Refinement Compatibility from Newton to Renormalization: Why Composition Forces Oscillatory Amplitudes

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Abstract

Many "continuum" constructions in physics are operational limits of iterative refinement: polygonal approximations in mechanics, time slicing in quantum mechanics, lattice and cutoff procedures in field theory. The central theme of this paper is a compatibility requirement: refinements should compose consistently and yield stable observable assignments. We argue that this requirement makes two structures hard to avoid. First, if weights compose multiplicatively under cutting while the underlying functional is additive, then exponential weighting is forced; the oscillatory weight $e^{\frac{i}{\hbar}S}$ is the canonical instance. Second, semigroup composition at short times forces rigid normalization exponents (the universal $t^{-d/2}$ scaling in d dimensions). These "square-root" normalization laws are naturally expressed at the amplitude/half-density level and become visible already in finite dimensions through stationary phase.

The paper is expository but theorem-facing: we state four representative theorem-grade packages in explicitly scoped settings (static stationary phase, time-sliced consistency under explicit gates, Gaussian semigroup normalization, and a screened-Abelian Yukawa kernel example). We also summarize the current boundary between proved scoped results and open global gates (continuum existence and reconstruction in interacting local QFT, and first-principles transfer control in non-Abelian confinement).

1 Introduction

The phrase "take the limit as the refinement parameter goes to zero" hides a genuine mathematical issue: the limit need not exist, need not be unique, and may depend on normalization or subtraction conventions. This is not a pathology of quantum theory alone. Newton's polygonal method in central-force motion is already a refinement construction: one first proves an exact finite-step invariant and only then defines the smooth statement as a controlled refinement limit.

This paper proposes a unifying viewpoint:

> *A physical law is an assignment of observables that is stable under controlled refinement and compatible with composition.*

In this view, quantization and renormalization are not treated as unrelated "add-ons". They are two distinct mechanisms for making refinement limits stable when naive limits fail:

1. **Oscillatory amplitudes** stabilize composition rules that localize on variational extrema in a distributional sense.

2. **Renormalization group (RG) flow** stabilizes divergent refinement procedures by explicit regulator dependence plus regulator-independent observables.

To avoid over-claiming, we maintain an explicit scope discipline:

- We do *not* claim a complete interacting continuum construction in $d = 4$.

- We do *not* treat geometric half-density language as a substitute for analytic convergence gates.
- When a statement depends on a model class or on dimension, we tag that dependency explicitly.

2 Refinement compatibility and composition

Definition 1 (Refinement compatibility principle (RCP))**.** *Fix a family of finite-resolution descriptions indexed by a refinement parameter Λ (for example, a time slicing step, a lattice spacing, or a cutoff scale), together with refinement maps $\rho_{\Lambda\to\Lambda'}$ for $\Lambda' \succcurlyeq \Lambda$. An observable assignment $\mathcal{O}_\Lambda \mapsto \langle \mathcal{O}_\Lambda \rangle_\Lambda$ is* refinement compatible *if:*

1. ***Projective consistency:*** $\langle \mathcal{O}_\Lambda \rangle_\Lambda = \langle \rho_{\Lambda\to\Lambda'} \mathcal{O}_\Lambda \rangle_{\Lambda'}$ *whenever both sides are defined.*
2. ***Stability:*** *for each $\mathcal{O}$ in a specified observable class, the refinement limit exists in a specified topology: $\langle \mathcal{O}_\Lambda \rangle_\Lambda \to \langle \mathcal{O} \rangle$ as $\Lambda \to \infty$ (or $\Lambda \to 0$, depending on convention).*

RCP is intentionally minimal: it does not assume an ontological "continuum". It is a compatibility requirement for *how* we define objects by refinement.

Why exponentials appear. Suppose a history weight W composes multiplicatively under cutting while the underlying functional S composes additively (as actions do under concatenation):

$$W(\mathrm{piece}_1 \circ \mathrm{piece}_2) = W(\mathrm{piece}_1)\, W(\mathrm{piece}_2), \qquad S(\mathrm{piece}_1 \circ \mathrm{piece}_2) = S(\mathrm{piece}_1) + S(\mathrm{piece}_2).$$

Under mild regularity assumptions, this forces $W \propto \exp(c_0 S)$ with c_0 of dimension 1/action. Dirac's phase-weighting and Feynman's time slicing are the best-known instances [2, 3].

3 Newton's polygonal refinement (finite invariants first)

In Newton's central-force polygonal construction, the key point is methodological: one proves a *finite-step invariant* (equal swept areas per equal time step under a central impulse) and then defines the smooth areal-velocity law by a refinement limit. In modern language, this is a template for how to keep refinement-to-zero statements honest. See [1] for the original and any standard mechanics text for the torque/areal velocity identity.

4 Static prototype: oscillatory amplitude and localization

The static prototype is finite-dimensional and theorem-grade under standard hypotheses.

Theorem 1 (Static localization from oscillatory amplitude (scoped))**.** *Let $f \in C^\infty(\mathbb{R})$ and $O \in C_c^\infty(\mathbb{R})$. Define the oscillatory amplitude*

$$A_\varepsilon(O) := \varepsilon^{-1/2} \int_{\mathbb{R}} \mathrm{e}^{\frac{i}{\varepsilon} f(x)}\, O(x)\, dx, \qquad \varepsilon > 0.$$

Assume f' has a unique simple zero $x_0 \in \operatorname{supp}(O)$, i.e. $f'(x_0) = 0$ and $f''(x_0) \neq 0$. Then, as $\varepsilon \to 0^+$,

$$|A_\varepsilon(O)|^2 \longrightarrow 2\pi \frac{|O(x_0)|^2}{|f''(x_0)|} = 2\pi \langle \delta(f'), |O|^2 \rangle,$$

up to Fourier-normalization convention.

Remark 1 (Multi-critical interference and averaging)**.** *If several critical points contribute, stationary phase produces a finite sum of oscillatory phases, and $|A_\varepsilon(O)|^2$ generally contains nonconvergent interference terms. In nonresonant settings (pairwise distinct critical values), these cross terms vanish under explicit coarse graining (e.g. a Cesàro average in $t = 1/\varepsilon$). When critical values coincide, coherent blocks must be grouped explicitly before averaging.*

Theorem 1 can be viewed as a toy witness for a broader pattern: an *amplitude*-level object is the compositionally natural one, while a squaring step produces a *density*-level object localized on classical extrema.

5 Dynamics prototype: time slicing and scoped consistency

For actions on time histories, the variational localization target is formally $\delta(\delta S/\delta q)$. Time slicing turns this into an exact finite-dimensional object $\delta(\nabla S_N)$ at fixed resolution. The remaining work is to control the refinement limit.

Theorem 2 (Scoped dynamic consistency (gate-form statement))**.** *Consider normalized finite-N states of the schematic form*

$$\omega_{c,N}(F) := \frac{\int_{\mathbb{R}^N} \mathrm{e}^{-cS_N(x)} F(x)\, dx}{\int_{\mathbb{R}^N} \mathrm{e}^{-cS_N(x)}\, dx}, \qquad c \in \mathbb{C},\ \Re c > 0,$$

on an explicit cylinder observable class. Assume the following gates on compact c-domains:

1. *projective compatibility under refinement $N \mapsto N'$,*
2. *uniform non-vanishing of denominators,*
3. *uniform Cauchy control for cylinder expectations,*
4. *finite-N Schwinger–Dyson identities,*
5. *c-invariance under a c-preserving reparameterization flow,*
6. *controlled de-regularization (a one-sided $\eta \to 0^+$ limit in an admissible class) commuting with the refinement limit.*

Then $\omega_{c,N}(F) \to \omega_c(F)$ on cylinders, Schwinger–Dyson identities pass to the limit in the same scope, and the refinement limit of the normalized time-sliced transition amplitude agrees with ω_c on the same observable class.

Theorem 2 is deliberately formulated as a gate checklist: it is a precise *proof target.* The foundational point is that time slicing does not by itself provide a continuum object; it provides a finite-resolution object to which RCP can be applied.

6 Newton-limit paradox support: semigroup normalization forces $t^{-d/2}$

One concrete place where refinement compatibility becomes rigid is short-time kernel composition. In Euclidean signature, the heat kernel is the standard example; here we highlight the abstract mechanism: *semigroup composition fixes the normalization exponent.*

Theorem 3 (Gaussian semigroup normalization exponent (Euclidean, kernel-level))**.** *Let $d \in \mathbb{N}$. Consider translation-invariant kernels $K_t : \mathbb{R}^d \to \mathbb{R}$ of the Gaussian form*

$$K_t(x) = a(t)\,\exp\!\left(-\frac{|x|^2}{4\,b(t)}\right), \qquad t > 0,$$

with $a(t) > 0$, $b(t) > 0$, and assume b is continuous on $(0, \infty)$. Assume further that:

1. ***Semigroup (convolution) composition:*** *$K_{t+s} = K_t * K_s$ for all $t, s > 0$.*
2. ***Mass normalization:*** *$\int_{\mathbb{R}^d} K_t(x)\,dx = 1$ for all $t > 0$.*

Then $b(t)$ is additive ($b(t+s) = b(t) + b(s)$), hence $b(t) = \sigma t$ for some $\sigma > 0$, and

$$a(t) = \big(4\pi b(t)\big)^{-d/2} \propto t^{-d/2}.$$

In particular, the diagonal normalization exhibits the universal exponent: $K_t(0) \propto t^{-d/2}$.

Proof sketch. Semigroup composition of Gaussians forces additivity of the variance parameter $b(t)$ (already visible in one dimension by explicit convolution). With $b(t) = \sigma t$, mass normalization fixes $a(t)$ by a direct Gaussian integral: $\int_{\mathbb{R}^d} \exp(-|x|^2/(4b))dx = (4\pi b)^{d/2}$. □

Theorem 3 is a minimal formal witness for the "square-root Jacobian" theme: composition fixes a half-power exponent ($d/2$) that is naturally attached to an amplitude kernel rather than to a probability density. In semiclassical settings, the same rigidity reappears through mixed-Hessian determinant prefactors (Van Vleck type formulas), where eliminating intermediate variables produces Schur-complement determinants.

7 Renormalization as limit control

In interacting field models, refinement limits may diverge. Renormalization is the disciplined version of the following template:

regulate → subtract divergence → fix finite ambiguity by a normalization condition → take the limit.

The RG expresses the requirement that changing the intermediate scale does not change physical observables; in operational terms, it is a compatibility condition for a family of regulated descriptions [4].

The key foundational separation is:

- **Kinematic coherence:** composition rules and coordinate invariances at fixed regulator (often expressible in geometric kernel language).
- **Dynamical convergence:** existence of regulator-removed limits in an explicit topology, plus persistence of identities (e.g. Schwinger–Dyson) under the limit.

8 Example: screened Abelian long-range behavior (dimension-explicit)

To illustrate the "tag dependencies explicitly" rule, consider a screened Abelian ($U(1)$) sector with a mass gap $m > 0$ in spatial dimension $n = d$ (spacetime dimension $D = n + 1$).

Theorem 4 (Yukawa kernel and asymptotic (screened Abelian, all D))**.** *Let $n \ge 1$ and $m > 0$. The unique decaying fundamental solution of $(-\Delta + m^2)G_{n,m} = \delta_0$ on $\mathbb{R}^n$ satisfies*

$$G_{n,m}(r) = \frac{1}{(2\pi)^{n/2}} \left(\frac{m}{r}\right)^{\nu} K_\nu(mr), \qquad \nu = \frac{n}{2} - 1,$$

and as $r \to \infty$,

$$G_{n,m}(r) = C_{n,m}\, r^{-(n-1)/2} \mathrm{e}^{-mr}\big(1 + O(r^{-1})\big),$$

hence the inter-source interaction contribution decays exponentially.

Proof sketch. Use the Fourier representation $\widehat{G}(k) = (|k|^2 + m^2)^{-1}$, reduce radially, and apply the standard Hankel/Bessel identity to obtain the modified Bessel K_ν form; then use $K_\nu(z) \sim \sqrt{\pi/(2z)}\, \mathrm{e}^{-z}$ as $z \to +\infty$. □

9 Status and open problems (date-anchored)

We close with a compact program scorecard as of 2026-02-10. Scores are internal maturity markers (0–10) used to prioritize proof work:

- 10: theorem-grade closure in intended global scope,
- 7–9: theorem-grade closure in a strong scoped model with explicit remaining gaps,
- 0–6: substantial structure but missing key closures.

Track	Score	Closure boundary (scoped) and open gap
Static amplitude localization	9	Theorem 1 is theorem-grade under nondegeneracy; multi-critical interference requires explicit averaging prescriptions.
Time-slicing consistency	8	Gate-form closure (Theorem 2); model-dependent discharge of denominator/tail/de-regularization gates remains the real frontier.
Kernel normalization lane	8	Semigroup normalization forces $t^{-d/2}$ in Gaussian settings (Theorem 3); extension to oscillatory kernels requires explicit regularization/analytic-continuation gates.
Gauge long-range taxonomy	8	Screened Abelian branch is theorem-closed (Theorem 4); non-Abelian confinement beyond strong-coupling windows requires first-principles transfer control and dynamical-matter string-breaking theorems.

Open problems (selected).

1. Close interacting local field-theory continuum existence and reconstruction gates in $d = 4$ in a theorem-grade framework.
2. Replace scoped transfer assumptions in non-Abelian confinement lanes by first-principles bounds and treat dynamical-matter crossover rigorously.
3. Make the relationship between amplitude-level half-density calculus and dynamical convergence gates fully explicit in representative models (beyond Gaussian witnesses).

References

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