Action–Angle Indeterminacy in Central Potentials: A Referee-Safe Witness

Alejandro Rivero

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Abstract

"Action–angle indeterminacy" should not be read as a force-range heuristic (in the style of energy–time slogans), but as a clean conjugacy statement: sharpening an action variable broadens the conjugate angle variable. For central potentials the safest, most explicit instance is the azimuthal pair (ϕ, L_z): an L_z eigenstate has ϕ-dependence $e^{im\phi}$, hence a uniform azimuthal probability distribution; conversely, any state localized in ϕ must involve a broad superposition of angular-momentum modes (Fourier on the circle). This note records that witness and explains its foundations-level message: classical orbit-phase/orientation pictures correspond to semiclassical packets/superpositions rather than single stationary eigenstates.

1. Purpose and scope

This dependent note isolates one specific "action–angle indeterminacy" statement that is both explicit and referee-safe in a central potential: **ϕ is delocalized in an L_z eigenstate**, and conversely **localizing ϕ requires a superposition over many m** modes.

We deliberately keep the scope bounded. We do **not** enter the self-adjoint "angle operator" debate; instead we use the standard circle/Fourier structure and the unitary phase variable $e^{i\phi}$. We also do **not** make any claims about the range of forces or potentials; the point here is about **which variables can be simultaneously sharp** in stationary states.

2. The safe conjugate pair on the circle: ϕ and L_z

In spherical coordinates the azimuthal angle is periodic, $\phi \sim \phi + 2\pi$. The generator of rotations about the z-axis is

$$L_z = -i\hbar \frac{\partial}{\partial \phi}.$$

The periodicity makes the naive commutator $[\phi, L_z] = i\hbar$ subtle if one insists on an everywhere-defined self-adjoint ϕ operator. A standard way to stay on safe ground is to use the unitary "phase" variable

$$E := e^{i\phi}.$$

Acting on 2π-periodic wavefunctions, E is well-defined and satisfies the canonical shift relation

$$[L_z, E] = \hbar\, E,$$

which already captures the operational content: sharp L_z implies maximal delocalization in the conjugate angle.

3. Central potentials: L_z eigenstates have uniform ϕ distribution

For a central potential (or any Hamiltonian commuting with L_z), one may choose simultaneous eigenstates of L_z. In the standard separation of variables, the azimuthal dependence of an angular-momentum eigenstate is the Fourier mode $e^{im\phi}$ with integer m (for example in the spherical-harmonic factor $Y_{\ell m}(\theta, \phi) \propto P_{\ell m}(\cos\theta)e^{im\phi}$) [TongQMLectures].

Thus an L_z eigenstate may be written as

$$\psi(r, \theta, \phi) = F(r, \theta)\, e^{im\phi}, \qquad m \in \mathbb{Z},$$

and therefore

$$|\psi(r, \theta, \phi)|^2 = |F(r, \theta)|^2,$$

independent of ϕ. In particular, the marginal distribution of ϕ is uniform on $[0, 2\pi)$. This is the minimal "angle indeterminacy" witness for central potentials.

4. Fourier tradeoff: localizing ϕ forces a broad m-superposition

Any square-integrable 2π-periodic function admits a Fourier series

$$\psi(\phi) = \sum_{m\in\mathbb{Z}} c_m e^{im\phi}, \qquad \sum_{m\in\mathbb{Z}} |c_m|^2 < \infty.$$

If only one Fourier mode is present (sharp m, hence sharp L_z), then $|\psi(\phi)|^2$ is constant; conversely, a state that is peaked in ϕ necessarily uses many Fourier modes (broad m-support).

`Example 4.1 (Dirichlet-kernel packet).` The normalized superposition of modes $-M \le m \le M$,

$$\psi_M(\phi) = \frac{1}{\sqrt{2\pi(2M+1)}} \sum_{m=-M}^{M} e^{im\phi},$$

is peaked near $\phi = 0$ with an angular width that scales like $1/M$, while its m-distribution is spread across $\{-M, \ldots, M\}$. This makes the "sharpening ϕ $\Rightarrow$ broadening L_z" tradeoff completely explicit without invoking any disputed angle-operator formalism.

The Fourier tradeoff above can be made into a sharp quantitative bound using only the self-adjoint observables $\cos\phi$ and $\sin\phi$:

`Proposition 4.2 (Circular uncertainty relation).` For any state on the circle, define the circular concentration $R = |\langle e^{i\phi}\rangle| \in [0, 1]$. Adding the Robertson inequalities for the two self-adjoint pairs $(L_z, \cos\phi)$ and $(L_z, \sin\phi)$ — using $[L_z, \cos\phi] = i\hbar \sin\phi$ and $[L_z, \sin\phi] = -i\hbar \cos\phi$ — and the identity $\mathrm{Var}(\cos\phi) + \mathrm{Var}(\sin\phi) = 1 - R^2$, gives

$$\mathrm{Var}(L_z) \cdot (1 - R^2) \geq \frac{\hbar^2}{4} R^2.$$

When $R = 0$ (uniform distribution, as in an L_z eigenstate) the bound is trivial. As $R \to 1$ (sharply localized angle) the bound forces $\mathrm{Var}(L_z) \to \infty$: angular localization requires spreading across many m-modes. This quantifies the Fourier tradeoff above without invoking a self-adjoint angle operator.

5. Foundations message: orbit pictures require packets/superpositions

This witness supports a simple interpretive guardrail for central-force intuition: a single stationary eigenstate (even when it carries classical-sounding quantum numbers) is typically **not** a localized classical orbit with a definite phase/orientation. Variables like the azimuthal phase ϕ (and, in more structured integrable cases, other angle variables on the invariant torus) become localized only in **coherent superpositions** of many stationary modes.

In other words, "classical orbit pictures" correspond to semiclassical packets and stationary-phase concentration, not to exact eigenstates that are sharp in the conserved actions.

6. A second witness: the harmonic oscillator

The same structure appears in the simplest one-dimensional integrable system.

`Example 6.1 (Harmonic oscillator: Fock states vs coherent states).` For a harmonic oscillator of frequency ω, define the classical action variable $J = E/\omega$. The quantum Fock states $|n\rangle$ are the action eigenstates ($J_n = (n + \frac{1}{2})\hbar$), and their phase-space (Husimi) distribution is a ring centered at the origin — the orbit phase θ is uniformly delocalized, exactly as ϕ is delocalized in an L_z eigenstate. Conversely, a coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \qquad \alpha = |\alpha|\, e^{i\theta_0},$$

is the closest quantum analog of a classical orbit with definite amplitude $|\alpha|$ and phase θ_0. Its Fock-state weights follow a Poisson distribution with mean $\bar{n} = |\alpha|^2$, so localizing the phase to width $\Delta\theta \sim 1/|\alpha|$ requires spreading the action over $\Delta n \sim |\alpha|$ modes. The tradeoff is the same as in Section 4: sharp action implies delocalized phase, and vice versa.

7. Outlook (kept minimal)

Beyond the (ϕ, L_z) sector and the harmonic oscillator above, integrable central problems admit a fuller action–angle description (with a radial action and additional angle variables on the invariant torus). EBK/WKB quantization makes the same structural point: the more sharply the actions are specified, the less information remains in the conjugate phases. Hardening that broader story into a standalone foundations claim would require a separate study cycle to avoid conflating (i) action–angle existence/global issues with (ii) semiclassical quantization conditions.

References

1. [TongQMLectures] David Tong, "Quantum Mechanics" (lecture notes, no DOI). OA: lecture-note PDF. (Contains $Y_{l,m}(\theta, \phi) = P_{l,m}(\cos\theta)e^{im\phi}$ as simultaneous eigenstates of L^2 and L_z.)