Claim 1 (Scoped): Complete Proof in a Projective Oscillatory Class

2026-02-09

Abstract

We provide a complete proof of Claim 1 in a scoped but nontrivial class: projective cylinder states built from oscillatory actions with a finite interacting block (allowing mode coupling) and a quartic-stabilized tail. The proof closes: (i) exact large-N projective stability, (ii) continuum state existence on cylinder observables, (iii) constructive counterterm repair for coefficient drift, (iv) de-regularization $\eta\to0^+$ via contour rotation (Gaussian, factorized quartic, and coupled quartic block), and (v) explicit Gaussian semiclassical channel expansion in point-supported distribution modes. It further closes: (vi) genuinely large-N mode-coupled lifts, including an explicit Gaussian-tail rate, a non-factorized quadratic-mixing determinant class, and a non-factorized quartic-tail class under log-derivative summability, (vii) explicit non-vanishing lower bounds for normalized oscillatory partition factors, (viii) extension of observables from Gaussian-exponential test families to Schwartz and weighted Sobolev classes, and (ix) finite-dimensional Schwinger-Dyson identities and exact τ_μ-type scale-flow covariance. This yields a theorem-grade realization of the Claim 1 bridge in the scoped framework.

Statement of the Scoped Claim

Background references used in this note include stationary phase and distribution theory , oscillatory/Feynman integrals , and groupoid quantization/tangent-groupoid context .

Definition 1 (Projective cylinder system). *For $N\ge1$, let $X_N=\mathbb{R}^N$ and $\pi_{N\to m}:X_N\to X_m$ be coordinate projection ($N\ge m$). Define $\mathrm{Cyl}:=\bigcup_{m\ge1}\{F=F_m\circ \pi_{\infty\to m}: F_m\in C_b^2(\mathbb{R}^m)\}$.*

Definition 2 (Block-tail action class). *Fix $b\in\mathbb{N}_0$, $g\ge0$, and parameters $0<\lambda_-\le\lambda_j\le\lambda_+,\qquad \kappa_j\in[0,\kappa_+]$. For $N\ge b$, define $S_N(x)=P_b(x_1,\dots,x_b)+\sum_{j=b+1}^N q_j(x_j), \quad q_j(u)=\frac{\lambda_j}{2}u^2+g\kappa_j u^4$. Assume:*

1. *P_b is a real polynomial with $P_b(0)=0$, $\nabla P_b(0)=0$.*

2. *There exist $c_4>0,c_2\ge0,C_0\ge0$ such that* $$P_b(z)\ge c_4\|z\|^4-c_2\|z\|^2-C_0,\qquad z\in\mathbb{R}^b.$$

For $\eta>0$ and $\varepsilon>0$, define the normalized oscillatory state $$\omega_{\varepsilon,\eta,N}(F_m) := \frac{\int_{\mathbb{R}^N}\mathrm{e}^{-(\eta-i/\varepsilon)S_N(x)}F_m(x_1,\dots,x_m)\,dx}{\int_{\mathbb{R}^N}\mathrm{e}^{-(\eta-i/\varepsilon)S_N(x)}\,dx}, \qquad N\ge m,$$ whenever the denominator is nonzero.

Theorem 3 (Scoped Claim 1, complete proof). *In the block-tail action class:*

1. ***Exact projective stability**: for every cylinder observable F_m and $N\ge M:=\max\{m,b\}$,* $$\omega_{\varepsilon,\eta,N}(F_m)=\omega_{\varepsilon,\eta,M}(F_m).$$

2. ***Continuum state**: for each (ε,η), there is a unique functional $\omega_{\varepsilon,\eta}:\mathrm{Cyl}\to\mathbb{C}$ with* $$\omega_{\varepsilon,\eta}(F_m\circ\pi_{\infty\to m}) := \omega_{\varepsilon,\eta,M}(F_m),\quad M=\max\{m,b\},$$ *and* $$|\omega_{\varepsilon,\eta}(F)|\le C_{\varepsilon,\eta,m}\,\|F\|_\infty,$$ *where, for $M=\max\{m,b\}$,* $$C_{\varepsilon,\eta,m}:= \frac{\int_{\mathbb{R}^M}\mathrm{e}^{-\eta S_M(u)}\,du}{\left|\int_{\mathbb{R}^M}\mathrm{e}^{-(\eta-i/\varepsilon)S_M(u)}\,du\right|}<\infty.$$

3. ***Counterterm repair**: explicit local quadratic/quartic counterterms can repair scale-dependent coefficient drift and restore exact projective stability.*

4. ***De-regularization**: for Gaussian-exponential cylinder observables $F_m(x)=p(x)\mathrm{e}^{-x^\top Bx}$ (polynomial p, $B\succeq0$), the limit $\omega_{\varepsilon,0}(F):=\lim_{\eta\to0^+}\omega_{\varepsilon,\eta}(F)$ exists (branch fixed by contour angle $\pi/8$).*

5. ***Semiclassical channels (Gaussian subcase)**: if $g=0$, $b=0$, then for $F_m\in\mathcal{S}(\mathbb{R}^m)$,* $$\omega_{\varepsilon,0}(F_m)=\left[\exp\!\left(\frac{i\varepsilon}{2}\mathcal{L}_m\right)F_m\right]_{x=0}, \quad \mathcal{L}_m:=\sum_{j=1}^m\lambda_j^{-1}\partial_{x_j}^2,$$ *hence* $$\omega_{\varepsilon,0}(F_m) = \sum_{k=0}^{K-1}\frac{1}{k!}\left(\frac{i\varepsilon}{2}\right)^k(\mathcal{L}_m^kF_m)(0)+R_{K,\varepsilon}(F_m),$$ *which is precisely a hierarchy of point-supported derivative channels at the extremum.*

Sections 2–5 prove each item.

Projective Stability and Continuum State

Lemma 4 (Tail factorization). *Let $M=\max\{m,b\}$ and $N\ge M$. Write $x=(u,v)$ with $u\in\mathbb{R}^M$, $v\in\mathbb{R}^{N-M}$. Then $S_N(u,v)=S_M(u)+\sum_{j=M+1}^N q_j(v_j)$.*

Proof. By construction, coordinates $1,\dots,b$ appear only in P_b, and each $j>b$ contributes only $q_j(x_j)$. For $N\ge M$, all interacting coordinates are contained in the u-block. □

Proposition 5 (Exact large-N stability). *Assume denominators are nonzero. Then $\omega_{\varepsilon,\eta,N}(F_m)=\omega_{\varepsilon,\eta,M}(F_m),\qquad N\ge M.$*

Proof. Using Lemma 4,

$$\begin{align*} &\int_{\mathbb{R}^N}\mathrm{e}^{-(\eta-i/\varepsilon)S_N(x)}F_m(x_1,\dots,x_m)\,dx\\ &= \left[\int_{\mathbb{R}^M}\mathrm{e}^{-(\eta-i/\varepsilon)S_M(u)}F_m(u_1,\dots,u_m)\,du\right]\prod_{j=M+1}^N \left[\int_{\mathbb{R}}\mathrm{e}^{-(\eta-i/\varepsilon)q_j(t)}\,dt\right]. \end{align*}$$

The denominator factorizes with the same tail product, which cancels in the ratio. □

Proposition 6 (Continuum functional on cylinders). *For fixed (ε,η), define $\omega_{\varepsilon,\eta}(F_m\circ\pi_{\infty\to m}) := \omega_{\varepsilon,\eta,M}(F_m),\quad M=\max\{m,b\}$. This is well-defined, linear on Cyl, and bounded by $|\omega_{\varepsilon,\eta}(F_m\circ\pi_{\infty\to m})| \le C_{\varepsilon,\eta,m}\|F_m\|_\infty$, with $C_{\varepsilon,\eta,m}$ as in Theorem 3.*

Proof. Well-definedness follows from Proposition 5. Linearity is immediate from linearity of finite-dimensional integrals. For the bound, write $Z_M:=\int_{\mathbb{R}^M}\mathrm{e}^{-(\eta-i/\varepsilon)S_M(u)}\,du,\qquad A_M:=\int_{\mathbb{R}^M}\mathrm{e}^{-\eta S_M(u)}\,du.$ Then $\left|\int_{\mathbb{R}^M}\mathrm{e}^{-(\eta-i/\varepsilon)S_M(u)}F_m(u)\,du \right| \le \|F_m\|_\infty A_M$, and therefore $|\omega_{\varepsilon,\eta}(F_m\circ\pi_{\infty\to m})| \le \frac{A_M}{|Z_M|}\|F_m\|_\infty =C_{\varepsilon,\eta,m}\|F_m\|_\infty.$ The constant is finite whenever $Z_M\neq0$. □

Counterterm Repair

Suppose bare coefficients drift with N: $\lambda_{j,N}^{\mathrm{bare}}=\lambda_j+r_{j,N},\qquad \kappa_{j,N}^{\mathrm{bare}}=\kappa_j+s_{j,N}$. Assume bounds $|r_{j,N}|\le\lambda_-/2,\quad|s_{j,N}|\le\kappa_+/2.$

$r_{j,N}|\le \lambda_-/2,\qquad |s_{j,N}|\le \kappa_+/2$. Define local counterterms $\delta S_N(x)=\sum_{j=1}^N\left[-\frac{r_{j,N}}{2}x_j^2-g\,s_{j,N}x_j^4\right]$. Then $S_N^{\mathrm{ren}}:=S_N^{\mathrm{bare}}+\delta S_N$ has coefficients exactly (λ_j,κ_j) and belongs to the stable block-tail class.

Proposition 7 (Constructive repair). *The renormalized family S_N^{ren} satisfies the hypotheses of Proposition 5; therefore projective stability is restored exactly.*

Proof. Direct substitution cancels all coefficient drifts coordinatewise. The restored action has N-independent coefficients and the same block-tail decomposition. Apply Proposition 5. □

De-Regularization $\eta\to0^+$

Lemma 8 (Rotated contour dominance). *Fix finite dimension d and polynomial action $\mathcal{S}(x)=Q_2(x)+gQ_4(x)$, where Q_2 is real quadratic and Q_4 is real quartic with $Q_4(y)\ge c\|y\|^4,\qquad c>0$. Let $x=\mathrm{e}^{i\pi/8}y$ and $\eta\in[0,\eta_0]$. For $F(y)=p(y)\mathrm{e}^{-y^\top B y}$ with polynomial p and $B\succeq0$, there exist constants $C,\tilde c_4>0,\tilde c_2\ge0$ such that $\left|\mathrm{e}^{-(\eta-i/\varepsilon)\mathcal{S}(\mathrm{e}^{i\pi/8}y)} F(\mathrm{e}^{i\pi/8}y) \right| \le C(1+\|y\|^k)\mathrm{e}^{-\tilde c_4\|y\|^4+\tilde c_2\|y\|^2}$.*

Proof. Under $x=\mathrm{e}^{i\pi/8}y$, quartic monomials acquire phase $\mathrm{e}^{i\pi/2}=i$. Hence $\Re\!\left(\frac{i}{\varepsilon}gQ_4(\mathrm{e}^{i\pi/8}y)\right) = -\frac{g}{\varepsilon}Q_4(y) \le -\frac{gc}{\varepsilon}\|y\|^4$. The remaining quadratic and η-terms contribute at most $+\tilde c_2\|y\|^2$. Polynomial prefactors produce $(1+\|y\|^k)$. The right side is integrable on $\mathbb{R}^d$. □

Proposition 9 (Finite-dimensional $\eta\to0^+$ limit). *In the setting of Lemma 8, define $I_\eta(F):=\int_{\mathbb{R}^d}\mathrm{e}^{-(\eta-i/\varepsilon)\mathcal{S}(x)}F(x)\,dx$, with contour branch fixed by angle $\pi/8$. Then $\lim_{\eta\to0^+}I_\eta(F)=I_0(F)$. If $I_\eta(1)\neq0$ for small η and $I_0(1)\neq0$, then $\lim_{\eta\to0^+}\frac{I_\eta(F)}{I_\eta(1)} = \frac{I_0(F)}{I_0(1)}$.*

Proof. For $\eta>0$, deform real contour to angle $\pi/8$ (entire integrand, quartic decay on connecting arcs). On that contour, pointwise convergence as $\eta\to0^+$ is immediate. Lemma 8 gives a common L^1 dominator. Apply dominated convergence to numerator and denominator. □

Corollary 10 (De-regularized cylinder state). *For Gaussian-exponential cylinder observables in Theorem 3, the limit $\omega_{\varepsilon,0}(F)=\lim_{\eta\to0^+}\omega_{\varepsilon,\eta}(F)$ exists and is independent of N.*

Proof. Reduce to stabilized finite dimension $M=\max\{m,b\}$ by Proposition 5. Then apply Proposition 9 in dimension M. □

Gaussian Channel Expansion

Now take the Gaussian subcase $g=0$, $b=0$: $S_m(x)=\frac12\sum_{j=1}^m\lambda_jx_j^2$. Define, for $F\in\mathcal{S}(\mathbb{R}^m)$,
$$\omega_{\varepsilon,0}(F) := \frac{\int_{\mathbb{R}^m}\mathrm{e}^{\frac{i}{\varepsilon}S_m(x)}F(x)\,dx}{\int_{\mathbb{R}^m}\mathrm{e}^{\frac{i}{\varepsilon}S_m(x)}\,dx}.$$

Proposition 11 (Exact operator form). *Let $\mathcal{L}_m=\sum_{j=1}^m\lambda_j^{-1}\partial_{x_j}^2$. Then $\omega_{\varepsilon,0}(F) = \left[\exp\!\left(\frac{i\varepsilon}{2}\mathcal{L}_m\right)F\right]_{x=0}$.*

Proof. Write $F(x)=\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\hat F(\xi)\mathrm{e}^{i\xi\cdot x}\,d\xi$. By Gaussian completion (Fresnel branch), $\frac{\int \mathrm{e}^{\frac{i}{2\varepsilon}\sum_j\lambda_j x_j^2}\mathrm{e}^{i\xi\cdot x}\,dx}{\int \mathrm{e}^{\frac{i}{2\varepsilon}\sum_j\lambda_j x_j^2}\,dx} = \exp\!\left(-\frac{i\varepsilon}{2}\sum_{j=1}^m\frac{\xi_j^2}{\lambda_j}\right)$. Therefore $\omega_{\varepsilon,0}(F) = \frac{1}{(2\pi)^m}\int \hat F(\xi) \exp\!\left(-\frac{i\varepsilon}{2}\sum_j\frac{\xi_j^2}{\lambda_j}\right)d\xi$. The multiplier is exactly that of $\exp((i\varepsilon/2)\mathcal{L}_m)$ evaluated at $x=0$. □

Corollary 12 (Point-supported channel hierarchy). *For $K\ge1$, $\omega_{\varepsilon,0}(F) = \sum_{k=0}^{K-1}\frac{1}{k!}\left(\frac{i\varepsilon}{2}\right)^k(\mathcal{L}_m^kF)(0) +R_{K,\varepsilon}(F)$, with $R_{K,\varepsilon}(F)=O(\varepsilon^K)$ as $\varepsilon\to0^+$. Thus channels are derivatives of F at the extremum $x=0$, i.e. point-supported distribution modes.*

Proof. Expand the exponential operator in power series and use Schwartz regularity. □

Static Extremum Localization and the Variational-Delta Ladder

Proposition 13 (Static Morse localization). *Let $f\in C^\infty(\mathbb{R}^d)$ with unique nondegenerate critical point $x_\star$: $\nabla f(x_\star)=0,\qquad \det \nabla^2 f(x_\star)\neq0$. For $O\in C_c^\infty(\mathbb{R}^d)$, $A_\varepsilon(O):=\varepsilon^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{\frac{i}{\varepsilon}f(x)}O(x)\,dx$*

satisfies $|A_\varepsilon(O)|^2 \to (2\pi)^d\frac{|O(x_\star)|^2}{|\det\nabla^2 f(x_\star)|}$. *Equivalently,* $|A_\varepsilon(O)|^2\to (2\pi)^d\langle \delta(\nabla f),|O|^2\rangle$.

Proof. Standard stationary phase at a single Morse critical point. □

Corollary 14 (Finite-dimensional QM/QFT truncations). *For any finite-dimensional discretization* S_N *of a variational action (time-sliced QM or lattice QFT), the same stationary-phase mechanism localizes on* $\nabla S_N=0$*, providing the finite-dimensional realization of* $\delta(\delta S)$ *as an extremum selector.*

Large-N Mode-Coupled Lift

We now pass from fixed interacting blocks to a genuinely growing mode-coupled family.

Theorem 15 (Large-N coupled Gaussian-tail convergence with rate). *Fix* $m\ge1$. *Let* $S_N(u,v)=P_m(u)+\sum_{j=m+1}^N\left(\frac{\lambda_j}{2}+\beta_j(u)\right)v_j^2, \quad \beta_j(u)=\sum_{i=1}^m a_{ij}u_i^2$, *with:*

1. $\lambda_j\ge \lambda_->0$,
2. $a_{ij}\ge0$ *and* $A_j:=\sum_{i=1}^m a_{ij}$ *satisfies* $\sum_{j=m+1}^\infty \frac{A_j}{\lambda_j}<\infty$,
3. $P_m(u)\ge c_4\|u\|^4-c_2\|u\|^2-C_0$.

For bounded F_m *and* $\eta>0,\varepsilon>0$, *define* $\omega_{\varepsilon,\eta,N}(F_m) := \frac{\int_{\mathbb{R}^N}e^{-(\eta-i/\varepsilon)S_N}F_m(u)\,du\,dv}{\int_{\mathbb{R}^N}e^{-(\eta-i/\varepsilon)S_N}\,du\,dv}$. *Then:*

1. $\omega_{\varepsilon,\eta,N}(F_m)$ *converges as* $N\to\infty$.
2. *There exists* $C_{F_m,\varepsilon,\eta}>0$ *such that for* $N'>N\ge m$, $|\omega_{\varepsilon,\eta,N'}(F_m)-\omega_{\varepsilon,\eta,N}(F_m)| \le C_{F_m,\varepsilon,\eta}\sum_{j=N+1}^{N'}\frac{A_j}{\lambda_j}$.

Proof. Integrate each Gaussian tail coordinate: $\int_\mathbb{R}e^{-(\eta-i/\varepsilon)\left(\frac{\lambda_j}{2}+\beta_j(u)\right)t^2}dt = \sqrt{\frac{2\pi}{\eta-i/\varepsilon}}\, \left(\lambda_j+2\beta_j(u)\right)^{-1/2}$. Constants independent of u cancel in the normalized ratio, giving $\omega_{\varepsilon,\eta,N}(F_m)= \frac{\mathcal N_N(F_m)}{\mathcal D_N}$, with $\mathcal N_N(F):=\int_{\mathbb{R}^m}e^{-(\eta-i/\varepsilon)P_m(u)}F(u)\Phi_N(u)\,du$, $\Phi_N(u):=\prod_{j=m+1}^N R_j(u),\qquad R_j(u):=\left(\frac{\lambda_j}{\lambda_j+2\beta_j(u)}\right)^{1/2}\in(0,1]$. Now $-\log R_j(u)=\frac12\log(1+2\beta_j(u)/\lambda_j)\le\beta_j(u)/\lambda_j\le\|u\|^2A_j/\lambda_j$.

$R_j(u)=\frac12\log\!\left(1+\frac{2\beta_j(u)}{\lambda_j}\right) \le \frac{\beta_j(u)}{\lambda_j} \le \|u\|^2\frac{A_j}{\lambda_j}$. Hence $\sum_j |\log R_j(u)|<\infty$, so $\Phi_N(u)\to\Phi_\infty(u)\in(0,1]$. By coercivity of P_m and $|\Phi_N|\le1$: $|\mathcal N_N(F)|\le \|F\|_\infty \int e^{-\eta P_m(u)} du<\infty$, thus dominated convergence gives $\mathcal N_N(F)\to\mathcal N_\infty(F)$ and $\mathcal D_N\to\mathcal D_\infty$. Assuming $\mathcal D_\infty\neq0$, ratios converge.

For the rate, write $\Phi_{N'}=\Phi_N\Psi_{N,N'}$, $\Psi_{N,N'}:=\prod_{j=N+1}^{N'}R_j$. Because $0<R_j\le1$, $1-\Psi_{N,N'}\le \sum_{j=N+1}^{N'}(1-R_j)$. Set $t_j=2\beta_j/\lambda_j\ge0$. Since $1-(1+t)^{-1/2}\le t$ for $t\ge0$, $1-R_j(u)\le \frac{2\beta_j(u)}{\lambda_j} \le 2\|u\|^2\frac{A_j}{\lambda_j}$. Therefore $|\Phi_{N'}(u)-\Phi_N(u)| \le 2\|u\|^2\sum_{j=N+1}^{N'}\frac{A_j}{\lambda_j}$. Insert this bound in $\mathcal N,\mathcal D$ differences and use $|e^{-(\eta-i/\varepsilon)P_m}|\le e^{-\eta P_m}$. Then for $C_1:=2\int e^{-\eta P_m}\|u\|^2du<\infty$: $|\mathcal N_{N'}(F)-\mathcal N_N(F)|\le \|F\|_\infty C_1\sum_{j=N+1}^{N'}\frac{A_j}{\lambda_j}$, $|\mathcal D_{N'}-\mathcal D_N| \le C_1\sum_{j=N+1}^{N'}\frac{A_j}{\lambda_j}$. For large N, $|\mathcal D_N|\ge d_*>0$, and $\left|\frac{a'}{b'}-\frac{a}{b}\right| \le \frac{|a'-a|}{|b'|}+\frac{|a|\,|b'-b|}{|b'||b|}$ gives the stated rate. □

Theorem 16 (Non-factorized quartic-tail large-N extension). *Let $S_N(u,v) = P_m(u)+\sum_{j=m+1}^N \Big(\big(\tfrac{\lambda_j}{2}+\beta_j(u)\big)v_j^2+\gamma_j v_j^4\Big)$, with $\lambda_j\ge\lambda_->0$, $\gamma_j\ge\gamma_->0$, coercive P_m, and $\beta_j(u)\le A_j\|u\|^2,\qquad A_j\ge0$. For $I_j(b):=\int_\mathbb{R}e^{-c((\lambda_j/2+b)t^2+\gamma_j t^4)}dt,\quad c=\eta-i/\varepsilon,\ b\ge0$, assume:*

1. *$I_j(b)\neq0$ for all $j,b\ge0$,*
2. *$\sup_{b\ge0}|\partial_b\log I_j(b)|\le L_j$ and $\sum_{j=m+1}^\infty L_jA_j<\infty$.*

Then for bounded cylinder observables F_m, $\omega_{\varepsilon,\eta,N}(F_m) := \frac{\int e^{-cS_N}F_m(u)\,du\,dv}{\int e^{-cS_N}\,du\,dv}$ converges as $N\to\infty$ (if the limiting denominator is nonzero), and satisfies the tail estimate $|\omega_{\varepsilon,\eta,N'}(F_m)-\omega_{\varepsilon,\eta,N}(F_m)| \le C_{F_m,\varepsilon,\eta}\sum_{j=N+1}^{N'}L_jA_j$.

Proof. Integrate each v_j:

$$\omega_{\varepsilon,\eta,N}(F_m)= \frac{\int_{\mathbb{R}^m}e^{-cP_m(u)}F_m(u)\Phi_N(u)\,du}{\int_{\mathbb{R}^m}e^{-cP_m(u)}\Phi_N(u)\,du}, \quad \Phi_N(u)=\prod_{j=m+1}^N\frac{I_j(\beta_j(u))}{I_j(0)}.$$

For each j, $\left|\log\frac{I_j(\beta_j(u))}{I_j(0)}\right| = \left|\int_0^{\beta_j(u)}\partial_b\log I_j(b)\,db\right| \le L_jA_j\|u\|$

^2. Hence ∑j=m+1∞|logIj(βj(u))Ij(0)|≤∥u∥2∑j=m+1∞LjAj<∞,$\sum_{j=m+1}^\infty\left|\log\frac{I_j(\beta_j(u))}{I_j(0)}\right| \le \|u\|^2\sum_{j=m+1}^\infty L_jA_j <\infty$, so ΦN(u)→Φ∞(u)$\Phi_N(u)\to\Phi_\infty(u)$ pointwise and |ΦN(u)|≤exp(B∥u∥2),B:=∑j=m+1∞LjAj.$|\Phi_N(u)|\le \exp(B\|u\|^2),\qquad B:=\sum_{j=m+1}^\infty L_jA_j$. Thus |e−cPm(u)ΦN(u)Fm(u)|≤∥Fm∥∞e−ηPm(u)eB∥u∥2,$|e^{-cP_m(u)}\Phi_N(u)F_m(u)| \le \|F_m\|_\infty e^{-\eta P_m(u)}e^{B\|u\|^2}$, integrable by quartic coercivity; dominated convergence yields numerator/denominator limits and ratio convergence.

For the rate, define ΔN,N′(u):=∑j=N+1N′logIj(βj(u))Ij(0),|ΔN,N′(u)|≤∥u∥2∑j=N+1N′LjAj.$\Delta_{N,N'}(u):=\sum_{j=N+1}^{N'}\log\frac{I_j(\beta_j(u))}{I_j(0)}, \quad |\Delta_{N,N'}(u)|\le \|u\|^2\sum_{j=N+1}^{N'}L_jA_j$. With ΦN′=ΦNeΔN,N′$\Phi_{N'}=\Phi_N e^{\Delta_{N,N'}}$: |ΦN′−ΦN|≤|ΦN||eΔN,N′−1|≤eB∥u∥2|ΔN,N′|e|ΔN,N′|.$|\Phi_{N'}-\Phi_N| \le |\Phi_N|\,|e^{\Delta_{N,N'}}-1| \le e^{B\|u\|^2}\,|\Delta_{N,N'}|e^{|\Delta_{N,N'}|}$. This gives |ΦN′−ΦN|≤e(B+B̃)∥u∥2∥u∥2∑j=N+1N′LjAj,$|\Phi_{N'}-\Phi_N| \le e^{(B+\tilde B)\|u\|^2}\|u\|^2\sum_{j=N+1}^{N'}L_jA_j$, for a finite B̃$\tilde B$ (tail-sum bound). Integrating against e−ηPm$e^{-\eta P_m}$ gives numerator/denominator Cauchy bounds, and the ratio estimate follows as in Theorem 15. □

Corollary 17 (Intrinsic sufficient conditions for Theorem 16). *For each jj, define block moments M¯j(1):=supb≥0𝔼νj,b[Sj,b],M¯j(2):=supb≥0𝔼νj,b[t2],*$\overline M^{(1)}_j:=\sup_{b\ge0}\mathbb E_{\nu_{j,b}}[S_{j,b}],\qquad \overline M^{(2)}_j:=\sup_{b\ge0}\mathbb E_{\nu_{j,b}}[t^2]$*, where νj,b(dt):=e−ηSj,b(t)∫e−ηSj,bdt,Sj,b(t)=(λj2+b)t2+γjt4.*$\nu_{j,b}(dt):=\frac{e^{-\eta S_{j,b}(t)}}{\int e^{-\eta S_{j,b}}}\,dt,\quad S_{j,b}(t)=\left(\frac{\lambda_j}{2}+b\right)t^2+\gamma_j t^4$*. If ε>supjM¯j(1),*$\varepsilon>\sup_j \overline M^{(1)}_j$*, and ∑j=m+1∞Aj|c|M¯j(2)1−M¯j(1)/ε<∞,*$\sum_{j=m+1}^\infty A_j\frac{|c|\,\overline M^{(2)}_j}{1-\overline M^{(1)}_j/\varepsilon} <\infty$*, then hypotheses (Q1)–(Q2) in Theorem 16 hold with Lj=|c|M¯j(2)1−M¯j(1)/ε.*$L_j= \frac{|c|\,\overline M^{(2)}_j}{1-\overline M^{(1)}_j/\varepsilon}$.

Proof. By Theorem 19 applied to each block Sj,b$S_{j,b}$: |Ij(b)|≥(∫e−ηSj,b)(1−M¯j(1)ε)>0,$|I_j(b)|\ge \left(\int e^{-\eta S_{j,b}}\right)\left(1-\frac{\overline M^{(1)}_j}{\varepsilon}\right)>0$, so (Q1) holds. Also ∂bIj(b)=−c∫t2e−cSj,b(t)dt,$\partial_b I_j(b)=-c\int t^2 e^{-cS_{j,b}(t)}dt$, thus |∂blogIj(b)|≤|c|∫t2e−ηSj,bdt∫e−ηSj,bdt(1−M¯j(1)/ε)≤|c|M¯j(2)1−M¯j(1)/ε.$\left|\partial_b\log I_j(b)\right| \le \frac{|c|\int t^2e^{-\eta S_{j,b}}dt} {\int e^{-\eta S_{j,b}}dt\,(1-\overline M^{(1)}_j/\varepsilon)} \le \frac{|c|\,\overline M^{(2)}_j}{1-\overline M^{(1)}_j/\varepsilon}$. This is (Q2). Summability is exactly the second assumption. □

Theorem 18 (Non-factorized quadratic-mixing large-NN extension). *Let SN(u,v)=Pm(u)+12v⊤(DN(u)+KN)v,*$S_N(u,v) = P_m(u)+\frac12\,v^\top\!\big(D_N(u)+K_N\big)v$*, where u∈ℝm*$u\in\mathbb{R}^m$*, v∈ℝN−m*$v\in\mathbb{R}^{N-m}$*, DN(u)=diag(dm+1(u),…,dN(u))*$D_N(u)=\mathrm{diag}(d_{m+1}(u),\dots,d_N(u))$ *with dj(u)=λj+2βj(u),βj(u)=∑i=1maijui2,aij≥0,λj≥λ−>0.*$d_j(u)=\lambda_j+2\beta_j(u),\quad \beta_j(u)=\sum_{i=1}^m a_{ij}u_i^2,\quad a_{ij}\ge0,\ \lambda_j\ge\lambda_->0$*. Assume coercive Pm*$P_m$ *and Aj:=∑i=1maij,∑j=m+1∞Ajλj<∞.*$A_j:=\sum_{i=1}^m a_{ij},\qquad \sum_{j=m+1}^\infty \frac{A_j}{\lambda_j}<\infty$*. Let K=(κjk)j,k>m*$K=(\kappa_{jk})_{j,k>m}$ *be real symmetric, KN*K_N *its principal truncation, and with Λ=diag(λj)*$\Lambda=\mathrm{diag}(\lambda_j)$ *define K̃:=Λ−1/2KΛ−1/2.*$\widetilde K:=\Lambda^{-1/2} K\Lambda^{-1/2}$*. Assume ∥K̃∥<θ<1,∥K̃∥1<∞,*$\|\widetilde K\|<\theta<1,\qquad$

$\|\widetilde K\|_1<\infty$, and $\tau_N:=\|\widetilde K-P_N\widetilde K P_N\|_1\to0$ (P_N: projection onto indices $m+1,\dots,N$).

Then for bounded cylinder F_m,

$$\omega_{\varepsilon,\eta,N}(F_m)= \frac{\int e^{-cS_N}F_m(u)\,du\,dv}{\int e^{-cS_N}\,du\,dv}, \qquad c=\eta-i/\varepsilon,\ \eta>0,$$

converges as $N\to\infty$ (if the limiting denominator is nonzero), and there exists $C_{F_m,\varepsilon,\eta}>0$ such that

$$|\omega_{\varepsilon,\eta,N'}(F_m)-\omega_{\varepsilon,\eta,N}(F_m)| \le C_{F_m,\varepsilon,\eta}\left(\sum_{j=N+1}^{N'}\frac{A_j}{\lambda_j} \;+\;\tau_N \right), \quad N'>N\ge m.$$

Proof. Integrate in v:

$$\int_{\mathbb{R}^{N-m}}e^{-c\frac12 v^\top(D_N+K_N)v}dv = C_N(c) \det(D_N+K_N)^{-1/2},$$

with u-independent $C_N(c)$ that cancels in normalized ratios. So

$$\omega_{\varepsilon,\eta,N}(F_m) = \frac{\int_{\mathbb{R}^m}e^{-cP_m(u)}F_m(u)\Phi_N(u)\,du} {\int_{\mathbb{R}^m}e^{-cP_m(u)}\Phi_N(u)\,du}, \quad \Phi_N(u):=\det(D_N(u)+K_N)^{-1/2}.$$

Write $\Phi_N(u)=\Phi_N^{\mathrm{diag}}(u)\,\Delta_N(u)$,

$$\Phi_N^{\mathrm{diag}}(u):=\prod_{j=m+1}^N d_j(u)^{-1/2}, \qquad \Delta_N(u):=\det(I+M_N(u))^{-1/2},$$

$$M_N(u):=D_N(u)^{-1/2}K_ND_N(u)^{-1/2}.$$

By Theorem 15, Φ_N^{diag} has Cauchy tail control by

$$\sigma_N:=\sum_{j=N+1}^\infty A_j/\lambda_j.$$

Since $D_N(u)\ge \Lambda_N$, we have

$$\|M_N(u)\|\le\|\widetilde K\|<\theta, \qquad \|M_N(u)\|_1\le\|\widetilde K\|_1=:K_1.$$

Thus $I+M_N(u)$ is invertible and

$$|\log\det(I+M_N(u))| \le \frac{1}{1-\theta}\|M_N(u)\|_1 \le \frac{K_1}{1-\theta},$$

hence $|\Delta_N(u)|\le C_\Delta$ uniformly.

Now set

$$Q(u):=\mathrm{diag}\!\left((\lambda_j/d_j(u))^{1/2}\right),\qquad 0<Q(u)\le I.$$

Then $M_\infty(u)=Q(u)\widetilde KQ(u)$ and $M_N(u)=P_NM_\infty(u)P_N$, so

$$\|M_\infty(u)-M_N(u)\|_1 \le \|\widetilde K-P_N\widetilde K P_N\|_1 =\tau_N.$$

On $\|A\|,\|B\|\le\theta<1$,

$$|\log\det(I+A)-\log\det(I+B)| \le \frac{1}{1-\theta}\|A-B\|_1,$$

thus $|\Delta_\infty(u)-\Delta_N(u)|\le C_\Delta'\tau_N$ uniformly in u.

Therefore, for $N'>N$,

$$\Phi_{N'}-\Phi_N = \Delta_{N'}(\Phi_{N'}^{\mathrm{diag}}-\Phi_N^{\mathrm{diag}}) +\Phi_N^{\mathrm{diag}}(\Delta_{N'}-\Delta_N),$$

and

$$|\Phi_{N'}(u)-\Phi_N(u)| \le C_1e^{B\|u\|^2}\big(\sigma_N+\tau_N\big),$$

for constants C_1,B independent of N,u. Multiplying by $|e^{-cP_m(u)}|=e^{-\eta P_m(u)}$ gives an integrable envelope by quartic coercivity. Dominated convergence plus the standard ratio-difference estimate yields convergence and the stated mixed tail rate. □

Partition-Factor Non-Vanishing Bounds

Theorem 19 (Moment criteria). *Let* $A_\eta=\int e^{-\eta S(x)}dx\in(0,\infty)$ *and* $Z_{\varepsilon,\eta}:=\int e^{-(\eta-i/\varepsilon)S(x)}dx = A_\eta\,\mathbb E_{\mu_\eta}[e^{iS/\varepsilon}], \quad \mu_\eta(dx):=\frac{e^{-\eta S(x)}}{A_\eta}\,dx$. *Define* $M_1:=\mathbb E_{\mu_\eta}|S|,\qquad M_2:=\mathbb E_{\mu_\eta}(S^2)$. *Then* $|Z_{\varepsilon,\eta}|\ge A_\eta\left(1-\frac{M_1}{\varepsilon}\right)$, $|Z_{\varepsilon,\eta}|\ge A_\eta\left(1-\frac{M_2}{2\varepsilon^2}\right)$. *Hence if* $\varepsilon>M_1$ *or* $\varepsilon^2>M_2/2$, *then* $Z_{\varepsilon,\eta}\neq0$.

Proof. First bound: $\left|\mathbb E[e^{iX}]\right| =|1+\mathbb E(e^{iX}-1)| \ge1-\mathbb E|e^{iX}-1|, \quad X=S/\varepsilon$. Since $|e^{it}-1|\le |t|$, $\left|\mathbb E[e^{iS/\varepsilon}]\right| \ge1-\frac{M_1}{\varepsilon}$. Multiply by A_η.

Second bound: $\Re\mathbb E[e^{iS/\varepsilon}] =\mathbb E[\cos(S/\varepsilon)] \ge1-\frac{1}{2}\mathbb E[(S/\varepsilon)^2] =1-\frac{M_2}{2\varepsilon^2}$. Now $|z|\ge \Re z$ gives the inequality for $|Z_{\varepsilon,\eta}|$. □

Observable-Class Extension

Theorem 20 (Continuity on Schwartz and weighted Sobolev classes). *Let* $\mathcal I(F)=\int_{\mathbb{R}^d}e^{i\Phi(y)}W(y)F(Ay)\,dy$, *with* $A\in GL(d,\mathbb{C})$ *and* $|W(y)|\le C_0 e^{-c_4\|y\|^4+c_2\|y\|^2},\qquad c_4>0$. *Then:*

1. *for every integer* $k>d$, *there exists* C_k *such that* $|\mathcal I(F)|\le C_k \sup_x (1+\|x\|)^k|F(x)|,\qquad F\in\mathcal S(\mathbb{R}^d)$;

2. *for every* $k>d/2$, *there exists* C_k' *such that* $|\mathcal I(F)|\le C_k' \|(1+\|x\|^2)^{k/2}F\|_{L^2},\qquad F\in H^{0,k}$.

Consequently, normalized functionals $\omega(F)=\mathcal I(F)/\mathcal I(1)$ *(when* $\mathcal I(1)\neq0$*) extend continuously from Gaussian-polynomial test families to both classes.*

Proof. For Schwartz: $|F(Ay)|\le C_A p_k(F)(1+\|y\|)^{-k}, \quad p_k(F):=\sup_x(1+\|x\|)^k|F(x)|$. Hence $|\mathcal I(F)| \le C_0C_A p_k(F)\int e^{-c_4\|y\|^4+c_2\|y\|^2}(1+\|y\|)^{-k}dy$, and the integral is finite.

For weighted Sobolev: $|\mathcal I(F)| \le \|W(\cdot)(1+\|\cdot\|^2)^{-k/2}\|_{L^2} \cdot \|(1+\|y\|^2)^{k/2} F(Ay)\|_{L^2_y}$. The first factor is finite by quartic decay; the second is bounded by $C_A'\|F\|_{H^{0,k}}$ after linear change of variables. □

Schwinger-Dyson and τ_μ Scale Covariance

Theorem 21 (Finite-dimensional Schwinger-Dyson identity). *Let $c=\eta-i/\varepsilon$ and $\mathcal I_c(F):=\int e^{-cS(x)}F(x)\,dx.$ Assume integrability and vanishing boundary flux for admissible F and vector field V. Then $\mathcal I_c(V\cdot\nabla S\,F) = \frac{1}{c}\mathcal I_c(\nabla\!\cdot(VF)).$ If $\mathcal I_c(1)\neq0$, then $\omega_c(V\cdot\nabla S\,F) = \frac{1}{c}\omega_c(\nabla\!\cdot(VF)).$ In particular, for constant $V=e_i$ and $F\equiv1$: $\omega_c(\partial_i S)=0.$*

Proof. $0=\int \nabla\!\cdot(e^{-cS}VF)\,dx =\int e^{-cS}(\nabla\!\cdot(VF)-c\,V\cdot\nabla S\,F)\,dx.$ Rearrange, then divide by $\mathcal I_c(1)$ for the normalized form. □

Theorem 22 (Exact τ_μ covariance). *For $\omega_{\kappa,\eta,h}(F):=\frac{\int e^{-(\eta-i/h)\kappa S(x)}F(x)\,dx}{\int e^{-(\eta-i/h)\kappa S(x)}\,dx},$ define $\tau_\mu:(\kappa,\eta,h)\mapsto(\mu\kappa,\eta/\mu,\mu h),\qquad \mu>0.$ Then $\omega_{\kappa,\eta,h}(F)=\omega_{\tau_\mu(\kappa,\eta,h)}(F).$*

Proof. Directly, $\left(\frac{\eta}{\mu}-\frac{i}{\mu h}\right)(\mu\kappa) =(\eta-i/h)\kappa.$ Hence numerator and denominator kernels are unchanged. □

Dependency Chain

The proofs in this manuscript now close in the following order:

1. Sections 2–5: core scoped Claim 1 closure.
2. Theorems 15, 18, 16, and Corollary 17: large-N coupled extensions (Gaussian-tail rate, non-factorized quadratic-mixing determinant class, non-factorized quartic-tail class, and intrinsic moment-based sufficient conditions).
3. Theorem 19: explicit non-vanishing criteria for partition factors.
4. Theorem 20: observable-class extension to Schwartz/Sobolev.
5. Theorems 21 and 22: Schwinger-Dyson identities and exact scale-flow covariance.

Conclusion

This manuscript now gives a theorem-grade chain from the static/finite-dimensional variational-delta picture to a broad scoped family of oscillatory states including:

1. exact projective closure and de-regularization,
2. explicit semiclassical channel expansion,
3. large-NN mode-coupled convergence with quantitative tail control (Gaussian-tail, non-factorized quadratic-mixing, and non-factorized quartic-tail classes), with intrinsic moment criteria for quartic-tail hypotheses,
4. non-vanishing control for normalization factors,
5. extension to large observable classes,
6. Schwinger-Dyson identities and scale-flow covariance.

Open frontier (outside this closure): full interacting, non-factorized, genuinely field-theoretic continuum limits with uniform renormalization control beyond the scoped polynomial block-tail class.

99

R. P. Feynman, *Space-Time Approach to Non-Relativistic Quantum Mechanics*, Rev. Mod. Phys. **20** (1948), 367–387. DOI: 10.1103/RevModPhys.20.367.

L. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, Springer, 2nd ed., 2003. DOI: 10.1007/978-3-642-61497-2.

I. M. Gel'fand and G. E. Shilov, *Generalized Functions, Vol. 1: Properties and Operations*, AMS Chelsea, 1964. DOI: 10.1090/chel/377.

S. Albeverio, R. J. Høegh-Krohn, and S. Mazzucchi, *Mathematical Theory of Feynman Path Integrals: An Introduction*, Lecture Notes in Mathematics 523, 2nd ed., Springer, 2008. DOI: 10.1007/978-3-540-76956-9.

A. Connes, *Noncommutative Geometry*, Academic Press, 1994. ISBN: 978-0-12-185860-5.

N. P. Landsman, *Mathematical Topics Between Classical and Quantum Mechanics*, Springer Monographs in Mathematics, Springer, 1998. DOI: 10.1007/978-1-4612-1680-3.

N. P. Landsman, *Lie Groupoid C*-Algebras and Weyl Quantization*, Commun. Math. Phys. **206** (1999), 367–381. DOI: 10.1007/s002200050709.

N. P. Landsman and B. Ramazan, *Quantization of Poisson algebras associated to Lie algebroids*, in *Groupoids in Analysis, Geometry, and Physics*, Contemporary Mathematics **282**, AMS, 2001, 159–192. DOI: 10.1090/conm/282.