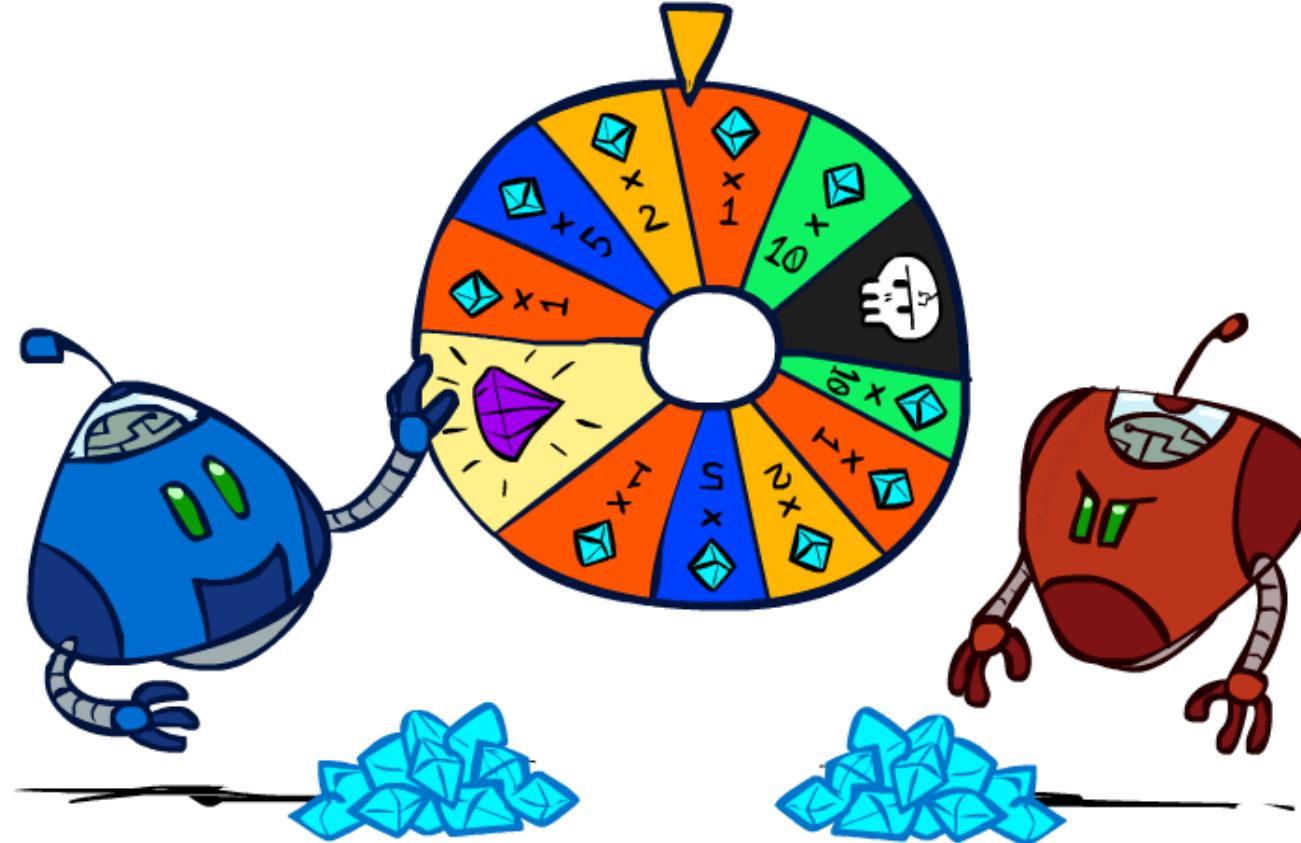


# CS 5522: Artificial Intelligence II

## Uncertainty and Utilities

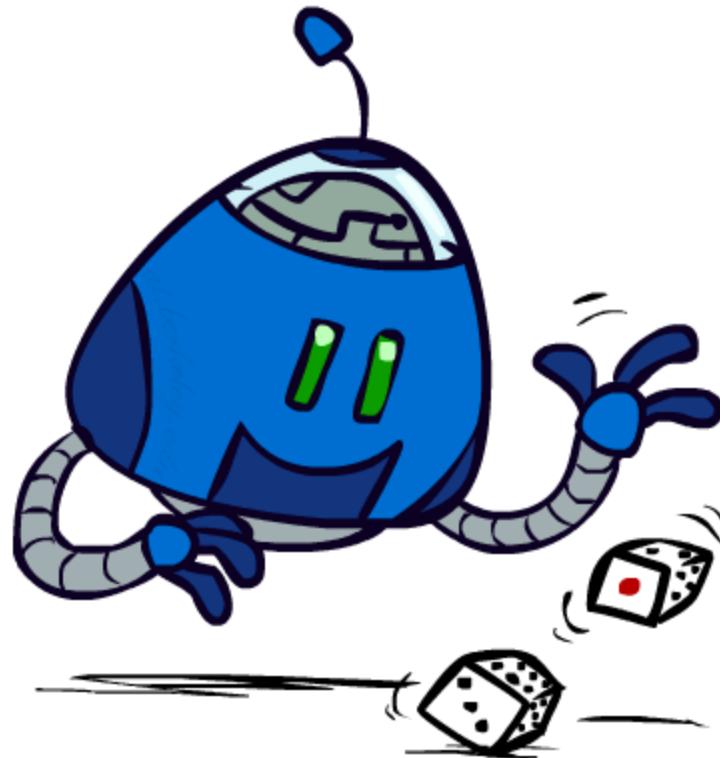


Instructor: Alan Ritter

Ohio State University

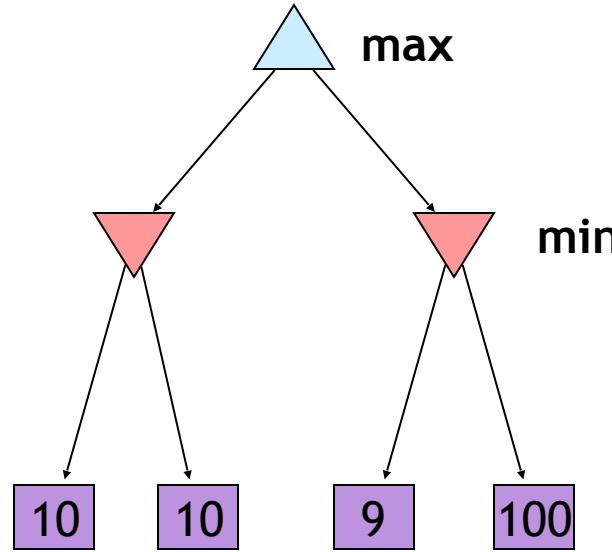
# Uncertain Outcomes

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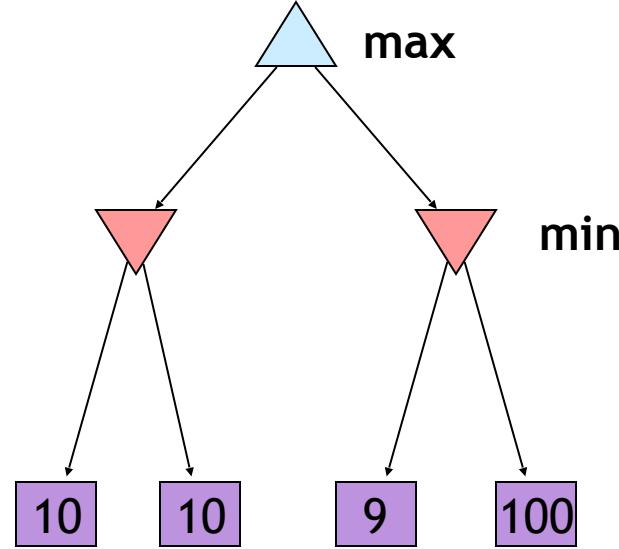
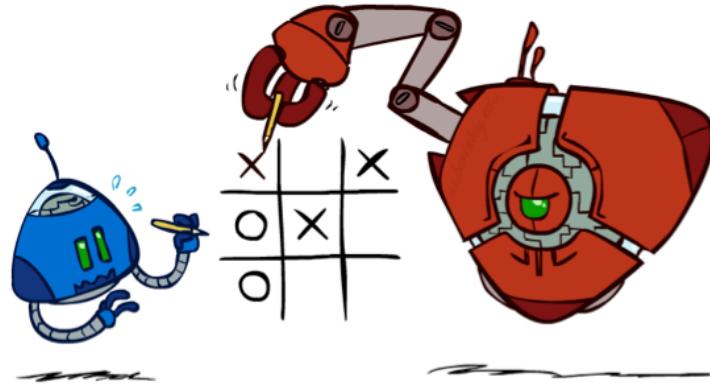
# Worst-Case vs. Average Case

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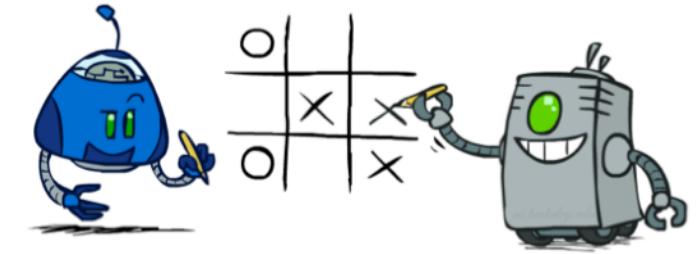
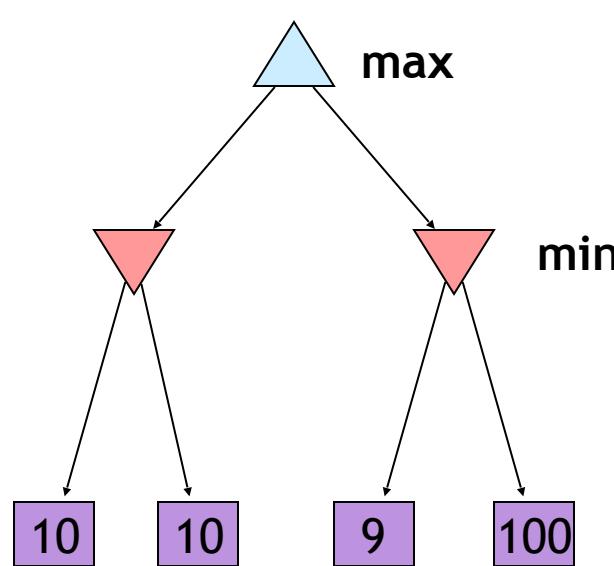
Idea: Uncertain outcomes controlled by chance, not an adversary!

# Worst-Case vs. Average Case



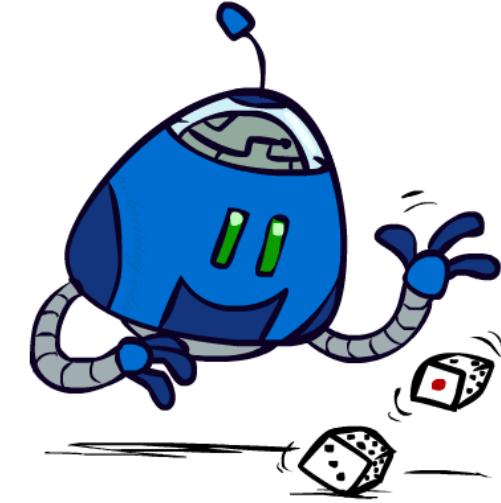
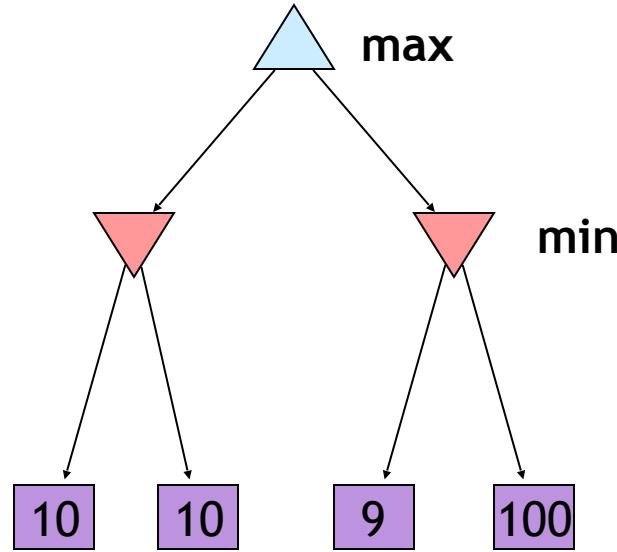
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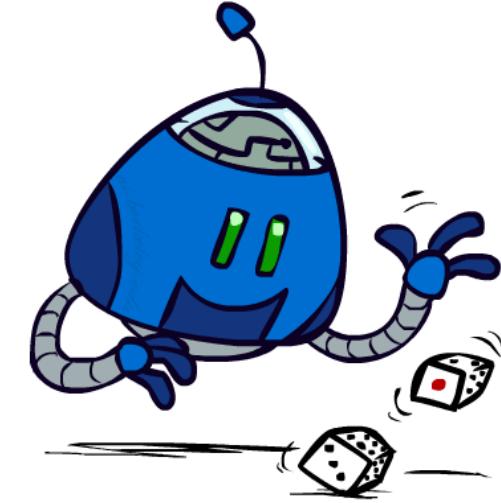
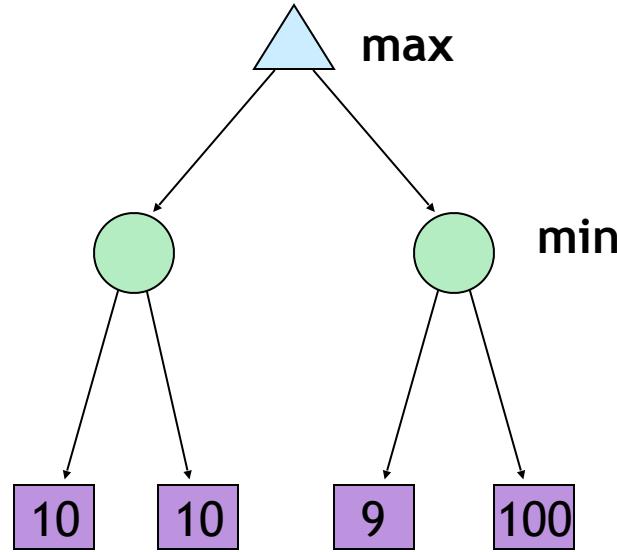
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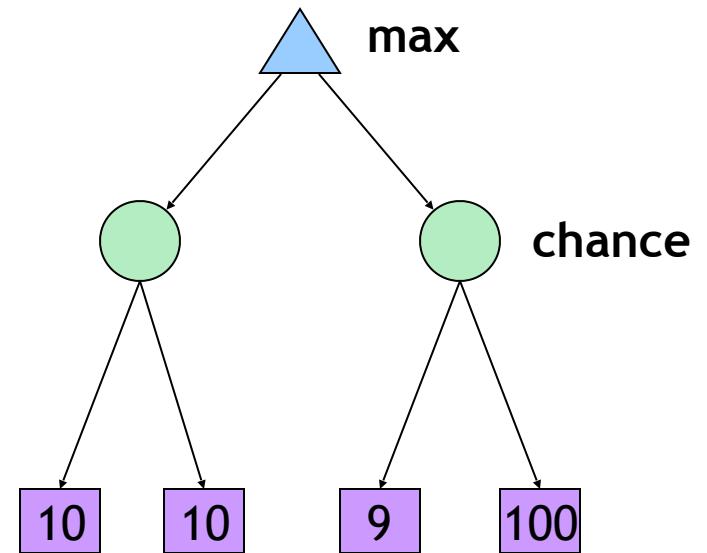
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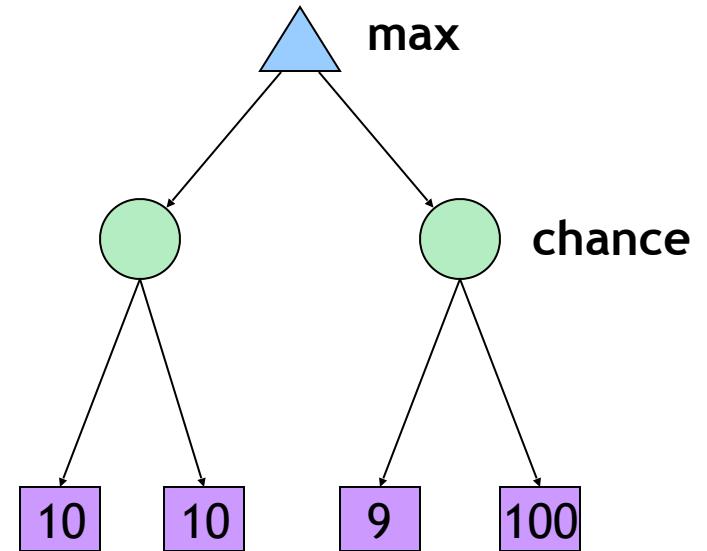
# Expectimax Search

- Why wouldn't we know what the result of an action will be?
  - Explicit randomness: rolling dice
  - Unpredictable opponents: the ghosts respond randomly
  - Actions can fail: when moving a robot, wheels might slip
- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes

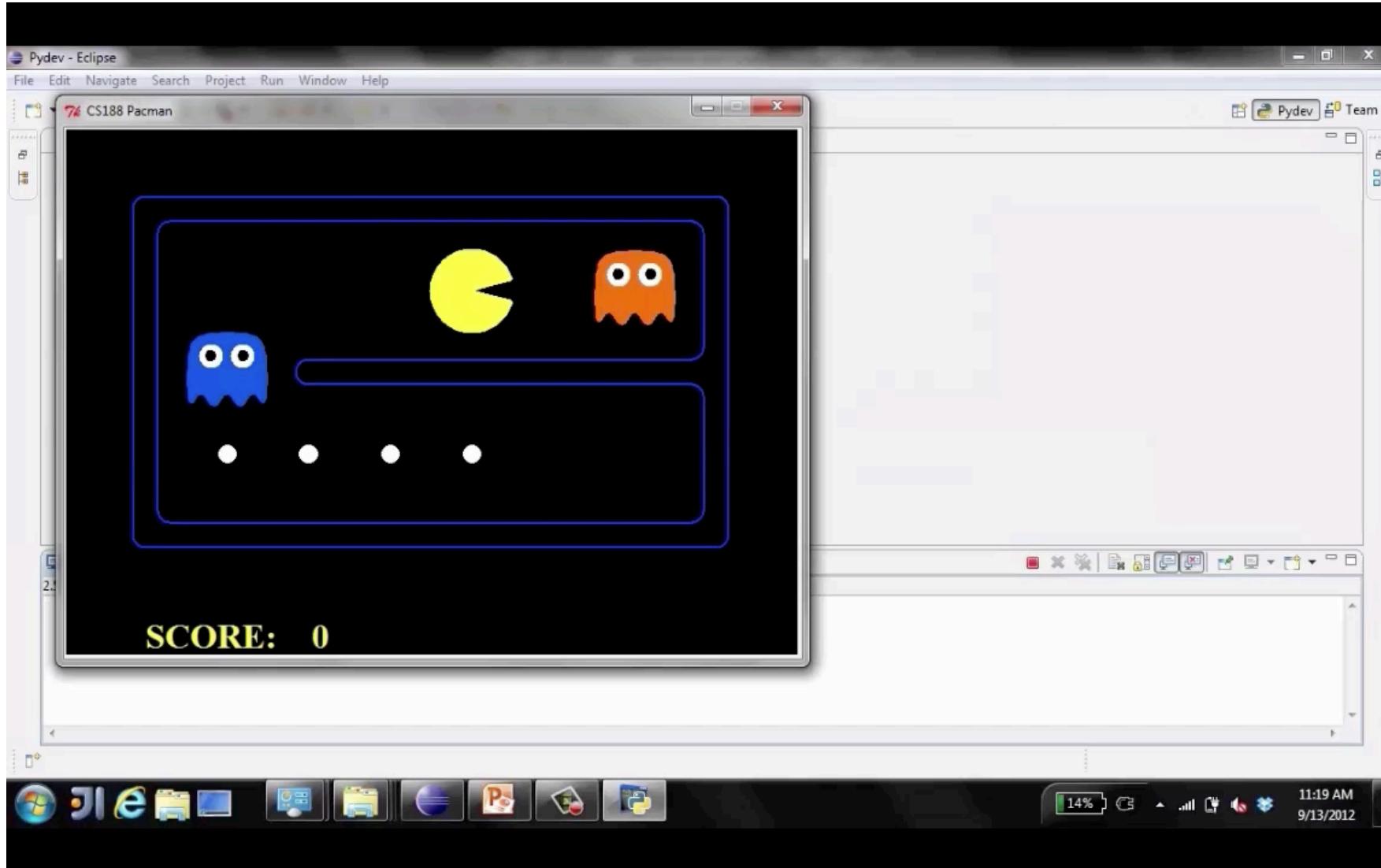


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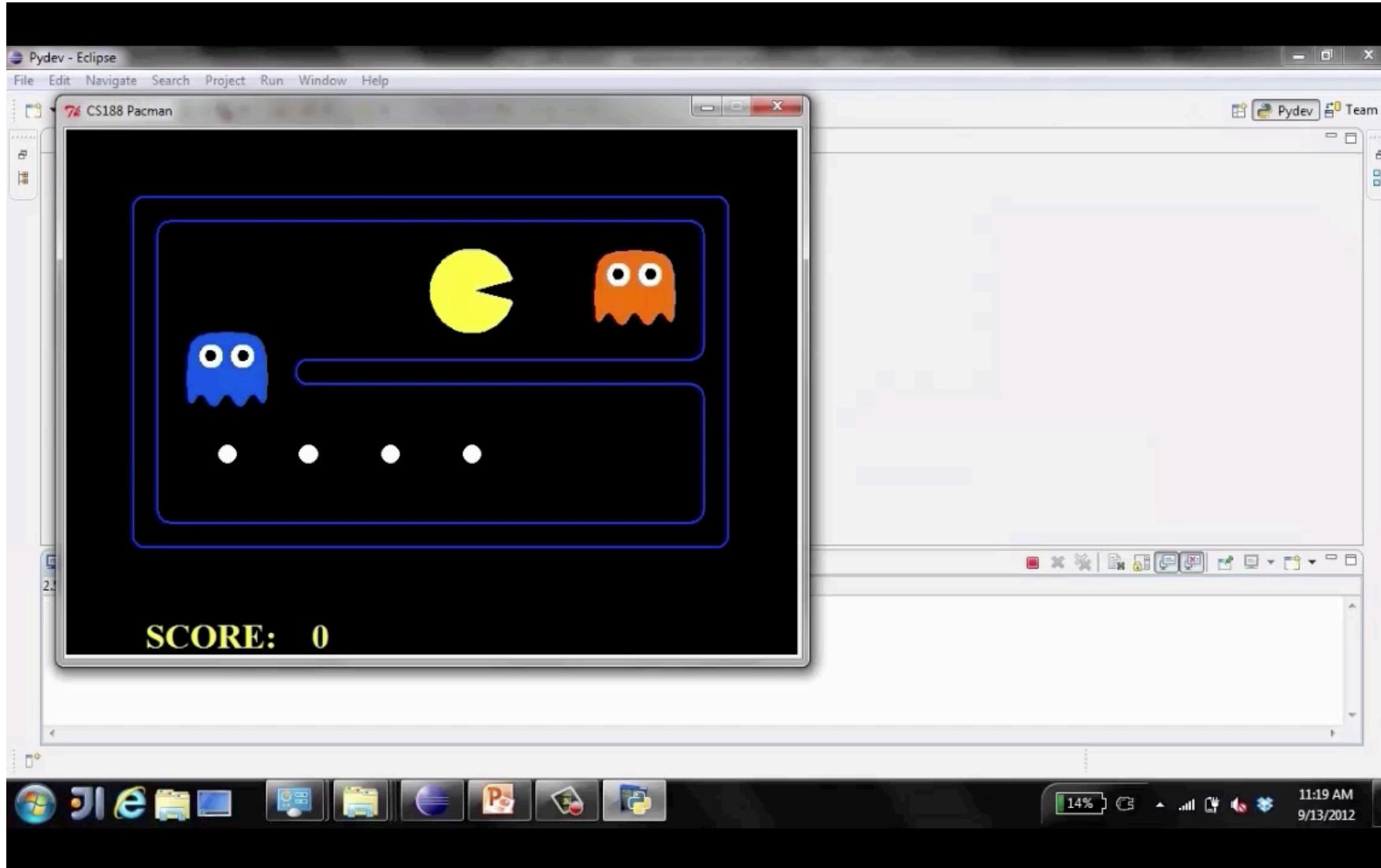
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  - Actions can fail: when moving a robot, wheels might slip
- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes
- **Expectimax search:** compute the average score under optimal play
  - Max nodes as in minimax search
  - Chance nodes are like min nodes but the outcome is uncertain
  - Calculate their **expected utilities**
  - I.e. take weighted average (expectation) of children
- Later, we'll learn how to formalize the underlying uncertain-result problems as **Markov Decision Processes**



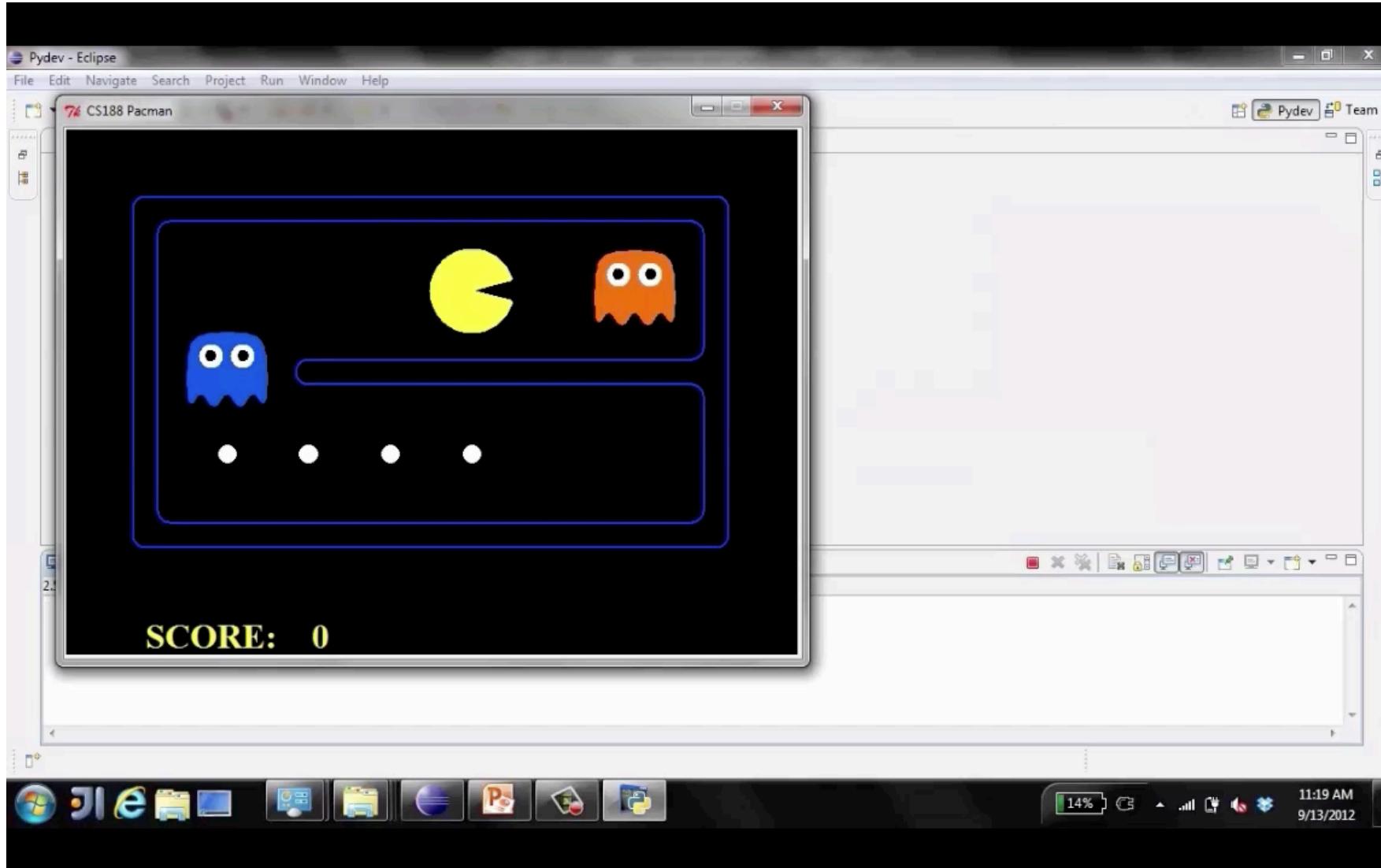
# Video of Demo Minimax vs Expectimax (Min)



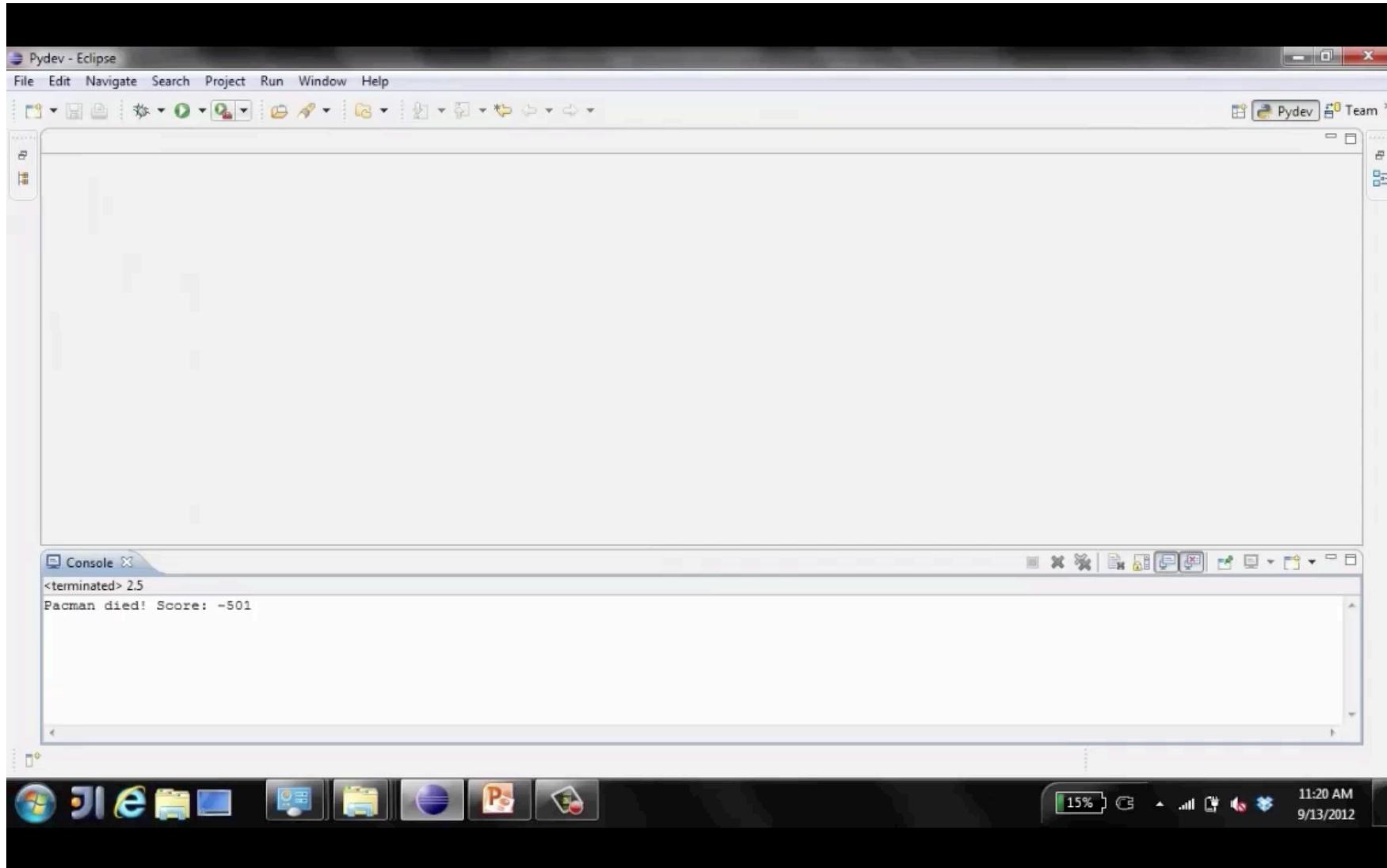
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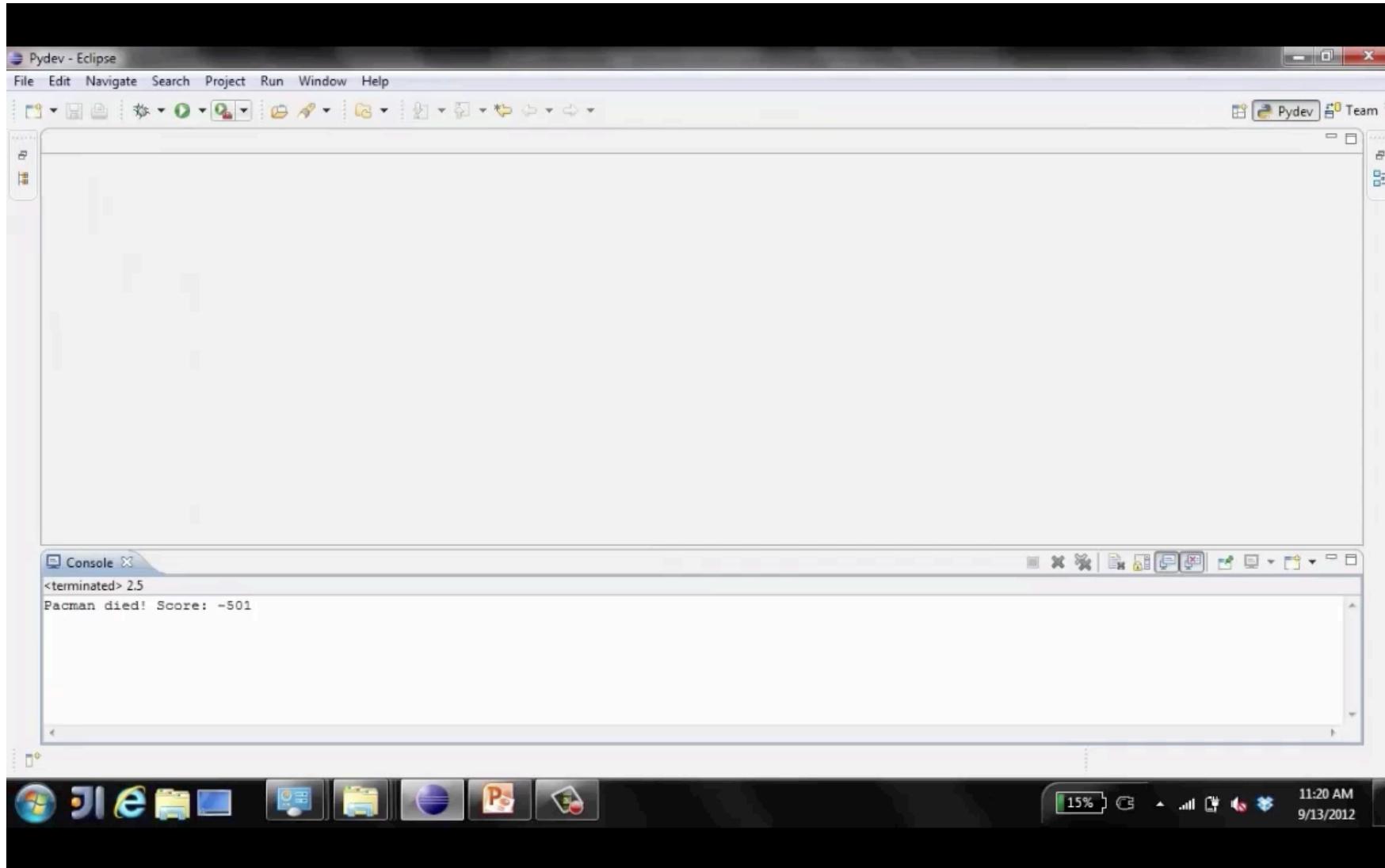
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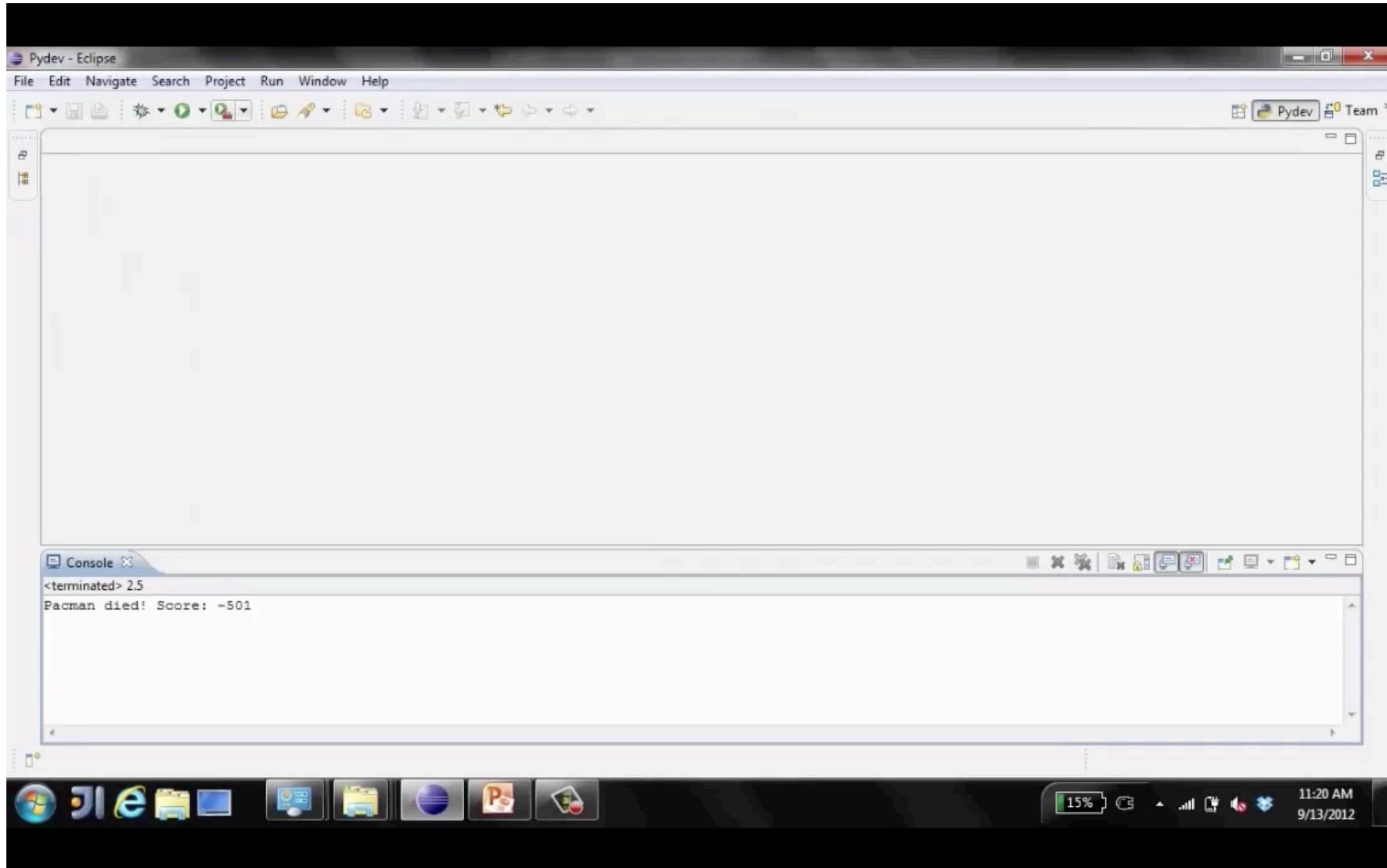
# Video of Demo Minimax vs Expectimax (Exp)



# Video of Demo Minimax vs Expectimax (Exp)



# Video of Demo Minimax vs Expectimax (Exp)



# Expectimax Pseudocode

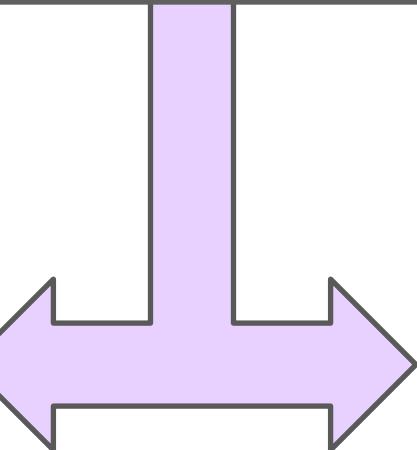
```
def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)
```

# Expectimax Pseudocode

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def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)
```

```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```



# Expectimax Pseudocode

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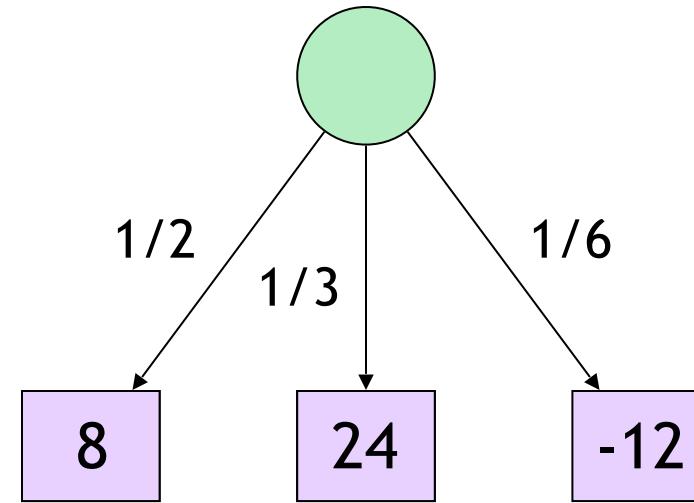
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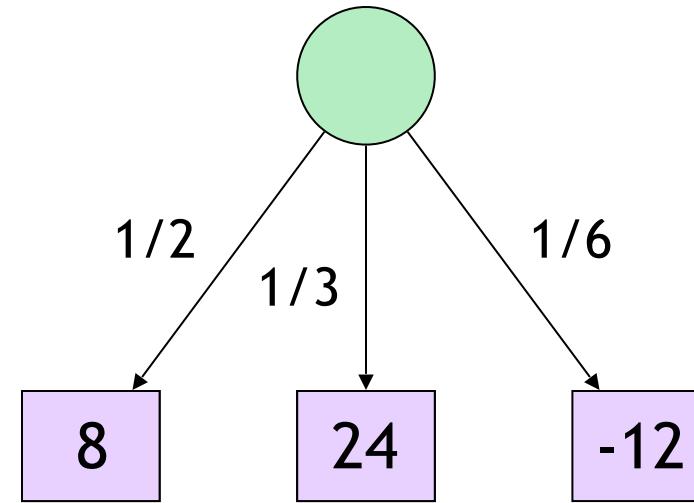
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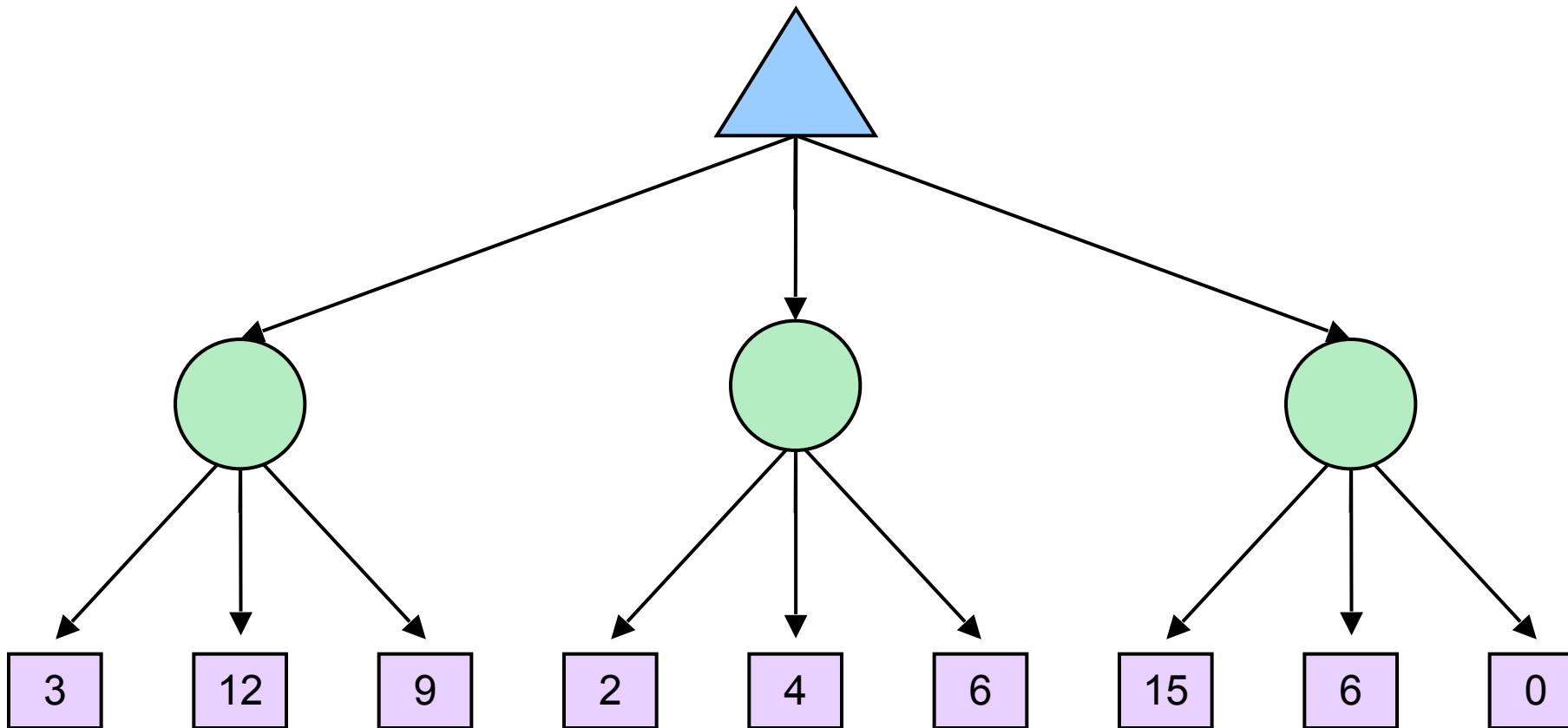
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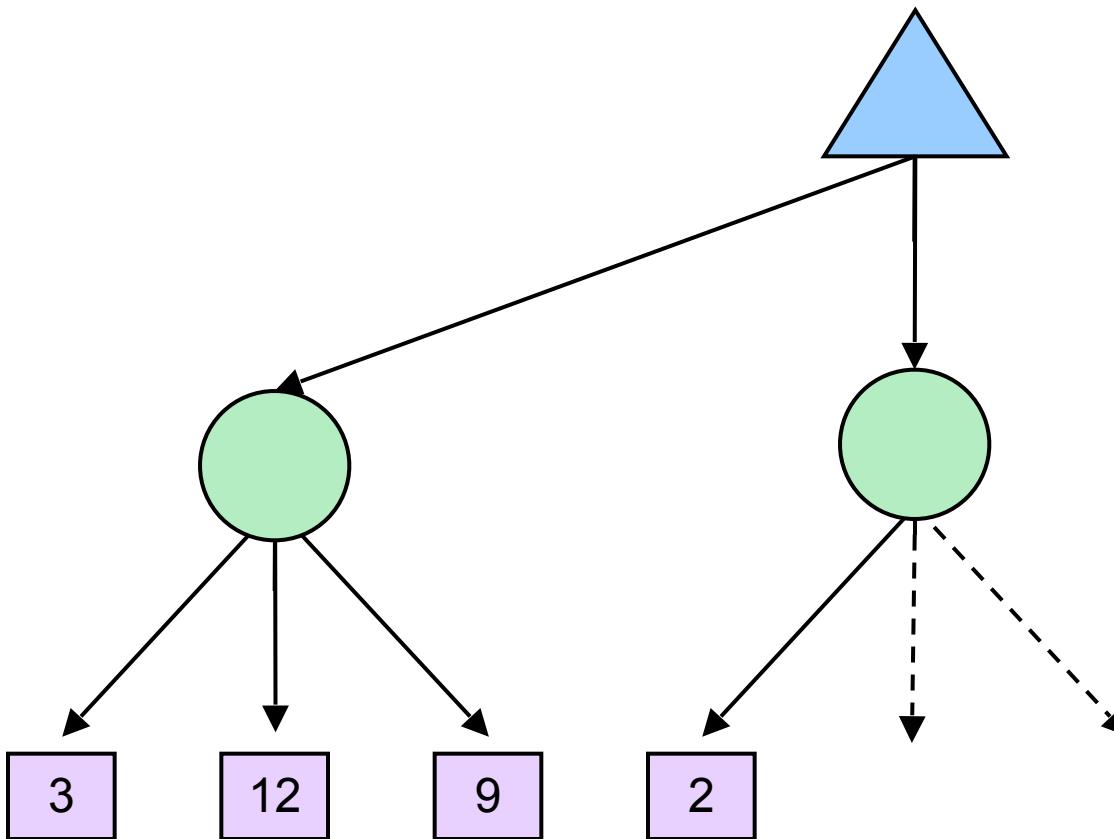


$$v = (1/2)(8) + (1/3)(24) + (1/6)(-12) = 10$$

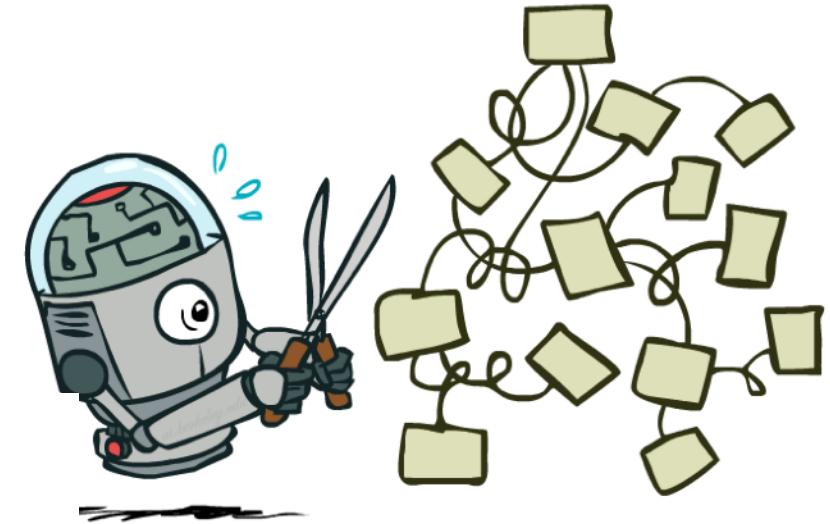
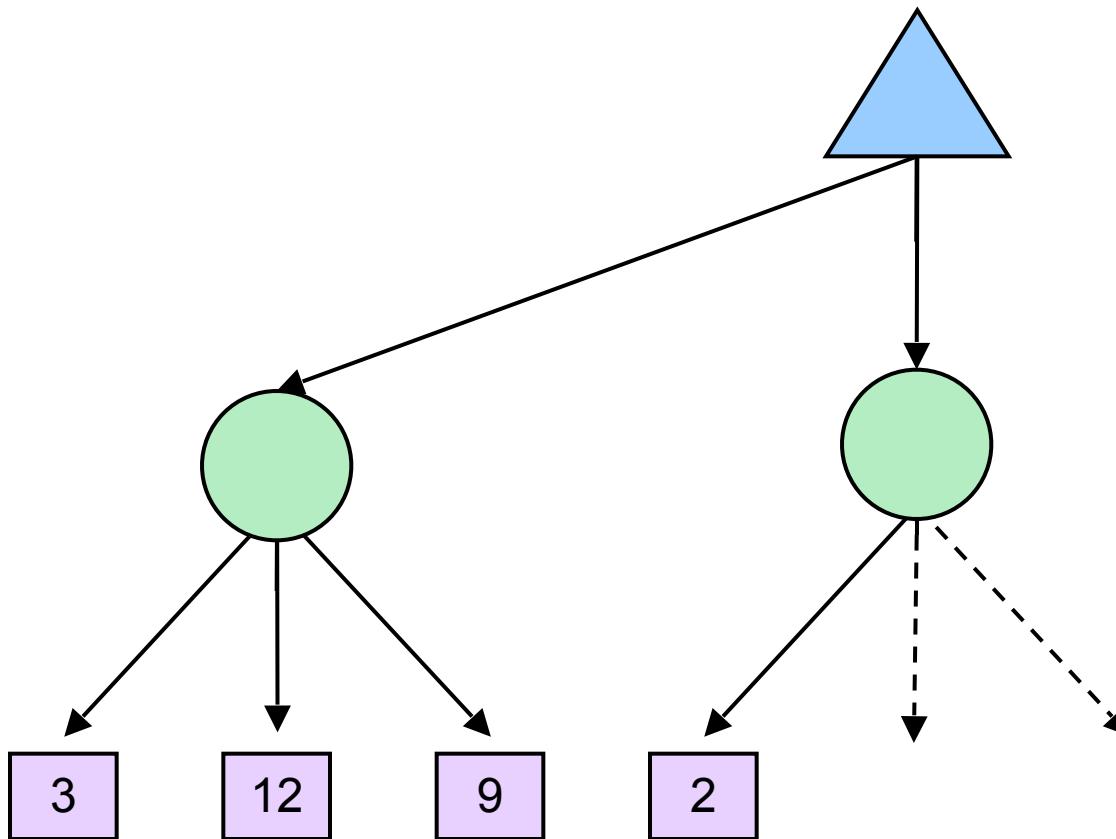
# Expectimax Example



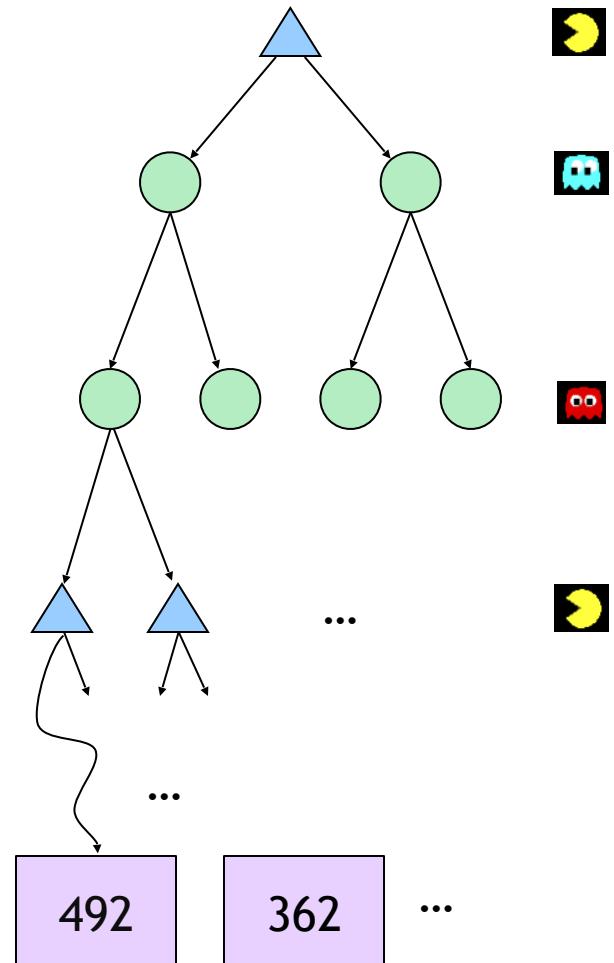
# Expectimax Pruning?



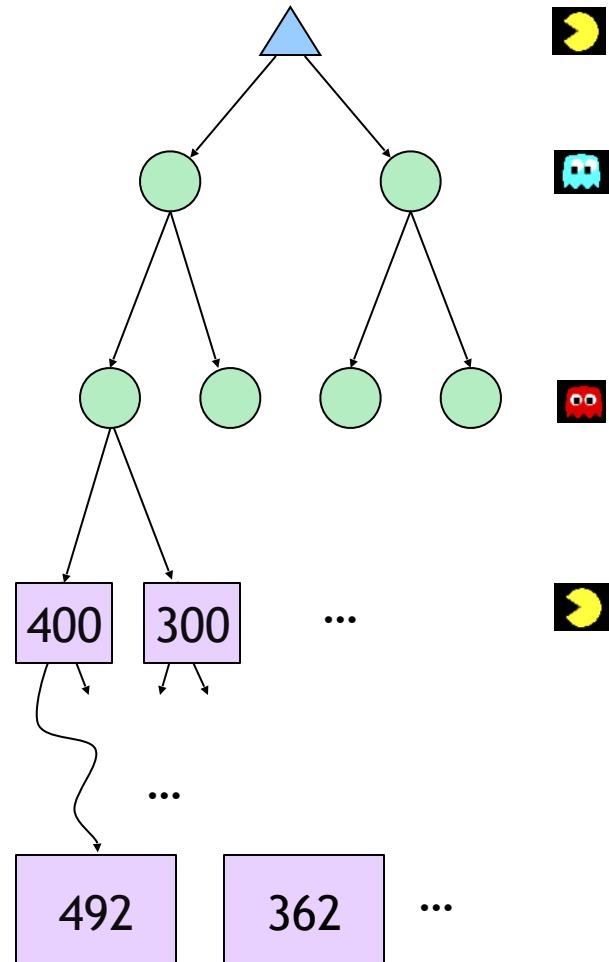
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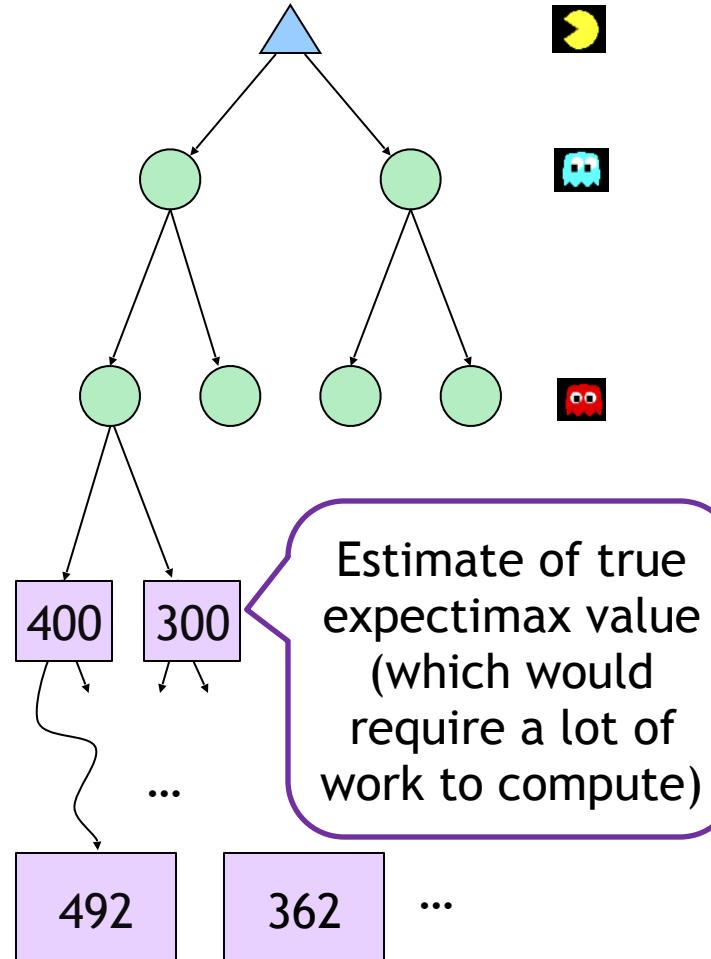
# Depth-Limited Expectimax



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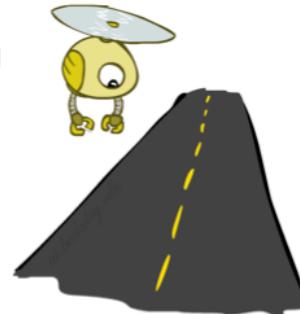
# Probabilities

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# Reminder: Probabilities

- A **random variable** represents an event whose outcome is unknown
- A **probability distribution** is an assignment of weights to outcomes
- Example: Traffic on freeway
  - Random variable:  $T$  = whether there's traffic
  - Outcomes:  $T \in \{\text{none}, \text{light}, \text{heavy}\}$
  - Distribution:  $P(T=\text{none}) = 0.25$ ,  $P(T=\text{light}) = 0.50$ ,  $P(T=\text{heavy}) = 0.25$
- Some laws of probability (more later):
  - Probabilities are always non-negative
  - Probabilities over all possible outcomes sum to one
- As we get more evidence, probabilities may change:
  - $P(T=\text{heavy}) = 0.25$ ,  $P(T=\text{heavy} \mid \text{Hour}=8\text{am}) = 0.60$
  - We'll talk about methods for reasoning and updating probabilities later



0.25



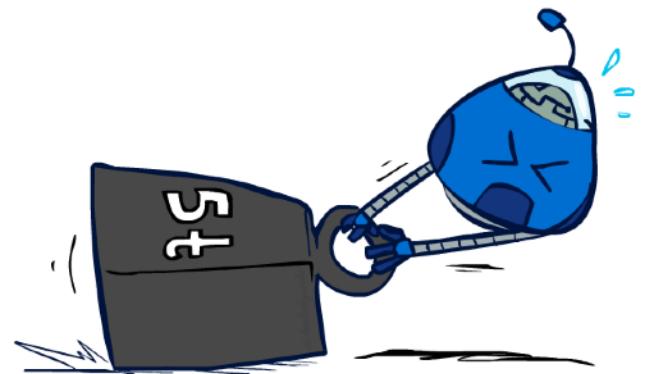
0.50



0.25

# Reminder: Expectations

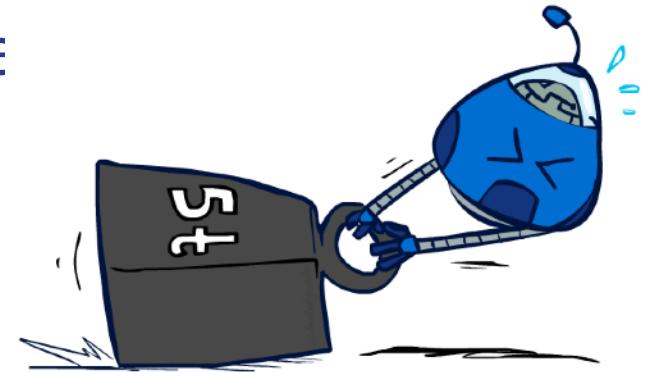
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# Reminder: Expectations

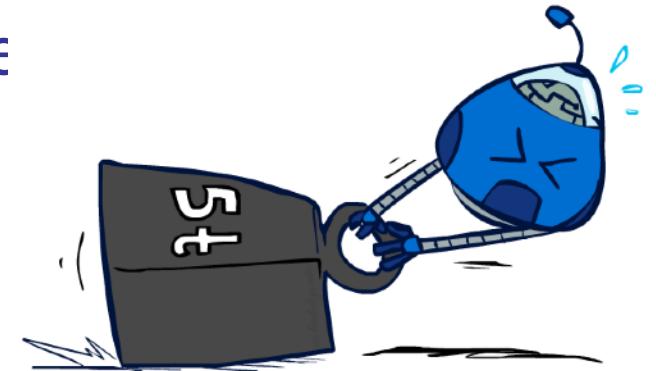
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- The expected value of a function of a random variable is the average, weighted by the probability distribution over outcomes
- Example: How long to get to the airport?

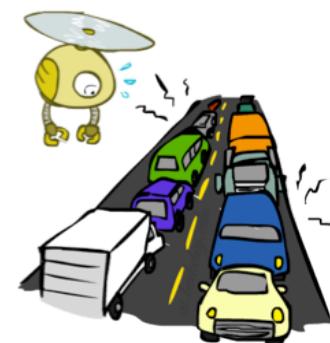
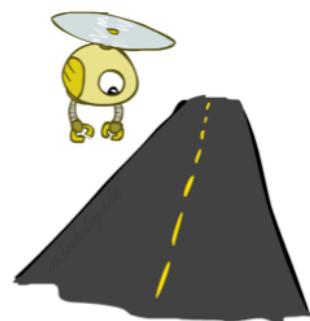


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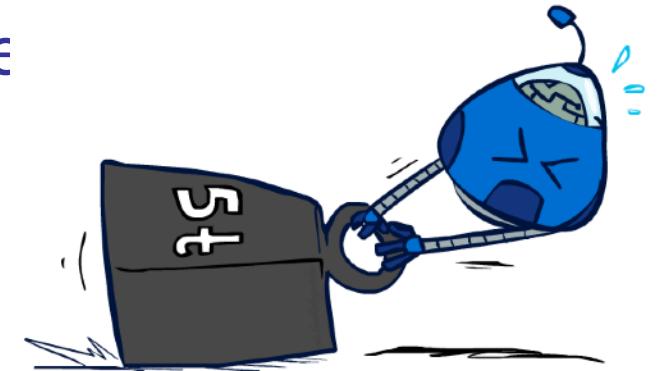


Probability:      0.25                          0.50                          0.25



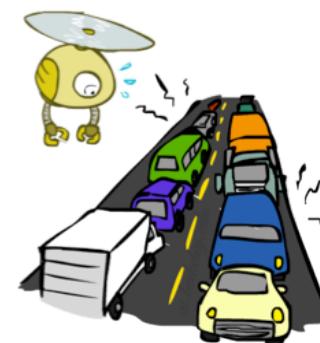
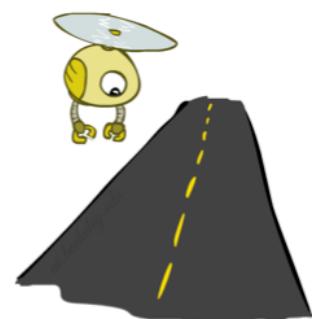
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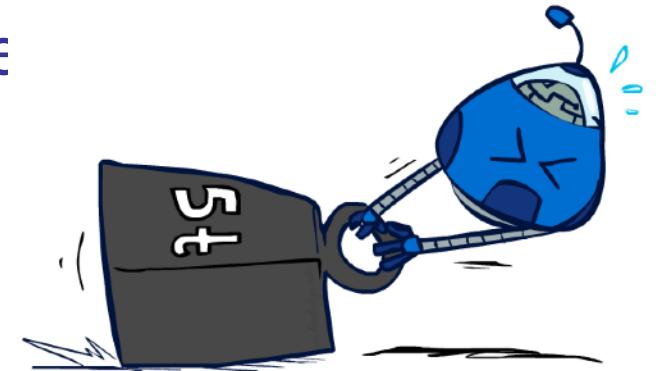
Time:	20 min	30 min	60 min
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Probability:	0.25	0.50	0.25
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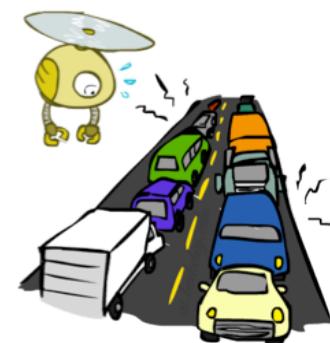
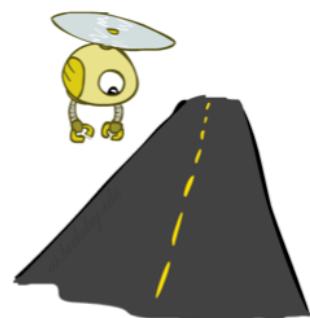


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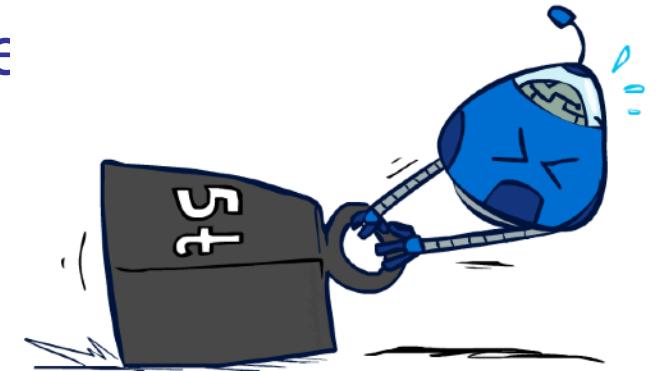


Time:	20 min	30 min	60 min
	x	x	x
Probability:	0.25	0.50	0.25

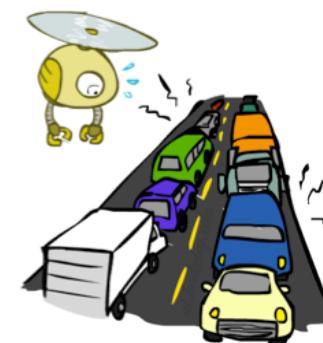
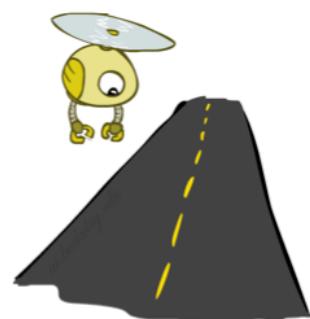


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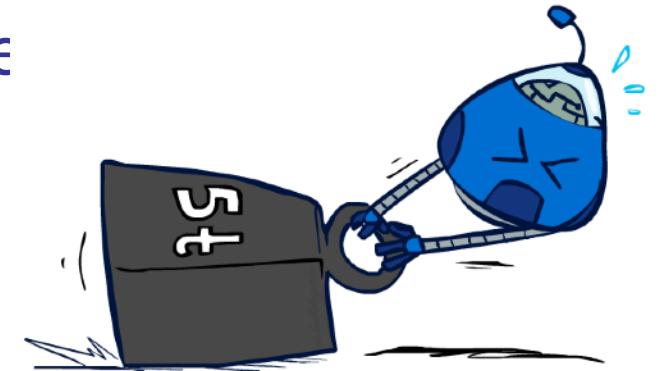


Time:	20 min	+	30 min	+	60 min
Probability:	x 0.25		x 0.50		x 0.25

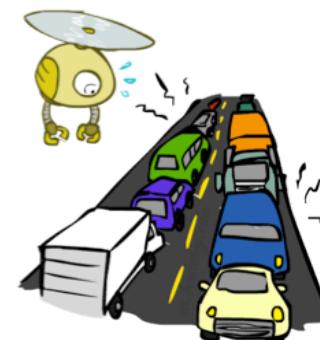
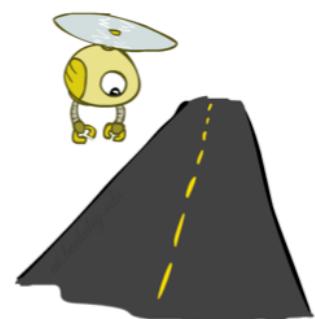


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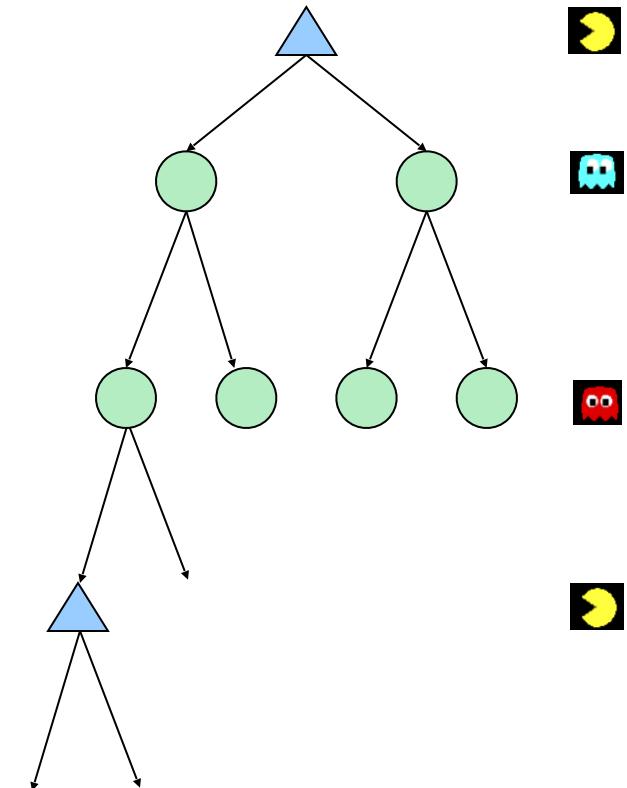


Time:	20 min	x	+	30 min	x	+	60 min	x	35 min
Probability:	0.25			0.50			0.25		



# What Probabilities to Use?

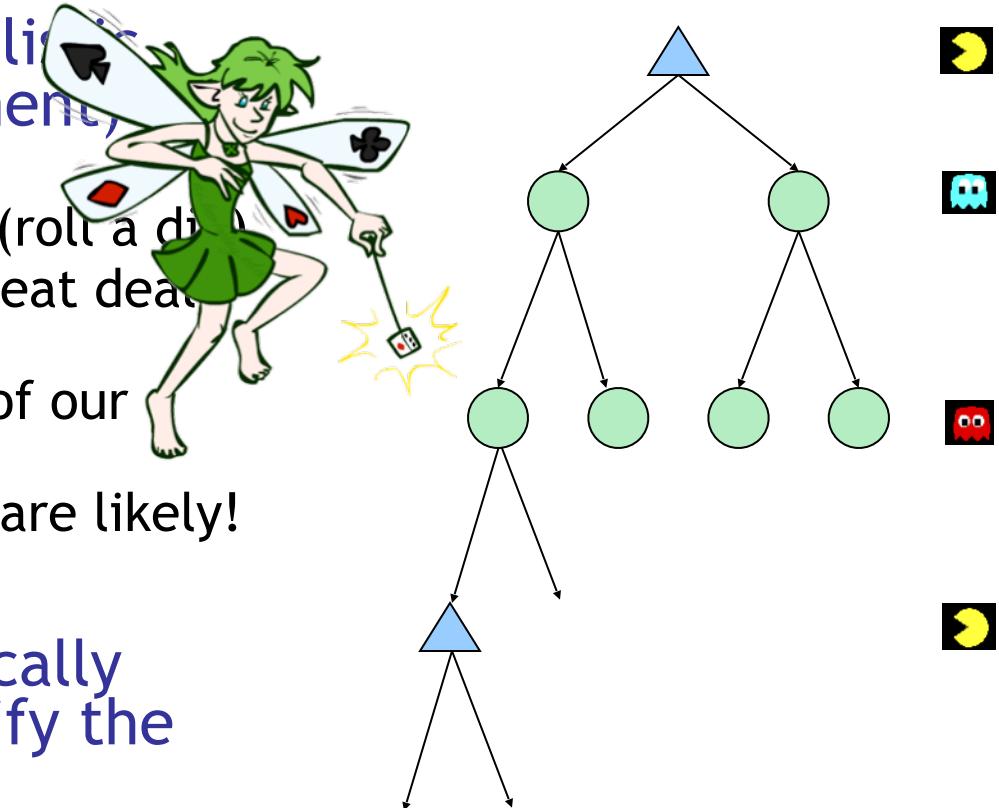
- In expectimax search, we have a probabilistic model of how the opponent (or environment) will behave in any state
  - Model could be a simple uniform distribution (roll a die)
  - Model could be sophisticated and require a great deal of computation
  - We have a chance node for any outcome out of our control: opponent or environment
  - The model might say that adversarial actions are likely!
- For now, assume each chance node magically comes along with probabilities that specify the distribution over its outcomes



*Having a probabilistic belief about another agent's action does not mean that the agent is flipping any coins!*

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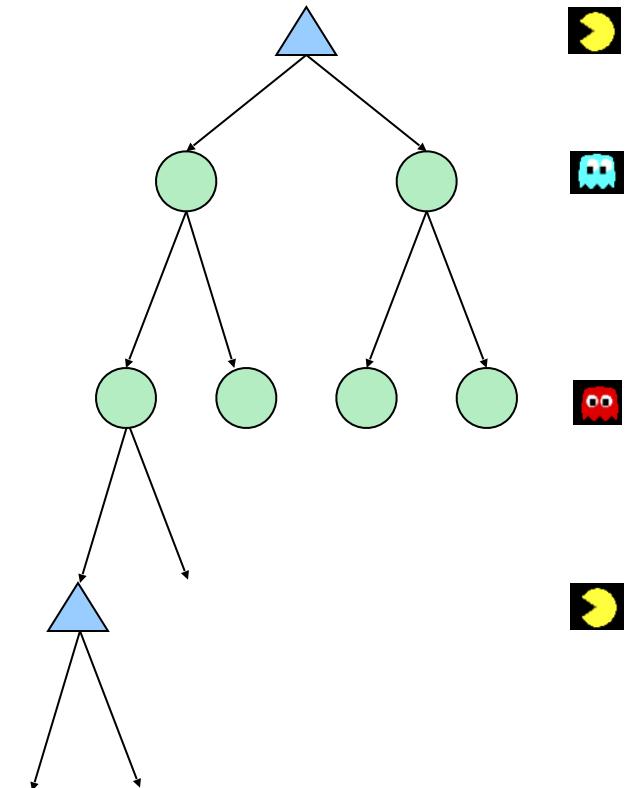
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# Quiz: Informed Probabilities

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- Let's say you know that your opponent is actually running a depth 2 minimax, using the result 80% of the time, and moving randomly otherwise
- Question: What tree search should you use?

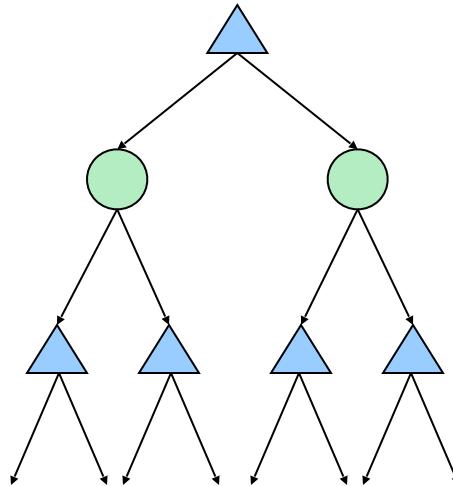
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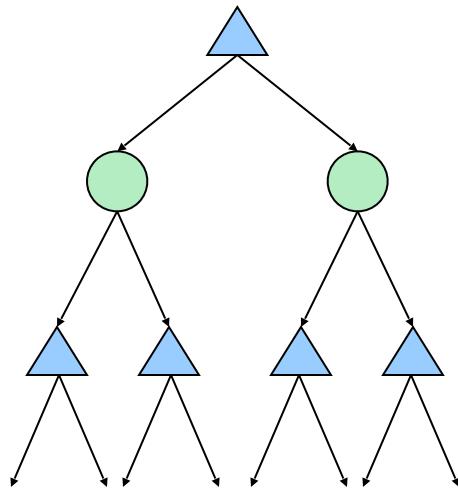
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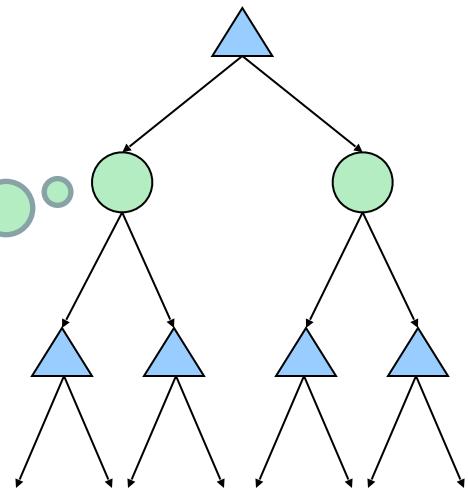
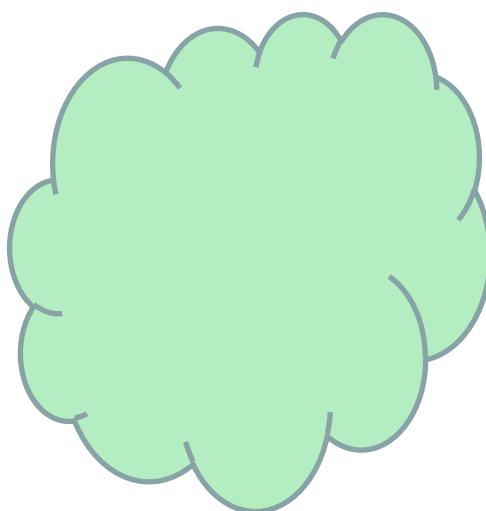
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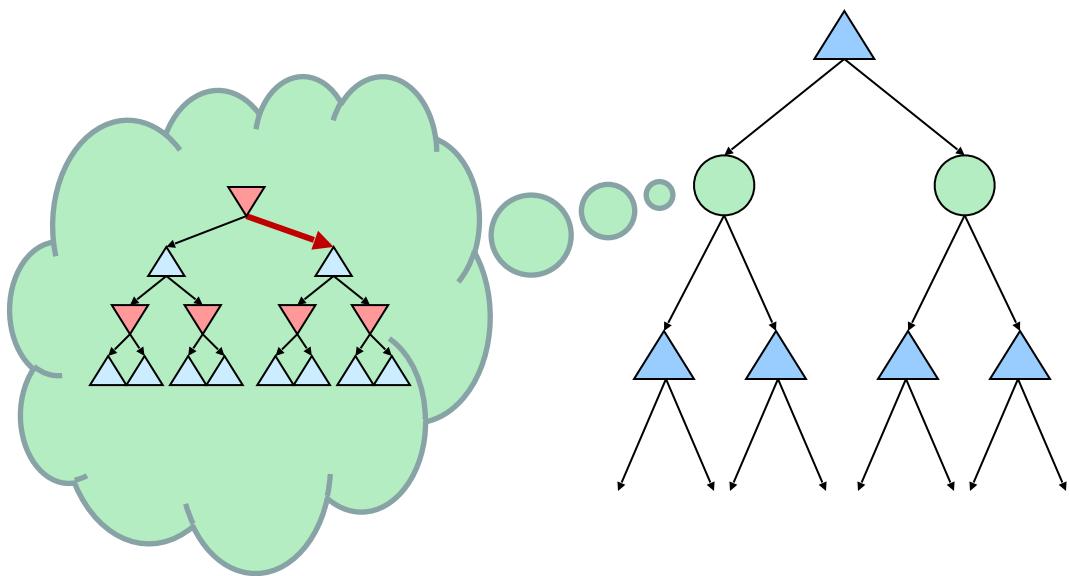


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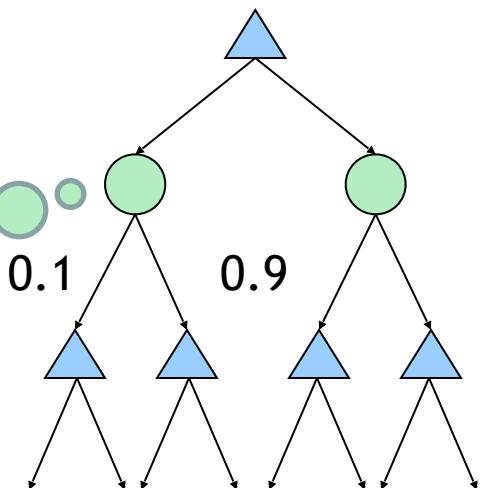
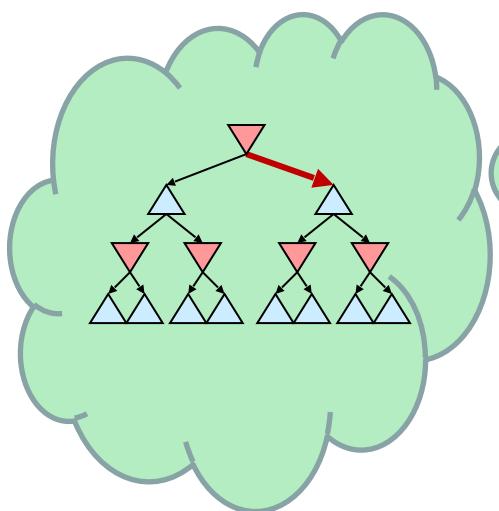


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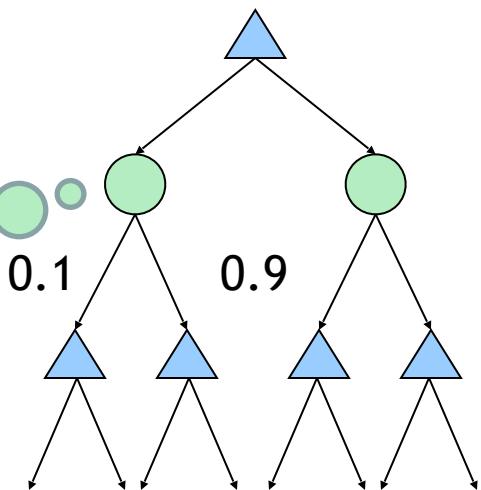
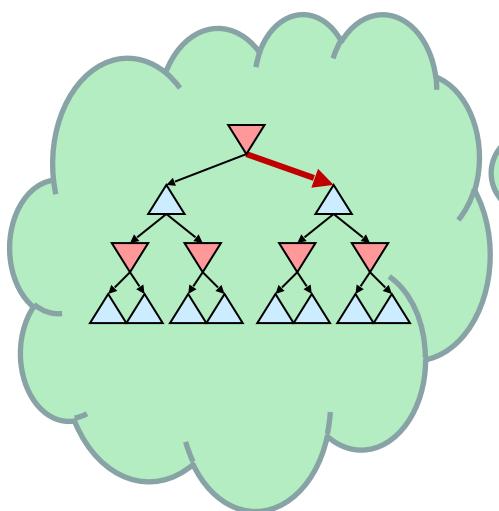


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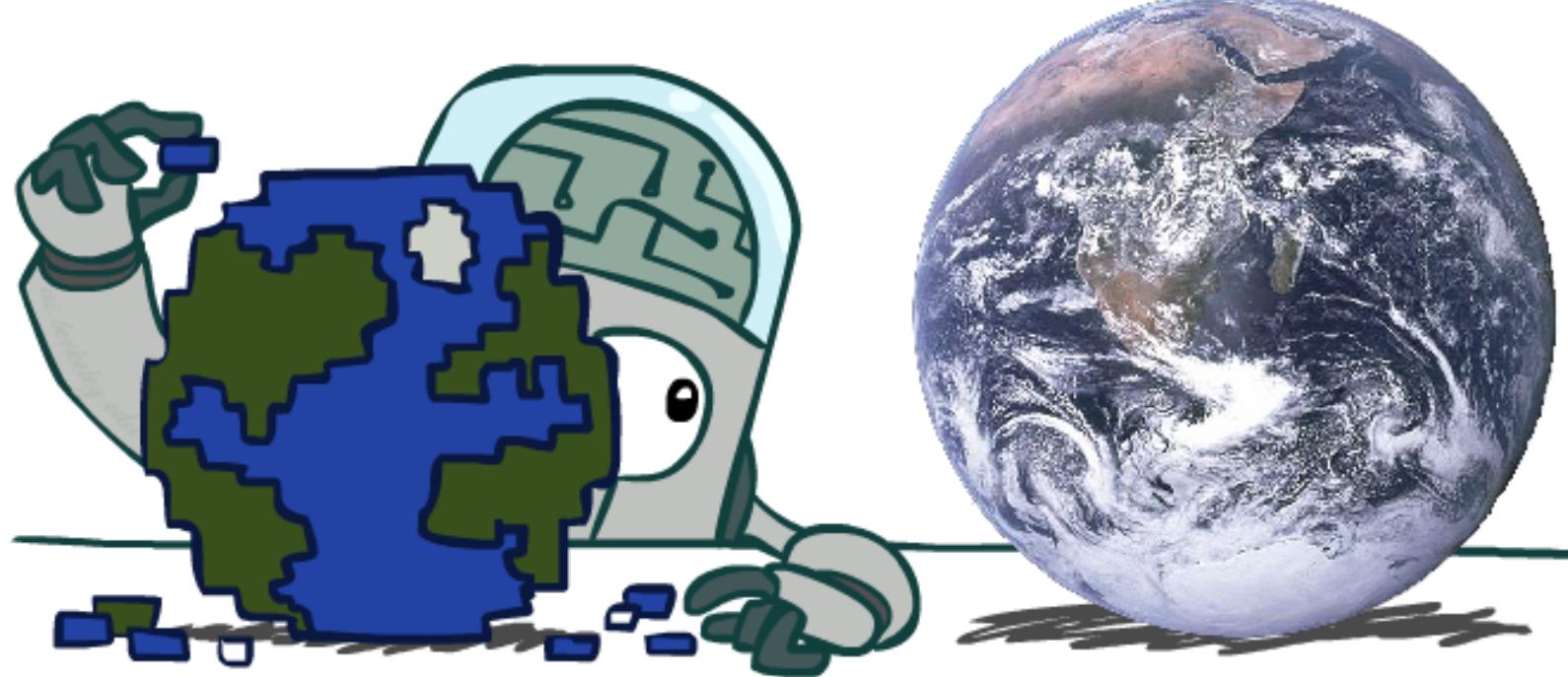


- Answer: Expectimax!

- To figure out EACH chance node's probabilities, you have to run a simulation of your opponent
- This kind of thing gets very slow very quickly
- Even worse if you have to simulate your opponent simulating you...
- ... except for minimax, which has the nice property that it all collapses into one game tree

# Modeling Assumptions

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# The Dangers of Optimism and Pessimism

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# The Dangers of Optimism and Pessimism

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## Dangerous Optimism

Assuming chance when the world is adversarial

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Assuming chance when the world is adversarial



# The Dangers of Optimism and Pessimism

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## Dangerous Optimism

Assuming chance when the world is adversarial



## Dangerous Pessimism

Assuming the worst case when it's not likely

# The Dangers of Optimism and Pessimism

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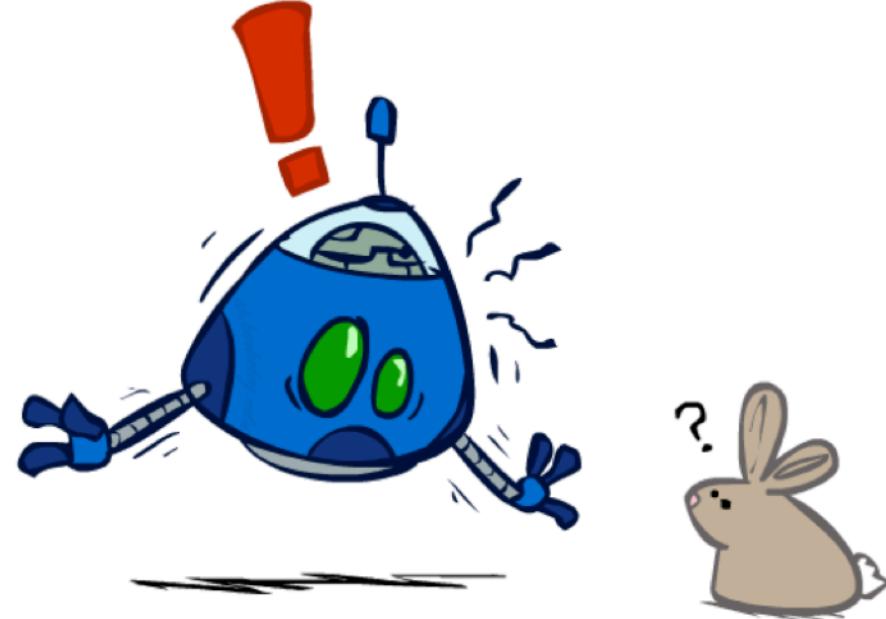
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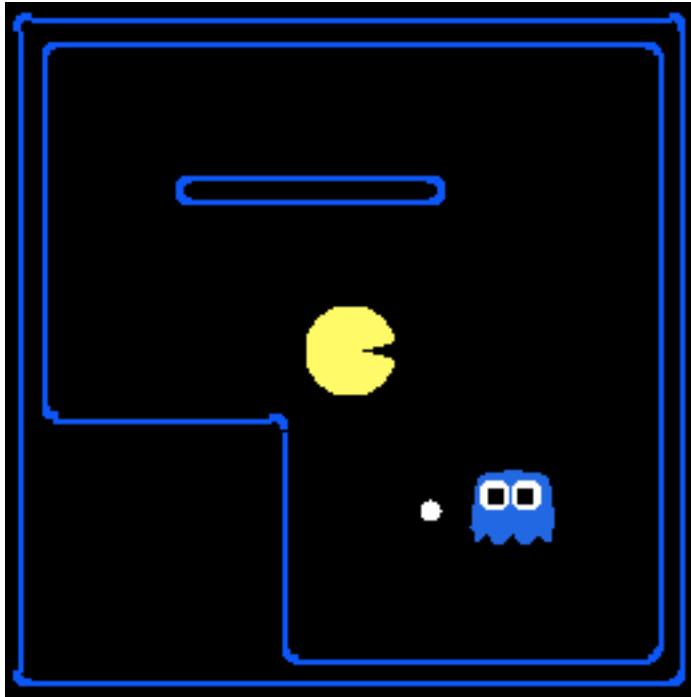


## Dangerous Pessimism

Assuming the worst case when it's not likely



# Assumptions vs. Reality



	Adversarial Ghost	Random Ghost
Minimax Pacman	Won 5/5 Avg. Score: 483	Won 5/5 Avg. Score: 493
Expectimax Pacman	Won 1/5 Avg. Score: -303	Won 5/5 Avg. Score: 503

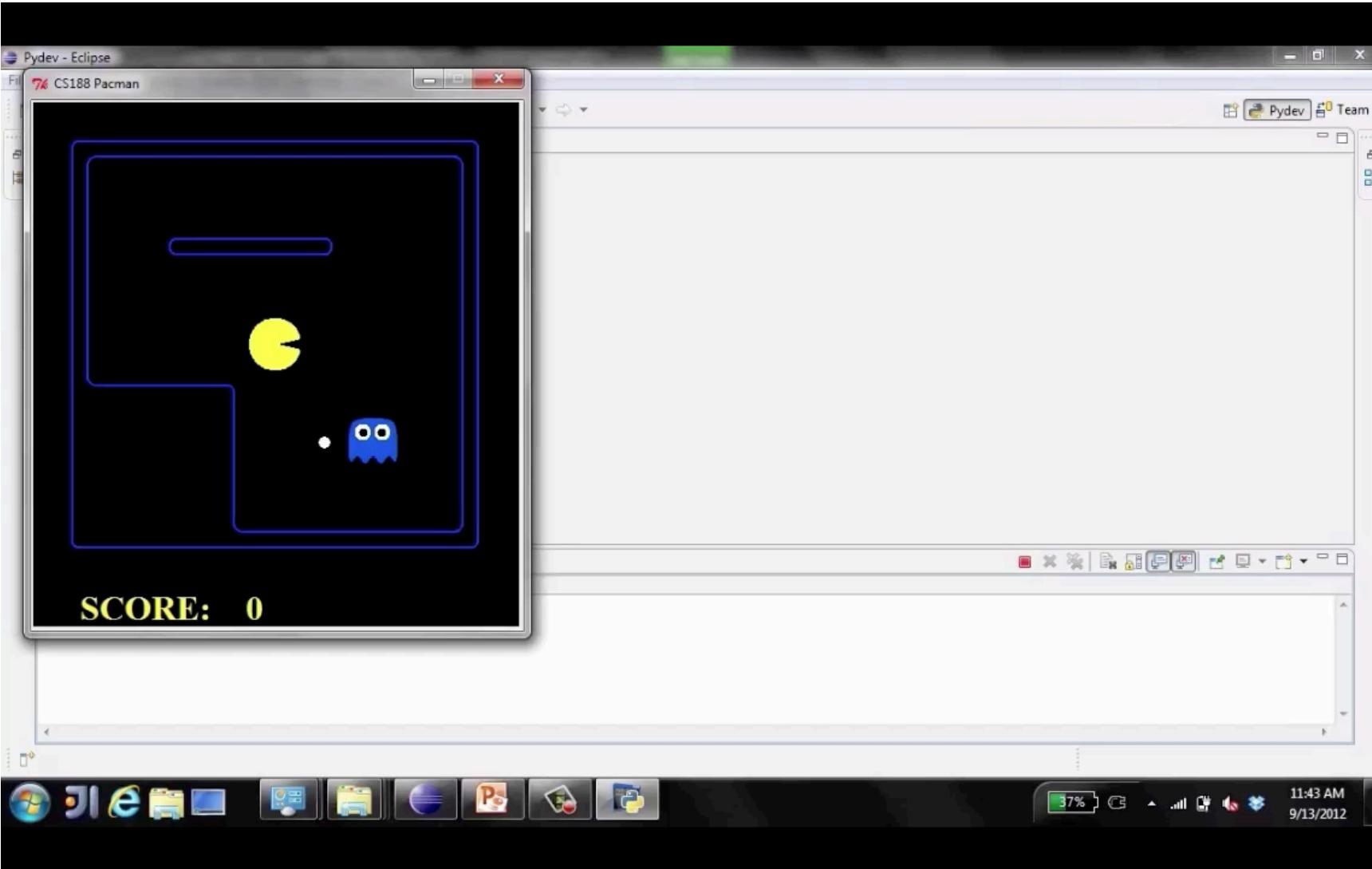
Results from playing 5 games

Pacman used depth 4 search with an eval function that avoids trouble  
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[Demos: world assumptions (L7D3,4,5,6)]

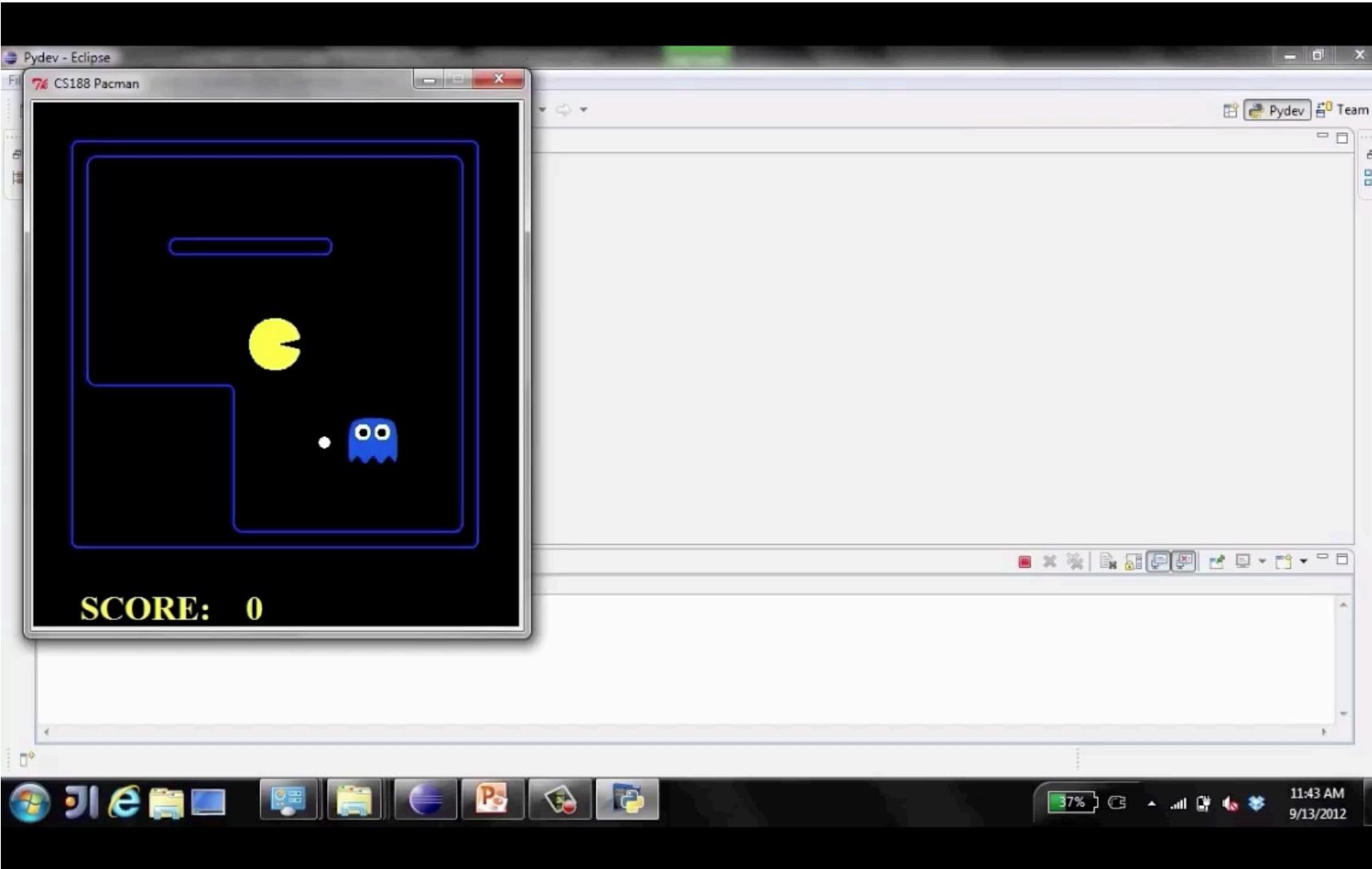
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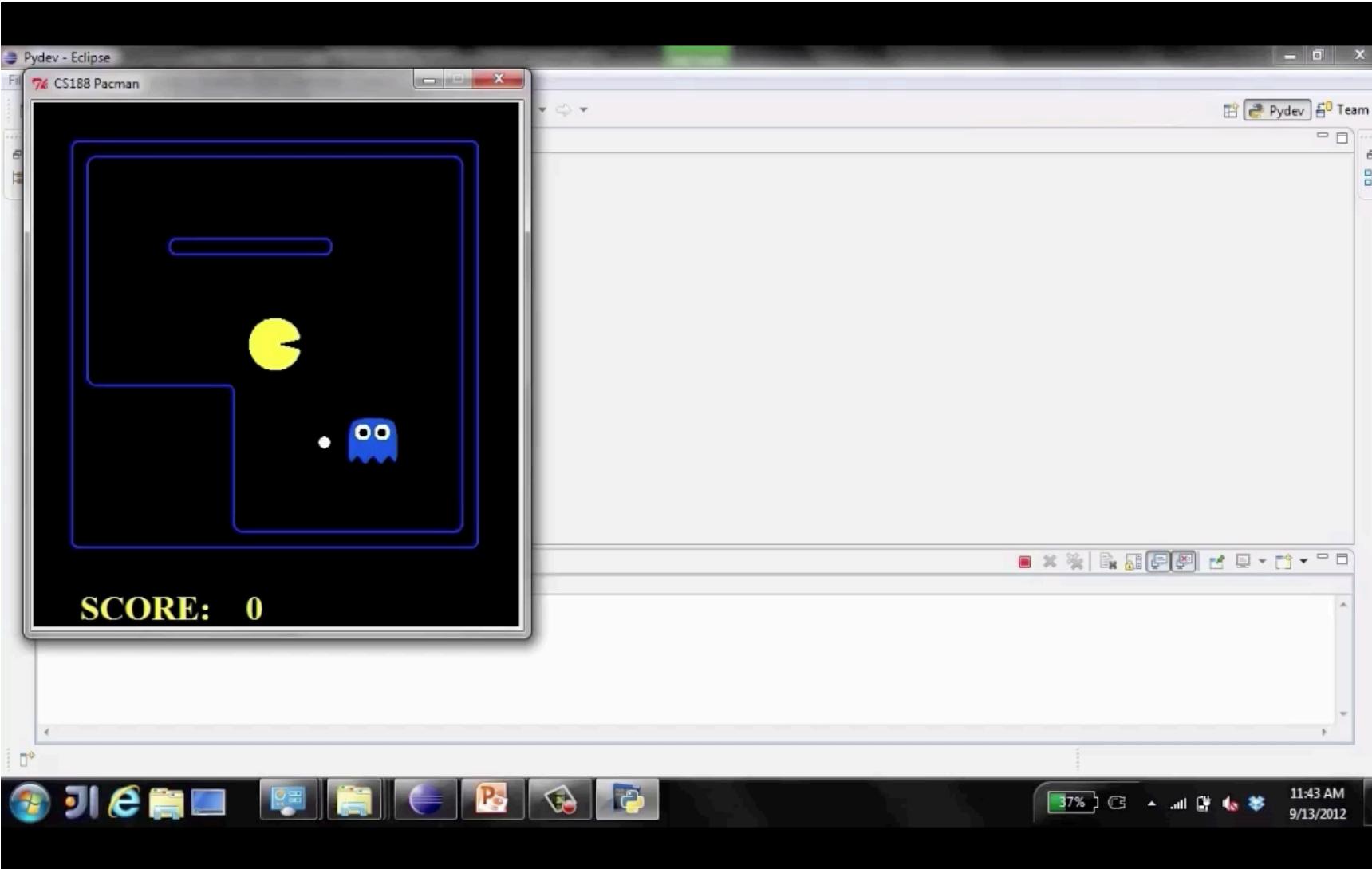
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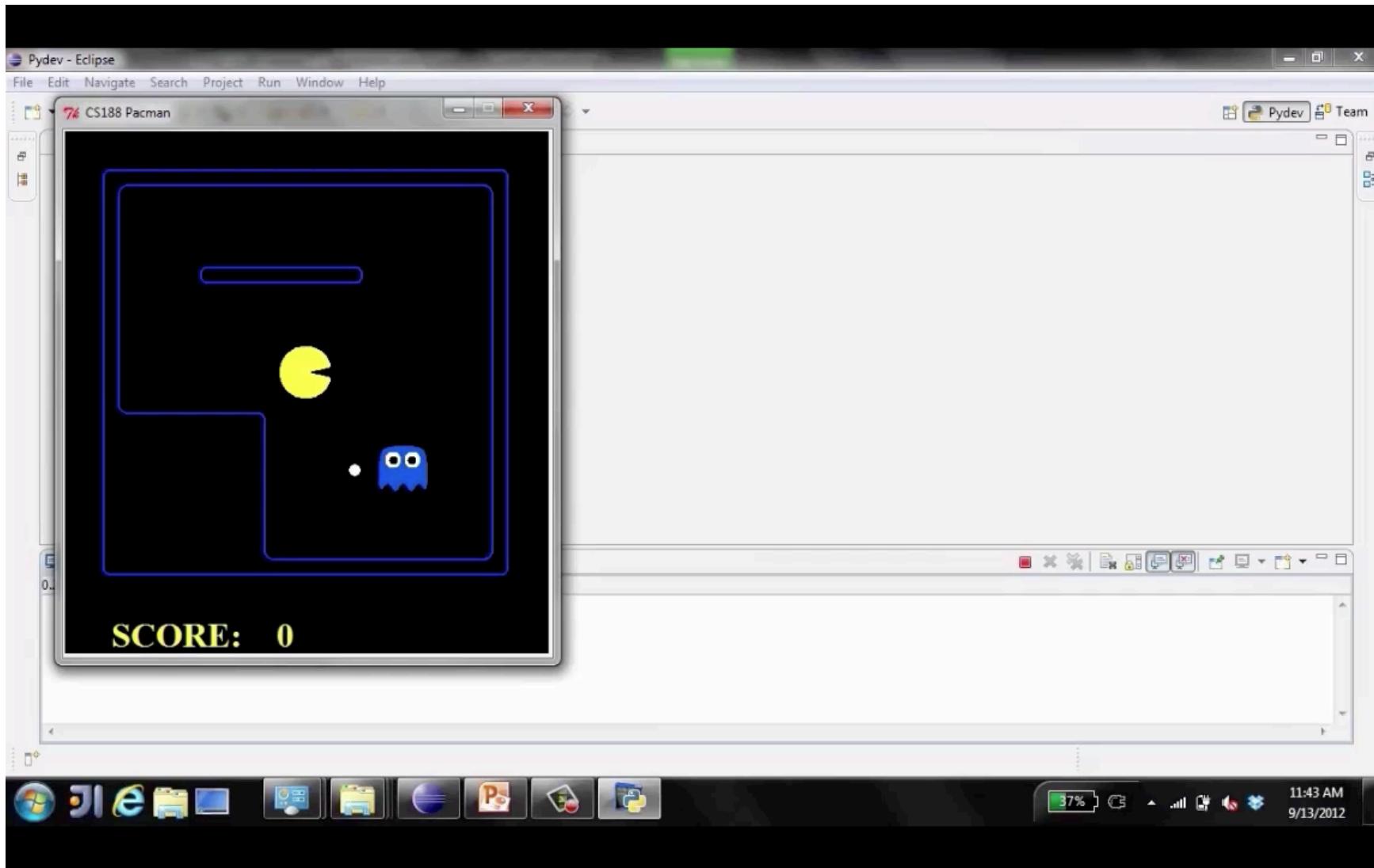
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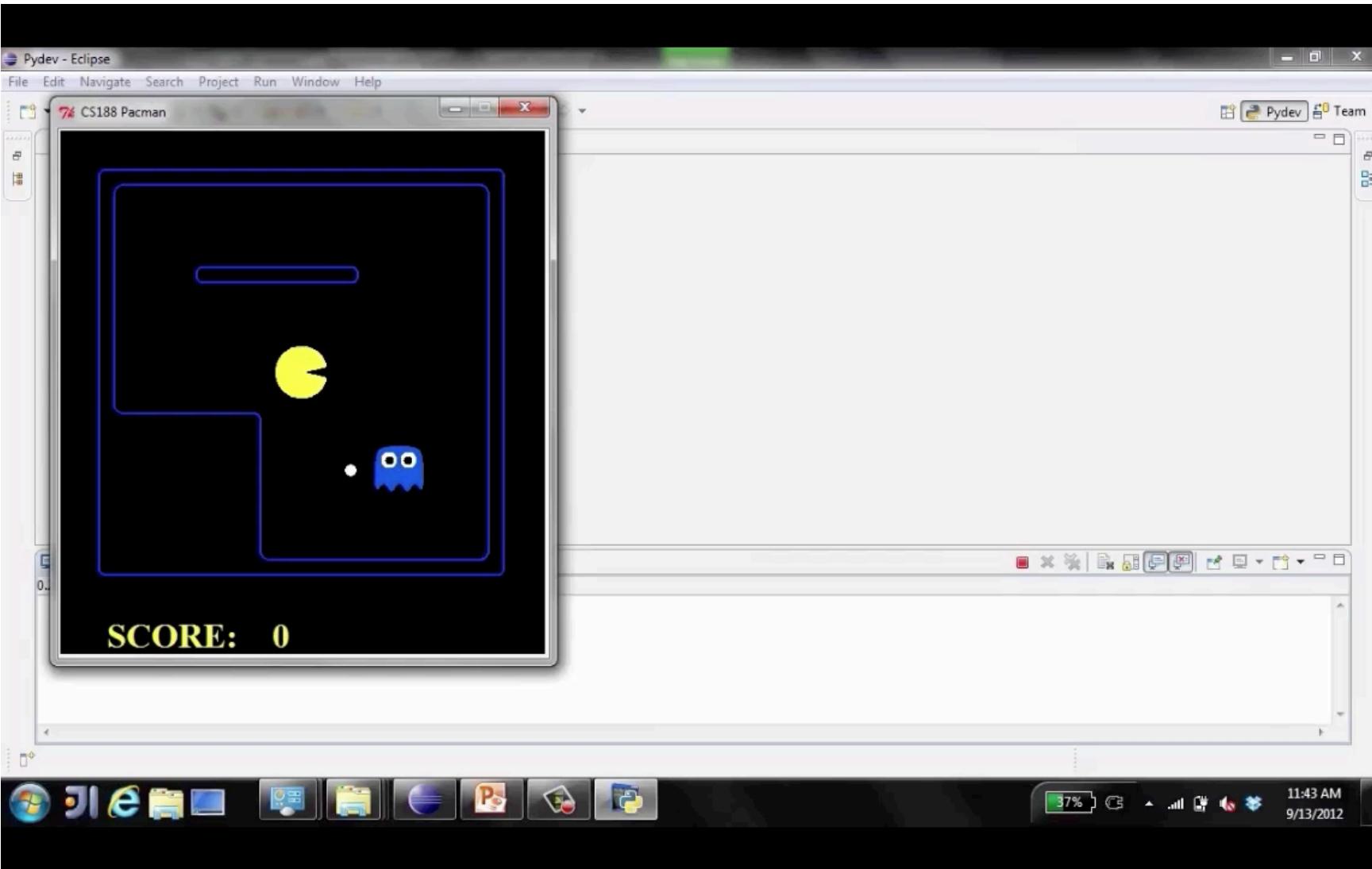
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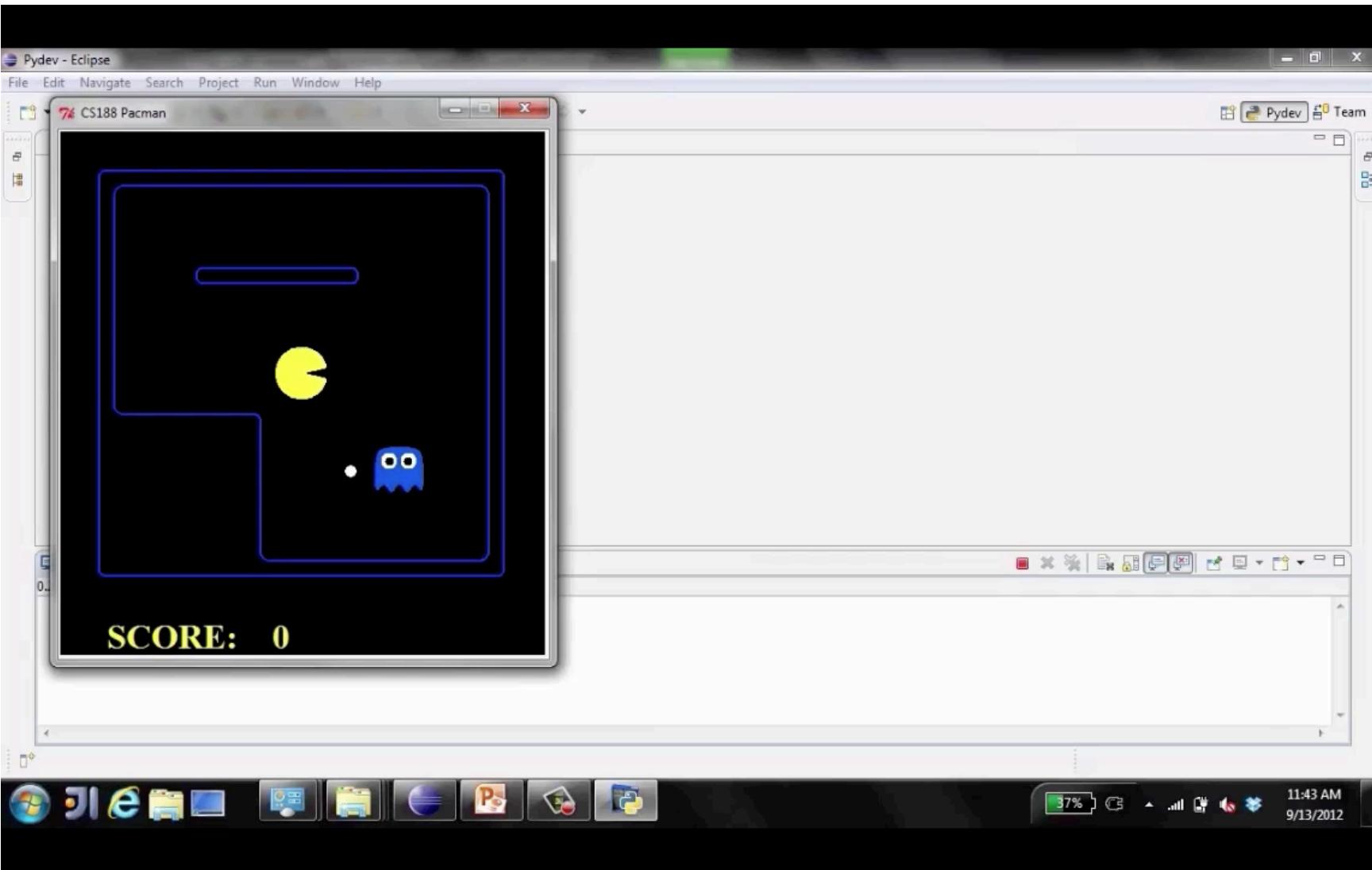
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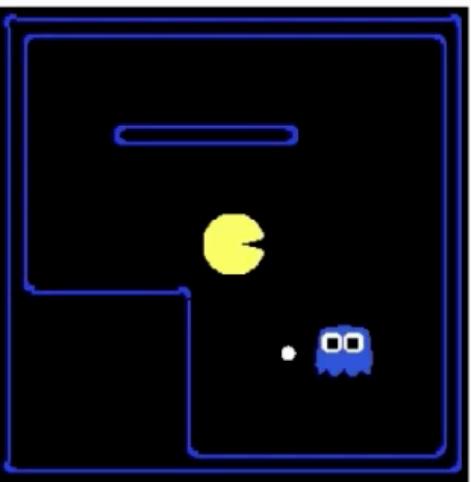
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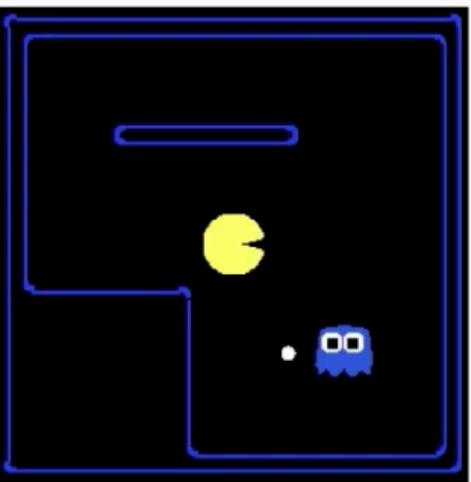
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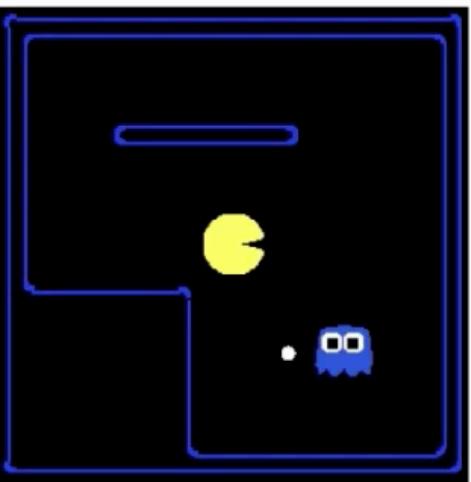
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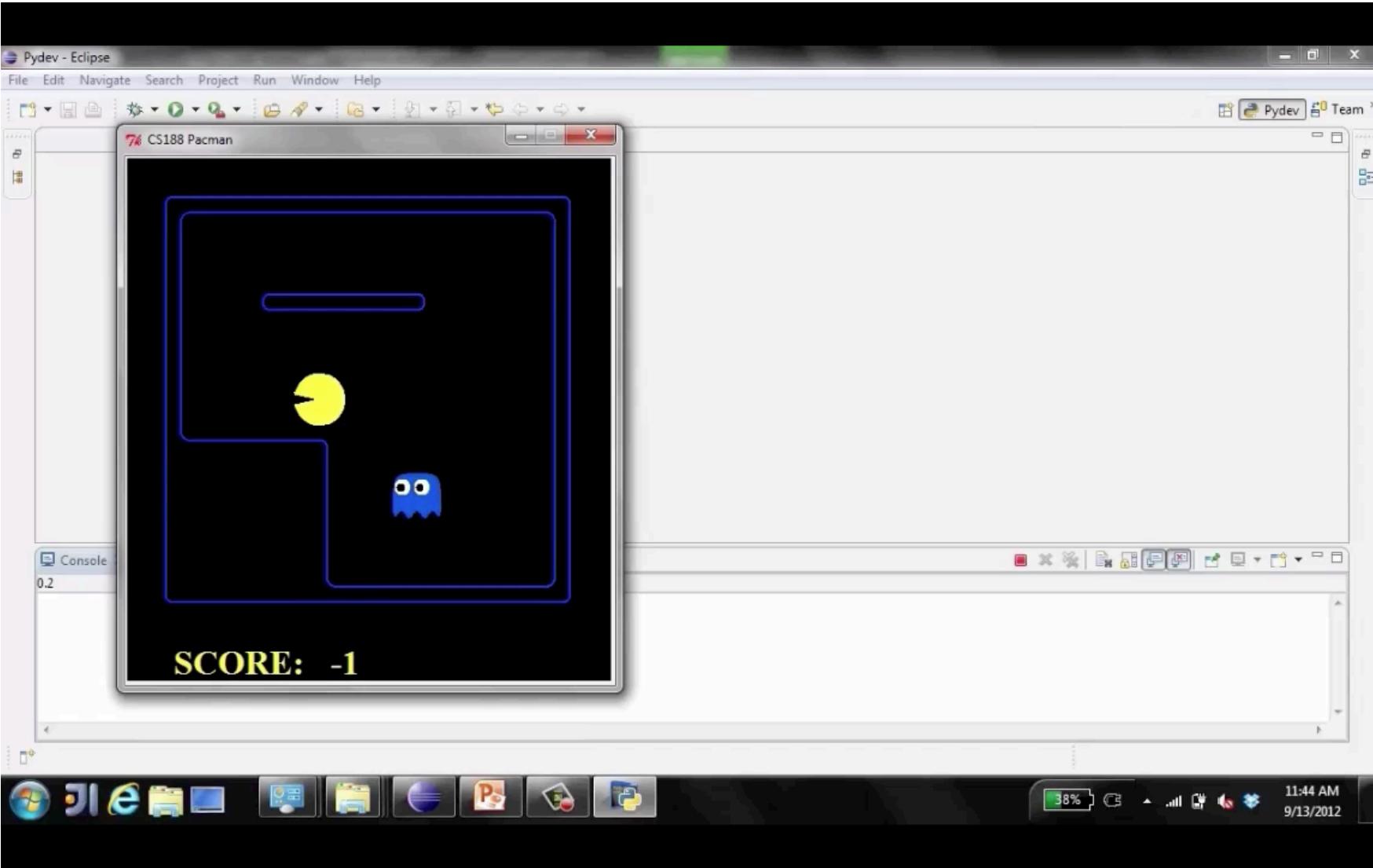
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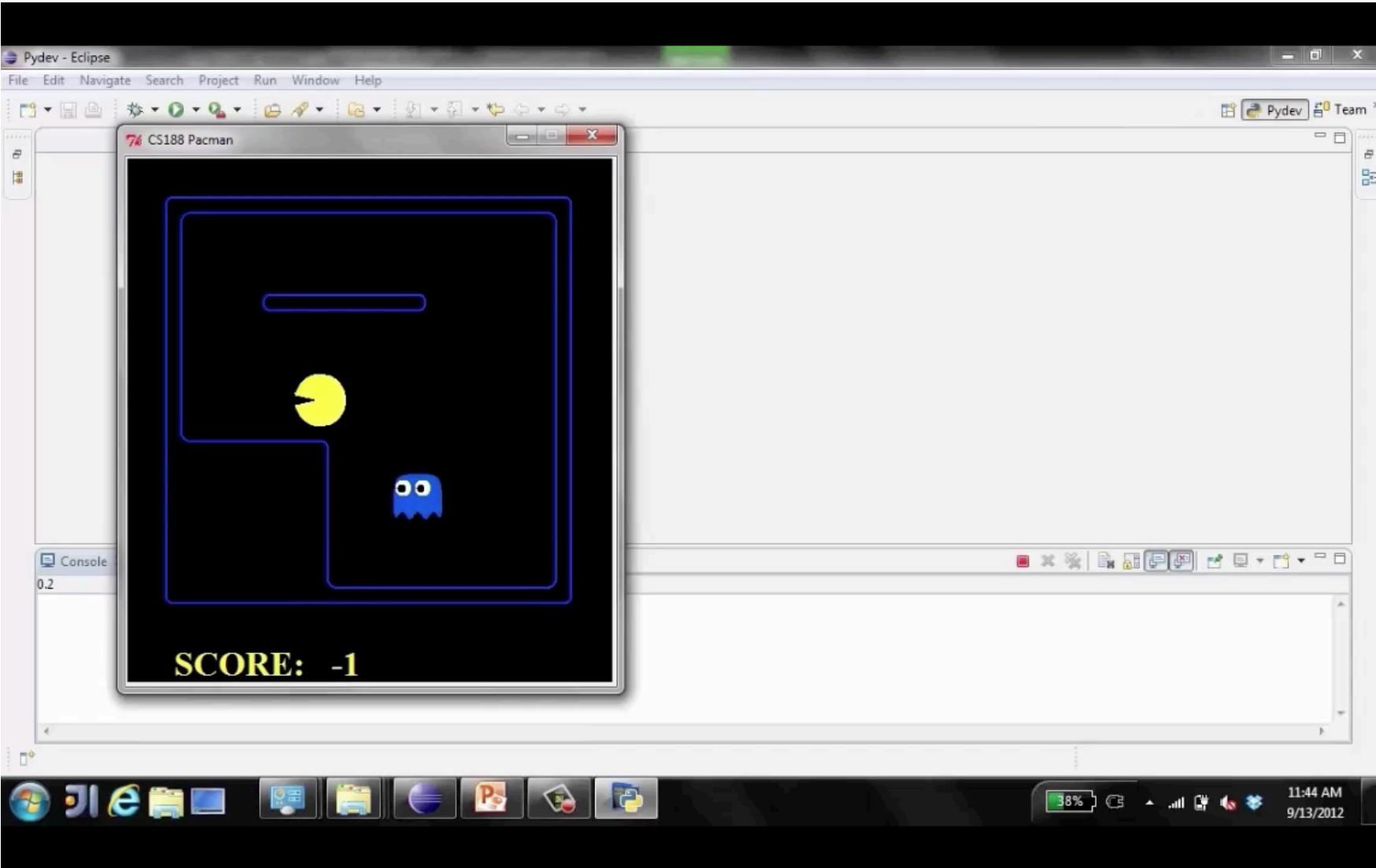
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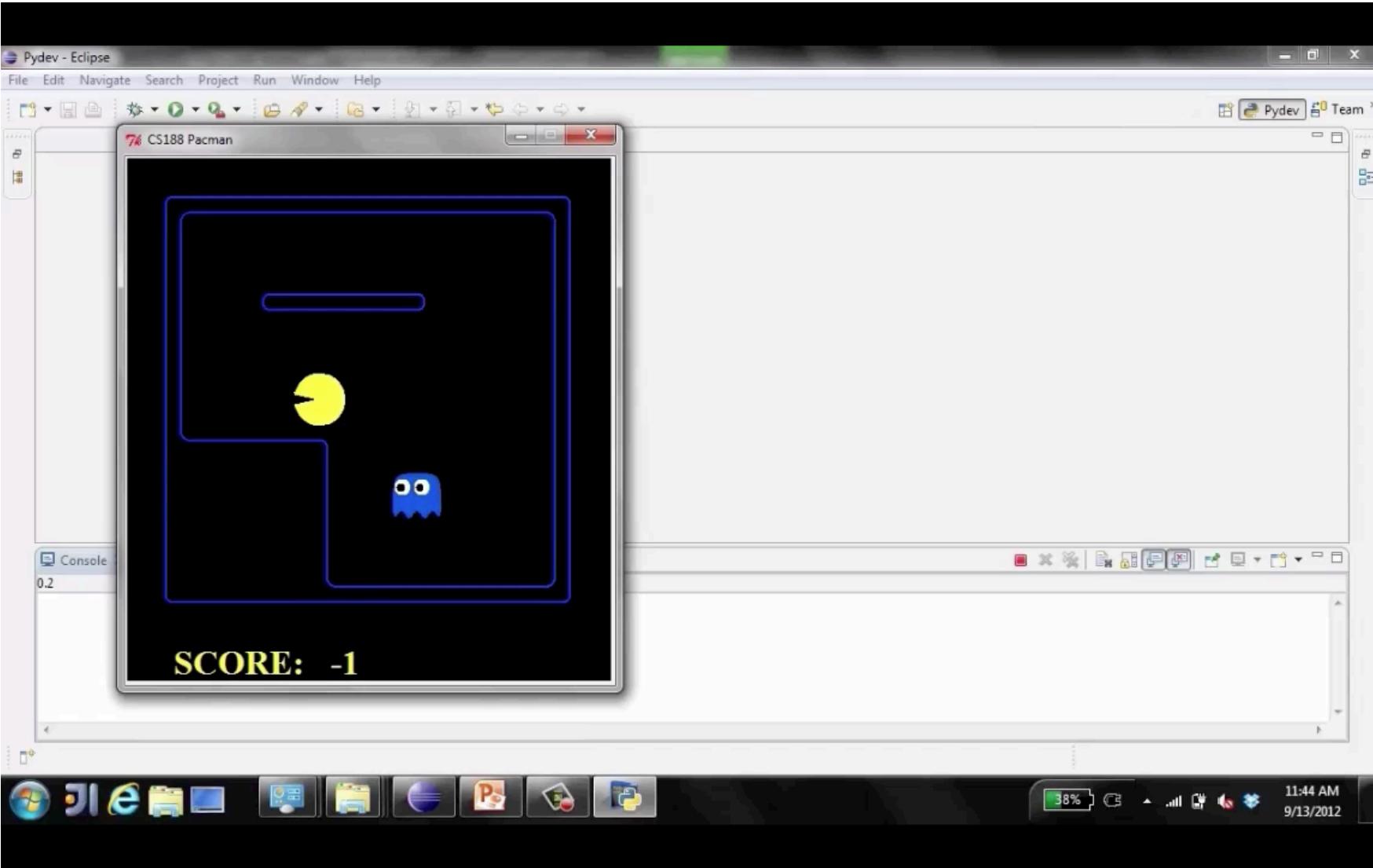
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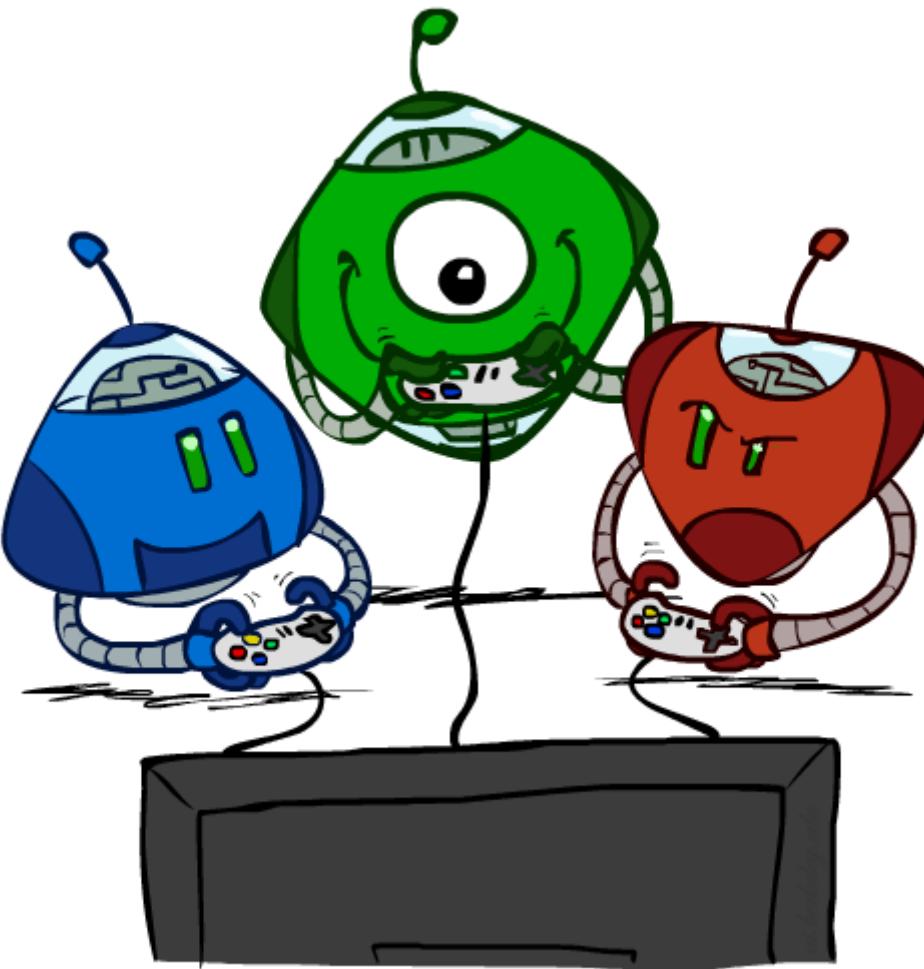
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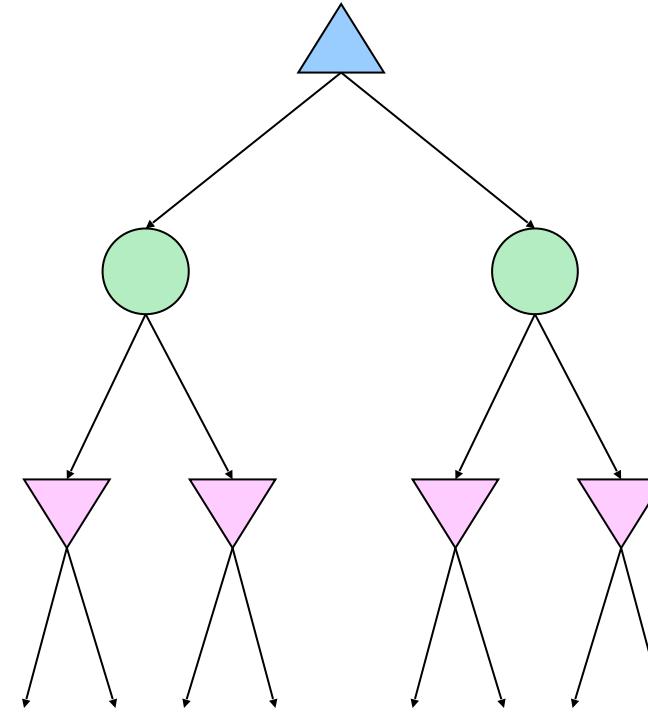
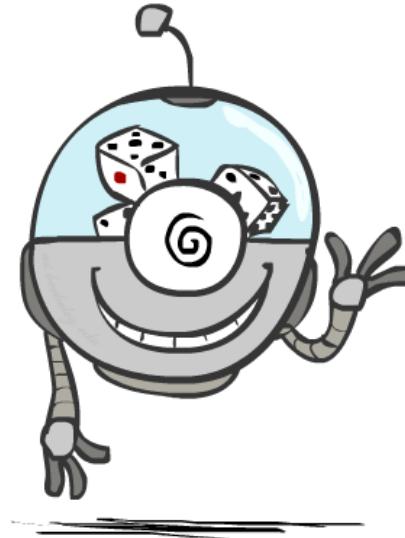
# Other Game Types

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# Mixed Layer Types

- E.g. Backgammon
- Expectiminimax
  - Environment is an extra “random agent” player that moves after each min/max agent
  - Each node computes the appropriate combination of its children



# Example: Backgammon

- Dice rolls increase  $b$ : 21 possible rolls with 2 dice
  - Backgammon  $\approx 20$  legal moves
  - Depth 2 =  $20 \times (21 \times 20)^3 = 1.2 \times 10^9$
- As depth increases, probability of reaching a given search node shrinks
  - So usefulness of search is diminished
  - So limiting depth is less damaging
  - But pruning is trickier...
- Historic AI: TDGammon uses depth-2 search + very good evaluation function + reinforcement learning: world-champion level play
- 1<sup>st</sup> AI world champion in any game!

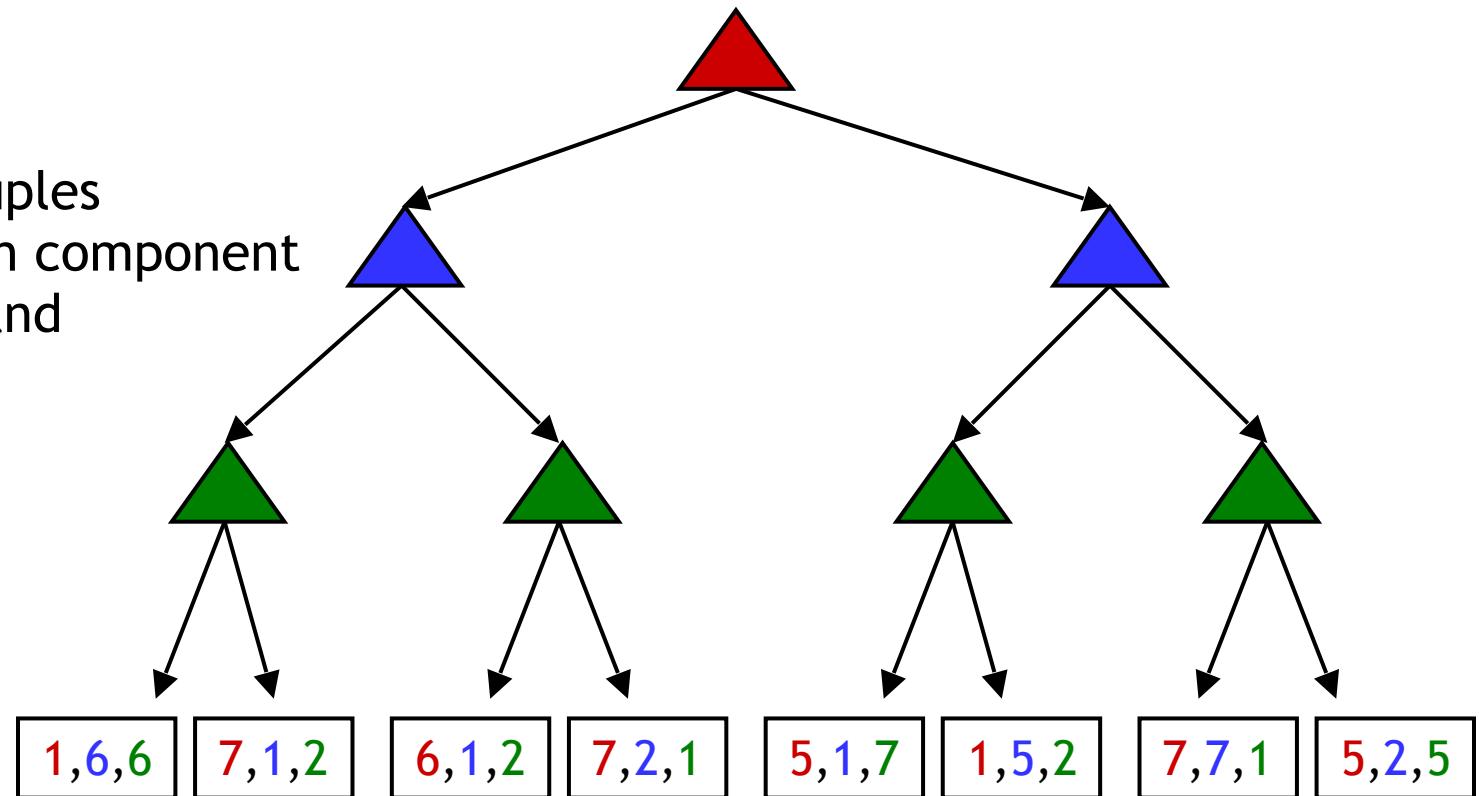


# Multi-Agent Utilities

- What if the game is not zero-sum, or has multiple players?

- Generalization of minimax:

- Terminals have utility tuples
- Node values are also utility tuples
- Each player maximizes its own component
- Can give rise to cooperation and competition dynamically...

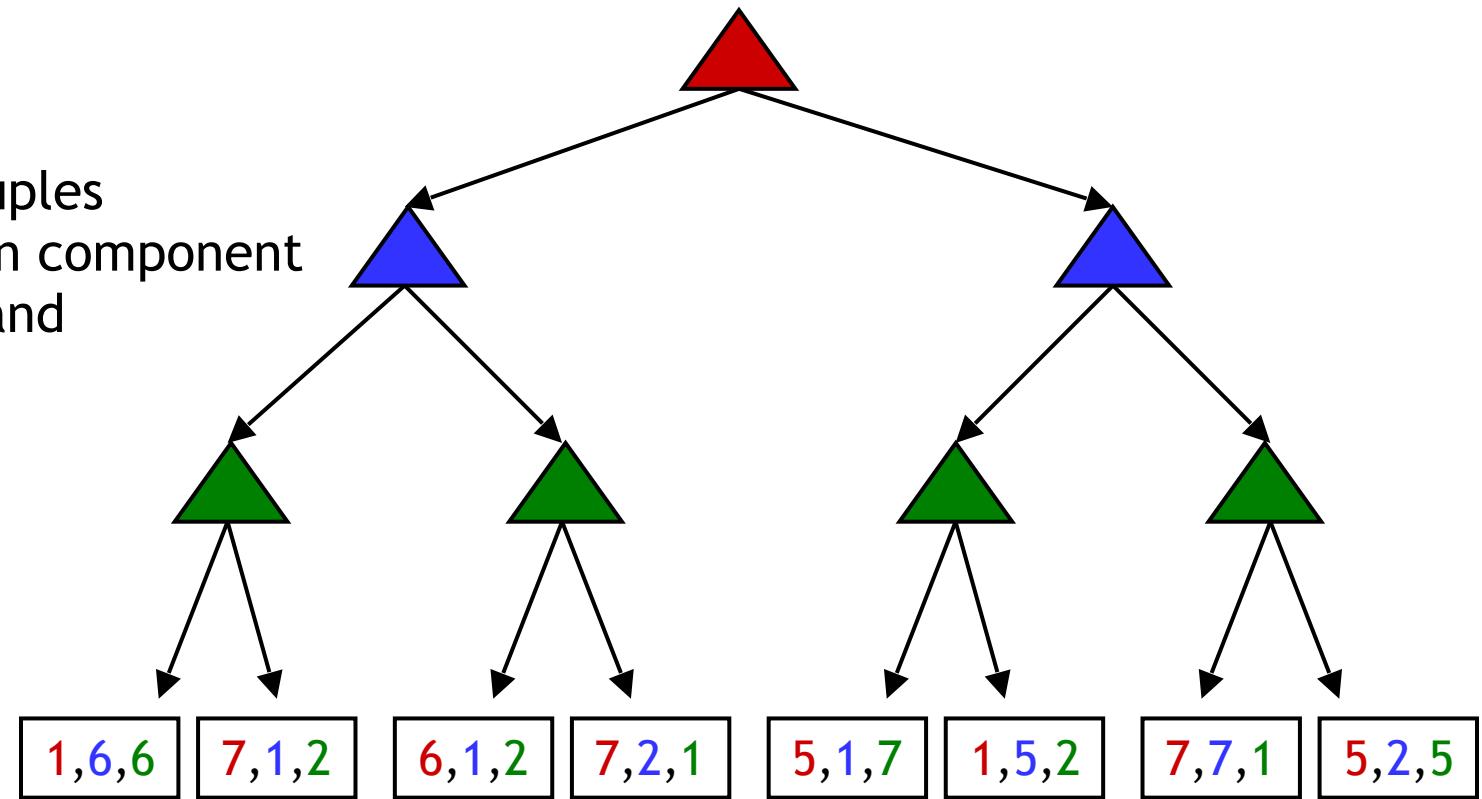
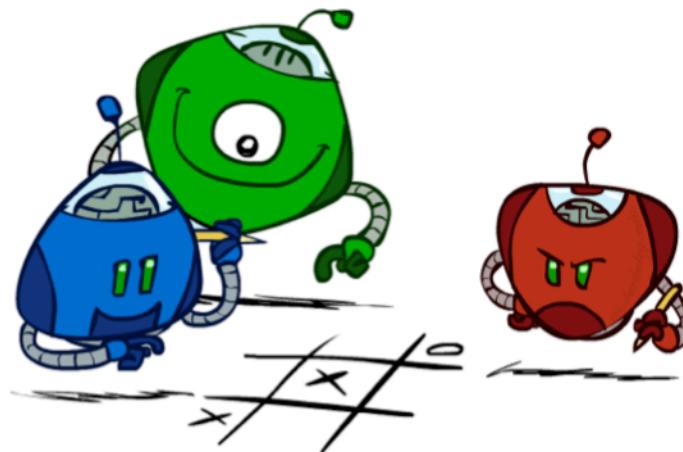


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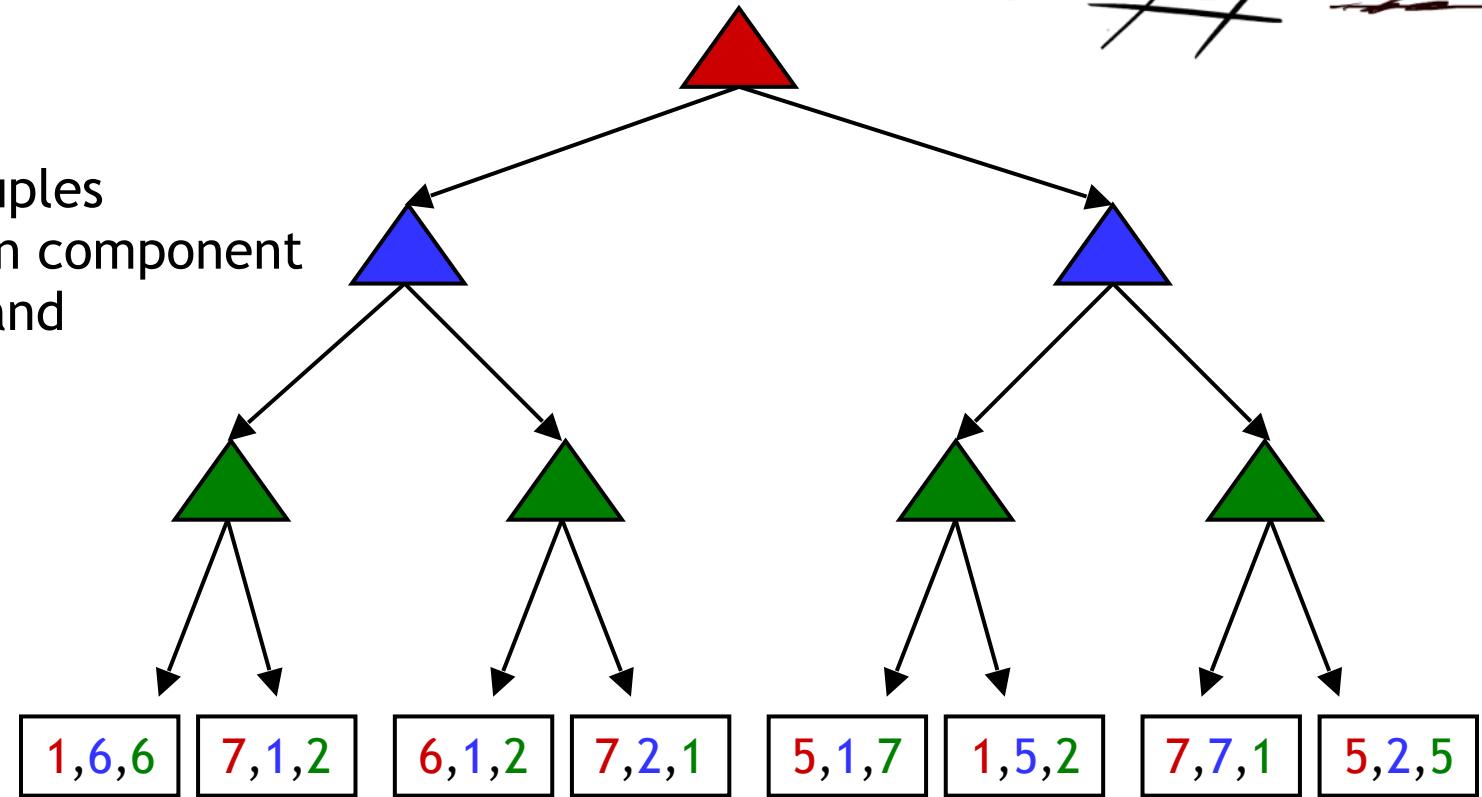
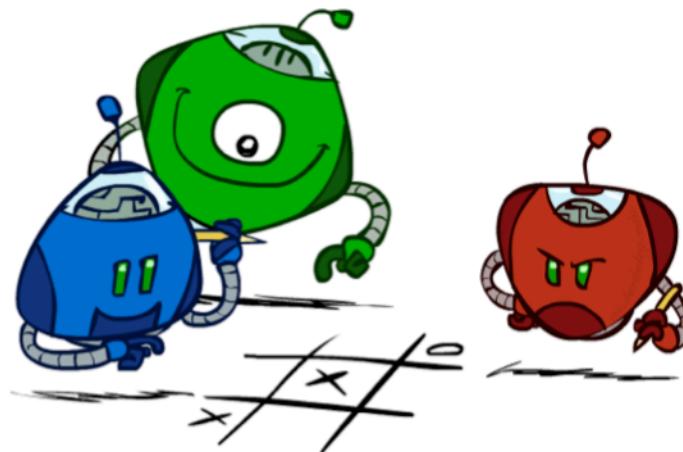
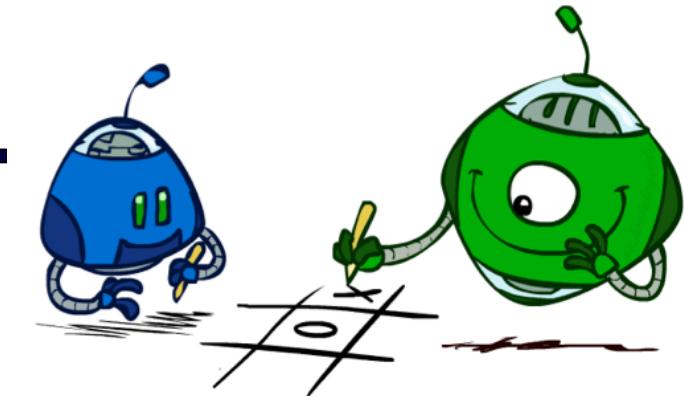


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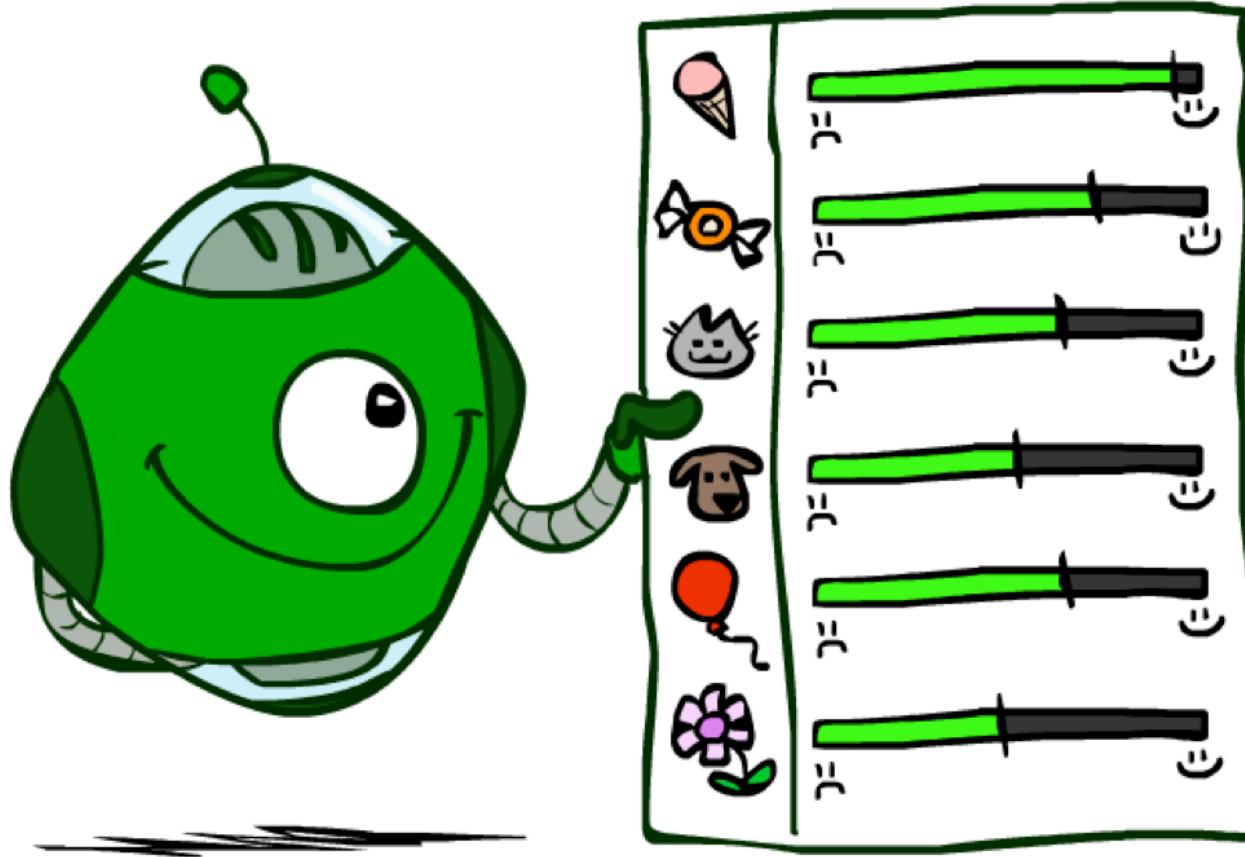
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# Utilities

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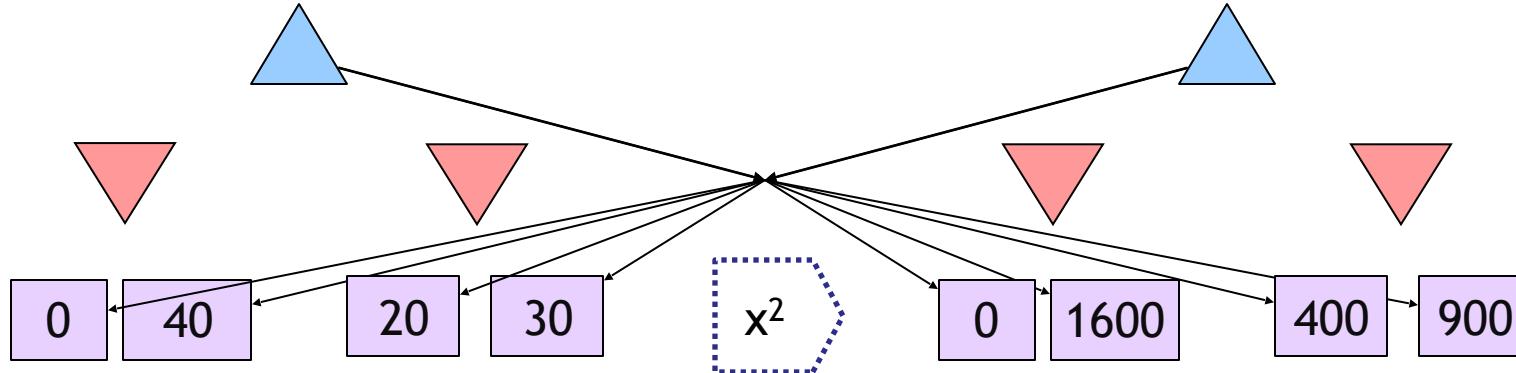
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# Maximum Expected Utility

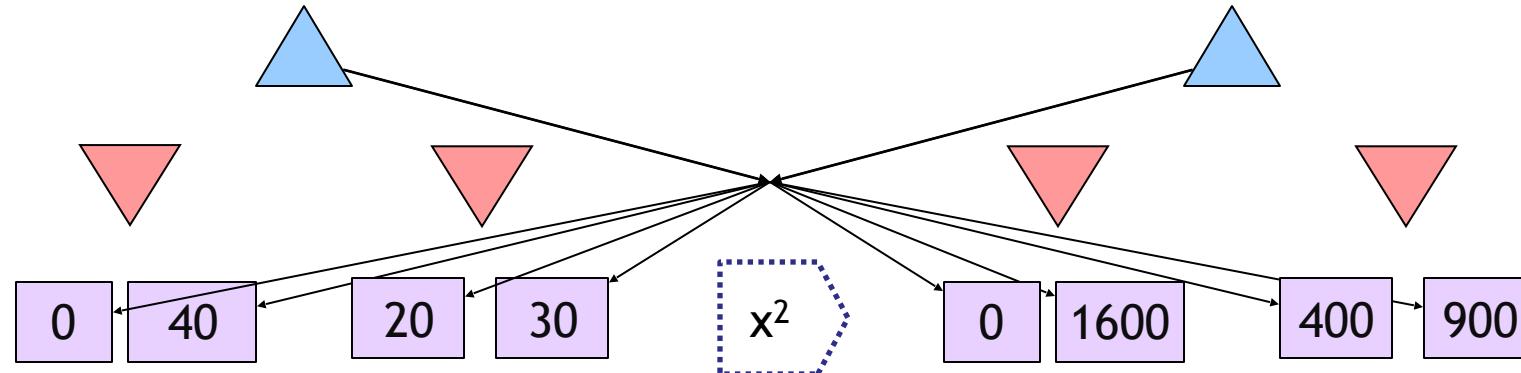
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- Why should we average utilities? Why not minimax?
- Principle of maximum expected utility:
  - A rational agent should chose the action that **maximizes its expected utility, given its knowledge**
- Questions:
  - Where do utilities come from?
  - How do we know such utilities even exist?
  - How do we know that averaging even makes sense?
  - What if our behavior (preferences) can't be described by utilities?

# What Utilities to Use?

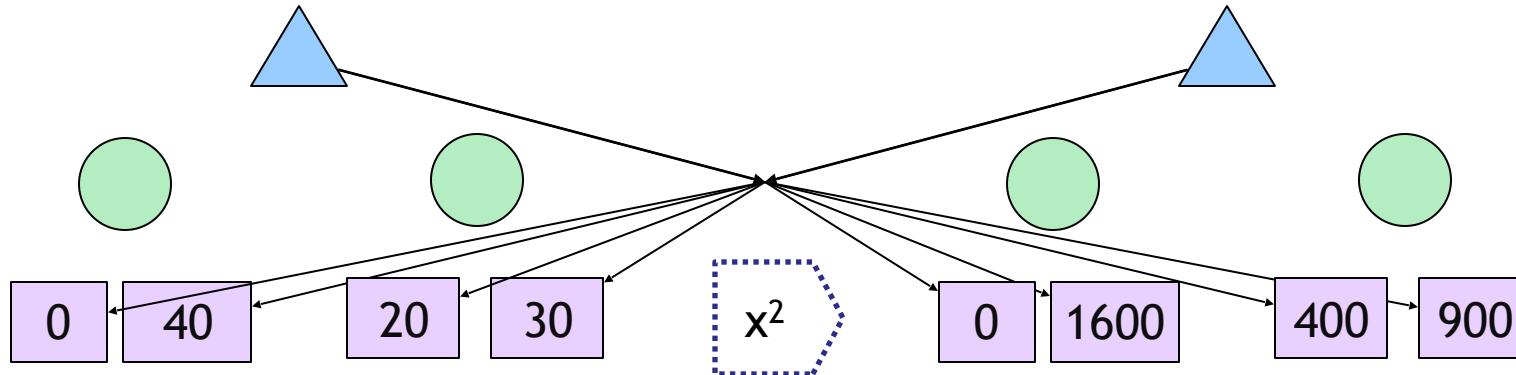


# What Utilities to Use?



- For worst-case minimax reasoning, terminal function scale doesn't matter
  - We just want better states to have higher evaluations (get the ordering right)
  - We call this **insensitivity to monotonic transformations**
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# Utilities

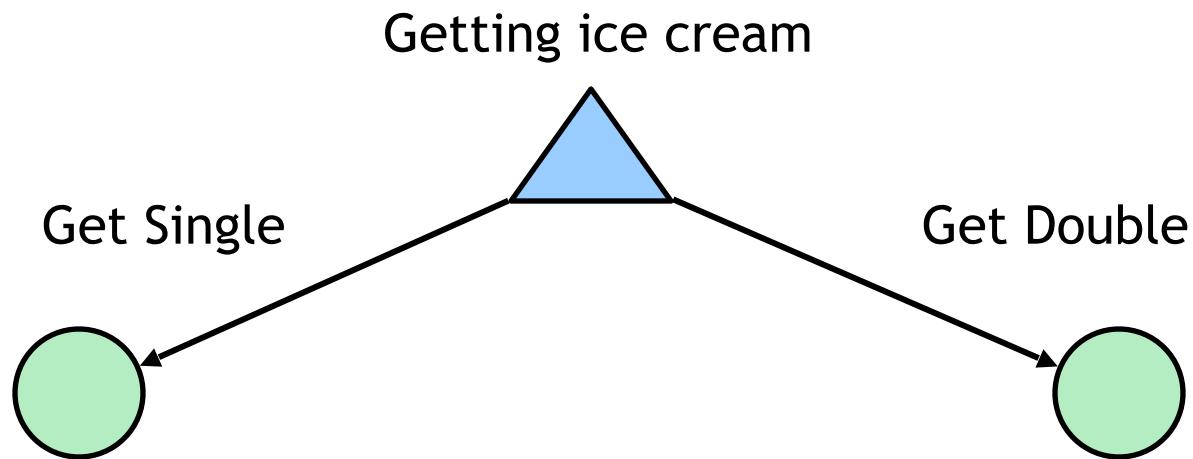
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- Utilities are functions from outcomes (states of the world) to real numbers that describe an agent's preferences
- Where do utilities come from?
  - In a game, may be simple (+1/-1)
  - Utilities summarize the agent's goals
  - Theorem: any “rational” preferences can be summarized as a utility function
- We hard-wire utilities and let behaviors emerge
  - Why don't we let agents pick utilities?
  - Why don't we prescribe behaviors?



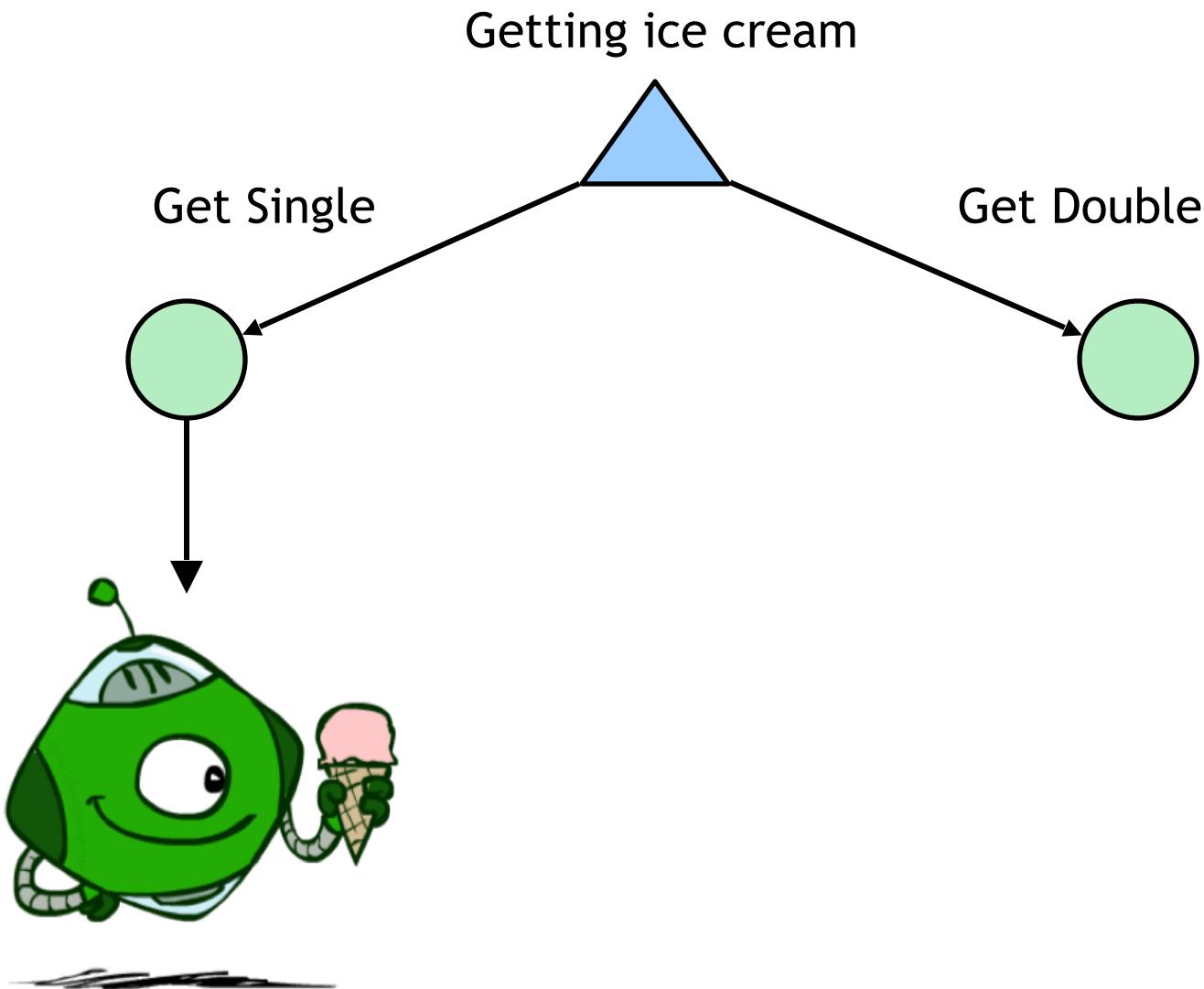
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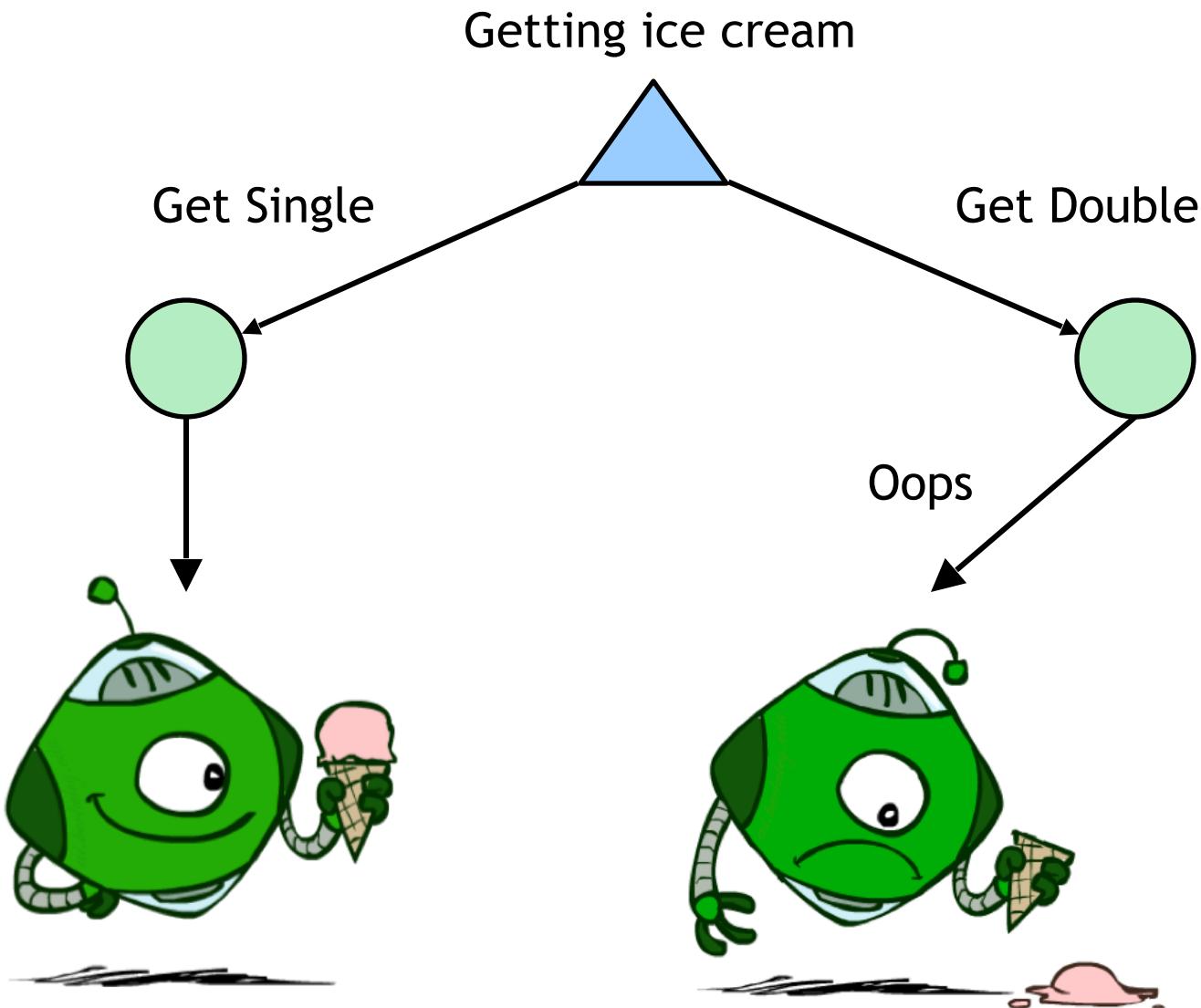


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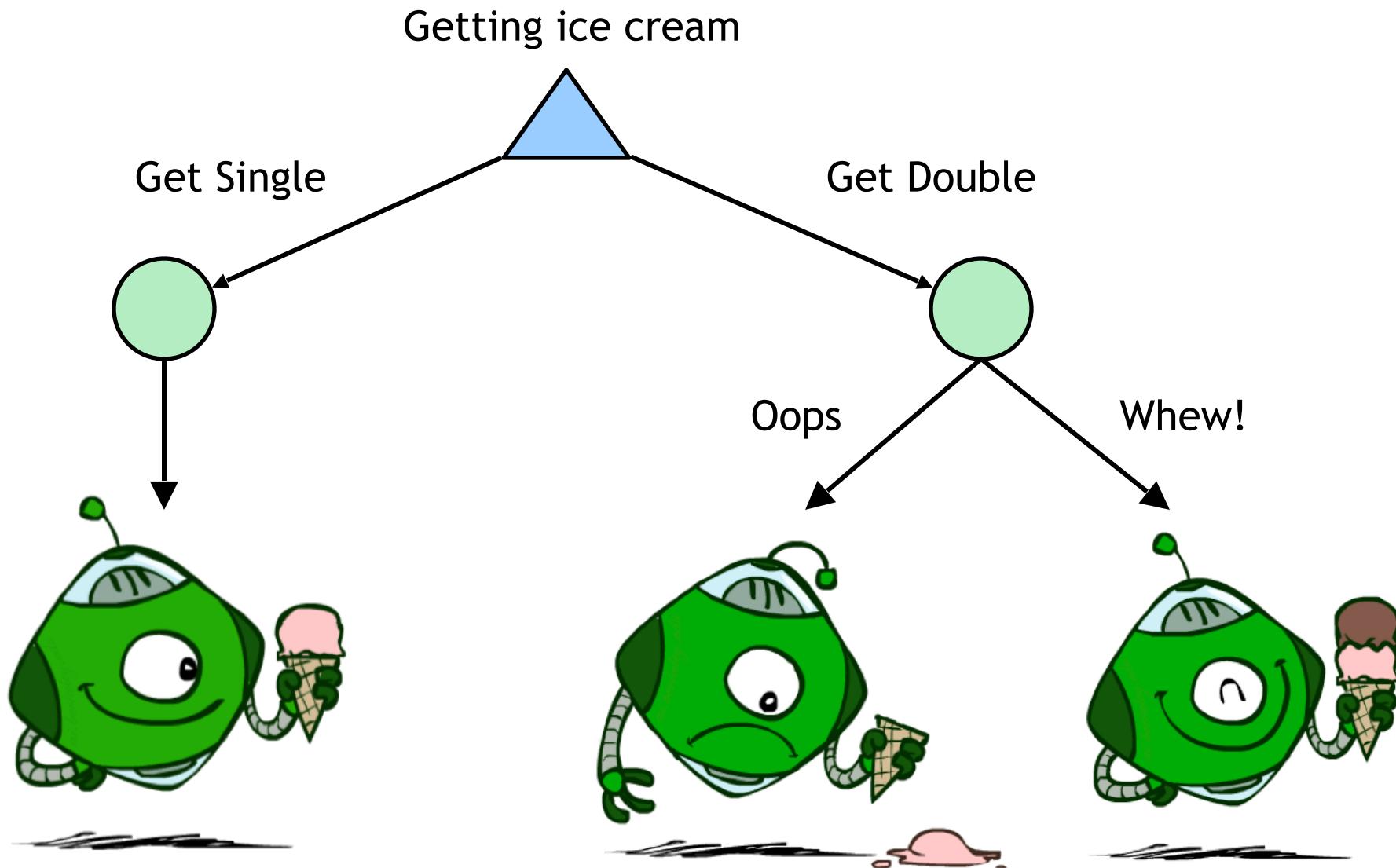
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# Utilities: Uncertain Outcomes



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# Preferences

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- An agent must have preferences among:
  - Prizes:  $A$ ,  $B$ , etc.
  - Lotteries: situations with uncertain prizes
$$L = [p, A; (1 - p), B]$$
- Notation:
  - Preference:  $A \succ B$
  - Preference:  $A \sim B$
  - Indifference:

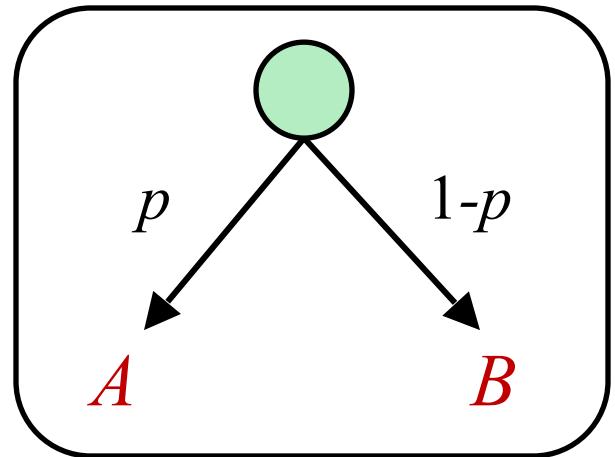
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A Lottery

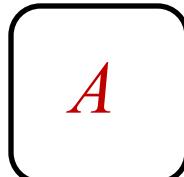


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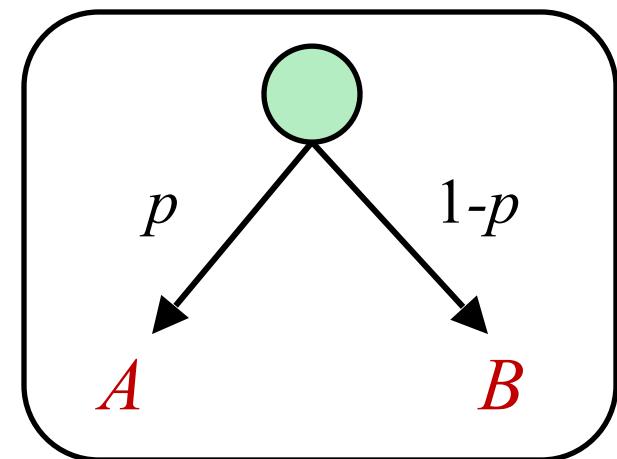
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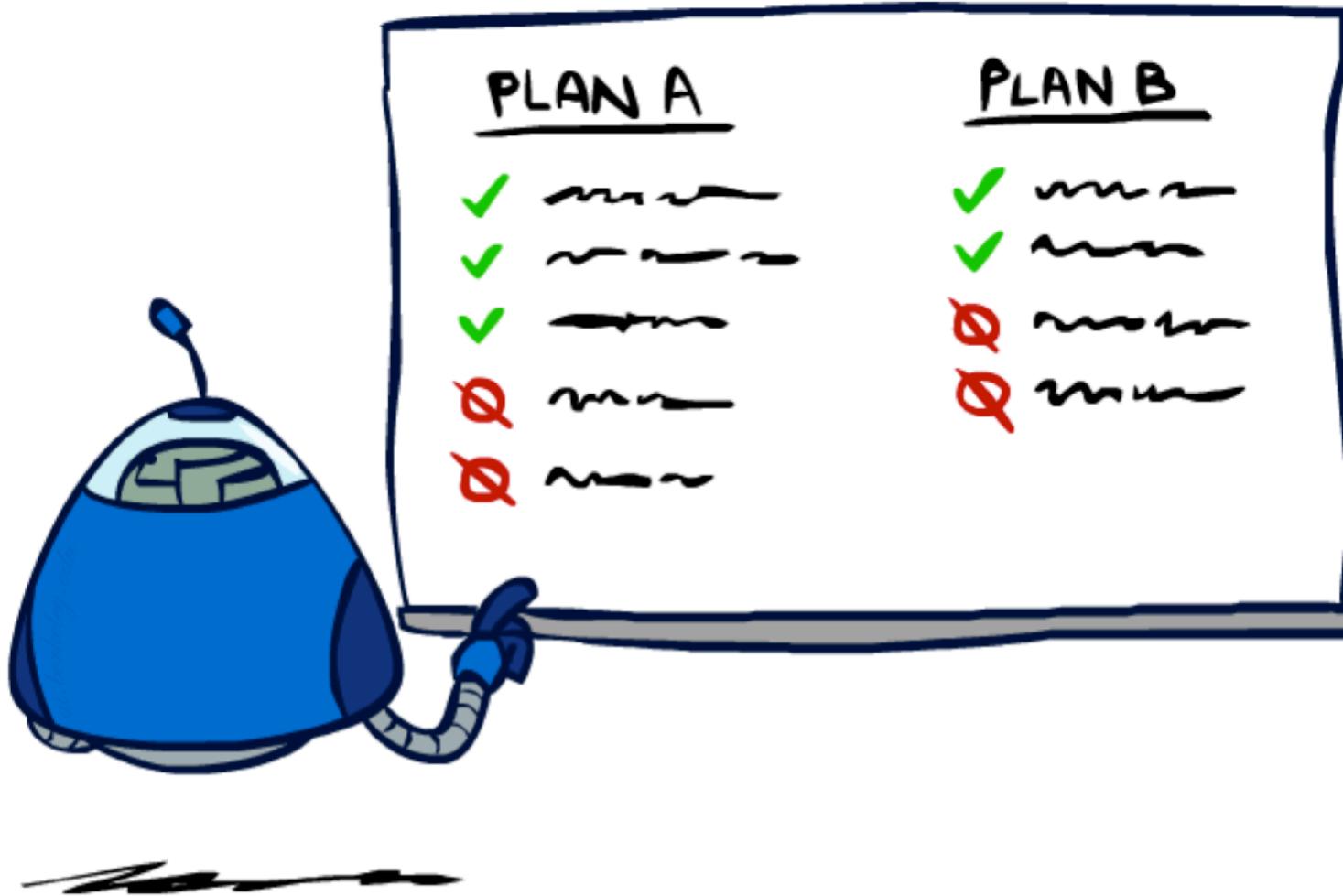


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# Rationality

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# Rational Preferences

## The Axioms of Rationality

### Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

### Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

### Continuity

$$A \succ B \succ C \Rightarrow \exists p [p, A; 1 - p, C] \sim B$$

### Substitutability

$$A \sim B \Rightarrow [p, A; 1 - p, C] \sim [p, B; 1 - p, C]$$

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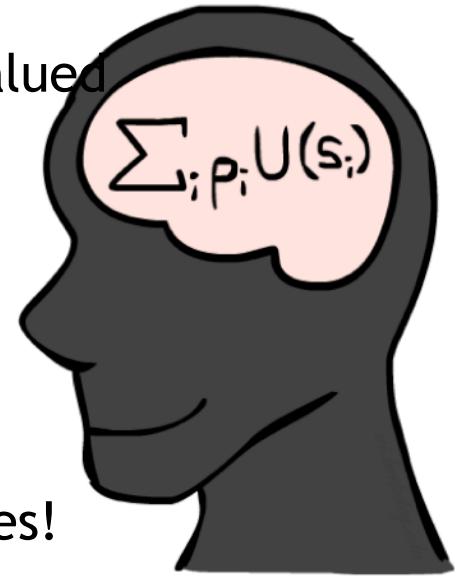
- Theorem [Ramsey, 1931; von Neumann & Morgenstern, 1944]

- Given any preferences satisfying these constraints, there exists a real-valued function  $U$  such that:

$$U(A) \geq U(B) \Leftrightarrow A \succeq B$$

$$U([p_1, S_1; \dots; p_n, S_n]) = \sum_i p_i U(S_i)$$

- I.e. values assigned by  $U$  preserve preferences of both prizes and lotteries!



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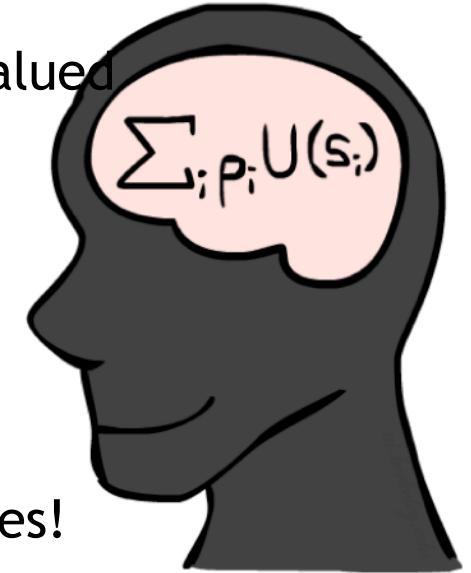
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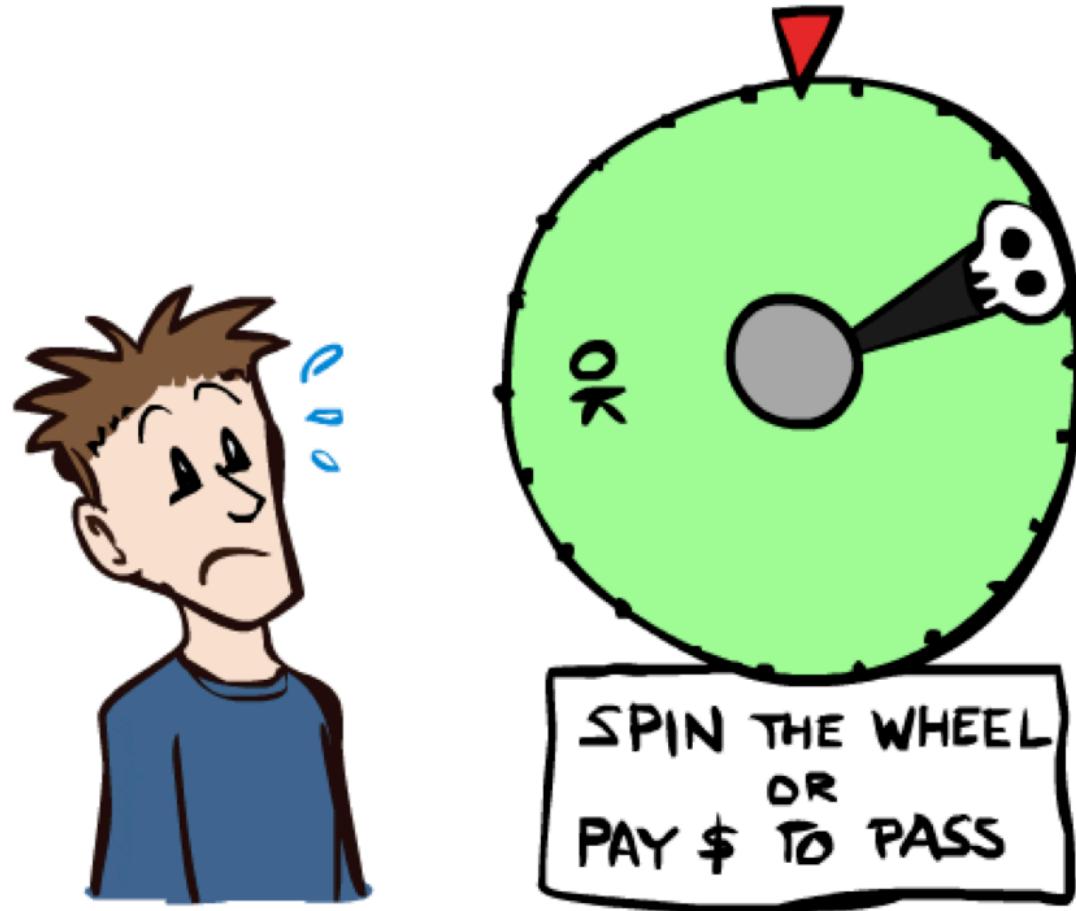


- Maximum expected utility (MEU) principle:

- Choose the action that maximizes expected utility
  - Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table for perfect tic-tac-toe, a reflex vacuum cleaner

# Human Utilities

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- With deterministic prizes only (no lottery choices), only ordinal utility can be determined, i.e., total order on prizes



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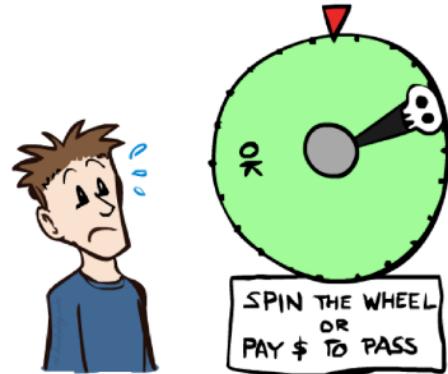
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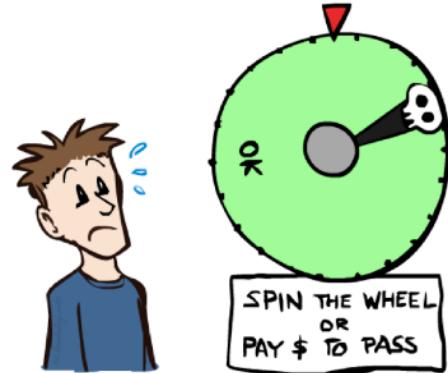
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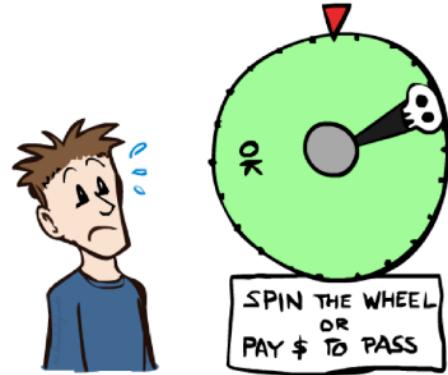
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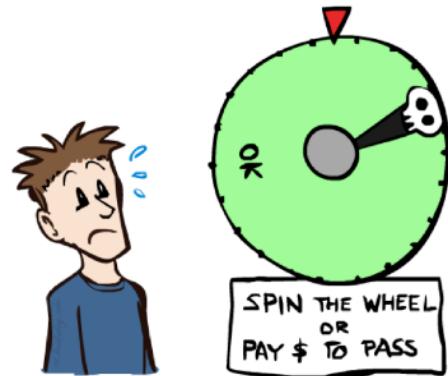
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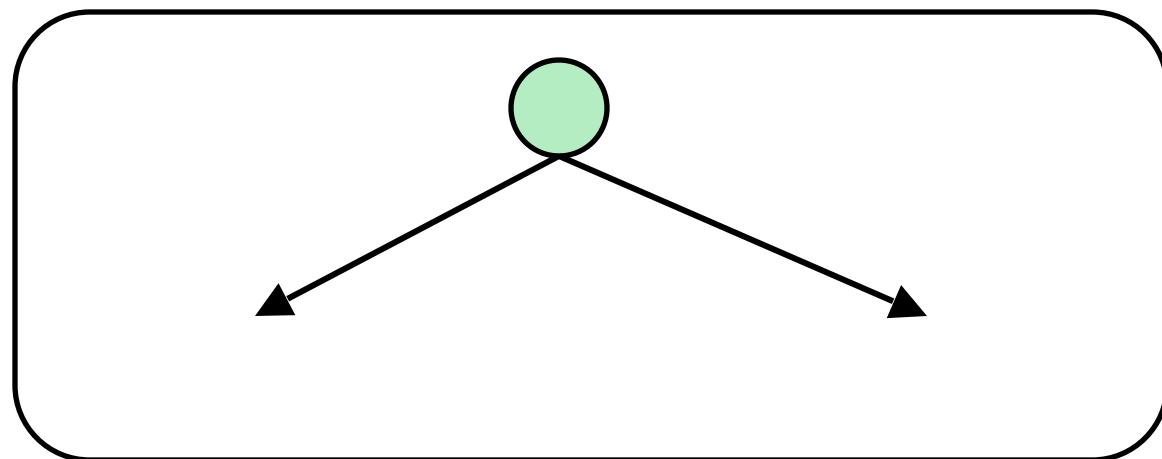
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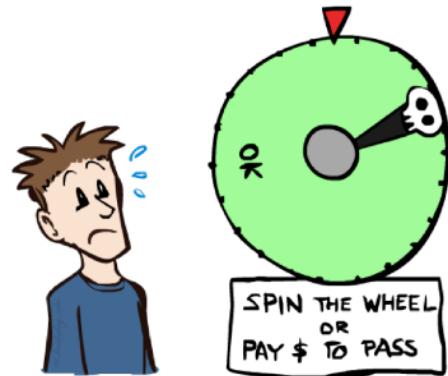
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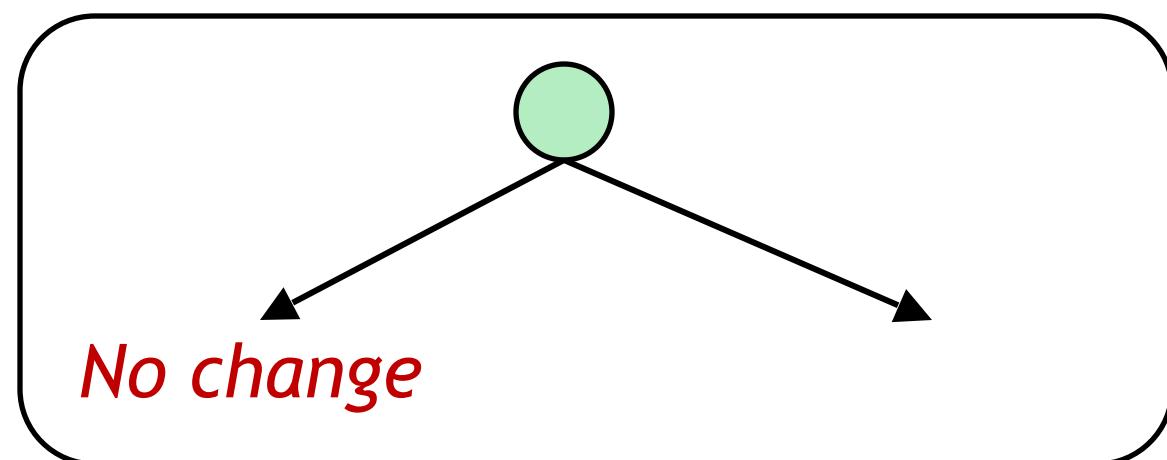
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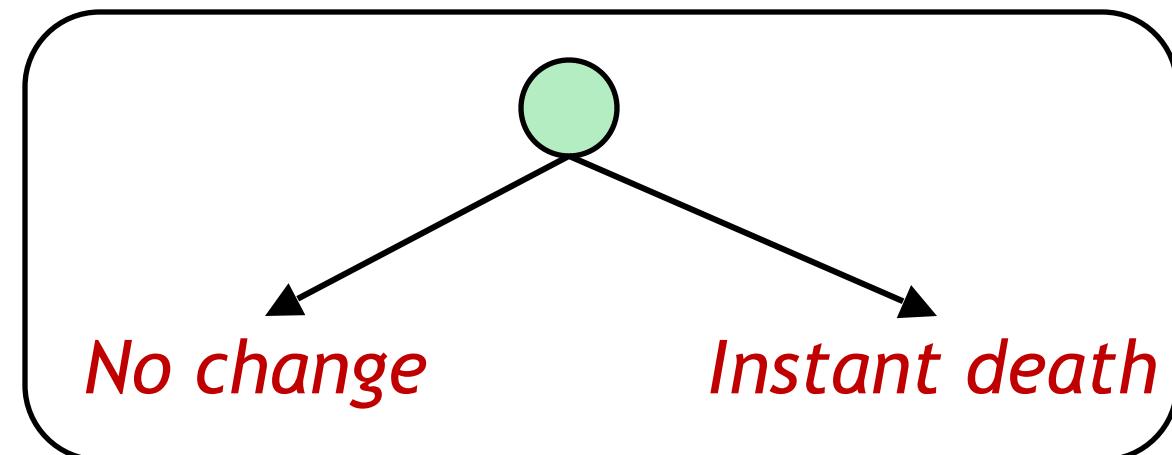
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- Utilities map states to real numbers. Which numbers?
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**Pay \$30**

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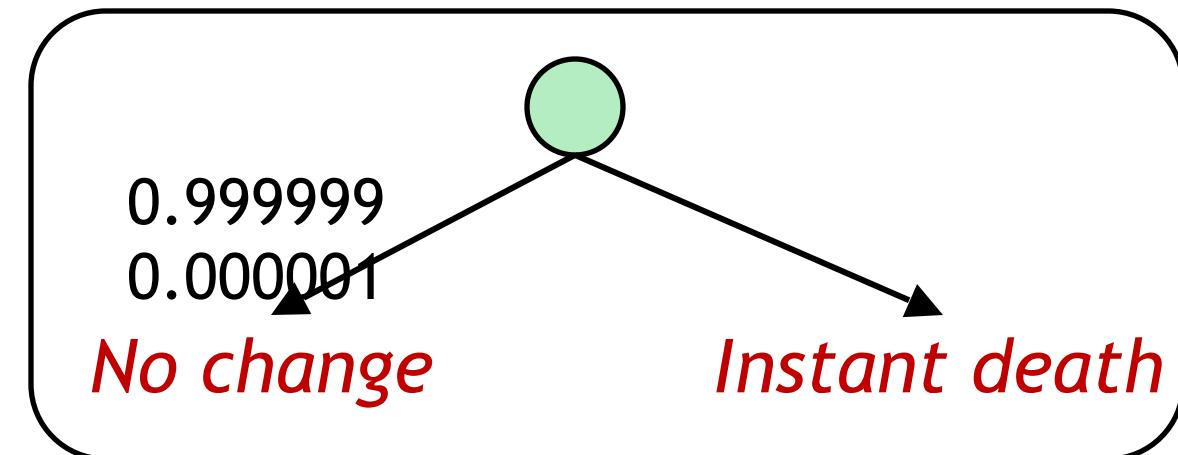
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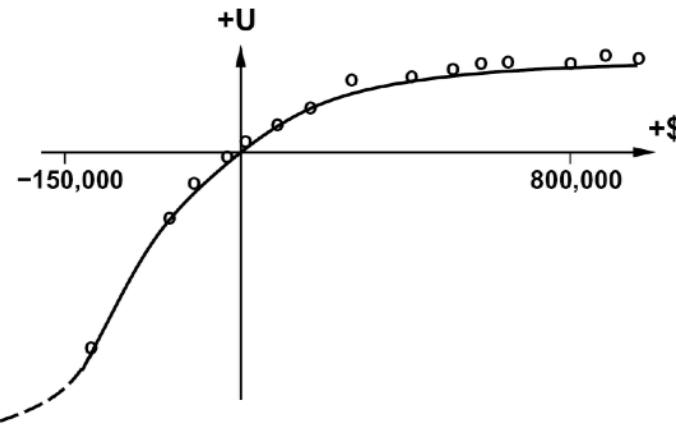
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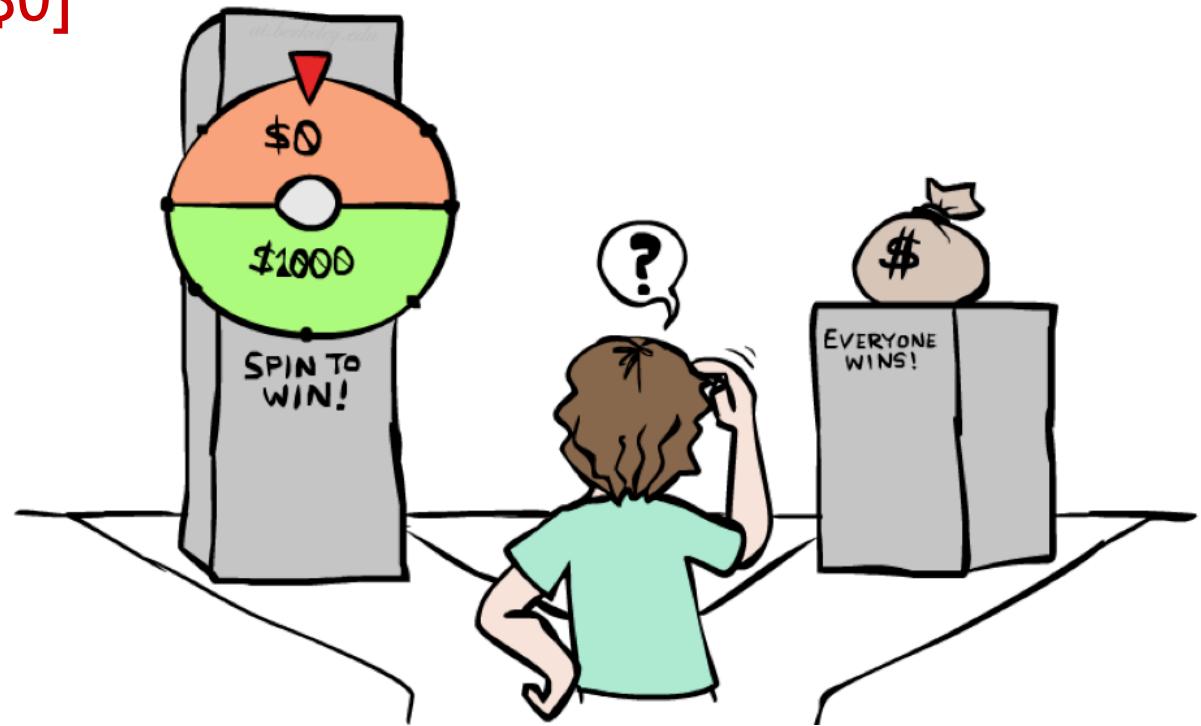
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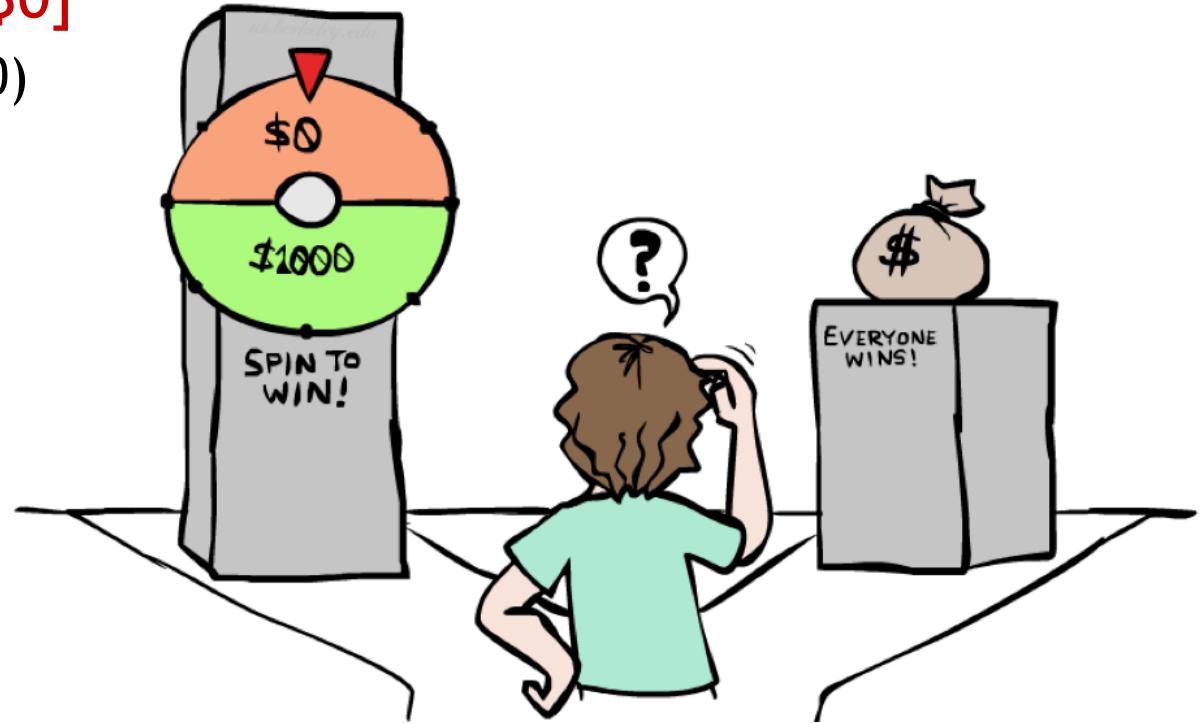
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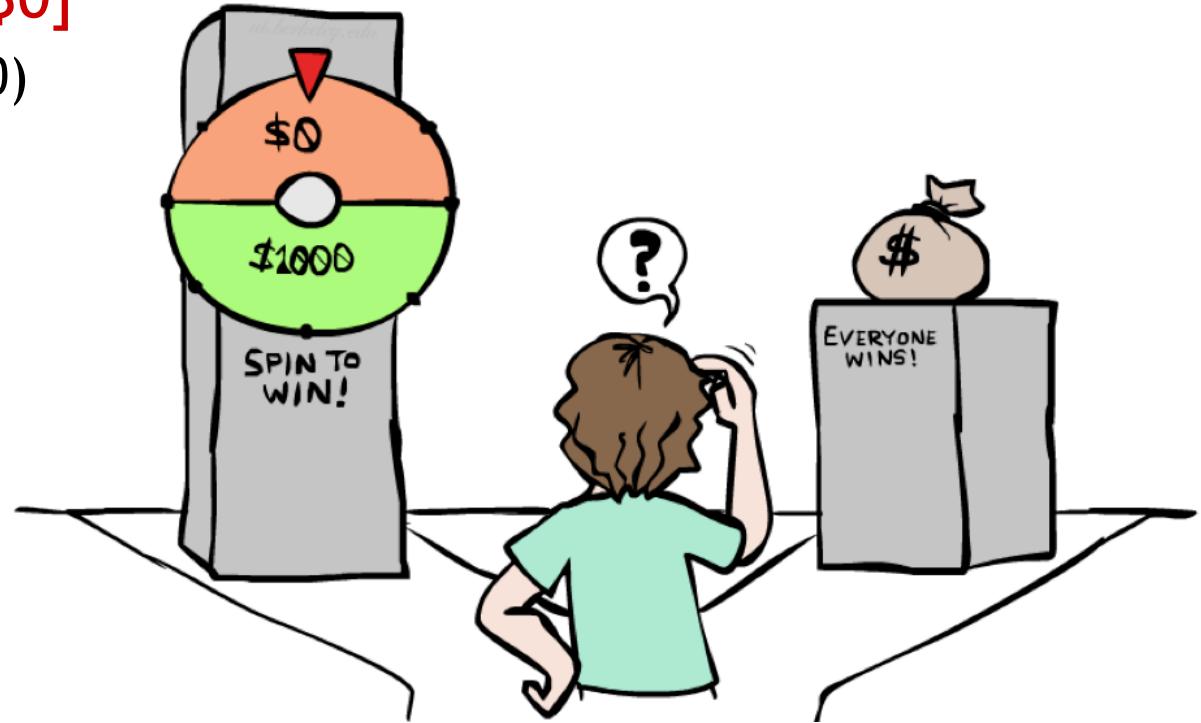
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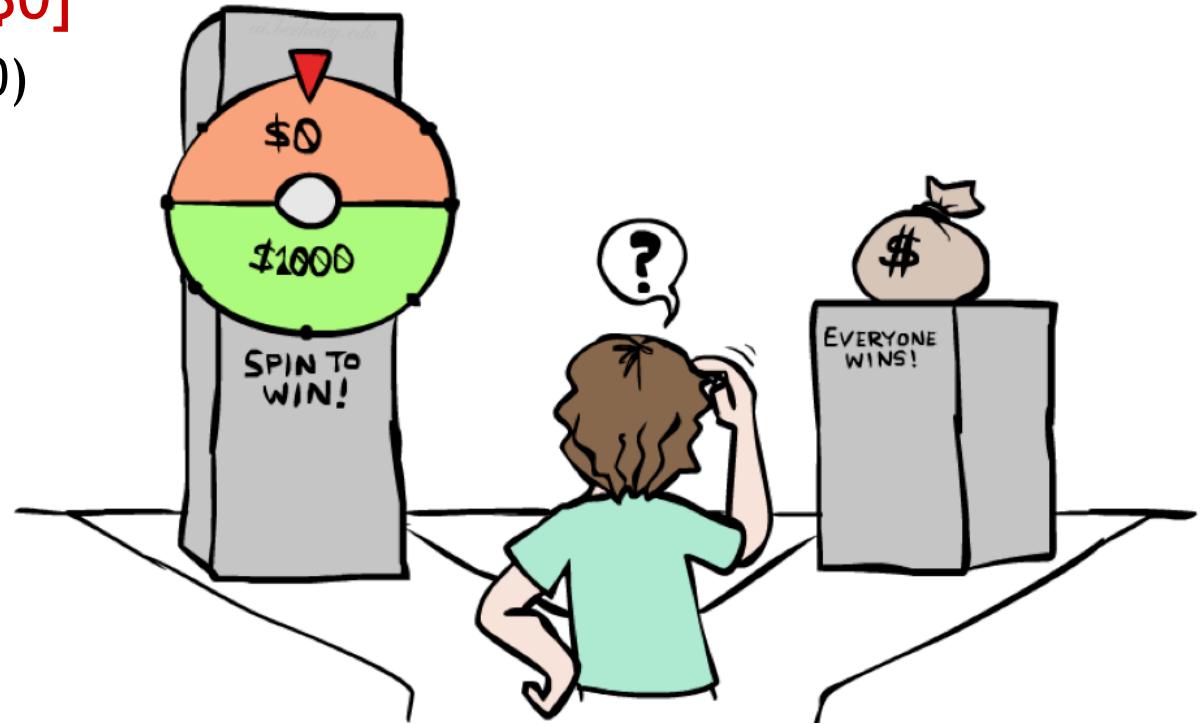
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- Consider the lottery  $[0.5, \$1000; 0.5, \$0]$ 
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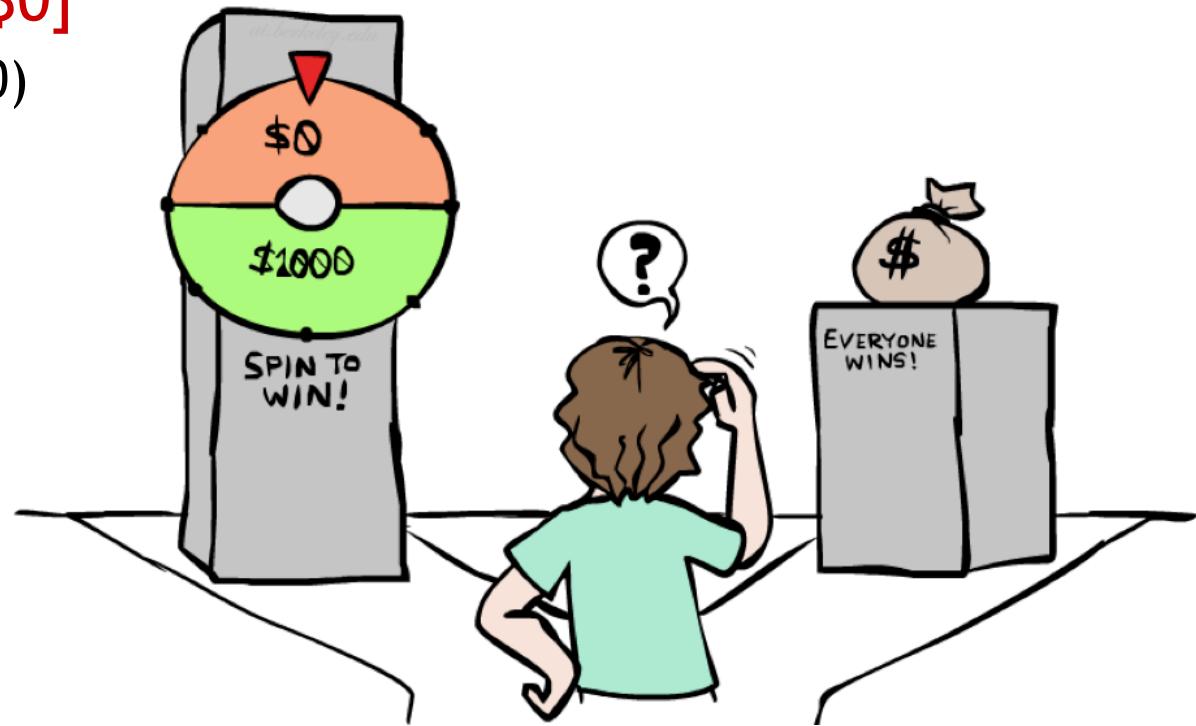
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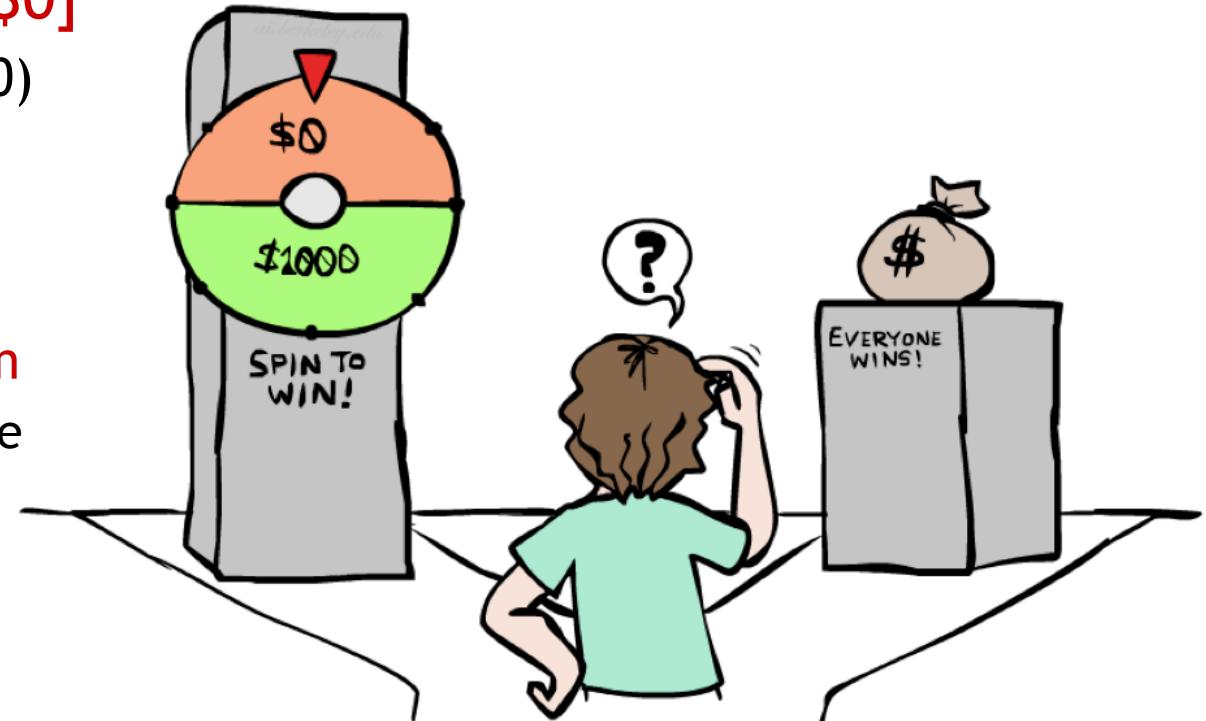
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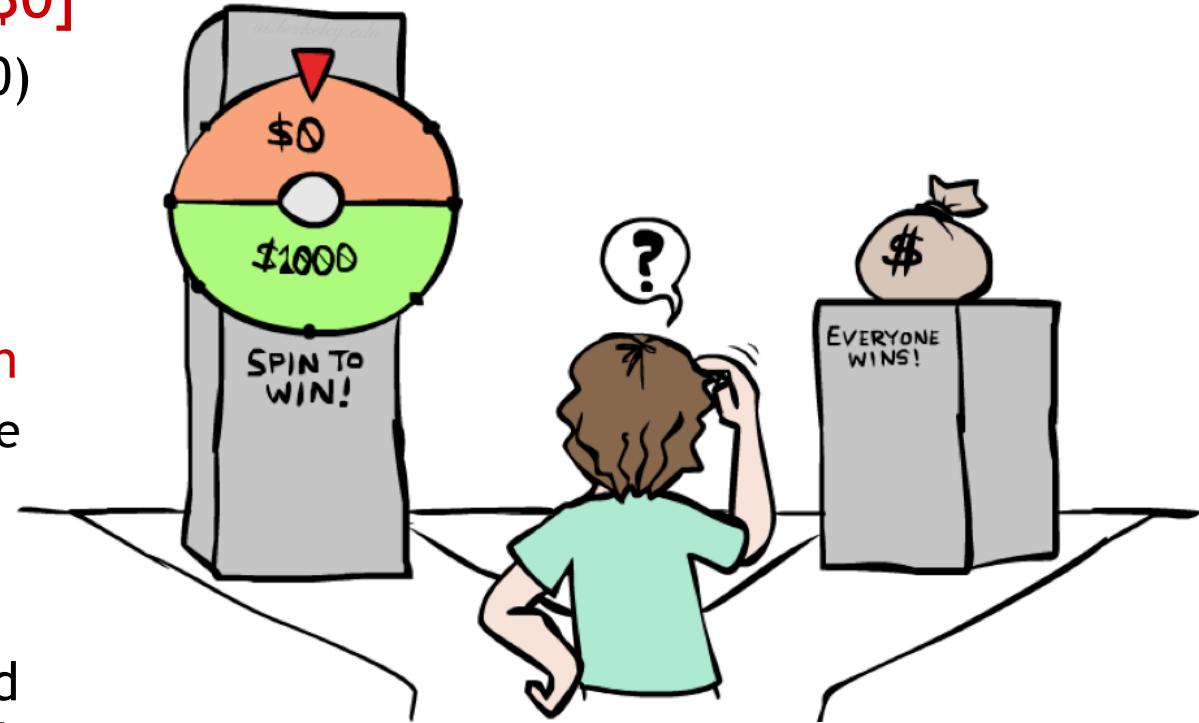
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    - If everyone were risk-neutral, no insurance needed!
  - It's win-win: you'd rather have the  $\$400$  and the insurance company would rather have the lottery (their utility curve is flat and they have many lotteries)



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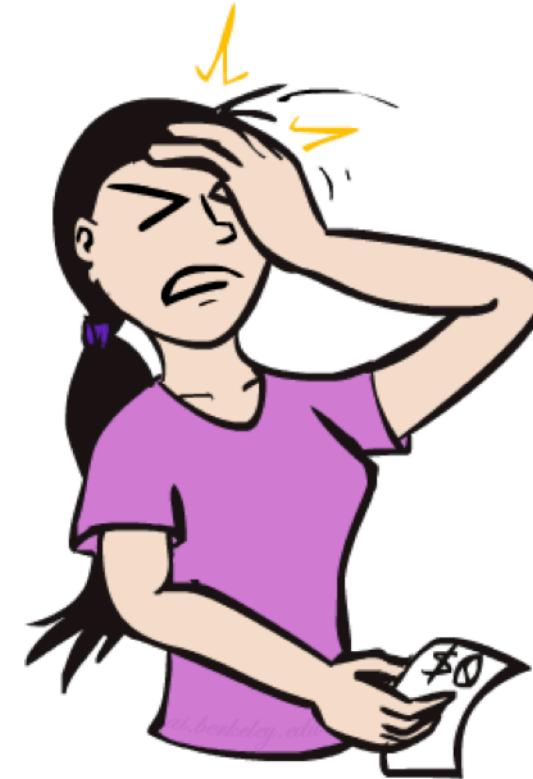
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# Next Time: MDPs!

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