

# Planck Area from Half-Density Normalization (Draft)

## Abstract

Half-densities are the natural “coordinate-free integrands” for composing kernels without choosing a background measure. But choosing a *universal* convention for turning half-density objects into dimensionless numerical amplitudes introduces a length $^{d/2}$  scale. In  $d = 4$ , this is an *area*. This note sharpens the hypothesis ladder needed for the claim “half-density normalization selects a universal area scale”, and isolates a simple dimension-matching condition under which the Planck area appears without fractional powers of couplings. A gravitational anchor based on a minimal-areal-speed principle is recorded as a separate heuristic thread [RiveroAreal] [RiveroSimple].

## 1. Purpose and Status

This is a dependent follow-up to `paper/main.md`. It is not yet a finished paper; its goal is to isolate one technical point that is only implicit in the main manuscript: the role of half-densities (and their scaling) in making composition laws coordinate-invariant *and* dimensionally well-defined.

Claims below are labeled as **Proposition** (math-precise under hypotheses) or **Heuristic** (programmatic bridge).

## 2. Half-Densities and Composition Kernels

Let  $M$  be a  $d$ -dimensional manifold. A (positive) density is a section of  $|\Lambda^d T^* M|$ , and a half-density is a section of  $|\Lambda^d T^* M|^{1/2}$ .

The key operational point is: when a kernel is a half-density in its integration variable, composition of kernels does not depend on an arbitrary choice of coordinate measure.

**Heuristic H1.1 (Why half-densities).** If  $K_1(x, y)$  and  $K_2(y, z)$  are chosen so that their product in the intermediate variable  $y$  is a density, then  $\int_M K_1(x, y)K_2(y, z)$  is coordinate-invariant without fixing a preferred  $dy$ . This matches the structural role of kernel composition used in `paper/main.md` (Section 6).

**Derivation D1.1 (Coordinate invariance of half-density pairing and composition).** In a local chart  $y = (y^1, \dots, y^d)$ , write a half-density as  $\psi(y) = \varphi(y)|dy|^{1/2}$ . Under a change of variables  $y = y(y')$ , one has  $|dy|^{1/2} = |\det(\partial y / \partial y')|^{1/2}|dy'|^{1/2}$ , so the coefficient transforms as  $\varphi'(y') = \varphi(y(y'))|\det(\partial y / \partial y')|^{1/2}$ .

Hence the product of two half-densities is a density:  $\psi_1\psi_2 = (\varphi_1\varphi_2)|dy|$ , and its integral is chart-independent:  $\int_M \psi_1\psi_2$  is well-defined without choosing a

background measure beyond the density bundle itself.

Kernel composition is the same mechanism: if  $K_1(x, y)$  and  $K_2(y, z)$  are half-densities in  $y$ , then  $K_1 K_2$  is a density in  $y$  and  $\int_M K_1 K_2$  is coordinate invariant.

### 3. Dimensional Analysis: Normalizing a Half-Density Requires a Scale

A density on  $M$  carries the units of length $^d$  once physical units are assigned to coordinates. A half-density therefore carries units length $^{d/2}$ .

**Proposition P1.1** (No canonical "half-density = function" identification). There is no canonical identification of a half-density  $\psi \in |\Lambda^d T^* M|^{1/2}$  with an ordinary scalar function  $f$  on  $M$ . Choosing such an identification is equivalent to choosing a nowhere-vanishing reference half-density  $\sigma_*$  (equivalently a positive density  $\rho_* = \sigma_*^2$ ) and writing  $\psi = f \sigma_*$ .

**Derivation D1.2** (Dilation makes the  $\text{length}^{d/2}$  weight explicit). On  $\mathbb{R}^d$ , consider a dilation  $y \mapsto y' = ay$  with  $a > 0$ . Then  $|dy'| = a^d |dy|$ , so  $|dy'|^{1/2} = a^{d/2} |dy|^{1/2}$ . Thus even in flat space, half-densities carry an inherent length $^{d/2}$  scaling weight.

**Derivation D1.2a** (Near-diagonal scaling forces the square-root Jacobian  $\varepsilon^{-d/2}$ ). On  $M = \mathbb{R}^d$ , introduce near-diagonal coordinates  $y = x + \varepsilon v$  with  $\varepsilon > 0$ . Then  $dy = \varepsilon^d dv$ , hence  $|dy|^{1/2} = \varepsilon^{d/2} |dv|^{1/2}$ . For a bi-half-density kernel written locally as

$$K_\varepsilon(x, y) = k_\varepsilon(x, y) |dx|^{1/2} |dy|^{1/2},$$

its pullback to  $(x, v)$  variables becomes

$$K_\varepsilon(x, x + \varepsilon v) = (\varepsilon^{d/2} k_\varepsilon(x, x + \varepsilon v)) |dx|^{1/2} |dv|^{1/2}.$$

Thus, any attempt to define a nontrivial " $\varepsilon \rightarrow 0$ " near-diagonal limit of kernels (the scaling step that tangent-groupoid quantization packages) inevitably produces an  $\varepsilon^{d/2}$  factor from the half-density Jacobian, and the corresponding scalar representative must be renormalized by  $\varepsilon^{-d/2}$  to stay finite. This is the same exponent as in the finite-dimensional "square-root delta" normalization: the half-density is the square root of the density Jacobian.

**Proposition P1.2** (Universal \*dimensionless\* amplitudes force a  $\text{length}^{d/2}$  constant). If one imposes the extra requirement that the scalar representative  $f$  in  $\psi = f \sigma_*$  be dimensionless in physical units, then the reference half-density  $\sigma_*$  must carry all of the length $^{d/2}$  dimension. In particular, a *constant* (field-independent) choice of  $\sigma_*$  is equivalent to choosing a universal length $^{d/2}$  scale.

In  $d = 4$ , this universal length $^{d/2}$  scale is a universal *area* scale.

**Heuristic H1.2 (Reciprocity claim).** Half-densities alone do not force a particular scale: the forced fact is that converting half-density objects into scalar

numerical amplitudes requires extra structure (a reference half-density). The “universal area scale” claim begins only after adding two further hypotheses: 1. the reference  $\sigma_*$  is taken to be *constant* (no dependence on background metric/fields), and 2. the constant is required to be fixed by universal constants/couplings of the theory.

Under these hypotheses,  $d = 4$  is the unique dimension in which the needed length $^{d/2}$  constant can be supplied by the gravitational coupling without fractional powers (Derivation D1.3).

**Derivation D1.3 (Gravity-only sieve: why  $\backslash(d=4\backslash)$  is singled out if only  $\backslash(G_d\backslash)$  is used).** In  $d$  spacetime dimensions, the Einstein–Hilbert action  $\frac{1}{16\pi G_d} \int d^d x \sqrt{|g|} R$  shows that (in  $c = \hbar = 1$  units) Newton’s constant has dimension  $[G_d] = \text{length}^{d-2}$ . Assume the only available dimensionful coupling used to build the universal normalization constant is  $G_d$  itself (no cosmological constant, no additional dimensionful scales), and impose H2.5 in the literal “no fractional powers of  $G_d$ ” sense. Then the normalization constant has dimension length $^{k(d-2)}$  for some integer  $k$ . Matching length $^{d/2}$  forces length $^{d/2} = \text{length}^{d-2}$ , which holds if and only if  $d = 4$ . In that case  $G_4$  itself has dimension of area, and the corresponding area scale is the Planck area  $L_P^2 \sim \hbar G_4/c^3$ .

### 3.1 Hypotheses as Separate Knobs (What Is Forced vs Chosen)

The discussion above mixes three different kinds of statements: 1. **Geometric facts** (what half-densities are, how they compose, how they scale), 2. **Representational choices** (how one turns half-density objects into scalar numbers), 3. **Universality/selection principles** (what choices are allowed if we demand “background-free” and “built from couplings”).

To study these separately, it is useful to keep the hypotheses explicit.

**Hypothesis H2.1 (Half-density formulation).** Quantum kernels are treated as bi-half-densities so that composition in intermediate variables is coordinate invariant (Section 2 and Derivation D1.4).

**Hypothesis H2.2 (Scalarization by a reference half-density).** To interpret half-density amplitudes as scalar numerical functions, we pick a nowhere-vanishing reference half-density  $\sigma_*$  and write  $\psi = f \sigma_*$  (Proposition P1.1).

**Hypothesis H2.3 (Dimensionless scalar representative).** The scalar representative  $f$  is required to be dimensionless in physical units (Proposition P1.2). This forces  $\sigma_*$  to carry the full length $^{d/2}$  weight.

**Hypothesis H2.4 (Background-free constancy).** The reference  $\sigma_*$  is taken to be constant/field-independent, rather than determined by background geometry (e.g. a Riemannian volume  $|g|^{1/4} |dx|^{1/2}$ ) or by dynamical fields (e.g. a dilaton-like factor). This is the first point where a *universal constant* enters.

**Hypothesis H2.5 (Analyticity / no fractional powers of couplings).** If the universal constant is required to be built from the theory’s couplings without fractional powers, then dimensional analysis becomes a *dimension sieve* rather than a tautology. This hypothesis has at least two distinct readings:

1. **Integrality (Diophantine) reading:** the constant is a monomial in the available couplings with integer exponents (possibly allowing negative powers).
2. **Perturbative analyticity reading (stronger):** the constant admits a Taylor expansion around zero couplings, so only nonnegative integer powers appear.

Derivation D1.3 is the simplest gravity-only instance under the integrality reading: “use  $G_d$  without fractional powers” singles out  $d = 4$ .

**Heuristic H2.6 (Where “special dimensions” can appear).** Special dimensions do not come from half-densities alone (Hypothesis H2.1). They appear only after adding a selection principle like H2.4–H2.5: the requirement that the scalarization choice be universal, background-free, and coupling-built in a restricted (e.g. analytic) way.

### 3.2 What Changes When a Hypothesis Is Relaxed?

This subsection records the main “branches” that need separate study.

1. **Drop H2.3 (allow dimensionful  $f$ ).** Then no universal length $^{d/2}$  constant is forced; the dimensional weight can be carried by the scalar representative itself (as in the usual statement “wavefunctions have dimension length $^{-d/2}$ ”).
2. **Drop H2.4 (allow background geometry).** Then  $\sigma_*$  can be chosen from a metric (or other structure), and the “universal constant” is replaced by background-dependent normalization.
3. **Drop H2.5 (allow fractional powers).** Then in any  $d > 2$  one can build a length $^{d/2}$  constant from gravity via  $G_d^{d/(2(d-2))}$  (in  $c = \hbar = 1$  units), so  $d = 4$  is no longer singled out; instead,  $d = 4$  is simply the unique case where the exponent is an integer.
4. **Change “which coupling supplies the scale”.** Using other dimensionful couplings (cosmological constant, string tension, gauge couplings in various dimensions, etc.) yields different “special-dimension” sieves. This is conceptually aligned with the observation that some dimensions are singled out by other structures (division algebras, special holonomy, supersymmetry), but those filters are separate from the half-density story and should not be conflated.

### 3.3 Starting with H2.5: Integrality as a Dimension Sieve

The point of H2.5 is not that dimensional analysis alone selects a unique scale (it does not), but that *restricting allowed functional dependence on couplings* can turn dimensional analysis into a selection principle.

**Derivation D1.6 (Diophantine form of "no fractional powers").**

Work in  $c = \hbar = 1$  units for dimension counting. Let the available couplings  $\{g_i\}$  have length dimensions  $[g_i] = \text{length}^{a_i}$ . Under the integrality reading of H2.5, the universal normalization constant is a monomial  $C = \prod_i g_i^{n_i}$  with integers  $n_i$ . Its length dimension is  $[C] = \text{length}^{\sum_i n_i a_i}$ . Requiring  $[C] = \text{length}^{d/2}$  is therefore the Diophantine condition

$$\sum_i n_i a_i = \frac{d}{2}.$$

Existence (and non-uniqueness) of solutions depends on: 1. which couplings are admitted as “universal” inputs, and 2. whether one allows negative exponents (non-analytic at zero coupling) or insists on perturbative analyticity (nonnegative exponents).

**Heuristic H2.7 (Why H2.5 needs a “what counts as a coupling” rule).** If one allows arbitrary redefinitions of couplings (e.g. adjoining a new symbol  $\tilde{G} = G_d^{1/(d-2)}$ ), then “no fractional powers” becomes vacuous: the forbidden root has simply been renamed as an allowed coupling. H2.5 is meaningful only together with a prior criterion for admissible coupling dependence (e.g. perturbative analyticity around a distinguished limit such as  $G_d \rightarrow 0$ ).

**Example E1 (Gravity-only).** With only  $G_d$  available,  $a_1 = d - 2$  and the condition becomes  $n(d - 2) = d/2$ . For integer  $d \geq 3$ , this has a solution only at  $d = 4$  with  $n = 1$ , reproducing Derivation D1.3.

**Example E2 (Gravity + cosmological constant).** If one also allows the cosmological constant  $\Lambda_d$  with  $[\Lambda_d] = \text{length}^{-2}$ , then the condition becomes  $n(d - 2) - 2m = d/2$  for integers  $n, m$ . A simple family of solutions exists for  $d$  divisible by 4: take  $n = 1$  and  $m = d/4 - 1$ , so

$$C \sim G_d \Lambda_d^{d/4-1},$$

has dimension  $\text{length}^{d/2}$ . Thus, even under H2.5,  $d = 4$  is not automatically unique once additional dimensional couplings are admitted; what is special about  $d = 4$  in this family is that it is the only case with  $m = 0$  (no need to involve  $\Lambda_d$ ).

**Example E3 (Yang--Mills coupling as an alternative sieve).** In  $d$  spacetime dimensions, the Yang--Mills action is typically written as  $\frac{1}{4g_d^2} \int d^d x F_{\mu\nu} F^{\mu\nu}$ , so  $[g_d^2] = \text{length}^{d-4}$  (equivalently  $[g_d] = \text{length}^{(d-4)/2}$ ). If we (hypothetically) allow the half-density normalization constant to be a pure monomial in  $g_d$ ,  $C \sim g_d^p$  with integer  $p \geq 0$ , then

$$[C] = \text{length}^{p(d-4)/2}.$$

Matching  $[C] = \text{length}^{d/2}$  gives the Diophantine condition

$$p(d-4) = d \implies d = \frac{4p}{p-1} = 4 + \frac{4}{p-1}.$$

Thus integer solutions occur only when  $p-1 \mid 4$ , i.e.  $p \in \{2, 3, 5\}$ , giving  $d \in \{8, 6, 5\}$  respectively.

In particular, in  $d=4$  the gauge coupling is dimensionless and cannot by itself supply the  $\text{length}^{d/2}$  factor needed for half-density scalarization; in that case the scale must come from another dimensionful coupling (e.g. gravity) or from a non-analytic mechanism (dimensional transmutation).

**Example E4 (String tension /  $\backslash(\backslash\alpha'\backslash)$  as a source of a universal area scale).** In perturbative string theory a fundamental length scale  $l_s$  is built in from the start; equivalently one has a parameter  $\alpha' = l_s^2$  with dimension  $\text{length}^2$  (string tension  $T \sim 1/\alpha'$ ). If one allows the half-density normalization constant to be built from  $\alpha'$  alone as a monomial  $C \sim (\alpha')^n$  with integer  $n$ , then

$$[C] = \text{length}^{2n},$$

and matching  $[C] = \text{length}^{d/2}$  forces  $2n = d/2$ , i.e.  $d = 4n$ . So  $\alpha'$  provides a “background-free” source of scale but does not single out  $d=4$  on its own; it selects dimensions divisible by 4 under the strict integrality reading of H2.5. In  $d=4$  it yields directly an area scale  $C \sim \alpha'$ .

**Heuristic H2.12 (Link to gravity/Planck length in a UV completion).** The gravity-only sieve (Derivation D1.3) uses  $G_d$  as the unique universal coupling supplying dimension. In a stringy UV completion,  $G_d$  is not independent: it is generated by the string sector and depends on  $\alpha'$  and a dimensionless coupling (e.g. the string coupling  $g_s$ ), and in compactified settings also on compactification volumes. At the level of scaling, one expects

$$G_d \propto g_s^2 (\alpha')^{(d-2)/2} \times (\text{volume factors}),$$

so the Planck length/area is *derived* from  $\alpha'$  and  $g_s$  rather than fundamental. In that framing the half-density “universal area scale” could naturally be  $\alpha'$  (or a simple function of it), while the Planck area is recovered as a consequence of how gravity emerges.

**Heuristic H2.8 (What H2.5 is really buying).** The value of H2.5 is comparative: it distinguishes dimensions in which the needed  $\text{length}^{d/2}$  factor can be supplied by *simple* coupling dependence (integer powers of the already-present couplings), versus dimensions in which any such factor requires either (i) introducing extra scales/couplings, (ii) taking fractional powers, or (iii) invoking non-analytic mechanisms (dimensional transmutation).

**Heuristic H2.13 (Dimensional transmutation as a scale-supplier).** If one relaxes H2.5’s “analytic monomial in couplings” requirement, then even a theory with only *dimensionless* couplings can generate a physical length

scale through RG invariance: a running coupling  $g(\mu)$  can be traded for an RG-invariant scale  $\kappa_*$  (or  $\Lambda$ ), typically of the form  $\kappa_* \sim \mu \exp(-\text{const}/g(\mu)^2)$  or  $\kappa_* \sim \mu \exp(-\text{const}/g(\mu))$ . In that branch, the half-density scalarization scale required by H2.4 can be supplied by  $\kappa_*^{-d/2}$  (or its square in  $d = 4$  as an area scale), but the scale is no longer an analytic monomial in the couplings: it is emergent and non-perturbative in the naive coupling expansion.

**Example E5 (2D delta: transmutation yields a length scale).** In the 2D delta interaction, the contact coupling is marginal and the renormalized theory is naturally parameterized by an RG-invariant inverse length  $\kappa_*$  rather than by the bare coupling. Concretely, one finds (up to conventions) a running coupling  $g_R(\mu)$  with beta function  $\beta(g_R) \propto g_R^2$ , and the RG invariant

$$\kappa_*^2 \equiv \mu^2 \exp\left(\frac{2\pi\hbar^2}{m} \frac{1}{g_R(\mu)}\right),$$

so  $\kappa_*$  is independent of the subtraction scale  $\mu$  and sets a bound-state/scattering scale [ManuelTarrach1994PertRenQM]. This is a minimal witness that “a scale is forced by compatibility” can occur even without a dimensionful coupling, via renormalization rather than via analytic monomials.

### 3.4 Running H2.3: Is “Dimensionless $f$ ” Physics or Convention?

The half-density formalism (H2.1) gives a canonical pairing  $\int \bar{\psi} \psi$  that does not require choosing a background measure. But when we write  $\psi = f \sigma_*$  (H2.2), we are choosing a *representation* of the same object as a scalar function with respect to a chosen positive density  $\rho_* = \sigma_*^2$ .

**Proposition P1.3 (Scalarization is a choice of measure, not new physics).** Choosing a reference half-density  $\sigma_*$  identifies the canonical Hilbert space of  $L^2$  half-densities on  $M$  with the scalar Hilbert space  $L^2(M, \rho_*)$ , where  $\rho_* = \sigma_*^2$ . Different choices  $\sigma_*$  yield unitarily equivalent scalar representations.

**Derivation D1.7 (Change of reference half-density acts by multiplication).** Let  $\sigma_1, \sigma_2$  be nowhere-vanishing half-densities on  $M$  and set  $r := \sigma_2/\sigma_1$ , a positive scalar function. Writing the same half-density state  $\psi$  as  $\psi = f_1 \sigma_1 = f_2 \sigma_2$  gives  $f_2 = r^{-1} f_1$ . Moreover,

$$\int_M \bar{\psi} \psi = \int_M |f_1|^2 \sigma_1^2 = \int_M |f_2|^2 \sigma_2^2 = \int_M |f_2|^2 r^2 \sigma_1^2,$$

so the two scalar pictures differ by a compensating change of measure and pointwise multiplication. In particular, if  $\sigma_2 = c \sigma_1$  is a constant rescaling, then  $f_2 = c^{-1} f_1$  is the familiar global wavefunction normalization freedom.

**Heuristic H2.9 (How H2.3 creates a scale).** In the usual “scalar wavefunction” presentation on  $\mathbb{R}^d$ , one implicitly chooses  $\sigma_* = |dx|^{1/2}$  and allows the scalar representative to carry dimension length $^{-d/2}$  so that  $|\psi|^2 d^d x$  is dimensionless probability. Requiring instead that the scalar representative  $f$  be

dimensionless (H2.3) shifts the length $^{-d/2}$  factor into the reference half-density:

$$\sigma_* \sim L_*^{-d/2} |dx|^{1/2},$$

so “dimensionless  $f$ ” is a convention unless the scale  $L_*^{d/2}$  is fixed by an additional universality principle (H2.4–H2.5).

### 3.5 Running H2.4: What Does “Background-Free Constancy” Mean?

From Derivation D1.7, changing the reference half-density  $\sigma_*$  by a positive function  $r(x)$  changes the scalar representative by  $f \mapsto r^{-1}f$  and changes the scalar measure by  $\rho_* \mapsto r^2\rho_*$ . So the raw half-density formulation has a large “scalarization gauge freedom”.

**Heuristic H2.10 (Constancy = no extra background function).** If we take “background-free” in the strong sense “no additional structure beyond the manifold and the theory’s couplings”, then allowing an arbitrary non-constant  $r(x)$  would amount to introducing a new background field/function by hand. In that strong sense, the only admissible changes are constant rescalings, and choosing  $\sigma_*$  becomes a choice of a single global scale (fixed or not fixed by couplings depending on H2.5).

**Derivation D1.8 (Three natural families of  $\sigma_*$  and what they mean).** On a configuration space  $M$ , the common ways to choose a reference half-density are:

1. **Flat/affine choice (when available):** on  $\mathbb{R}^d$  with its affine structure, translation invariance picks  $|dx|^{1/2}$  uniquely up to a constant factor. This is “constant” in the sense of being homogeneous under translations.
2. **Metric-derived choice:** given a Riemannian/Lorentzian metric  $g$ , one can take  $\sigma_g := |g|^{1/4} |dx|^{1/2}$ , so that  $\rho_g = \sigma_g^2 = \sqrt{|g|} |dx|$  is the familiar invariant volume density. This makes the scalar representative  $f$  a genuine scalar field but makes the scalarization depend on background geometry.
3. **Field-derived (dilaton-like) choice:** given a scalar field  $\Phi$  (background or dynamical), one can take  $\sigma_\Phi := e^{-\Phi} \sigma_g$ . In the scalar picture this is a local rescaling of the measure, and it is the natural way to encode “local units” or Weyl factors.

H2.4 asserts that the theory supplies (or selects) a choice of type (1) with no  $x$ -dependent factor: a fixed reference  $\sigma_*$  whose only remaining ambiguity is an overall constant scale.

**Heuristic H2.11 (RG as scale dependence of scalarization).** If refinement/coarse-graining forces an  $x$ -independent but scale-dependent choice  $\sigma_*(\mu)$  (equivalently a scale-dependent constant  $L_*(\mu)$ ), then H2.4 is replaced by an RG statement: the scalarization convention becomes part of the renormalization scheme (a “wavefunction renormalization” for the scalar representative). In that case, a universal area/length scale can still appear, but typically as an RG invariant (dimensional transmutation scale) rather than as a fixed analytic monomial in couplings.

## 4. Stationary Phase Produces Half-Density Prefactors (Short-Time Kernel)

The main manuscript uses stationary phase to explain why classical extremals dominate refinement limits. Here we add the complementary kernel-level fact: stationary phase does not only pick the extremal; it also produces a determinant prefactor that transforms as a half-density, i.e. the object needed for coordinate-free kernel composition.

**Derivation D1.4 (Van Vleck prefactor is a bi-half-density).** Let  $S_{\text{cl}}(x, z; t)$  be the classical action as a function of endpoints and time, treated as a generating function. The standard short-time/stationary-phase approximation to the propagator has the form

$$K(x, z; t) \approx \frac{1}{(2\pi i\hbar)^{d/2}} \left| \det \left( -\frac{\partial^2 S_{\text{cl}}}{\partial x \partial z} \right) \right|^{1/2} \exp \left( \frac{i}{\hbar} S_{\text{cl}}(x, z; t) \right).$$

Under a change of coordinates  $x = x(x')$ ,  $z = z(z')$ , the mixed Hessian transforms by the chain rule, and its determinant acquires Jacobian factors:

$$\det \left( -\frac{\partial^2 S_{\text{cl}}}{\partial x' \partial z'} \right) = \det \left( \frac{\partial x}{\partial x'} \right) \det \left( \frac{\partial z}{\partial z'} \right) \det \left( -\frac{\partial^2 S_{\text{cl}}}{\partial x \partial z} \right).$$

Taking square roots shows that the prefactor transforms with  $|\det(\partial x/\partial x')|^{1/2} |\det(\partial z/\partial z')|^{1/2}$ , i.e. exactly as a half-density factor at each endpoint. Thus the stationary-phase prefactor is naturally interpreted as making  $K$  a half-density in each variable, so that kernel composition does not depend on a background measure choice.

**Derivation D1.9 (Square-root delta normalization has half-density weight).** In finite dimension, the “localize on critical points” distribution is  $\delta(\nabla f)$ , supported on  $\text{Crit}(f)$ . A concrete way it appears is via a “halved” oscillatory integral with a normalization exponent fixed by dimension.

Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  be smooth and define, for  $\varepsilon > 0$ ,

$$A_\varepsilon(O) := \varepsilon^{-N/2} \int_{\mathbb{R}^N} e^{\frac{i}{\varepsilon} f(x)} O(x) dx.$$

Then

$$|A_\varepsilon(O)|^2 = \varepsilon^{-N} \iint e^{\frac{i}{\varepsilon} (f(x) - f(y))} O(x) \overline{O(y)} dx dy.$$

Applying the near-diagonal scaling  $y = x + \varepsilon z$  (so  $dy = \varepsilon^N dz$ ) gives

$$|A_\varepsilon(O)|^2 = \iint e^{-\frac{i}{\varepsilon} (f(x + \varepsilon z) - f(x))} O(x) \overline{O(x + \varepsilon z)} dz dx.$$

Formally letting  $\varepsilon \rightarrow 0$  yields

$$|A_\varepsilon(O)|^2 \rightarrow \iint e^{-iz \cdot \nabla f(x)} |O(x)|^2 dz dx = (2\pi)^N \int \delta(\nabla f(x)) |O(x)|^2 dx.$$

The exponent  $N/2$  in the prefactor is exactly the half-density scaling: it cancels the Jacobian  $dy = \varepsilon^N dz$  under near-diagonal rescaling, and it is the “square root” of the density normalization that produces  $\delta(\nabla f)$ .

**Heuristic H1.4 (Where Planck area can enter, minimally).** Derivation D1.3 isolates one minimal route by which a Planck-scale quantity can enter: if the theory supplies a single universal coupling with dimension of length (Newton’s constant) and one demands that the half-density normalization constant be built from that coupling *without fractional powers*, then  $d = 4$  is singled out and the resulting constant has the dimension of an area, naturally identified with the Planck area  $L_P^2 \sim \hbar G_4/c^3$ .

## 5. A Gravitational Anchor: Minimal Areal Speed and the $D = 4$ Cancellation

Rivero’s “Planck areal speed” observation gives a concrete route by which Planck-scale discreteness reappears at Compton scales in inverse-square gravity [RiveroAreal] [RiveroSimple].

**Heuristic H1.3 (Areal-speed selection).** In  $3 + 1$  Newtonian gravity (inverse-square), imposing a discrete areal-speed/area-time condition at a Planck scale can yield characteristic radii proportional to a reduced Compton length, with Newton’s constant canceling when expressed in Planck units. This is a non-trivial indication that “a universal area scale” can be operationally meaningful at low energies in  $D = 4$ .

**Derivation D1.5 (Inverse-square circular orbit + Planck areal speed  $\Rightarrow$  Compton radius).** For a circular orbit under an inverse-square central force  $F(r) = K/r^2$  (with coupling  $K > 0$ ), the centripetal balance is  $mv^2/r = K/r^2$ . The areal speed is  $\dot{A} = \frac{1}{2}rv$ , so  $v = 2\dot{A}/r$ . Substituting into the force balance gives

$$m \left( \frac{2\dot{A}}{r} \right)^2 = \frac{K}{r} \implies r = \frac{4m\dot{A}^2}{K}.$$

For Newtonian gravity between a source mass  $M$  and test mass  $m$ ,  $K = GMm$ , hence

$$r = \frac{4\dot{A}^2}{GM},$$

independent of the test mass  $m$ . If one now imposes  $\dot{A} = k\dot{A}_P$ , where Rivero’s Planck areal speed is  $\dot{A}_P = cL_P$  [RiveroAreal], then using  $L_P^2 = G\hbar/c^3$  yields

$$r = \frac{4k^2(cL_P)^2}{GM} = \frac{4k^2(G\hbar/c)}{GM} = 4k^2 \frac{\hbar}{cM}.$$

Thus  $r$  becomes a multiple of the reduced Compton length  $L_M = \hbar/(cM)$ , with Newton's constant canceled out. In particular,  $k = \frac{1}{2}$  gives  $r = L_M$ . This is the “Planck area per Planck time  $\Rightarrow$  Compton scale” cancellation highlighted in [RiveroAreal] and summarized in [RiveroSimpler].

## 6. Interface with the Main Paper

The main manuscript argues that: 1. classical dynamics are recovered from quantum composition by stationary-phase concentration, and 2. refinement across scales forces RG-style consistency conditions when naive limits diverge.

This draft adds a complementary ingredient: the kernel side is most naturally formulated in half-density language, and stationary phase produces the bi-half-density prefactor directly. A universal convention for turning those half-densities into scalar amplitudes then requires a length $^{d/2}$  scale; in  $d = 4$  this is an area scale.

## 7. Open Problems (Needed for a Real Paper)

1. Make the half-density normalization argument precise for a concrete groupoid or kernel model (tangent-groupoid or short-time propagator model).
2. Show how the area scale enters stationary-phase prefactors and how this interacts with RG scaling.
3. General-dimension analysis: clarify what replaces “area” in odd dimensions and whether a universal normalization is still defensible.
4. Identify minimal hypotheses under which “need of half-density scale  $\Rightarrow$  Planck area” is more than dimensional bookkeeping.