From Newton to the Path Integral

Abstract

This paper develops a single structural thesis across classical and quantum theory: physically meaningful laws arise as controlled limits of composable local refinements. We begin with Newton's polygonal approximation of central-force motion and its limit to continuous dynamics, then re-express the same logic in modern variational form through additive action functionals. We treat the path integral as a composition law over refined time slices, not as an isolated quantum postulate, and we frame deformation quantization and renormalization as two mathematically distinct control mechanisms for limit consistency. The narrative is constructive: each stage retains the previous one as a limiting or compatibility condition rather than replacing it. Within this architecture we reserve a dedicated role for point-like (Dirac-supported) probes in weak formulations of the action principle, emphasizing where they are mathematically valid and where regularization is mandatory. The result is a staged program from Newtonian limit methods to quantum amplitudes in which the classical equations are recovered as stationary limits of a broader compositional framework.

1. Introduction

The historical and technical problem addressed here is not merely "how to quantize," but "how to define a stable continuum theory from iterative refinement." The paper therefore treats Newtonian mechanics, action principles, path integration, deformation quantization, and renormalization as parts of one continuity problem.

The first anchor is Newton's geometric method in central-force motion: replace a curve by a sequence of short segments, impose a local update rule, and pass to a limit while controlling what is meant by "vanishing" quantities. In modern language, the key object is not a smallest geometric piece but a refinement procedure with invariant content [Newton1687].

The second anchor is the action formulation. Action is additive under temporal partitioning, and that additivity is exactly the algebraic structure needed to compare coarse and fine descriptions. This creates the bridge to quantum composition: if local contributions compose multiplicatively while the underlying functional is additive, exponential weighting is structurally natural [Dirac1933] [Feynman1948].

The third anchor is methodological. In this manuscript, deformation quantization and renormalization are not presented as detached specialist topics. They are two ways to control limits: 1. Deformation quantization controls the classical-to-quantum passage through algebraic deformation and recovery of Poisson structure in the small-parameter limit [Landsman1998] [Connes1994]. 2. Renormalization controls divergent refinement procedures by regulator-dependent intermediate steps and regulator-independent observables [ConnesKreimer2000].

Section 2 fixes the formal vocabulary and claim taxonomy used in later sections. It also narrows one foundational ambiguity: the paper does not assume that continuum limits are ontological statements about nature. It assumes only that they are operational definitions of stable predictive objects. This narrowed statement will be stress-tested in later sections.

2. Notation and Claim Taxonomy

2.1 Core Objects

Let $q : [t_i, t_f] \to \mathbb{R}^d$ be a configuration-space trajectory and $\mathcal{L}(q, \dot{q}, t)$ a Lagrangian density. Define the action:

$$S[q] = \int_{t_i}^{t_f} \mathcal{L}(q, \dot{q}, t)\, dt.$$

For a partition $t_i = t_0 < t_1 < \cdots < t_N = t_f$ with $\Delta t_k = t_{k+1} - t_k$, define the discrete action functional:

$$S_N[q] = \sum_{k=0}^{N-1} \mathcal{L}\left(q_k, \frac{q_{k+1} - q_k}{\Delta t_k}, t_k\right) \Delta t_k.$$

For planar central motion $q = (r, \theta)$, define areal velocity and angular momentum:

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta}, \qquad L_{\text{ang}} = m r^2 \dot{\theta} = 2m\dot{A}.$$

These definitions are used as the Newtonian-to-variational bridge in Section 3 and Section 4.

2.2 Weak-Form Preliminaries for Point-Like Probes

Let $\eta \in C_c^\infty((t_i, t_f); \mathbb{R}^d)$ be a smooth compactly supported test variation. The first variation is written $\delta S[q; \eta]$, and stationarity means $\delta S[q; \eta] = 0$ for all admissible η.

To model point-supported probes later, introduce a mollifier family ρ_ε with $\rho_\varepsilon \rightharpoonup \delta$ in distributions as $\varepsilon \to 0^+$. Any use of Dirac-supported variations in this manuscript is understood as a mollified limit unless explicitly labeled heuristic.

2.3 Claim Taxonomy

Every substantive claim is marked by one of: 1. `Proposition`: statement intended as mathematically valid under explicit assumptions. 2. `Derivation`: explicit calculation from stated premises. 3. `Heuristic`: physically motivated bridge that is not presented as full proof.

2.4 Seed Claims for the Program

`Proposition P0.1 (Additive refinement structure).` Given a partition of $[t_i, t_f]$, the discrete action S_N is additive over concatenated subintervals by construction. Therefore action is a natural candidate for refinement comparison.

`Derivation D0.1 (Composition-compatible exponential form).` Suppose a weight map W on time-sliced paths satisfies: 1. $W[\gamma_1 \circ \gamma_2] = W[\gamma_1]W[\gamma_2]$ for composable segments. 2. $\log W$ is local in the slice contributions. 3. The corresponding additive functional is proportional to S_N in the refinement limit.

Then there exists a scale κ and constant c_0 such that, up to normalization, $W[\gamma] \propto \exp(c_0 S[\gamma]/\kappa)$. Choosing $c_0 = i$ and $\kappa = \hbar$ recovers the standard oscillatory quantum weighting form.

`Heuristic H0.1 (Classical recovery as concentration).` When the phase scale is small relative to action variation, dominant contributions concentrate near stationary-action trajectories. This is the structural claim later made precise through stationary-phase analysis.

2.5 Scope Boundary Established

This section fixes notation and methodological boundaries: 1. Historical statements are used only as source-anchored motivation. 2. Mathematical validity requires explicit assumptions and, for singular objects, explicit regularization. 3. Quantum and QFT-level statements are introduced only after the composition law and refinement language are fixed.

Transition to Section 3: with notation fixed, the next section derives the Newtonian area-law refinement argument in modern symbols and links it to L_{ang} conservation.

3. Newtonian Refinement and Area Law

3.1 Source-Critical Framing

In Book I, Proposition I of the *Principia*, Newton proves that a centripetal forcing rule implies equal areas swept in equal times by the radius vector. The historical proof is polygonal and limit-based: one constructs a piecewise-linear trajectory with impulses directed to a fixed center, then passes to a continuous curve by refinement [Newton1687].

This section uses that structure directly and only then translates to modern vector notation. Source-critically, the statements below distinguish: 1. Newton's geometric argument about polygons and limits. 2. A modern reformulation via torque and angular momentum.

The reformulation is mathematically equivalent under the same assumptions, but it is an interpretive translation, not a verbatim historical rendering.

3.2 Discrete Refinement Model

Fix equal time steps $\Delta t > 0$, times $t_k = t_0 + k\Delta t$, and a fixed center O. Let $\mathbf{r}_k$ be the position vector at t_k. The stepwise model is: 1. Free inertial drift between t_k and t_{k+1}. 2. Instantaneous impulse at each vertex t_k, directed along $\mathbf{r}_k$ (centripetal/central).

Let $\mathbf{v}_k^-$ be velocity just before the impulse at t_k, and $\mathbf{v}_k^+$ just after. The impulse condition is

$$m\big(\mathbf{v}_k^+ - \mathbf{v}_k^-\big) = J_k\,\hat{\mathbf{r}}_k, \qquad \hat{\mathbf{r}}_k = \frac{\mathbf{r}_k}{\|\mathbf{r}_k\|}.$$

Drift then gives

$$\mathbf{r}_{k+1} = \mathbf{r}_k + \mathbf{v}_k^+\Delta t, \qquad \mathbf{v}_{k+1}^- = \mathbf{v}_k^+.$$

`Derivation D1.1 (Finite-step angular momentum invariance).` Define $\mathbf{L}_k^- = m\,\mathbf{r}_k \times \mathbf{v}_k^-$, $\mathbf{L}_k^+ = m\,\mathbf{r}_k \times \mathbf{v}_k^+$.

At impulse:

$$\mathbf{L}_k^+ - \mathbf{L}_k^- = m\,\mathbf{r}_k \times (\mathbf{v}_k^+ - \mathbf{v}_k^-) = \mathbf{r}_k \times (J_k\hat{\mathbf{r}}_k) = \mathbf{0}.$$

During drift:

$$\mathbf{L}_{k+1}^- = m\,\mathbf{r}_{k+1} \times \mathbf{v}_{k+1}^- = m(\mathbf{r}_k + \mathbf{v}_k^+\Delta t) \times \mathbf{v}_k^+ = m\,\mathbf{r}_k \times \mathbf{v}_k^+ = \mathbf{L}_k^+.$$

Hence $\mathbf{L}_{k+1}^- = \mathbf{L}_k^-$: angular momentum is exactly conserved at every finite step in this refinement model.

`Derivation D1.2 (Equal areas in equal times, discrete form).` The swept area in step k is the triangle area

$$\Delta A_k = \frac{1}{2}\,\|\mathbf{r}_k \times (\mathbf{r}_{k+1} - \mathbf{r}_k)\| = \frac{1}{2}\,\big\|\mathbf{r}_k \times \mathbf{v}_k^+\big\|\,\Delta t = \frac{\|\mathbf{L}\|}{2m}\Delta t.$$

Therefore for fixed Δt, ΔA_k is independent of k. This is the equal-areas statement at finite polygonal level.

3.3 Continuum Passage and Central-Force Generality

`Proposition P1.1 (Refinement limit of areal velocity).` If $\max_k \Delta t_k \to 0$ under consistent refinement, the finite-step law above yields

$$\frac{dA}{dt} = \frac{\|\mathbf{L}\|}{2m},$$

for the limiting trajectory whenever the limit exists in the standard differentiable sense.

For a smooth central force $\mathbf{F}(\mathbf{r}) = f(r)\hat{\mathbf{r}}$, this same invariant follows from torque:

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{0}.$$

So the areal law is independent of the inverse-power index n in $\mathbf{F} = -(K/r^n)\hat{\mathbf{r}}$: n changes radial dynamics and orbit families, but not the areal-velocity conservation mechanism itself.

`Heuristic H1.1 (Impulse-to-continuous interpretation).` The impulse model is a refinement scaffold for continuous forcing, not a literal claim that nature acts by discrete kicks. Its value is structural: invariants proven exactly at finite step survive controlled refinement.

3.4 Closed Question from the Section 2 Setup

Section 2 left one key ambiguity open: is Newton's area law a small-step approximation or a genuine invariant statement? The derivations above close that point: within the polygonal central-impulse model, the equal-area law is exact at each finite step and only the curve interpolation is a limiting passage.

Transition to Section 4: with the Newtonian invariant fixed in modern notation, the next section derives Euler-Lagrange equations and Noether charge conservation to show the same structure directly in action language.

4. Action as Additive Invariant

4.1 Stationarity Setup

The Section 3 invariant was derived from a refinement model in configuration geometry. We now restate the same physics through stationarity of action.

Assume: 1. $q : [t_i, t_f] \to \mathbb{R}^d$ is at least C^2, and variations η are C^1 (or smooth with compact support). 2. $\mathcal{L}(q, \dot{q}, t)$ is C^1 in t and C^2 in $(q, \dot{q})$ on the region reached by $(q(t), \dot{q}(t))$.

Let the action be

$$S[q] = \int_{t_i}^{t_f} \mathcal{L}\big(q(t), \dot{q}(t), t\big)\, dt,$$

and define $q_\varepsilon = q + \varepsilon\eta$ for an admissible variation η, with either: 1. fixed endpoints $\eta(t_i) = \eta(t_f) = 0$, or 2. compact support in (t_i, t_f).

Stationarity means

$$\delta S[q;\eta] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S[q_\varepsilon] = 0 \quad \text{for all admissible } \eta.$$

`Proposition P2.0 (Fundamental lemma, vector form).` If $F : [t_i, t_f] \to \mathbb{R}^d$ is continuous and $\int_{t_i}^{t_f} F(t) \cdot \eta(t)\, dt = 0$ for all $\eta \in C_c^\infty((t_i, t_f); \mathbb{R}^d)$, then $F(t) = 0$ for all $t \in (t_i, t_f)$.

4.2 Euler-Lagrange Derivation

`Derivation D2.1 (Euler-Lagrange equation).` Differentiate under the integral sign (justified by the smoothness assumptions). By the chain rule, $\frac{d}{d\varepsilon}\big|_0 \mathcal{L}(q + \varepsilon\eta, \dot{q} + \varepsilon\dot{\eta}, t) = \frac{\partial \mathcal{L}}{\partial q} \cdot \eta + \frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \dot{\eta}$. Therefore:

$$\delta S[q;\eta] = \int_{t_i}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial q} \cdot \eta + \frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \dot{\eta} \right) dt.$$

Integrating the second term by parts:

$$\delta S[q;\eta] = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \eta \right]_{t_i}^{t_f} + \int_{t_i}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \cdot \eta \, dt.$$

Endpoint or compact-support conditions remove the boundary term. By `Proposition P2.0`, stationarity for all admissible η implies:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0.$$

This is the Euler-Lagrange equation.

4.3 Rotational Symmetry and Angular Momentum

For planar central motion with

$$\mathcal{L}(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - V(r),$$

θ is cyclic ($\partial \mathcal{L} / \partial \theta = 0$). Applying Euler-Lagrange to θ gives:

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} = L_{\text{ang}} \quad \Rightarrow \quad \frac{dL_{\text{ang}}}{dt} = 0,$$

which is the rotational Noether conservation law [Noether1918].

In full vector form for $\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{m}{2}\|\dot{\mathbf{r}}\|^2 - V(\|\mathbf{r}\|)$, the canonical momentum is $\mathbf{p} = \partial\mathcal{L}/\partial\dot{\mathbf{r}} = m\dot{\mathbf{r}}$ and rotational invariance yields the conserved angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

`Proposition P2.1 (Geometric-variational invariant equivalence).` Under the regularity assumptions above, the Section 3 area-law invariant and the Noether charge are the same quantity in different language:

$$\dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{L_{\text{ang}}}{2m}.$$

Thus Section 3 and Section 4 do not provide competing derivations; they provide geometric and variational presentations of one conserved structure.

`Proposition P2.2 (Energy for autonomous central motion).` If $\mathcal{L}$ has no explicit time dependence, then the energy function

$$E = \dot{q} \cdot \frac{\partial\mathcal{L}}{\partial\dot{q}} - \mathcal{L}$$

is conserved (time-translation symmetry, another Noether law) [Noether1918]. For the central-motion Lagrangian above,

$$E = \frac{m}{2}\dot{r}^2 + \frac{L_{\text{ang}}^2}{2mr^2} + V(r),$$

showing the standard reduction to one-dimensional radial motion with effective potential $V_{\text{eff}}(r) = V(r) + L_{\text{ang}}^2/(2mr^2)$.

4.4 Additivity and Composition Pre-Bridge

Recall the discrete action functional from the refinement viewpoint:

$$S_N[q] = \sum_{k=0}^{N-1} \mathcal{L}\left(q_k, \frac{q_{k+1} - q_k}{\Delta t_k}, t_k\right) \Delta t_k.$$

It is additive under interval concatenation by construction. This additivity is the structural input used later for composition-based quantum weighting in Section 6.

`Heuristic H2.1 (Toward distributional probes).` Point-like probes of extrema can be expressed in distributional language; the idea appears in explicit form in [Rivero9803035]. In this manuscript, technical use of such probes is deferred to Section 5, where mollifier limits and admissibility are stated explicitly.

Transition to Section 5: with Euler-Lagrange and Noether structure fixed, we next extend stationarity analysis to weak/distributional settings and clarify where Dirac-supported constructions are valid.

5. Dirac Distributions and Extremal Action

5.1 Why Weak Formulations Appear Here

The story so far treated trajectories as classically smooth. Two themes force a more careful formulation: 1. Refinement limits often produce objects that are only piecewise smooth (corners) or are best handled by weak limits. 2. The "point-like probe" idea (Dirac-supported localization) is naturally stated in distribution theory.

We keep the role of distributions narrow and explicit: distributions are used as linear functionals on test functions and as limits of smooth approximations. Nonlinear operations on distributions are not assumed unless regularized.

5.2 Weak Euler-Lagrange Equation

Let $q \in C^1([t_i, t_f]; \mathbb{R}^d)$ be a candidate trajectory and assume $\mathcal{L}(q, \dot{q}, t)$ is smooth enough that $\partial_q \mathcal{L}$ and $\partial_{\dot{q}} \mathcal{L}$ are well-defined along q.

`Proposition P3.1 (Weak stationarity statement).` If $\delta S[q; \eta] = 0$ for all $\eta \in C_c^\infty((t_i, t_f); \mathbb{R}^d)$, then the Euler-Lagrange operator

$$F[q](t) \equiv \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}, t) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}, t)$$

vanishes as a distribution on (t_i, t_f): for all test η,

$$\int_{t_i}^{t_f} F[q](t) \cdot \eta(t)\, dt = 0.$$

Equivalently, $F[q] = 0$ in $\mathcal{D}'((t_i, t_f); \mathbb{R}^d)$, where $\mathcal{D}'$ is the dual of C_c^∞.

`Derivation D3.1 (Weak form from first variation).` Start from the first-variation identity (as in Section 4):

$$\delta S[q; \eta] = \int_{t_i}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial q} \cdot \eta + \frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \dot{\eta} \right) dt.$$

Integrate the second term by parts. Compact support eliminates the boundary term and yields the stated distributional identity.

5.3 Point-Like Probes via Mollifiers (Not Raw Deltas)

Pick a nonnegative mollifier $\rho \in C_c^\infty(\mathbb{R})$ with $\int \rho = 1$, and define $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$.

`Proposition P3.2 (Localized probing under continuity).` Assume $F[q](t)$ is continuous at a time $t_0 \in (t_i, t_f)$. Then weak stationarity implies the pointwise condition $F[q](t_0) = 0$.

`Derivation D3.2.` For any fixed vector $u \in \mathbb{R}^d$, choose a localized test function $\eta_\varepsilon(t) = \rho_\varepsilon(t - t_0)\, u$. Then the weak identity gives

$$0 = \int_{t_i}^{t_f} F[q](t) \cdot \rho_\varepsilon(t - t_0)\, u\, dt = u \cdot \int_{t_i}^{t_f} \rho_\varepsilon(t - t_0)\, F[q](t)\, dt.$$

As $\varepsilon \to 0^+$, the convolution integral tends to $F[q](t_0)$ by continuity. Since u was arbitrary, $F[q](t_0) = 0$.

This is the precise sense in which "Dirac-supported probes" recover pointwise Euler-Lagrange equations: they do so through mollifier limits, not by inserting nonlinear expressions involving $\delta(t - t_0)$.

5.4 Corners and Impulses: Jump Conditions

There are two distinct phenomena that look "singular" in time: 1. **Corners**: q is continuous but $\dot{q}$ has a jump at t_0, with no delta forcing. 2. **Impulses**: the dynamics includes a delta force at t_0, producing a momentum jump.

We record both conditions explicitly.

`Proposition P3.3 (Corner condition without impulse).` Assume q is piecewise C^2 with a velocity discontinuity at t_0, and satisfies the unforced Euler-Lagrange equation on each side of t_0. Then:

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0^-}^{t_0^+} = 0.$$

`Derivation D3.3 (Corner condition).` Integrate the unforced Euler-Lagrange equation on $[t_0 - \varepsilon, t_0 + \varepsilon]$:

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0-\varepsilon}^{t_0+\varepsilon} = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \frac{\partial \mathcal{L}}{\partial q}\, dt.$$

Let $\varepsilon \to 0^+$. Under local boundedness of $\partial_q \mathcal{L}$, the right-hand side vanishes, yielding the jump condition above. This is the local corner continuity of canonical momentum (Weierstrass-Erdmann form in this one-corner setting).

`Proposition P3.4 (Impulse force implies momentum jump).` Consider the forced Euler-Lagrange equation in distribution form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = J\,\delta(t - t_0),$$

for a fixed impulse vector $J \in \mathbb{R}^d$. If $\partial_{\dot{q}}\mathcal{L}$ has one-sided limits at t_0, then

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0^-}^{t_0^+} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}}(t_0^+) - \frac{\partial \mathcal{L}}{\partial \dot{q}}(t_0^-) = J.$$

`Derivation D3.4.` Integrate the equation on $[t_0 - \varepsilon, t_0 + \varepsilon]$. The integral of the smooth term $\partial_q \mathcal{L}$ tends to 0 as $\varepsilon \to 0$. The derivative term integrates to the jump in $\partial_{\dot{q}}\mathcal{L}$. The right-hand side integrates to J.

For the standard mechanical Lagrangian $\mathcal{L} = \frac{m}{2}\|\dot{q}\|^2 - V(q)$, this reduces to the familiar momentum jump:

$$m\big(\dot{q}(t_0^+) - \dot{q}(t_0^-)\big) = J.$$

This connects directly to the Section 3 impulse scaffold: central impulses preserve angular momentum because they change momentum only in the radial direction.

5.5 Extremal Measures: Finite-Dimensional Analogy and Limits

The phrase "Dirac distributions to calculate extrema" is unambiguous in finite dimensions. For a smooth $f : \mathbb{R} \to \mathbb{R}$, the distribution $\delta(f'(x))$ is supported on the critical points of f. In higher dimensions one analogously uses $\delta(\nabla f)$.

In infinite-dimensional settings (paths), one is tempted to write "formal measures" supported on stationary-action trajectories. A concrete version of this idea, and its relation to exponential weighting and consistency, is developed in [Rivero9803035]. In this manuscript we treat such expressions as roadmap heuristics until they are regularized and made compatible with composition (Section 6).

5.6 Caveats (Nonlinear Distribution Pitfalls)

1. Products like $\delta(t)^2$ are not defined in standard distribution theory; any appearance requires a regularization scheme and a proof of scheme-independence for claimed observables.
2. "Evaluate at a point" is only legitimate for quantities known to be continuous (or otherwise well-defined) at that point; mollifier probing must state this assumption explicitly.
3. Stationarity ($\delta S = 0$) is not the same as minimality; second variation and convexity conditions are separate and are not assumed here.

Transition to Section 6: we now have a controlled notion of "extremal classical dynamics" (including impulses and corners) and a precise language for refinement-local probes. The next section uses composition under time slicing to motivate amplitude weights and the path integral.

6. Composition and Path Integral

6.1 Composition Postulate for Amplitudes

Let $K(q_f, t_f; q_i, t_i)$ denote the transition amplitude. The structural postulate is composition on intermediate time slices:

$$K(q_f, t_f; q_i, t_i) = \int dq\, K(q_f, t_f; q, t)\, K(q, t; q_i, t_i), \qquad t_i < t < t_f.$$

`Derivation D4.1 (Time slicing from repeated composition).` Iterating the composition law over a partition $t_i = t_0 < \cdots < t_N = t_f$ gives

$$K(q_f, t_f; q_i, t_i) = \int \prod_{k=1}^{N-1} dq_k \prod_{k=0}^{N-1} K_\Delta(q_{k+1}, q_k; t_k),$$

with $q_0 = q_i,\ q_N = q_f,\ \Delta t_k = t_{k+1} - t_k$, and K_Δ the short-time kernel.

6.2 From Additive Action to Multiplicative Weights

The Section 4/Section 5 structure gives an additive discrete action:

$$S_N[q] = \sum_{k=0}^{N-1} \mathcal{L}\left(q_k, \frac{q_{k+1} - q_k}{\Delta t_k}, t_k\right) \Delta t_k.$$

Assume short-time locality: each slice contributes a factor depending only on local step data. Write

$$K_\Delta(q_{k+1}, q_k; t_k) = \mathcal{N}_k\, W_k.$$

`Proposition P4.1 (Exponential form under locality + composition).` If 1. total path weight is multiplicative across concatenated slices, and 2. $\log W_k$ is additive in Δt_k to first order,

then, up to normalization and higher-order slicing corrections,

$$\prod_{k=0}^{N-1} W_k \propto \exp\big(c_0\, S_N[q]\big),$$

for a constant c_0 with dimensions $[\text{action}]^{-1}$.

Choosing oscillatory quantum time evolution gives $c_0 = i/\hbar$, hence the standard phase factor $\exp(iS_N/\hbar)$ [Dirac1933] [Feynman1948].

6.3 Ordering, Discretization, and Quantum Ambiguity

Different short-time discretizations (left/right/midpoint or more general α-schemes) typically correspond to different operator orderings. In deformation language, this is the same ambiguity as choosing a star-product representative; these constructions agree in the classical limit but can differ at subleading quantum order [Rivero0302285] [Landsman1998].

`Heuristic H4.1 (Same classical limit, different quantum corrections).` Two discretizations that differ by $O(\Delta t)$ in each slice can produce equivalent classical equations while shifting $O(\hbar)$ terms in quantum generators. Thus ordering is a controlled modeling choice, not a contradiction.

6.4 Formal Continuum Limit and Stationary Phase

Formally, as mesh size $\max_k \Delta t_k \to 0$:

$$K(q_f, t_f; q_i, t_i) \sim \int_{q_i}^{q_f} \mathcal{D}q \, \exp\left(\frac{i}{\hbar} S[q]\right).$$

This expression is formal at this stage: we do not claim a countably additive measure construction on full path space.

`Derivation D4.2 (Classical recovery mechanism).` Let $q = q_{\rm cl} + \xi$, where $q_{\rm cl}$ is stationary: $\delta S[q_{\rm cl}; \eta] = 0$. Expand:

$$S[q_{\rm cl} + \xi] = S[q_{\rm cl}] + \frac{1}{2}\langle \xi, \mathcal{H}_{q_{\rm cl}} \xi \rangle + O(\xi^3).$$

Fast phase oscillations cancel nonstationary contributions, while neighborhoods of stationary paths contribute coherently. This is the precise sense in which the classical equations reappear as $\hbar \to 0$ asymptotics.

6.5 Link Back to Section 5 Singular Dynamics

The composition picture naturally includes piecewise-smooth trajectories. At impulses, the dominant classical skeleton must satisfy the jump laws from Section 5:

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0^-}^{t_0^+} = J \quad \text{(impulse)} \qquad \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0^-}^{t_0^+} = 0 \quad \text{(corner, unforced)}.$$

So the "extremal set" entering semiclassical evaluation is broader than globally smooth trajectories; it includes admissible broken trajectories obeying the correct matching conditions.

Transition to Section 7: with composition, weighting, and classical-recovery logic in place, we can now present quantization as deformation of algebraic products, linking path-integral discretization choices to tangent/cotangent groupoid deformation structure.

7. Deformation Quantization Bridge

7.1 From Path Weights to Product Deformation

Section 6 established that discretized composition introduces nonunique short-time prescriptions (left/right/midpoint and related schemes). The algebraic restatement is: quantization should deform the classical product of observables rather than replace classical mechanics by unrelated objects [Landsman1998] [Connes1994].

Let M be phase space with Poisson bracket $\{\cdot,\cdot\}$, and let $\mathcal{A}_0$ be a commutative algebra of classical observables (e.g., smooth functions with suitable decay/domain conditions). A deformation quantization is a family of associative products $\star_\hbar$ on $\mathcal{A}_0$ such that:

$$f \star_\hbar g = fg + \sum_{n\geq 1} \hbar^n B_n(f,g),$$

where B_n are bilinear operators, with $f \star_0 g = fg$.

`Proposition P5.1 (Classical compatibility conditions).` If $\star_\hbar$ is associative for each $\hbar$ and depends smoothly/formally on $\hbar$, then the antisymmetric part of B_1 controls the leading noncommutativity:

$$[f,g]_{\star_\hbar} \equiv f \star_\hbar g - g \star_\hbar f = \hbar\big(B_1(f,g) - B_1(g,f)\big) + O(\hbar^2).$$

So first-order noncommutativity is fully determined by B_1^{anti}.

7.2 Commutator-to-Poisson Recovery

`Derivation D5.1 (Correspondence limit).` Impose the correspondence requirement that first-order antisymmetry matches the Poisson bracket:

$$B_1(f,g) - B_1(g,f) = i\,\{f,g\}.$$

Then

$$[f,g]_{\star_\hbar} = i\hbar\,\{f,g\} + O(\hbar^2),$$

and therefore

$$\lim_{\hbar\to 0}\frac{1}{i\hbar}[f,g]_{\star_\hbar}=\{f,g\}.$$

Dimensional closure: $[\hbar]=[\text{action}]$, while $\{f,g\}$ carries one inverse action factor relative to fg in canonical coordinates, so $i\hbar\{f,g\}$ has the same physical dimension as fg. This is the same unit-consistency condition already used in Section 6 for $\exp(iS/\hbar)$.

7.3 Concrete Model and Ordering Content

For flat phase space, the Moyal product provides an explicit representative:

$$(f\star_M g)(q,p)=f(q,p)\exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial_q}\overrightarrow{\partial_p}-\overleftarrow{\partial_p}\overrightarrow{\partial_q}\right)\right]g(q,p),$$

which reproduces the Poisson bracket at leading order and higher quantum corrections at higher orders [Landsman1998].

`Heuristic H5.1 (Ordering as deformation gauge choice).` The Section 6 discretization ambiguity is naturally interpreted as choosing different but deformation-equivalent star products; they share the same classical bracket data but differ in $O(\hbar)$ and higher corrections [Rivero0302285].

7.4 Equivalence Classes and Groupoid Viewpoint

`Proposition P5.2 (Equivalent star products, same classical limit).` If two products $\star_\hbar$ and $\star'_\hbar$ are related by a formal automorphism

$$T_\hbar=\mathrm{id}+\hbar T_1+O(\hbar^2),\qquad f\star'_\hbar g=T_\hbar^{-1}\big((T_\hbar f)\star_\hbar(T_\hbar g)\big),$$

then they define the same Poisson bracket in the $\hbar\to 0$ limit, while generally differing in subleading quantum terms.

This is the algebraic side of the same continuity narrative: quantization data are organized into equivalence classes compatible with one classical limit. Geometric deformation programs (including tangent-groupoid viewpoints) encode the same bridge from commutative classical data to noncommutative quantum products [Connes1994] [Rivero9710026].

7.5 Formal Deformation Boundary

In this section we use formal/asymptotic deformation language for local bridge statements. We do not require the full C^*-algebraic deformation-quantization program for the manuscript's main argument; the needed ingredient is compatibility of the classical limit and quantum corrections under the stated assumptions [Landsman1998].

Transition to Section 8: with the deformation bridge in place, the remaining problem is not how to define first-order quantum corrections, but how to keep refined predictions finite and scale-consistent when naive limits diverge. That control problem is precisely the renormalization step.

8. Renormalization as Controlled Refinement

8.1 Why Renormalization Appears in Refinement Limits

The previous sections treated refinement as benign: polygonal refinement in Section 3, time-slicing in Section 6, and deformation parameter limits in Section 7. In quantum field theory and in several singular quantum-mechanical models (e.g. contact interactions), the same refinement step can instead *diverge* [ManuelTarrach1994PertRenQM] [BoyaRivero1994Contact]: as the cutoff scale is removed, intermediate quantities blow up even when low-energy physics is expected to remain finite.

Renormalization is the mechanism that restores the program's central thesis in the divergent case: it provides a controlled rule for taking refinement limits so that observables remain stable. Operationally, it accepts that intermediate expressions depend on a regulator (cutoff), but requires that properly defined observables do not.

To keep the discussion aligned with the paper's composition language, we treat renormalization as an invariance/consistency condition across composed refinement steps.

`Heuristic H6.1 (Renormalization as part of "what a theory is").` In benign refinement problems, one can often send the refinement parameter to zero without further choices. In divergent refinement problems, the renormalization prescription (what is regulated, what is held fixed, and how parameters are re-expressed as the cutoff moves) is not optional bookkeeping: it is part of the definition of the continuum theory, because it specifies which composed/refined predictions are declared physically stable.

8.2 Regulator, Bare Data, and Renormalized Observables

Let Λ denote a refinement cutoff (e.g., momentum cutoff $|k| < \Lambda$ or lattice spacing a with $\Lambda \sim 1/a$). Let $g_B(\Lambda)$ denote the cutoff-dependent *bare* parameters of the regulated theory (couplings, masses, field normalizations), and let O_Λ be a regulated prediction for some observable O.

`Proposition P6.1 (Renormalized observable as cutoff-stable limit).` If there exists a choice of cutoff-dependent bare parameters $g_B(\Lambda)$ such that the limit

$$O_{\text{phys}} \equiv \lim_{\Lambda\to\infty} O_\Lambda\big(g_B(\Lambda)\big)$$

exists and is finite (or has a controlled asymptotic expansion) for the observables of interest, then the refinement limit is *defined* by this prescription.

This statement is intentionally operational: it does not assume that the cutoff-free object exists without tuning. It states that "physical theory" means a stable target under refinement.

It is often convenient to introduce a renormalization scale μ (a reference resolution) and a renormalization map $R_{\Lambda\to\mu}$ from bare to renormalized parameters:

$$g_R(\mu) = R_{\Lambda\to\mu}\big(g_B(\Lambda)\big).$$

The composition viewpoint suggests a compatibility condition: renormalizing from Λ down to μ should be the same as renormalizing from Λ to an intermediate scale κ and then from κ to μ:

$$R_{\Lambda\to\mu} = R_{\kappa\to\mu} \circ R_{\Lambda\to\kappa}.$$

This is the renormalization-group (RG) semigroup property in refinement language.

8.3 RG Equation from Cutoff Independence

`Derivation D6.1 (RG equation as consistency under refinement).` Assume a regulated observable depends on the cutoff Λ both explicitly and through the bare parameters $g_B(\Lambda)$:

$$O_{\text{phys}} = O_\Lambda\big(g_B(\Lambda)\big),$$

and impose cutoff-independence of the physical prediction:

$$\frac{d}{d\ln\Lambda} O_\Lambda\big(g_B(\Lambda)\big) = 0.$$

By the chain rule,

$$0 = \frac{\partial O_\Lambda}{\partial \ln\Lambda} + \sum_a \frac{dg_B^a}{d\ln\Lambda}\frac{\partial O_\Lambda}{\partial g_B^a},$$

where a ranges over the components of the parameter vector. Defining the beta functions $\beta_B^a(g_B) \equiv \frac{dg_B^a}{d\ln\Lambda}$, we obtain the RG equation:

$$\left(\frac{\partial}{\partial \ln \Lambda} + \sum_a \beta_B^a(g_B)\frac{\partial}{\partial g_B^a}\right) O_\Lambda(g_B) = 0.$$

In the μ-parametrized form with renormalized parameters $g_R(\mu)$, the same reasoning yields a flow equation $\mu \frac{d}{d\mu} g_R(\mu) = \beta(g_R)$ plus corresponding RG invariance equations for renormalized observables. The key point for this manuscript is structural: RG is not extra physics added after quantization; it is the *consistency condition* that makes composed refinement meaningful when naive limits diverge.

`Proposition P6.2 (Flow generator from refinement semigroup).` Let $W_{\Lambda\to\kappa}$ be the map sending effective parameters at scale Λ to effective parameters at scale $\kappa < \Lambda$, and assume: 1. $W_{\Lambda\to\Lambda} = \mathrm{id}$, 2. $W_{\kappa\to\mu} \circ W_{\Lambda\to\kappa} = W_{\Lambda\to\mu}$, 3. differentiability with respect to $\ln \Lambda$.

Then the infinitesimal generator defines beta functions:

$$\beta^a(g) = \frac{d}{dt}\, W^a_{e^t\mu\to\mu}(g)\bigg|_{t=0},$$

and finite scale changes are recovered by integrating this vector field on parameter space. So RG flow is the differential form of composed refinement.

`Derivation D6.2 (Toy logarithmic divergence and subtraction).` Consider a single-coupling situation with a logarithmic cutoff dependence in a regulated prediction:

$$O_\Lambda(g_B;\mu) = g_B + c\, g_B^2 \ln\left(\frac{\Lambda}{\mu}\right) + O(g_B^3),$$

where c is a dimensionless constant determined by the model and by the chosen renormalization convention. Define the renormalized coupling at scale μ by a renormalization condition $g_R(\mu) \equiv O_\Lambda(g_B(\Lambda);\mu)$ which is held fixed as $\Lambda \to \infty$. Cutoff-independence then enforces:

$$0 = \frac{d}{d\ln\Lambda} g_R(\mu) = \frac{dg_B}{d\ln\Lambda} + c\, g_B^2 + O(g_B^3),$$

so $\beta_B(g_B) \equiv \frac{dg_B}{d\ln\Lambda} = -c\, g_B^2 + O(g_B^3)$. Equivalently, at fixed bare coupling one finds the running with μ:

$$\beta(g_R) \equiv \mu \frac{dg_R}{d\mu} = -c\, g_R^2 + O(g_R^3),$$

illustrating how renormalization turns the divergent $\ln \Lambda$ refinement effect into a scale-dependent coupling consistent with stable observables.

8.4 Refinement Control Before QFT: Scale-Halving as a Model

One can see the same logic in purely classical numerical refinement. Consider an evolution operator Φ_ε representing "one step" at resolution ε. Composition gives $\Phi_{2\varepsilon} \approx \Phi_\varepsilon \circ \Phi_\varepsilon$. A refinement-control question is then: what correction to Φ_ε makes the two-step composition agree with a one-step method after rescaling back to the same reference resolution?

`Heuristic H6.2 (Rooted trees as refinement bookkeeping).` In Runge-Kutta and related integrators, the comparison between composed steps and a single step organizes into rooted-tree expansions; the corresponding composition law forms a group (the Butcher group). Rivero observes that "scale-halving then rescaling back" is an RG-like operation on method coefficients, and the same rooted-tree/Hopf-algebra combinatorics appears in perturbative renormalization bookkeeping [RiveroOde2002] [Brouder1999] [McLachlan2017] [ConnesKreimer2000].

This example is included not to replace QFT renormalization, but to reinforce the paper's thesis with a clean model: renormalization is what you do when "refine and compare" is not automatically stable. The Butcher *group* concerns formal method composition, whereas Wilsonian coarse-graining is generally a *semigroup* because information is discarded at each coarse-graining step.

8.5 Counterterms as Refinement Corrections

In field theory language, refinement is implemented by a regulated action S_Λ with cutoff-dependent parameters. Schematically,

$$S_\Lambda[\phi] = \int d^d x \left(\frac{Z(\Lambda)}{2}(\partial\phi)^2 + \frac{m^2(\Lambda)}{2}\phi^2 + \frac{\lambda(\Lambda)}{4!}\phi^4 + \cdots \right),$$

where the "$\cdots$" stands for additional operators allowed by symmetries and by the desired accuracy. The counterterm viewpoint is simply the statement that $Z, m^2, \lambda, \ldots$ must be chosen as functions of Λ so that the cutoff-stable limits of observables exist. In this compositional narrative, counterterms are refinement corrections required to keep the "same theory" after integrating out short scales.

`Derivation D6.3 (Difference quotient as counterterm subtraction).` Let $f \in C^1$ and $\varepsilon \to 0^+$. The two regulated quantities $f(x+\varepsilon)/\varepsilon$ and $f(x)/\varepsilon$ each diverge like $1/\varepsilon$. Subtracting the local counterterm $f(x)/\varepsilon$ produces a finite limit:

$$\frac{f(x+\varepsilon)}{\varepsilon} - \frac{f(x)}{\varepsilon} = \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \xrightarrow{\varepsilon\to 0} f'(x).$$

This is a minimal model of renormalization: divergent regulated expressions become finite after subtracting a divergence that depends only on local data, and the renormalized quantity is the cutoff-stable remainder. (This "derivative as renormalized limit" viewpoint is stated explicitly in [RiveroOde2002].)

The Connes-Kreimer framework makes this consistency structural by encoding perturbative renormalization as a factorization problem with a Hopf-algebra organization of divergences [ConnesKreimer2000]. For this manuscript, the take-away is not the full machinery but the alignment: renormalization is a principled method for producing regulator-independent predictions from composable local pieces when refinement alone does not converge.

8.6 Assumptions and Boundaries Audit

`Proposition P6.3 (Closure assumption for finite-parameter flow).` Finite-dimensional beta-function systems are closed only after choosing a truncation/ansatz for allowed operators (or a complete effective basis). If new operators are generated by refinement and omitted from the parameter vector, the reduced flow is only approximate.

This caveat is essential for interpreting section 8 correctly: the RG equations written here are exact at the level of the chosen variable set, but practical truncations can make them approximate. The main thesis is unaffected: renormalization remains the rule that restores cross-scale consistency under composed refinement.

Transition to Section 9: we now have the full chain Newtonian refinement $\to$ action additivity $\to$ path-integral composition $\to$ deformation compatibility $\to$ renormalized refinement control. The final synthesis section stress-tests the transitions, labels what remains heuristic, and consolidates the manuscript into a single coherent argument.

9. Unified Perspective and Open Problems

9.1 End-to-End Claim Graph

The manuscript has built one chain across seven technical steps: 1. Section 3: refinement geometry in central-force motion yields an exact finite-step invariant (equal areas / angular momentum preservation). 2. Section 4: action stationarity and Noether symmetry express the same invariant in variational language. 3. Section 5: weak/distributional formulation extends stationarity to singular limits (mollifiers, corners, impulses) with explicit admissibility boundaries. 4. Section 6: composition under slicing plus additive action yields exponential weighting and stationary-phase classical recovery. 5. Section 7: ordering ambiguity is recast as deformation-equivalence data with a shared Poisson classical limit. 6. Section 8: divergent refinement is controlled by renormalization maps and RG semigroup consistency.

The unifying thesis is therefore not "classical then quantum then QFT" as disconnected layers, but "one refinement/composition problem under progressively stricter consistency requirements."

`Proposition P7.1 (Compatibility chain of limits).` Under the regularity and admissibility assumptions stated in sections 3–8, the following compatibility conditions can be imposed simultaneously:

$$\delta S[q;\eta] = 0 \iff \text{Euler-Lagrange in weak form},$$

$$K \sim \int \mathcal{D}q\, e^{iS[q]/\hbar} \implies \hbar \to 0 \text{ concentrates on } \delta S = 0,$$

$$\lim_{\hbar\to 0} \frac{1}{i\hbar}[f,g]_{\star_\hbar} = \{f,g\}, \qquad (\partial_{\ln\Lambda} + \beta\cdot\partial_g)\, O_\Lambda = 0.$$

These equations are not identical statements; they are compatibility constraints on one staged construction: classical extremals, quantum composition, algebraic deformation, and scale consistency must match in their overlap domains.

9.2 Transition Stress Test

`Derivation D7.1 (No hidden leap audit across transitions).` The manuscript can be stress-tested by checking each transition against one explicit closure condition:

1. `Section 3 -> Section 4` closure: finite-step angular momentum invariance and variational Noether charge agree through $\dot{A} = \frac{L_{\text{ang}}}{2m}$. This closes the geometry-to-variational bridge.

2. `Section 4 -> Section 5` closure: Euler-Lagrange equations in classical smooth form imply weak distributional form when tested against C_c^∞, and mollifier localization recovers pointwise equations under continuity assumptions. This closes smooth-to-weak extension.

3. `Section 5 -> Section 6` closure: the admissible classical set in semiclassics includes smooth and piecewise-smooth trajectories satisfying matching laws $[\partial_{\dot{q}}\mathcal{L}]_-^+ = 0$ (corner) or $= J$ (impulse). This closes singular dynamics into composition.

4. `Section 6 -> Section 7` closure: discretization/ordering freedom in short-time kernels maps to star-product representatives that share the same Poisson boundary at $\hbar \to 0$. This closes path-integral ambiguity into deformation language.

5. `Section 7 -> Section 8` closure: deformation handles classical/quantum compatibility at fixed scale; renormalization handles cross-scale consistency when refinement diverges. This closes fixed-scale quantization into multi-scale consistency.

The audit criterion is simple: every bridge states its assumptions and carries an explicit invariant or equation through the transition. Where this fails, the claim is downgraded to heuristic.

9.3 What Is Proven vs Heuristic

The section-level status map requested by Section 9 is:

1. Section 3: polygonal central-impulse refinement preserves angular momentum and equal areas at finite step. Status: `Proposition` + `Derivation`. Main boundary: central impulses and consistent refinement limit.

2. Section 4: Euler-Lagrange plus Noether recover angular-momentum and energy invariants. Status: `Derivation` + `Proposition`. Main boundary: C^2 trajectory regularity and standard variational hypotheses.

3. Section 5: weak Euler-Lagrange, mollifier probes, and jump laws for corners/impulses. Status: `Proposition` + `Derivation`. Main boundary: distribution linearity and no undefined nonlinear products.

4. Section 6: composition plus additivity imply exponential weighting; stationary phase yields classical recovery. Status: mixed (`Proposition` + `Derivation` + `Heuristic`). Main boundary: formal path-integral usage and local stationary-phase assumptions.

5. Section 7: deformation products recover the Poisson bracket; ordering appears as deformation-equivalence choice. Status: mixed (`Proposition` + `Derivation` + `Heuristic`). Main boundary: formal/asymptotic deformation setting and scope of equivalence.

6. Section 8: RG appears as semigroup consistency under composed refinement; counterterms appear as refinement corrections. Status: mixed (`Proposition` + `Derivation` + `Heuristic`). Main boundary: closure/truncation assumptions and regulator-scheme choice.

`Heuristic H7.1 (Programmatic interpretation).` The Newton-to-path-integral narrative is best read as a *compatibility program* rather than a single theorem: each layer adds new consistency constraints while preserving prior invariants in its domain of validity.

9.4 Residual Vulnerabilities

1. The path integral remains formal at full measure-theoretic level; constructive control is not provided here.
2. Deformation equivalence is stated at the structural level; explicit model-by-model operator-domain analysis is deferred.
3. RG flow is derived structurally; no full one-loop or higher-loop field-theory computation is included in the main text.
4. Truncation closure in section 8 is identified but not benchmarked by an explicit truncation-error study.

These are not hidden defects; they are explicit scope boundaries. The manuscript now separates proven derivations from heuristic bridges, which was a core objective of the staged design.

9.5 Next-Paper Agenda

The remaining extensions beyond the continuation appendices are: 1. Add a fully model-specific loop computation (beyond the template-level calculation in section `10.1`). 2. Add a short dimensional-aside appendix (from `[RiveroSimple]`) clearly labeled as heuristic/exploratory. 3. Add one diagram-level summary figure of the compatibility chain for publication readability. 4. Add a final notation-normalization pass for submission formatting.

Roadmap status: with Section 9, the planned section sequence through Section 9 is complete.

10. Technical Appendices (Continuation Stages)

This section executes the continuation appendices announced at the end of Section 9. Each subsection is a compact worked extension tied to one residual vulnerability.

10.1 Worked Renormalization Template (Single Coupling)

The objective is to replace purely structural RG language with one explicit subtraction-and-running calculation.

Assume a regulated quantity has the perturbative form

$$F_\Lambda(g_B;\mu) = g_B + c\, g_B^2 \ln\left(\frac{\Lambda}{\mu}\right) + d\, g_B^2 + O(g_B^3),$$

with dimensionless constants c, d. Define a renormalized coupling by a subtraction condition at scale μ:

$$g_R(\mu) \equiv g_B + c\, g_B^2 \ln\left(\frac{\Lambda}{\mu}\right).$$

`Derivation D8.1 (Finite renormalized prediction at fixed subtraction scale).` Invert the definition to second order:

$$g_B = g_R - c\, g_R^2 \ln\left(\frac{\Lambda}{\mu}\right) + O(g_R^3).$$

Substitute into F_Λ:

$$F_\Lambda = g_R + d\, g_R^2 + O(g_R^3),$$

so the explicit logarithmic cutoff dependence cancels at this order. This is the concrete implementation of the Section 8 rule: tune bare data so predictions at reference scale remain stable.

`Derivation D8.2 (Running from cutoff-independence).` At fixed bare coupling g_B, differentiating the renormalization condition gives

$$\mu \frac{dg_R}{d\mu} = -c\, g_R^2 + O(g_R^3) \equiv \beta(g_R).$$

This turns divergent cutoff dependence into controlled scale dependence.

`Proposition P8.1 (Leading beta coefficient under analytic scheme change).` For a reparametrization $g'_R = g_R + a\, g_R^2 + O(g_R^3)$, the leading coefficient of β is unchanged:

$$\beta'(g'_R) = -c\, {g'_R}^2 + O({g'_R}^3).$$

So scheme changes shift higher-order terms while preserving the first nontrivial flow coefficient in this template.

10.2 Ordering/Discretization Pair with Same Classical Limit

This appendix gives an explicit example of the Section 6/Section 7 claim that discretization choice changes $O(\hbar)$ terms while preserving classical dynamics.

Take the classical symbol $A(q,p) = f(q)p$, with smooth f. Consider two quantizations: 1. Left ordering: $Q_L(A) = f(\hat{q})\hat{p}$. 2. Weyl (symmetric) ordering: $Q_W(A) = \frac{1}{2}\left(f(\hat{q})\hat{p} + \hat{p}f(\hat{q})\right)$.

Using $[\hat{p}, f(\hat{q})] = -i\hbar f'(\hat{q})$:

$$Q_W(A) = f(\hat{q})\hat{p} - \frac{i\hbar}{2} f'(\hat{q}) = Q_L(A) - \frac{i\hbar}{2} f'(\hat{q}).$$

`Derivation D9.1 (Classical agreement, quantum shift).` The difference operator is $O(\hbar)$:

$$Q_W(A) - Q_L(A) = -\frac{i\hbar}{2} f'(\hat{q}).$$

Therefore

$$\lim_{\hbar \to 0} \big(Q_W(A) - Q_L(A)\big) = 0$$

in the formal classical limit, while quantum generators differ at subleading order.

`Proposition P9.1 (Discretization-ordering equivalence class statement).` If two short-time kernel prescriptions map to Q_L-type and Q_W-type representatives of the same classical symbol algebra, then they define the same classical equations and differ only by controlled $O(\hbar)$ corrections. This is the worked version of the Section 6 to Section 7 transition claim [Rivero0302285] [Landsman1998].

10.3 Foundational Compatibility Principle

This appendix replaces the earlier continuation draft with a foundational concept aimed at the manuscript's core objective.

`Proposition P10.1 (Refinement Compatibility Principle, RCP).` A dynamical framework is admissible when three compatibility conditions hold simultaneously: 1. **Partition compatibility**: composition across temporal subdivisions preserves the same action-based extremal equations in the refinement limit. 2. **Representation compatibility**: alternative quantum representations (ordering/discretization choices) agree in the classical limit and differ only by controlled subleading corrections. 3. **Scale compatibility**: observable predictions remain stable under composed coarse/fine scale changes after parameter flow adjustment.

In compact form, for any prediction map $\mathcal{O}$,

$$\mathcal{O} = \mathcal{O} \circ \mathcal{C}_t = \mathcal{O} \circ \mathcal{Q}_\hbar = \mathcal{O} \circ \mathcal{R}_\Lambda,$$

where $\mathcal{C}_t$ is temporal composition/refinement, $\mathcal{Q}_\hbar$ is representation change within a fixed classical-limit class, and $\mathcal{R}_\Lambda$ is scale-refinement/renormalization flow.

`Derivation D10.1 (Bridge to sections 3--8).` 1. Section 3/Section 4 provide partition compatibility via area-law and Noether-action equivalence. 2. Section 6/Section 7 provide representation compatibility via ordering/deformation equivalence with identical $\hbar \to 0$ limit. 3. Section 8 provides scale compatibility via RG semigroup consistency.

Therefore the Newton-to-path-integral narrative is an implementation of RCP rather than a sequence of disconnected formalisms.

`Heuristic H10.1 (Foundational reading).` RCP can be interpreted as a candidate foundational postulate: physical laws are those statements that survive controlled changes of partition, representation, and scale.

10.4 Continuation Outcome

Continuation appendices in sections 10.1–10.3 close the three technical gaps identified in Section 9: 1. explicit renormalization subtraction and running, 2.

explicit ordering/discretization $O(\hbar)$ shift with fixed classical limit, 3. explicit foundational compatibility principle unifying the full chain.

These additions do not alter the thesis; they increase computational accountability of the existing chain. Operational note: four bibliography items are intentionally deferred as `PENDING` for later local-PDF ingestion (`Dirac1933`, `Feynman1948`, `Connes1994`, `Landsman1998`).

10.5 Singular Contact Interaction as an Explicit RG Computation (2D Delta)

Section 8 argues that RG is the scale-compatibility condition required when refinement limits diverge. This appendix supplies a fully explicit example in a singular quantum-mechanical model where the continuum theory is defined only after a renormalization prescription is chosen. For a perturbative-QFT-style treatment of this mechanism in quantum mechanics (including the 2D delta interaction), see [ManuelTarrach1994PertRenQM].

Consider the two-dimensional contact interaction

$$H = -\frac{\hbar^2}{2m}\Delta + g\,\delta^{(2)}(x) \quad \text{on } \mathbb{R}^2.$$

The interaction is Dirac-supported and the naive continuum limit is ill-defined: loop integrals diverge logarithmically.

`Derivation D11.1 (Cutoff evaluation of the contact loop).` Let $E > 0$ and write $E = \hbar^2 k^2/(2m)$. The Lippmann–Schwinger equation yields an algebraic T-matrix

$$T(E;\Lambda) = \frac{1}{g_B(\Lambda)^{-1} - I(E;\Lambda)},$$

where the loop integral is the free resolvent at the origin with a wavevector cutoff $|q| < \Lambda$:

$$I(E;\Lambda) = \int_{|q|<\Lambda} \frac{d^2q}{(2\pi)^2}\,\frac{1}{E - \frac{\hbar^2 q^2}{2m} + i0} = -\frac{m}{2\pi\hbar^2}\left[\ln\left(\frac{\Lambda^2}{k^2}\right) + i\pi\right] + O\left(\frac{k^2}{\Lambda^2}\right).$$

Thus the regulated theory contains a logarithmic divergence $\sim -\frac{m}{2\pi\hbar^2}\ln\Lambda^2$.

`Derivation D11.2 (Renormalized coupling and beta function).` Define a renormalized coupling at subtraction scale μ by

$$\frac{1}{g_R(\mu)} \equiv \frac{1}{g_B(\Lambda)} + \frac{m}{2\pi\hbar^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right).$$

Substituting into $T(E;\Lambda)$ cancels the explicit cutoff dependence and gives a finite amplitude:

$$T(E) = \frac{1}{\frac{1}{g_R(\mu)} + \frac{m}{2\pi\hbar^2}\ln\left(\frac{\mu^2}{k^2}\right) + i\,\frac{m}{2\hbar^2}}.$$

Since μ is arbitrary, physical predictions must satisfy $dT/d\ln\mu = 0$. This yields the RG equation

$$\mu\frac{d}{d\mu}\left(\frac{1}{g_R(\mu)}\right) = -\frac{m}{\pi\hbar^2}, \qquad \beta(g_R) \equiv \mu\frac{dg_R}{d\mu} = \frac{m}{\pi\hbar^2}\,g_R^2.$$

This is the explicit "scale-compatibility vector field" promised by Section 8, obtained from the demand that the subtraction scale not affect the composed prediction.

`Proposition P11.1 (Dimensional transmutation: an RG-invariant bound-state scale).` For $E < 0$, write $E = -\hbar^2\kappa^2/(2m)$. The bound state corresponds to a pole of T, which occurs when

$$\frac{1}{g_R(\mu)} + \frac{m}{2\pi\hbar^2}\ln\left(\frac{\mu^2}{\kappa^2}\right) = 0.$$

Define

$$\kappa_*^2 \equiv \mu^2\exp\left(\frac{2\pi\hbar^2}{m}\frac{1}{g_R(\mu)}\right).$$

Using the RG equation for $1/g_R(\mu)$, one checks $d\kappa_*/d\mu = 0$. Thus the renormalized delta interaction trades the regulator-dependent coupling for a physical scale κ_* (equivalently a bound-state energy $E_B = \hbar^2\kappa_*^2/(2m)$).

`Derivation D11.3 (Scheme dependence as rescaling of the transmutation scale).` The subtraction defining $g_R(\mu)$ is not unique: one may shift it by a finite constant C by defining

$$\frac{1}{g_R^{(C)}(\mu)} \equiv \frac{1}{g_R(\mu)} + \frac{m}{2\pi\hbar^2}C.$$

Differentiation in $\ln\mu$ removes the constant, so the beta function is unchanged. However, the RG-invariant scale rescales:

$$\kappa_*^{(C)\,2} \equiv \mu^2\exp\left(\frac{2\pi\hbar^2}{m}\frac{1}{g_R^{(C)}(\mu)}\right) = e^C\,\kappa_*^2.$$

Thus, in this one-scale model, "scheme dependence" is precisely the freedom to rescale the single physical scale. Fixing one physical datum (e.g. E_B) fixes κ_* and removes the ambiguity from predictions.