

# Dirac-Supported Probes, Corners, and Impulses: A Variational Note

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## Abstract

Variational principles routinely invoke “point-like probes” of extrema, yet the precise hypotheses under which such probes are safe are often left implicit. This note collects the functional-analytic conditions that make mollifier-based localization of the Euler–Lagrange equation rigorous, states them as an explicit theorem, and works through a complete model — the free particle with a single delta-kick — to illustrate corners, impulse jumps, and the role of distributional forcing. A clean separation is maintained between *Dirac-supported variations* (always safe under stated regularity) and *delta potentials* (which in dimension  $d \geq 2$  require renormalization and are a distinct mathematical object).

This note is a companion to the cornerstone manuscript. It expands the content of Section 5 there into a self-contained treatment with sharper hypotheses and a worked model.

## 1. Motivation

The cornerstone manuscript (Section 5) introduces weak stationarity, mollifier probing, and corner/impulse conditions as Propositions P3.1–P3.4. Those statements are sufficient for the structural chain developed there, but they compress the hypotheses and omit worked computations. This satellite note serves three purposes:

1. State the mollifier localization result as a formal theorem with explicit, numbered hypotheses (Section 2).
2. Work through a complete model — the delta-kick free particle — showing trajectory, momentum jump, and action evaluation in full detail (Section 4).
3. Separate two superficially similar but logically distinct uses of the Dirac delta in variational mechanics (Section 5).

## 2. Mollifier Localization Theorem

We work on a time interval  $[t_i, t_f]$  with Lagrangian  $\mathcal{L}(q, \dot{q}, t)$  and candidate trajectory  $q : [t_i, t_f] \rightarrow \mathbb{R}^d$ .

**Theorem 2.1 (Mollifier localization of the Euler-Lagrange equation).** Assume:

(H1)  $q \in C^1([t_i, t_f]; \mathbb{R}^d)$  and  $\mathcal{L}$  is  $C^2$  in  $(q, \dot{q})$  and  $C^0$  in  $t$ .

(H2) The first variation satisfies  $\delta S[q; \eta] = 0$  for every  $\eta \in C_c^\infty((t_i, t_f); \mathbb{R}^d)$ .

(H3) The Euler-Lagrange expression

$$F[q](t) := \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}, t) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}, t)$$

is continuous at a point  $t_0 \in (t_i, t_f)$ .

Then  $F[q](t_0) = 0$ .

**Proof.** Fix a nonnegative mollifier  $\rho \in C_c^\infty(\mathbb{R})$  with  $\int \rho = 1$  and set  $\rho_\varepsilon(s) = \varepsilon^{-1} \rho(s/\varepsilon)$ . For any unit vector  $u \in \mathbb{R}^d$ , the test variation  $\eta_\varepsilon(t) = \rho_\varepsilon(t - t_0) u$  is in  $C_c^\infty$  for  $\varepsilon$  small enough. By (H2):

$$0 = \delta S[q; \eta_\varepsilon] = \int_{t_i}^{t_f} F[q](t) \cdot \rho_\varepsilon(t - t_0) u \, dt = u \cdot \int_{t_i}^{t_f} \rho_\varepsilon(t - t_0) F[q](t) \, dt.$$

By (H3) the convolution converges to  $F[q](t_0)$  as  $\varepsilon \rightarrow 0^+$ . Since  $u$  is arbitrary,  $F[q](t_0) = 0$ .  $\square$

**Remark 2.2 (Role of each hypothesis).** (H1) ensures  $F[q]$  is locally integrable so the distributional pairing makes sense. (H2) is the global stationarity input. (H3) is the local regularity gate: without it, mollifier limits may fail to converge or may converge to an averaged value rather than a pointwise one. If  $F[q]$  is continuous on all of  $(t_i, t_f)$ , iteration of Theorem 2.1 recovers the classical Euler-Lagrange equation everywhere.

## 3. Corners and Impulses: Formal Statements

When hypothesis (H3) fails — because  $\dot{q}$  or external forcing is discontinuous — two distinct situations arise.

### 3.1 Corners (unforced velocity jump)

**Theorem 3.1 (Corner condition / Weierstrass-Erdmann).** Assume  $q$  is piecewise  $C^2$  with a single velocity discontinuity at  $t_0$ , satisfying the unforced Euler-Lagrange equation on  $(t_i, t_0)$  and  $(t_0, t_f)$  separately. Then the canonical

momentum is continuous at  $t_0$ :

$$\left[ \frac{\partial \mathcal{L}}{\partial \dot{q}} \right]_{t_0^-}^{t_0^+} = 0.$$

**Proof.** Integrate the Euler–Lagrange equation over  $[t_0 - \varepsilon, t_0 + \varepsilon]$ . The integral of  $\partial_q \mathcal{L}$  vanishes as  $\varepsilon \rightarrow 0$  by boundedness; the derivative term yields the momentum jump.  $\square$

### 3.2 Impulses (delta forcing)

**Theorem 3.2 (Impulse jump condition).** Consider the forced distributional equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = J \delta(t - t_0), \quad J \in \mathbb{R}^d.$$

If  $\partial_{\dot{q}} \mathcal{L}$  has one-sided limits at  $t_0$ , then

$$\frac{\partial \mathcal{L}}{\partial \dot{q}}(t_0^+) - \frac{\partial \mathcal{L}}{\partial \dot{q}}(t_0^-) = J.$$

**Proof.** Same integration argument: the delta integrates to  $J$ , the smooth remainder vanishes.  $\square$

The distinction is structural: corners arise from variational boundary conditions (matching at a junction), while impulses arise from external forcing (a source term in the equation of motion).

## 4. Worked Model: Free Particle with a Single Delta-Kick

We give a complete computation that illustrates both Theorem 3.2 and the evaluation of action on a kinked trajectory.

### 4.1 Setup

Consider a particle of mass  $m$  in one dimension with Lagrangian  $\mathcal{L} = \frac{m}{2} \dot{q}^2$  and an external impulsive force  $J \delta(t - t_0)$  applied at time  $t_0 \in (0, T)$ . The equation of motion is

$$m \ddot{q} = J \delta(t - t_0).$$

### 4.2 Solution

The trajectory is piecewise linear:

$$q(t) = \begin{cases} q_i + v_- t & 0 \leq t < t_0, \\ q_i + v_- t_0 + v_+ (t - t_0) & t_0 \leq t \leq T, \end{cases}$$

with the velocity jump  $v_+ - v_- = J/m$  from Theorem 3.2.

Boundary conditions  $q(0) = q_i$ ,  $q(T) = q_f$  fix the velocities. Writing  $\Delta v = J/m$ :

$$v_- = \frac{q_f - q_i - \Delta v (T - t_0)}{T}, \quad v_+ = v_- + \Delta v.$$

### 4.3 Action evaluation

The action splits across the kink:

$$S = \frac{m}{2} (v_-^2 t_0 + v_+^2 (T - t_0)).$$

In the unforced limit ( $J = 0$ , so  $\Delta v = 0$ ):

$$S_0 = \frac{m}{2} \frac{(q_f - q_i)^2}{T},$$

the standard free-particle result. The impulse adds a positive-definite kinetic energy cost:

$$S - S_0 = \frac{m}{2} \frac{t_0(T - t_0)}{T} (\Delta v)^2 > 0 \quad (J \neq 0).$$

This confirms that the delta-kick raises the action above the free minimum — the impulsive trajectory is not an extremum of the unforced problem.

### 4.4 Angular momentum preservation under central impulses

For a central force in the plane, the impulse is radial:  $J = J_r \hat{r}$ . Since angular momentum depends only on the transverse velocity component,

$$L = m r \dot{\theta},$$

a purely radial impulse leaves  $\dot{\theta}$  (and hence  $L$ ) unchanged across the kick, recovering the equal-area property of Newton's polygon at the distributional level.

### 4.5 From $N$ impulses to the time-sliced path integral

The single-impulse model extends naturally to a sequence of  $N$  impulses. This extension bridges the distributional mechanics of Sections 3–4 to the path-integral composition framework of the cornerstone manuscript (Section 6 there).

Partition  $[0, T]$  into  $N+1$  equal intervals of length  $\Delta t = T/(N+1)$ , with junction times  $t_k = k \Delta t$  for  $k = 1, \dots, N$ . Fix the endpoints  $q_0 = q_i$ ,  $q_{N+1} = q_f$ , and let  $q_1, \dots, q_N$  be free intermediate positions. On each segment the particle is free, so the trajectory is piecewise linear with velocities

$$v_k = \frac{q_{k+1} - q_k}{\Delta t}, \quad k = 0, \dots, N.$$

The discrete action is

$$S_N[\{q_k\}] = \sum_{k=0}^N \frac{m}{2} \frac{(q_{k+1} - q_k)^2}{\Delta t}.$$

At each junction  $t_k$ , the velocity jumps from  $v_{k-1}$  to  $v_k$ . By Theorem 3.2, each jump requires an impulse  $J_k = m(v_k - v_{k-1})$ . The *classical* stationary condition  $\partial S_N / \partial q_k = 0$  imposes  $v_k = v_{k-1}$  for all  $k$  — that is, Theorem 3.1’s corner condition (momentum continuity) at every junction — and the path collapses to a single straight line.

In the quantum theory, one instead sums over all intermediate configurations with amplitude weights:

$$K(q_f, q_i; T) = \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi i \hbar \Delta t} \right)^{(N+1)/2} \int \prod_{k=1}^N dq_k \exp\left(\frac{i}{\hbar} S_N[\{q_k\}]\right).$$

There are  $N + 1$  segments and  $N$  intermediate integrations; each segment contributes one factor of  $\sqrt{m/(2\pi i \hbar \Delta t)}$ , giving the exponent  $(N + 1)/2$ . This is precisely the half-density normalization required for the composition law to hold at each intermediate integration — a point treated systematically in the cornerstone’s half-density framework. The distributional impulse-matching of Theorem 3.2 thus connects, through this  $N \rightarrow \infty$  limit, to the composition postulate for transition amplitudes.

## 5. Safe vs Unsafe Uses of the Dirac Delta in Variational Mechanics

The preceding sections involve two related but *logically distinct* mathematical objects. Conflating them is a common source of error.

### 5.1 Dirac-supported variations (safe under regularity)

Using mollifier sequences  $\rho_\varepsilon \rightarrow \delta$  as *test functions* against a continuous integrand is always safe — it is standard distribution theory. This is Theorem 2.1. No renormalization or regularization ambiguity arises; the  $\varepsilon \rightarrow 0$  limit is unique and controlled by continuity.

### 5.2 Delta potentials (require renormalization)

A point interaction  $V(q) = g\delta(q)$  in the Hamiltonian is a different object. In dimensions  $d \geq 2$ , the naive coupling constant  $g$  requires renormalization (the resolvent acquires a logarithmic or power-law divergence depending on  $d$ ). In  $d = 1$  the delta potential is well-defined without renormalization, but this is an accident of low dimension, not a general principle. The companion note on delta objects treats the half-density kernel structure of point interactions in detail.

### 5.3 Summary table

Object	Math status	Renormalization?
Mollifier probe of $F[q]$ (Thm 2.1)	Rigorous	No
Corner/impulse matching (Thms 3.1–3.2)	Rigorous	No
$\delta$ potential, $d = 1$	Well-defined	No
$\delta$ potential, $d \geq 2$	Requires care	Yes
Products $\delta(t)^2$	Undefined	Always

## 6. Outlook

1. The stochastic-forcing interpretation of Section 4.5’s  $N$ -impulse model — random impulses with prescribed statistics — remains open as a bridge to stochastic mechanics.
2. Treat the piecewise-smooth trajectory as a weak solution and examine whether the Hamilton–Jacobi equation acquires viscosity-solution structure at the kink.
3. Connect the corner-condition analysis to broken geodesics in Riemannian geometry (Synge’s world function approach).

## References

1. [Gelfand1963] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, 1963. (Reprinted by Dover, 2000.)
2. [Giaquinta1996] M. Giaquinta and S. Hildebrandt, *Calculus of Variations I: The Lagrangian Formalism*, Springer, 1996.
3. [AlbeverioGesztesyHoeghKrohnHolden2005] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed., AMS Chelsea Publishing, 2005.
4. [FeynmanHibbs1965] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, 1965. (Path integral as refinement limit of time-sliced amplitudes.)