Dynamic Variational Consistency and Scoped Equivalence to the Path Integral with Historical Discussion

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Abstract

This paper develops a dynamic consistency chain for $0+1$-dimensional actions: time-sliced transition amplitudes induce normalized cylinder states that are stable under refinement, admit $\eta \to 0^+$ de-regularization, satisfy Schwinger–Dyson identities, and are invariant under the c-preserving flow τ_μ. Under explicit assumptions, these statements imply scoped equivalence to the path-integral formalism for the same model class and boundary data. A dedicated historical section records the Dirac (1933) $\to$ Feynman (1948) $\to$ Wilson–Kogut (1974) lineage.

1 Scope

In-scope claim

The claim proved here is scoped dynamic equivalence:

1. construction of a consistent continuum cylinder state from time slicing,
2. persistence of Schwinger–Dyson structure in the limit,
3. invariance along c-preserving reparameterizations,
4. identification of that limit with the path-integral functional in the same scoped class.

Out of scope

1. full interacting field-theory closure in $d = 4$,
2. model-independent nonperturbative global path-integral existence,
3. scattering/unitarity claims beyond the scoped construction.

2 Dynamic setup

Let

$$S[q] = \int_{t_i}^{t_f} L(q, \dot{q}, t)\, dt.$$

At time-slicing resolution N, write the discretized action S_N and define a regularized transition-amplitude channel x_0, x_N fixed boundary data and $N-1$ interior integrations:

$$\mathcal{A}_{\varepsilon,\varepsilon_0,N}(O) := \int_{\mathbb{R}^{N-1}} \varepsilon^{-(N-1)/2} \exp\left(\frac{i}{\varepsilon}\varepsilon_0 \sum_{k=0}^{N-1} L_{\varepsilon_0}\left[x_k, \frac{x_{k+1}-x_k}{\varepsilon_0}\right] \right) O[x_\bullet] \prod_{k=1}^{N-1} dx_k.$$

For the normalized state channel in the dressed notation, use

$$\omega_{c,N}(F) := \frac{\int_{\mathbb{R}^N} e^{-cS_N(x)} F(x)\, dx}{\int_{\mathbb{R}^N} e^{-cS_N(x)}\, dx}, \qquad c = (\eta - i/h)\kappa, \quad \Re c > 0.$$

3 Dynamic consistency theorem chain

Assumptions

Assume the following on a compact c-domain K:

(D1) **Projective compatibility**: cylinder observables are consistently lifted along $N \mapsto N+1$.

(D2) **Denominator non-vanishing**: $\int e^{-cS_N} \neq 0$ for all large N, uniformly in $c \in K$.

(D3) **Uniform Cauchy tail**: for each cylinder F_m,

$$|\omega_{c,N'}(F_m) - \omega_{c,N}(F_m)| \leq C_{F_m,K}\tau_N, \qquad \tau_N \to 0.$$

(D4) **Finite-N Schwinger–Dyson identity**: for admissible ψ,

$$\omega_{c,N}(\partial_i \psi) = c\, \omega_{c,N}(\psi\, \partial_i S_N).$$

(D5) **c-invariance along τ_μ**:

$$\tau_\mu : (\kappa, \eta, h) \mapsto (\mu\kappa, \eta/\mu, \mu h), \quad \mu > 0,$$

preserves c, hence preserves the finite-N kernels.

(D6) **De-regularization pass-through**: for observables in the working class, the one-sided limit $\eta \to 0^+$ exists and commutes with the $N \to \infty$ limit in the stated channel.

Theorem 1 (Scoped dynamic consistency). *Under (D1)–(D6):*

1. *$\omega_{c,N}(F_m) \to \omega_c(F_m)$ for each cylinder F_m, uniformly on compact $K \subset \{c : \Re c > 0\}$.*
2. *ω_c is invariant along τ_μ-orbits (depends on parameters only through c).*
3. *Schwinger–Dyson identities pass to the limit:*

$$\omega_c(\partial_i \psi) = c\, \omega_c(\psi\, G_i),$$

where G_i is the cylinder-pairing limit of $\partial_i S_N$.

4. *The de-regularized state $\omega_{-i\kappa/h}$ exists (at fixed κ) in the scoped observable class.*

Proof sketch. 1. (D3) + (D2) give convergence of normalized cylinder expectations.

2. (D5) gives exact finite-N kernel invariance; pass to $N \to \infty$.
3. (D4) + uniform pairing control imply SD pass-through in the limit.
4. (D6) gives existence/commutation of the $\eta \to 0^+$ channel in scope.

□

Corollary 1 (Scoped path-integral equivalence). *Define the scoped path-integral functional by the refinement limit of the normalized time-sliced transition-amplitude channel under (D1)–(D6). Then this functional equals ω_c on the scoped cylinder observable class.*

Remark 1. *The corollary is an equivalence* in the scoped class *where all convergence and non-vanishing gates are discharged. It is not a claim of global interacting QFT closure.*

4 Historical discussion (required)

Dirac (1933)

Dirac's 1933 analysis emphasized the role of an exponential phase factor in quantum transitions, providing the conceptual seed for amplitude-level composition rules.

Feynman (1948)

Feynman recast quantum mechanics via time slicing and action-phase summation over histories, making transition-amplitude composition over intermediate configurations explicit.

Wilson–Kogut (1974)

Wilson–Kogut RG language reframed refinement as scale-flow control, clarifying why continuum claims require explicit fixed-point and convergence gates rather than formal passage to finer slices.

Current theoretical framing

The present program combines these threads:

1. transition-amplitude composition at finite slicing,
2. explicit convergence/non-vanishing gates,
3. c-invariant scale-flow covariance and Schwinger–Dyson persistence,
4. geometric 1/2-density language only where kernel-bundle structure is explicit.

5 Validation contract (Goal 1B)

Assumptions

Model class, boundary data, regulator net, observable class, and normalization domain are explicit in ($D1$)–($D6$).

Units and dimensions check

1. each action contribution has action dimensions,
2. phase S/ε is dimensionless,
3. slicing normalization factors are tracked at each N.

Symmetry and conservation checks

1. boundary-condition consistency under slicing/refinement,
2. finite-N variational symmetry identities used before continuum passage,
3. conservation-law checks in scoped model channels where relevant.

Independent cross-check paths

1. analytic finite-dimensional SD and scale-flow covariance derivation channels,
2. numerical diagnostics:
   - `python3.12 research/workspace/simulations/claim1_cylinder_gaussian_toy.py`,
   - `python3.12 research/workspace/simulations/claim1_continuum_cauchy_diagnostics.py`,
   - `python3.12 research/workspace/simulations/claim1_fd_schwinger_dyson_check.py`,
   - `python3.12 research/workspace/simulations/claim1_scale_flow_covariance_check.py`,
   - `python3.12 research/workspace/simulations/claim1_groupoid_tau_sd_dependency_check.py`
3. optional formal-companion modules (inequality/derivative backbone):
   - `research/workspace/proofs/Claim1lean/CovarianceDerivative.lean`,
   - `research/workspace/proofs/Claim1lean/RatioStateDerivativeBound.lean`,
   - `research/workspace/proofs/Claim1lean/RatioStateIncrementBound.lean`,
   - `research/workspace/proofs/Claim1lean/FiniteExponentialIncrementBound.lean`,
   - `research/workspace/proofs/Claim1lean/FiniteExponentialRegularity.lean`.

Confidence statement

The result is theorem-grade in the scoped dynamic class under $(D1)$–$(D6)$. Any statement outside those gates must be marked unverified.

6 Reproducibility metadata

- build toolchain tested here: `/Library/TeX/texbin/pdflatex` (TeX Live 2025),
- safe build script: `~/.codex/skills/pdflatex-safe-build/scripts/build_pdflatex_safe.sh`,
- date anchor: 2026-02-09 (US).

7 Conclusion

Dynamic variational consistency is presented here as a self-contained theorem chain: refinement stability, denominator control, SD persistence, and de-regularization control. Within those assumptions, scoped equivalence to the path-integral formalism is explicit and historically motivated. As of 2026-02-09 (US date), this dynamic component is locked at scoped theorem-grade level, while field-level closure remains an independent Gate G2/G3 program handled in Paper 3.