

Half-Densities in QFT: Propagators as Bi-Half-Density Kernels

Abstract

In QFT, the basic free object is the inverse of a kinetic operator, i.e. a propagator/Green kernel. On a manifold, writing “ $P_x G(x, y) = \delta(x, y)$ ” hides conventions: which volume form defines the adjoint, and which delta normalization realizes the identity. This note adopts an organizing choice consistent with the repo’s main paper: treat fields (or kernels) as **half-densities**, so the identity kernel is canonical and kernel composition is coordinate-invariant without choosing a background measure. A worked computation shows how a scalar field on (M, g) becomes a half-density $\psi = |g|^{1/4} \phi$, with kinetic operator $\tilde{P} = |g|^{1/4} P |g|^{-1/4}$ symmetric in the coordinate pairing. We also record a kernel-level remark: local counterterms/contact terms appear as distributions supported on the diagonal $(x = y)$ (delta kernels and their derivatives), which are most naturally expressed using the canonical bi-half-density delta.

This note is a dependent follow-up to the main manuscript and relates to companion notes on scalarization scales (the Planck-area program) and on RG as compatibility.

1. Purpose and Scope

This note is intentionally narrow: 1. establish the “kernel as bi-half-density” semantics for spacetime propagators in QFT, 2. isolate what is **canonical** (half-density kernels, identity delta kernel) versus what is **a convention** (scalarization choices such as $\sqrt{|g|}$), 3. give one explicit computation that can later be promoted (densitized scalar field).

BV/BRST/field-space half-densities are only flagged as outlook here; a full treatment would require additional dedicated sources and is beyond scope.

2. Kernels on a Manifold: half-densities make the identity canonical

Let M be a D -dimensional manifold. A half-density is a section of $|\Lambda^D T^* M|^{1/2}$.

The key operational point (as in the main paper’s kernel-composition spine) is: if an operator acts on half-densities, then its Schwartz kernel is naturally a **bi-half-density**

$$K_A(x, y) = a(x, y) |dx|^{1/2} |dy|^{1/2},$$

and composition is intrinsic:

$$(A \circ B)(x, z) = \int_M K_A(x, y) K_B(y, z),$$

because the product in the intermediate variable y is a density.

Derivation D1.1 (Identity kernel). The identity operator on half-densities has kernel

$$K_{\text{Id}}(x, y) = \delta^{(D)}(x - y) |dx|^{1/2} |dy|^{1/2},$$

which is canonical: it does not require choosing a background density/volume form.

This is the same object used repeatedly in the project’s half-density scaling thread: the diagonal delta kernel is the universal distributional witness of the $\varepsilon^{-D/2}$ exponent (see the companion Planck-area note, Derivation D1.2b).

3. Worked computation: densitized scalar field

$$\psi = |g|^{1/4} \phi$$

Consider a real scalar field on a fixed Lorentzian/Euclidean background (M, g) with quadratic action

$$S[\phi] = \frac{1}{2} \int_M d^D x \sqrt{|g|} \phi P \phi, \quad P = -\nabla^2 + m^2 + \xi R \quad (\text{example}).$$

The pairing for which P is symmetric is

$$(\phi_1, \phi_2)_g = \int d^D x \sqrt{|g|} \phi_1 \phi_2.$$

Define the densitized field (a half-density in coordinates)

$$\psi := |g|^{1/4} \phi, \quad \text{so} \quad \phi = |g|^{-1/4} \psi.$$

Then the action becomes

$$S[\phi] = \frac{1}{2} \int d^D x \psi \tilde{P} \psi, \quad \tilde{P} := |g|^{1/4} P |g|^{-1/4},$$

so the pairing is now just the coordinate density $d^D x$.

Derivation D1.2 (Explicit form of the conjugated kinetic operator). For $P_{\text{kin}} = -\nabla^2$ one has in coordinates

$$\nabla^2 \phi = |g|^{-1/2} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi \right),$$

hence

$$\tilde{P}_{\text{kin}} = -|g|^{1/4} \nabla^2 |g|^{-1/4} = -|g|^{-1/4} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu (|g|^{-1/4} \cdot) \right).$$

Assuming compact support (or boundary conditions killing boundary terms),

$$\int d^D x \psi_1 \tilde{P}_{\text{kin}} \psi_2 = \int d^D x \sqrt{|g|} g^{\mu\nu} \partial_\mu (|g|^{-1/4} \psi_1) \partial_\nu (|g|^{-1/4} \psi_2),$$

which is manifestly symmetric under $(1 \leftrightarrow 2)$.

Derivation D1.3 (Conformal metric expansion; D=4 simplification in the conformal class). As a worked expansion, take a conformally flat background $g_{\mu\nu} = e^{2\sigma(x)} \delta_{\mu\nu}$ (Euclidean for simplicity). Then $\sqrt{|g|} = e^{D\sigma}$, $|g|^{1/4} = e^{D\sigma/2}$, $g^{\mu\nu} = e^{-2\sigma} \delta^{\mu\nu}$, and one finds

$$\tilde{\Delta} \psi := |g|^{1/4} \Delta_g |g|^{-1/4} \psi = e^{-2\sigma} \left(\partial^2 \psi - 2 \partial \sigma \cdot \partial \psi - \frac{D}{2} (\partial^2 \sigma) \psi + \frac{D(4-D)}{4} (\partial \sigma)^2 \psi \right),$$

so the kinetic operator $\tilde{P}_{\text{kin}} = -\tilde{\Delta}$ contains a term proportional to $D(4-D)(\partial \sigma)^2$, which cancels at $D = 4$ in this conformal ansatz. This is recorded only as a concrete simplification in a natural metric class; its broader meaning (if any) is a separate question. A symbolic coefficient/sign check (SymPy) confirms this expansion.

Interpretation: - the metric half-density $|g|^{1/4} |dx|^{1/2}$ is a **scalarization gauge** (a choice of reference half-density) on a fixed background, - writing the field as ψ makes the “half-density priority” viewpoint explicit: both the field and the kernels live naturally as half-density objects.

4. Propagators/Green functions as bi-half-density kernels

Let \tilde{G} be the inverse kernel of \tilde{P} with respect to the coordinate pairing:

$$(\tilde{P}^{-1} f)(x) = \int \tilde{G}(x, y) f(y) d^D y, \quad \tilde{P}_x \tilde{G}(x, y) = \delta^{(D)}(x - y).$$

Then the corresponding canonical bi-half-density kernel is

$$K_{\tilde{P}^{-1}}(x, y) = \tilde{G}(x, y) |dx|^{1/2} |dy|^{1/2}.$$

Equivalently, if $G_g(x, y)$ denotes the usual **scalar** Green function for P defined with respect to the metric pairing $\int \sqrt{|g|} d^D y$ (so $(P^{-1} J)(x) = \int G_g(x, y) J(y) \sqrt{|g(y)|} d^D y$), then the kernels are related by

$$\tilde{G}(x, y) = |g(x)|^{1/4} G_g(x, y) |g(y)|^{1/4}.$$

This is exactly the same “kernel as bi-half-density” structure used for QM propagators in the main manuscript, now applied to spacetime Green functions in QFT.

Remark D4.1 (Doubling: densities live on $M \times M$). Half-density kernels also make the amplitude-vs-density doubling completely explicit. Let U_t be a (unitary) evolution operator on half-densities with kernel $K_t(x, y)$. Then a density operator $\rho_t = U_t \rho_0 U_t^{-1}$ has a kernel satisfying

$$K_{\rho_t}(x, y) = \int_{M \times M} K_t(x, x') K_{\rho_0}(x', y') \overline{K_t(y, y')}.$$

So densities naturally propagate by a kernel on the doubled space $M \times M$, built from the forward kernel and its conjugate. This is the kernel-level origin of bra/ket (forward/backward) doubling in real expectation values; a fuller discussion is beyond this note’s scope.

5. Contact terms and counterterms as diagonal delta kernels (kernel-level remark)

In QFT, divergences are removed by adding local counterterms to the action, e.g. $\delta m^2 \phi^2$, $\delta Z (\partial\phi)^2$, curvature couplings, etc. At the operator/kernel level this means: the kinetic operator P is modified by local (differential) operators, and therefore its kernel acquires **distributions supported on the diagonal** $x = y$.

In the half-density kernel language the diagonal object is canonical:

$$K_{\text{Id}}(x, y) = \delta^{(D)}(x - y) |dx|^{1/2} |dy|^{1/2}.$$

Multiplication counterterms correspond to $c(x) K_{\text{Id}}(x, y)$, and derivative counterterms correspond to derivatives acting on the delta kernel (e.g. $\partial_x \delta^{(D)}(x - y)$ and higher).

Remark D5.1 (Derivative of the diagonal delta kernel; coordinate-free identity). The slogan “ $\partial_x \delta(x - y) = -\partial_y \delta(x - y)$ ” has a clean, connection-free formulation in the half-density kernel calculus. The identity kernel K_{Id} is invariant under diagonal diffeomorphisms $(\Phi \times \Phi)$, so for any vector field V on M one has

$$(\mathcal{L}_{V_x} + \mathcal{L}_{V_y}) K_{\text{Id}} = 0, \quad \text{hence} \quad \mathcal{L}_{V_x} K_{\text{Id}} = -\mathcal{L}_{V_y} K_{\text{Id}},$$

where \mathcal{L} is the Lie derivative acting on half-densities. In local coordinates, taking $V = \partial/\partial x^\mu$ recovers $\partial_{x^\mu} \delta^{(D)}(x - y) = -\partial_{y^\mu} \delta^{(D)}(x - y)$. This is the kernel-level mechanism behind “moving derivatives between slots” in integration by parts and in contact-term identities.

Remark D5.2 (Point splitting produces \u03b4' contact terms).
Point splitting makes the simplest derivative contact term explicit: in one dimension,

$$\frac{\delta(x + \varepsilon) - \delta(x)}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \delta'(x), \quad \langle \delta', \varphi \rangle = -\varphi'(0).$$

This limit is the distributional companion of the “difference quotient as divergence + subtraction” toy model.

This framing is useful for two reasons: 1. it makes “locality = diagonal support” literal at the kernel level, 2. it separates the canonical distributional objects from scheme-dependent scalarizations and finite-subtraction conventions.

6. Link to the half-density scale program (where Planck-area enters conditionally)

On a fixed background (M, g) , the metric provides a natural reference half-density $|g|^{1/4}|dx|^{1/2}$. The Planck-area program begins only when we ask for a **background-free** scalarization convention that turns half-density amplitudes into universal dimensionless numbers. That stronger hypothesis ladder is developed in the companion Planck-area note.

This paper’s role is only to show that half-densities are not a QM quirk: the same kernel semantics is already present in standard QFT propagator definitions, once the hidden measure conventions are made explicit.

7. Outlook: BV half-densities

Gauge theories suggest a second, deeper appearance of half-densities: the BV formalism treats the integrand as a (half-)density on an (odd) symplectic space of fields/antifields, and the quantum master equation expresses independence of gauge-fixing choices. This note does not develop BV beyond this remark; doing so responsibly would require additional authoritative sources and a separate dedicated treatment.