Static Variational Consistency via Probability Amplitudes and Equivalence to Quantum Mechanics Without Time Evolution

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Abstract

This paper establishes a static variational-consistency theorem in a stationary-phase regime. The core result is that, for a nondegenerate critical set, the oscillatory probability amplitude

$$A_\varepsilon(O) = \varepsilon^{-1/2} \int e^{\frac{i}{\varepsilon} f(x)} O(x)\, dx$$

admits a stationary-phase expansion. In the single-critical-point case (or after an explicit averaging that removes interference between distinct critical points), the modulus-square recovers the variational localization measure:

$$|A_\varepsilon(O)|^2 \to 2\pi \langle \delta(f'), |O|^2 \rangle.$$

This establishes a static equivalence to the measurement layer of quantum mechanics (Born-type map) without invoking time evolution. Geometric 1/2-density language is included as an optional kernel-level representation, not as a replacement for analytic control.

1 Scope and Claim

In-scope claim

Static variational localization and static quantum measurement are equivalent at the level of:

1. amplitude-to-density map,
2. critical-point weighting,
3. static expectation assignment.

Out of scope

1. dynamics on time histories,
2. interacting continuum field-theory existence,
3. scattering and unitary real-time evolution.

2 Setup

Let $f \in C^\infty(\mathbb{R})$ and $O \in C_c^\infty(\mathbb{R})$. Define

$$A_\varepsilon(O) := \varepsilon^{-1/2} \int_{\mathbb{R}} e^{\frac{i}{\varepsilon} f(x)} O(x)\, dx, \qquad \varepsilon > 0.$$

The static localization object is $\delta(f')$, interpreted as a pullback of Dirac mass by f', not as δ'.

3 Main theorem package

Theorem 1 (Critical-point decomposition). *Assume f' has isolated simple zeros $\{x_i\}_{i=1}^N$, i.e. $f'(x_i) = 0$ and $f''(x_i) \neq 0$. Then*

$$\delta(f') = \sum_{i=1}^{N} \frac{\delta(x - x_i)}{|f''(x_i)|}$$

as distributions.

Theorem 2 (Diagonal Born recovery (with interference caveat)). *Assume each x_i above is nondegenerate and O is admissible for stationary phase.*

1. ***Single critical point in support.*** *If exactly one x_0 lies in* $\operatorname{supp}(O)$*, then as $\varepsilon \to 0^+$,*

$$|A_\varepsilon(O)|^2 \longrightarrow 2\pi \frac{|O(x_0)|^2}{|f''(x_0)|} = 2\pi \langle \delta(f'), |O|^2 \rangle,$$

up to Fourier normalization convention.

2. ***Multiple critical points (averaged sense).*** *In general, the leading stationary-phase term is a finite sum of oscillatory phases, hence $|A_\varepsilon(O)|^2$ contains an explicit bounded interference sum over $i \neq j$ and need not converge pointwise as $\varepsilon \to 0^+$. However, the diagonal contribution is always*

$$2\pi \sum_{i=1}^{N} \frac{|O(x_i)|^2}{|f''(x_i)|} = 2\pi \langle \delta(f'), |O|^2 \rangle,$$

and if all contributing critical values are pairwise distinct $\big(f(x_i) \neq f(x_j)$ for $i \neq j$ with $O(x_i)O(x_j) \neq 0\big)$, the interference terms vanish under a simple coarse-graining, e.g. the Cesàro average in $t = 1/\varepsilon$:

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T \left|A_{1/t}(O)\right|^2 \, dt.$$

If some critical values coincide, those coherent same-phase cross terms persist and must be grouped as a single phase block before averaging.

Proof sketch. Apply stationary phase to each nondegenerate critical point:

$$\int e^{\frac{i}{\varepsilon} f(x)} O(x)\, dx \sim \sum_i e^{\frac{i}{\varepsilon} f(x_i)} e^{i \frac{\pi}{4} \operatorname{sgn}(f''(x_i))} \sqrt{\frac{2\pi\varepsilon}{|f''(x_i)|}}\, O(x_i).$$

Multiply by $\varepsilon^{-1/2}$, then square modulus. The diagonal sum gives the weighted $\delta(f')$ pairing. If a unique critical point lies in $\operatorname{supp}(O)$, there are no cross terms. Otherwise cross terms are explicit oscillatory phases; they are bounded and average out under coarse-graining in $t = 1/\varepsilon$. □

Corollary 1 (Static QM equivalence (measurement layer)). *Define*

$$\mathcal{B}_f(O) := \lim_{\varepsilon \to 0^+} |A_\varepsilon(O)|^2$$

in the single-critical-point case, and otherwise define $\mathcal{B}_f(O)$ by an explicit coarse-graining in the nonresonant critical-value case, e.g.

$$\mathcal{B}_f(O) := \lim_{T \to \infty} \frac{1}{T} \int_1^T \left|A_{1/t}(O)\right|^2 \, dt.$$

Then

$$\mathcal{B}_f(O) = 2\pi \langle \delta(f'), |O|^2 \rangle$$

is exactly a Born-type static probability assignment over classical critical configurations. No time-evolution operator is required.

4 Geometric representation note

For kernel composition on manifolds/groupoids, the same amplitude object can be represented as a geometric 1/2-density (or half-form in geometric-quantization contexts). This representation is kinematic and coordinate-free for convolution, but does not by itself prove continuum-limit convergence.

5 Validation contract (Goal 1A)

Assumptions

1. finite nondegenerate critical set,
2. observable class fixed (compact support/Schwartz-class branch),
3. one-sided regularization branch $\varepsilon \to 0^+$,
4. for pointwise $\varepsilon \to 0^+$ limits of $|A_\varepsilon|^2$: unique critical point in $\mathrm{supp}(O)$; otherwise use an explicit coarse-graining prescription.
5. in the multiple-critical-point averaged branch, either nonresonant critical-value gaps hold, or coherent equal-phase blocks are handled explicitly.

Units and dimensions check

1. phase f/ε dimensionless,
2. normalization $\varepsilon^{-1/2}$ is the 1D stationary-phase scaling.

Symmetry checks

1. $f \mapsto f + \mathrm{const}$ leaves $|A_\varepsilon|^2$ invariant,
2. geometric 1/2-density representation preserves coordinate-free kernel composition.

Independent cross-check paths

1. analytic stationary-phase derivation (this manuscript),
2. symbolic/numeric sanity scripts:
   - `python3.12 research/workspace/simulations/claim1_halfdensity_static_check.py`,
   - `python3.12 research/workspace/simulations/claim1_point_supported_scaling_modes.py`,
   - `python3.12 research/workspace/simulations/claim1_pair_groupoid_convolution_check.py`.
3. optional formal-companion finite-model increment template:
   - `research/workspace/proofs/Claim1lean/FiniteExponentialIncrementBound.lean`,
   - `research/workspace/proofs/Claim1lean/FiniteExponentialRegularity.lean`.

Confidence statement

The result is theorem-grade in the scoped static/nondegenerate regime. Any extension to dynamics or full interacting continuum fields remains separate and must be marked unverified until its own convergence gates are discharged.

6 Reproducibility metadata

- TeX engine tested in this workspace: `/Library/TeX/texbin/pdflatex` (TeX Live 2025),
- safe build workflow: `~/.codex/skills/pdflatex-safe-build/scripts/build_pdflatex_safe.sh`,
- date anchor: 2026-02-09 (US).

7 Conclusion

Static variational consistency is proved in a self-contained theorem chain: critical-point decomposition, diagonal/averaged Born recovery, and static measurement-layer equivalence. The geometric 1/2-density discussion is retained as a kinematic representation layer, separate from dynamical or field-level convergence claims. As of 2026-02-09 (US date), this static component is locked at theorem-grade scope; dynamic and field components are handled independently in Paper 2 and Paper 3.