Field-Theory Consistency Across Spacetime Dimension: General-Dimension Program, Gates, and Literature Anchors

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Abstract

This manuscript starts Goal 1C (fields) as a general-dimension paper track. It does not claim full interacting closure in all dimensions. Instead it fixes a dimension-indexed theorem program with explicit gates: regulated existence, continuum existence, and reconstruction. Current progress includes an explicit $d = 2$ interacting closure in an ultralocal class, plus a $d = 3$ local-interaction scoped closure in a compact-spin Euclidean subclass. AN-24 removes the hard cutoff in that same branch inside a local-renormalized channel. AN-26 plus AN-26B close SD test-side C_b^1 extension with explicit tail-control and insertion-moment verification in-branch. AN-27 then transfers the widened local class to the oscillatory/de-regularized branch under explicit non-vanishing and contour-envelope hypotheses. AN-28 extends that oscillatory branch to disconnected nonlocal cylinders, and AN-29 adds explicit refinement-Cauchy rates plus denominator bookkeeping for their continuum extraction in the same scoped lane. AN-30 extends this further to finite graph-indexed multi-block families with explicit combinatorial constants and projective-consistency closure. AN-31 then lifts AN-30 to uniformly locally finite exhaustion families with summability-weighted tail control.

1 Scope

In scope

1. one paper-level framework that treats $d = 2$, $d = 3$, $d = 4$, and $d > 4$ in a unified way,
2. explicit assumptions and failure gates by dimension,
3. source-backed baseline claims from standard literature.

Out of scope

1. claiming a fully closed interacting $d = 4$ continuum theorem in this pass,
2. replacing constructive estimates with formal kernel language alone.

2 Field Setup

For a local scalar prototype on $\mathbb{R}^d$,

$$S_d[\Phi] = \int_{\mathbb{R}^d} \left[\frac{1}{2}|\nabla\Phi|^2 + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 \right] d^d x.$$

At finite cutoff/volume level (symbolically Λ, a, L),

$$\omega_{c,\Lambda,a,L}(F) = \frac{\int e^{-cS_{d,\Lambda,a,L}[\Phi]} F[\Phi]\, D\Phi}{\int e^{-cS_{d,\Lambda,a,L}[\Phi]}\, D\Phi}, \qquad \Re c > 0.$$

3 Three Gates (Mandatory)

For any dimension branch, the paper only upgrades to "closed" when all three gates are met:

1. **G1 Regulated existence**: finite cutoff and finite volume channel is well-defined.
2. **G2 Continuum existence**: limits in cutoff/volume exist with nontrivial content.
3. **G3 Reconstruction**: Euclidean channel maps to the intended physical QFT object class.

4 Dimension-Indexed Program Claim

Proposition 1 (General-dimension program schema)**.** *Let $d \geq 2$. For each dimension branch $\mathcal{B}_d$, define closure status*

$$\mathrm{Closed}(\mathcal{B}_d) \iff G1(\mathcal{B}_d) \wedge G2(\mathcal{B}_d) \wedge G3(\mathcal{B}_d).$$

Then:

1. *$d = 2$ is the first full-closure candidate branch.*
2. *$d = 3$ is the next branch with superrenormalizable control targets.*
3. *$d = 4$ remains a frontier branch where theorem-grade scoped statements and explicit open assumptions must be separated.*
4. *$d > 4$ is treated primarily as EFT/mean-field/triviality-guided branch unless a UV-complete model class is fixed.*

Remark 1. *This proposition is a program statement, not a universal theorem of existence. Its role is to prevent overclaiming and to force assumption-explicit upgrades.*

5 Current Status by Dimension

1. $d = 2$**:** scoped interacting closure is already available in an ultralocal ϕ^4 class (existence of the cylinder limit state, SD pass-through, and exact c-invariance in that class).
2. $d = 3$**:** nearest-neighbor local interactions now have scoped closure in a compact-spin Euclidean subclass (B1-B4 + renormalized B5 input closed there), while hard-cutoff removal is now closed in a local-renormalized channel and class-widening is closed in this scoped Euclidean branch through AN-25/AN-26/AN-26B. AN-27 transfers this widened class to the oscillatory/de-regularized branch; AN-28/AN-29 then extend to disconnected nonlocal cylinders with explicit refinement-Cauchy and denominator-rate control in that same scoped branch. AN-30 upgrades this to finite graph-indexed multi-block families with explicit combinatorial rates and projective consistency in the refinement limit. AN-31 lifts this to exhaustion families with explicit summable-tail Cauchy and projective-defect control.
3. $d = 4$**:** frontier branch; no full interacting closure claim is made here.
4. $d > 4$**:** treated as EFT/triviality-guided unless a UV-complete model is fixed.

6 What the Current Lean Chain Supports

Machine-checked finite-model modules provide reusable inequality templates for small-parameter increment control and regularity bookkeeping.

Current limitation: these modules do not by themselves discharge field-level G2/G3. They feed the field program only when translated into dimension-indexed bound propositions.

7 Current $d = 3$ Interacting Branch (Scoped Closure + Open Gap)

Work in the local periodic finite-volume ϕ^4 lattice class with nearest-neighbor coupling, Euclidean $c \in [c_0, c_1] \subset (0, \infty)$, bounded local cylinders $F(\phi) = f(\phi|_B)$, and local edge insertion

$$G_B = \sum_{\langle x,y \rangle \in E_B} (\phi_x - \phi_y)^2.$$

A first finite-volume estimate layer gives explicit constants

$$K_F = 2\|F\|_\infty, \qquad M_{B,a} = \frac{4|E_B|}{c_0 m_0^2 a^3},$$

with

$$|F - \omega(F)| \leq K_F, \qquad \omega(G_B) \leq M_{B,a},$$

uniform in L and $\kappa \in [0, \kappa_*]$.

A renormalized insertion channel removes explicit a^{-3} growth from this B5b input:

$$G_{B,a}^{\mathrm{ren}} := a^3 G_B, \qquad \omega(G_{B,a}^{\mathrm{ren}}) \leq M_B^{\mathrm{ren}}, \qquad M_B^{\mathrm{ren}} = \frac{4|E_B|}{c_0 m_0^2},$$

uniform in a, L, κ.

The AN-23 compact-spin branch discharges these four obligations in one explicit interacting Euclidean subclass:

- **B1** uniform local moment bounds in (a, L),
- **B2** local tightness/precompactness of cylinder marginals,
- **B3** denominator non-vanishing on the working c-domain,
- **B4** SD insertion-control pass-through to the continuum limit.

Thus the AN-22 candidate is upgraded to scoped closure in that subclass. AN-24 then removes the hard compact-spin cutoff $R \to \infty$ while keeping B1-B4 in the local-renormalized compact-support channel. AN-25 closes observable-side widening $C_c \to C_b$. AN-26 + AN-26B close SD test-side widening $C_c^1 \to C_b^1$ by combining the Holder/Markov tail criterion with an explicit $q = 4/3$ insertion-moment bound.

8 Literature Anchors for the Dimension Ladder

Reconstruction and Euclidean axioms

1. Osterwalder–Schrader I (1973), Euclidean axioms: doi:10.1007/BF01645738.
2. Osterwalder–Schrader II (1975), reconstruction continuation: doi:10.1007/BF01608978.

Scale/refinement framework

1. Wilson–Kogut (1974), renormalization-group framing: doi:10.1016/0370-1573(74)90023-4.

Constructive $d = 2$ baseline

1. Guerra–Rosen–Simon (1975), $P(\phi)_2$ Euclidean construction: doi:10.2307/1970985.

$d = 4$ frontier and triviality anchors

1. Aizenman (1981), ϕ_d^4 triviality/mean-field features for $d > 4$: doi:10.1103/PhysRevLett.47.1.
2. Aizenman–Duminil-Copin (2021), marginal triviality in critical $4d$ scaling limits: doi:10.4007/annals.2021.

9 Immediate Next Scientific Step

The next theorem target is:

1. extend AN-31 from cylinder observables to weighted-local test classes with explicit exhaustion-uniform insertion estimates (AN-32),
2. keep summability-weighted constants explicit while preserving projective-consistency bookkeeping and de-regularization compatibility.

10 Validation Contract

Assumptions

1. model class and dimension branch are explicit in each statement,
2. closure status is always reported against G1/G2/G3, never implied.

Units and dimensions

1. S_d has action dimension,
2. phase/weight argument is dimensionless.

Independent checks

1. theorem/estimate channel (Lean where feasible, analytic where not),
2. bibliography check against standard references above,
3. executable finite surrogate checks: `python3.12 research/workspace/simulations/claim1_d3_finite_`
4. renormalized-channel check: `python3.12 research/workspace/simulations/claim1_d3_renormalize`
5. continuum-branch proxy check: `python3.12 research/workspace/simulations/claim1_d3_an22_cont`
6. AN-23 compact-spin closure diagnostics: `python3.12 research/workspace/simulations/claim1_d3_a`
7. AN-24 hard-cutoff lift diagnostics: `python3.12 research/workspace/simulations/claim1_d3_an24_cu`
8. AN-25 local class-extension diagnostics: `python3.12 research/workspace/simulations/claim1_d3_an`
9. AN-26 SD-test tail insertion-control diagnostics: `python3.12 research/workspace/simulations/clai`

10. AN-26B insertion $L^{4/3}$-moment diagnostics: `python3.12 research/workspace/simulations/claim1_d3`

11. AN-27 oscillatory/de-regularized transfer diagnostics: `python3.12 research/workspace/simulations/`

12. AN-28 two-block nonlocal transfer diagnostics: `python3.12 research/workspace/simulations/claim1`

13. AN-29 nonlocal refinement-Cauchy diagnostics: `python3.12 research/workspace/simulations/claim`

14. AN-30 multiblock projective-consistency diagnostics: `python3.12 research/workspace/simulations/c`

15. AN-31 exhaustion-summability lift diagnostics: `python3.12 research/workspace/simulations/claim1`

Confidence statement

This document is a theorem-program and source-anchored roadmap. It now includes scoped $d = 3$ closure with hard-cutoff lift and AN-25/AN-26/AN-26B class-extension closure in a scoped Euclidean branch. AN-27 closes oscillatory/de-regularized transfer; AN-28/AN-29 extend this to disconnected nonlocal cylinders with explicit refinement-Cauchy bookkeeping in the same scoped branch; AN-30 extends further to finite graph-indexed multi-block projective consistency with explicit combinatorial constants. AN-31 lifts this to exhaustion-family summability control. Full global continuum interacting closure remains open.

11 Reproducibility Metadata

- date anchor: 2026-02-09 (US),
- build toolchain: `/Library/TeX/texbin/pdflatex` (TeX Live 2025),
- safe build script: `~/.codex/skills/pdflatex-safe-build/scripts/build_pdflatex_safe.sh`.