“Uncuttable” as Controlled Refinement

Alejandro Rivero

2026

Abstract

This note fixes a meaning of “uncuttable” aligned with the refinement-compatibility thesis of the cornerstone manuscript. Here **uncuttable** does not mean “indivisible.” It means: the quantity of interest is not determined by any *finite* dissection alone; it is a **limit object** whose definition requires a refinement rule and a comparison structure across refinements.

The point is structural and mathematical: once a theory is built from composable local pieces, the continuum theory is the stable target of a refinement limit, and extra control data may be required for that limit to exist or be unique.

1. Definition

Call a quantity Q **uncuttable** (in this note’s sense) if: 1. there exists a family of finite approximants Q_N produced by a finite dissection/refinement scheme of depth N, but 2. the value of interest is not any finite Q_N; it is a controlled limit $Q = \lim_{N\to\infty} Q_N$, and 3. specifying the *rule of refinement* and the *comparison across refinements* is part of the definition of Q.

The historical resonance is deliberate. The Greek $\acute{\alpha}\tau o\mu o\varsigma$ (“a-tomos,” uncuttable) was coined by Leucippus and Democritus to denote indivisible substance — matter that cannot be divided further. The shift proposed here is from ontology to procedure: what is “uncuttable” is not a smallest piece of stuff, but a limit object that no single finite dissection captures. The indivisibility is not in the substance but in the definition: you cannot “cut” the limit into finitely many pieces and recover it without specifying how the pieces are to be reassembled and refined.

This is the ordinary situation in analysis: finite partitions approximate, but the object is defined by a limiting procedure together with hypotheses that ensure convergence/uniqueness.

2. Toy model: an integral is already a refinement limit

Let $f : [a, b] \to \mathbb{R}$. A prototypical refinement family is a partition $P_N = \{a = t_0 < \cdots < t_N = b\}$ with mesh $\|P_N\| := \max_k (t_{k+1} - t_k) \to 0$. Define the Riemann-sum approximants

$$Q_N := \sum_{k=0}^{N-1} f(\xi_k)\,(t_{k+1} - t_k), \qquad \xi_k \in [t_k, t_{k+1}].$$

In good cases, $Q_N \to \int_a^b f(t)\,dt$ as $\|P_N\| \to 0$, and the limit is independent of the tags ξ_k. But this is not a tautology: the limit can fail to exist or can depend on the refinement rule unless hypotheses are stated.

A concrete failure case appears already in calculus. Consider the difference quotient $f(x + \varepsilon)/\varepsilon$: for each finite ε, the quantity diverges as $\varepsilon \to 0$. The "refinement limit" exists only after subtracting a counterterm $f(x)/\varepsilon$, yielding the derivative $f'(x)$. The subtraction is part of the definition of the limit — without it, the refinement procedure does not converge. This is the simplest model of renormalization: a "cut" at finite ε is not the answer; the answer requires a controlled $\varepsilon \to 0$ limit with explicit subtraction rules.

In the present program, this is the basic moral: finite cuts approximate, but the value is defined by **controlled refinement**.

2.5 Worked example: the derivative as a renormalized refinement limit

The paragraph above describes the difference-quotient subtraction informally. We now make it explicit as a minimal model of renormalization.

`Example 2.1 (Derivative as counterterm-subtracted limit).` Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at x. The naive "refinement" approximant at scale ε is

$$R(\varepsilon) := \frac{f(x + \varepsilon)}{\varepsilon}.$$

This diverges as $\varepsilon \to 0$ whenever $f(x) \neq 0$. To extract a finite limit, subtract a counterterm:

$$R_{\text{ren}}(\varepsilon) := \frac{f(x + \varepsilon) - f(x)}{\varepsilon}.$$

Now $\lim_{\varepsilon \to 0} R_{\text{ren}}(\varepsilon) = f'(x)$, which is finite and independent of the "regulator" ε. The structure is:

1. **Regulated quantity**: $R(\varepsilon)$, finite for each $\varepsilon > 0$ but divergent as $\varepsilon \to 0$.
2. **Counterterm**: $f(x)/\varepsilon$, which absorbs the divergence.
3. **Renormalized observable**: $f'(x)$, the ε-independent limit.

This is formally identical to the pattern in perturbative renormalization: a bare quantity diverges as the regulator is removed, but a systematic subtraction yields a finite, physically meaningful result. The subtraction rule is not ad hoc; it is forced by the requirement that the result depend smoothly on the data and be independent of the regulator.

`Remark 2.2 (Higher-order counterterms).` If f is C^n, the Taylor expansion

$$f(x+\varepsilon) = f(x) + f'(x)\varepsilon + \tfrac{1}{2}f''(x)\varepsilon^2 + \cdots + \tfrac{1}{n!}f^{(n)}(x)\varepsilon^n + o(\varepsilon^n)$$

provides a systematic tower of subtractions, one per order. Removing terms through order $k-1$ and dividing by ε^k gives the k-th Taylor coefficient $f^{(k)}(x)/k!$ plus vanishing corrections — or equivalently, dividing by $\varepsilon^k/k!$ recovers the derivative $f^{(k)}(x)$ itself. Each order is a "counterterm-subtracted refinement limit" — the analog of loop-by-loop renormalization in QFT.

3. Dynamics: action, stationarity, and the need for control data

The cornerstone manuscript uses the same template in mechanics. Given a partition of time, the discrete action

$$S_N[q] = \sum_k \mathcal{L}\left(q_k, \frac{q_{k+1}-q_k}{\Delta t_k}, t_k\right)\Delta t_k$$

is a finite refinement approximant. The continuum action $S[q] = \int \mathcal{L}\, dt$ is a refinement limit.

Two "uncuttable" features appear immediately when one pushes beyond smooth classical paths: 1. **Singular probes and corners:** stationarity must be interpreted in weak/distributional form; point-supported variations require mollification. 2. **Non-uniqueness of refinement schemes:** different discretization conventions (even if classically equivalent) can produce distinct refined objects unless an equivalence or control map is specified.

These are exactly the obstructions discussed in the cornerstone manuscript: the point is not indivisible atoms, but limit control.

In the quantum setting, the "uncuttable" character becomes sharper. The path-integral amplitude

$$K(q_f, t_f; q_i, t_i) = \int \prod_{k=1}^{N-1} dq_k \prod_{k=0}^{N-1} K_\Delta(q_{k+1}, q_k; t_k)$$

is a product of short-time kernels composed over a time partition of depth N. No finite N gives the exact propagator; the propagator is the $N \to \infty$ refinement limit. Crucially, the control parameter $\hbar$ enters the short-time kernels as $\exp(iS_\Delta/\hbar)$, and different discretization conventions (left-point, midpoint, symmetric) can produce distinct $O(\hbar)$ corrections even though they share the same classical $\hbar \to 0$ limit [FeynmanHibbs1965]. Thus the quantum amplitude is doubly "uncuttable": it requires both a refinement rule (time-slicing prescription) and a comparison/equivalence structure (ordering convention or half-density normalization) before the limit is well-defined.

4. Outlook: refinement compatibility as "the extra structure"

In the companion papers, the "extra structure" used to control refinement limits is made explicit: - half-densities make kernel composition coordinate-free without hidden measure choices, - control maps τ encode how parameters must flow under refinement to maintain stability, - renormalization is the compatibility rule when naive refinement limits diverge.

This note is therefore a small conceptual bridge: it isolates an early, analysis-level instance of the same meta-problem that reappears in quantization and in QFT.

References

1. [BatesWeinstein1997] Sean Bates and Alan Weinstein, *Lectures on the Geometry of Quantization*, Berkeley Mathematics Lecture Notes, vol. 8, AMS, 1997. OA: https://math.berkeley.edu/~alanw/GofQ.pdf. (Half-density formalism for coordinate-free kernel composition.)
2. [FeynmanHibbs1965] Richard P. Feynman and Albert R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, 1965. (Path integral as a refinement limit of time-sliced amplitudes.)