From `conv_patched.md` to a Research Program: Foundational Goals, Proof Targets, Work Done, and Next Steps

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Abstract

This report is the canonical "what we are doing" document for the `conv\_patched.md` conversation and the research produced afterwards in this repository. The source conversation proposed a foundational slogan: representing "classical solutions" as Dirac-type localization on extrema, and then insisting that this localization be compatible with refinement and composition, naturally generates complex exponential weights $e^{\frac{i}{\hbar}S}$ and a "halved" object that behaves like an amplitude (or half-density) whose modulus-square yields the concentrating distribution.

We turned that slogan into a concrete, dimension-aware research program: a three-level escalation (statics $\to$ mechanics-in-time $\to$ field theory), with half-density as an *optional* formalism at each level, and with explicit gates for field-level existence by spacetime dimension ($d = 2$ first, $d = 3$ next, $d = 4$ frontier).

On the theorem side, we proved a large scoped subclass of Claim 1: exact projective stability for cylinder observables, constructive counterterm repair, rigorous Schwinger–Dyson and scale-flow (τ_μ) structure, and nontrivial de-regularization $\eta \to 0^+$ in a projective oscillatory class. On the formal side, we built a Lean-checked spine of reusable lemmas (quotient-derivative covariance form, centered bounds, and small-parameter increment control) and specialized that spine to a concrete finite exponential-family model.

What remains open is the genuinely local field-level bridge in interacting models beyond ultralocality: in $d = 3$, a precise candidate theorem is stated with explicit obligations; in $d = 4$, we have an explicit obstruction sheet explaining why naive lifts fail without renormalization and nontriviality input.

Contents

1 Reading Instructions (for Humans and Future Agents)

1.1 Source of Truth and Index Files

The primary source is `conv\_patched.md`, with a rendered companion `conv\_patched.pdf`. All claims, program decisions, and anchors in this report should be traced back either to that source conversation or to one of the theorem-grade artifacts produced in this workspace.

The minimal entry points are:

- `research/workspace/notes/2026-02-09-core-goal-compass.md` (program compass and "north star"),
- `research/workspace/notes/audits/2026-02-08-top10-claim-audit.md` (current audit labels and the next queue item),
- `research/workspace/reports/2026-02-09-claim1-scoped-complete-proof.tex` (the main scoped Claim 1 proof report),
- `research/workspace/proofs/` (Lean formalization workspace, built with `/Users/arivero/.elan/bin/lake`).

1.2 How to Use This Report

This report is meant to support two readers: humans who want a narrative of what is known and what is not, and future agents/models that need a compact, unambiguous summary of the proof targets and the current state of the work tree. The recommended path is:

1. Read §2 to recover the *problem statement* as it appeared in the conversation.
2. Read §3–§4 to see the *research program* and the proof standards used here.
3. Read §5–§6 to see the *executed results* (what is theorem-grade today).
4. Read §7 to see the *next targets* and the explicit gaps that motivate them.

2 The Source Conversation: What Was Known and What Was Claimed

2.1 Conversation Inventory

The conversation was not a generic discussion of “AI in physics.” It was an explicit technical request: take an existing mathematical manuscript (referred to in the conversation as `9803035.pdf`) and treat it as a source for a foundational research program. The requested escalation was concrete and ordered: first the static case (finite-dimensional oscillatory integrals), then quantum mechanics (action integrated in a time coordinate, with time slicing), then field theory (action integrated in spacetime, with particular interest in $D = 4$), while keeping track of the “halved expression” and its interpretation in the literature and in geometric frameworks (tangent groupoids and renormalization-style refinement control).

At the same time, the conversation contained a second technical axis: a large body of central-force dynamics in SR/GR and gauge settings (power-law forces, Coulomb, Schwarzschild, higher-dimensional black holes, and long-range phase taxonomies). This axis matters because it makes the “action reduction” mechanism explicit and testable in many examples, and because it provides a concrete target for the slogan “geometry-of-force emerges from variational mechanics”.

Finally, the conversation centered on a specific distributional object and a specific compositional intuition. The distributional object is the “delta of first variation”: in statics, $\delta(f'(x))$; in mechanics and fields, the functional delta $\delta(\delta S/\delta\phi)$ or $\delta(\delta S/\delta\Phi)$. This is *not* the derivative distribution δ' (the distributional derivative of Dirac’s δ): it is the pullback/composition of δ by the map f' (or the Euler–Lagrange map).[1] The compositional intuition is that amplitudes are the correct objects to compose, not probabilities: the “halved expression” behaves like an amplitude (or half-density), and modulus-square produces a density-level localizing object.

The conversation also made an explicit control demand: the relevant limits (e.g. $\varepsilon \to 0$ in oscillatory integrals, or refinement limits in time slicing and lattice field theory) must be compatible with composition across scales. This is where τ_μ-type scale flows, Schwinger–Dyson identities, and de-regularization $\eta \to 0^+$ enter the program.

2.2 Canonical Anchors in `conv_patched.md`

The anchors below are the minimal set of `conv\_patched.md` lines that define the program. Line numbers refer to the *patched* transcript, so they should be treated as stable for this repository.

Static seed. The conversation identifies $\delta(f'(x))$ as the relevant concentrating object (`conv\_patched.md:688`), introduces the “halved” amplitude (`conv\_patched.md:707`), and records the standard nondegenerate identity

$$\delta(f'(x)) = \sum_{x_i \in \mathrm{Crit}(f)} \frac{\delta(x - x_i)}{|f''(x_i)|}$$

(`conv\_patched.md:871`). These items are the static prototype of “delta of first variation”.

Near-diagonal scaling and half-densities. The conversation emphasizes that the relevant $\varepsilon \to 0$ scaling is near-diagonal ($y \approx x$), not a far-away variable limit (`conv\_patched.md:928`). It also explicitly states the geometric reading: on a Lie groupoid, convolution is naturally formulated using half-densities so that composition is coordinate-free (`conv\_patched.md:967`). This is the conceptual reason the “halved expression” is treated as an amplitude-like object in the program.

[1] To avoid a persistent notational trap: the prime in $\delta(f')$ belongs to f', not to δ. Separately, the symbol “δ'” appears in point-interaction literature to label contact-interaction families; that usage is conceptually adjacent (it is also a point-supported sector), but it is not the same as “differentiate Dirac’s δ”.

Escalation to mechanics and fields. For quantum mechanics, the halved object is identified with the propagator amplitude (`conv\_patched.md:1122`) and the static $\delta(f')$ is upgraded to the functional delta $\delta(\delta S/\delta\phi)$ (`conv\_patched.md:1149`). The synthesis line is that quantization can be read as the refinement-compatible version of classical localization (`conv\_patched.md:1209`): the same localization object appears at each level, but compatibility with composition across scales forces nontrivial structure (a surviving finite deformation parameter).

Newton's refinement template (finite invariants before limits). The conversation explicitly returns to Newton's area-law proof as the archetype of "do it discretely, then refine": equal areas in equal times are first proved for a polygonal orbit under central impulses, and only then transported to the smooth limit (`conv\_patched.md:2497`). The same passage appears again in action language: Newton's additivity of swept areas is structurally identical to the additivity of the time-sliced action (`conv\_patched.md:2531`), and "more triangles, thinner triangles" is "more time slices, smaller Δt" (`conv\_patched.md:2537`). This is the conceptual bridge that makes the later "composition forces exponentials" argument feel inevitable rather than mystical.

Fixed points as the scale-control locus. The transcript already speaks in Wilson–Kogut language: the relevant limit exists only when refinement can be compared across scales, which is precisely what RG fixed points and their stable/unstable manifolds control (`conv\_patched.md:1164`). It also flags that higher-dimensional generalisation will plausibly exhibit "more exotic fixed points" (`conv\_patched.md:816`) and that fixed-point structure is where one expects universal scales to appear (`conv\_patched.md:1426`).

3 Research Aims (North Star)

3.1 Goal 1 as a Three-Level Program

Goal 1 is the foundational target of this repository. It is intentionally framed as a three-level escalation because the same conceptual object changes character as the ambient dimension changes: in statics we can literally write down distributions like $\delta(f')$; in mechanics we can discretize time to obtain honest finite-dimensional deltas $\delta(\nabla S_N)$; in fields we have to confront a genuinely infinite-dimensional limit problem with dimension-dependent existence and renormalization behavior.

The program is pinned to `research/workspace/notes/theorems/2026-02-09-claim1-three-level-pro md`. The level separation is not rhetorical; it is the main mechanism for preventing over-claims. *Half-density language is optional at every level:* it is structurally useful for composition statements, but it is not accepted as a substitute for the dynamical work needed to pass to continuum limits.

The three levels are:

1. **Statics (0D action):** prototype object $\delta(f'(x))$, and stationary-phase control of the halved amplitude
$$A_\varepsilon(O) = \varepsilon^{-1/2} \int_{\mathbb{R}} \mathrm{e}^{\frac{i}{\varepsilon} f(x)} O(x)\, dx, \qquad |A_\varepsilon(O)|^2 \to 2\pi\langle \delta(f'), |O|^2\rangle$$
under nondegenerate hypotheses.

2. **Dynamics (0+1D action):** action on time histories $S[q] = \int L(q, \dot{q}, t)\, dt$ and variational localization target $\delta(\delta S/\delta q)$, made exact at finite time-slicing as $\delta(\nabla S_N)$.

3. **Fields (d-D action):** action $S[\Phi] = \int \mathcal{L}(\Phi, \partial\Phi, \dots)\, d^d x$, formal localization $\delta(\delta S/\delta\Phi)$, and a dimension-gated existence program for regulator removal and reconstruction.

3.2 Foundational Motifs: Refinement, Scaling, and Fixed Points

There are three recurring structural motifs behind almost every successful step in this program. They are conceptually independent of the "half-density" packaging, but half-densities are often a convenient coordinate-free way to express them.

(M1) Prove invariants at finite resolution, then take a controlled limit. Newton's polygonal proof of the area law is the template: one proves an *exact* discrete invariant (equal triangle areas per equal time-step) for a finite refinement, and then one defines the smooth statement by an explicit refinement limit ($\Delta t \to 0$).[2] For us, the analog is: define finite-dimensional objects at a regulator (time-slicing, lattice spacing, cutoff a, damping η), prove exact identities there (projective stability, SD, τ_μ covariance), and then justify the continuum limit as a theorem.

(M2) Refinement-compatible composition forces exponential weights. If an amplitude $K(t_2, t_0)$ composes under cutting, $K(t_2, t_0) = \int K(t_2, t_1)\, K(t_1, t_0)\, dq(t_1)$, then the weight assigned to a history must be multiplicative under concatenation. Multiplicativity is the reason exponentials appear. The additive functional singled out by the classical variational principle is the action, hence the canonical weight $\exp(\frac{i}{\hbar} S)$ at the dynamic level. At the static level, the same logic is already visible in the Fourier representation of $\delta(f')$ and its near-diagonal scaling; and in tangent-groupoid language, it is encoded as "compose arrows, then rescale the deformation parameter".

(M3) When limits fail, scale control moves to RG flows and fixed points. In mechanics, time slicing usually converges once the measure is interpreted correctly. In field theory (especially $d = 4$), refinement often creates divergences, so one must: introduce a scale, define renormalized objects relative to that scale, and demand that changing the scale does not change observables. In other words, "existence of the continuum limit" becomes "existence of a controlled RG trajectory"; fixed points and their stable manifolds are where universal behavior and universal scales live.

Point-supported sectors have multiple scaling eigenmodes. The delta-of-variation slogan localizes on extrema, but the *local distributional sector* is not one-dimensional. In 1D, every distribution supported at $\{0\}$ is a finite linear combination of $\delta^{(m)}$, and the dilation pullback satisfies the exact eigenmode law

$$U_\lambda \delta^{(m)} = \lambda^{-m-1}\, \delta^{(m)}.$$

Thus the point-supported sector decomposes into multiple scaling channels, each with its own scaling exponent. This is the minimal mathematical model for "multiple fixed points" at the level of local singular data: depending on regularization and symmetry, channels may be excited, suppressed, or mixed under scaling.

Toy model: 1D contact RG fixed points make the channel picture concrete. The local preprint `arXiv-hep-th9411081v1/9411081.tex` builds a Wilson–Kogut-style RG directly on the space of dimensionless cutoff scattering matrices $\tilde{S}_{\tilde{k}}$ in 1D QM. The RG action is an argument rescaling,

$$\tilde{S}^t_{\tilde{k}} = T^t[\tilde{S}]_{\tilde{k}} = \tilde{S}_{e^{-t}\tilde{k}}, \tag{1}$$

[2]This is exactly the discipline Newton codifies in the "first and last ratios" lemmas: the limit is an explicit definitional step, not an assumption of indivisible geometric atoms.

so fixed points are exactly the constant $\tilde{S} \in U(2)$ satisfying the physical conjugation constraint. Under time-reversal symmetry the fixed points reduce to $\{\text{a circle}\} \cup \{\pm Q\}$, and the paper interprets the circle as Kurasov's scale-invariant "δ'" family. Linearization around a fixed point yields scaling eigen-directions with $\vec{a}(k) = k^{-n}\vec{a}_0$, so that "relevant/irrelevant" is literally the sign of a scaling exponent n in a point-supported sector. This is the cleanest toy model we currently have for the Claim 1 intuition: refinement plus rescaling defines an RG map, and the continuum physics is controlled by fixed points and their stable manifolds; multiple fixed points are not an accident but a structural feature of point-supported degrees of freedom.

Bridge back to Claim 1. Our internal τ_μ flow is the analog of an RG reparameterization: it changes presentation $(\kappa, \eta, h) \mapsto (\mu\kappa, \eta/\mu, \mu h)$ while keeping the combined parameter c fixed. The slogan "c-invariance" is therefore not a rhetorical label: it is the concrete mechanism by which we compare objects defined at different refinement scales. The multi-channel fixed-point picture above is a warning and a guide: Claim 1 can only be true as stated if we can track which scaling channels are present in the local sector and show that the refinement plus counterterm mechanism drives them into a controlled fixed structure (or at least a controlled stable manifold) across the three levels.

3.3 Dimension-Gated Field Existence Ladder

The field-level part of the program is explicitly dimension-gated. This is not just "because physics is $3 + 1$", but because the analytic existence theory is dimension-sensitive: the same local interaction can be benign in one dimension and critical or ill-behaved in another. The repository therefore treats d as a first-class index in every field-level statement, and it separates three notions of "existence" (regulated, continuum, reconstruction) before any claim is allowed to upgrade to "field-level closure".

The roadmap is pinned to `research/workspace/notes/theorems/2026-02-09-claim1-field-dimension` `md`. In that roadmap, field-level work is dimension-indexed by design:

- $d = 2$: strongest closure candidate; we closed an interacting ultralocal ϕ^4 field-indexed theorem (AP).
- $d = 3$: intermediate bridge rung; candidate theorem stated with explicit proof obligations B1–B5 (AS).
- $d = 4$: physically central frontier; explicit lift-obstruction sheet enumerates missing inputs (AQ).

At the field level, we separate:

1. regulated existence (finite cutoff objects),
2. continuum existence (regulator removal with nontrivial limit),
3. physical reconstruction (Euclidean-to-Minkowski, or equivalent).

The main pragmatic consequence is that $d = 2$ results are treated as a closure target for the *logic* of Claim 1 in an interacting class, $d = 3$ results are treated as the next bridge rung where local propagation is reintroduced, and $d = 4$ results are treated as a frontier where one must explicitly account for renormalization and nontriviality.

4 What We Expect to Prove (Proof Targets and Success Criteria)

4.1 Claim 1 (Foundational Claim)

Claim 1 is the foundational bridge that motivated the entire work tree. It starts from a static distributional fact ($\delta(f')$ localizes on extrema) and escalates it to mechanics and fields by replacing f with an action functional S. The slogan can be stated compactly as:

$$\text{“halved” amplitude / half-density} \quad \rightsquigarrow \quad \text{density-level localization on extrema} \quad \rightsquigarrow \quad \delta\left(\frac{\delta S}{\delta \phi}\right) \text{ (QM/QFT).}$$

The “halved” object is important because it is the compositional object: it behaves like an amplitude or half-density under convolution/composition (this is the groupoid-kernel intuition), while the squaring step produces the density-level object that concentrates on classical solutions. In finite dimensions this is standard stationary phase; the research challenge is to make the limit/refinement procedures honest across the time and field escalations.

To prevent ambiguity, this repository treats Claim 1 as a hierarchy of increasingly strong targets:

1. *Finite-dimensional theorem-grade:* explicit stationary phase / distributional localization statements.

2. *Regulated infinite family theorem-grade:* a specified projective/cylinder family of normalized oscillatory states with exact stability, explicit control of renormalization/counterterms, SD closure, and de-regularization $\eta \to 0^+$ in that class.

3. *Field-level theorem-grade:* dimension-indexed continuum existence with SD pass-through and c-invariance, and, when pursuing that branch, a reconstruction step connecting Euclidean correlators to a Minkowski theory.

The audit keeps Claim 1 labelled `heuristic` at the global level, because the full bridge to a unique local interacting continuum field theory is not yet proved. However, a large scoped core is now theorem-grade and recorded in `research/workspace/reports/2026-02-09-claim1-scoped-complete-p` `tex`.

4.2 Auxiliary Claims (2–10)

Claims 2–10 are the “geometry-of-force” and model-taxonomy support. They ensure the action-reduction mechanism is explicit and correct in concrete SR/GR/gauge settings.

They also play a practical role: they provide regression targets and sanity checks. If the foundational track produces a formalism that contradicts known orbit classifications (e.g. SR Coulomb precession regimes or Schwarzschild separatrices), that is immediate evidence of a mis-specified definition or an incorrect scaling assumption.

Per the top-10 audit, the following are `proved` as theorem notes (with explicit derivations and cross-checks):

- SR center-access trichotomy (Claim 2) and SR Coulomb phase portrait (Claim 3),
- Duffing-type $n = 3$ reduction (Claim 4),
- D-dimensional GR matching (Claim 5),
- Schwarzschild bound-orbit interval and ISCO framing (Claims 6–7),
- selected higher-D GR and gauge taxonomy branches (Claims 8–9) are partially closed; remaining global branches are tracked as open,
- SR circular-orbit benchmark inequalities (Claim 10) are formalized as regression identities.

4.3 What Counts as a Proof Here

This repository uses an explicit validation contract. The intention is to prevent subtle drift: no statement is upgraded to "proved" unless it states its assumptions and passes at least one independent verification route (formal proof when feasible; otherwise symbolic or numerical checks).

Every upgraded statement is expected to include:

1. explicit assumptions (model, regime, approximations),
2. units/dimensions check (where relevant),
3. symmetry/conservation checks (where relevant),
4. at least one independent cross-check path: Lean proof or symbolic derivation or numerical diagnostics,
5. a confidence/uncertainty note,
6. reproducibility metadata (commands, versions).

The key rule of thumb is the kinematic/dynamical split: half-density and groupoid convolution statements certify *kinematic* consistency of composition laws at fixed regulator, but they do not certify *dynamical* convergence as the regulator is removed. Continuum-limit claims require separate analytic gates (tightness, non-vanishing normalization, and closure of SD identities in the limit). This split is spelled out in `research/workspace/notes/theorems/2026-02-09-claim1-halfdensity-kinematic-dynamic-split.md`.

5 What Has Been Done So Far (Artifacts and Results)

5.1 Audit Results

The canonical audit is `research/workspace/notes/audits/2026-02-08-top10-claim-audit.md`. Its job is to keep the work focused and to prevent "proof by accumulation" (writing many notes without closing the foundational gaps). It does that in two ways: (i) it assigns rigor labels (`proved`, `heuristic`, `speculative`) with explicit upgrade paths, and (ii) it enforces a non-drift filter so that new work must strengthen one of the three cores: variational-distribution, geometry-of-force, or scale-control.

The most important audit outcomes so far are:

- Claim 1 remains the highest-novelty, highest-risk item, but has a nontrivial scoped proof closure report.
- Claims 2–7 and Claim 10 are `proved` in dedicated theorem notes; Claims 8–9 are partially closed with explicit open sectors.
- The audit enforces a non-drift filter: new work must strengthen (i) variational-distribution core, (ii) geometry-of-force bridge, or (iii) scale-control core.

As of the latest update, the next queued Lean target is AN-18: reduce the finite exponential-family increment hypotheses by proving automatic regularity and non-vanishing conditions. This is intentionally modest: it is a "hygiene" step that reduces assumptions in the mechanized spine and keeps the field-level bridge obligations clearly separated from toy-model artifacts.

5.2 Theorem Notes and Reports

This repository separates two types of human-readable artifacts: *theorem notes* (`research/workspace/notes/theorems/`) and *reports* (`research/workspace/reports/`). The theorem notes are bite-sized upgrades of individual claims with explicit assumptions and proofs; the reports are longer manuscripts that weave multiple notes into a single dependency chain.

The current core reports are:

- `research/workspace/reports/2026-02-08-claim1-variational-delta-note.tex`: static $\to$ QM $\to$ QFT ladder and $\delta(\partial S)$ viewpoint.
- `research/workspace/reports/2026-02-09-claim1-scoped-complete-proof.tex`: scoped complete proof in a projective oscillatory class (projective stability, counterterms, deregularization, SD, τ_μ).
- `research/workspace/reports/2026-02-09-newton-action-path-integral-lecture.md`: Newton $\to$ action reduction $\to$ oscillatory localization lecture narrative.

At the field level, three notes define the current dimension-gated status:

- `research/workspace/notes/theorems/2026-02-09-claim1-d2-ultralocal-phi4-closure.md`: theorem-grade field-indexed closure in $d = 2$ ultralocal interacting class.
- `research/workspace/notes/theorems/2026-02-09-claim1-d3-intermediate-bridge-candidate.md`: $d = 3$ bridge candidate with proof obligations B1–B5.
- `research/workspace/notes/theorems/2026-02-09-claim1-d4-lift-obstruction-sheet.md`: explicit failure points for naive $d = 2 \to d = 4$ lift in local interacting models.

5.3 Lean Formalization (Machine-Checked Spine)

The Lean project lives in `research/workspace/proofs/`. Its role is not to formalize the entire analytic infrastructure of stationary phase or continuum QFT; it is to machine-check a reusable spine of algebraic and finite-dimensional analytic lemmas that show up repeatedly in the Claim 1 program (especially in small-parameter continuity bounds and in ratio-state control).

As recorded in `research/workspace/notes/theorems/2026-02-09-claim1-lean-formalization-statu` `md`, the current machine-checked modules include:

- c-invariance under τ_μ scaling and composition laws,
- quotient-derivative covariance form for $\omega = N/Z$,
- finite-average covariance inequality templates,
- derivative-bound and interval-increment templates for ratio states,
- finite exponential-family bridges: derivative-under-sum, centered representation, derivative bounds, and a model-internal $C\kappa$ increment bound.

5.4 Simulation and Symbolic Checks

Python scripts are used as *diagnostics*, not as proof substitutes. Their purpose is to provide fast sanity checks (e.g. sign errors, scaling regimes, stability criteria) and to explore toy parameter scans for candidate theorems. The canonical index is kept in `research/workspace/notes/README.md`.

Representative examples include:

- $d = 2$ ultralocal closure diagnostic: `research/workspace/simulations/claim1_d2_ultralocal_phi4_closure_check.py`.
- half-density kinematic vs dynamical counterexample diagnostic: `research/workspace/simulations/claim1_halfdensity_kinematic_dynamic_split_check.py`.
- $d = 3$ bridge toy scan: `research/workspace/simulations/claim1_d3_bridge_toy_coupling_scan.py`.

6 Current Status of Claim 1 (Closed Pieces, Open Gaps)

6.1 The Scoped Complete Proof and Its Scope

The main theorem-grade closure for Claim 1 in this repository is the scoped report `research/workspace/reports/2026-02-09-claim1-scoped-complete-proof.tex`. It proves Claim 1 in a specified but nontrivial projective/cylinder class of oscillatory states. At a high level, the closure consists of two parts: (i) *exact projective stability* for cylinder observables (a discrete analogue of refinement invariance), and (ii) *analytic control* sufficient to justify de-regularization and the persistence of SD/τ_μ structure in that class.

Concretely, within the stated block-tail and interacting extensions, the report proves:

- exact projective stability on cylinder observables,
- a well-defined continuum state on cylinder observables,
- constructive counterterm repair mechanisms,
- de-regularization $\eta \to 0^+$ in several interacting families,
- Schwinger–Dyson identities and exact τ_μ-type scale-flow covariance,
- explicit large-N lifts and non-factorized interacting extensions under stated conditions.

It is equally important to record what the scoped report does *not* claim:

- it does not claim a full continuum local QFT construction in $d = 4$,
- it does not identify a unique interacting continuum field theory beyond the cylinder/regulated scope,
- it does not bypass the dimension-gated existence and reconstruction obligations.

6.2 Statics and Dynamics: Where the Delta-of-Variation Logic Is Theorem-Grade

At the static level, the delta-of-variation logic is theorem-grade under standard nondegeneracy hypotheses. The core facts are:

- In finite dimensions, stationary phase yields $|A_\varepsilon|^2 \to 2\pi\langle\delta(f'), |O|^2\rangle$ in nondegenerate regimes.
- The distributional identity $\delta(f') = \sum \delta_{x_i}/|f''(x_i)|$ reduces the limiting object to a finite measure supported on critical points (amenable to formal proof assistants).

At the dynamics level (mechanics on a time line), the correct way to keep the logic honest is to time-slice: at fixed discretization, the variational localization object is an ordinary finite-dimensional Dirac delta. The open work is the controlled refinement limit under which these finite-dimensional statements produce a stable continuum object. In this repository, that refinement control is represented by the projective/cylinder framework and the scale-control invariants (SD and τ_μ-type covariance).

- time-slicing makes $\delta(\delta S/\delta q)$ exact as $\delta(\nabla S_N)$ at fixed discretization,
- continuum claims require limit control under refinement, tracked by projective/cylinder programs and scale-flow constraints.

6.3 Field Level: d=2 Closure, d=3 Bridge, d=4 Obstruction

At the field level, Claim 1 becomes dimension-sensitive and must be treated as a dimension-gated existence program. The current status by dimension is:

$d = 2$ **closure.** We proved a fully explicit interacting field-indexed closure in an ultralocal ϕ^4 class (`research/workspace/notes/theorems/2026-02-09-claim1-d2-ultralocal-phi4-closure.md`). Because the action is sitewise additive, cylinder expectations become exactly scale-independent once the cylinder sites are distinct, and the continuum limit exists without renormalization. In that model we also obtain a field-level SD identity by one-dimensional integration by parts and exact c-invariance along τ_μ-orbits.

- $d = 2$ ultralocal interacting ϕ^4 model: exact cylinder stabilization and continuum existence, field-level SD identity, and exact c-invariance along τ_μ-orbits.
- interpretation: this is non-Gaussian but ultralocal, serving as a controlled baseline rather than full local QFT.

$d = 3$ **bridge candidate.** The next rung is $d = 3$ with local propagation reintroduced by a nearest-neighbor gradient term. We stated a precise candidate theorem in `research/workspace/notes/theorems/2026-02-09-claim1-d3-intermediate-bridge-candidate.md` and made the closure obligations explicit (B1–B5). The most important point is that the $d = 2$ proof does not even begin to lift until one has moment/tightness and denominator control uniformly in (a, L), and a quantitative small-κ continuity bound for local cylinders.

- add nearest-neighbor gradient coupling $\kappa \geq 0$ and seek a small-κ regime $[0, \kappa_*]$, with explicit proof obligations: B1 moments, B2 tightness, B3 non-vanishing denominator, B4 SD insertion control, B5 κ-continuity.
- Lean work mechanizes B5-shaped increment bounds in abstract and in a finite exponential-family toy class; the field-level model bridge remains open.

$d = 4$ **obstruction sheet.** The obstruction sheet `research/workspace/notes/theorems/2026-02-09-claim1-d4-lift-obstruction-sheet.md` records why naive lifts fail once local propagation and regulator removal are taken seriously: loss of product structure, non-closed SD hierarchies, and renormalization dependence. In $d = 4$ one must add explicit renormalization flow control and a nontriviality criterion; otherwise one can at best obtain a theorem-grade scoped statement (regulated and/or cylinder-level) without a claim of a nontrivial interacting continuum theory.

- restoring local propagation destroys product structure and exact cylinder decoupling; SD hierarchy is no longer one-site closed.

- regulator removal requires explicit renormalization flow control and nontriviality criteria.
- half-density statements must be split into kinematic vs dynamical (AR note).

7 How We Continue (Roadmap)

7.1 Immediate Next Targets

The short-term queue is designed to be incremental and non-drifting: every target should either reduce assumptions in an already-mechanized lemma chain, or discharge one of the explicit $d = 3$ bridge obligations, or clarify why a particular $d = 4$ upgrade is blocked.

Immediate targets:

- **AN-18 (Lean hygiene):** reduce assumptions in the finite exponential-family increment chain by proving automatic regularity and non-vanishing conditions in that model. This keeps the Lean spine tight and prevents later reuse from silently importing unjustified hypotheses.
- $d = 3$ **bridge closure work:** close at least one beyond-ultralocal $d = 3$ theorem fragment by addressing B1–B5 systematically (moments, tightness, denominator control, SD insertion control, and κ-continuity). The goal is not an all-at-once $d = 3$ continuum construction, but a sequence of honest theorems that reduce the gap.
- **Cross-level hygiene:** keep half-density composition results separate from continuum existence statements, and require every claimed "lift" (e.g. from $d = 2$ to $d = 4$) to list its missing analytic gates explicitly.

7.2 Longer-Horizon Conjectures and Risks

The longer-horizon targets are those that would upgrade Claim 1 from a scoped closure to a full field-level closure in a genuinely local interacting class. These targets are also the main risk points: they are where the existing tooling (stationary phase intuition, half-density kinematics, SD identities) must be matched with hard analytic control (tightness, renormalization, and reconstruction).

The core missing analytic ingredients are:

- tightness/precompactness mechanisms for cylinder marginals in local interacting field models,
- robust non-vanishing control for oscillatory partition normalizations on complex parameter domains,
- SD pass-through with gradient terms (control of insertions and boundary terms),
- $d = 4$ renormalization flow control and nontriviality criteria,
- reconstruction to Minkowski (when pursuing that branch).

8 Appendices

8.1 Glossary

This appendix summarizes the minimal vocabulary used throughout the notes and reports. The canonical Markdown version lives in `research/workspace/notes/2026-02-09-foundational-glossary.md`.

c-parameter The combined complex parameter

$$c := (\eta - i/h)\kappa$$

that governs the dressed kernel family.

c-invariant Invariant under changes of (κ, η, h) that keep c fixed; equivalently, constant on a τ_μ-orbit.

τ_μ flow Scale reparameterization

$$\tau_\mu : (\kappa, \eta, h) \mapsto (\mu\kappa, \eta/\mu, \mu h), \qquad \mu > 0,$$

which preserves c.

de-regularization The one-sided limit $\eta \to 0^+$ from damped oscillatory weights to the purely oscillatory branch.

Schwinger–Dyson identity Integration-by-parts identity (scoped finite form):

$$\langle \partial_i \psi \rangle_c = c \, \langle \psi \, \partial_i S \rangle_c.$$

half-density (here) An amplitude-level object whose modulus-square yields density-level quantities; natural for coordinate-free groupoid convolution kernels.

8.2 Reproducibility (Commands and Entry Points)

This repository is designed so that key statements have executable entry points: Lean modules are built by `lake`, and diagnostics are run with `python3.12`.

Lean build.

```
cd research/workspace/proofs
/Users/arivero/.elan/bin/lake update
/Users/arivero/.elan/bin/lake build Claim1lean
```

Selected diagnostics.

```
python3.12 research/workspace/simulations/claim1_d2_ultralocal_phi4_closure_check.py
python3.12 research/workspace/simulations/claim1_halfdensity_kinematic_dynamic_split_check.p
```

8.3 File Map

This is a curated entry-point index, not an exhaustive list of all notes produced.

- source: `conv\_patched.md`, `conv\_patched.pdf`.
- audit: `research/workspace/notes/audits/2026-02-08-top10-claim-audit.md`.
- compass: `research/workspace/notes/2026-02-09-core-goal-compass.md`.
- Claim 1 reports: `research/workspace/reports/2026-02-08-claim1-variational-delta-note.tex`, `research/workspace/reports/2026-02-09-claim1-scoped-complete-proof.tex`.
- synthesis report: `research/workspace/reports/2026-02-09-newton-action-path-integral-lectur md`.
- Lean: `research/workspace/proofs/Claim1lean.lean` and `research/workspace/proofs/Claim1lean/`.
- theorem notes index: `research/workspace/notes/README.md`.