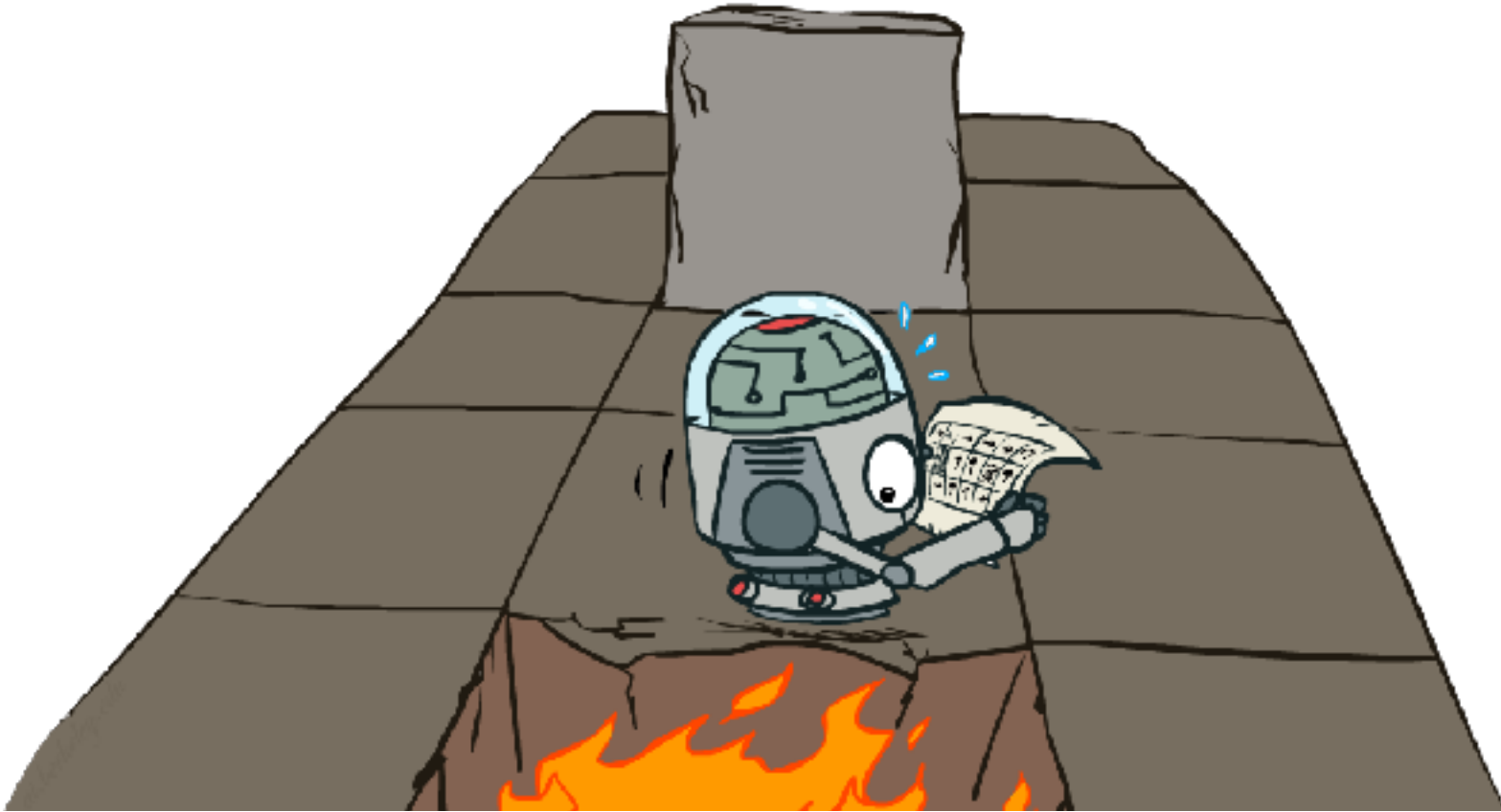


CS 5522: Artificial Intelligence II

Markov Decision Processes II



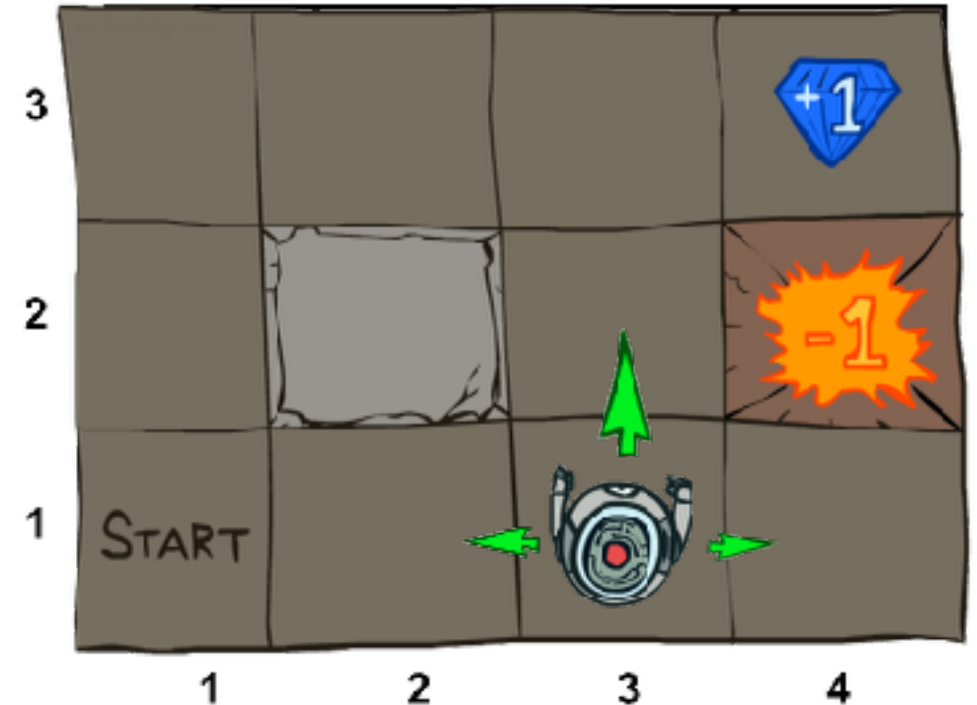
Instructor: Alan Ritter

Ohio State University

[These slides were adapted from CS188 Intro to AI at UC Berkeley. All materials available at <http://ai.berkeley.edu>.]

Example: Grid World

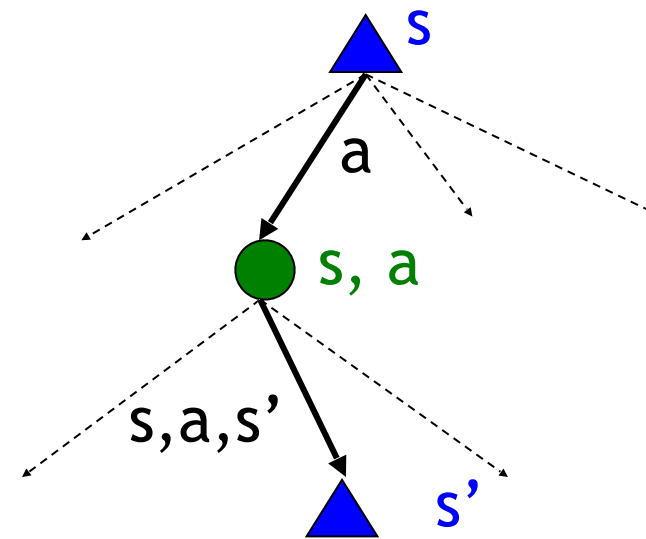
- A maze-like problem
 - The agent lives in a grid
 - Walls block the agent's path
- Noisy movement: actions do not always go as planned
 - 80% of the time, the action North takes the agent North
 - 10% of the time, North takes the agent West; 10% East
 - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
 - Small “living” reward each step (can be negative)
 - Big rewards come at the end (good or bad)
- Goal: maximize sum of (discounted) rewards



Recap: MDPs

- Markov decision processes:

- States S
- Actions A
- Transitions $P(s' | s, a)$ (or $T(s, a, s')$)
- Rewards $R(s, a, s')$ (and discount γ)
- Start state s_0

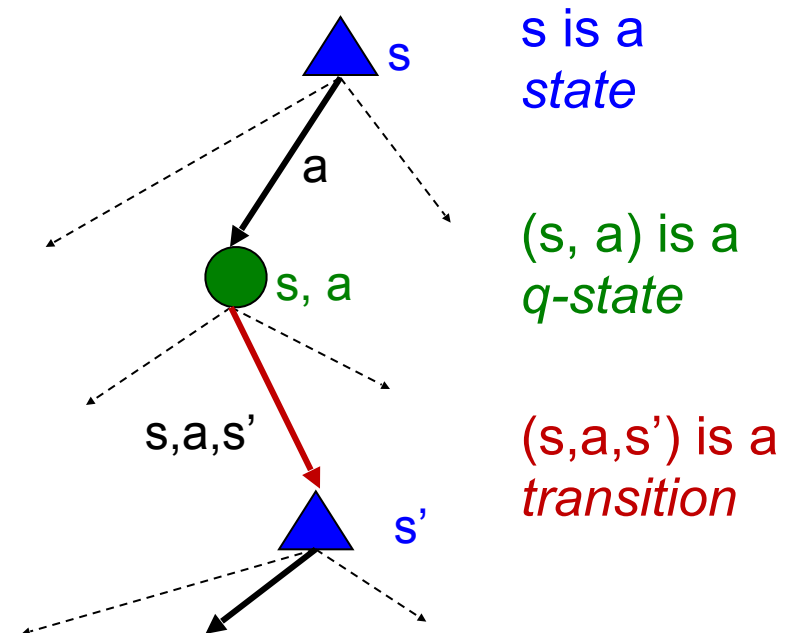


- Quantities:

- Policy = map of states to actions
- Utility = sum of discounted rewards
- Values = expected future utility from a state (max node)
- Q-Values = expected future utility from a q-state (chance node)

Optimal Quantities

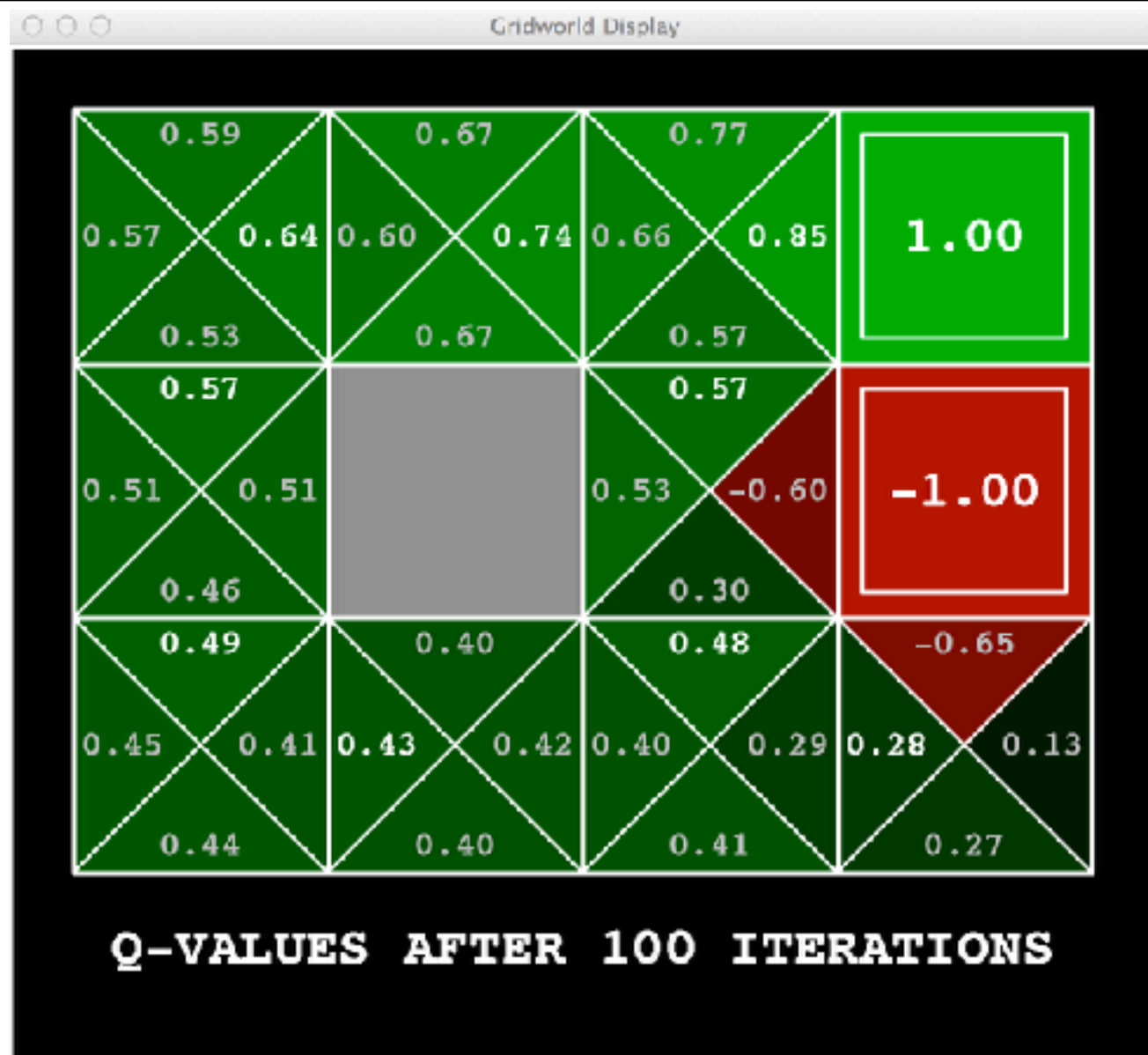
- The value (utility) of a state s :
 $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a) :
 $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
 $\pi^*(s)$ = optimal action from state s



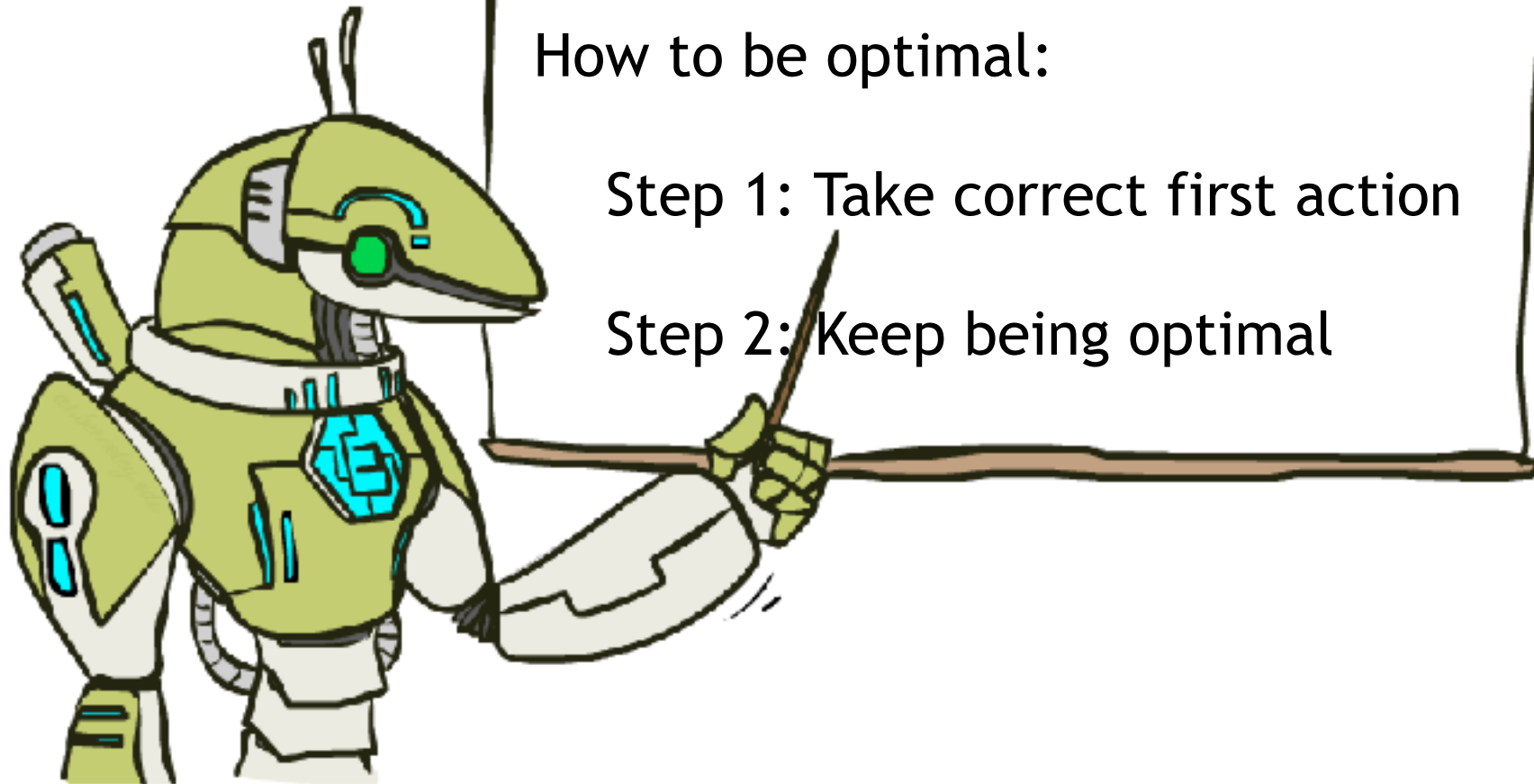
Gridworld Values V^*



Gridworld: Q^*

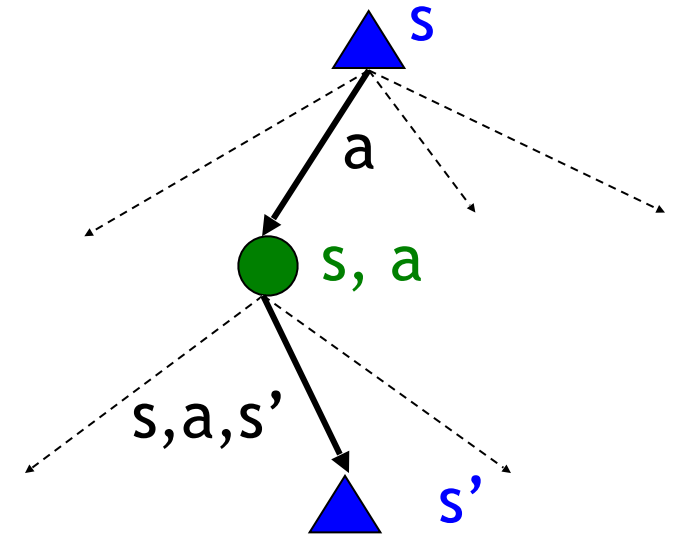


The Bellman Equations



The Bellman Equations

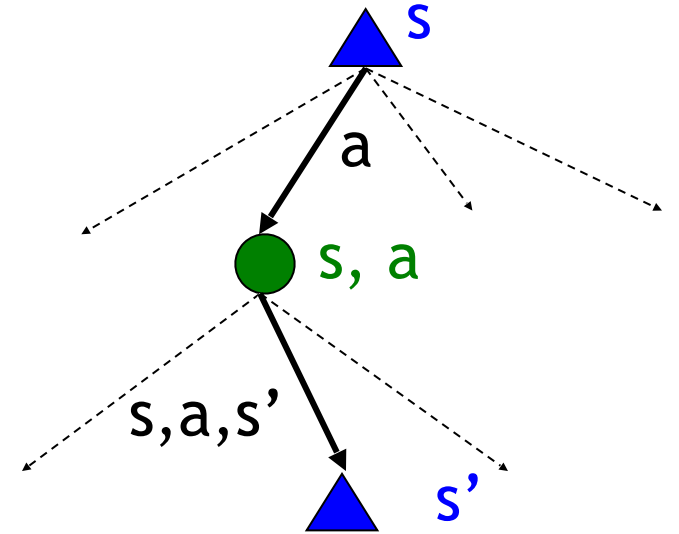
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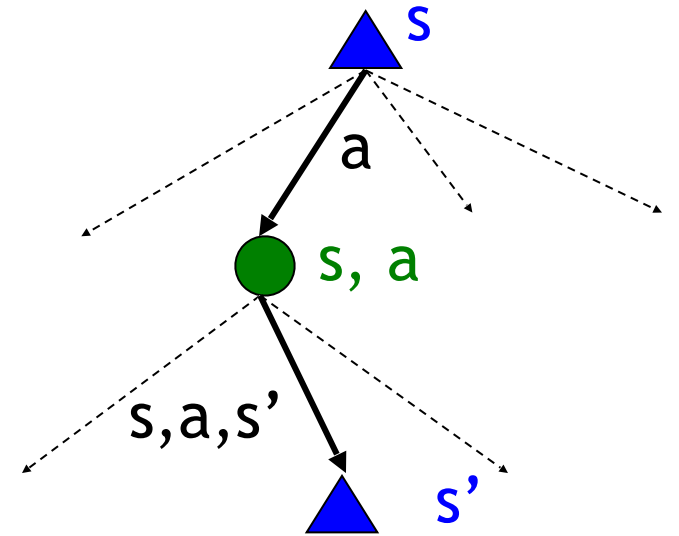


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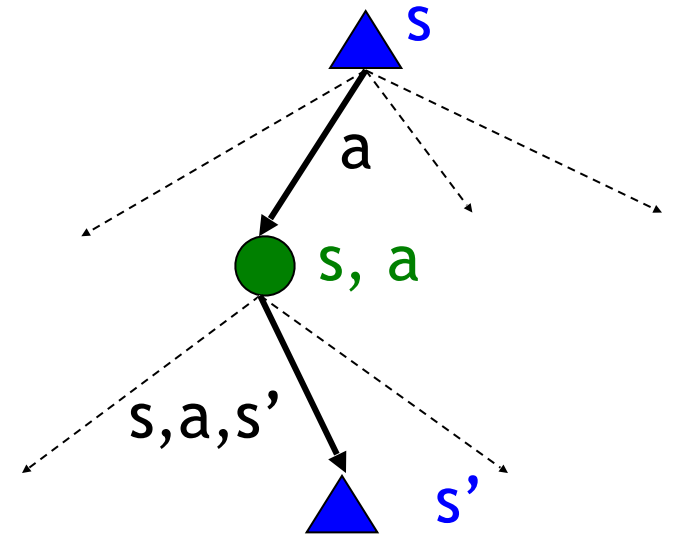
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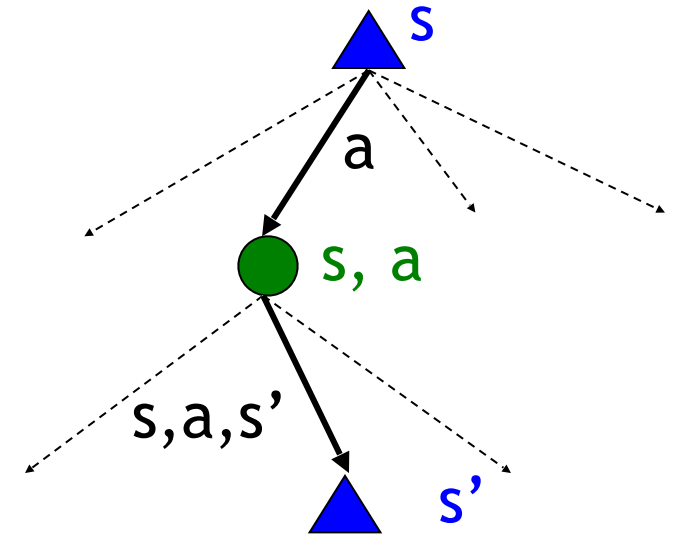
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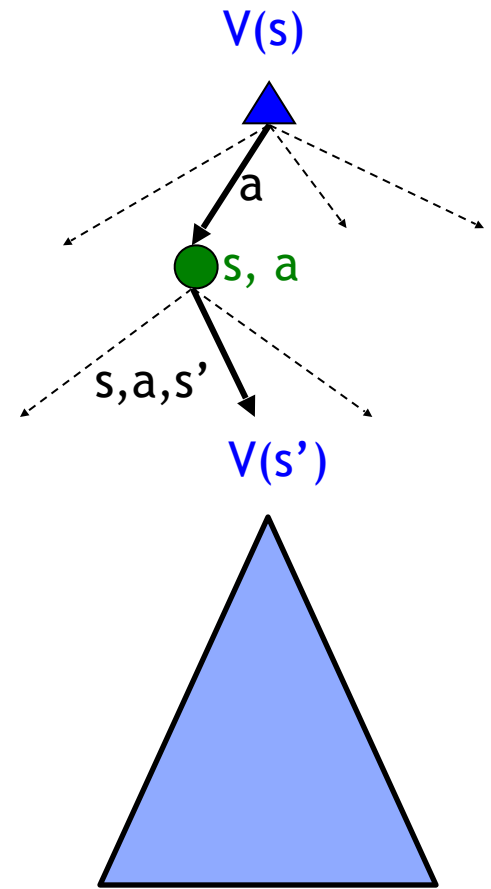
- These are the Bellman equations, and they characterize optimal values in a way we'll use over and over



Value Iteration

Value Iteration

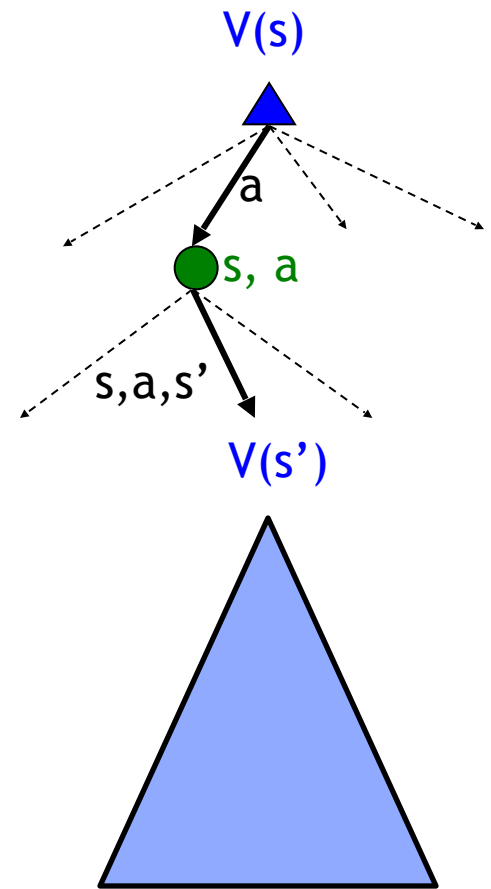
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Value Iteration

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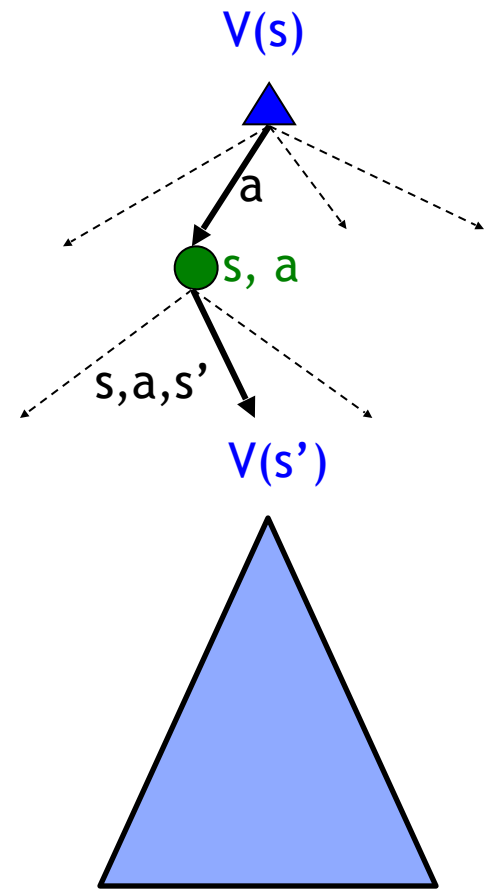


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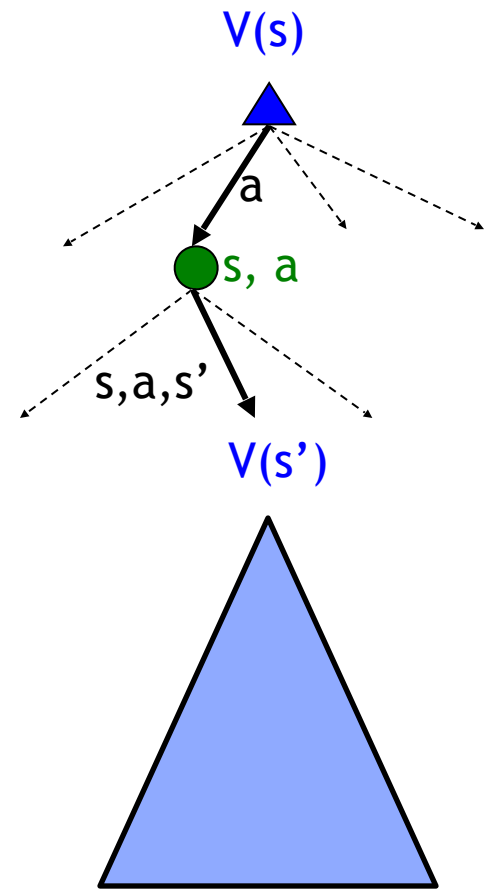
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Value Iteration

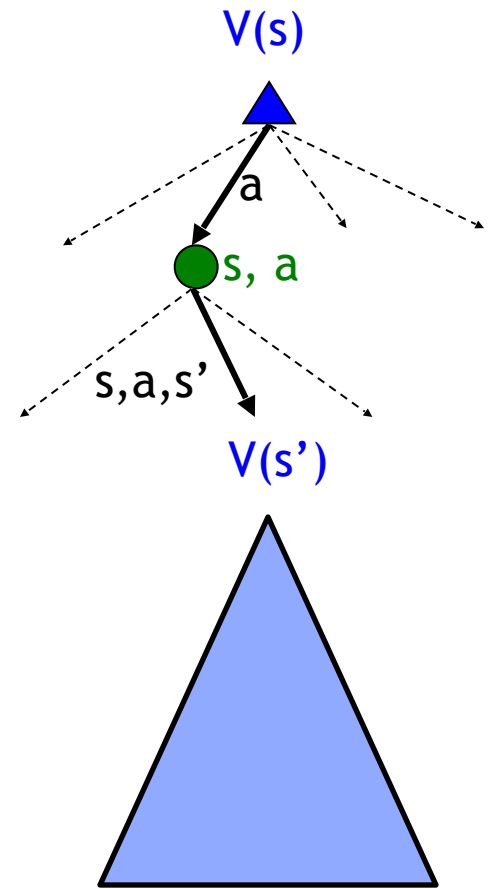
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- Value iteration is just a fixed point solution method
 - ... though the V_k vectors are also interpretable as time-limited values



Convergence*

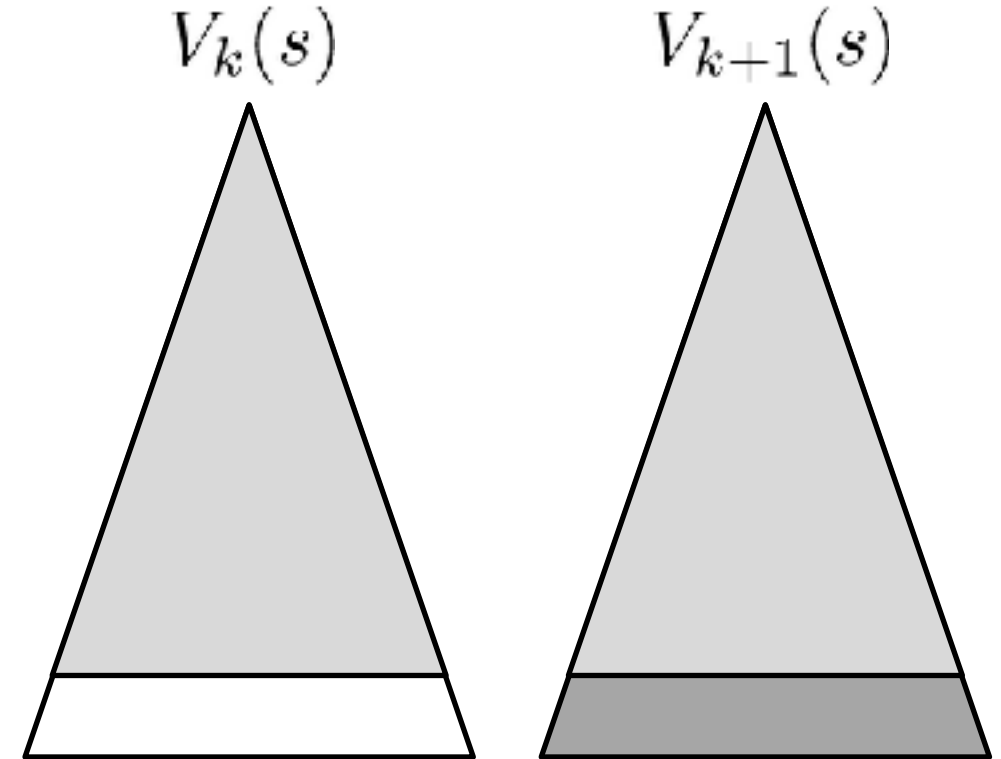
- How do we know the V_k vectors are going to converge?

Convergence*

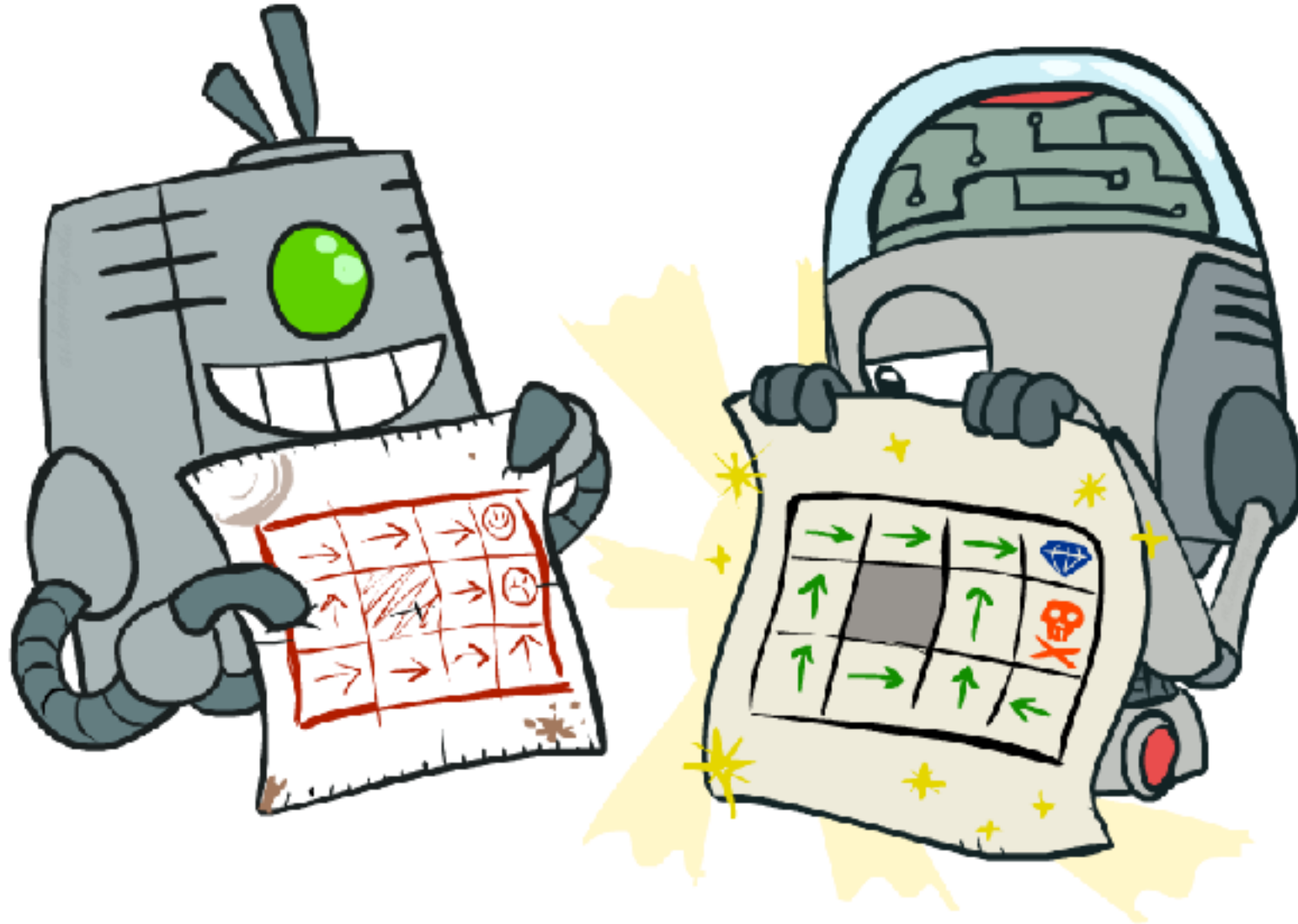
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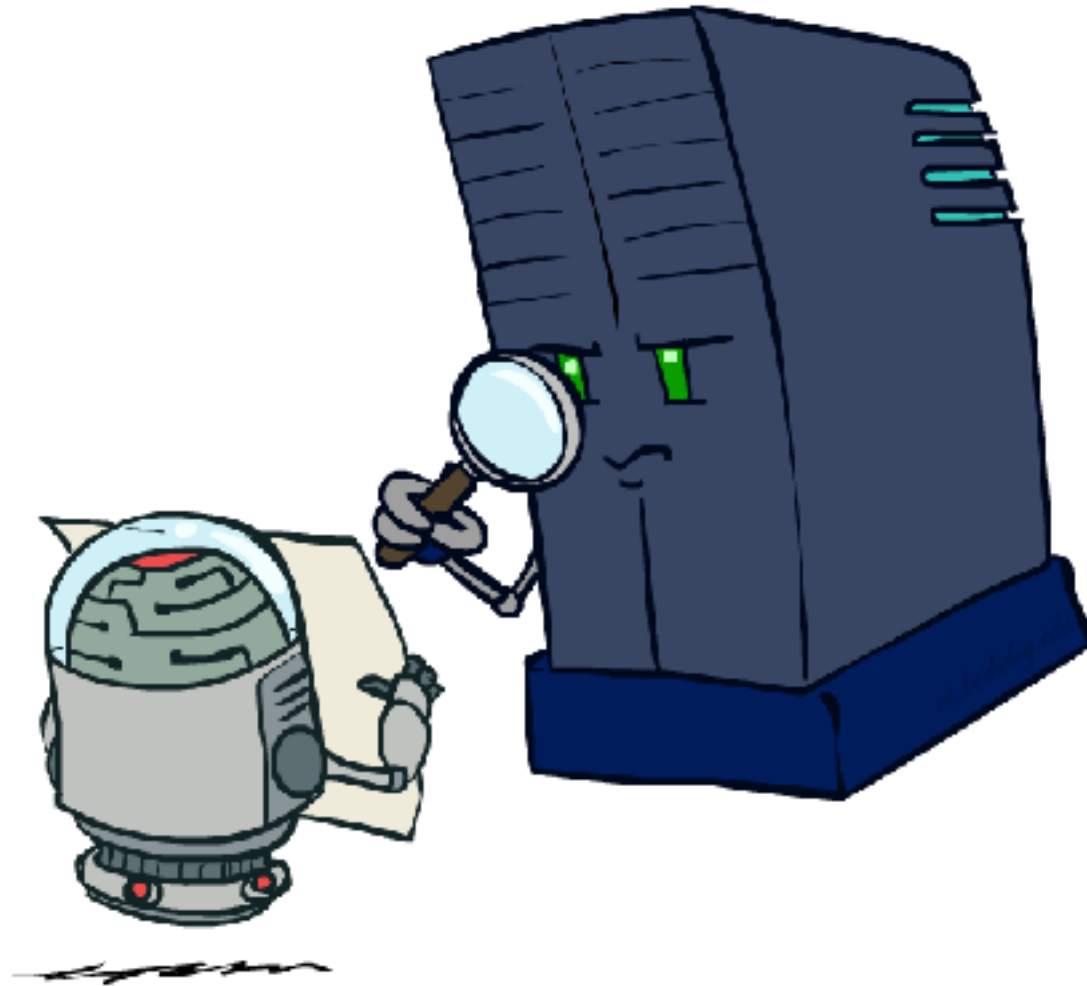
- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M , then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
 - Sketch: For any state V_k and V_{k+1} can be viewed as depth $k+1$ expectimax results in nearly identical search trees
 - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
 - That last layer is at best all R_{MAX}
 - It is at worst R_{MIN}
 - But everything is discounted by γ^k that far out
 - So V_k and V_{k+1} are at most $\gamma^k \max |R|$ different
 - So as k increases, the values converge



Policy Methods

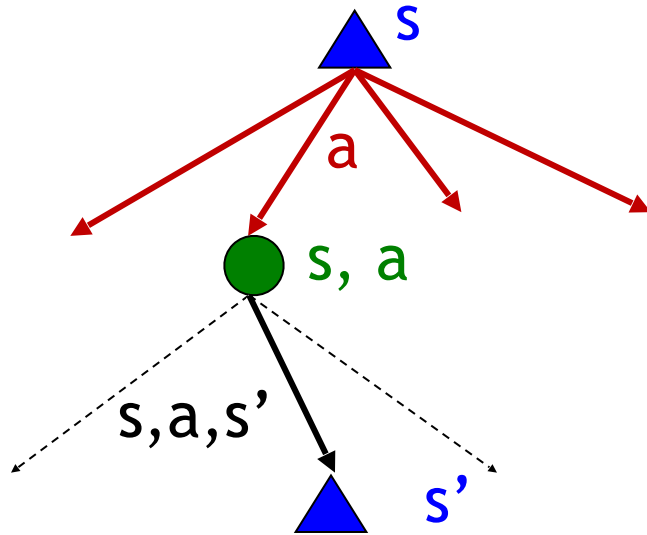


Policy Evaluation



Fixed Policies

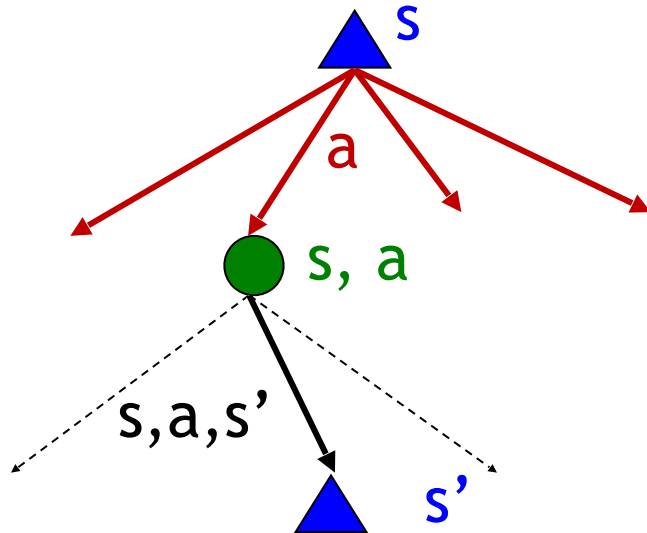
Do the optimal action



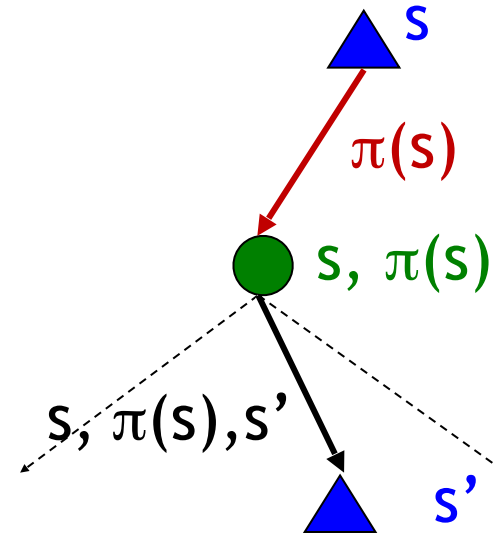
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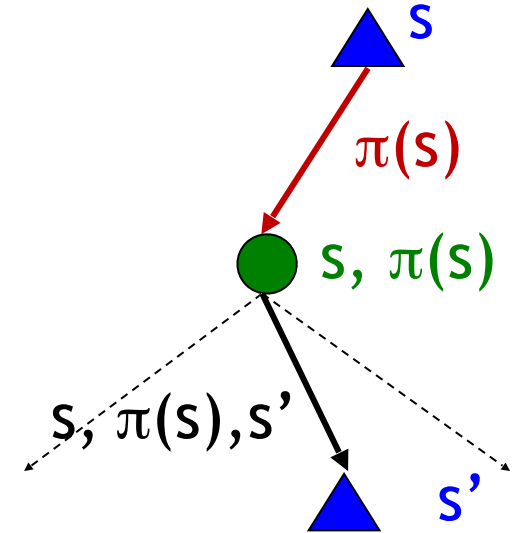


Do what π says to do



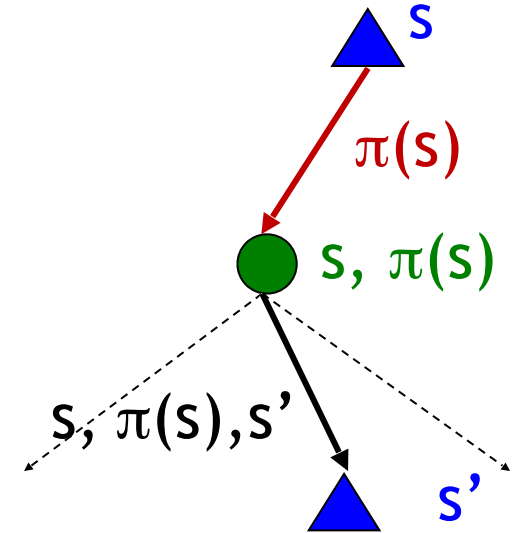
- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler - only one action per state
 - ... though the tree's value would depend on which policy we fixed

Utilities for a Fixed Policy



Utilities for a Fixed Policy

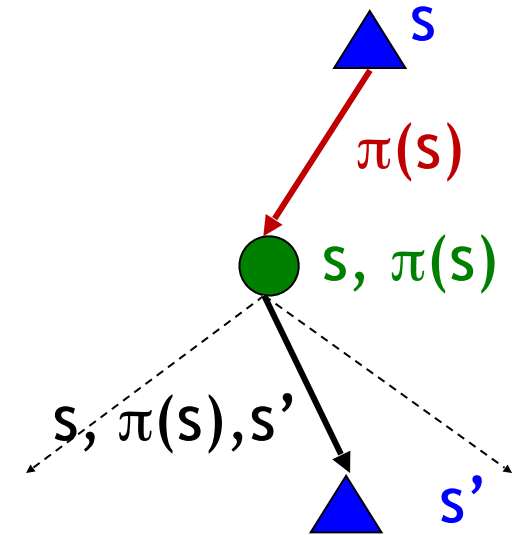
- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s , under a fixed policy π :
 $V_\pi(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):



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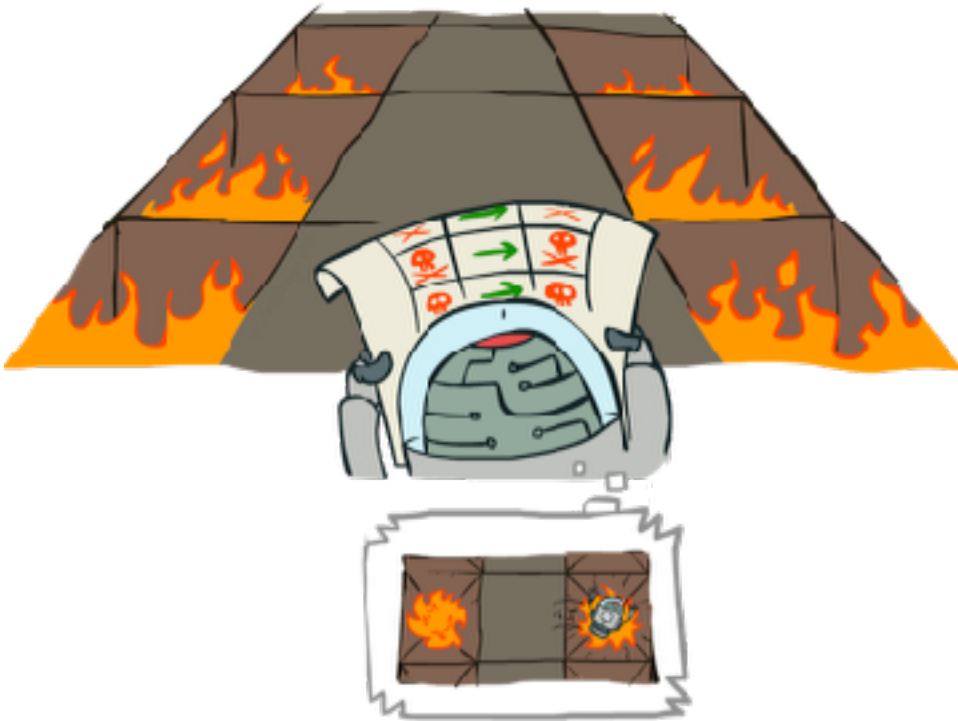
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Example: Policy Evaluation

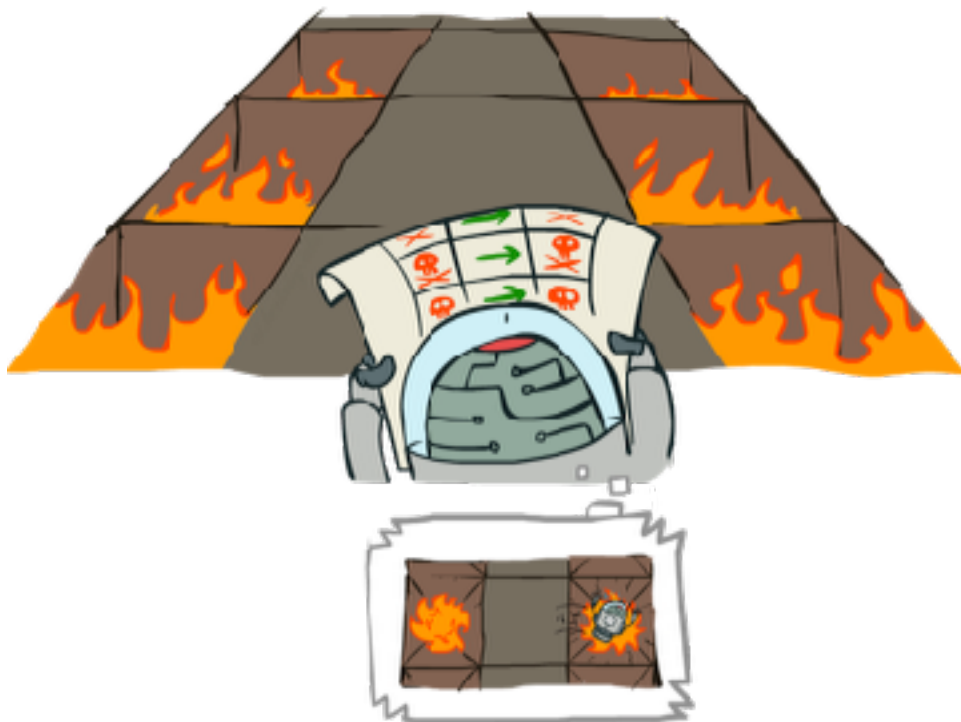
Example: Policy Evaluation

Always Go Right

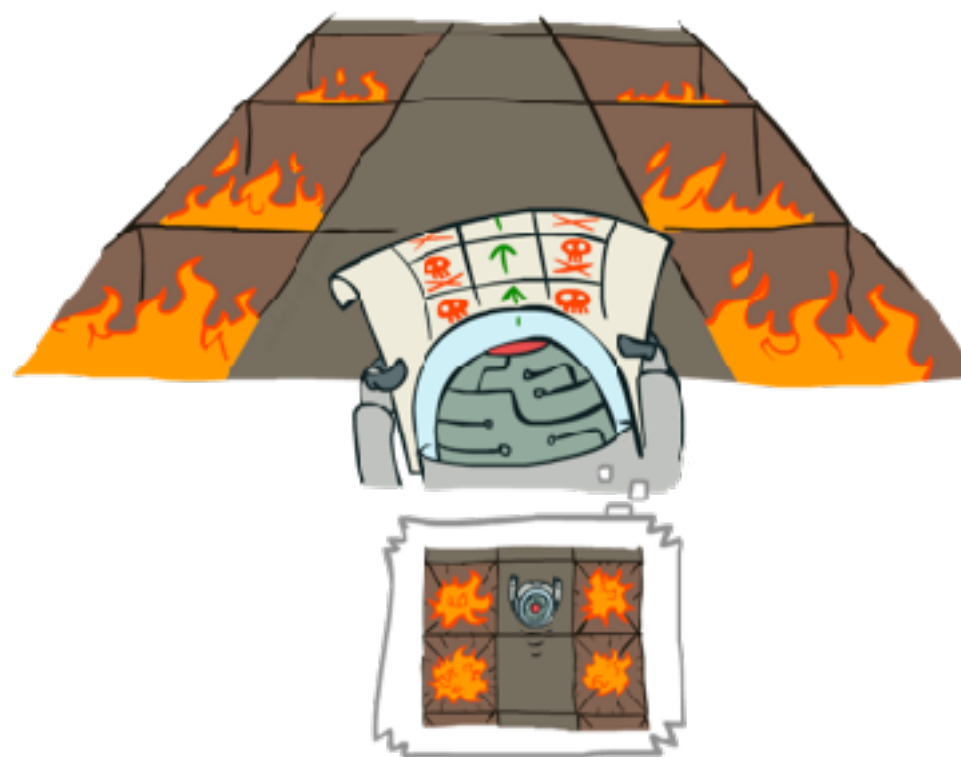


Example: Policy Evaluation

Always Go Right



Always Go Forward



Example: Policy Evaluation

Always Go Right

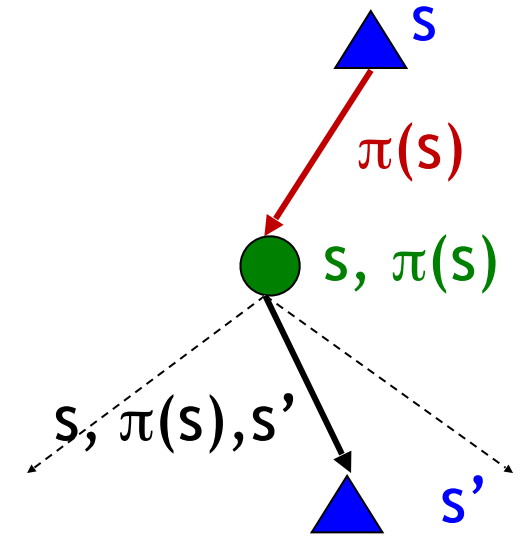


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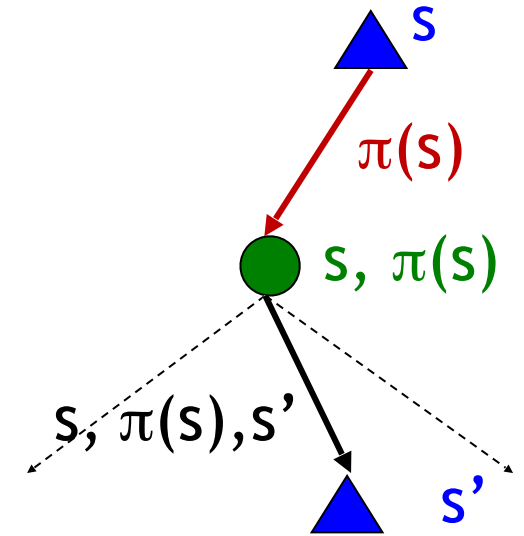
Policy Evaluation

- How do we calculate the V 's for a fixed policy π ?



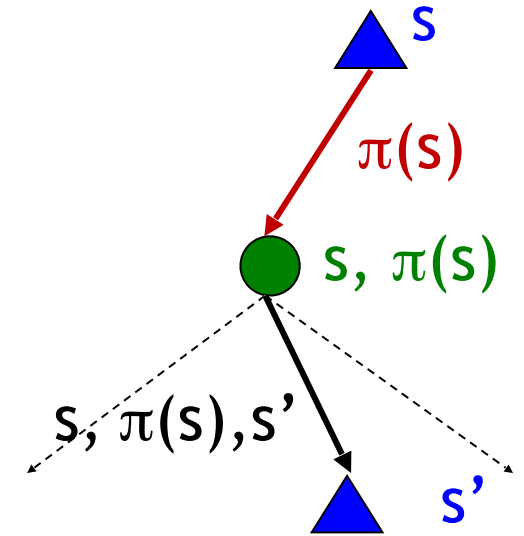
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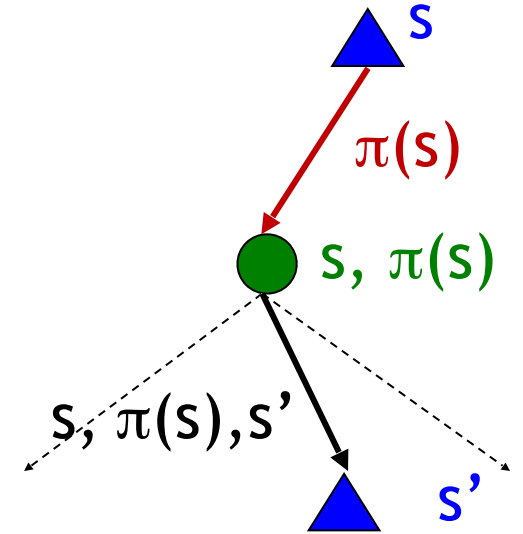
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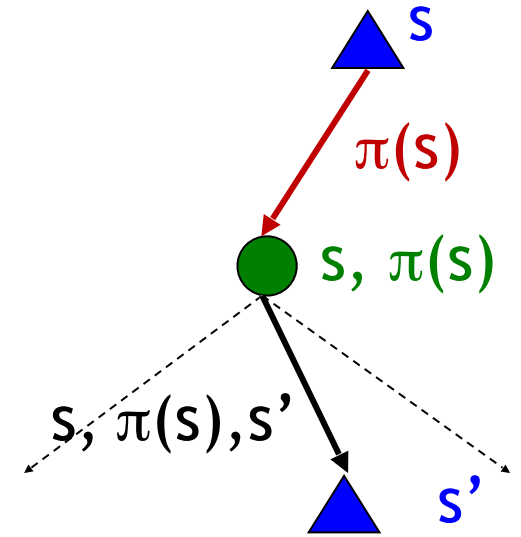


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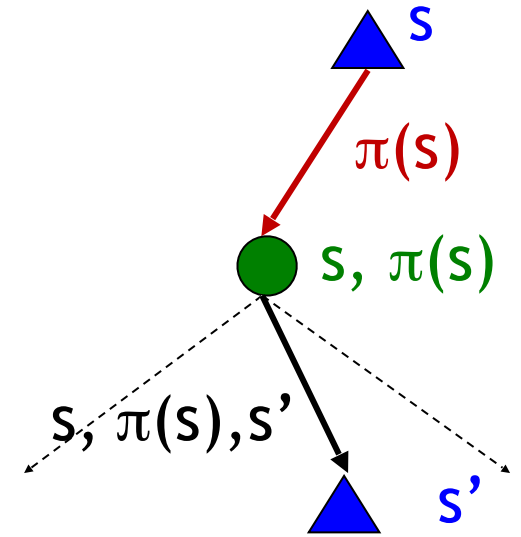
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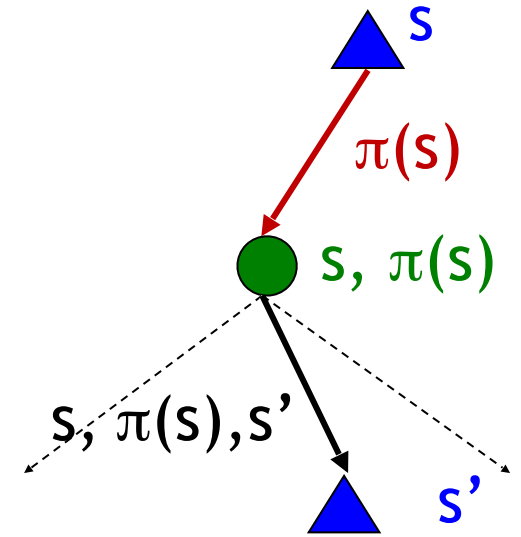
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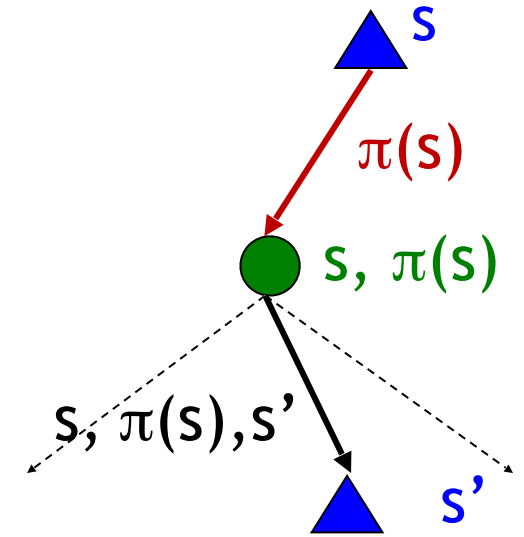
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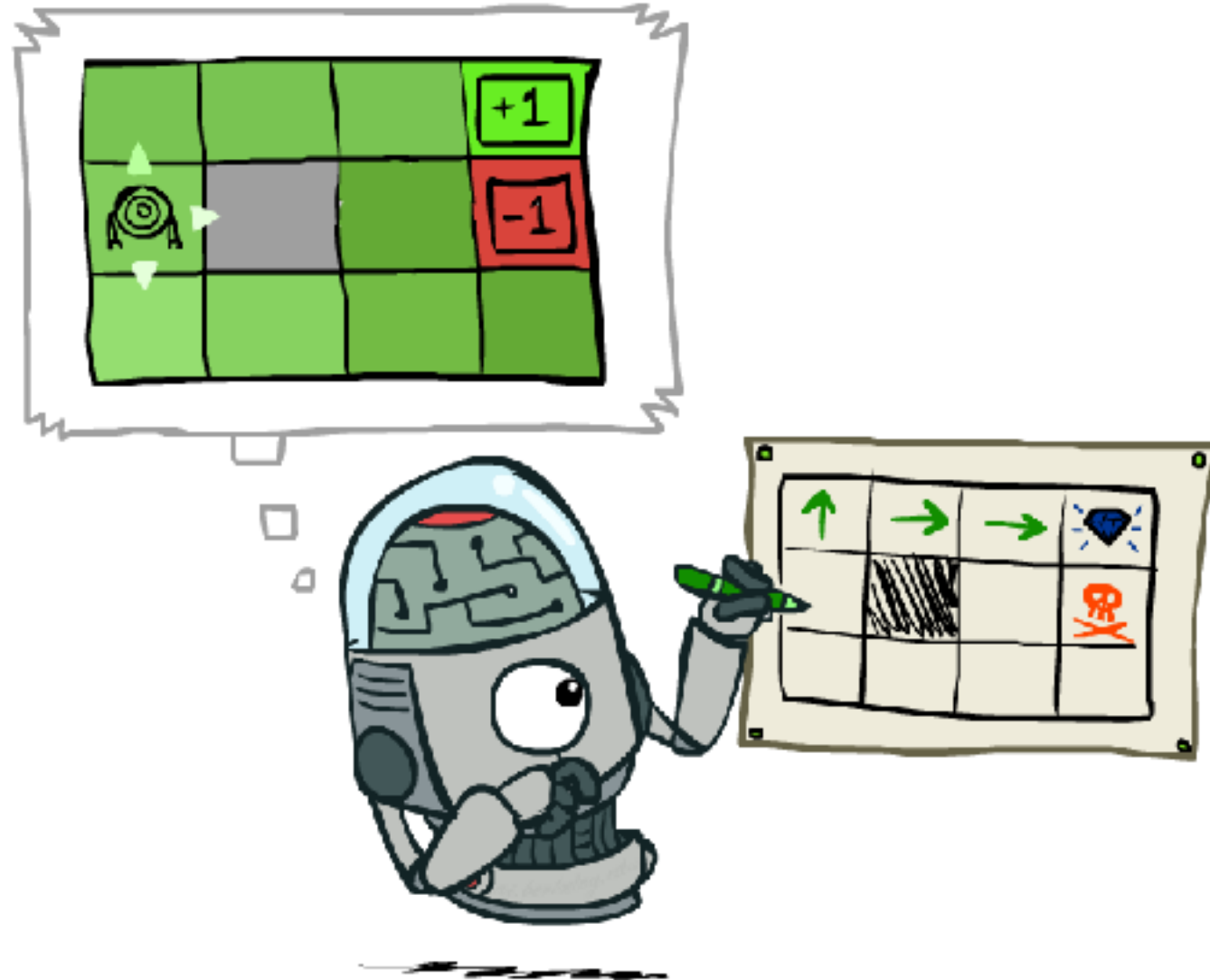
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 - Solve with Matlab (or your favorite linear system solver)

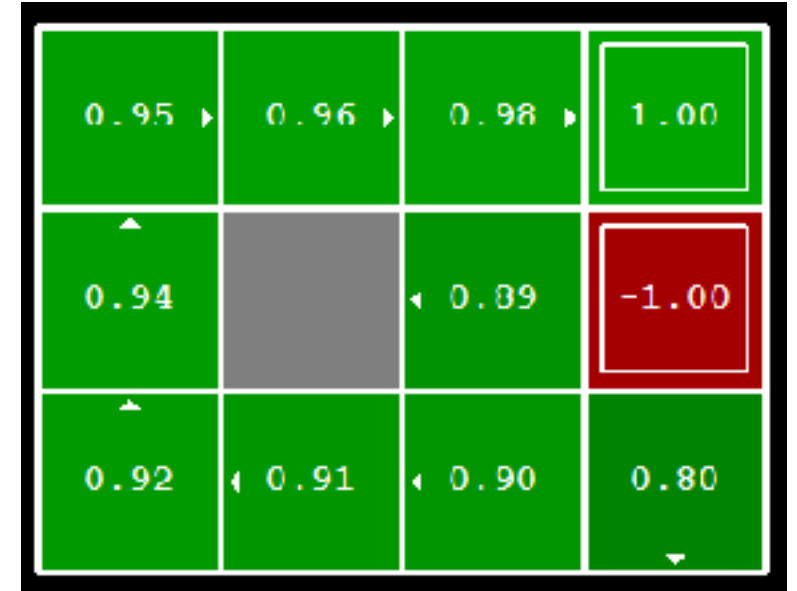


Policy Extraction



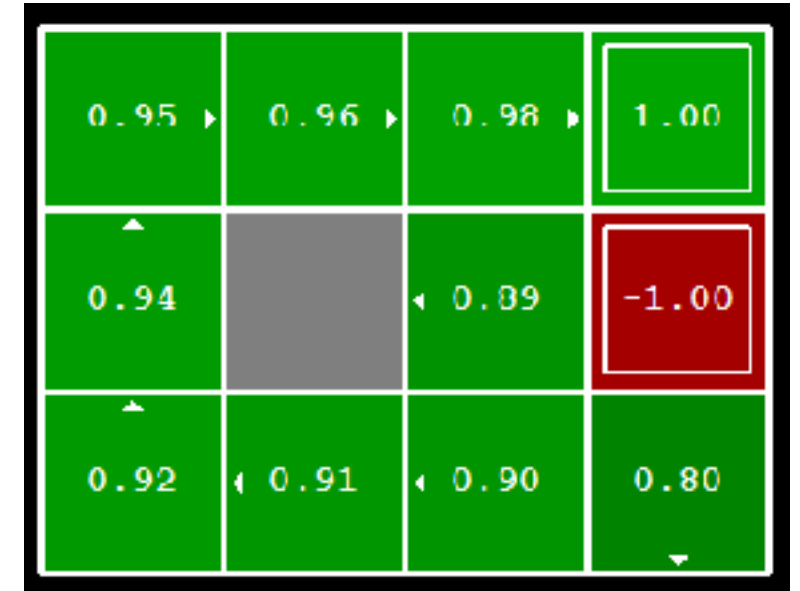
Computing Actions from Values

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Computing Actions from Values

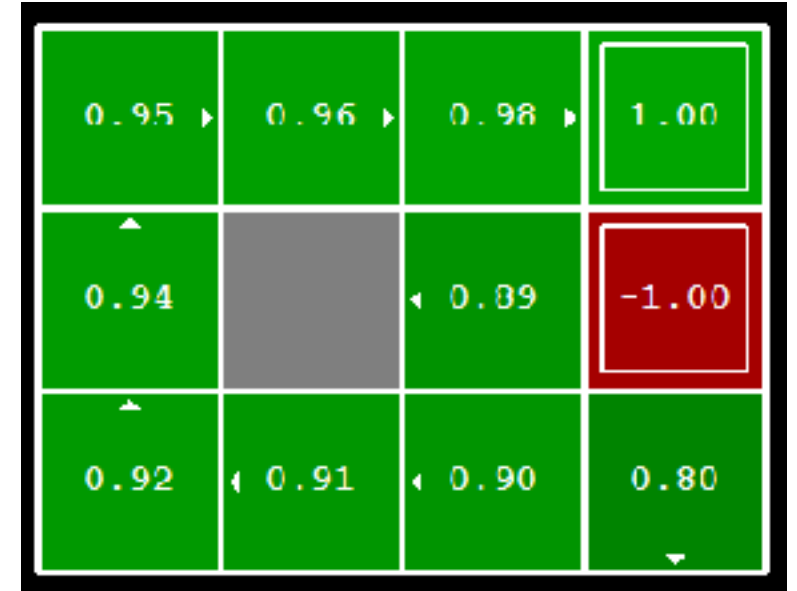
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Computing Actions from Values

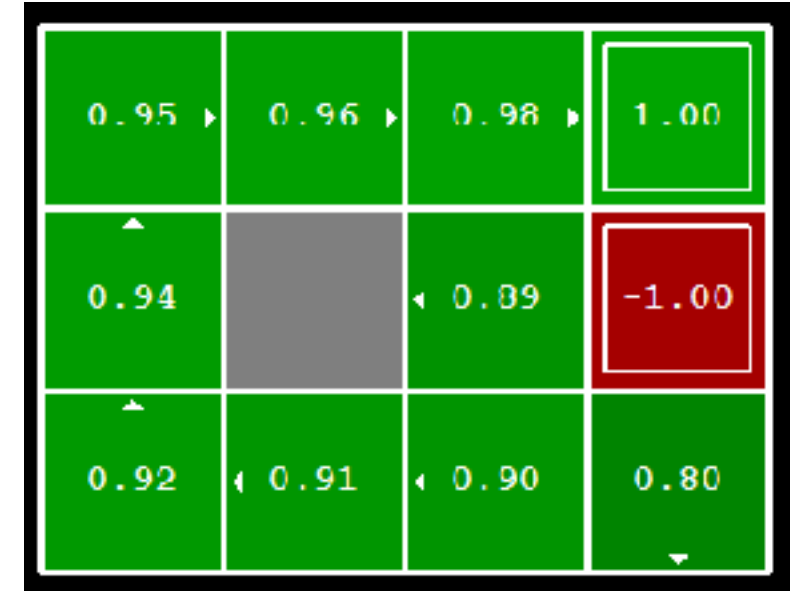
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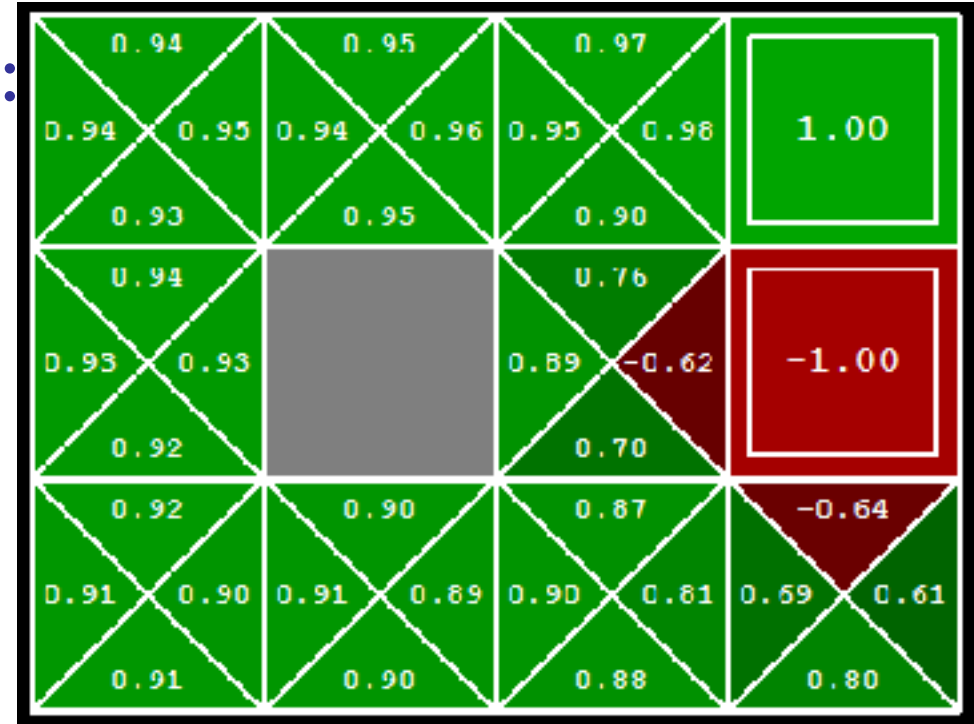


$$\pi^*(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- This is called **policy extraction**, since it gets the policy implied by the values

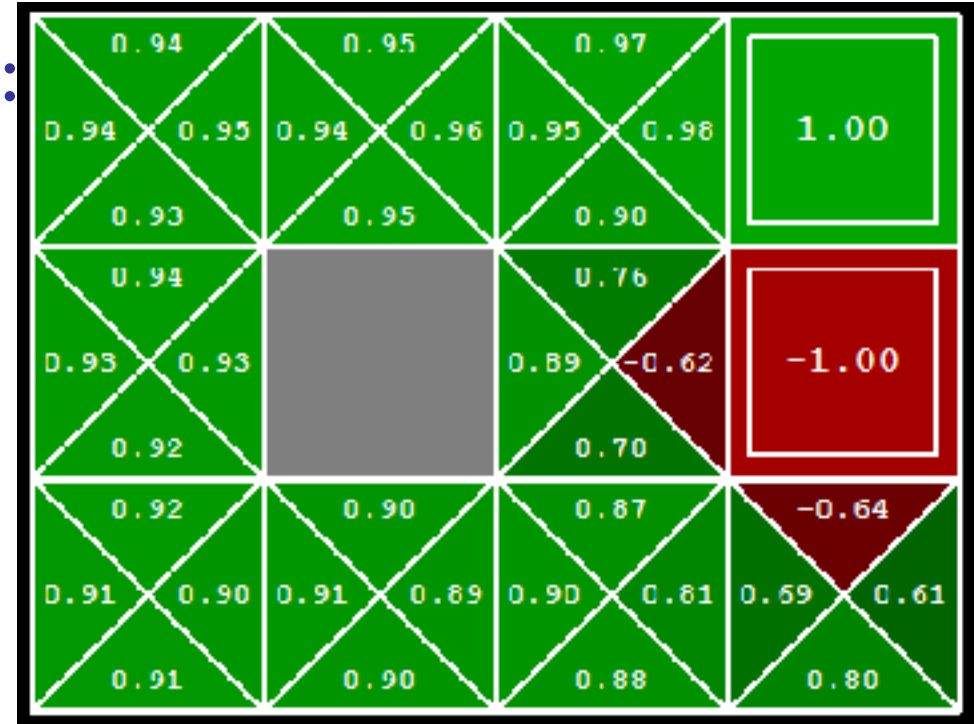
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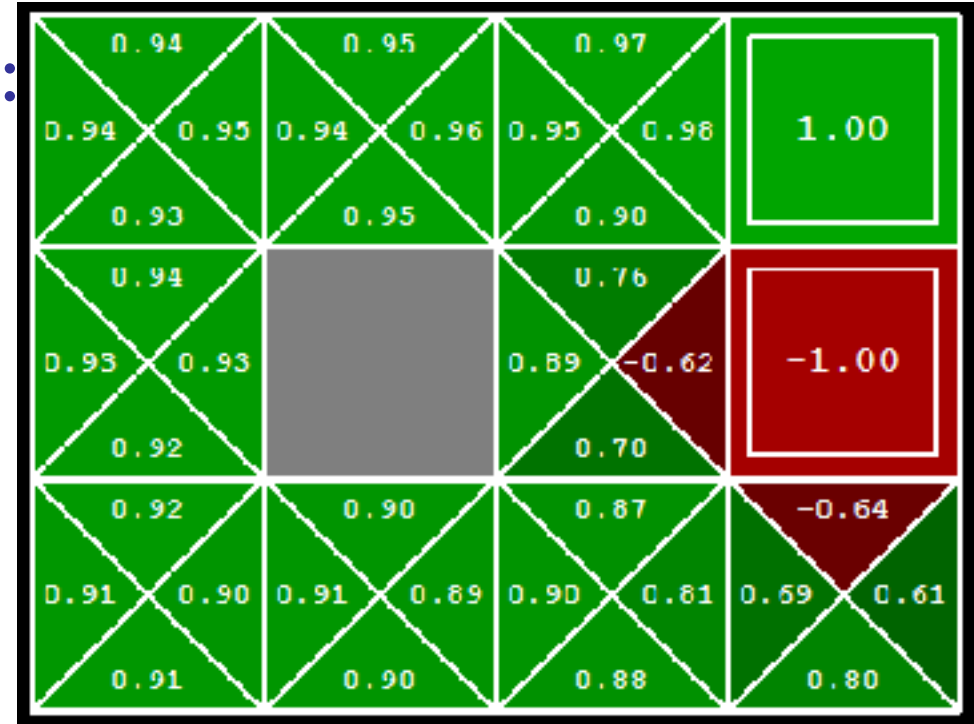
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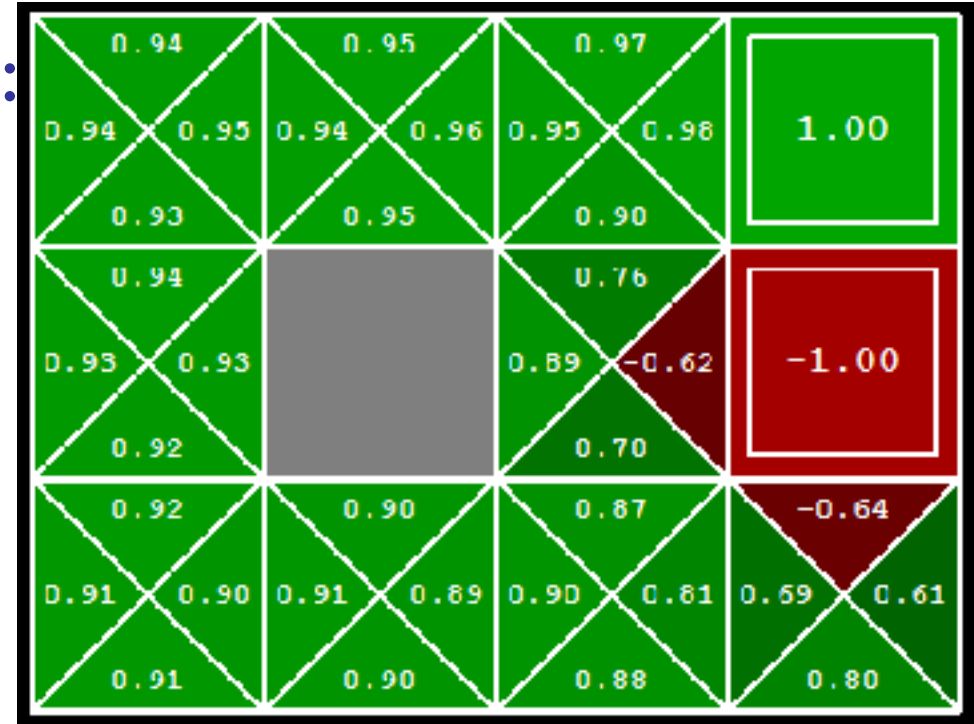
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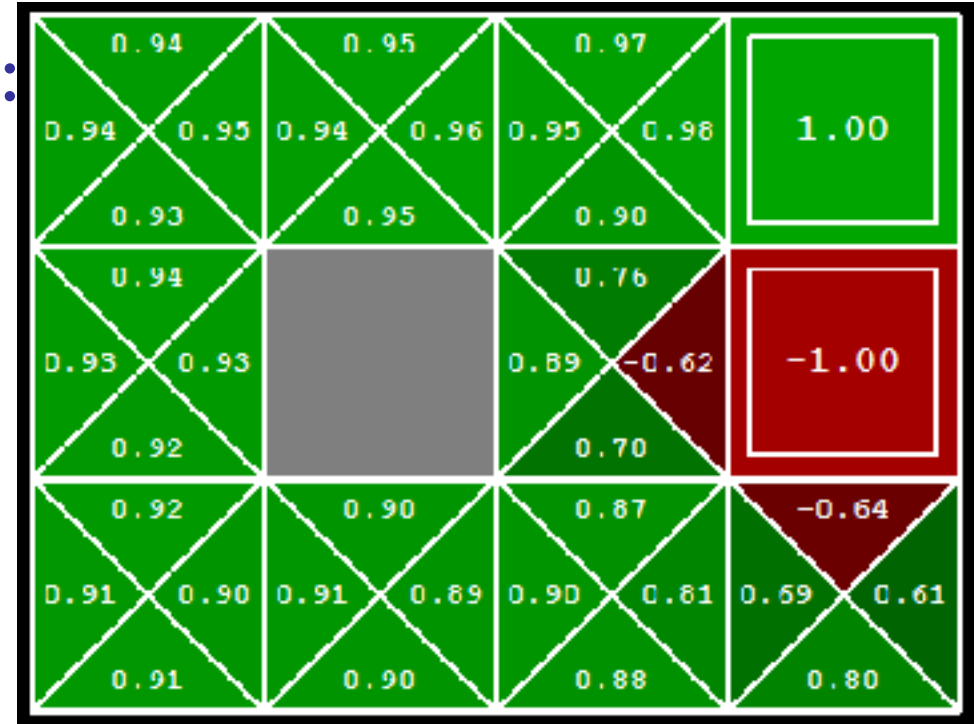
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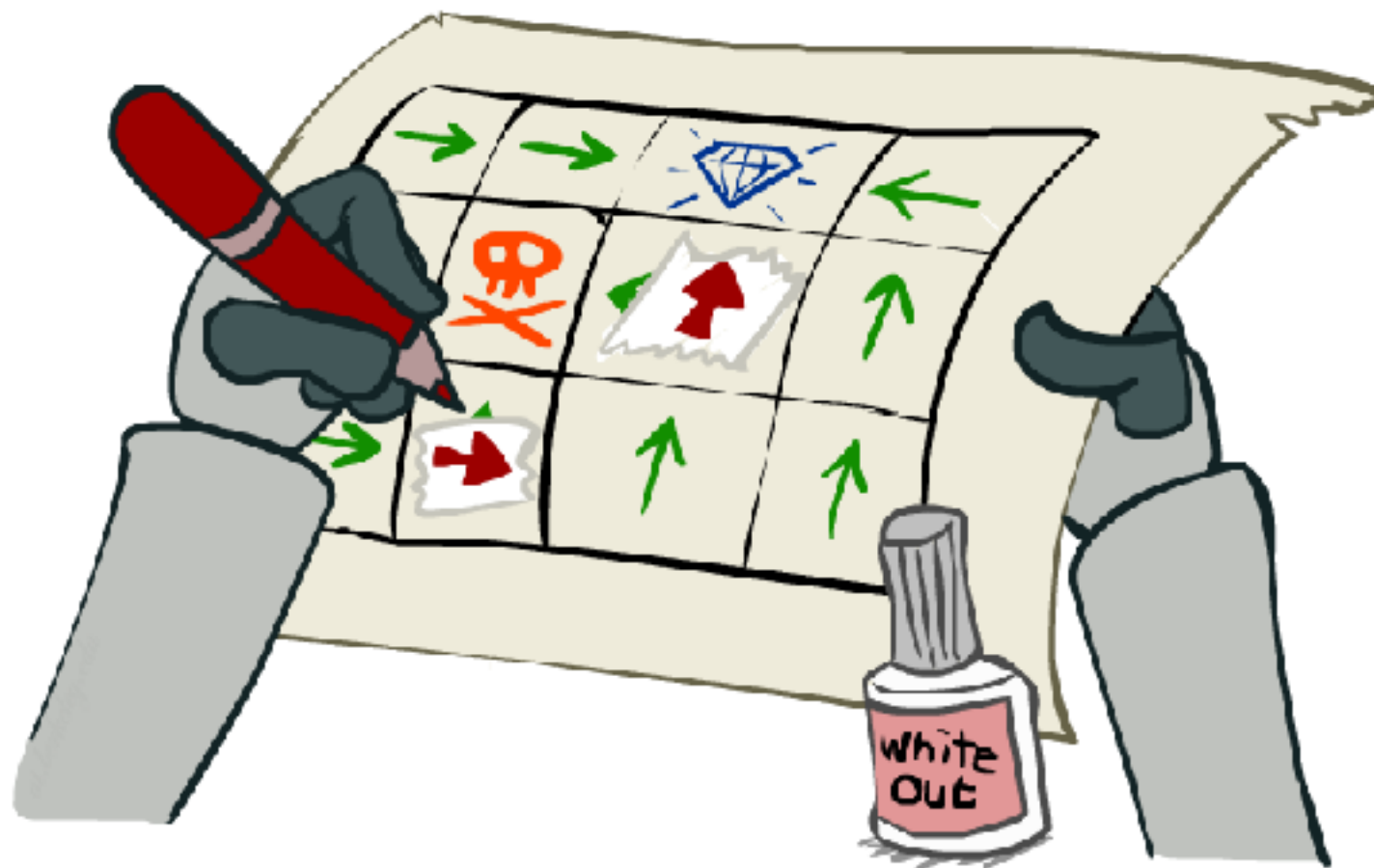
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$$\pi^*(s) = \arg \max_a Q^*(s, a)$$



- Important lesson: actions are easier to select from q-values than values!

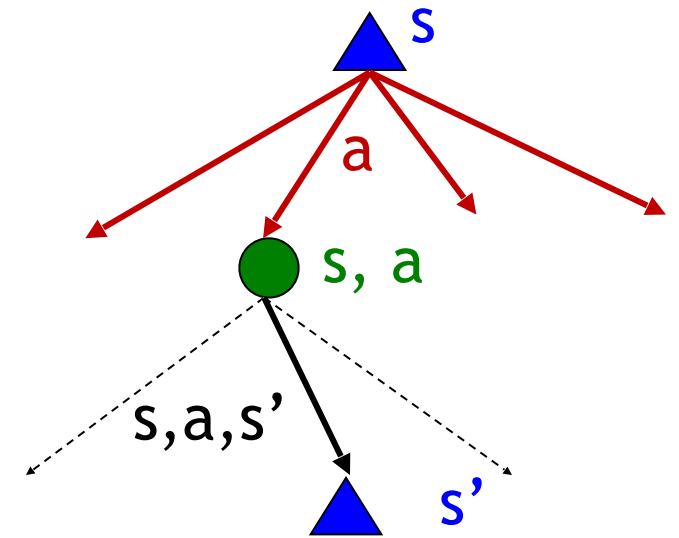
Policy Iteration



Problems with Value Iteration

- Value iteration repeats the Bellman updates:

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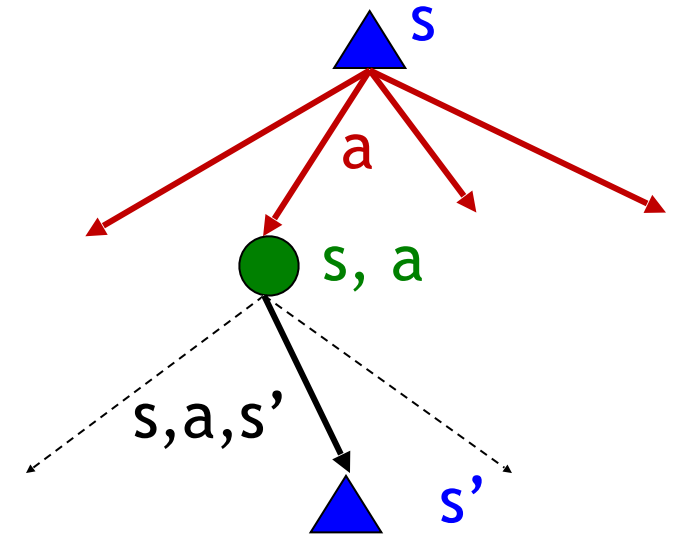


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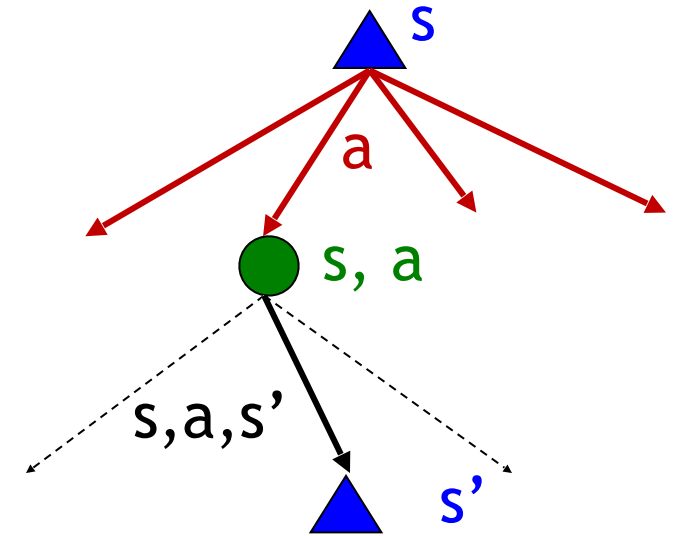


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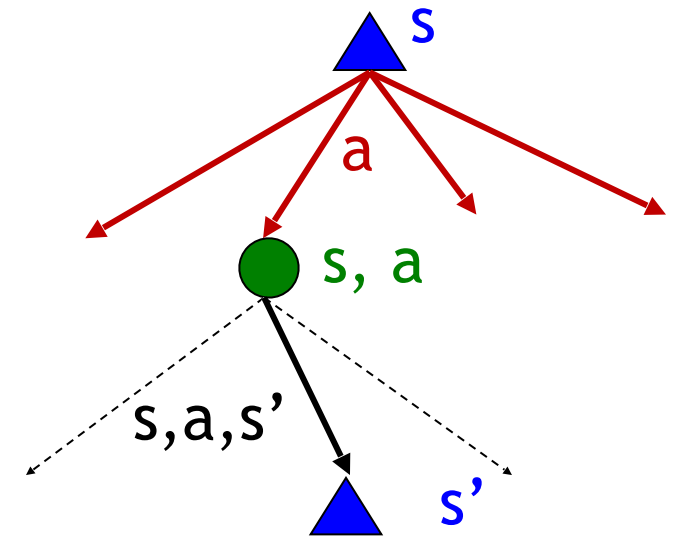


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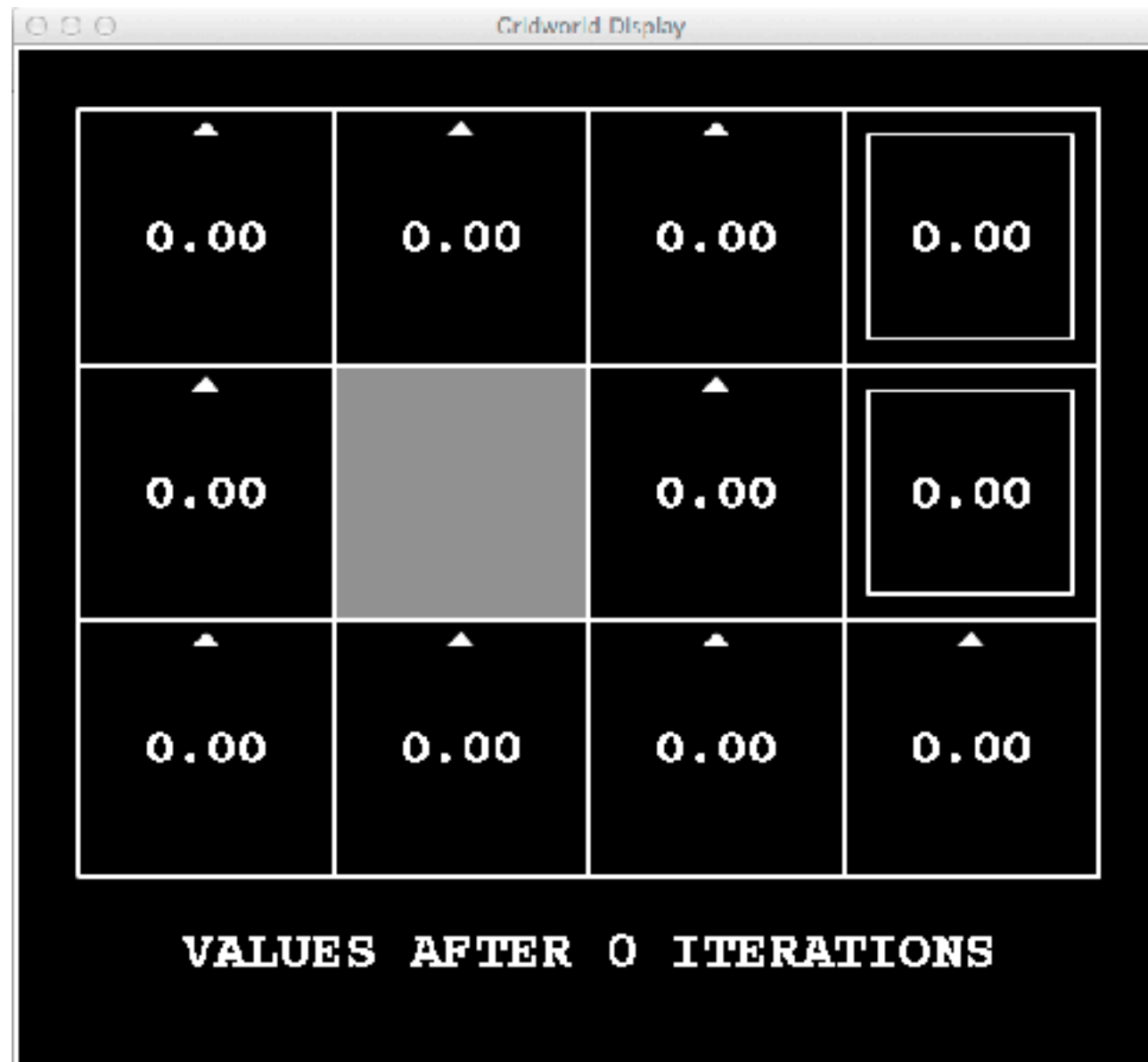
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- Problem 2: The “max” at each state rarely changes
- Problem 3: The policy often converges long before the values

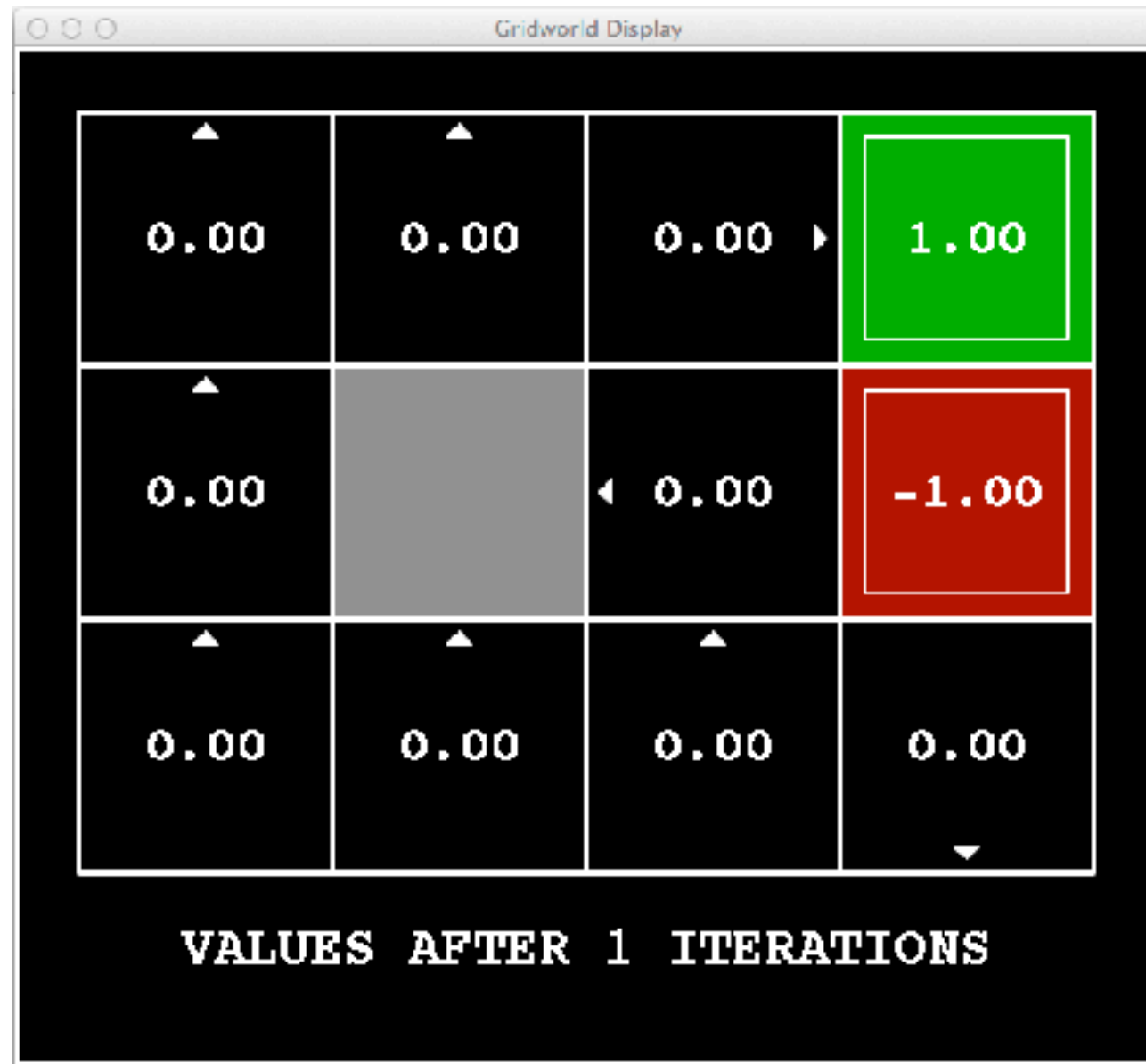


$k=0$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=1$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=2$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=3



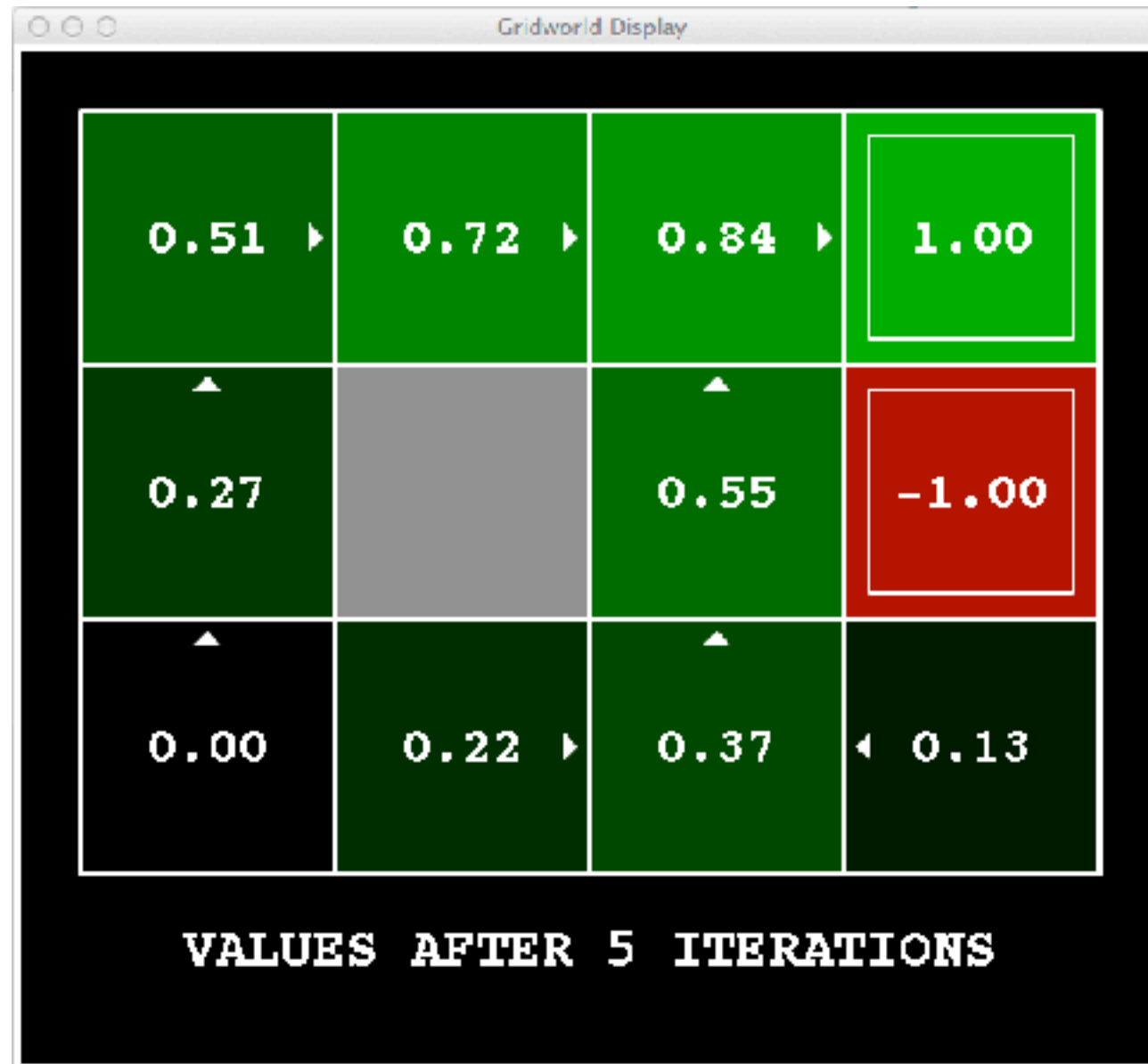
Noise = 0.2
Discount = 0.9
Living reward = 0

$k=4$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



Noise = 0.2
Discount = 0.9
Living reward = 0

k=6



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=7$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=8$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=9$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=10



Noise = 0.2
Discount = 0.9
Living reward = 0

k=11



Noise = 0.2
Discount = 0.9
Living reward = 0

k=12



Noise = 0.2
Discount = 0.9
Living reward = 0

k=100



Noise = 0.2
Discount = 0.9
Living reward = 0

Policy Iteration

- Alternative approach for optimal values:

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- This is **policy iteration**
 - It's still optimal!
 - Can converge (much) faster under some conditions

Policy Iteration

- Evaluation: For fixed current policy π , find values with policy evaluation:
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Policy Iteration

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$$\pi_{i+1}(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^{\pi_i}(s')]$$

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- Both are dynamic programs for solving MDPs

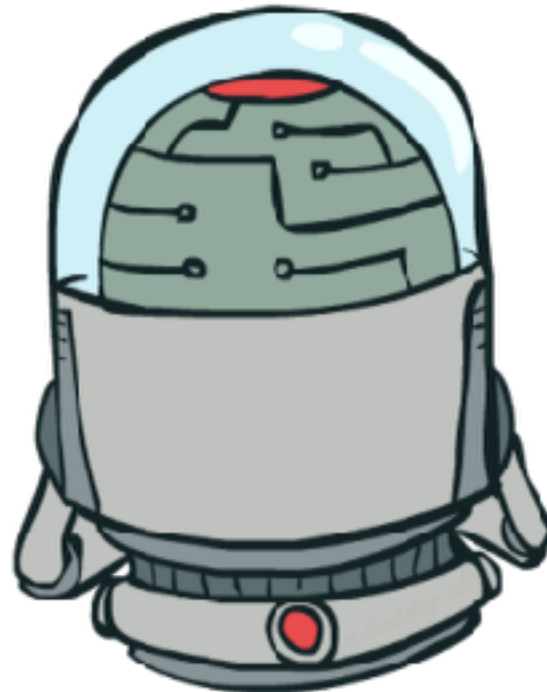
Summary: MDP Algorithms

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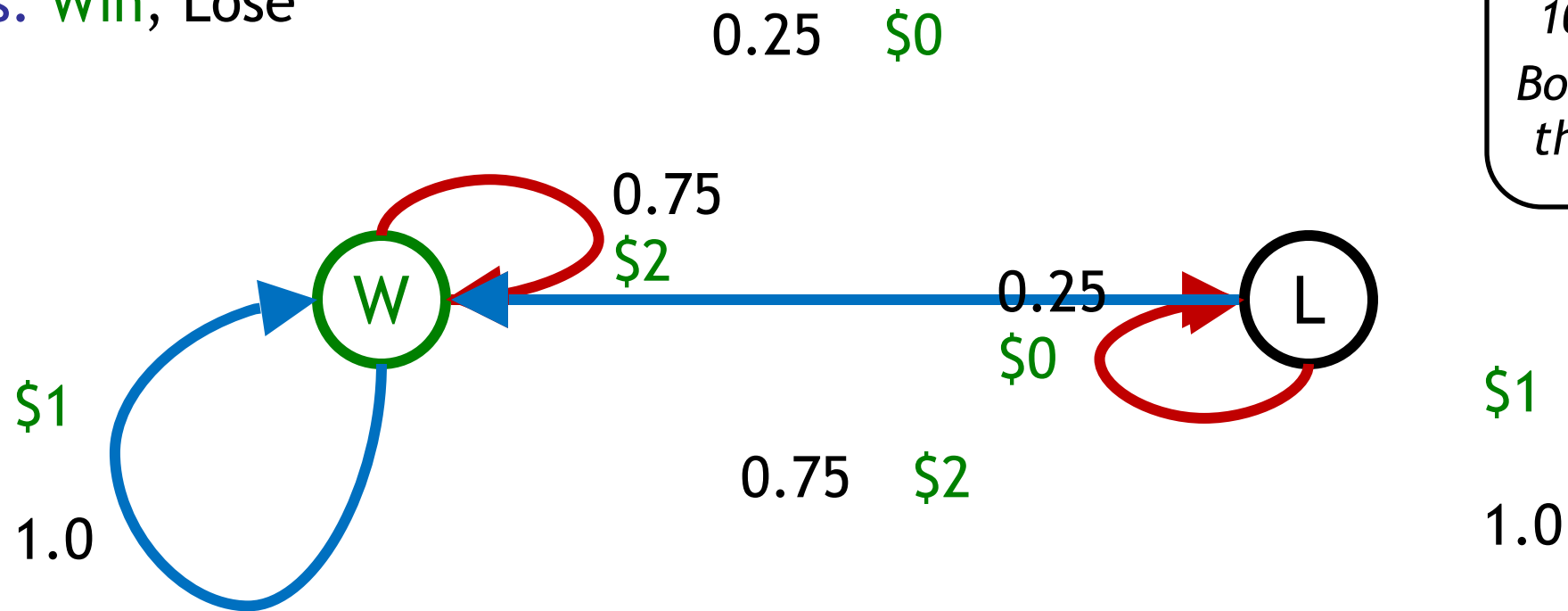
- So you want to....
 - Compute optimal values: use value iteration or policy iteration
 - Compute values for a particular policy: use policy evaluation
 - Turn your values into a policy: use policy extraction (one-step lookahead)
- These all look the same!
 - They basically are - they are all variations of Bellman updates
 - They all use one-step lookahead expectimax fragments
 - They differ only in whether we plug in a fixed policy or max over actions

Double Bandits



Double-Bandit MDP

- Actions: *Blue*, *Red*
- States: *Win*, Lose



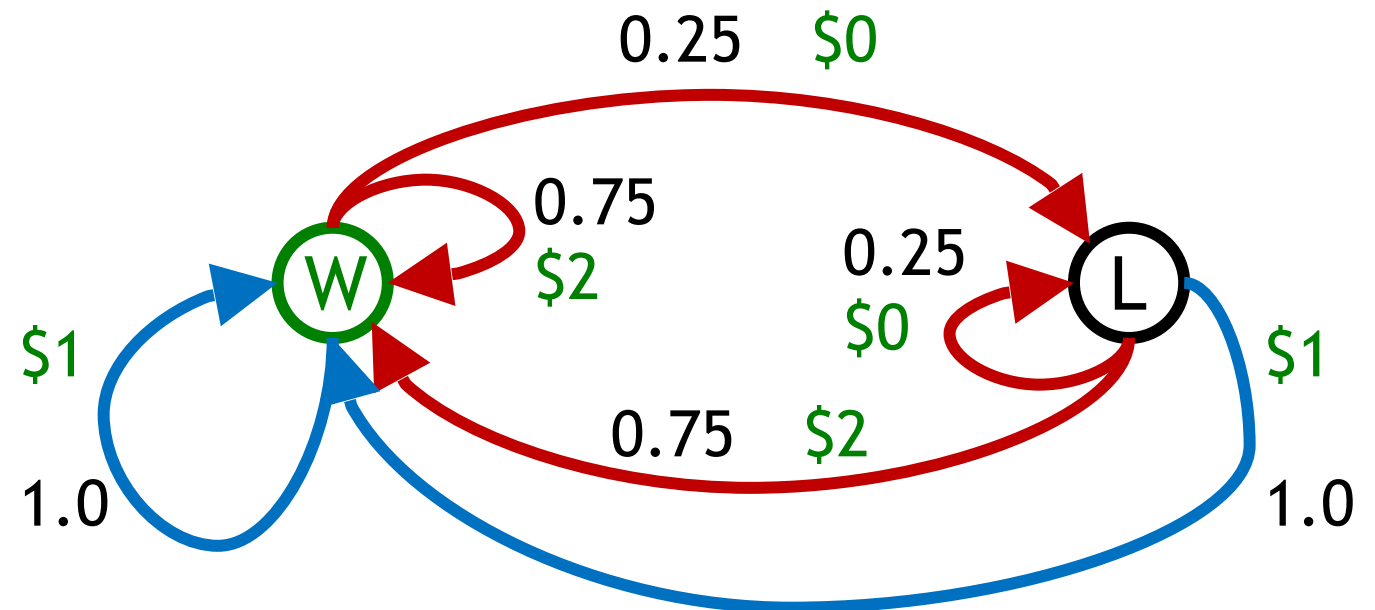
No discount
100 time steps
Both states have the same value

Offline Planning

- Solving MDPs is offline planning
 - You determine all quantities through computation
 - You need to know the details of the MDP
 - You do not actually play the game!

*No discount
100 time steps
Both states have
the same value*

	Value
Play Red	150
Play Blue	100



Let's Play!



Let's Play!



\$2

Let's Play!



\$2 \$2

Let's Play!



\$2 \$2 \$0

Let's Play!



\$2 \$2 \$0 \$2

Let's Play!



\$2 \$2 \$0 \$2 \$2

Let's Play!



\$2 \$2 \$0 \$2 \$2
\$2

Let's Play!



\$2 \$2 \$0 \$2 \$2
\$2 \$2

Let's Play!



\$2 \$2 \$0 \$2 \$2
\$2 \$2 \$0

Let's Play!



\$2 \$2 \$0 \$2 \$2
\$2 \$2 \$0 \$0

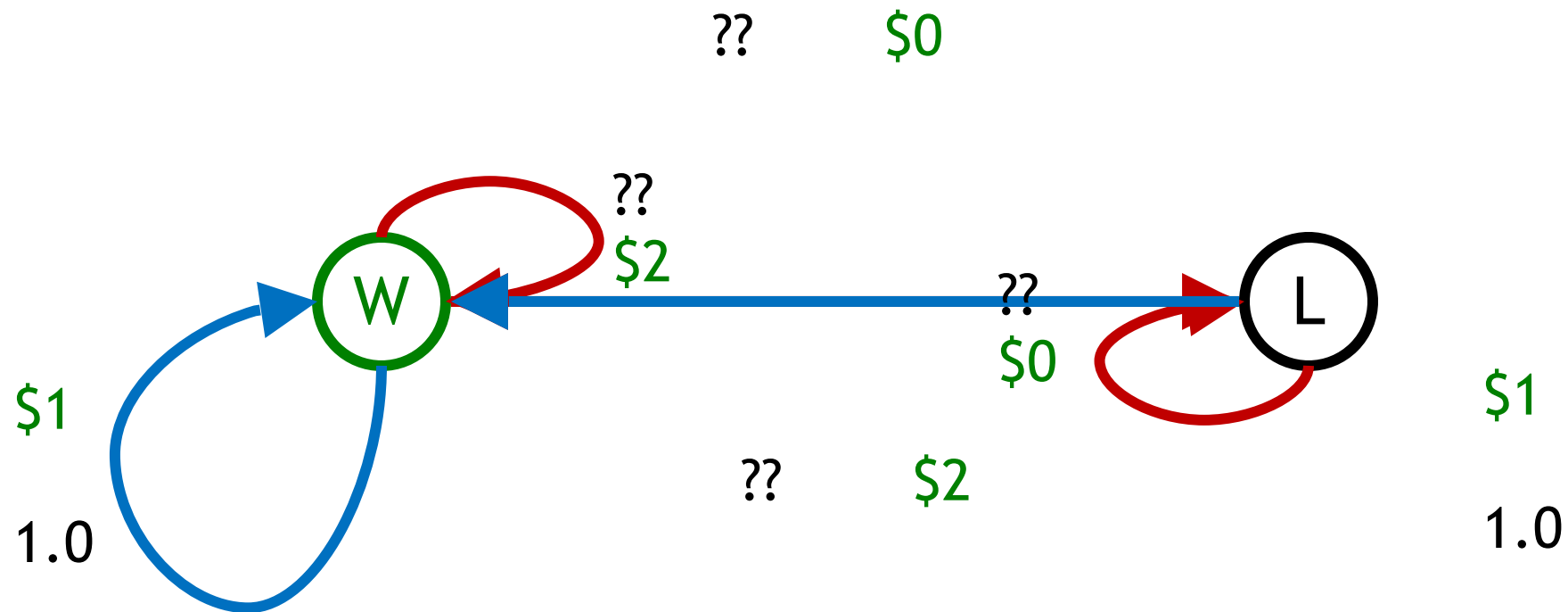
Let's Play!



\$2 \$2 \$0 \$2 \$2
\$2 \$2 \$0 \$0 \$0

Online Planning

- Rules changed! Red's win chance is different.



Let's Play!



Let's Play!



\$0

Let's Play!



\$0 \$0

Let's Play!



\$0 \$0 \$0

Let's Play!



\$0 \$0 \$0 \$2

Let's Play!



\$0 \$0 \$0 \$2 \$0

Let's Play!



\$0 \$0 \$0 \$2 \$0
\$2

Let's Play!



\$0 \$0 \$0 \$2 \$0
\$2 \$0

Let's Play!



\$0 \$0 \$0 \$2 \$0
\$2 \$0 \$0

Let's Play!



\$0 \$0 \$0 \$2 \$0
\$2 \$0 \$0 \$0

Let's Play!



\$0 \$0 \$0 \$2 \$0
\$2 \$0 \$0 \$0 \$0

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- That wasn't planning, it was learning!
 - Specifically, reinforcement learning
 - There was an MDP, but you couldn't solve it with just computation
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- That wasn't planning, it was learning!
 - Specifically, reinforcement learning
 - There was an MDP, but you couldn't solve it with just computation
 - You needed to actually act to figure it out
- Important ideas in reinforcement learning that came up
 - Exploration: you have to try unknown actions to get information
 - Exploitation: eventually, you have to use what you know
 - Regret: even if you learn intelligently, you make mistakes
 - Sampling: because of chance, you have to try things repeatedly
 - Difficulty: learning can be much harder than solving a known MDP



Next Time: Reinforcement Learning!
