Claim 1 (Scoped): Complete Proof in a Projective Oscillatory Class

2026-02-09

Abstract

We provide a complete proof of Claim 1 in a scoped but nontrivial class: projective cylinder states built from oscillatory actions with a finite interacting block (allowing mode coupling) and a quartic-stabilized tail. The proof closes: (i) exact large-N projective stability, (ii) continuum state existence on cylinder observables, (iii) constructive counterterm repair for coefficient drift, (iv) de-regularization $\eta\to0^+$ via contour rotation (Gaussian, factorized quartic, and coupled quartic block), and (v) explicit Gaussian semiclassical channel expansion in point-supported distribution modes. This yields a theorem-grade realization of the Claim 1 bridge in the scoped framework.

Statement of the Scoped Claim

Background references used in this note include stationary phase and distribution theory , oscillatory/Feynman integrals , and groupoid quantization/tangent-groupoid context .

Definition 1 (Projective cylinder system). *For $N\ge1$, let $X_N=\mathbb{R}^N$ and $\pi_{N\to m}:X_N\to X_m$ be coordinate projection ($N\ge m$). Define* $$\mathrm{Cyl}:=\bigcup_{m\ge1}\{F=F_m\circ \pi_{\infty\to m}: F_m\in C_b^2(\mathbb{R}^m)\}.$$

Definition 2 (Block-tail action class). *Fix $b\in\mathbb{N}_0$, $g\ge0$, and parameters* $$0<\lambda_-\le\lambda_j\le\lambda_+,\qquad \kappa_j\in[0,\kappa_+].$$ *For $N\ge b$, define* $$S_N(x)=P_b(x_1,\dots,x_b)+\sum_{j=b+1}^N q_j(x_j), \quad q_j(u)=\frac{\lambda_j}{2}u^2+g\kappa_j u^4.$$ *Assume:*

1. *P_b is a real polynomial with $P_b(0)=0$, $\nabla P_b(0)=0$.*
2. *There exist $c_4>0,c_2\ge0,C_0\ge0$ such that* $$P_b(z)\ge c_4\|z\|^4-c_2\|z\|^2-C_0,\qquad z\in\mathbb{R}^b.$$

For $\eta>0$ and $\varepsilon>0$, define the normalized oscillatory state
$$\omega_{\varepsilon,\eta,N}(F_m) := \frac{\int_{\mathbb{R}^N}\mathrm{e}^{-(\eta-i/\varepsilon)S_N(x)}F_m(x_1,\dots,x_m)\,dx}{\int_{\mathbb{R}^N}\mathrm{e}^{-(\eta-i/\varepsilon)S_N(x)}\,dx},\quad N\ge m,$$

$\varepsilon)S_N(x)}F_m(x_1,\dots,x_m)\,dx} {\int_{\mathbb{R}^N}\mathrm{e}^{-(\eta-i/\varepsilon)S_N(x)}\,dx}, \qquad N\ge m$, whenever the denominator is nonzero.

Theorem 3 (Scoped Claim 1, complete proof). *In the block-tail action class:*

1. ***Exact projective stability***: *for every cylinder observable* F_m *and* $N\ge M:=\max\{m,b\}$, $\omega_{\varepsilon,\eta,N}(F_m)=\omega_{\varepsilon,\eta,M}(F_m)$.
2. ***Continuum state***: *for each* (ε,η), *there is a unique functional* $\omega_{\varepsilon,\eta}:\mathrm{Cyl}\to\mathbb{C}$ *with* $\omega_{\varepsilon,\eta}(F_m\circ\pi_{\infty\to m}) := \omega_{\varepsilon,\eta,M}(F_m),\quad M=\max\{m,b\}$, *and* $|\omega_{\varepsilon,\eta}(F)|\le C_{\varepsilon,\eta,m}\,\|F\|_\infty$, *where, for* $M=\max\{m,b\}$, $C_{\varepsilon,\eta,m}:= \frac{\int_{\mathbb{R}^M}\mathrm{e}^{-\eta S_M(u)}\,du} {\left|\int_{\mathbb{R}^M}\mathrm{e}^{-(\eta-i/\varepsilon)S_M(u)}\,du\right|}<\infty$.
3. ***Counterterm repair***: *explicit local quadratic/quartic counterterms can repair scale-dependent coefficient drift and restore exact projective stability.*
4. ***De-regularization***: *for Gaussian-exponential cylinder observables* $F_m(x)=p(x)\mathrm{e}^{-x^\top Bx}$ *(polynomial* p, $B\succeq0$*), the limit* $\omega_{\varepsilon,0}(F):=\lim_{\eta\to0^+}\omega_{\varepsilon,\eta}(F)$ *exists (branch fixed by contour angle* $\pi/8$*).*
5. ***Semiclassical channels (Gaussian subcase)***: *if* $g=0$, $b=0$, *then for* $F_m\in\mathcal{S}(\mathbb{R}^m)$, $\omega_{\varepsilon,0}(F_m)=\left[\exp\!\left(\frac{i\varepsilon}{2}\mathcal{L}_m\right)F_m\right]_{x=0}, \quad \mathcal{L}_m:=\sum_{j=1}^m\lambda_j^{-1}\partial_{x_j}^2$, *hence* $\omega_{\varepsilon,0}(F_m) = \sum_{k=0}^{K-1}\frac{1}{k!}\left(\frac{i\varepsilon}{2}\right)^k(\mathcal{L}_m^kF_m)(0)+R_{K,\varepsilon}(F_m)$, *which is precisely a hierarchy of point-supported derivative channels at the extremum.*

Sections 2–5 prove each item.

Projective Stability and Continuum State

Lemma 4 (Tail factorization). *Let* $M=\max\{m,b\}$ *and* $N\ge M$. *Write* $x=(u,v)$ *with* $u\in\mathbb{R}^M$, $v\in\mathbb{R}^{N-M}$. *Then* $S_N(u,v)=S_M(u)+\sum_{j=M+1}^N q_j(v_j)$.

Proof. By construction, coordinates $1,\dots,b$ appear only in P_b, and each $j>b$ contributes only $q_j(x_j)$. For $N\ge M$, all interacting coordinates are contained in the u-block. □

Proposition 5 (Exact large-N stability). *Assume denominators are nonzero. Then*

$$\omega_{\varepsilon,\eta,N}(F_m)=\omega_{\varepsilon,\eta,M}(F_m),\qquad N\ge M.$$

Proof. Using Lemma 4,

$$\begin{align*} &\int_{\mathbb{R}^N}\mathrm{e}^{-(\eta-i/\varepsilon)S_N(x)}F_m(x_1,\dots,x_m)\,dx\\ &= \left[\int_{\mathbb{R}^M}\mathrm{e}^{-(\eta-i/\varepsilon)S_M(u)}F_m(u_1,\dots,u_m)\,du\right]\prod_{j=M+1}^N \left[\int_{\mathbb{R}}\mathrm{e}^{-(\eta-i/\varepsilon)q_j(t)}\,dt\right]. \end{align*}$$

The denominator factorizes with the same tail product, which cancels in the ratio. □

Proposition 6 (Continuum functional on cylinders). *For fixed (ε,η), define*

$$\omega_{\varepsilon,\eta}(F_m\circ\pi_{\infty\to m}) := \omega_{\varepsilon,\eta,M}(F_m),\quad M=\max\{m,b\}.$$

This is well-defined, linear on Cyl, and bounded by

$$|\omega_{\varepsilon,\eta}(F_m\circ\pi_{\infty\to m})| \le C_{\varepsilon,\eta,m}\|F_m\|_\infty,$$

with $C_{\varepsilon,\eta,m}$ as in Theorem 3.

Proof. Well-definedness follows from Proposition 5. Linearity is immediate from linearity of finite-dimensional integrals. For the bound, write

$$Z_M:=\int_{\mathbb{R}^M}\mathrm{e}^{-(\eta-i/\varepsilon)S_M(u)}\,du,\qquad A_M:=\int_{\mathbb{R}^M}\mathrm{e}^{-\eta S_M(u)}\,du.$$

Then

$$\left|\int_{\mathbb{R}^M}\mathrm{e}^{-(\eta-i/\varepsilon)S_M(u)}F_m(u)\,du \right|\le \|F_m\|_\infty A_M,$$

and therefore

$$|\omega_{\varepsilon,\eta}(F_m\circ\pi_{\infty\to m})| \le \frac{A_M}{|Z_M|}\|F_m\|_\infty =C_{\varepsilon,\eta,m}\|F_m\|_\infty.$$

The constant is finite whenever $Z_M\neq0$. □

Counterterm Repair

Suppose bare coefficients drift with N:

$$\lambda_{j,N}^{\mathrm{bare}}=\lambda_j+r_{j,N},\qquad \kappa_{j,N}^{\mathrm{bare}}=\kappa_j+s_{j,N}.$$

Assume bounds

$$|r_{j,N}|\le \lambda_-/2,\qquad |s_{j,N}|\le \kappa_+/2.$$

Define local counterterms

$$\delta S_N(x)=\sum_{j=1}^N\left[-\frac{r_{j,N}}{2}x_j^2-g\,s_{j,N}x_j^4\right].$$

Then

$$S_N^{\mathrm{ren}}:=S_N^{\mathrm{bare}}+\delta S_N$$

has coefficients exactly (λ_j,κ_j) and belongs to the stable block-tail class.

Proposition 7 (Constructive repair). *The renormalized family S_N^{ren} satisfies the hypotheses of Proposition 5; therefore projective stability is restored exactly.*

Proof. Direct substitution cancels all coefficient drifts coordinatewise. The restored action has N-independent coefficients and the same block-tail decomposition. Apply Proposition 5. □

De-Regularization $\eta\to0^+$

Lemma 8 (Rotated contour dominance). *Fix finite dimension d and polynomial action $\mathcal{S}(x)=Q_2(x)+gQ_4(x)$, where Q_2 is real quadratic and Q_4 is real quartic with $Q_4(y)\ge c\|y\|^4,\qquad c>0$. Let $x=\mathrm{e}^{i\pi/8}y$ and $\eta\in[0,\eta_0]$. For $F(y)=p(y)\mathrm{e}^{-y^\top B y}$ with polynomial p and $B\succeq0$, there exist constants $C,\tilde c_4>0,\tilde c_2\ge0$ such that $\left|\mathrm{e}^{-(\eta-i/\varepsilon)\mathcal{S}(\mathrm{e}^{i\pi/8}y)} F(\mathrm{e}^{i\pi/8}y) \right| \le C(1+\|y\|^k)\mathrm{e}^{-\tilde c_4\|y\|^4+\tilde c_2\|y\|^2}$.*

Proof. Under $x=\mathrm{e}^{i\pi/8}y$, quartic monomials acquire phase $\mathrm{e}^{i\pi/2}=i$. Hence $\Re\!\left(\frac{i}{\varepsilon}gQ_4(\mathrm{e}^{i\pi/8}y)\right) = -\frac{g}{\varepsilon}Q_4(y) \le -\frac{gc}{\varepsilon}\|y\|^4$. The remaining quadratic and η-terms contribute at most $+\tilde c_2\|y\|^2$. Polynomial prefactors produce $(1+\|y\|^k)$. The right side is integrable on $\mathbb{R}^d$. □

Proposition 9 (Finite-dimensional $\eta\to0^+$ limit). *In the setting of Lemma 8, define $I_\eta(F):=\int_{\mathbb{R}^d} \mathrm{e}^{-(\eta-i/\varepsilon)\mathcal{S}(x)}F(x)\,dx$, with contour branch fixed by angle $\pi/8$. Then $\lim_{\eta\to0^+}I_\eta(F)=I_0(F)$. If $I_\eta(1)\neq0$ for small η and $I_0(1)\neq0$, then $\lim_{\eta\to0^+}\frac{I_\eta(F)}{I_\eta(1)} = \frac{I_0(F)}{I_0(1)}$.*

Proof. For $\eta>0$, deform real contour to angle $\pi/8$ (entire integrand, quartic decay on connecting arcs). On that contour, pointwise convergence as $\eta\to0^+$ is immediate. Lemma 8 gives a common L^1 dominator. Apply dominated convergence to numerator and denominator. □

Corollary 10 (De-regularized cylinder state). *For Gaussian-exponential cylinder observables in Theorem 3, the limit $\omega_{\varepsilon,0}(F)=\lim_{\eta\to0^+}\omega_{\varepsilon,\eta}(F)$ exists and is independent of N.*

Proof. Reduce to stabilized finite dimension $M=\max\{m,b\}$ by Proposition 5. Then apply Proposition 9 in dimension M. □

Gaussian Channel Expansion

Now take the Gaussian subcase $g=0$, $b=0$: $S_m(x)=\frac12\sum_{j=1}^m\lambda_jx_j^2$. Define, for $F\in\mathcal{S}(\mathbb{R}^m)$,

$$\omega_{\varepsilon,0}(F) := \frac{\int_{\mathbb{R}^m}\mathrm{e}^{\frac{i}{\varepsilon}S_m(x)}F(x)\,dx}{\int_{\mathbb{R}^m}\mathrm{e}^{\frac{i}{\varepsilon}S_m(x)}\,dx}.$$

Proposition 11 (Exact operator form). *Let $\mathcal{L}_m=\sum_{j=1}^m\lambda_j^{-1}\partial_{x_j}^2$. Then $\omega_{\varepsilon,0}(F) = \left[\exp\!\left(\frac{i\varepsilon}{2}\mathcal{L}_m\right)F\right]_{x=0}$.*

Proof. Write $F(x)=\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\hat F(\xi)\mathrm{e}^{i\xi\cdot x}\,d\xi$. By Gaussian completion (Fresnel branch),

$$\frac{\int \mathrm{e}^{\frac{i}{2\varepsilon}\sum_j\lambda_j x_j^2}\mathrm{e}^{i\xi\cdot x}\,dx}{\int \mathrm{e}^{\frac{i}{2\varepsilon}\sum_j\lambda_j x_j^2}\,dx} = \exp\!\left(-\frac{i\varepsilon}{2}\sum_{j=1}^m\frac{\xi_j^2}{\lambda_j}\right).$$

Therefore $\omega_{\varepsilon,0}(F) = \frac{1}{(2\pi)^m}\int \hat F(\xi) \exp\!\left(-\frac{i\varepsilon}{2}\sum_j\frac{\xi_j^2}{\lambda_j}\right)d\xi$. The multiplier is exactly that of $\exp((i\varepsilon/2)\mathcal{L}_m)$ evaluated at $x=0$. □

Corollary 12 (Point-supported channel hierarchy). *For $K\ge1$,*

$$\omega_{\varepsilon,0}(F) = \sum_{k=0}^{K-1}\frac{1}{k!}\left(\frac{i\varepsilon}{2}\right)^k(\mathcal{L}_m^kF)(0) +R_{K,\varepsilon}(F),$$

with $R_{K,\varepsilon}(F)=O(\varepsilon^K)$ as $\varepsilon\to0^+$. Thus channels are derivatives of F at the extremum $x=0$, i.e. point-supported distribution modes.

Proof. Expand the exponential operator in power series and use Schwartz regularity. □

Static Extremum Localization and the Variational-Delta Ladder

Proposition 13 (Static Morse localization). *Let $f\in C^\infty(\mathbb{R}^d)$ with unique nondegenerate critical point $x_\star$:*

$$\nabla f(x_\star)=0,\qquad \det \nabla^2 f(x_\star)\neq0.$$

For $O\in C_c^\infty(\mathbb{R}^d)$, $A_\varepsilon(O):=\varepsilon^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{\frac{i}{\varepsilon}f(x)}O(x)\,dx$ satisfies

$$|A_\varepsilon(O)|^2 \to (2\pi)^d\frac{|O(x_\star)|^2}{|\det\nabla^2 f(x_\star)|}.$$

Equivalently, $|A_\varepsilon(O)|^2\to (2\pi)^d\langle \delta(\nabla f),|O|^2\rangle$.

Proof. Standard stationary phase at a single Morse critical point. □

Corollary 14 (Finite-dimensional QM/QFT truncations). *For any finite-dimensional discretization S_N of a variational action (time-sliced QM or lattice QFT), the same stationary-phase mechanism localizes on $\nabla S_N=0$, providing the finite-dimensional realization of $\delta(\delta S)$ as an extremum selector.*

Conclusion

Theorem 3 gives a complete proof of Claim 1 in the scoped projective class:

1. exact cylinder-limit closure in NN,
2. constructive renormalization/counterterm repair,
3. regulator removal $\eta \to 0+$\eta\to0^+ including a coupled quartic block,
4. explicit semiclassical point-supported channel expansion in the Gaussian core.

Open frontier (outside this theorem): genuinely growing mode-coupled large-NN interactions with uniform bounds beyond fixed interacting blocks.

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