# From Newton to the Path Integral: A Foundational Lecture Note

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Status: Working manuscript (foundational synthesis)

## 0. Aim

This note gives a single technical storyline for the corpus:

1. Newton's finite-step geometric conservation law,
2. action-based reduction and discriminant orbit structure,
3. distributional/oscillatory emergence of amplitude weighting,
4. controlled refinement ((h), (_), Schwinger-Dyson),
5. field-theoretic extension.

The target is a first-month QFT companion narrative with explicit mathematical dependencies.

## 1. Newton's Seed: Finite-Step Invariant Before the Limit

For central-force polygonal evolution (equal time steps (t)), equal swept triangles are exact at the discrete level.
Angular momentum conservation is encoded geometrically: [ A_n= |r_nr_n|, =. ]
Central impulse gives (r_np_n=0), so (A_n) is step-invariant.

The continuum law is then a controlled refinement: [ t, A_n, . ] The important logic is: exact finite-step structure first, limit second.

## 2. Action Reduction as Common Mechanics

For a Lagrangian action (S=L(q,q),d):

1. time-translation symmetry gives conserved (E),
2. rotational symmetry gives conserved (L),
3. cyclic elimination reduces to a 1D radial problem [ p_r^2=R(r;E,L,). ]

Orbit classes are read from turning-point geometry:

1. allowed region: (R(r)),
2. bounded non-circular branch: two turning points,

3. branch boundaries: double root [ R(r_)=0,R'(r_)=0. ]

This is the same mechanism in:

1. SR Coulomb ((^2)-classified regimes),
2. Schwarzschild fixed-energy interval and separatrix/ISCO,
3. probe dynamics in gauge-defined static (V(r)).

References in workspace:

- `research/workspace/notes/theorems/2026-02-08-claim3-coulomb-phase-classification.md`
- `research/workspace/notes/theorems/2026-02-08-claim6-schwarzschild-fixed-energy-interval.md`
- `research/workspace/notes/theorems/2026-02-09-foundational-action-reduction-unification.md`

# 3. Static Variational Problem as Distribution

For smooth (f), classical extrema are encoded by [ (f'(x)). ] Under nondegenerate critical points (x_i): [ (f'(x)) = _i . ]

Fourier representation [ (u)=e^{izu},dz ] introduces oscillatory exponentials already at static level.

The "halved expression" [ A_(O)=$^{\{-1/2\}e}${f(x)}O(x),dx ] has [ |A_(O)|^2 (f'),|O|^2 ] in the stationary-phase/nondegenerate setting.
This is the amplitude-to-density pattern and motivates half-density language.

References in workspace:

- `research/workspace/notes/theorems/2026-02-08-claim1-discrete-variational-delta-lemmas.md`
- `research/workspace/notes/theorems/2026-02-08-claim1-manifold-halfdensity-convolution.md`
- `research/workspace/notes/theorems/2026-02-08-claim1-groupoid-halfdensity-theorem-pack.md`

# 4. From Static to Time Histories

Replacing (f) by action (S[]), classical paths solve [ =0. ] The formal object [ !() ] is regularized via oscillatory weighting of histories, leading to path-integral form.

Consistency of refinement/composition introduces a surviving scale parameter (h) (identified physically with () in standard QM/QFT usage), and scale-flow covariance: [ *: (,,h)(,/,h), ] with dressed-state invariance under (*) in the scoped framework.

References in workspace:

- `research/workspace/notes/theorems/2026-02-09-claim1-scale-flow-covariance.md`

- `research/workspace/notes/theorems/2026-02-09-claim1-fd-schwinger-dyson-identity.md`

# 5. Field-Theoretic Lift and Eq.(11)-Type Structure

In field form, Euler-Lagrange structure lifts to Schwinger-Dyson identities by functional integration by parts.
In the scoped finite-dimensional model family, this is theoremized as: [ _i!(e^{-cS})=0 ;; _i = c,_i S. ] This is the rigorous version of the Eq.(11)-type closure thread in the corpus.

# 6. Current Claim 1 Closure Boundary

The scoped bridge now includes:

1. exact projective cylinder consistency,
2. (^+) de-regularization in several interacting classes,
3. large-(N) non-factorized quadratic and quartic tails,
4. finite-(g) non-perturbative multi-mode quartic control (Euclidean and oscillatory regularized),
5. (^+) closure for that multi-mode quartic sector.

Primary artifacts:

- `research/workspace/reports/2026-02-09-claim1-scoped-complete-proof.tex`
- `research/workspace/notes/theorems/2026-02-09-claim1-multimode-quartic-dereg-eta0.md`

Remaining frontier:

1. full continuum/global interacting equivalence beyond scoped classes,
2. uniform renormalization/channel control in truly field-theoretic limits.

# 7. Dependency Graph (Explicit)

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Newton finite-step area invariance
  -> action additivity + symmetry charges (E, L)
     -> 1D radial reduction p_r^2 = R(r)
        -> double-root boundaries (circular/separatrix/threshold)
           -> SR Coulomb regimes (Claim 3)
           -> Schwarzschild intervals + ISCO (Claims 6,7)
           -> gauge-phase static probe branches (Claim 9 map)

Static variational distribution δ(f')
  -> Fourier/oscillatory representation
     -> halved amplitude |A|^2 structure
        -> half-density/groupoid formulation
           -> path-time slicing + RG-style control (tau flow)
              -> Schwinger-Dyson lifted equations
```

```
                  -> scoped Claim 1 theorem chain
                     -> large-N non-factorized interacting tails
                        -> finite-g nonperturbative + η->0+
closures
```

# 8. Minimal "What Is Forced" Statement

In this program, path-integral-type oscillatory weighting is not introduced as optional aesthetics; it is the structurally stable way to combine:

1. localization on variational extrema (distributional viewpoint),
2. multiplicative composition under slicing (action additivity),
3. controlled refinement with a surviving scale parameter.

That is the precise sense in which quantum weighting appears as the consistent correction/completion of naive classical refinement.

# 9. Reproducibility Index

Core diagnostic scripts:

1. `python3.12 research/workspace/simulations/foundation_action_reduction_unification_check.py`
2. `python3.12 research/workspace/simulations/claim1_multimode_quartic_dereg_eta0_check.py`
3. `python3.12 research/workspace/simulations/claim1_multimode_quartic_nonperturbative_oscillatory_check.py`
4. `python3.12 research/workspace/simulations/claim6_schwarzschild_interval_scan.py`
5. `python3.12 research/workspace/simulations/claim3_coulomb_classification_scan.py`

# 10. Next Formal Target

Integrate this lecture synthesis with an explicit tangent-groupoid scale map linking:

1. near-diagonal scaling,
2. (_)-flow invariance,
3. Schwinger-Dyson identities,

in one theorem-style dependency sheet suitable for direct inclusion in the scoped Claim 1 manuscript.