

# Claim 1 (Scoped): Complete Proof in a Projective Oscillatory Class

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## Abstract

We provide a complete proof of Claim 1 in a scoped but nontrivial class: projective cylinder states built from oscillatory actions with a finite interacting block (allowing mode coupling) and a quartic-stabilized tail. The proof closes: (i) exact large-NN projective stability, (ii) continuum state existence on cylinder observables, (iii) constructive counterterm repair for coefficient drift, (iv) de-regularization  $\eta \rightarrow 0^+$  via contour rotation (Gaussian, factorized quartic, and coupled quartic block), and (v) explicit Gaussian semiclassical channel expansion in point-supported distribution modes. It further closes: (vi) genuinely large-NN mode-coupled lifts, including an explicit Gaussian-tail rate and a non-factorized quartic-tail class under log-derivative summability, (vii) explicit non-vanishing lower bounds for normalized oscillatory partition factors, (viii) extension of observables from Gaussian-exponential test families to Schwartz and weighted Sobolev classes, and (ix) finite-dimensional Schwinger-Dyson identities and exact  $\tau_\mu$   $\mu$ -type scale-flow covariance. This yields a theorem-grade realization of the Claim 1 bridge in the scoped framework.

## Statement of the Scoped Claim

Background references used in this note include stationary phase and distribution theory, oscillatory/Feynman integrals, and groupoid quantization/tangent-groupoid context.

**Definition 1** (Projective cylinder system). For  $N \geq 1, N \geq 1$ , let  $X_N = \mathbb{R}^N$ ,  $X_N = \mathbb{R}^N$  and  $\pi_N \rightarrow m: X_N \rightarrow X_m$ ,  $\pi_N \rightarrow m: X_N \rightarrow X_m$  be coordinate projection ( $N \geq m \geq 1$ ). Define  $\text{Cyl} := \bigcup_{m \geq 1} \{F = F_m \circ \pi_N \rightarrow m: F_m \in C_b^2(\mathbb{R}^m)\}$ .  $\text{Cyl} := \bigcup_{m \geq 1} \{F = F_m \circ \pi_N \rightarrow m: F_m \in C_b^2(\mathbb{R}^m)\}$ .

**Definition 2** (Block-tail action class). Fix  $b \in \mathbb{N}_{>0}$ ,  $g \geq 0$ , and parameters  $0 < \lambda_- \leq \lambda_j \leq \lambda_+$ ,  $\kappa_j \in [0, \kappa_+]$ ,  $0 < \lambda_- \leq \lambda_j \leq \lambda_+$ ,  $\kappa_j \in [0, \kappa_+]$ . For  $N \geq b$ , define  $S_N(x) = P_b(x_1, \dots, x_b) + \sum_{j=b+1}^N q_j(x_j)$ ,  $q_j(u) = \lambda_j^2 u^2 + g \kappa_j u^4$ .  $S_N(x) = P_b(x_1, \dots, x_b) + \sum_{j=b+1}^N q_j(x_j)$ ,  $q_j(u) = \lambda_j^2 u^2 + g \kappa_j u^4$ . Assume:

- $P_b$  is a real polynomial with  $P_b(0) = 0$ ,  $P'_b(0) = 0$ ,  $\nabla P_b(0) = 0$ .
- There exist  $c_4 > 0$ ,  $c_2 \geq 0$ ,  $C_0 \geq 0$ ,  $c_4 > 0$ ,  $c_2 \geq 0$ ,  $C_0 \geq 0$  such that  $P_b(z) \geq c_4 \|z\|^4 - c_2 \|z\|^2 - C_0$ ,  $z \in \mathbb{R}^b$ .  $P_b(z) \geq c_4 \|z\|^4 - c_2 \|z\|^2 - C_0$ ,  $z \in \mathbb{R}^b$ .



# Projective Stability and Continuum State

**Lemma 4** (Tail factorization). *Let  $M = \max\{m, b\}$  and  $N \geq MN \geq M$ . Write  $x = (u, v)$  with  $u \in \mathbb{R}^M$ ,  $v \in \mathbb{R}^{N-M}$ . Then  $S_N(u, v) = S_M(u) + \sum_{j=M+1}^N q_j(v_j)$ .*

*Proof.* By construction, coordinates  $1, \dots, b$  appear only in  $P_b$ , and each  $j > b$  contributes only  $q_j(x_j)$ . For  $N \geq MN \geq M$ , all interacting coordinates are contained in the  $u$ -block.  $\square$

**Proposition 5** (Exact large-NN stability). *Assume denominators are nonzero. Then  $\omega_{\varepsilon, \eta, N}(F_m) = \omega_{\varepsilon, \eta, M}(F_m)$ ,  $N \geq M$ .  $\omega_{\varepsilon, \eta, N}(F_m) = \omega_{\varepsilon, \eta, M}(F_m)$ ,  $N \geq M$ .*

*Proof.* Using Lemma 4,  $\int_{\mathbb{R}^N} e^{-(\eta - i/\varepsilon) S_N(x)} F_m(x_1, \dots, x_m) dx = \int_{\mathbb{R}^M} e^{-(\eta - i/\varepsilon) S_M(u)} F_m(u_1, \dots, u_m) du \prod_{j=M+1}^N \int_{\mathbb{R}} e^{-(\eta - i/\varepsilon) q_j(t)} dt$ .  $\begin{aligned} & \int_{\mathbb{R}^N} e^{-(\eta - i/\varepsilon) S_N(x)} F_m(x_1, \dots, x_m) dx \\ &= \left[ \int_{\mathbb{R}^M} e^{-(\eta - i/\varepsilon) S_M(u)} F_m(u_1, \dots, u_m) du \right] \prod_{j=M+1}^N \left[ \int_{\mathbb{R}} e^{-(\eta - i/\varepsilon) q_j(t)} dt \right] \end{aligned}$  The denominator factorizes with the same tail product, which cancels in the ratio.  $\square$

**Proposition 6** (Continuum functional on cylinders). *For fixed  $(\varepsilon, \eta)$ , define  $\omega_{\varepsilon, \eta}(F_m \circ \pi_\infty \rightarrow m) := \omega_{\varepsilon, \eta, M}(F_m)$ ,  $M = \max\{m, b\}$ .  $\omega_{\varepsilon, \eta}(F_m \circ \pi_\infty \rightarrow m) := \omega_{\varepsilon, \eta, M}(F_m)$ ,  $M = \max\{m, b\}$ . This is well-defined, linear on  $\text{Cyl}$ , and bounded by  $|\omega_{\varepsilon, \eta}(F_m \circ \pi_\infty \rightarrow m)| \leq C_{\varepsilon, \eta, m} \|F_m\|_\infty$ ,  $\omega_{\varepsilon, \eta}(F_m \circ \pi_\infty \rightarrow m) \leq C_{\varepsilon, \eta, m} \|F_m\|_\infty$ ,  $\omega_{\varepsilon, \eta}(F_m \circ \pi_\infty \rightarrow m) \leq C_{\varepsilon, \eta, m} \|F_m\|_\infty$ , with  $C_{\varepsilon, \eta, m}$  as in Theorem 3.*

*Proof.* Well-definedness follows from Proposition 5. Linearity is immediate from linearity of finite-dimensional integrals. For the bound, write  $Z_M := \int_{\mathbb{R}^M} e^{-(\eta - i/\varepsilon) S_M(u)} du$ ,  $A_M := \int_{\mathbb{R}^M} e^{-\eta S_M(u)} du$ ,  $Z_M := \int_{\mathbb{R}^M} e^{-(\eta - i/\varepsilon) S_M(u)} du$ ,  $A_M := \int_{\mathbb{R}^M} e^{-\eta S_M(u)} du$ . Then  $|\int_{\mathbb{R}^M} e^{-(\eta - i/\varepsilon) S_M(u)} F_m(u) du| \leq \|F_m\|_\infty A_M$ ,  $|\int_{\mathbb{R}^M} e^{-(\eta - i/\varepsilon) S_M(u)} F_m(u) du| \leq \|F_m\|_\infty A_M$ , and therefore  $|\omega_{\varepsilon, \eta}(F_m \circ \pi_\infty \rightarrow m)| \leq A_M |Z_M| / \|F_m\|_\infty = C_{\varepsilon, \eta, m} \|F_m\|_\infty$ .  $\omega_{\varepsilon, \eta}(F_m \circ \pi_\infty \rightarrow m) \leq C_{\varepsilon, \eta, m} \|F_m\|_\infty$ . The constant is finite whenever  $Z_M \neq 0$ .  $\square$

## Counterterm Repair

Suppose bare coefficients drift with NN:  $\lambda_j, N_{\text{bare}} = \lambda_j + r_j, N, \kappa_j, N_{\text{bare}} = \kappa_j + s_j, N$ .  $\lambda_{j, N}^{\text{bare}} = \lambda_j + r_j, N, \kappa_{j, N}^{\text{bare}} = \kappa_j + s_j, N$ . Assume bounds  $|r_j, N| \leq \lambda_-/2, |s_j, N| \leq \kappa_-/2$ .

$r_{\{j,N\}}|\leq \lambda_{-}/2, \text{quad } |s_{\{j,N\}}|\leq \kappa_{+}/2$ . Define local counterterms  $\delta S_N(x)=\sum_{j=1}^N[-r_j N^2 x_j^2 - g s_{\{j,N\}} x_j^4]$ .  $\delta S_N(x)=\sum_{j=1}^N \left[ -\frac{r_{\{j,N\}}}{2} x_j^2 - g s_{\{j,N\}} x_j^4 \right]$ . Then  $S_N^{\text{ren}}:=S_N^{\text{bare}}+\delta S_N$   $S_N^{\text{ren}}:=S_N^{\text{bare}}+\delta S_N$  has coefficients exactly  $(\lambda_j, \kappa_j)$  and belongs to the stable block-tail class.

**Proposition 7** (Constructive repair). *The renormalized family  $S_N^{\text{ren}}$  satisfies the hypotheses of Proposition 5; therefore projective stability is restored exactly.*

*Proof.* Direct substitution cancels all coefficient drifts coordinatewise. The restored action has NN-independent coefficients and the same block-tail decomposition. Apply Proposition 5.  $\square$

## De-Regularization $\eta \rightarrow 0^+, \eta \rightarrow 0^+$

**Lemma 8** (Rotated contour dominance). *Fix finite dimension  $d$  and polynomial action  $\mathcal{S}(x)=Q_2(x)+gQ_4(x)$ ,  $\mathcal{S}(x)=Q_2(x)+gQ_4(x)$ , where  $Q_2$  is real quadratic and  $Q_4$  is real quartic with  $Q_4(y)\geq c|y|^4, c>0$ .  $Q_4(y)\geq c|y|^4, c>0$ . Let  $x=e^{i\pi/8}y$  and  $\eta\in[0,\eta_0]$ . For  $F(y)=p(y)e^{-y\top B y}$  with polynomial  $p$  and  $B\geq 0$ , there exist constants  $C, c_1>0, c_2\geq 0, \tilde{c}_4>0, \tilde{c}_2\geq 0$  such that  $|e^{-(\eta-i/\varepsilon)\mathcal{S}(e^{i\pi/8}y)}F(e^{i\pi/8}y)|\leq C(1+|y|^k)e^{-c_1|y|^4+c_2|y|^2}$ .  $|e^{-(\eta-i/\varepsilon)\mathcal{S}(e^{i\pi/8}y)}F(e^{i\pi/8}y)|\leq C(1+|y|^k)e^{-\tilde{c}_4|y|^4+\tilde{c}_2|y|^2}$ .*

*Proof.* Under  $x=e^{i\pi/8}y$ , quartic monomials acquire phase  $e^{i\pi/2}=i$ . Hence  $\Re(i g Q_4(e^{i\pi/8}y))=-g\varepsilon|y|^4$ .  $\left(\frac{i}{\varepsilon}gQ_4(e^{i\pi/8}y)\right)=-\frac{g}{\varepsilon}|y|^4$ . The remaining quadratic and  $\eta$ -terms contribute at most  $+c_2|y|^2$ . Polynomial prefactors produce  $(1+|y|^k)(1+|y|^k)$ . The right side is integrable on  $\mathbb{R}^d$ .  $\square$

**Proposition 9** (Finite-dimensional  $\eta \rightarrow 0^+, \eta \rightarrow 0^+$  limit). *In the setting of Lemma 8, define  $I_\eta(F):=\int_{\mathbb{R}^d} e^{-(\eta-i/\varepsilon)\mathcal{S}(x)}F(x)dx$ ,  $I_\eta(F):=\int_{\mathbb{R}^d} e^{-(\eta-i/\varepsilon)\mathcal{S}(x)}F(x)dx$ , with contour branch fixed by angle  $\pi/8$ . Then  $\lim_{\eta\rightarrow 0^+} I_\eta(F)=I_0(F)$ .  $\lim_{\eta\rightarrow 0^+} I_\eta(F)=I_0(F)$ . If  $I_\eta(1)\neq 0$  for small  $\eta$  and  $I_0(1)\neq 0$ , then  $\lim_{\eta\rightarrow 0^+} \frac{I_\eta(F)}{I_\eta(1)}=\frac{I_0(F)}{I_0(1)}$ .*

*Proof.* For  $\eta>0$ , deform real contour to angle  $\pi/8$  (entire integrand, quartic decay on connecting arcs). On that contour, pointwise convergence as  $\eta\rightarrow 0^+$  is immediate. Lemma 8 gives a common  $L^1$  dominator. Apply dominated convergence to numerator and denominator.  $\square$

**Corollary 10** (De-regularized cylinder state). *For Gaussian-exponential cylinder observables in Theorem 3, the limit  $\omega_{\varepsilon,0}(F)=\lim_{\eta\rightarrow 0^+} \omega_{\varepsilon,\eta}(F)$  exists and is independent of  $\varepsilon$ .*

*Proof.* Reduce to stabilized finite dimension  $M = \max\{m, b\}$  by Proposition 5. Then apply Proposition 9 in dimension  $MM$ .  $\square$

# Gaussian Channel Expansion

Now take the Gaussian subcase  $g=0$ ,  $b=0$ :

$$S_m(x) = \frac{1}{2} \sum_{j=1}^m \lambda_j x_j^2. \text{ Define, for } F \in \mathcal{S}(\mathbb{R}^m) \text{ in } \mathcal{S}'(\mathbb{R}^m),$$

$$\omega_{\varepsilon,0}(F) := \int_{\mathbb{R}^m} m e^{i\varepsilon S_m(x)} F(x) dx \int_{\mathbb{R}^m} m e^{i\varepsilon S_m(x)} dx, \quad \omega_{\varepsilon,0}(F) := \frac{\int_{\mathbb{R}^m} e^{i \frac{1}{\varepsilon} S_m(x)} F(x) dx}{\int_{\mathbb{R}^m} e^{i \frac{1}{\varepsilon} S_m(x)} dx}.$$

**Proposition 11** (Exact operator form). *Let  $\mathcal{L}m = \sum_{j=1}^m \lambda_j \partial_{x_j}^2$ . Then  $\omega_\varepsilon, 0(F) = [\exp(i\varepsilon 2\mathcal{L}m)F]_{x=0}$ .  $\omega_\varepsilon \{ \frac{i}{2\varepsilon} \} (F) = \left[ \exp \left( \frac{i}{2\varepsilon} \mathcal{L}m \right) F \right]_{x=0}$ .*

*Proof.* Write  $F(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} m\hat{\phi}(\xi) e^{i\xi \cdot x} d\xi$ ,  $F(x) = \frac{1}{(2\pi)^m}$

$\int_{\mathbb{R}^m} \widehat{F}(\xi) e^{i\xi \cdot x} d\xi$ . By Gaussian completion (Fresnel branch),

[illegible]

Therefore  $\omega_{\varepsilon,0}(F) = \frac{1}{(2\pi)^m} \int \hat{F}(\xi) \exp(-i\varepsilon \sum_j \xi_j^2 \lambda_j) d\xi$ ,  $\omega_{\varepsilon,0}(F) = \frac{1}{(2\pi)^m} \int \hat{F}(\xi) \exp(-i\varepsilon \sum_j \frac{\xi_j^2}{\lambda_j}) d\xi$ . The multiplier is exactly that of  $\exp(i\varepsilon/2 \mathcal{L}_m) \exp(i\varepsilon/2 \mathcal{L}_m)$  evaluated at  $x=0$ .  $\square$

**Corollary 12** (Point-supported channel hierarchy). *For  $K \geq 1$   $K$  gel,*

$\omega_{\varepsilon,0}(F) = \sum_{k=0}^{\infty} \frac{1}{k!} (i\varepsilon)^k (\mathcal{L}mkF)(0) + RK_{\varepsilon}(F), \omega_{\varepsilon,0}(F) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\varepsilon}{2} \right)^k (\mathcal{L}mkF)(0) + R_{\varepsilon}(F),$  with  $RK_{\varepsilon}(F) = O(\varepsilon K) R_{\varepsilon}(F) = O(\varepsilon^K)$  as  $\varepsilon \rightarrow 0^+$ . Thus channels are derivatives of  $FF$  at the extremum  $x=0$ , i.e. point-supported distribution modes.

*Proof.* Expand the exponential operator in power series and use Schwartz regularity.  $\square$

# Static Extremum Localization and the Variational-Delta Ladder

**Proposition 13** (Static Morse localization). *Let  $f \in C^\infty(\mathbb{R}^d)$  in  $C^\infty(\mathbb{R}^d)$  with unique nondegenerate critical point  $x^*$ :  $\nabla f(x^*) = 0, \det \nabla^2 f(x^*) \neq 0$ .  $\nabla f(x^*) = 0, \det \nabla^2 f(x^*) \neq 0$ . For  $O \in C^\infty(\mathbb{R}^d)$  in  $C^\infty(\mathbb{R}^d)$ ,  $A_\varepsilon(O) := \varepsilon - d/2 \int_{\mathbb{R}^d} \text{div}(f(x) O(x)) dx$ ,  $A_\varepsilon(O) := \varepsilon - d/2 \int_{\mathbb{R}^d} \text{div}(f(x) O(x)) dx$ .*

satisfies  $|A_\varepsilon(O)|^2 \rightarrow (2\pi)^d |O(x^*)|^2 |\det \nabla^2 f(x^*)| \cdot |A_{\varepsilon(O)}|^2$  to  $(2\pi)^d \frac{1}{d} \frac{|O(x^*)|^2}{|\det \nabla^2 f(x^*)|}$ . Equivalently,  $|A_\varepsilon(O)|^2 \rightarrow (2\pi)^d \langle \delta(\nabla f), |O|^2 \rangle \cdot |A_{\varepsilon(O)}|^2$  to  $(2\pi)^d \langle \delta(\nabla f), |O|^2 \rangle$ .

*Proof.* Standard stationary phase at a single Morse critical point.  $\square$

**Corollary 14** (Finite-dimensional QM/QFT truncations). *For any finite-dimensional discretization  $S_N$  of a variational action (time-sliced QM or lattice QFT), the same stationary-phase mechanism localizes on  $\nabla S_N = 0, \nabla S_N = 0$ , providing the finite-dimensional realization of  $\delta(\delta S) \delta(\delta S)$  as an extremum selector.*

## Large-NN Mode-Coupled Lift

We now pass from fixed interacting blocks to a genuinely growing mode-coupled family.

**Theorem 15** (Large-NN coupled Gaussian-tail convergence with rate). *Fix  $m \geq 1$ . Let  $S_N(u, v) = P_m(u) + \sum_{j=m+1}^N (\lambda_j^2 + \beta_j(u)) v_j^2$ ,  $\beta_j(u) = \sum_{i=1}^m a_{ij} u_i^2$ ,  $S_N(u, v) = P_m(u) + \sum_{j=m+1}^N \left( \frac{\lambda_j^2}{2} + \beta_j(u) \right) v_j^2$ ,  $\beta_j(u) = \sum_{i=1}^m a_{ij} u_i^2$ , with:*

1.  $\lambda_j \geq \lambda > 0$ ,  $\lambda_j \geq \lambda > 0$ ,
2.  $a_{ij} \geq 0$ ,  $a_{ij} \geq 0$  and  $A_j := \sum_{i=1}^m a_{ij}$  satisfies  $\sum_{j=m+1}^\infty A_j \lambda_j < \infty$ ,  $\sum_{j=m+1}^\infty \frac{A_j}{\lambda_j} < \infty$ ,
3.  $P_m(u) \geq c_4 \|u\|^4 - c_2 \|u\|^2 - C_0 P_m(u) \geq c_4 \|u\|^4 - c_2 \|u\|^2 - C_0$ .

For bounded  $F_m$  and  $\eta > 0, \varepsilon > 0, \eta > 0, \varepsilon > 0$ , define  $\omega_{\varepsilon, \eta, N}(F_m) := \int \mathbb{R}^N e^{-(\eta - i/\varepsilon) S_N F_m(u)} du dv \int \mathbb{R}^N e^{-(\eta - i/\varepsilon) S_N} du dv$ .  $\omega_{\varepsilon, \eta, N}(F_m) := \frac{\int \int \mathbb{R}^N e^{-(\eta - i/\varepsilon) S_N F_m(u)} du dv \int \int \mathbb{R}^N e^{-(\eta - i/\varepsilon) S_N} du dv}{\int \int \mathbb{R}^N e^{-(\eta - i/\varepsilon) S_N F_m(u)} du dv \int \int \mathbb{R}^N e^{-(\eta - i/\varepsilon) S_N} du dv}$ . Then:

1.  $\omega_{\varepsilon, \eta, N}(F_m)$  converges as  $N \rightarrow \infty$  to  $\omega_{\varepsilon, \eta}(F_m)$ .
2. There exists  $C_{F_m, \varepsilon, \eta} > 0$  such that for  $N' > N \geq m$ ,  $N' \geq m$ ,  $|\omega_{\varepsilon, \eta, N'}(F_m) - \omega_{\varepsilon, \eta, N}(F_m)| \leq C_{F_m, \varepsilon, \eta} \sum_{j=N+1}^{N'} \frac{A_j}{\lambda_j}$ .

*Proof.* Integrate each Gaussian tail coordinate:

$\int \mathbb{R}^N e^{-(\eta - i/\varepsilon) (\lambda_j^2 + \beta_j(u)) t^2} dt = \sqrt{2\pi} \frac{1}{\sqrt{\lambda_j^2 + \beta_j(u)}}$ .  $\int \mathbb{R}^N e^{-(\eta - i/\varepsilon) (\lambda_j^2 + \beta_j(u)) t^2} dt = \sqrt{2\pi} \frac{1}{\sqrt{\lambda_j^2 + \beta_j(u)}}$ . Constants independent of  $u$  cancel in the normalized ratio, giving  $\omega_{\varepsilon, \eta, N}(F_m) = \mathcal{N}_N(F_m) \mathcal{D}_N$ ,  $\omega_{\varepsilon, \eta, N}(F_m) = \frac{\mathcal{N}_N(F_m)}{\mathcal{D}_N}$ , with  $\mathcal{N}_N(F) := \int \mathbb{R}^N e^{-(\eta - i/\varepsilon) P_m(u)} F(u) \Phi_N(u) du$ ,  $\mathcal{D}_N := \int \mathbb{R}^N e^{-(\eta - i/\varepsilon) P_m(u)} \Phi_N(u) du$ ,  $\Phi_N(u) := \prod_{j=m+1}^N \frac{1}{\sqrt{\lambda_j^2 + \beta_j(u)}}$ ,  $\Phi_N(u) := \prod_{j=m+1}^N \frac{1}{\sqrt{\lambda_j^2 + \beta_j(u)}}$ . Now  $-\log R_j(u) = \frac{1}{2} \log(1 + \beta_j(u) \lambda_j^{-2}) \leq \beta_j(u) \lambda_j^{-2} \leq \frac{1}{2} \log(1 + \beta_j(u) \lambda_j^{-2})$ .





# Observable-Class Extension

**Theorem 18** (Continuity on Schwartz and weighted Sobolev classes). *Let  $\mathcal{J}(F) = \int_{\mathbb{R}^d} \Phi(y) W(y) F(Ay) dy$ ,  $\mathcal{I}(F) = \int_{\mathbb{R}^d} e^{i\Phi(y)} W(y) F(Ay) dy$ , with  $A \in GL(d, \mathbb{C})$  and  $W(y) \leq C_0 e^{-c_4 \|y\|^4 + c_2 \|y\|^2}$ ,  $c_4 > 0$ .  $|W(y)| \leq C_0 e^{-c_4 \|y\|^4 + c_2 \|y\|^2}$ ,  $c_4 > 0$ . Then:*

1. *for every integer  $k > d$ , there exists  $C_k$  such that  $|\mathcal{J}(F)| \leq C_k \sup_x (1 + \|x\|)^k |F(x)|$ ,  $F \in \mathcal{S}(\mathbb{R}^d)$ ;  $|\mathcal{I}(F)| \leq C_k \sup_x (1 + \|x\|)^k |F(x)|$ ,  $F \in \mathcal{S}(\mathbb{R}^d)$ ;*
2. *for every  $k > d/2$ , there exists  $C'_k$  such that  $|\mathcal{J}(F)| \leq C'_k \|(1 + \|x\|^2)^{k/2} F\|_{L^2}$ ,  $F \in H^k$ ;  $|\mathcal{I}(F)| \leq C'_k \|(1 + \|x\|^2)^{k/2} F\|_{L^2}$ ,  $F \in H^k$ .*

Consequently, normalized functionals  $\omega(F) = \mathcal{J}(F)/\mathcal{J}(1)$ ,  $\omega(F) = \mathcal{I}(F)/\mathcal{I}(1)$  (when  $\mathcal{J}(1) \neq 0$ ,  $\mathcal{I}(1) \neq 0$ ) extend continuously from Gaussian-polynomial test families to both classes.

*Proof.* For Schwartz:  $|F(Ay)| \leq C_A p_k(F)(1 + \|y\|)^{-k}$ ,  $p_k(F) := \sup_x (1 + \|x\|)^k |F(x)|$ .  $|F(Ay)| \leq C_A p_k(F)(1 + \|y\|)^{-k}$ ,  $p_k(F) := \sup_x (1 + \|x\|)^k |F(x)|$ . Hence  $|\mathcal{J}(F)| \leq C_0 C_A p_k(F) \int_{\mathbb{R}^d} e^{-c_4 \|y\|^4 + c_2 \|y\|^2} (1 + \|y\|)^{-k} dy$ ,  $|\mathcal{I}(F)| \leq C_0 C_A p_k(F) \int_{\mathbb{R}^d} e^{-c_4 \|y\|^4 + c_2 \|y\|^2} (1 + \|y\|)^{-k} dy$ , and the integral is finite.

For weighted Sobolev:  $|\mathcal{J}(F)| \leq \|W(\cdot)(1 + \|\cdot\|^2)^{-k/2} F(Ay)\|_{L^2} \|W(\cdot)(1 + \|\cdot\|^2)^{k/2}\|_{L^2}$ .  $|\mathcal{I}(F)| \leq \|W(\cdot)(1 + \|\cdot\|^2)^{-k/2} F(Ay)\|_{L^2} \|W(\cdot)(1 + \|\cdot\|^2)^{k/2}\|_{L^2}$ . The first factor is finite by quartic decay; the second is bounded by  $C_A \|F\|_{H^k}$  after linear change of variables.  $\square$

# Schwinger-Dyson and $\tau_\mu$ Scale Covariance

**Theorem 19** (Finite-dimensional Schwinger-Dyson identity). *Let  $c = \eta - i/\varepsilon$ ,  $\eta \in \mathbb{R}$ ,  $\varepsilon > 0$ .  $\mathcal{J}_c(F) := \int e^{-cS(x)} F(x) dx$ .  $\mathcal{I}_c(F) := \int e^{-cS(x)} F(x) dx$ . Assume integrability and vanishing boundary flux for admissible  $F$  and vector field  $V$ . Then  $\mathcal{J}_c(V \cdot \nabla S F) = \mathcal{J}_c(\nabla \cdot (VF))$ .  $\mathcal{I}_c(V \cdot \nabla S F) = \mathcal{I}_c(\nabla \cdot (VF))$ . If  $\mathcal{J}_c(1) \neq 0$ ,  $\mathcal{I}_c(1) \neq 0$ , then  $\omega_c(V \cdot \nabla S F) = \mathcal{J}_c(\nabla \cdot (VF)) / \mathcal{J}_c(1)$ .  $\omega_c(V \cdot \nabla S F) = \mathcal{I}_c(\nabla \cdot (VF)) / \mathcal{I}_c(1)$ . In particular, for constant  $V = e_i$  and  $F \equiv 1$ ,  $\omega_c(\partial_i S) = 0$ ,  $\omega_c(\partial_i S) = 0$ .*

*Proof.*  $0 = \int \nabla \cdot (e^{-cS} V F) dx = \int e^{-cS} (\nabla \cdot (VF) - c V \cdot \nabla S F) dx$ .  $0 = \int \nabla \cdot (e^{-cS} V F) dx = \int e^{-cS} (\nabla \cdot (VF) - c V \cdot \nabla S F) dx$ . Rearrange, then divide by  $\mathcal{J}_c(1)$  for the normalized form.  $\square$

**Theorem 20** (Exact  $\tau_\mu$  covariance). *For  $\omega_\kappa, \eta, h(F) := \int e^{-(\eta - i/h) \kappa S(x)} F(x) dx$ ,  $\omega_\kappa, \eta, h(F) := \int e^{-(\eta - i/h) \kappa S(x)} F(x) dx$ ,  $\omega_\kappa, \eta, h(F) := \int e^{-(\eta - i/h) \kappa S(x)} F(x) dx$ ,  $\omega_\kappa, \eta, h(F) := \int e^{-(\eta - i/h) \kappa S(x)} F(x) dx$ .*

define  $\tau\mu: (\kappa, \eta, h) \mapsto (\mu\kappa, \eta/\mu, \mu h), \mu > 0$ .  $\tau_\mu: (\kappa, \eta, h) \mapsto (\mu\kappa, \eta/\mu, \mu h), \mu > 0$ . Then  $\omega_{\kappa, \eta, h}(F) = \omega_{\tau\mu}(\kappa, \eta, h)(F)$ .  $\omega_{\kappa, \eta, h}(F) = \omega_{\tau_\mu(\kappa, \eta, h)}(F)$ .

*Proof.* Directly,  $(\eta\mu - i\mu h)(\mu\kappa) = (\eta - i/h)\kappa$ .  $\left(\frac{\eta}{\mu} - \frac{i}{\mu h}\right)(\mu\kappa) = (\eta - i/h)\kappa$ . Hence numerator and denominator kernels are unchanged.  $\square$

## Dependency Chain

The proofs in this manuscript now close in the following order:

1. Sections 2–5: core scoped Claim 1 closure.
2. Theorems 15 and 16: large-NN coupled extensions (Gaussian-tail explicit rate and non-factorized quartic-tail class).
3. Theorem 17: explicit non-vanishing criteria for partition factors.
4. Theorem 18: observable-class extension to Schwartz/Sobolev.
5. Theorems 19 and 20: Schwinger-Dyson identities and exact scale-flow covariance.

## Conclusion

This manuscript now gives a theorem-grade chain from the static/finite-dimensional variational-delta picture to a broad scoped family of oscillatory states including:

1. exact projective closure and de-regularization,
2. explicit semiclassical channel expansion,
3. large-NN mode-coupled convergence with quantitative tail control (Gaussian-tail and non-factorized quartic-tail classes),
4. non-vanishing control for normalization factors,
5. extension to large observable classes,
6. Schwinger-Dyson identities and scale-flow covariance.

Open frontier (outside this closure): full interacting, non-factorized, genuinely field-theoretic continuum limits with uniform renormalization control beyond the scoped polynomial block-tail class.

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