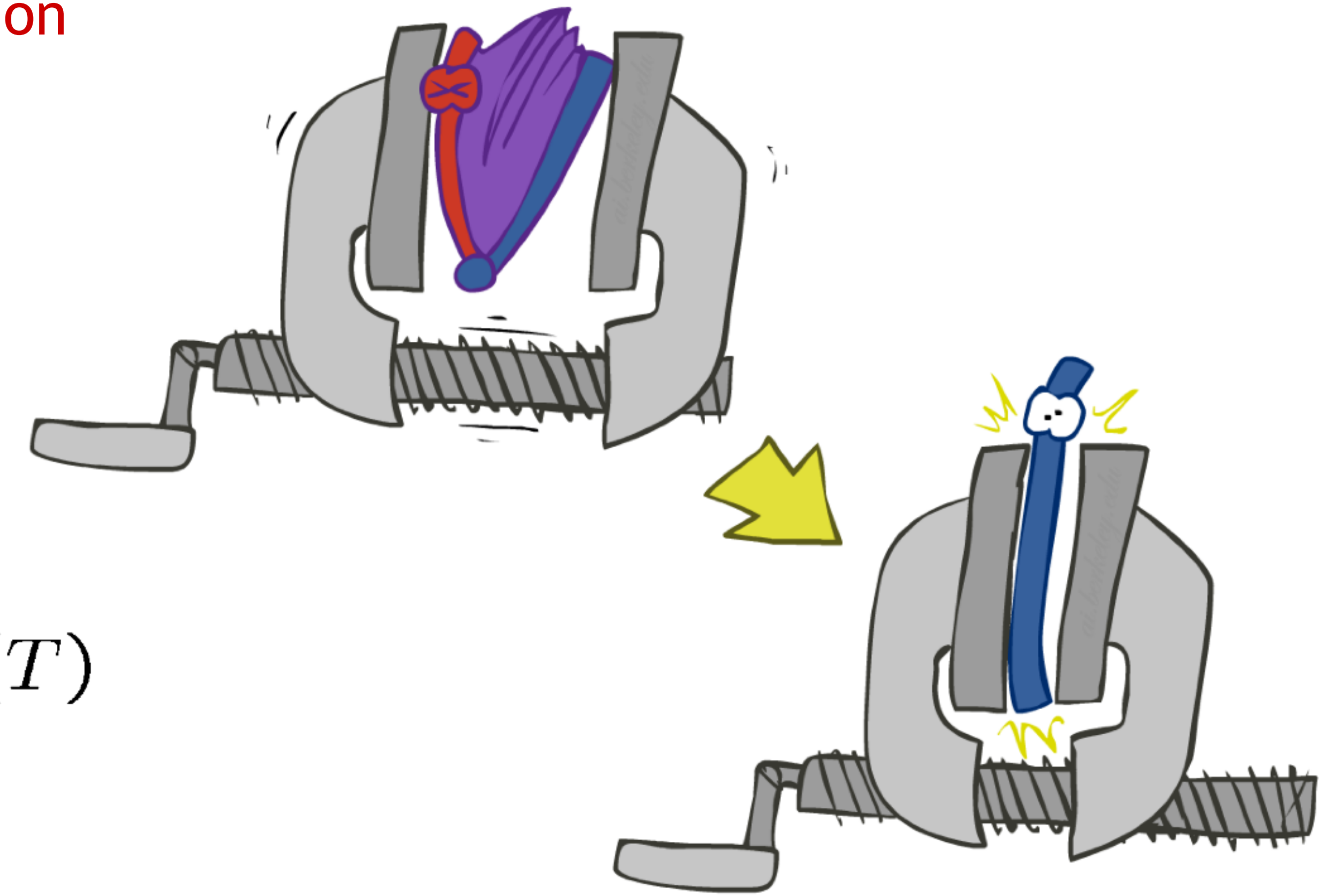
$$\xrightarrow{\text{sum } R}$$

$$P(T)$$



Operation 2: Eliminate

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- Example:

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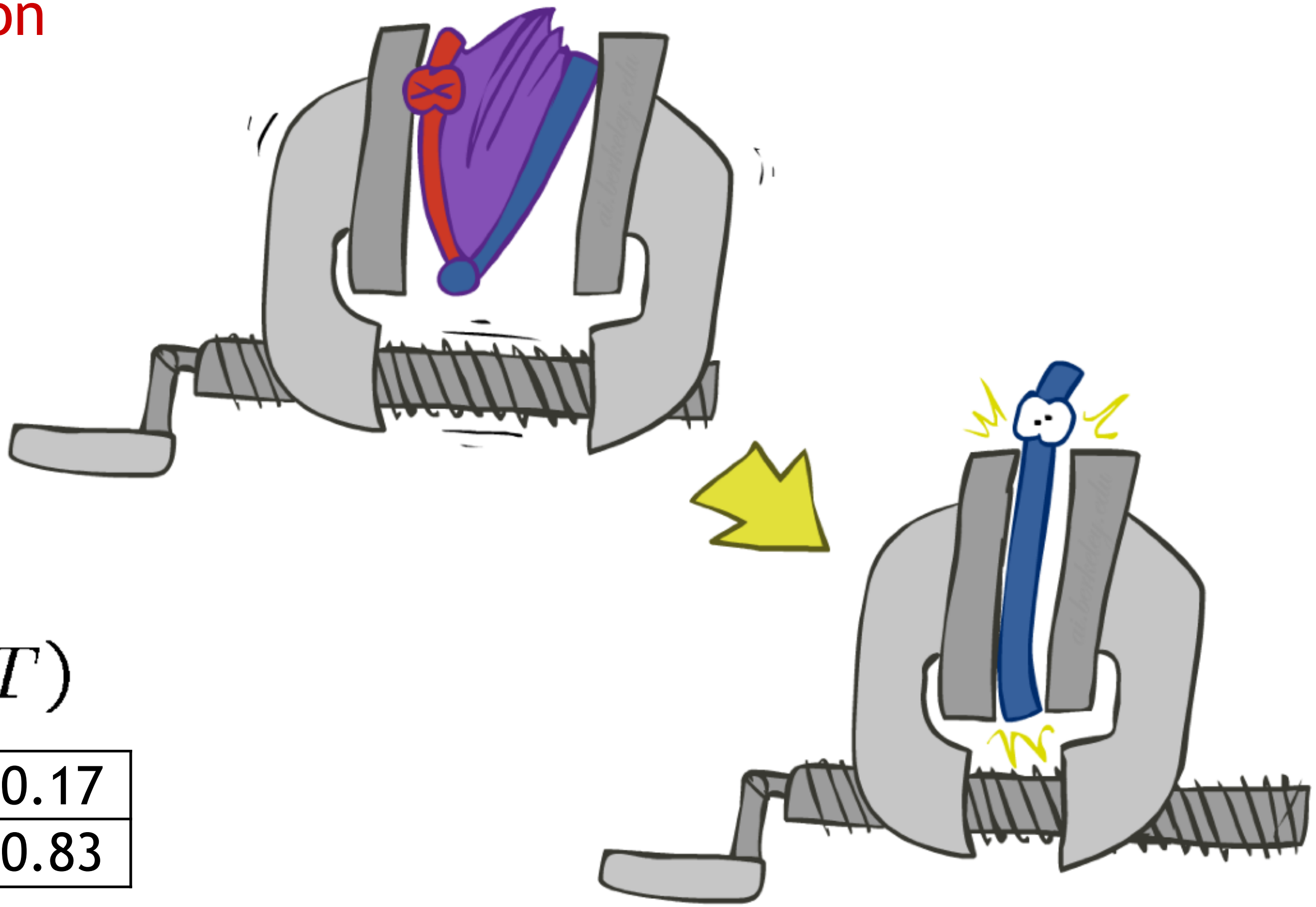
sum R →

$P(T)$

+t	0.17
-t	0.83



Multiple Elimination

R, T, L

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+r	+t	-l	0.056
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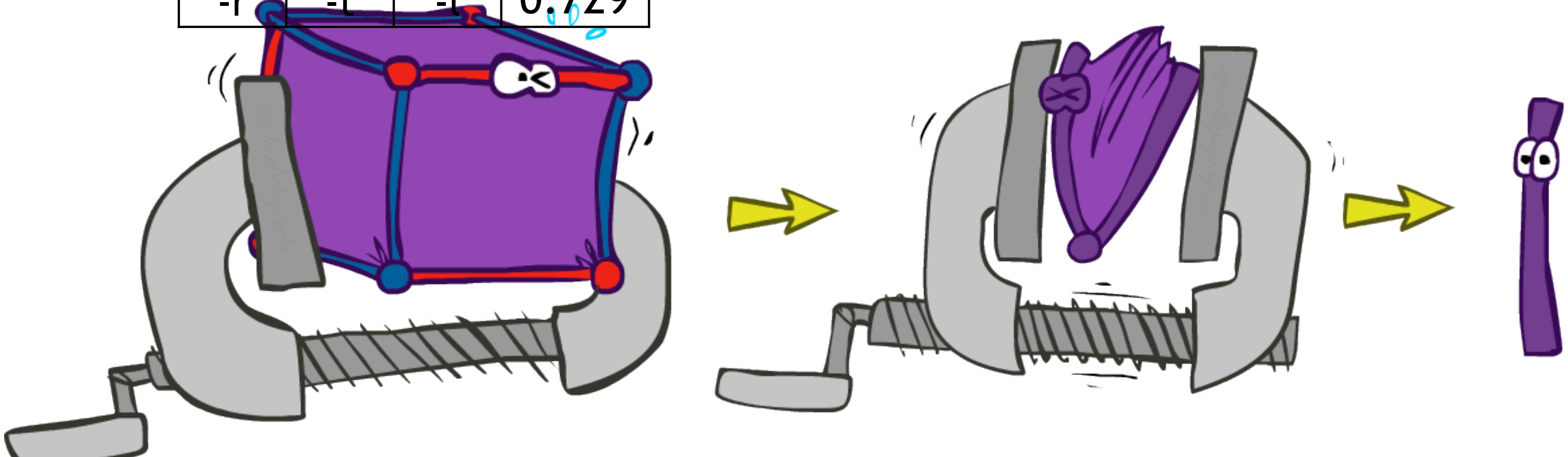

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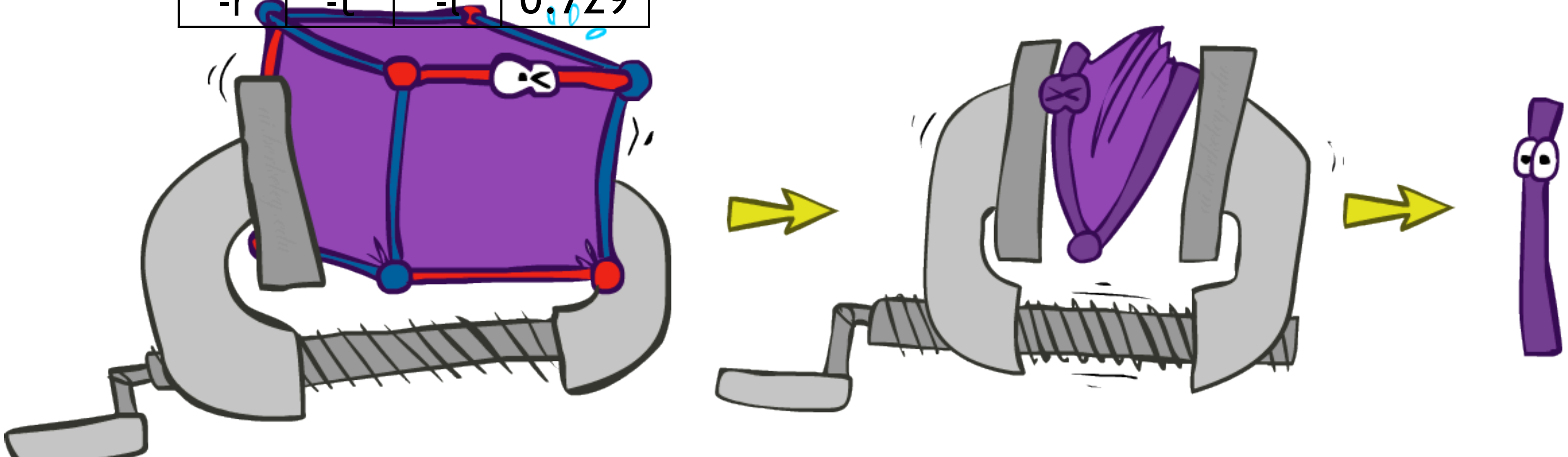
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

Sum out R



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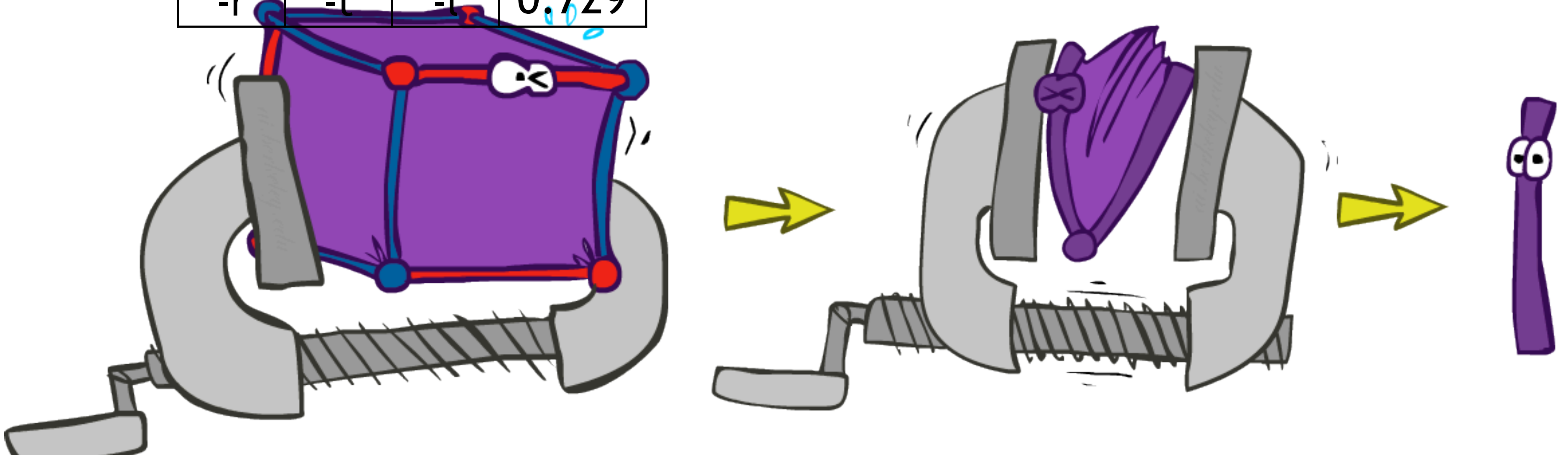
-r	-t	+l	0.081
-r	-t	-l	0.729

Sum out R

T, L

$P(T,L)$

+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747



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+t	-l	0.119
-t	+l	0.083
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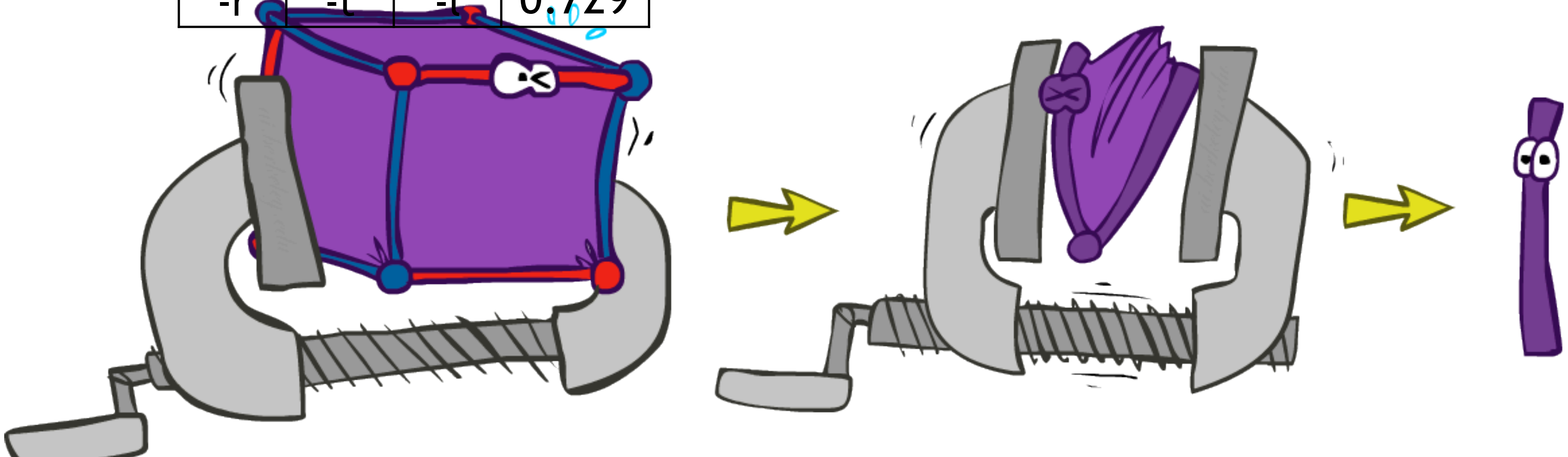
Sum out T



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-r	-t	+l	0.081
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Sum out R $\rightarrow$

T, L

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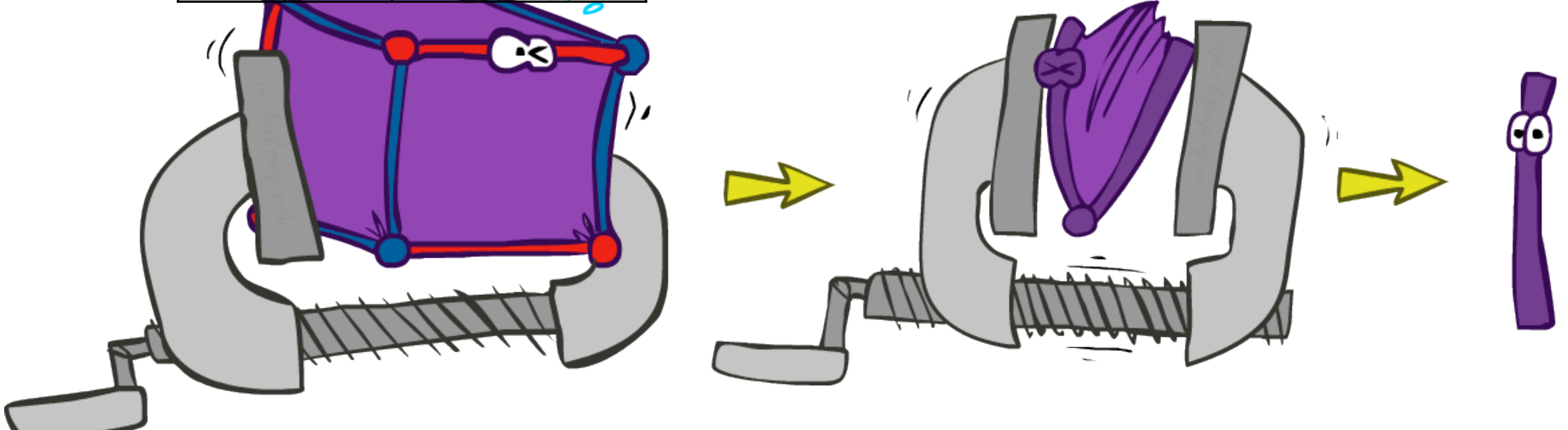
+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

Sum out T $\rightarrow$

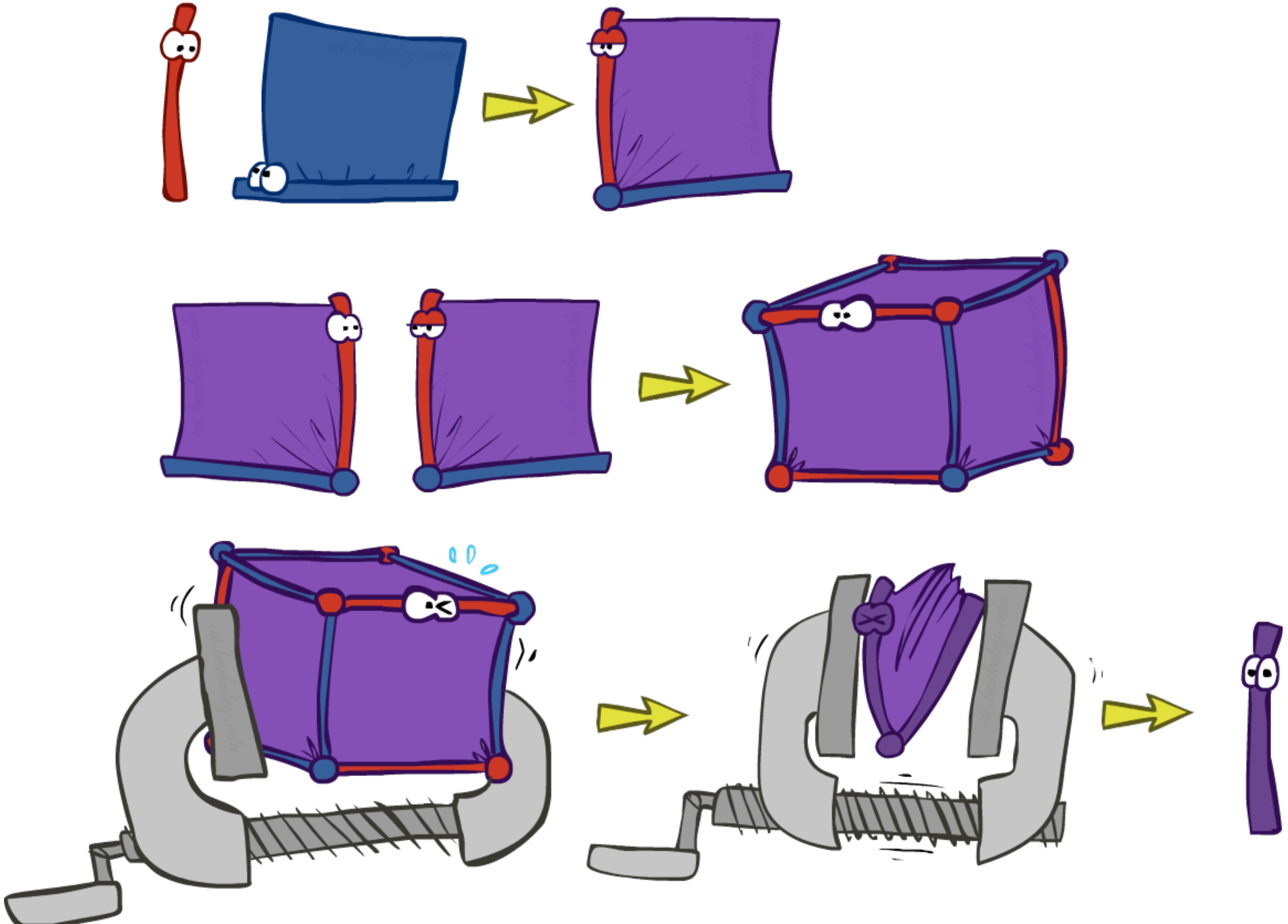
L

$P(L)$

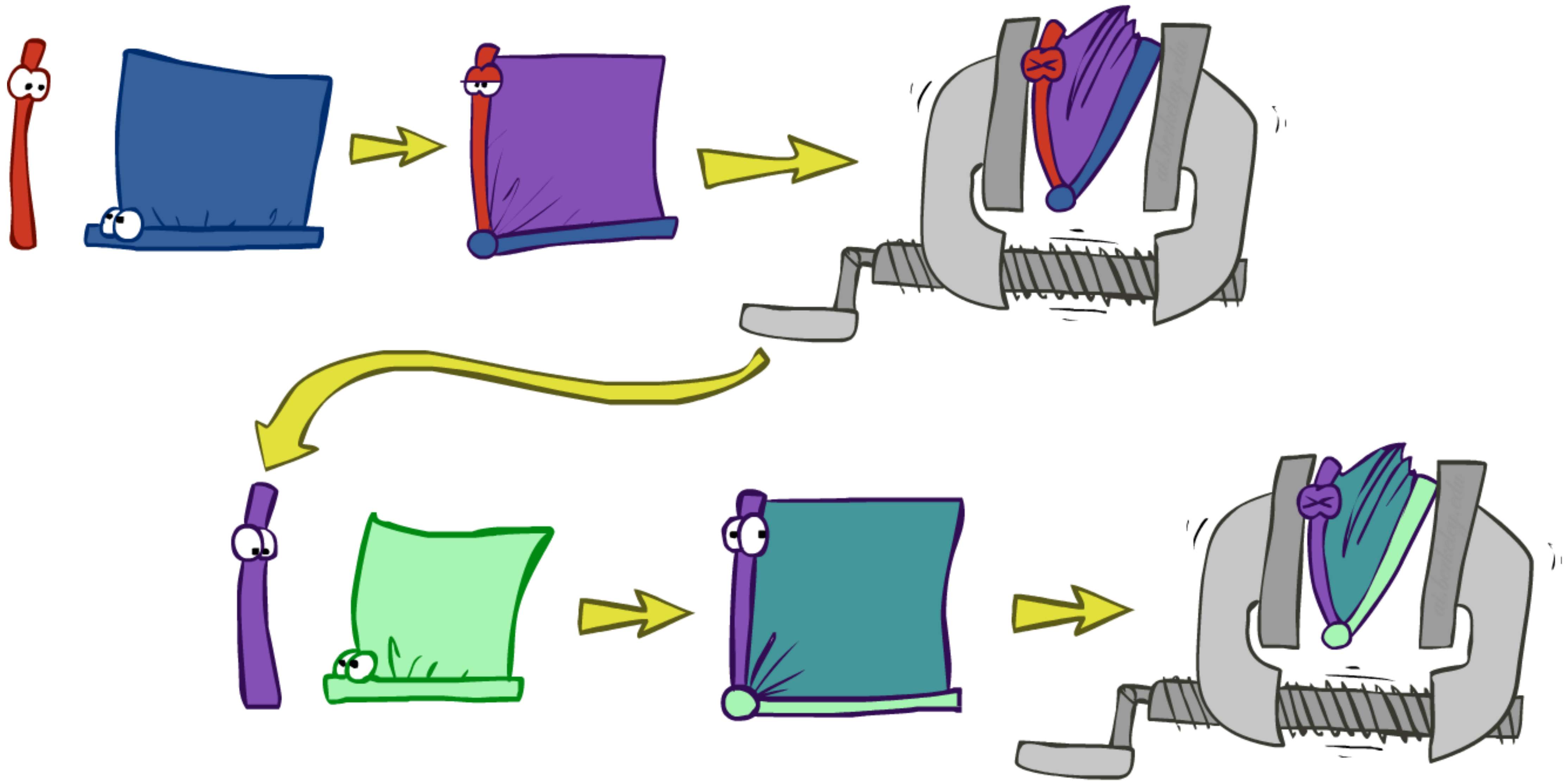
+l	0.134
-l	0.886



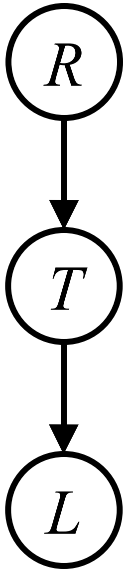
Thus Far: Multiple Join, Multiple Eliminate (= Inference by Enumeration)



Marginalizing Early (= Variable Elimination)



Traffic Domain



$$P(L) = ?$$

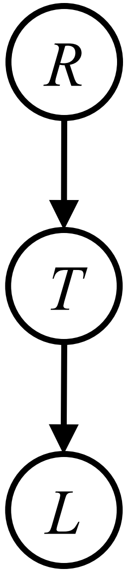
- Inference by Enumeration

$$= \underbrace{\sum_t \underbrace{\sum_r \underbrace{P(L|t)\underbrace{P(r)P(t|r)}_{\text{Join on r}}}_{\text{Join on t}}}_{\text{Eliminate r}}}_{\text{Eliminate t}}$$

- Variable Elimination

$$= \underbrace{\sum_t \underbrace{P(L|t)\underbrace{\sum_r \underbrace{P(r)P(t|r)}_{\text{Join on r}}}_{\text{Eliminate r}}}_{\text{Join on t}}}_{\text{Eliminate t}}$$

Traffic Domain



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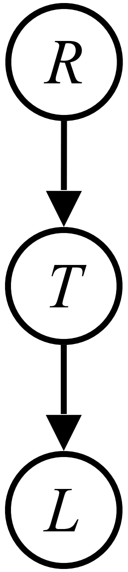
Marginalizing Early! (aka VE)

$P(R)$

+r	0.1
-r	0.9



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Marginalizing Early! (aka VE)

Join R

$P(R)$

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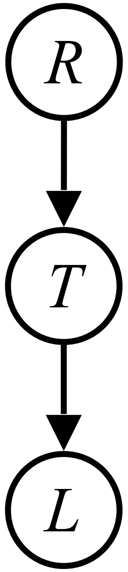

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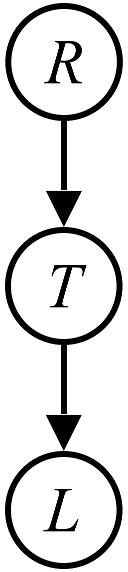

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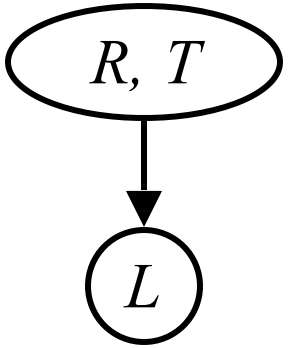

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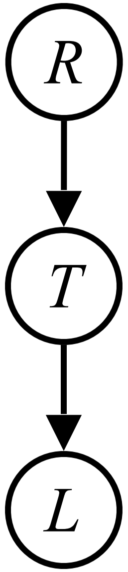

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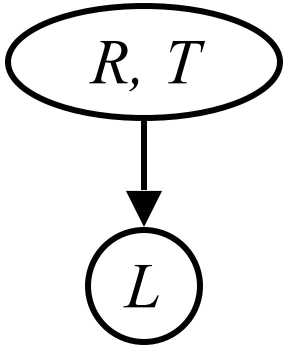

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Sum out R

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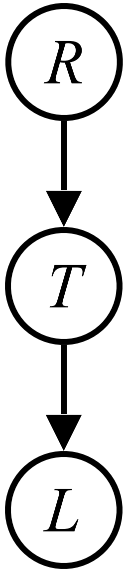

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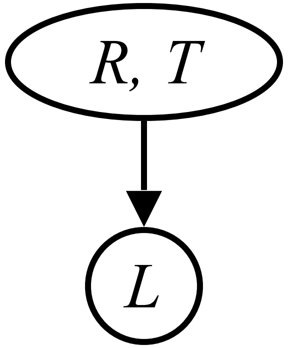

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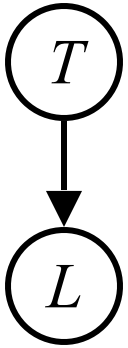

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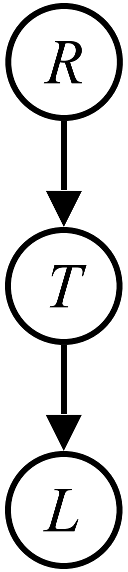

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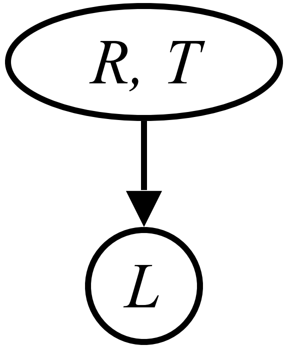

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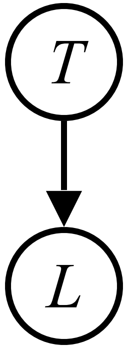

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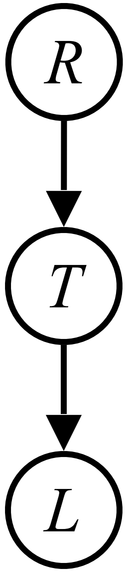

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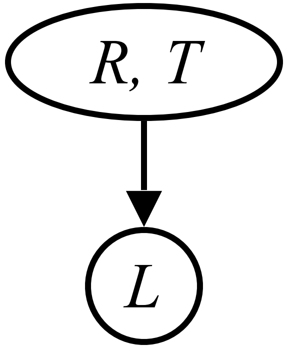

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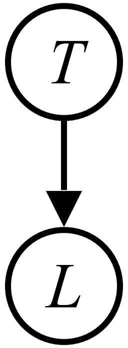

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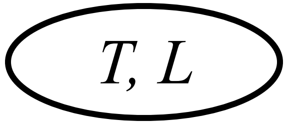

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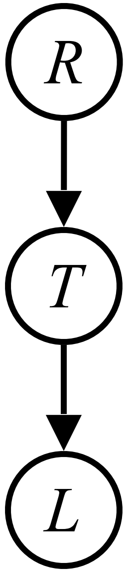
+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

Marginalizing Early! (aka VE)



$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

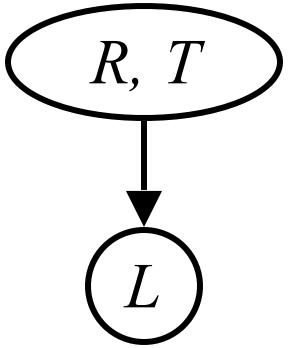
$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join R

$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81



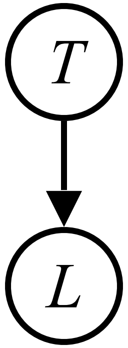
$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Sum out R

$P(T)$

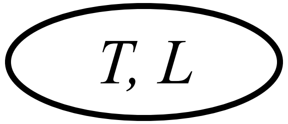
+t	0.17
-t	0.83



$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join T



$P(T,L)$

+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

Sum out T

Marginalizing Early! (aka VE)

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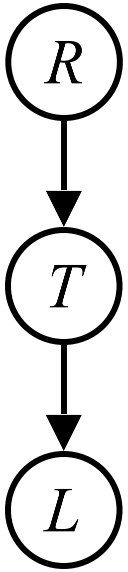

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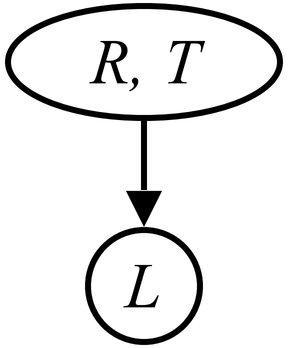

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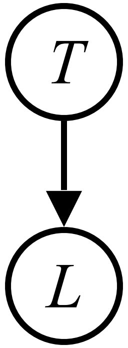

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+t	+l	0.3
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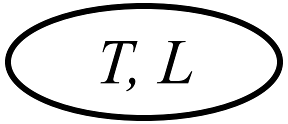
Join T



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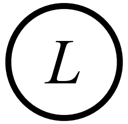
-t	-l	0.747

Sum out T

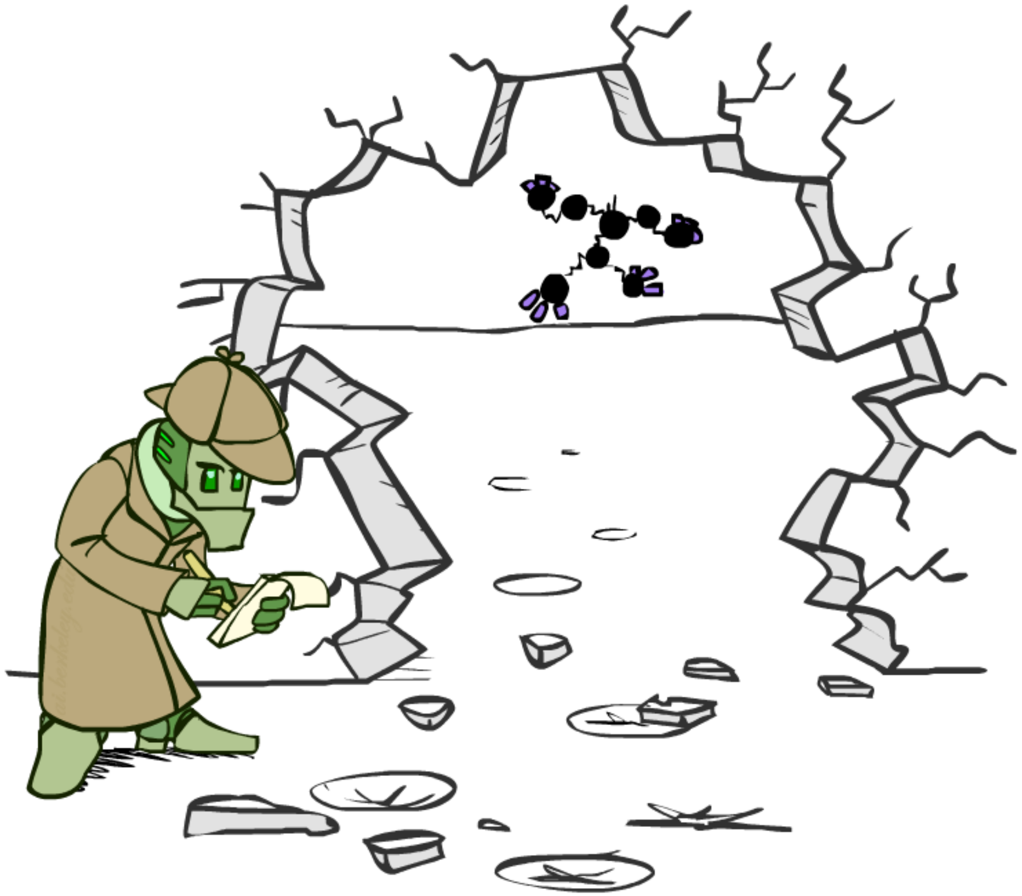


$P(L)$

+l	0.134
-l	0.866

Evidence

- If evidence, start with factors that select that evidence



Evidence

- If evidence, start with factors that select that evidence
  - No evidence uses these initial factors:

$P(R)$

+r	0.1
-r	0.9

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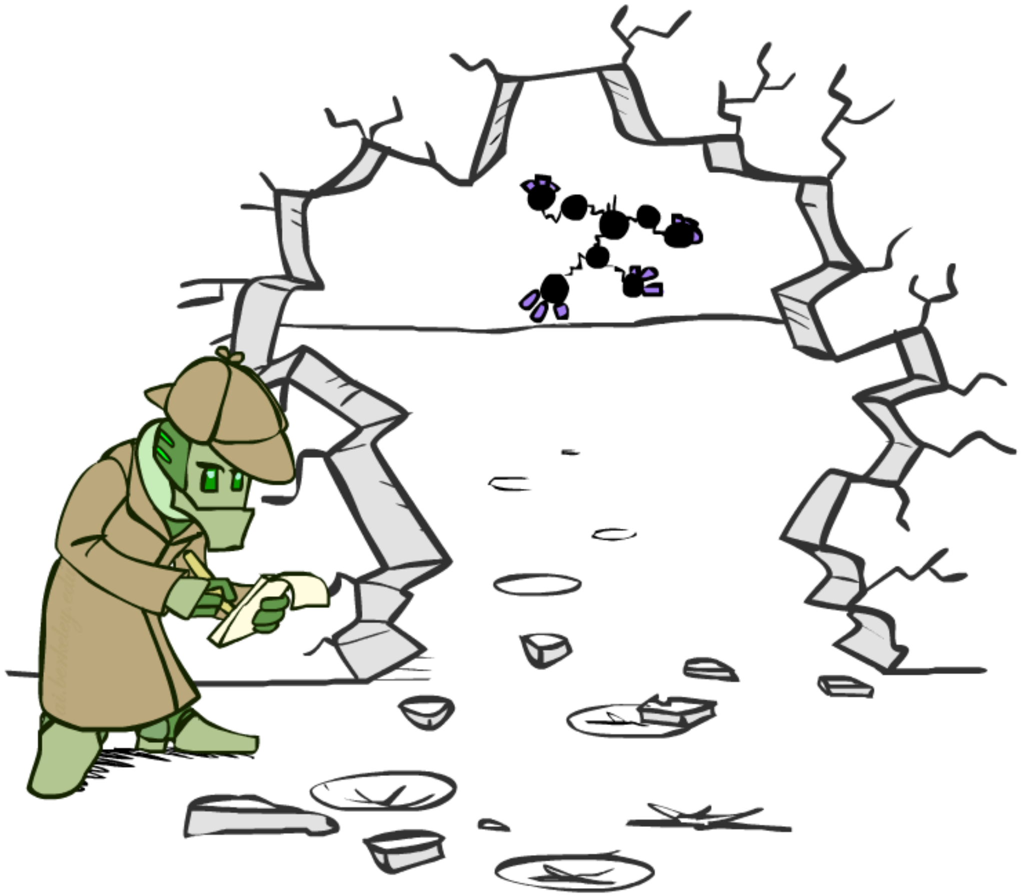

Evidence

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+t	-l	0.7
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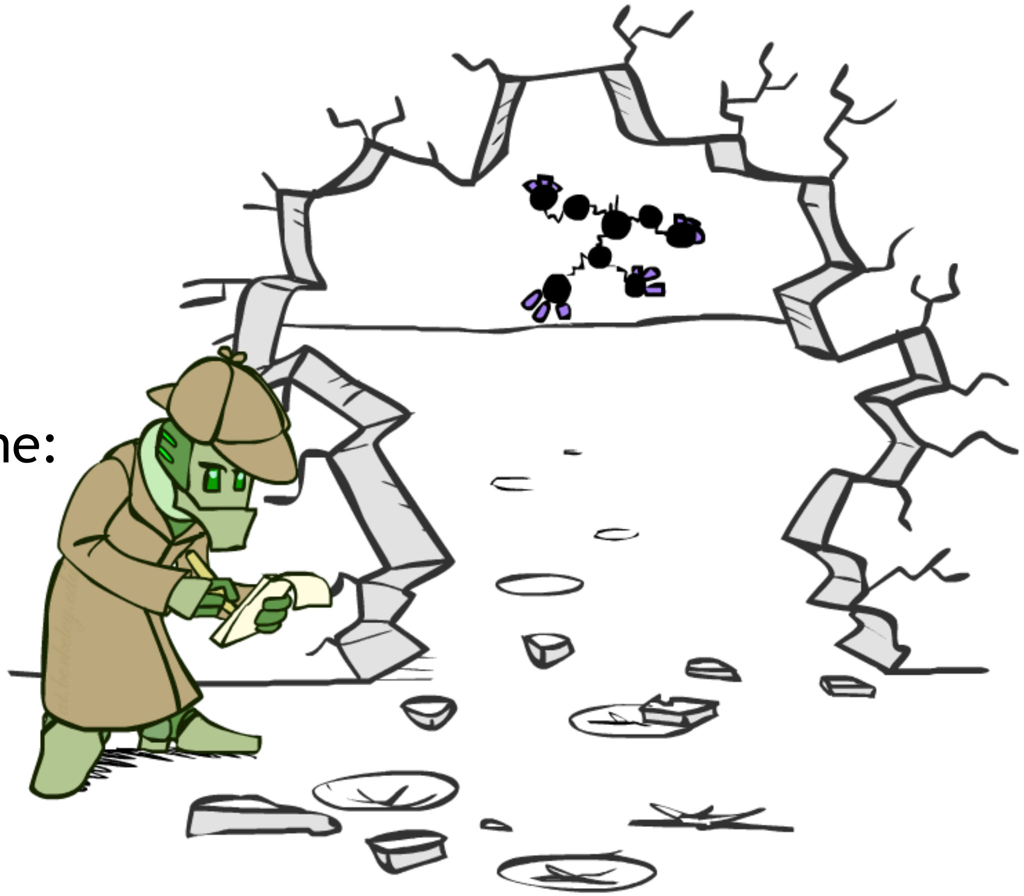

  - Computing $P(L|+r)$, the initial factors become:

Evidence

- If evidence, start with factors that select that evidence
  - No evidence uses these initial factors:

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+t	+l	0.3
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-t	+l	0.1
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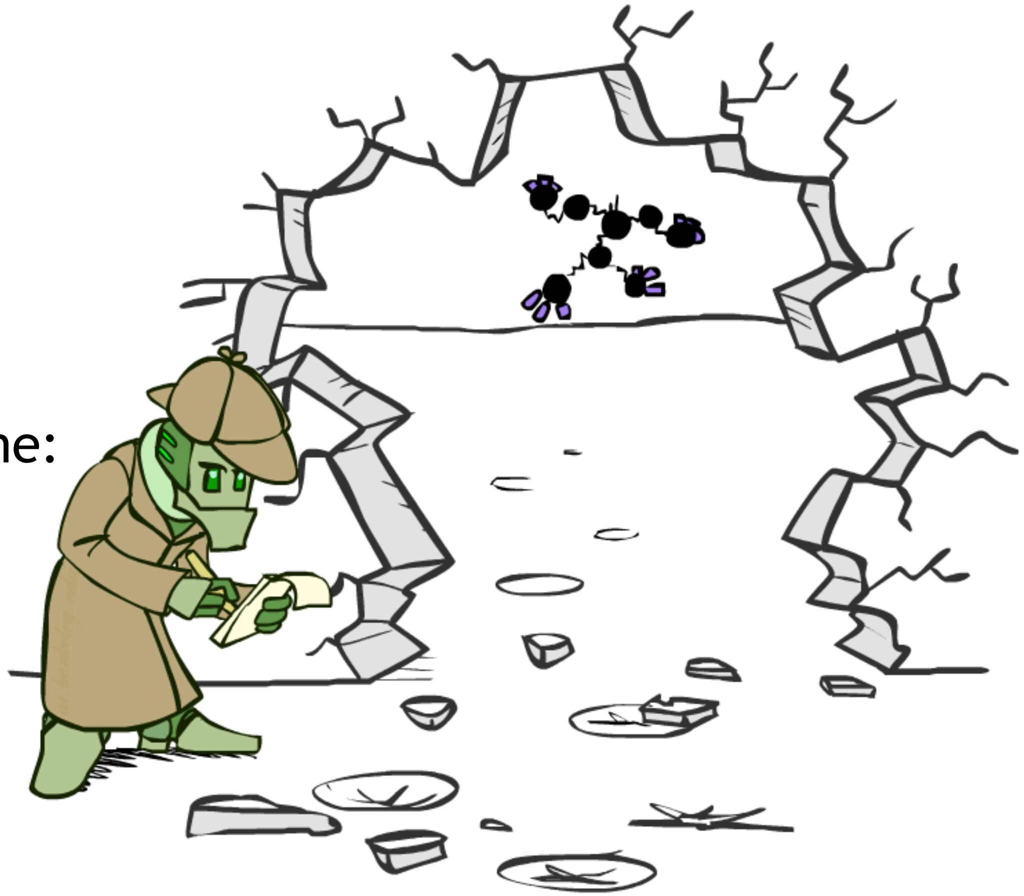

  - Computing $P(L|+r)$, the initial factors become:

$P(+r)$

+r	0.1

Evidence

- If evidence, start with factors that select that evidence
  - No evidence uses these initial factors:

$P(R)$

+r	0.1
-r	0.9

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+r	+t	0.8
+r	-t	0.2
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+r	+t	0.8
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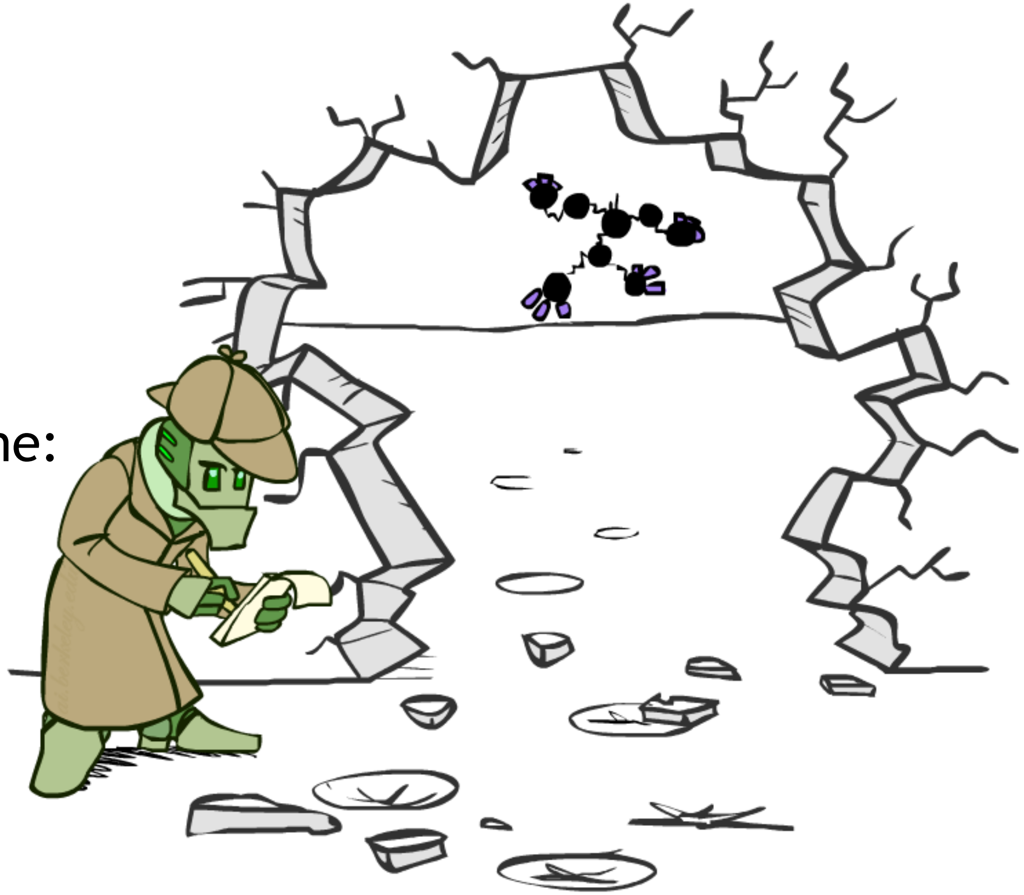

Evidence

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  - No evidence uses these initial factors:

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-r	0.9

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+r	+t	0.8
+r	-t	0.2
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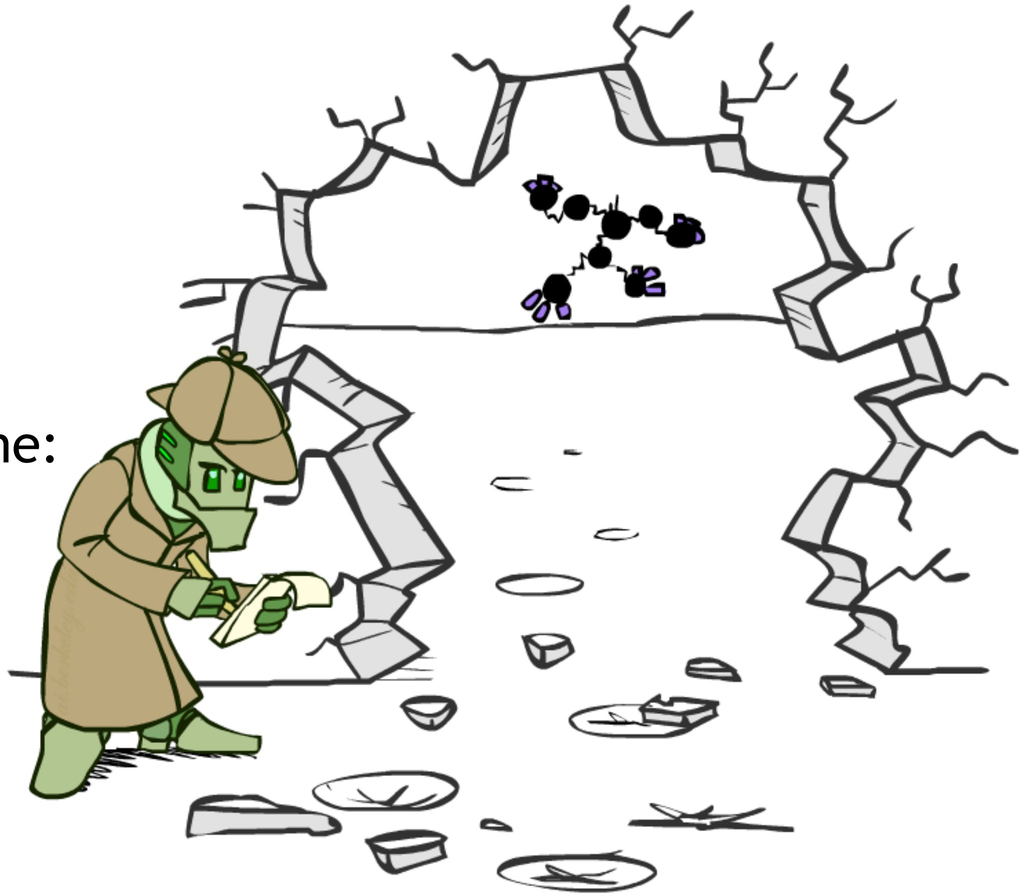

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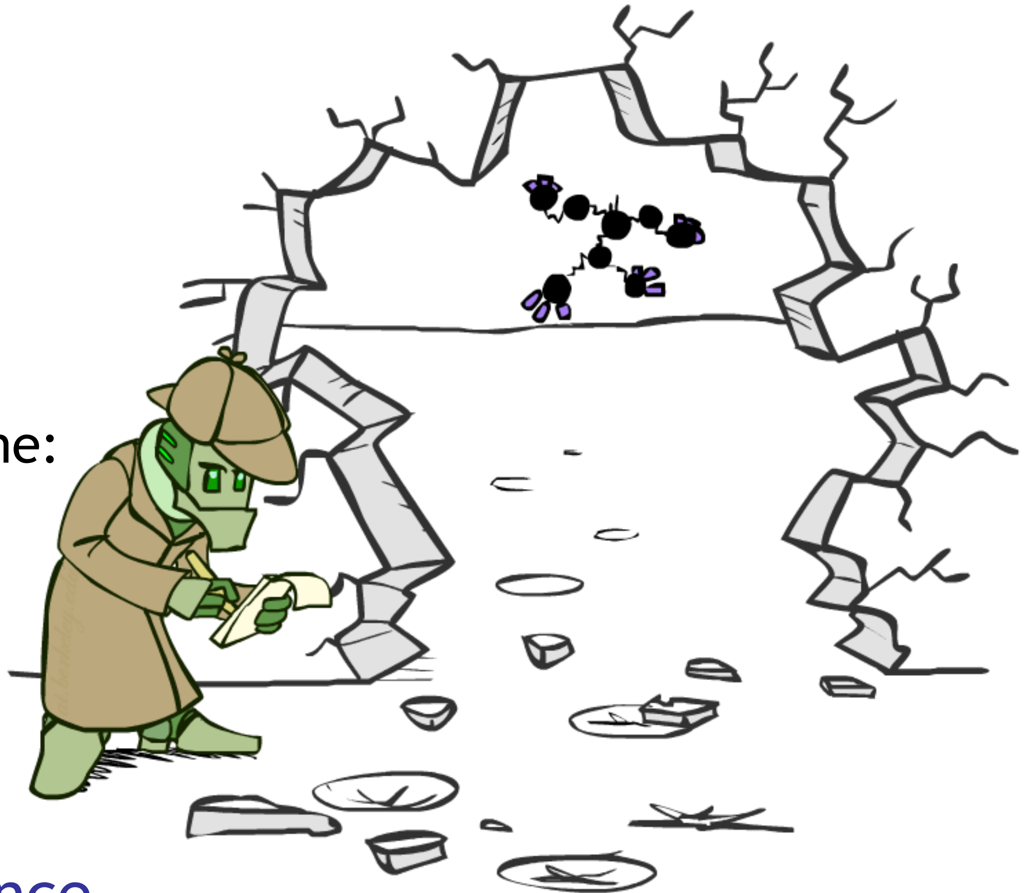
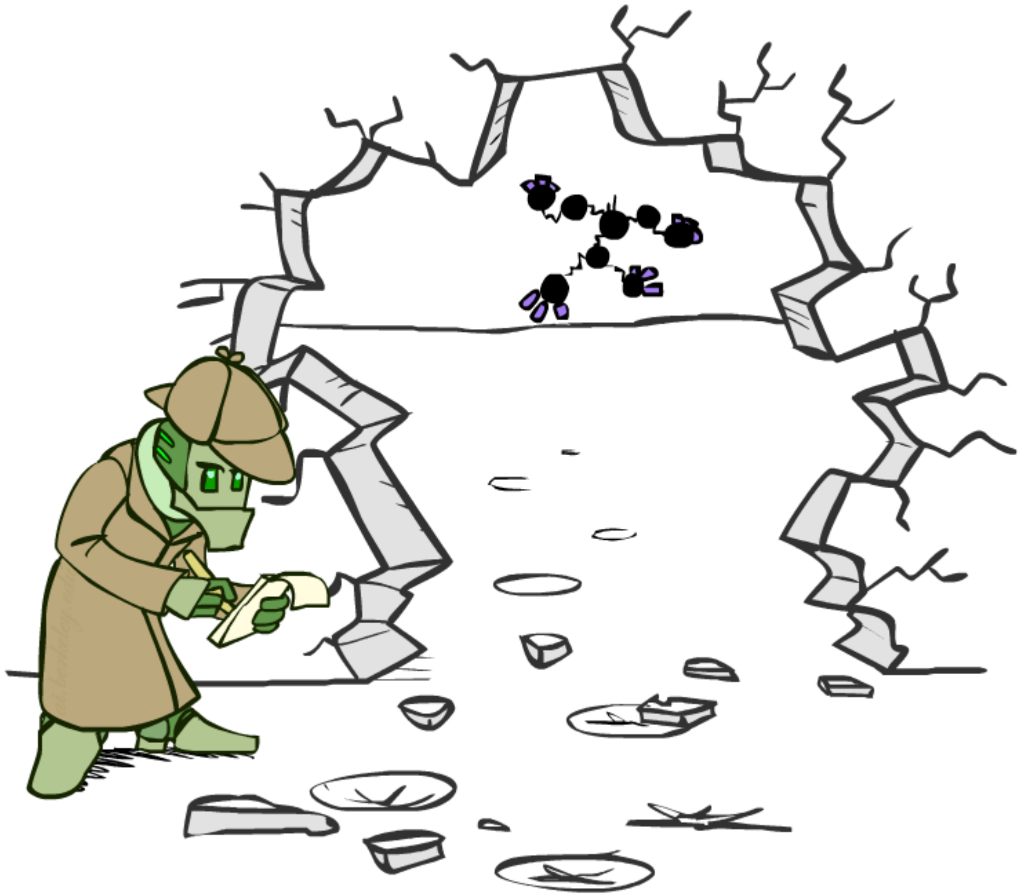


- We eliminate all vars other than query + evidence

Evidence II

- Result will be a selected joint of query and evidence
  - E.g. for $P(L \mid +r)$, we would end up with:

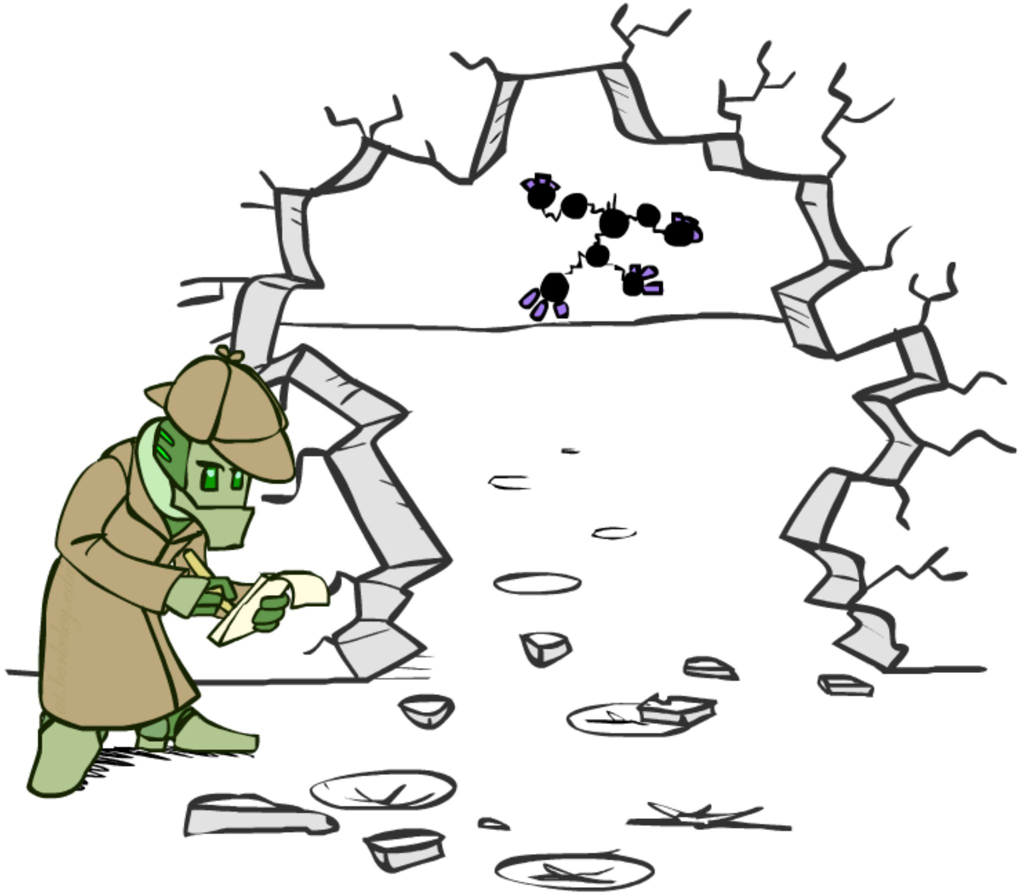


Evidence II

- Result will be a selected joint of query and evidence
  - E.g. for P(L | +r), we would end up with:

$$P(+r, L)$$

+r	+l	0.026
+r	-l	0.074



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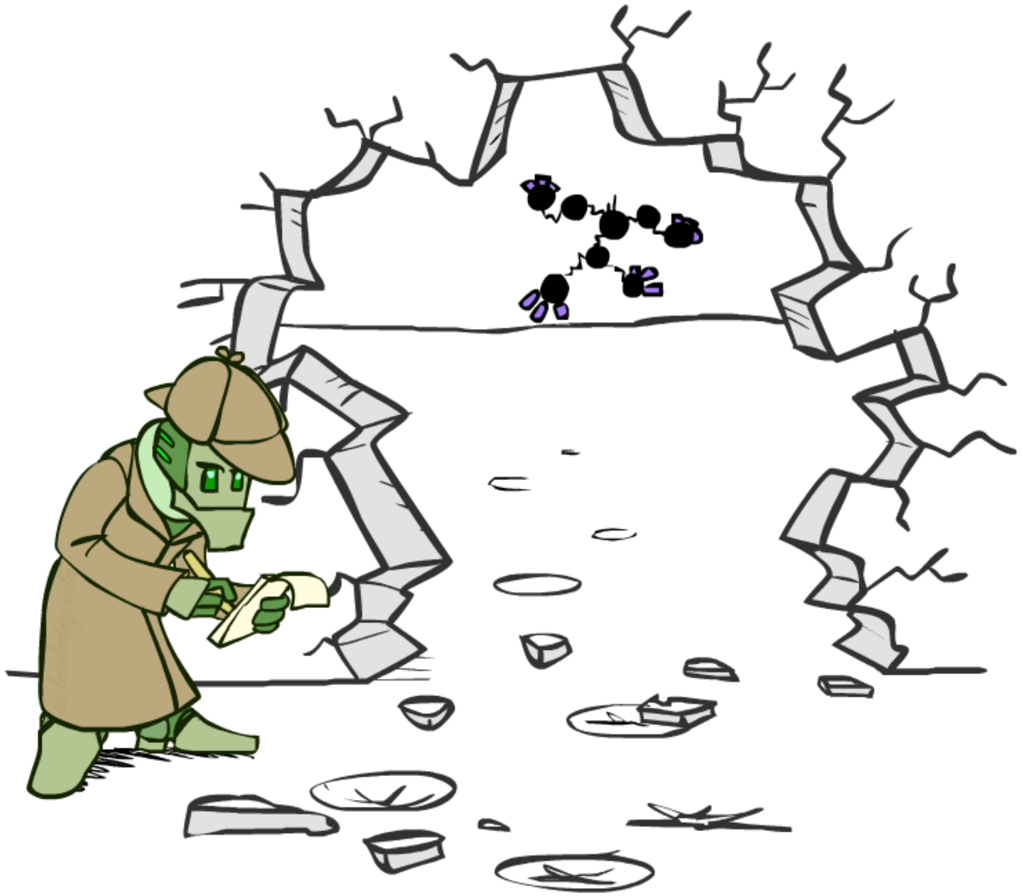
  - E.g. for P(L | +r), we would end up with:

$$P(+r, L)$$

+r	+l	0.026
+r	-l	0.074

- To get our answer, just normalize this!



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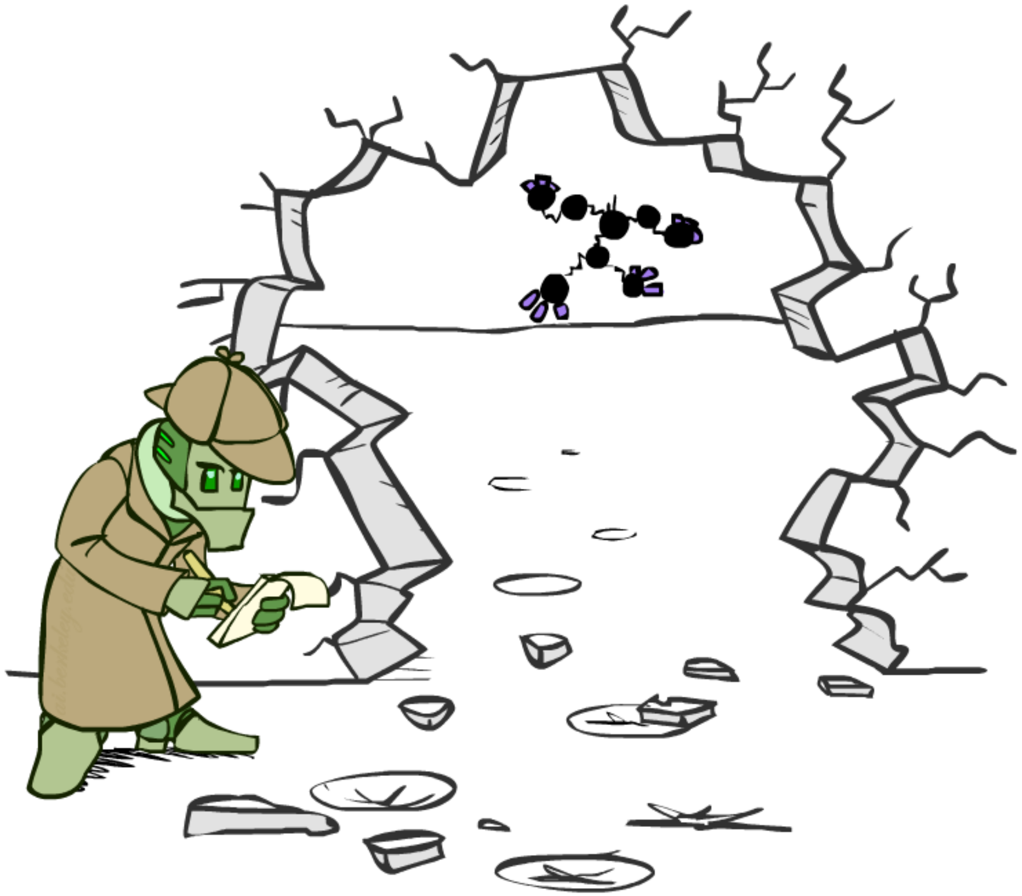

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  - E.g. for P(L | +r), we would end up with:

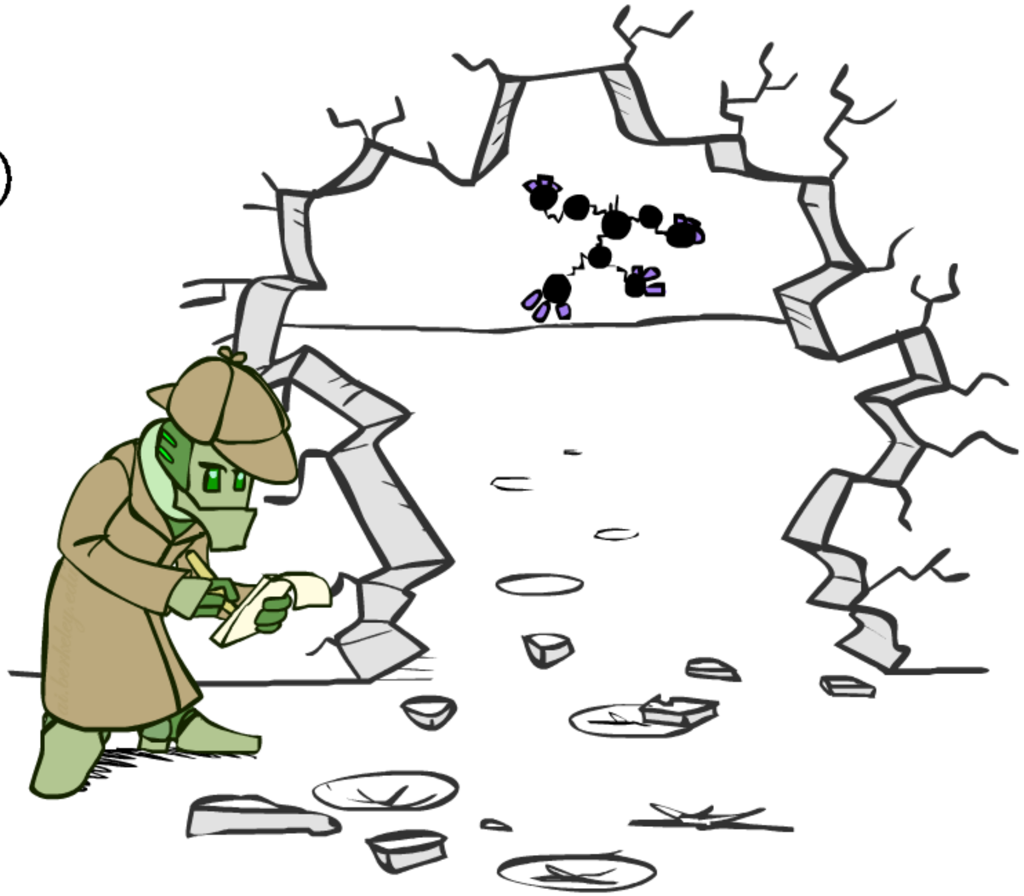
$P(+r, L)$

+r	+l	0.026
+r	-l	0.074

Normalize $\longrightarrow$

$P(L | +r)$

+l	0.26
-l	0.74



- To get our answer, just normalize this!

Evidence II

- Result will be a selected joint of query and evidence
  - E.g. for P(L | +r), we would end up with:

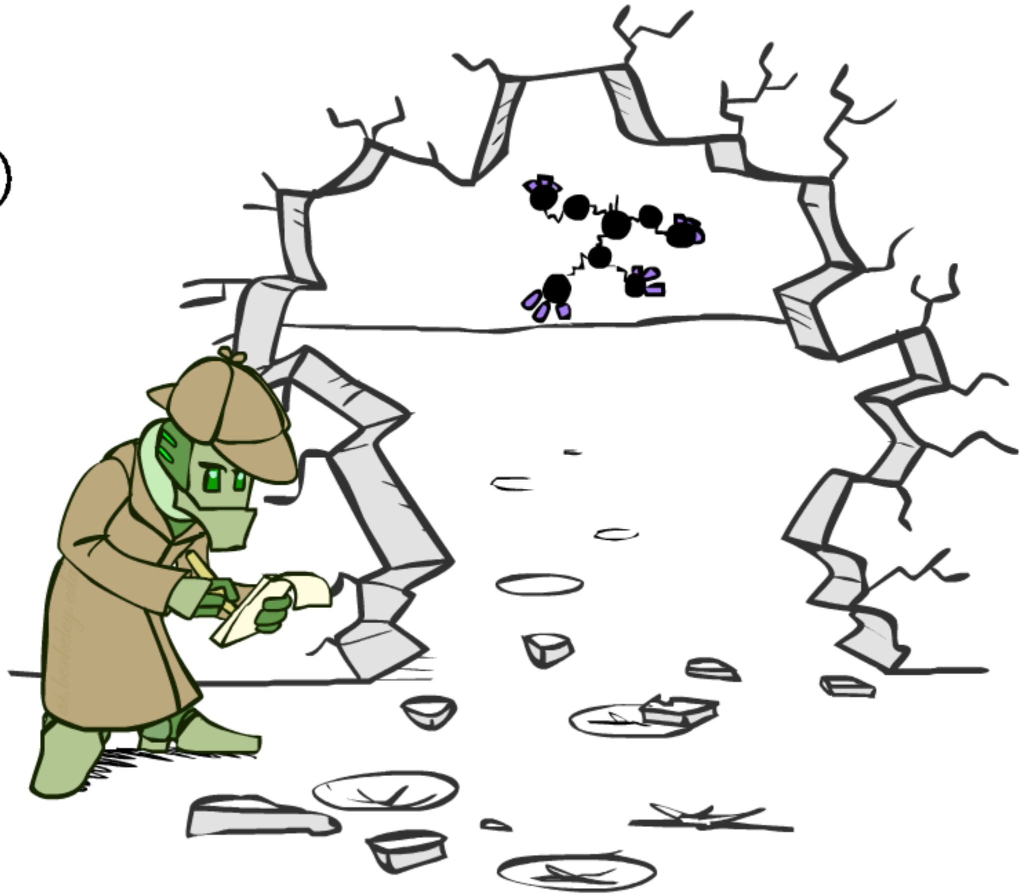
$P(+r, L)$

+r	+l	0.026
+r	-l	0.074

Normalize

$P(L | +r)$

+l	0.26
-l	0.74

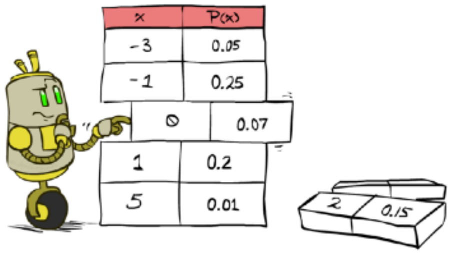


- To get our answer, just normalize this!

- That 's it!

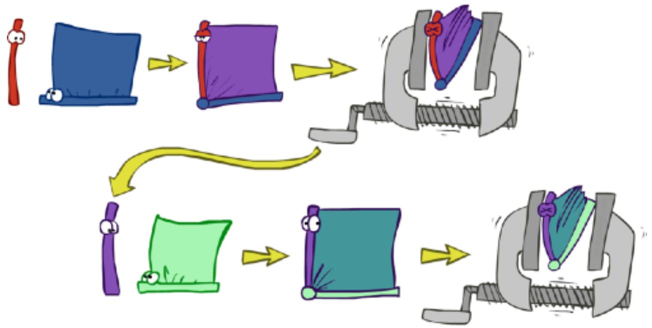
General Variable Elimination

- Query: $P(Q|E_1 = e_1, \dots E_k = e_k)$



- Start with initial factors:
  - Local CPTs (but instantiated by evidence)

- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H



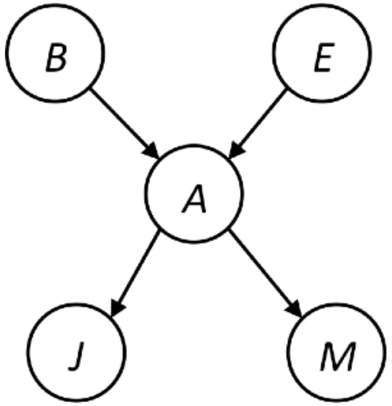
- Join all remaining factors and normalize

 $\times \frac{1}{Z}$

Example

$$P(B|j,m) \propto P(B,j,m)$$

$$\boxed{P(B) \qquad P(E) \qquad P(A|B,E) \qquad P(j|A) \qquad P(m|A)}$$

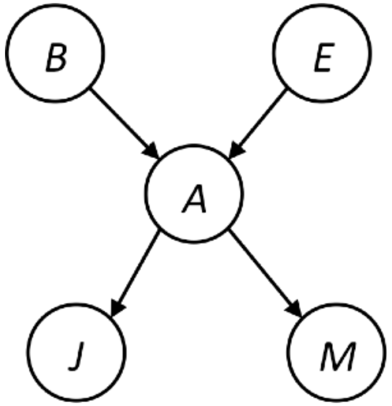


Choose A

Example

$$P(B|j,m) \propto P(B,j,m)$$

$$\boxed{P(B) \qquad P(E) \qquad P(A|B,E) \qquad P(j|A) \qquad P(m|A)}$$

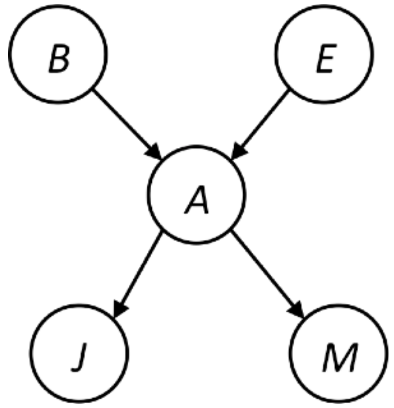


Choose A

$$P(A|B,E)$$
$$P(j|A)$$
$$P(m|A)$$

Example

$$P(B|j,m) \propto P(B,j,m)$$



$P(B)$	$P(E)$	$P(A|B,E)$	$P(j|A)$	$P(m|A)$

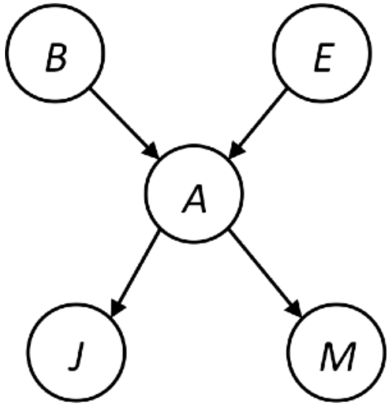
Choose A

$P(A|B,E)$

$P(j|A)$ $\times$

$P(m|A)$

Example



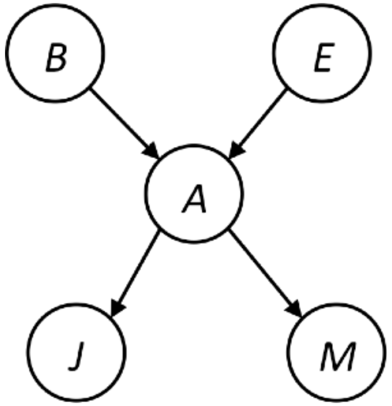
$$P(B|j,m) \propto P(B,j,m)$$

$$\boxed{P(B) \qquad P(E) \qquad P(A|B,E) \qquad P(j|A) \qquad P(m|A)}$$

Choose A

$$\begin{array}{l} P(A|B,E) \\ P(j|A) \\ P(m|A) \end{array} \quad \xrightarrow{\times} \quad P(j,m,A|B,E)$$

Example



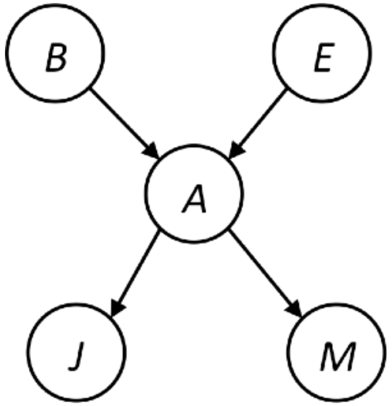
$$P(B|j,m) \propto P(B,j,m)$$

$$\boxed{P(B) \qquad P(E) \qquad P(A|B,E) \qquad P(j|A) \qquad P(m|A)}$$

Choose A

$$P(A|B,E)$$
$$P(j|A) \quad \xrightarrow{\times} \quad P(j,m,A|B,E) \quad \xrightarrow{\sum}$$
$$P(m|A)$$

Example



$$P(B|j,m) \propto P(B,j,m)$$

$$\boxed{P(B) \qquad P(E) \qquad P(A|B,E) \qquad P(j|A) \qquad P(m|A)}$$

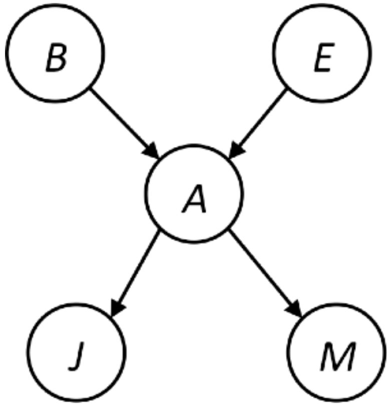
Choose A

$$P(A|B,E)$$
$$P(j|A) \quad \xrightarrow{\times} \quad P(j,m,A|B,E) \quad \xrightarrow{\sum} \quad P(j,m|B,E)$$
$$P(m|A)$$

Example

$$P(B|j,m) \propto P(B,j,m)$$

$$\boxed{P(B) \qquad P(E) \qquad P(A|B,E) \qquad P(j|A) \qquad P(m|A)}$$



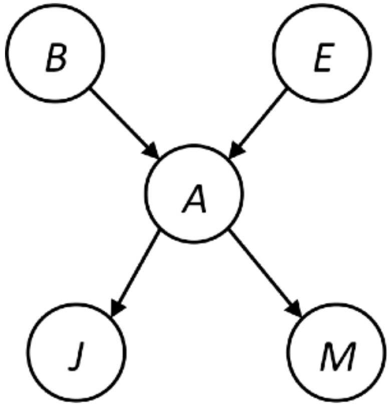
Choose A

$$\left.\begin{array}{l} P(A|B,E) \\ P(j|A) \\ P(m|A) \end{array}\right. \xrightarrow{\times} P(j,m,A|B,E) \xrightarrow{\sum} P(j,m|B,E)$$

$$\boxed{P(B) \qquad P(E) \qquad P(j,m|B,E)}$$

Example

$$\boxed{P(B) \qquad P(E) \qquad P(j,m|B,E)}$$



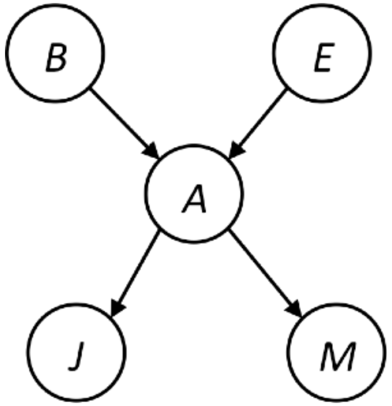
Choose E

$$P(E)$$

$$P(j,m|B,E)$$

$$\xrightarrow{\times} \quad P(j,m,E|B) \quad \xrightarrow{\sum} \quad P(j,m|B)$$

Example



$$P(B) \qquad P(E) \qquad P(j, m|B, E)$$

Choose E

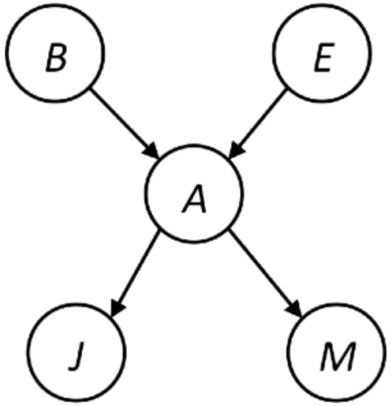
$$P(E)$$
$$P(j, m|B, E)$$

$$\xrightarrow{\times} \quad P(j, m, E|B) \quad \xrightarrow{\sum} \quad P(j, m|B)$$

$$P(B) \qquad P(j, m|B)$$

Example

$$P(B) \qquad P(E) \qquad P(j, m|B, E)$$



Choose E

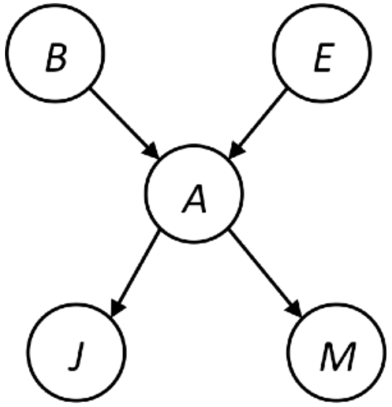
$$P(E), \; P(j, m|B, E) \xrightarrow{\times} P(j, m, E|B) \xrightarrow{\sum} P(j, m|B)$$

$$P(B) \qquad P(j, m|B)$$

Finish with B

$$P(B), \; P(j, m|B) \xrightarrow{\times} P(j, m, B)$$

Example



$$\boxed{P(B) \qquad P(E) \qquad P(j, m|B, E)}$$

Choose E

$$P(E) \quad P(j, m|B, E) \xrightarrow{\times} P(j, m, E|B) \xrightarrow{\sum} P(j, m|B)$$

$$\boxed{P(B) \qquad P(j, m|B)}$$

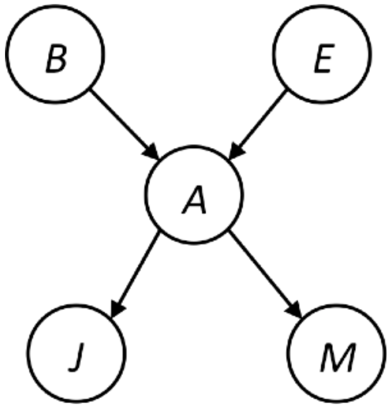
Finish with B

$$P(B) \quad P(j, m|B) \xrightarrow{\times} P(j, m, B) \xrightarrow{\text{Normalize}} P(B|j, m)$$

Same Example in Equations

$$P(B|j,m) \propto P(B,j,m)$$

$$\boxed{P(B) \qquad P(E) \qquad P(A|B,E) \qquad P(j|A) \qquad P(m|A)}$$



$$\begin{aligned}
P(B|j,m) &\propto P(B,j,m) \\
&= \sum_{e,a} P(B,j,m,e,a) \\
&= \sum_{e,a} P(B)P(e)P(a|B,e)P(j|a)P(m|a) \\
&= \sum_{e} P(B)P(e)\sum_{a} P(a|B,e)P(j|a)P(m|a) \\
&= \sum_{e} P(B)P(e)f_1(B,e,j,m) \\
&= P(B)\sum_{e} P(e)f_1(B,e,j,m) \\
&= P(B)f_2(B,j,m)
\end{aligned}$$

marginal can be obtained from joint by summing out

use Bayes' net joint distribution expression

use $x*(y+z) = xy + xz$

joining on a, and then summing out gives f_1

use $x*(y+z) = xy + xz$

joining on e, and then summing out gives f_2

All we are doing is exploiting uwy + uwz + uxy + uxz + vwy + vwz + vxy +vxz = (u+v)(w+x)(y+z) to improve computational efficiency!

Another Variable Elimination Example

Query: $P(X_3|Y_1 = y_1, Y_2 = y_2, Y_3 = y_3)$

Start by inserting evidence, which gives the following initial factors:

$$p(Z)p(X_1|Z)p(X_2|Z)p(X_3|Z)p(y_1|X_1)p(y_2|X_2)p(y_3|X_3)$$

Eliminate X_1, this introduces the factor $f_1(Z, y_1) = \sum_{x_1} p(x_1|Z)p(y_1|x_1)$, and we are left with:

$$p(Z)f_1(Z, y_1)p(X_2|Z)p(X_3|Z)p(y_2|X_2)p(y_3|X_3)$$

Eliminate X_2, this introduces the factor $f_2(Z, y_2) = \sum_{x_2} p(x_2|Z)p(y_2|x_2)$, and we are left with:

$$p(Z)f_1(Z, y_1)f_2(Z, y_2)p(X_3|Z)p(y_3|X_3)$$

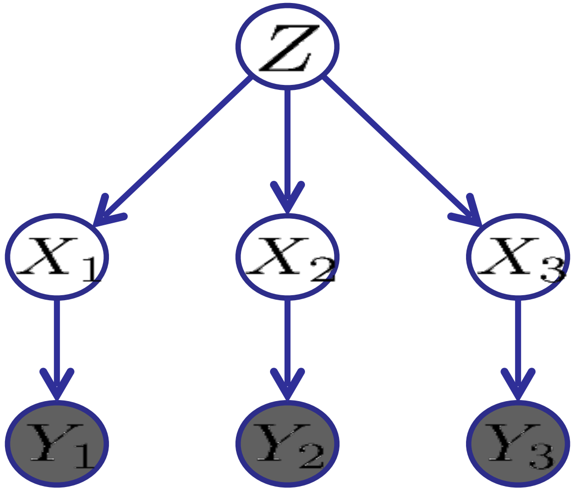
Eliminate Z, this introduces the factor $f_3(y_1, y_2, X_3) = \sum_z p(z)f_1(z, y_1)f_2(z, y_2)p(X_3|z)$, and we are left:

$$p(y_3|X_3), f_3(y_1, y_2, X_3)$$

No hidden variables left. Join the remaining factors to get:

$$f_4(y_1, y_2, y_3, X_3) = P(y_3|X_3)f_3(y_1, y_2, X_3).$$

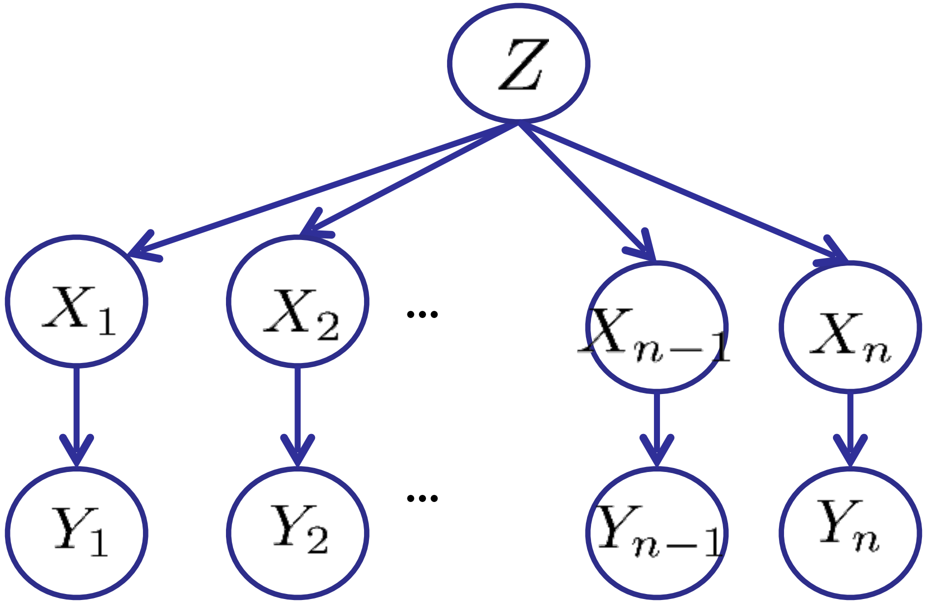
Normalizing over X_3 gives $P(X_3|y_1, y_2, y_3)$.



Computational complexity critically depends on the largest factor being generated in this process. Size of factor = number of entries in table. In example above (assuming binary) all factors generated are of size 2 --- as they all only have one variable (Z, Z, and X_3 respectively).

Variable Elimination Ordering

- For the query $P(X_n | y_1,\ldots,y_n)$ work through the following two different orderings as done in previous slide: $Z, X_1, \ldots, X_{n-1}$ and $X_1, \ldots, X_{n-1}, Z$. What is the size of the maximum factor generated for each of the orderings?



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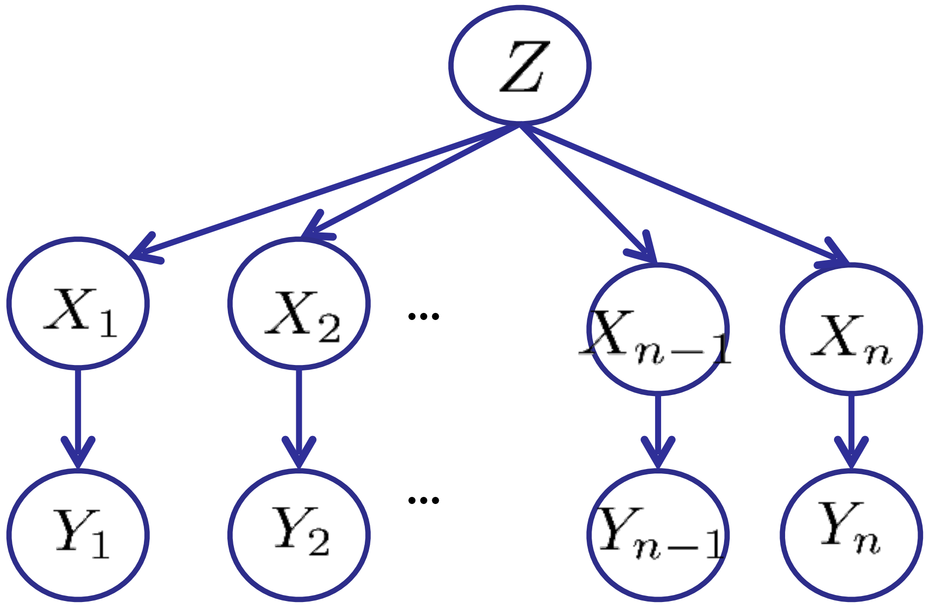

- Answer: 2^{n+1} versus 2^2 (assuming binary)

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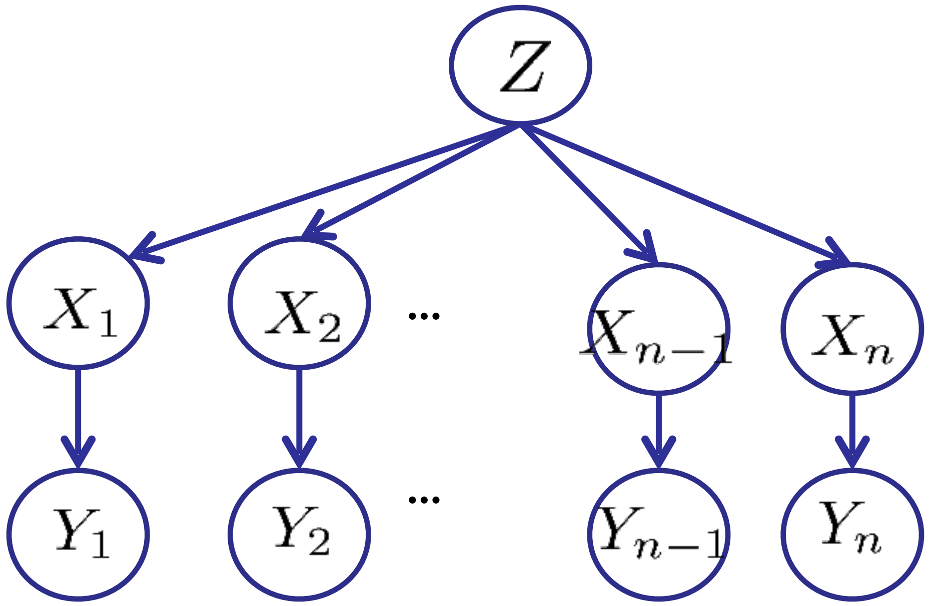

- Answer: 2^{n+1} versus 2^2 (assuming binary)

- In general: the ordering can greatly affect efficiency.

VE: Computational and Space Complexity

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- Does there always exist an ordering that only results in small factors?
  - No!

Worst Case Complexity?

- CSP:

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_2 \vee x_4) \wedge (\neg x_3 \vee \neg x_4 \vee \neg x_5) \wedge (x_2 \vee x_5 \vee x_7) \wedge (x_4 \vee x_5 \vee x_6) \wedge (\neg x_5 \vee x_6 \vee \neg x_7) \wedge (\neg x_5 \vee \neg x_6 \vee x_7)$$

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$$\dots$$

$$Y_8 = \neg X_5 \vee X_6 \vee X_7$$

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$$\dots$$

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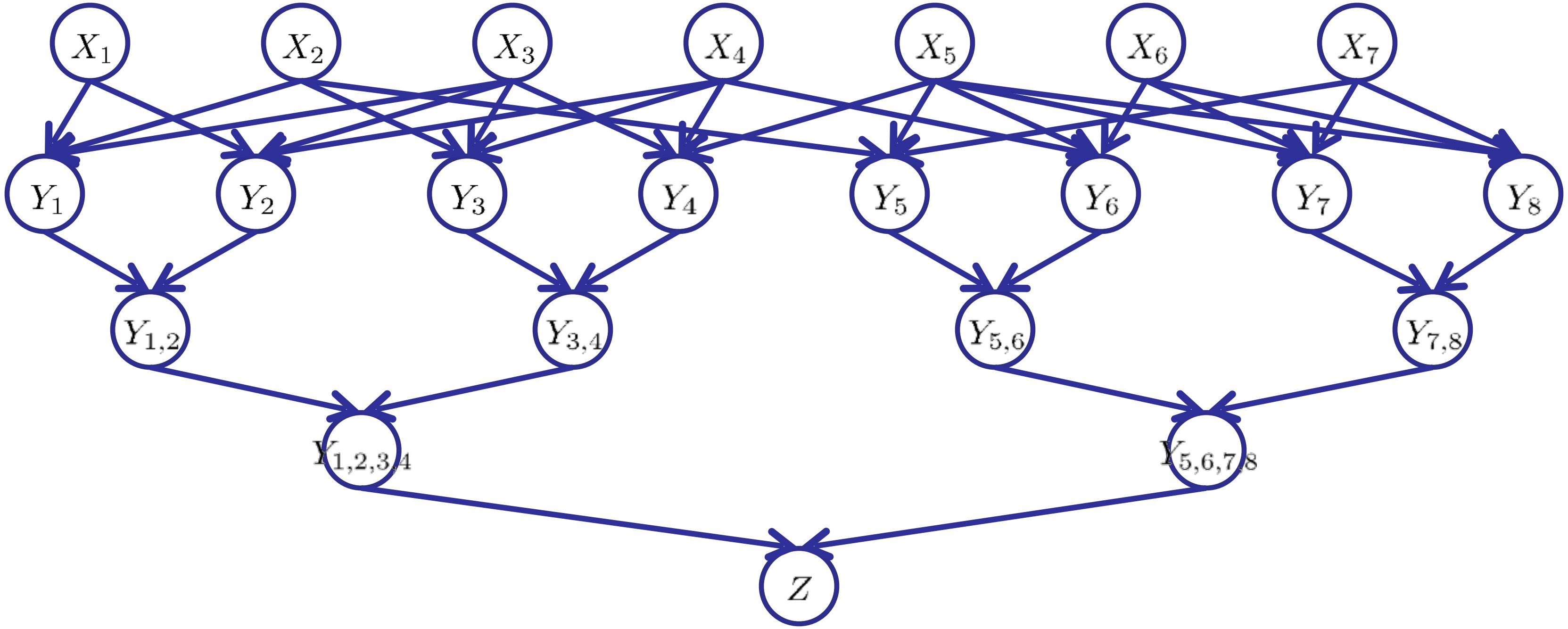

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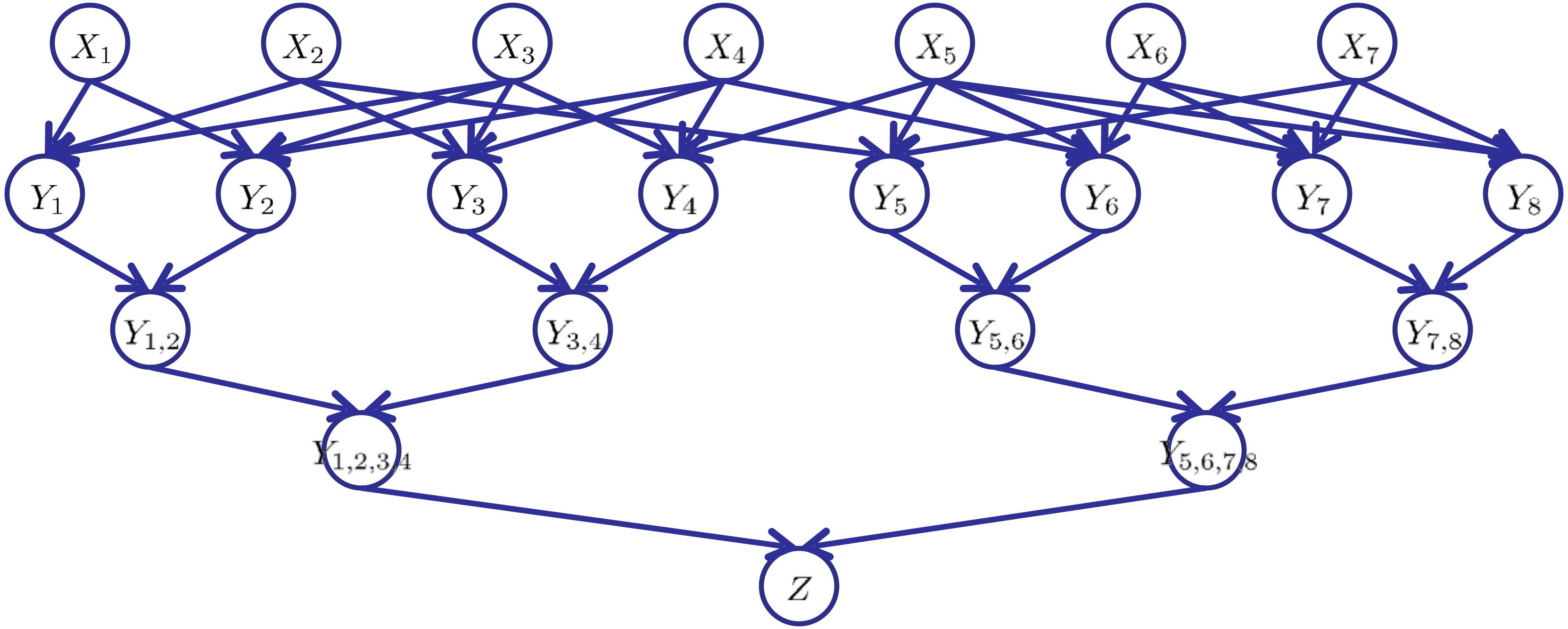

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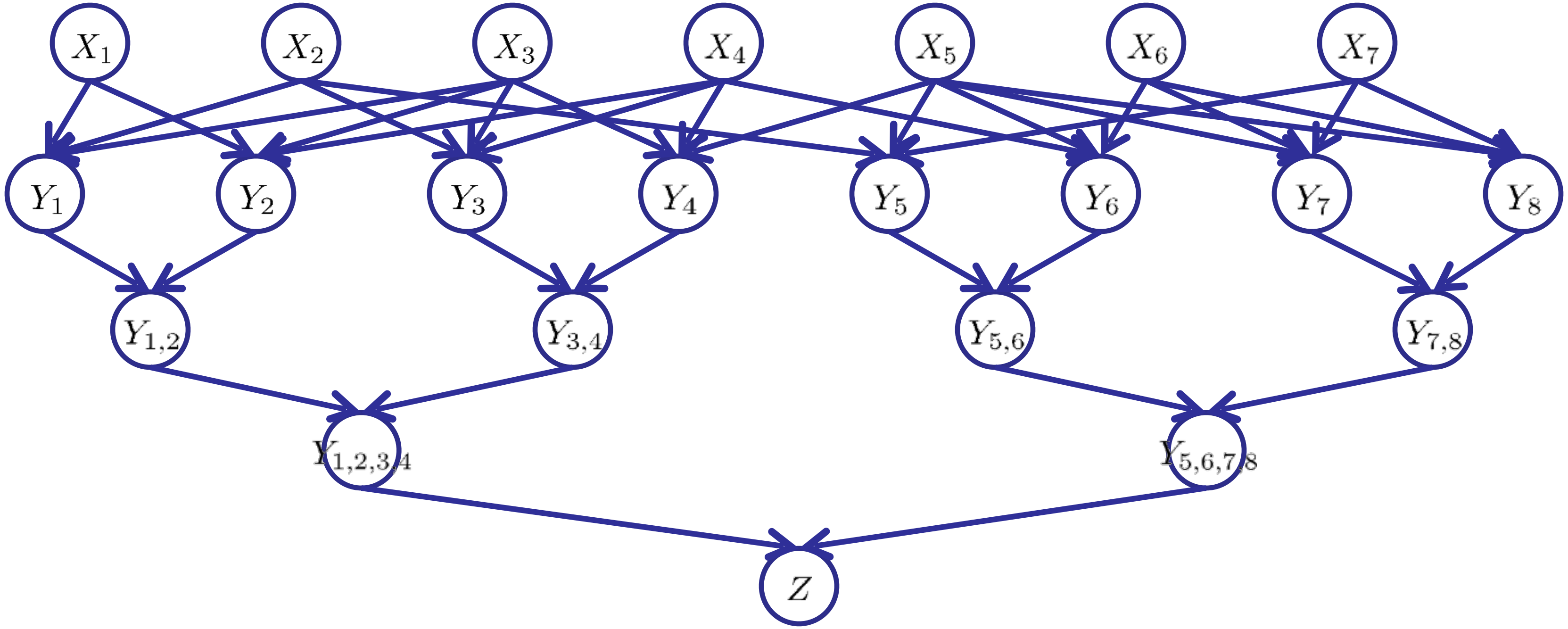

- If we can answer P(z) equal to zero or not, we answered whether the 3-SAT problem has a solution.
- Hence inference in Bayes' nets is NP-hard. No known efficient probabilistic inference in general.

Polytrees

- A polytree is a directed graph with no undirected cycles

- For poly-trees you can always find an ordering that is efficient
  - Try it!!

- Cut-set conditioning for Bayes' net inference
  - Choose set of variables such that if removed only a polytree remains
  - Exercise: Think about how the specifics would work out!

Bayes' Nets

- ✓ Representation
- ✓ Conditional Independences
- Probabilistic Inference
  - ✓ Enumeration (exact, exponential complexity)
  - ✓ Variable elimination (exact, worst-case exponential complexity, often better)
  - Inference is NP-complete
  - ✓ Sampling (approximate)
- Learning Bayes' Nets from Data