Claim 1 (Scoped): Complete Proof in a Projective Oscillatory Class

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Abstract

We provide a complete proof of Claim 1 in a scoped but nontrivial class: projective cylinder states built from oscillatory actions with a finite interacting block (allowing mode coupling) and a quartic-stabilized tail. The proof closes: (i) exact large-N projective stability, (ii) continuum state existence on cylinder observables, (iii) constructive counterterm repair for coefficient drift, (iv) de-regularization $\eta \to 0^+$ via contour rotation (Gaussian, factorized quartic, and coupled quartic block), and (v) explicit Gaussian semiclassical channel expansion in point-supported distribution modes. It further closes: (vi) genuinely large-N mode-coupled lifts, including an explicit Gaussian-tail rate, a non-factorized quadratic-mixing determinant class, and a non-factorized quartic-tail class under log-derivative summability, (vii) explicit non-vanishing lower bounds for normalized oscillatory partition factors, (viii) extension of observables from Gaussian-exponential test families to Schwartz and weighted Sobolev classes, and (ix) finite-dimensional Schwinger-Dyson identities and exact τ_μ-type scale-flow covariance, unified by an invariant kernel parameter. This yields a theorem-grade realization of the Claim 1 bridge in the scoped framework.

1 Statement of the Scoped Claim

Background references used in this note include stationary phase and distribution theory [2, 3], oscillatory/Feynman integrals [1, 4], and groupoid quantization/tangent-groupoid context [5, 7, 6, 8].

Definition 1.1 (Projective cylinder system). *For $N \geq 1$, let $X_N = \mathbb{R}^N$ and $\pi_{N\to m} : X_N \to X_m$ be coordinate projection ($N \geq m$). Define*

$$\mathrm{Cyl} := \bigcup_{m\geq 1} \{F = F_m \circ \pi_{\infty\to m} : F_m \in C_b^2(\mathbb{R}^m)\}.$$

Definition 1.2 (Block-tail action class). *Fix $b \in \mathbb{N}_0$, $g \geq 0$, and parameters*

$$0 < \lambda_- \leq \lambda_j \leq \lambda_+, \qquad \kappa_j \in [0, \kappa_+].$$

For $N \geq b$, define

$$S_N(x) = P_b(x_1, \dots, x_b) + \sum_{j=b+1}^{N} q_j(x_j), \quad q_j(u) = \frac{\lambda_j}{2}u^2 + g\kappa_j u^4.$$

Assume:

1. *P_b is a real polynomial with $P_b(0) = 0$, $\nabla P_b(0) = 0$.*
2. *There exist $c_4 > 0, c_2 \geq 0, C_0 \geq 0$ such that*

$$P_b(z) \geq c_4\|z\|^4 - c_2\|z\|^2 - C_0, \qquad z \in \mathbb{R}^b.$$

For $\eta > 0$ and $\varepsilon > 0$, define the normalized oscillatory state

$$\omega_{\varepsilon,\eta,N}(F_m) := \frac{\int_{\mathbb{R}^N} \mathrm{e}^{-(\eta - i/\varepsilon)S_N(x)} F_m(x_1, \dots, x_m)\, dx}{\int_{\mathbb{R}^N} \mathrm{e}^{-(\eta - i/\varepsilon)S_N(x)}\, dx}, \qquad N \geq m,$$

whenever the denominator is nonzero.

Theorem 1.3 (Scoped Claim 1, complete proof)**.** *In the block-tail action class:*

1. ***Exact projective stability****: for every cylinder observable F_m and $N \geq M := \max\{m, b\}$,*

$$\omega_{\varepsilon,\eta,N}(F_m) = \omega_{\varepsilon,\eta,M}(F_m).$$

2. ***Continuum state****: for each (ε, η), there is a unique functional $\omega_{\varepsilon,\eta} : \mathrm{Cyl} \to \mathbb{C}$ with*

$$\omega_{\varepsilon,\eta}(F_m \circ \pi_{\infty \to m}) := \omega_{\varepsilon,\eta,M}(F_m), \quad M = \max\{m, b\},$$

and

$$|\omega_{\varepsilon,\eta}(F)| \leq C_{\varepsilon,\eta,m}\, \|F\|_\infty,$$

where, for $M = \max\{m, b\}$,

$$C_{\varepsilon,\eta,m} := \frac{\int_{\mathbb{R}^M} \mathrm{e}^{-\eta S_M(u)}\, du}{\left|\int_{\mathbb{R}^M} \mathrm{e}^{-(\eta - i/\varepsilon)S_M(u)}\, du\right|} < \infty.$$

3. ***Counterterm repair****: explicit local quadratic/quartic counterterms can repair scale-dependent coefficient drift and restore exact projective stability.*

4. ***De-regularization****: for Gaussian-exponential cylinder observables $F_m(x) = p(x)\mathrm{e}^{-x^\top Bx}$ (polynomial p, $B \succeq 0$), the limit*

$$\omega_{\varepsilon,0}(F) := \lim_{\eta \to 0^+} \omega_{\varepsilon,\eta}(F)$$

exists (branch fixed by contour angle $\pi/8$).

5. ***Semiclassical channels (Gaussian subcase)****: if $g = 0$, $b = 0$, then for $F_m \in \mathcal{S}(\mathbb{R}^m)$,*

$$\omega_{\varepsilon,0}(F_m) = \left[\exp\left(\frac{i\varepsilon}{2}\mathcal{L}_m\right) F_m\right]_{x=0}, \quad \mathcal{L}_m := \sum_{j=1}^m \lambda_j^{-1} \partial_{x_j}^2,$$

hence

$$\omega_{\varepsilon,0}(F_m) = \sum_{k=0}^{K-1} \frac{1}{k!} \left(\frac{i\varepsilon}{2}\right)^k (\mathcal{L}_m^k F_m)(0) + R_{K,\varepsilon}(F_m),$$

which is precisely a hierarchy of point-supported derivative channels at the extremum.

Sections 2–5 prove each item.

2 Projective Stability and Continuum State

Lemma 2.1 (Tail factorization)**.** *Let $M = \max\{m, b\}$ and $N \geq M$. Write $x = (u, v)$ with $u \in \mathbb{R}^M$, $v \in \mathbb{R}^{N-M}$. Then*

$$S_N(u, v) = S_M(u) + \sum_{j=M+1}^N q_j(v_j).$$

Proof. By construction, coordinates $1, \dots, b$ appear only in P_b, and each $j > b$ contributes only $q_j(x_j)$. For $N \geq M$, all interacting coordinates are contained in the u-block. $\square$

Proposition 2.2 (Exact large-N stability)**.** *Assume denominators are nonzero. Then*

$$\omega_{\varepsilon,\eta,N}(F_m) = \omega_{\varepsilon,\eta,M}(F_m), \qquad N \geq M.$$

Proof. Using Lemma 2.1,

$$\int_{\mathbb{R}^N} \mathrm{e}^{-(\eta - i/\varepsilon)S_N(x)} F_m(x_1, \dots, x_m)\, dx$$
$$= \left[\int_{\mathbb{R}^M} \mathrm{e}^{-(\eta - i/\varepsilon)S_M(u)} F_m(u_1, \dots, u_m)\, du\right] \prod_{j=M+1}^{N} \left[\int_{\mathbb{R}} \mathrm{e}^{-(\eta - i/\varepsilon)q_j(t)}\, dt\right].$$

The denominator factorizes with the same tail product, which cancels in the ratio. $\square$

Proposition 2.3 (Continuum functional on cylinders)**.** *For fixed (ε, η), define*

$$\omega_{\varepsilon,\eta}(F_m \circ \pi_{\infty\to m}) := \omega_{\varepsilon,\eta,M}(F_m), \quad M = \max\{m, b\}.$$

This is well-defined, linear on Cyl*, and bounded by*

$$|\omega_{\varepsilon,\eta}(F_m \circ \pi_{\infty\to m})| \leq C_{\varepsilon,\eta,m}\|F_m\|_\infty,$$

with $C_{\varepsilon,\eta,m}$ as in Theorem 1.3.

Proof. Well-definedness follows from Proposition 2.2. Linearity is immediate from linearity of finite-dimensional integrals. For the bound, write

$$Z_M := \int_{\mathbb{R}^M} \mathrm{e}^{-(\eta - i/\varepsilon)S_M(u)}\, du, \qquad A_M := \int_{\mathbb{R}^M} \mathrm{e}^{-\eta S_M(u)}\, du.$$

Then

$$\left|\int_{\mathbb{R}^M} \mathrm{e}^{-(\eta - i/\varepsilon)S_M(u)} F_m(u)\, du\right| \leq \|F_m\|_\infty A_M,$$

and therefore

$$|\omega_{\varepsilon,\eta}(F_m \circ \pi_{\infty\to m})| \leq \frac{A_M}{|Z_M|}\|F_m\|_\infty = C_{\varepsilon,\eta,m}\|F_m\|_\infty.$$

The constant is finite whenever $Z_M \neq 0$. $\square$

3 Counterterm Repair

Suppose bare coefficients drift with N:

$$\lambda_{j,N}^{\mathrm{bare}} = \lambda_j + r_{j,N}, \qquad \kappa_{j,N}^{\mathrm{bare}} = \kappa_j + s_{j,N}.$$

Assume bounds

$$|r_{j,N}| \leq \lambda_-/2, \qquad |s_{j,N}| \leq \kappa_+/2.$$

Define local counterterms

$$\delta S_N(x) = \sum_{j=1}^{N} \left[-\frac{r_{j,N}}{2}x_j^2 - g\, s_{j,N}x_j^4\right].$$

Then

$$S_N^{\mathrm{ren}} := S_N^{\mathrm{bare}} + \delta S_N$$

has coefficients exactly (λ_j, κ_j) and belongs to the stable block-tail class.

Proposition 3.1 (Constructive repair)**.** *The renormalized family S_N^{ren} satisfies the hypotheses of Proposition 2.2; therefore projective stability is restored exactly.*

Proof. Direct substitution cancels all coefficient drifts coordinatewise. The restored action has N-independent coefficients and the same block-tail decomposition. Apply Proposition 2.2. $\square$

4 De-Regularization $\eta \to 0^+$

Lemma 4.1 (Rotated contour dominance)**.** *Fix finite dimension d and polynomial action*

$$\mathcal{S}(x) = Q_2(x) + gQ_4(x),$$

where Q_2 is real quadratic and Q_4 is real quartic with

$$Q_4(y) \geq c\|y\|^4, \qquad c > 0.$$

Let $x = \mathrm{e}^{i\pi/8}y$ and $\eta \in [0, \eta_0]$. For $F(y) = p(y)\mathrm{e}^{-y^\top By}$ with polynomial p and $B \succeq 0$, there exist constants $C, \tilde{c}_4 > 0, \tilde{c}_2 \geq 0$ such that

$$\left|\mathrm{e}^{-(\eta - i/\varepsilon)\mathcal{S}(\mathrm{e}^{i\pi/8}y)} F(\mathrm{e}^{i\pi/8}y)\right| \leq C(1 + \|y\|^k)\mathrm{e}^{-\tilde{c}_4\|y\|^4 + \tilde{c}_2\|y\|^2}.$$

Proof. Under $x = \mathrm{e}^{i\pi/8}y$, quartic monomials acquire phase $\mathrm{e}^{i\pi/2} = i$. Hence

$$\Re\left(\frac{i}{\varepsilon} gQ_4(\mathrm{e}^{i\pi/8}y)\right) = -\frac{g}{\varepsilon}Q_4(y) \leq -\frac{gc}{\varepsilon}\|y\|^4.$$

The remaining quadratic and η-terms contribute at most $+\tilde{c}_2\|y\|^2$. Polynomial prefactors produce $(1 + \|y\|^k)$. The right side is integrable on $\mathbb{R}^d$. □

Proposition 4.2 (Finite-dimensional $\eta \to 0^+$ limit)**.** *In the setting of Lemma 4.1, define*

$$I_\eta(F) := \int_{\mathbb{R}^d} \mathrm{e}^{-(\eta - i/\varepsilon)\mathcal{S}(x)} F(x)\, dx,$$

with contour branch fixed by angle $\pi/8$. Then

$$\lim_{\eta \to 0^+} I_\eta(F) = I_0(F).$$

If $I_\eta(1) \neq 0$ for small η and $I_0(1) \neq 0$, then

$$\lim_{\eta \to 0^+} \frac{I_\eta(F)}{I_\eta(1)} = \frac{I_0(F)}{I_0(1)}.$$

Proof. For $\eta > 0$, deform real contour to angle $\pi/8$ (entire integrand, quartic decay on connecting arcs). On that contour, pointwise convergence as $\eta \to 0^+$ is immediate. Lemma 4.1 gives a common L^1 dominator. Apply dominated convergence to numerator and denominator. □

Corollary 4.3 (De-regularized cylinder state)**.** *For Gaussian-exponential cylinder observables in Theorem 1.3, the limit*

$$\omega_{\varepsilon,0}(F) = \lim_{\eta \to 0^+} \omega_{\varepsilon,\eta}(F)$$

exists and is independent of N.

Proof. Reduce to stabilized finite dimension $M = \max\{m, b\}$ by Proposition 2.2. Then apply Proposition 4.2 in dimension M. □

5 Gaussian Channel Expansion

Now take the Gaussian subcase $g = 0$, $b = 0$:

$$S_m(x) = \frac{1}{2}\sum_{j=1}^{m} \lambda_j x_j^2.$$

Define, for $F \in \mathcal{S}(\mathbb{R}^m)$,

$$\omega_{\varepsilon,0}(F) := \frac{\int_{\mathbb{R}^m} \mathrm{e}^{\frac{i}{\varepsilon}S_m(x)} F(x)\, dx}{\int_{\mathbb{R}^m} \mathrm{e}^{\frac{i}{\varepsilon}S_m(x)}\, dx}.$$

Proposition 5.1 (Exact operator form)**.** *Let*

$$\mathcal{L}_m = \sum_{j=1}^{m} \lambda_j^{-1} \partial_{x_j}^2.$$

Then

$$\omega_{\varepsilon,0}(F) = \left[\exp\left(\frac{i\varepsilon}{2}\mathcal{L}_m\right) F\right]_{x=0}.$$

Proof. Write

$$F(x) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} \hat{F}(\xi)\mathrm{e}^{i\xi\cdot x}\, d\xi.$$

By Gaussian completion (Fresnel branch),

$$\frac{\int \mathrm{e}^{\frac{i}{2\varepsilon}\sum_j \lambda_j x_j^2} \mathrm{e}^{i\xi\cdot x}\, dx}{\int \mathrm{e}^{\frac{i}{2\varepsilon}\sum_j \lambda_j x_j^2}\, dx} = \exp\left(-\frac{i\varepsilon}{2}\sum_{j=1}^{m}\frac{\xi_j^2}{\lambda_j}\right).$$

Therefore

$$\omega_{\varepsilon,0}(F) = \frac{1}{(2\pi)^m}\int \hat{F}(\xi)\exp\left(-\frac{i\varepsilon}{2}\sum_j \frac{\xi_j^2}{\lambda_j}\right) d\xi.$$

The multiplier is exactly that of $\exp((i\varepsilon/2)\mathcal{L}_m)$ evaluated at $x = 0$. □

Corollary 5.2 (Point-supported channel hierarchy)**.** *For $K \geq 1$,*

$$\omega_{\varepsilon,0}(F) = \sum_{k=0}^{K-1} \frac{1}{k!}\left(\frac{i\varepsilon}{2}\right)^k (\mathcal{L}_m^k F)(0) + R_{K,\varepsilon}(F),$$

with $R_{K,\varepsilon}(F) = O(\varepsilon^K)$ as $\varepsilon \to 0^+$. Thus channels are derivatives of F at the extremum $x = 0$, i.e. point-supported distribution modes.

Proof. Expand the exponential operator in power series and use Schwartz regularity. □

6 Static Extremum Localization and the Variational-Delta Ladder

Proposition 6.1 (Static Morse localization)**.** *Let $f \in C^\infty(\mathbb{R}^d)$ with unique nondegenerate critical point $x_\star$:*

$$\nabla f(x_\star) = 0, \qquad \det \nabla^2 f(x_\star) \neq 0.$$

For $O \in C_c^\infty(\mathbb{R}^d)$,

$$A_\varepsilon(O) := \varepsilon^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\frac{i}{\varepsilon} f(x)} O(x)\, dx$$

satisfies

$$|A_\varepsilon(O)|^2 \to (2\pi)^d \frac{|O(x_\star)|^2}{|\det \nabla^2 f(x_\star)|}.$$

Equivalently,

$$|A_\varepsilon(O)|^2 \to (2\pi)^d \langle \delta(\nabla f), |O|^2 \rangle.$$

Proof. Standard stationary phase at a single Morse critical point. □

Corollary 6.2 (Finite-dimensional QM/QFT truncations)**.** *For any finite-dimensional discretization S_N of a variational action (time-sliced QM or lattice QFT), the same stationary-phase mechanism localizes on*

$$\nabla S_N = 0,$$

providing the finite-dimensional realization of $\delta(\delta S)$ as an extremum selector.

7 Large-N Mode-Coupled Lift

We now pass from fixed interacting blocks to a genuinely growing mode-coupled family.

Theorem 7.1 (Large-N coupled Gaussian-tail convergence with rate)**.** *Fix $m \geq 1$. Let*

$$S_N(u, v) = P_m(u) + \sum_{j=m+1}^{N} \left(\frac{\lambda_j}{2} + \beta_j(u) \right) v_j^2, \quad \beta_j(u) = \sum_{i=1}^{m} a_{ij} u_i^2,$$

with:

1. $\lambda_j \geq \lambda_- > 0$,
2. $a_{ij} \geq 0$ *and* $A_j := \sum_{i=1}^m a_{ij}$ *satisfies*

$$\sum_{j=m+1}^{\infty} \frac{A_j}{\lambda_j} < \infty,$$

3. $P_m(u) \geq c_4 \|u\|^4 - c_2 \|u\|^2 - C_0$.

For bounded F_m and $\eta > 0, \varepsilon > 0$, define

$$\omega_{\varepsilon,\eta,N}(F_m) := \frac{\int_{\mathbb{R}^N} e^{-(\eta - i/\varepsilon) S_N} F_m(u)\, du\, dv}{\int_{\mathbb{R}^N} e^{-(\eta - i/\varepsilon) S_N}\, du\, dv}.$$

Then:

1. $\omega_{\varepsilon,\eta,N}(F_m)$ *converges as* $N \to \infty$.
2. *There exists* $C_{F_m,\varepsilon,\eta} > 0$ *such that for* $N' > N \geq m$,

$$|\omega_{\varepsilon,\eta,N'}(F_m) - \omega_{\varepsilon,\eta,N}(F_m)| \leq C_{F_m,\varepsilon,\eta} \sum_{j=N+1}^{N'} \frac{A_j}{\lambda_j}.$$

Proof. Integrate each Gaussian tail coordinate:

$$\int_{\mathbb{R}} e^{-(\eta-i/\varepsilon)\left(\frac{\lambda_j}{2}+\beta_j(u)\right)t^2}\,dt=\sqrt{\frac{2\pi}{\eta-i/\varepsilon}}\,\left(\lambda_j+2\beta_j(u)\right)^{-1/2}.$$

Constants independent of u cancel in the normalized ratio, giving

$$\omega_{\varepsilon,\eta,N}(F_m)=\frac{\mathcal{N}_N(F_m)}{\mathcal{D}_N},$$

with

$$\mathcal{N}_N(F):=\int_{\mathbb{R}^m} e^{-(\eta-i/\varepsilon)P_m(u)}F(u)\Phi_N(u)\,du,$$

$$\Phi_N(u):=\prod_{j=m+1}^{N}R_j(u),\qquad R_j(u):=\left(\frac{\lambda_j}{\lambda_j+2\beta_j(u)}\right)^{1/2}\in(0,1].$$

Now

$$-\log R_j(u)=\frac{1}{2}\log\left(1+\frac{2\beta_j(u)}{\lambda_j}\right)\le\frac{\beta_j(u)}{\lambda_j}\le\|u\|^2\frac{A_j}{\lambda_j}.$$

Hence $\sum_j|\log R_j(u)|<\infty$, so $\Phi_N(u)\to\Phi_\infty(u)\in(0,1]$. By coercivity of P_m and $|\Phi_N|\le 1$:

$$|\mathcal{N}_N(F)|\le\|F\|_\infty\int e^{-\eta P_m(u)}du<\infty,$$

thus dominated convergence gives $\mathcal{N}_N(F)\to\mathcal{N}_\infty(F)$ and $\mathcal{D}_N\to\mathcal{D}_\infty$. Assuming $\mathcal{D}_\infty\neq 0$, ratios converge.

For the rate, write $\Phi_{N'}=\Phi_N\Psi_{N,N'}$, $\Psi_{N,N'}:=\prod_{j=N+1}^{N'}R_j$. Because $0<R_j\le 1$,

$$1-\Psi_{N,N'}\le\sum_{j=N+1}^{N'}(1-R_j).$$

Set $t_j=2\beta_j/\lambda_j\ge 0$. Since $1-(1+t)^{-1/2}\le t$ for $t\ge 0$,

$$1-R_j(u)\le\frac{2\beta_j(u)}{\lambda_j}\le 2\|u\|^2\frac{A_j}{\lambda_j}.$$

Therefore

$$|\Phi_{N'}(u)-\Phi_N(u)|\le 2\|u\|^2\sum_{j=N+1}^{N'}\frac{A_j}{\lambda_j}.$$

Insert this bound in $\mathcal{N},\mathcal{D}$ differences and use $|e^{-(\eta-i/\varepsilon)P_m}|\le e^{-\eta P_m}$. Then for $C_1:=2\int e^{-\eta P_m}\|u\|^2du<\infty$:

$$|\mathcal{N}_{N'}(F)-\mathcal{N}_N(F)|\le\|F\|_\infty C_1\sum_{j=N+1}^{N'}\frac{A_j}{\lambda_j},$$

$$|\mathcal{D}_{N'}-\mathcal{D}_N|\le C_1\sum_{j=N+1}^{N'}\frac{A_j}{\lambda_j}.$$

For large N, $|\mathcal{D}_N|\ge d_*>0$, and

$$\left|\frac{a'}{b'}-\frac{a}{b}\right|\le\frac{|a'-a|}{|b'|}+\frac{|a|\,|b'-b|}{|b'||b|}$$

gives the stated rate. $\square$

Theorem 7.2 (Non-factorized quartic-tail large-N extension)**.** *Let*

$$S_N(u,v) = P_m(u) + \sum_{j=m+1}^{N} \Big(\big(\tfrac{\lambda_j}{2} + \beta_j(u)\big) v_j^2 + \gamma_j v_j^4 \Big),$$

with $\lambda_j \geq \lambda_- > 0$, $\gamma_j \geq \gamma_- > 0$, coercive P_m, and

$$\beta_j(u) \leq A_j \|u\|^2, \qquad A_j \geq 0.$$

For

$$I_j(b) := \int_{\mathbb{R}} e^{-c((\lambda_j/2+b)t^2+\gamma_j t^4)} dt, \quad c = \eta - i/\varepsilon, \; b \geq 0,$$

assume:

1. $I_j(b) \neq 0$ *for all* $j, b \geq 0$,
2. $\sup_{b\geq 0} |\partial_b \log I_j(b)| \leq L_j$ *and*
$$\sum_{j=m+1}^{\infty} L_j A_j < \infty.$$

Then for bounded cylinder observables F_m,

$$\omega_{\varepsilon,\eta,N}(F_m) := \frac{\int e^{-cS_N} F_m(u)\, du\, dv}{\int e^{-cS_N}\, du\, dv}$$

converges as $N \to \infty$ (if the limiting denominator is nonzero), and satisfies the tail estimate

$$|\omega_{\varepsilon,\eta,N'}(F_m) - \omega_{\varepsilon,\eta,N}(F_m)| \leq C_{F_m,\varepsilon,\eta} \sum_{j=N+1}^{N'} L_j A_j.$$

Proof. Integrate each v_j:

$$\omega_{\varepsilon,\eta,N}(F_m) = \frac{\int_{\mathbb{R}^m} e^{-cP_m(u)} F_m(u) \Phi_N(u)\, du}{\int_{\mathbb{R}^m} e^{-cP_m(u)} \Phi_N(u)\, du}, \quad \Phi_N(u) = \prod_{j=m+1}^{N} \frac{I_j(\beta_j(u))}{I_j(0)}.$$

For each j,

$$\left| \log \frac{I_j(\beta_j(u))}{I_j(0)} \right| = \left| \int_0^{\beta_j(u)} \partial_b \log I_j(b)\, db \right| \leq L_j A_j \|u\|^2.$$

Hence

$$\sum_{j=m+1}^{\infty} \left| \log \frac{I_j(\beta_j(u))}{I_j(0)} \right| \leq \|u\|^2 \sum_{j=m+1}^{\infty} L_j A_j < \infty,$$

so $\Phi_N(u) \to \Phi_\infty(u)$ pointwise and

$$|\Phi_N(u)| \leq \exp(B\|u\|^2), \qquad B := \sum_{j=m+1}^{\infty} L_j A_j.$$

Thus

$$|e^{-cP_m(u)} \Phi_N(u) F_m(u)| \leq \|F_m\|_\infty e^{-\eta P_m(u)} e^{B\|u\|^2},$$

integrable by quartic coercivity; dominated convergence yields numerator/denominator limits and ratio convergence.

For the rate, define

$$\Delta_{N,N'}(u) := \sum_{j=N+1}^{N'} \log \frac{I_j(\beta_j(u))}{I_j(0)}, \quad |\Delta_{N,N'}(u)| \le \|u\|^2 \sum_{j=N+1}^{N'} L_j A_j.$$

With $\Phi_{N'} = \Phi_N e^{\Delta_{N,N'}}$:

$$|\Phi_{N'} - \Phi_N| \le |\Phi_N|\,|e^{\Delta_{N,N'}} - 1| \le e^{B\|u\|^2}\,|\Delta_{N,N'}|e^{|\Delta_{N,N'}|}.$$

This gives

$$|\Phi_{N'} - \Phi_N| \le e^{(B+\tilde{B})\|u\|^2}\|u\|^2 \sum_{j=N+1}^{N'} L_j A_j,$$

for a finite $\tilde{B}$ (tail-sum bound). Integrating against $e^{-\eta P_m}$ gives numerator/denominator Cauchy bounds, and the ratio estimate follows as in Theorem 7.1. □

Corollary 7.3 (Intrinsic sufficient conditions for Theorem 7.2)**.** *For each j, define block moments*

$$\overline{M}_j^{(1)} := \sup_{b\ge 0} \mathbb{E}_{\nu_{j,b}}[S_{j,b}], \qquad \overline{M}_j^{(2)} := \sup_{b\ge 0} \mathbb{E}_{\nu_{j,b}}[t^2],$$

where

$$\nu_{j,b}(dt) := \frac{e^{-\eta S_{j,b}(t)}}{\int e^{-\eta S_{j,b}}}\,dt, \quad S_{j,b}(t) = \left(\frac{\lambda_j}{2} + b\right)t^2 + \gamma_j t^4.$$

If

$$\varepsilon > \sup_j \overline{M}_j^{(1)},$$

and

$$\sum_{j=m+1}^{\infty} A_j \frac{|c|\,\overline{M}_j^{(2)}}{1 - \overline{M}_j^{(1)}/\varepsilon} < \infty,$$

then hypotheses (Q1)–(Q2) in Theorem 7.2 hold with

$$L_j = \frac{|c|\,\overline{M}_j^{(2)}}{1 - \overline{M}_j^{(1)}/\varepsilon}.$$

Proof. By Theorem 8.1 applied to each block $S_{j,b}$:

$$|I_j(b)| \ge \left(\int e^{-\eta S_{j,b}}\right)\left(1 - \frac{\overline{M}_j^{(1)}}{\varepsilon}\right) > 0,$$

so (Q1) holds. Also

$$\partial_b I_j(b) = -c\int t^2 e^{-cS_{j,b}(t)}dt,$$

thus

$$|\partial_b \log I_j(b)| \le \frac{|c|\int t^2 e^{-\eta S_{j,b}}dt}{\int e^{-\eta S_{j,b}}dt\,(1 - \overline{M}_j^{(1)}/\varepsilon)} \le \frac{|c|\,\overline{M}_j^{(2)}}{1 - \overline{M}_j^{(1)}/\varepsilon}.$$

This is (Q2). Summability is exactly the second assumption. □

Theorem 7.4 (Non-factorized quadratic-mixing large-N extension)**.** *Let*

$$S_N(u,v) = P_m(u) + \frac{1}{2}\, v^\top \big(D_N(u) + K_N\big) v,$$

where $u \in \mathbb{R}^m$*,* $v \in \mathbb{R}^{N-m}$*,* $D_N(u) = \operatorname{diag}(d_{m+1}(u), \dots, d_N(u))$ *with*

$$d_j(u) = \lambda_j + 2\beta_j(u), \quad \beta_j(u) = \sum_{i=1}^m a_{ij} u_i^2, \quad a_{ij} \ge 0,\ \lambda_j \ge \lambda_- > 0.$$

Assume coercive P_m *and*

$$A_j := \sum_{i=1}^m a_{ij}, \qquad \sum_{j=m+1}^\infty \frac{A_j}{\lambda_j} < \infty.$$

Let $K = (\kappa_{jk})_{j,k>m}$ *be real symmetric,* K_N *its principal truncation, and with* $\Lambda = \operatorname{diag}(\lambda_j)$ *define*

$$\widetilde{K} := \Lambda^{-1/2} K \Lambda^{-1/2}.$$

Assume

$$\|\widetilde{K}\| < \theta < 1, \qquad \|\widetilde{K}\|_1 < \infty,$$

and

$$\tau_N := \|\widetilde{K} - P_N \widetilde{K} P_N\|_1 \to 0$$

(P_N*: projection onto indices* $m+1, \dots, N$*).*

Then for bounded cylinder F_m*,*

$$\omega_{\varepsilon,\eta,N}(F_m) = \frac{\int e^{-cS_N} F_m(u)\, du\, dv}{\int e^{-cS_N}\, du\, dv}, \qquad c = \eta - i/\varepsilon,\ \eta > 0,$$

converges as $N \to \infty$ *(if the limiting denominator is nonzero), and there exists* $C_{F_m,\varepsilon,\eta} > 0$ *such that*

$$|\omega_{\varepsilon,\eta,N'}(F_m) - \omega_{\varepsilon,\eta,N}(F_m)| \le C_{F_m,\varepsilon,\eta} \left(\sum_{j=N+1}^{N'} \frac{A_j}{\lambda_j} + \tau_N \right), \quad N' > N \ge m.$$

Proof. Integrate in v:

$$\int_{\mathbb{R}^{N-m}} e^{-c\frac{1}{2}v^\top (D_N + K_N) v} dv = C_N(c) \det(D_N + K_N)^{-1/2},$$

with u-independent $C_N(c)$ that cancels in normalized ratios. So

$$\omega_{\varepsilon,\eta,N}(F_m) = \frac{\int_{\mathbb{R}^m} e^{-cP_m(u)} F_m(u) \Phi_N(u)\, du}{\int_{\mathbb{R}^m} e^{-cP_m(u)} \Phi_N(u)\, du}, \quad \Phi_N(u) := \det(D_N(u) + K_N)^{-1/2}.$$

Write

$$\Phi_N(u) = \Phi_N^{\text{diag}}(u)\, \Delta_N(u),$$

$$\Phi_N^{\text{diag}}(u) := \prod_{j=m+1}^N d_j(u)^{-1/2}, \qquad \Delta_N(u) := \det(I + M_N(u))^{-1/2},$$

$$M_N(u) := D_N(u)^{-1/2} K_N D_N(u)^{-1/2}.$$

By Theorem 7.1, Φ_N^{diag} has Cauchy tail control by

$$\sigma_N := \sum_{j=N+1}^\infty A_j/\lambda_j.$$

Since $D_N(u) \geq \Lambda_N$, we have

$$\|M_N(u)\| \leq \|\widetilde{K}\| < \theta, \qquad \|M_N(u)\|_1 \leq \|\widetilde{K}\|_1 =: K_1.$$

Thus $I + M_N(u)$ is invertible and

$$|\log\det(I + M_N(u))| \leq \frac{1}{1-\theta}\|M_N(u)\|_1 \leq \frac{K_1}{1-\theta},$$

hence $|\Delta_N(u)| \leq C_\Delta$ uniformly.

Now set

$$Q(u) := \operatorname{diag}\Big((\lambda_j/d_j(u))^{1/2}\Big), \qquad 0 < Q(u) \leq I.$$

Then $M_\infty(u) = Q(u)\widetilde{K}Q(u)$ and $M_N(u) = P_N M_\infty(u) P_N$, so

$$\|M_\infty(u) - M_N(u)\|_1 \leq \|\widetilde{K} - P_N\widetilde{K}P_N\|_1 = \tau_N.$$

On $\|A\|, \|B\| \leq \theta < 1$,

$$|\log\det(I + A) - \log\det(I + B)| \leq \frac{1}{1-\theta}\|A - B\|_1,$$

thus

$$|\Delta_\infty(u) - \Delta_N(u)| \leq C'_\Delta \tau_N$$

uniformly in u.

Therefore, for $N' > N$,

$$\Phi_{N'} - \Phi_N = \Delta_{N'}(\Phi_{N'}^{\text{diag}} - \Phi_N^{\text{diag}}) + \Phi_N^{\text{diag}}(\Delta_{N'} - \Delta_N),$$

and

$$|\Phi_{N'}(u) - \Phi_N(u)| \leq C_1 e^{B\|u\|^2}\big(\sigma_N + \tau_N\big),$$

for constants C_1, B independent of N, u. Multiplying by $|e^{-cP_m(u)}| = e^{-\eta P_m(u)}$ gives an integrable envelope by quartic coercivity. Dominated convergence plus the standard ratio-difference estimate yields convergence and the stated mixed tail rate. $\square$

8 Partition-Factor Non-Vanishing Bounds

Theorem 8.1 (Moment criteria)**.** *Let $A_\eta = \int e^{-\eta S(x)}dx \in (0, \infty)$ and*

$$Z_{\varepsilon,\eta} := \int e^{-(\eta - i/\varepsilon)S(x)}dx = A_\eta\, \mathbb{E}_{\mu_\eta}[e^{iS/\varepsilon}], \quad \mu_\eta(dx) := \frac{e^{-\eta S(x)}}{A_\eta}\, dx.$$

Define

$$M_1 := \mathbb{E}_{\mu_\eta}|S|, \qquad M_2 := \mathbb{E}_{\mu_\eta}(S^2).$$

Then

$$|Z_{\varepsilon,\eta}| \geq A_\eta\left(1 - \frac{M_1}{\varepsilon}\right),$$

$$|Z_{\varepsilon,\eta}| \geq A_\eta\left(1 - \frac{M_2}{2\varepsilon^2}\right).$$

Hence if $\varepsilon > M_1$ or $\varepsilon^2 > M_2/2$, then $Z_{\varepsilon,\eta} \neq 0$.

Proof. First bound:

$$\left|\mathbb{E}[e^{iX}]\right| = |1 + \mathbb{E}(e^{iX} - 1)| \geq 1 - \mathbb{E}|e^{iX} - 1|, \quad X = S/\varepsilon.$$

Since $|e^{it} - 1| \leq |t|$,

$$\left|\mathbb{E}[e^{iS/\varepsilon}]\right| \geq 1 - \frac{M_1}{\varepsilon}.$$

Multiply by A_η.

Second bound:

$$\Re\mathbb{E}[e^{iS/\varepsilon}] = \mathbb{E}[\cos(S/\varepsilon)] \geq 1 - \frac{1}{2}\mathbb{E}[(S/\varepsilon)^2] = 1 - \frac{M_2}{2\varepsilon^2}.$$

Now $|z| \geq \Re z$ gives the inequality for $|Z_{\varepsilon,\eta}|$. □

9 Observable-Class Extension

Theorem 9.1 (Continuity on Schwartz and weighted Sobolev classes)**.** *Let*

$$\mathcal{I}(F) = \int_{\mathbb{R}^d} e^{i\Phi(y)} W(y) F(Ay)\, dy,$$

with $A \in GL(d, \mathbb{C})$ and

$$|W(y)| \leq C_0 e^{-c_4\|y\|^4 + c_2\|y\|^2}, \qquad c_4 > 0.$$

Then:

1. *for every integer $k > d$, there exists C_k such that*

$$|\mathcal{I}(F)| \leq C_k \sup_x (1 + \|x\|)^k |F(x)|, \qquad F \in \mathcal{S}(\mathbb{R}^d);$$

2. *for every $k > d/2$, there exists C'_k such that*

$$|\mathcal{I}(F)| \leq C'_k \|(1 + \|x\|^2)^{k/2} F\|_{L^2}, \qquad F \in H^{0,k}.$$

Consequently, normalized functionals $\omega(F) = \mathcal{I}(F)/\mathcal{I}(1)$ (when $\mathcal{I}(1) \neq 0$) extend continuously from Gaussian-polynomial test families to both classes.

Proof. For Schwartz:

$$|F(Ay)| \leq C_A p_k(F)(1 + \|y\|)^{-k}, \quad p_k(F) := \sup_x (1 + \|x\|)^k |F(x)|.$$

Hence

$$|\mathcal{I}(F)| \leq C_0 C_A p_k(F) \int e^{-c_4\|y\|^4 + c_2\|y\|^2} (1 + \|y\|)^{-k} dy,$$

and the integral is finite.

For weighted Sobolev:

$$|\mathcal{I}(F)| \leq \|W(\cdot)(1 + \|\cdot\|^2)^{-k/2}\|_{L^2} \cdot \|(1 + \|y\|^2)^{k/2} F(Ay)\|_{L^2_y}.$$

The first factor is finite by quartic decay; the second is bounded by $C'_A \|F\|_{H^{0,k}}$ after linear change of variables. □

10 Schwinger-Dyson and τ_μ Scale Covariance

Definition 10.1 (c-invariant quantity)**.** *For parameter triples (κ, η, h), define*

$$c := (\eta - i/h)\kappa.$$

A quantity is called c-invariant if it is unchanged under any parameter change that leaves c fixed (equivalently, unchanged along τ_μ-orbits).

Theorem 10.2 (Finite-dimensional Schwinger-Dyson identity)**.** *Let $c = \eta - i/\varepsilon$ and*

$$\mathcal{I}_c(F) := \int e^{-cS(x)} F(x)\, dx.$$

Assume integrability and vanishing boundary flux for admissible F and vector field V. Then

$$\mathcal{I}_c(V \cdot \nabla S\, F) = \frac{1}{c}\mathcal{I}_c(\nabla\cdot (VF)).$$

If $\mathcal{I}_c(1) \neq 0$, then

$$\omega_c(V \cdot \nabla S\, F) = \frac{1}{c}\omega_c(\nabla\cdot (VF)).$$

In particular, for constant $V = e_i$ and $F \equiv 1$:

$$\omega_c(\partial_i S) = 0.$$

Proof.

$$0 = \int \nabla\cdot (e^{-cS} VF)\, dx = \int e^{-cS}(\nabla\cdot (VF) - c\, V \cdot \nabla S\, F)\, dx.$$

Rearrange, then divide by $\mathcal{I}_c(1)$ for the normalized form. □

Theorem 10.3 (Exact τ_μ covariance)**.** *For*

$$\omega_{\kappa,\eta,h}(F) := \frac{\int e^{-(\eta - i/h)\kappa S(x)} F(x)\, dx}{\int e^{-(\eta - i/h)\kappa S(x)}\, dx},$$

define

$$\tau_\mu : (\kappa, \eta, h) \mapsto (\mu\kappa, \eta/\mu, \mu h), \qquad \mu > 0.$$

Then

$$\omega_{\kappa,\eta,h}(F) = \omega_{\tau_\mu(\kappa,\eta,h)}(F).$$

Proof. Directly,

$$\left(\frac{\eta}{\mu} - \frac{i}{\mu h}\right)(\mu\kappa) = (\eta - i/h)\kappa.$$

Hence numerator and denominator kernels are unchanged. □

Proposition 10.4 (Kernel-parameter unification of τ_μ and Schwinger-Dyson)**.** *Set*

$$c := (\eta - i/h)\kappa.$$

Then:

1. *τ_μ preserves c,*
2. *the Schwinger-Dyson identity in Theorem 10.2 depends only on c.*

Hence two parameter triples on the same τ_μ-orbit define identical Schwinger-Dyson relations for all admissible observables.

Proof. Part (i) is exactly the computation in Theorem 10.3. For part (ii), Theorem 10.2 is written purely in terms of $\mathcal{I}_c(\cdot)$, hence only through c. Therefore τ_μ-related parameter triples yield the same SD equations. □

11 Dependency Chain

The proofs in this manuscript now close in the following order:

1. Sections 2–5: core scoped Claim 1 closure.
2. Theorems 7.1, 7.4, 7.2, and Corollary 7.3: large-N coupled extensions (Gaussian-tail rate, non-factorized quadratic-mixing determinant class, non-factorized quartic-tail class, and intrinsic moment-based sufficient conditions).
3. Theorem 8.1: explicit non-vanishing criteria for partition factors.
4. Theorem 9.1: observable-class extension to Schwartz/Sobolev.
5. Theorems 10.2, 10.3, and Proposition 10.4: Schwinger-Dyson identities, exact scale-flow covariance, and their shared invariant kernel parameter.

12 Conclusion

This manuscript now gives a theorem-grade chain from the static/finite-dimensional variational-delta picture to a broad scoped family of oscillatory states including:

1. exact projective closure and de-regularization,
2. explicit semiclassical channel expansion,
3. large-N mode-coupled convergence with quantitative tail control (Gaussian-tail, non-factorized quadratic-mixing, and non-factorized quartic-tail classes), with intrinsic moment criteria for quartic-tail hypotheses,
4. non-vanishing control for normalization factors,
5. extension to large observable classes,
6. Schwinger-Dyson identities and scale-flow covariance with a shared c-invariance structure.

Open frontier (outside this closure): full interacting, non-factorized, genuinely field-theoretic continuum limits with uniform renormalization control beyond the scoped polynomial block-tail class.

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