Claim 1 Note: Variational Delta from Static Integrals to QM and QFT

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Abstract

This note organizes Claim 1 into a three-level ladder: static finite-dimensional oscillatory integrals, quantum mechanics in time, and quantum field theory in spacetime. At each level, the same object appears: a Dirac delta of the first variation, supported on extrema of the corresponding action. The static level is theorem-grade; the full infinite-dimensional bridge to continuum path integrals remains conjectural.

1 Notation and Scope

We use the shorthand

$$\delta(\partial S)$$

for "delta of first variation" (or gradient in finite dimensions), i.e. a Dirac concentration on critical points. This is *not* the distribution derivative $\partial_x \delta$ (often denoted δ'): it is composition/pullback by the map ∂S.

At the same time, both belong the point-supported-distribution framework: in one dimension, any point-supported distribution is a finite sum

$$\sum_{m=0}^{N} c_m \, \delta^{(m)}.$$

Under dilation $x \mapsto \lambda x$ ($\lambda > 0$),

$$\delta^{(m)}(\lambda x) = \lambda^{-m-1} \delta^{(m)}(x),$$

so different derivatives of δ carry different scaling weights, i.e. multiple scaling modes/fixed channels.

2 Level 0: Static Oscillatory Integral

Let $f \in C^\infty(\mathbb{R})$ and $O \in C_c^\infty(\mathbb{R})$. Define

$$A_\varepsilon(O) := \varepsilon^{-1/2} \int_{\mathbb{R}} e^{\frac{i}{\varepsilon} f(x)} \, O(x) \, dx, \qquad \varepsilon > 0.$$

Proposition 1 (Single nondegenerate critical point)**.** *Assume $f'(x_0) = 0$, $f''(x_0) \neq 0$, and x_0 is the unique critical point. Then, as $\varepsilon \to 0^+$,*

$$|A_\varepsilon(O)|^2 = 2\pi \, \frac{|O(x_0)|^2}{|f''(x_0)|} + o(1),$$

up to Fourier-normalization convention.

Proof sketch. Apply stationary phase to A_ε:

$$\int e^{\frac{i}{\varepsilon}f(x)}O(x)\,dx \sim e^{\frac{i}{\varepsilon}f(x_0)}e^{i\frac{\pi}{4}\operatorname{sgn}(f''(x_0))}\sqrt{\frac{2\pi\varepsilon}{|f''(x_0)|}}\,O(x_0).$$

Multiply by $\varepsilon^{-1/2}$, then take modulus squared. $\square$

Distributionally, this is the finite-dimensional template

$$|A_\varepsilon(O)|^2 \to 2\pi\,\langle\delta(f'),|O|^2\rangle.$$

Remark 1 (Multiple critical points)**.** *With several nondegenerate critical points, cross terms carry phases $e^{i(f(x_i)-f(x_j))/\varepsilon}$. Pointwise limits of $|A_\varepsilon|^2$ can fail without averaging/weak-limit prescriptions.*

3 Level 1: Quantum Mechanics (Action in Time)

For paths $\phi : [t_0, t_1] \to M$ and action

$$S[\phi] = \int_{t_0}^{t_1} L(\phi,\dot{\phi})\,dt,$$

the critical-point equation is Euler–Lagrange:

$$\frac{\delta S}{\delta \phi} = 0.$$

The direct analogue of $\delta(f')$ is

$$\Delta_{\mathrm{QM}} := \delta\left(\frac{\delta S}{\delta\phi}\right),$$

formally supported on classical trajectories.

Discrete approximation

Time slicing yields finite-dimensional variables $(q_1,\dots,q_N)$ and discrete action $S_N(q_1,\dots,q_N)$. Then

$$\delta\left(\frac{\delta S}{\delta\phi}\right) \rightsquigarrow \delta(\nabla S_N),$$

where $\nabla S_N = 0$ are the discrete Euler–Lagrange equations. This is the exact finite-dimensional support statement at each fixed N.

4 Level 2: Quantum Field Theory (Action in Spacetime)

For a field Φ in D dimensions,

$$S[\Phi] = \int \mathcal{L}(\Phi,\partial\Phi)\,d^D x, \qquad \frac{\delta S}{\delta\Phi} = 0.$$

The corresponding object is

$$\Delta_{\mathrm{QFT}} := \delta\left(\frac{\delta S}{\delta\Phi}\right),$$

formally concentrated on classical field equations.

Lattice regularization

On a lattice with finitely many variables $\Phi_1, \dots, \Phi_N$,

$$S[\Phi] \rightsquigarrow S_N(\Phi_1, \dots, \Phi_N), \qquad \Delta_{\mathrm{QFT}} \rightsquigarrow \delta(\nabla S_N),$$

again making the "support on extrema" statement exact at fixed cutoff.

5 Amplitude and Geometric Representation

The oscillatory probability-amplitude viewpoint is:

- amplitudes can be represented geometrically as 1/2-density-level objects,
- modulus square produces density-level concentration on critical sets.

In finite dimensions this is standard stationary phase/distribution theory. The open step is to prove, with full control, the continuum limit from finite-dimensional regularizations to a rigorous infinite-dimensional object compatible with tangent-groupoid composition and renormalized QFT limits.

6 Current Program Status (Date-Anchored)

As of 2026-02-09, the ladder has the following state:

- **Statics:** theorem-grade closure in nondegenerate stationary-phase regimes (probability-amplitude to Born-density map).
- **Dynamics:** scoped consistency chain established for time-sliced transition amplitudes with explicit refinement/de-regularization/SD assumptions.
- **Fields:** $d = 2$ interacting ultralocal closure is established; $d = 3$ has a renormalized finite-volume bound channel and a scoped continuum-branch candidate; $d = 4$ remains frontier-sensitive and requires explicit renormalization/nontriviality input.

7 Summary

The core structural claim survives escalation:

$$\text{static } \delta(\nabla S) \;\longrightarrow\; \text{QM } \delta\!\left(\frac{\delta S}{\delta \phi}\right) \;\longrightarrow\; \text{QFT } \delta\!\left(\frac{\delta S}{\delta \Phi}\right).$$

At every stage, the delta of first variation localizes on extrema of the corresponding action.