

# Delta Objects as Half-Density Kernels: Identity, Stationary-Set Concentration, and Point Interactions

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## Abstract

Three seemingly different uses of the Dirac delta share one geometric meaning when amplitudes are treated as **half-densities**: 1. the delta as the Schwartz kernel of the identity operator, 2. the delta as a density supported on stationary points ( $\delta(\nabla f)$ ), 3. the delta as a rank-one kernel defining a point interaction ( $g|0\rangle\langle 0|$ ).

In each case, the amplitude-level object carries **square-root Jacobian** weights (half-density weights), while the corresponding “probability”/density-level object carries the unsquared Jacobians. This note collects the finite-dimensional identities and scaling computations that make this pattern explicit, and isolates where a physical length scale may enter when one insists on scalar representatives.

This note is a companion to the cornerstone manuscript. Statements are kept finite-dimensional unless explicitly labeled heuristic.

## 1. Half-densities and kernels (coordinate free)

Let  $M$  be a  $d$ -dimensional manifold and  $|\Omega|^{1/2}$  the half-density bundle [BatesWeinstein1997]. An operator  $K : \Gamma_c(|\Omega|^{1/2}) \rightarrow \Gamma(|\Omega|^{1/2})$  has a natural Schwartz kernel [Hormander2003]

$$K \in \mathcal{D}'(M \times M; |\Omega|^{1/2} \boxtimes |\Omega|^{1/2}),$$

so that

$$(K\psi)(x) = \int_M K(x, y) \psi(y),$$

is coordinate invariant:  $K(x, y)\psi(y)$  is a density in  $y$  valued in a half-density at  $x$ .

Scalarizing kernels (writing  $\int dy$  with a scalar integrand) implicitly chooses a reference density/half-density; the half-density formalism keeps this choice explicit.

## 2. Delta as the identity kernel (and near-diagonal scaling)

The identity operator on half-densities has Schwartz kernel

$$K_{\text{Id}}(x, y) = \delta^{(d)}(x - y) |dx|^{1/2} |dy|^{1/2}.$$

### Worked scaling computation (the $d/2$ exponent)

Introduce near-diagonal coordinates  $y = x + \varepsilon v$ . Then  $\delta^{(d)}(x - y) = \delta^{(d)}(\varepsilon v) = \varepsilon^{-d} \delta^{(d)}(v)$  and  $|dy|^{1/2} = \varepsilon^{d/2} |dv|^{1/2}$ , so

$$K_{\text{Id}}(x, x + \varepsilon v) = \varepsilon^{-d/2} \delta^{(d)}(v) |dx|^{1/2} |dv|^{1/2}.$$

Thus the universal  $\varepsilon^{-d/2}$  normalization exponent is already present in the identity delta kernel, once kernels are treated as half-densities.

## 3. Delta on the stationary set: (f) and determinant weights

### 3.1 One-dimensional identity ((f))

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  have finitely many nondegenerate critical points  $x_i$  (so  $f'(x_i) = 0$ ,  $f''(x_i) \neq 0$ ). Then, as distributions,

$$\delta(f'(x)) = \sum_i \frac{\delta(x - x_i)}{|f''(x_i)|}.$$

So  $\delta(f') dx$  is a density supported at stationary points with weights  $1/|f''|$ .

#### 3.1a (f) versus ': delta of a derivative vs derivative of delta

The notation  $\delta(f')$  above means: apply the Dirac delta distribution  $\delta(\cdot)$  to the **function**  $f'(x)$ , thereby localizing to the stationary set  $f'(x) = 0$ . It should not be confused with  $\delta'$ , the **distributional derivative** of  $\delta$ , defined by duality:

$$\langle \delta', \varphi \rangle := -\langle \delta, \varphi' \rangle = -\varphi'(0).$$

So  $\delta'$  is the distribution that probes derivatives of test functions at a point (“value of the derivative at zero”, up to sign), whereas  $\delta(f')$  is a stationary-set localization distribution.

#### 3.1b ' from point splitting (difference quotient of shifted deltas)

The distribution  $\delta'$  can be realized as a regulated point-splitting limit. Let  $\varepsilon \rightarrow 0$  and consider the shifted delta  $\delta(x + \varepsilon)$ . For any test function  $\varphi$ ,

$$\left\langle \frac{\delta(\cdot + \varepsilon) - \delta}{\varepsilon}, \varphi \right\rangle = \frac{\varphi(-\varepsilon) - \varphi(0)}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} -\varphi'(0) = \langle \delta', \varphi \rangle.$$

Hence, in the sense of distributions,

$$\frac{\delta(x + \varepsilon) - \delta(x)}{\varepsilon} \xrightarrow[\varepsilon \rightarrow 0]{} \delta'(x).$$

This gives a clean dictionary item for “probing the derivative at a point”:

$$f'(0) = \langle -\delta', f \rangle.$$

For the parallel smooth-function toy model (“difference quotient as divergence + subtraction”) and further remarks, see the companion notes.

### 3.2 Multi-dimensional identity ((f))

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  have finitely many nondegenerate critical points  $x_i$  (so  $\nabla f(x_i) = 0$  and  $\det(\text{Hess } f)(x_i) \neq 0$ ). Then

$$\delta^{(n)}(\nabla f(x)) = \sum_i \frac{\delta^{(n)}(x - x_i)}{|\det(\text{Hess } f)(x_i)|}.$$

### 3.3 Stationary phase and square-root weights (amplitudes vs densities)

For the oscillatory integral

$$I(\hbar) = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} f(x)} a(x) dx, \quad \hbar \rightarrow 0^+,$$

stationary phase gives amplitude contributions weighted by

$$\frac{1}{\sqrt{|\det(\text{Hess } f)(x_i)|}},$$

up to a universal  $\hbar$ -dependent factor and a signature phase. Squaring amplitude weights produces the density weights in  $\delta^{(n)}(\nabla f)$ . This is the finite-dimensional prototype of the slogan: **amplitudes are half-densities; probabilities are densities**.

### 3.4 Extremals in weak form: where ' and ' appear in Euler–Lagrange

For an action  $S[q] = \int L(q, \dot{q}, t) dt$ , the extremal condition is naturally distributional: for test variations  $\eta(t)$  of compact support,

$$\delta S[q; \eta] = \int \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \eta(t) dt.$$

If  $\delta S[q; \eta] = 0$  for all  $\eta$ , then the Euler–Lagrange expression vanishes as a distribution. Approximating  $\eta$  by bump functions converging to  $\delta(t - t_*)$  localizes the equation at  $t_*$  under regularity.

When  $\partial L/\partial \dot{q}$  has jumps (corners/impulses), the distributional derivative produces delta terms automatically; more generally, point-supported singularities are encoded by delta kernels and their derivatives  $(\delta, \delta', \dots)$ , depending on distributional order.

## 4. Delta at a point: point interactions as rank-one kernels

A point interaction [AlbeverioGesztesyHoeghKrohnHolden2005] is naturally the rank-one operator

$$V = g |0\rangle\langle 0|.$$

In the half-density kernel calculus this is written as the bi-half-density distribution supported at  $(0, 0)$ :

$$K_V(x, y) = g \delta^{(d)}(x) \delta^{(d)}(y) |dx|^{1/2} |dy|^{1/2}.$$

This is the “projector-like delta” object underlying contact interactions.

## 5. Where scales enter upon scalarization (and why RG invariants are natural candidates)

Half-density kernels are canonical; scalar representatives are not. Choosing a reference half-density  $\sigma_*$  identifies any half-density  $\psi$  with a scalar  $f$  via  $\psi = f \sigma_*$ . If one insists that scalar representatives be dimensionless, then  $\sigma_*$  must carry a length $^{d/2}$  constant.

In marginal cases (notably the 2D point interaction), renormalization generates an RG-invariant scale  $\kappa_*$  (dimensional transmutation). This suggests a conditional identification: if one adds a universality hypothesis that scalarization scales must be built from physical invariants, then RG-invariant scales are natural candidates to supply the missing length $^{d/2}$  factors required by scalarization.

This note treats that identification as an organizing perspective, not as a theorem.

## 6. Outlook

1. Relate the determinant weights in  $\delta(\nabla f)$  to the mixed Hessian determinants (Van Vleck type [Morette1951]) that appear after eliminating intermediate variables in time slicing (Schur complement template).
2. Clarify which parts of the “functional delta  $\delta(\delta S)$ ” story survive rigorous regularization and which remain heuristic.

## References

1. [BatesWeinstein1997] Sean Bates and Alan Weinstein, *Lectures on the Geometry of Quantization*, Berkeley Mathematics Lecture Notes, vol. 8, AMS, 1997. OA: <https://math.berkeley.edu/~alanw/GofQ.pdf>.
2. [Hormander2003] Lars Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, 2nd ed., Springer, 2003.
3. [AlbeverioGesztesyHoeghKrohnHolden2005] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed., AMS Chelsea Publishing, 2005.
4. [Morette1951] C. Morette, “On the Definition and Approximation of Feynman’s Path Integrals,” *Phys. Rev.* **81**, 848–852 (1951).