Dirac-Supported Probes, Corners, and Impulses: A Variational Note

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2026

Abstract

Variational principles routinely invoke "point-like probes" of extrema, yet the precise hypotheses under which such probes are safe are often left implicit. This note collects the functional-analytic conditions that make mollifier-based localization of the Euler–Lagrange equation rigorous, states them as an explicit theorem, and works through a complete model — the free particle with a single delta-kick — to illustrate corners, impulse jumps, and the role of distributional forcing. A clean separation is maintained between *Dirac-supported variations* (always safe under stated regularity) and *delta potentials* (which require renormalization and are a distinct mathematical object).

This note is a companion to the cornerstone manuscript. It expands the content of Section 5 there into a self-contained treatment with sharper hypotheses and a worked model.

1. Motivation

The cornerstone manuscript (Section 5) introduces weak stationarity, mollifier probing, and corner/impulse conditions as Propositions P3.1–P3.4. Those statements are sufficient for the structural chain developed there, but they compress the hypotheses and omit worked computations. This satellite note serves three purposes:

1. State the mollifier localization result as a formal theorem with explicit, numbered hypotheses (Section 2).
2. Work through a complete model — the delta-kick free particle — showing trajectory, momentum jump, and action evaluation in full detail (Section 4).
3. Separate two superficially similar but logically distinct uses of the Dirac delta in variational mechanics (Section 5).

2. Mollifier Localization Theorem

We work on a time interval $[t_i, t_f]$ with Lagrangian $\mathcal{L}(q, \dot{q}, t)$ and candidate trajectory $q : [t_i, t_f] \to \mathbb{R}^d$.

`Theorem 2.1 (Mollifier localization of the Euler-Lagrange equation).` Assume:

(H1) $q \in C^1([t_i, t_f]; \mathbb{R}^d)$ and $\mathcal{L}$ is C^2 in $(q, \dot{q})$ and C^0 in t.

(H2) The first variation satisfies $\delta S[q; \eta] = 0$ for every $\eta \in C_c^\infty((t_i, t_f); \mathbb{R}^d)$.

(H3) The Euler–Lagrange expression

$$F[q](t) := \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}, t) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}, t)$$

is continuous at a point $t_0 \in (t_i, t_f)$.

Then $F[q](t_0) = 0$.

`Proof.` Fix a nonneg mollifier $\rho \in C_c^\infty(\mathbb{R})$ with $\int \rho = 1$ and set $\rho_\varepsilon(s) = \varepsilon^{-1}\rho(s/\varepsilon)$. For any unit vector $u \in \mathbb{R}^d$, the test variation $\eta_\varepsilon(t) = \rho_\varepsilon(t - t_0)\, u$ is in C_c^∞ for ε small enough. By (H2):

$$0 = \delta S[q; \eta_\varepsilon] = \int_{t_i}^{t_f} F[q](t) \cdot \rho_\varepsilon(t - t_0)\, u\, dt = u \cdot \int_{t_i}^{t_f} \rho_\varepsilon(t - t_0)\, F[q](t)\, dt.$$

By (H3) the convolution converges to $F[q](t_0)$ as $\varepsilon \to 0^+$. Since u is arbitrary, $F[q](t_0) = 0$. □

`Remark 2.2 (Role of each hypothesis).` (H1) ensures $F[q]$ is locally integrable so the distributional pairing makes sense. (H2) is the global stationarity input. (H3) is the local regularity gate: without it, mollifier limits may fail to converge or may converge to an averaged value rather than a pointwise one. If $F[q]$ is continuous on all of (t_i, t_f), iteration of Theorem 2.1 recovers the classical Euler–Lagrange equation everywhere.

3. Corners and Impulses: Formal Statements

When hypothesis (H3) fails — because $\dot{q}$ or external forcing is discontinuous — two distinct situations arise.

3.1 Corners (unforced velocity jump)

`Theorem 3.1 (Corner condition / Weierstrass-Erdmann).` Assume q is piecewise C^2 with a single velocity discontinuity at t_0, satisfying the unforced Euler–Lagrange equation on (t_i, t_0) and (t_0, t_f) separately. Then the canonical momentum is continuous at t_0:

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0^-}^{t_0^+} = 0.$$

`Proof.` Integrate the Euler–Lagrange equation over $[t_0-\varepsilon, t_0+\varepsilon]$. The integral of $\partial_q \mathcal{L}$ vanishes as $\varepsilon \to 0$ by boundedness; the derivative term yields the momentum jump. □

3.2 Impulses (delta forcing)

`Theorem 3.2 (Impulse jump condition).` Consider the forced distributional equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = J\,\delta(t-t_0), \quad J \in \mathbb{R}^d.$$

If $\partial_{\dot{q}}\mathcal{L}$ has one-sided limits at t_0, then

$$\frac{\partial \mathcal{L}}{\partial \dot{q}}(t_0^+) - \frac{\partial \mathcal{L}}{\partial \dot{q}}(t_0^-) = J.$$

`Proof.` Same integration argument: the delta integrates to J, the smooth remainder vanishes. □

The distinction is structural: corners arise from variational boundary conditions (matching at a junction), while impulses arise from external forcing (a source term in the equation of motion).

4. Worked Model: Free Particle with a Single Delta-Kick

We give a complete computation that illustrates both Theorem 3.2 and the evaluation of action on a kinked trajectory.

4.1 Setup

Consider a particle of mass m in one dimension with Lagrangian $\mathcal{L} = \frac{m}{2}\dot{q}^2$ and an external impulsive force $J\,\delta(t-t_0)$ applied at time $t_0 \in (0, T)$. The equation of motion is

$$m\ddot{q} = J\,\delta(t-t_0).$$

4.2 Solution

The trajectory is piecewise linear:

$$q(t) = \begin{cases} q_i + v_-\,t & 0 \le t < t_0, \\ q_i + v_-\,t_0 + v_+\,(t-t_0) & t_0 \le t \le T, \end{cases}$$

with the velocity jump $v_+ - v_- = J/m$ from Theorem 3.2.

Boundary conditions $q(0) = q_i$, $q(T) = q_f$ fix the velocities. Writing $\Delta v = J/m$:

$$v_- = \frac{q_f - q_i - \Delta v\,(T - t_0)}{T}, \qquad v_+ = v_- + \Delta v.$$

4.3 Action evaluation

The action splits across the kink:

$$S = \frac{m}{2}\big(v_-^2\, t_0 + v_+^2\,(T - t_0)\big).$$

In the unforced limit ($J = 0$, so $\Delta v = 0$):

$$S_0 = \frac{m}{2}\frac{(q_f - q_i)^2}{T},$$

the standard free-particle result. The impulse adds a positive-definite kinetic energy cost:

$$S - S_0 = \frac{m}{2}\frac{t_0(T - t_0)}{T}\,(\Delta v)^2 > 0 \quad (J \neq 0).$$

This confirms that the delta-kick raises the action above the free minimum — the impulsive trajectory is not an extremum of the unforced problem.

4.4 Angular momentum preservation under central impulses

For a central force in the plane, the impulse is radial: $J = J_r\,\hat{r}$. Since angular momentum depends only on the transverse velocity component,

$$L = m\,r\,\dot{\theta},$$

a purely radial impulse leaves $\dot{\theta}$ (and hence L) unchanged across the kick, recovering the equal-area property of Newton's polygon at the distributional level.

5. Safe vs Unsafe Uses of the Dirac Delta in Variational Mechanics

The preceding sections involve two related but *logically distinct* mathematical objects. Conflating them is a common source of error.

5.1 Dirac-supported variations (safe under regularity)

Using mollifier sequences $\rho_\varepsilon \to \delta$ as *test functions* against a continuous integrand is always safe — it is standard distribution theory. This is Theorem 2.1. No renormalization or regularization ambiguity arises; the $\varepsilon \to 0$ limit is unique and controlled by continuity.

5.2 Delta potentials (require renormalization)

A point interaction $V(q) = g\,\delta(q)$ in the Hamiltonian is a different object. In dimensions $d \geq 2$, the naive coupling constant g requires renormalization (the resolvent acquires a logarithmic or power-law divergence depending on d). In $d = 1$ the delta potential is well-defined without renormalization, but this is an accident of low dimension, not a general principle. The companion note on delta objects treats the half-density kernel structure of point interactions in detail.

5.3 Summary table

Object	Math status	Renormalization?
Mollifier probe of $F[q]$ (Thm 2.1)	Rigorous	No
Corner/impulse matching (Thms 3.1–3.2)	Rigorous	No
δ potential, $d = 1$	Well-defined	No
δ potential, $d \geq 2$	Requires care	Yes
Products $\delta(t)^2$	Undefined	Always

6. Outlook

1. Extend the single-impulse model to a sequence of N impulses and take the continuum limit, recovering stochastic forcing or the path-integral time-slicing prescription.
2. Treat the piecewise-smooth trajectory as a weak solution and examine whether the Hamilton–Jacobi equation acquires viscosity-solution structure at the kink.
3. Connect the corner-condition analysis to broken geodesics in Riemannian geometry (Synge's world function approach).

References

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