Delta Objects as Half-Density Kernels: Identity, Stationary-Set Concentration, and Point Interactions

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Abstract

Three seemingly different uses of the Dirac delta share one geometric meaning when amplitudes are treated as **half-densities**: 1. the delta as the Schwartz kernel of the identity operator, 2. the delta as a density supported on stationary points ($\delta(\nabla f)$), 3. the delta as a rank-one kernel defining a point interaction ($g|0\rangle\langle 0|$).

In each case, the amplitude-level object carries **square-root Jacobian** weights (half-density weights), while the corresponding "probability"/density-level object carries the unsquared Jacobians. This note collects the finite-dimensional identities and scaling computations that make this pattern explicit, and isolates where a physical length scale may enter when one insists on scalar representatives.

This note is a companion to the cornerstone manuscript. Statements are kept finite-dimensional unless explicitly labeled heuristic.

1. Half-densities and kernels (coordinate free)

Let M be a d-dimensional manifold and $|\Omega|^{1/2}$ the half-density bundle [BatesWeinstein1997]. An operator $K : \Gamma_c(|\Omega|^{1/2}) \to \Gamma(|\Omega|^{1/2})$ has a natural Schwartz kernel [Hormander2003]

$$\mathsf{K} \in \mathcal{D}'(M \times M;\ |\Omega|^{1/2} \boxtimes |\Omega|^{1/2}),$$

so that

$$(K\psi)(x) = \int_M \mathsf{K}(x,y)\,\psi(y),$$

is coordinate invariant: $\mathsf{K}(x,y)\psi(y)$ is a density in y valued in a half-density at x.

Scalarizing kernels (writing $\int dy$ with a scalar integrand) implicitly chooses a reference density/half-density; the half-density formalism keeps this choice explicit.

2. Delta as the identity kernel (and near-diagonal scaling)

The identity operator on half-densities has Schwartz kernel

$$\mathsf{K}_{\mathrm{Id}}(x,y) = \delta^{(d)}(x-y)\,|dx|^{1/2}|dy|^{1/2}.$$

Worked scaling computation (the $d/2$ exponent)

Introduce near-diagonal coordinates $y = x + \varepsilon v$. Then $\delta^{(d)}(x-y) = \delta^{(d)}(\varepsilon v) = \varepsilon^{-d}\delta^{(d)}(v)$ and $|dy|^{1/2} = \varepsilon^{d/2}|dv|^{1/2}$, so

$$\mathsf{K}_{\mathrm{Id}}(x, x+\varepsilon v) = \varepsilon^{-d/2}\,\delta^{(d)}(v)\,|dx|^{1/2}|dv|^{1/2}.$$

Thus the universal $\varepsilon^{-d/2}$ normalization exponent is already present in the identity delta kernel, once kernels are treated as half-densities.

3. Delta on the stationary set: $\delta(\nabla f)$ and determinant weights

3.1 One-dimensional identity ($\delta(f')$)

Let $f : \mathbb{R} \to \mathbb{R}$ have finitely many nondegenerate critical points x_i (so $f'(x_i) = 0$, $f''(x_i) \neq 0$). Then, as distributions,

$$\delta(f'(x)) = \sum_i \frac{\delta(x - x_i)}{|f''(x_i)|}.$$

So $\delta(f')\,dx$ is a density supported at stationary points with weights $1/|f''|$.

3.1a $\delta(f')$ versus δ': delta of a derivative vs derivative of delta

The notation $\delta(f')$ above means: apply the Dirac delta distribution $\delta(\cdot)$ to the **function** $f'(x)$, thereby localizing to the stationary set $f'(x) = 0$. It should not be confused with δ', the **distributional derivative** of δ, defined by duality:

$$\langle \delta', \varphi \rangle := -\langle \delta, \varphi' \rangle = -\varphi'(0).$$

So δ' is the distribution that probes derivatives of test functions at a point ("value of the derivative at zero", up to sign), whereas $\delta(f')$ is a stationary-set localization distribution.

3.1b δ' from point splitting (difference quotient of shifted deltas)

The distribution δ' can be realized as a regulated point-splitting limit. Let $\varepsilon \to 0$ and consider the shifted delta $\delta(x+\varepsilon)$. For any test function φ,

$$\left\langle \frac{\delta(\,\cdot+\varepsilon)-\delta}{\varepsilon}, \varphi \right\rangle = \frac{\varphi(-\varepsilon)-\varphi(0)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} -\varphi'(0) = \langle \delta', \varphi \rangle.$$

Hence, in the sense of distributions,

$$\frac{\delta(x+\varepsilon)-\delta(x)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} \delta'(x).$$

This gives a clean dictionary item for "probing the derivative at a point":

$$f'(0) = \langle -\delta', f \rangle.$$

For the parallel smooth-function toy model ("difference quotient as divergence + subtraction") and further remarks, see the companion notes.

3.2 Multi-dimensional identity ($\delta(\nabla f)$)

Let $f : \mathbb{R}^n \to \mathbb{R}$ have finitely many nondegenerate critical points x_i (so $\nabla f(x_i) = 0$ and $\det(\operatorname{Hess} f)(x_i) \neq 0$). Then

$$\delta^{(n)}(\nabla f(x)) = \sum_i \frac{\delta^{(n)}(x - x_i)}{|\det(\operatorname{Hess} f)(x_i)|}.$$

3.3 Stationary phase and square-root weights (amplitudes vs densities)

For the oscillatory integral

$$I(\hbar) = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} f(x)}\, a(x)\, dx, \qquad \hbar \to 0^+,$$

stationary phase gives amplitude contributions weighted by

$$\frac{1}{\sqrt{|\det(\operatorname{Hess} f)(x_i)|}},$$

up to a universal $\hbar$-dependent factor and a signature phase. Squaring amplitude weights produces the density weights in $\delta^{(n)}(\nabla f)$. This is the finite-dimensional prototype of the slogan: **amplitudes are half-densities; probabilities are densities.**

3.4 Extremals in weak form: where δ and δ' appear in Euler–Lagrange

For an action $S[q] = \int L(q, \dot{q}, t)\, dt$, the extremal condition is naturally distributional: for test variations $\eta(t)$ of compact support,

$$\delta S[q;\eta] = \int \Big(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \Big) \eta(t)\, dt.$$

If $\delta S[q;\eta] = 0$ for all η, then the Euler–Lagrange expression vanishes as a distribution. Approximating η by bump functions converging to $\delta(t - t_*)$ localizes the equation at t_* under regularity.

When $\partial L/\partial \dot{q}$ has jumps (corners/impulses), the distributional derivative produces delta terms automatically; more generally, point-supported singularities are encoded by delta kernels and their derivatives $(\delta, \delta', \ldots)$, depending on distributional order.

3.5 Van Vleck determinant: the propagator instance of the square-root Hessian

The square-root Hessian weight of Section 3.3 has a distinguished physical instance: the Van Vleck determinant [VanVleck1928Correspondence] [Morette1951] in the semiclassical propagator.

For the short-time quantum propagator between positions q_i and q_f with time interval Δt, stationary-phase evaluation of the path integral gives

$$K(q_f, q_i; \Delta t) \;\propto\; \sqrt{D(q_f, q_i; \Delta t)}\; e^{(i/\hbar)\, S_{\rm cl}(q_f, q_i; \Delta t)},$$

where $S_{\rm cl}$ is the classical action on the extremal path and

$$D(q_f, q_i; \Delta t) \;:=\; \left| \det\!\left(-\frac{\partial^2 S_{\rm cl}}{\partial q_f^a\, \partial q_i^b} \right) \right|$$

is the Van Vleck determinant — a *mixed* Hessian (derivatives at the two endpoints of the classical path), as opposed to the full Hessian of f that appears in $\delta(\nabla f)$. Despite this difference, it arises by the same stationary-phase mechanism: square-root Hessian weights at the amplitude level, confirming the "amplitudes are half-densities" pattern.

`Example 3.5a (Free particle).` For the free particle in d dimensions, $S_{\rm cl} = m|q_f - q_i|^2/(2\Delta t)$, so

$$D = (m/\Delta t)^d, \qquad \sqrt{D} = (m/\Delta t)^{d/2},$$

reproducing the $(\Delta t)^{-d/2}$ normalization of Section 2.

`Example 3.5b (Harmonic oscillator).` For the harmonic oscillator ($V = \frac{1}{2}m\omega^2 q^2$) in $d = 1$, the classical action between q_i and q_f in time Δt is $S_{\rm cl} = \frac{m\omega}{2\sin\omega\Delta t}\left[(q_f^2 + q_i^2)\cos\omega\Delta t - 2q_f q_i\right]$, giving

$$D = \left|\frac{m\omega}{\sin\omega\Delta t}\right|, \qquad \sqrt{D} = \sqrt{\frac{m\omega}{|\sin\omega\Delta t|}}.$$

As $\omega\Delta t \to 0$, $\sin\omega\Delta t \approx \omega\Delta t$, recovering the free-particle result $\sqrt{D} \to \sqrt{m/\Delta t}$. At $\omega\Delta t = \pi$ (half-period), $\sin\omega\Delta t \to 0$ and $\sqrt{D} \to \infty$: this is the familiar caustic (focal point) where the semiclassical approximation breaks down because the classical flow focuses all initial momenta onto a single final point.

4. Delta at a point: point interactions as rank-one kernels

A point interaction [AlbeverioGesztesyHoeghKrohnHolden2005] is naturally the rank-one operator

$$V = g\,|0\rangle\langle 0|.$$

In the half-density kernel calculus this is written as the bi-half-density distribution supported at $(0, 0)$:

$$\mathsf{K}_V(x, y) = g\,\delta^{(d)}(x)\,\delta^{(d)}(y)\,|dx|^{1/2}|dy|^{1/2}.$$

This is the "projector-like delta" object underlying contact interactions.

`Example 4.1 (1D delta potential: resolvent as a rank-one perturbation).` In one dimension the rank-one structure is particularly explicit. For $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + g\,\delta(x)$ with $g < 0$ (attractive), the free resolvent at energy $E = -\hbar^2\kappa^2/(2m)$ is $G_0(x, x'; E) = -\frac{m}{\hbar^2\kappa}\,e^{-\kappa|x-x'|}$. The rank-one perturbation formula gives

$$G(x, x'; E) = G_0(x, x'; E) + \frac{g\,G_0(x, 0; E)\,G_0(0, x'; E)}{1 - g\,G_0(0, 0; E)}.$$

The correction term factors as $f(x)\cdot f(x')$ with $f(x) = G_0(x, 0; E)$ — this is the rank-one kernel in action: the point interaction contributes a term proportional to $|0\rangle\langle 0|$ in the resolvent. The denominator vanishes at $\kappa = |g|m/\hbar^2$, yielding the unique bound state $E = -mg^2/(2\hbar^2)$, and the residue at this pole factors as $\psi_b(x)\,\psi_b(x')$ with $\psi_b(x) = \sqrt{\kappa}\,e^{-\kappa|x|}$ — a rank-one projector $|\psi_b\rangle\langle\psi_b|$ [AlbeverioGesztesyHoeghKrohnHolden2005]. In the half-density kernel language, the factored piece reads $(\sqrt{\kappa}\,e^{-\kappa|x|}|dx|^{1/2}) \otimes (\sqrt{\kappa}\,e^{-\kappa|x'|}|dx'|^{1/2})$, manifestly a product of half-densities.

5. Where scales enter upon scalarization (and why RG invariants are natural candidates)

Half-density kernels are canonical; scalar representatives are not. Choosing a reference half-density σ_* identifies any half-density ψ with a scalar f via $\psi = f\,\sigma_*$. If one insists that scalar representatives be dimensionless, then σ_* must carry a length$^{d/2}$ constant.

In marginal cases (notably the 2D point interaction), renormalization generates an RG-invariant scale κ_* (dimensional transmutation). This suggests a conditional identification: if one adds a universality hypothesis that scalarization scales must be built from physical invariants, then RG-invariant scales are natural candidates to supply the missing length$^{d/2}$ factors required by scalarization.

This note treats that identification as an organizing perspective, not as a theorem.

6. Outlook

1. ~~Relate determinant weights to Van Vleck type.~~ Addressed: Section 3.5 makes the connection explicit.
2. Clarify which parts of the "functional delta $\delta(\delta S)$" story survive rigorous regularization and which remain heuristic.
3. Extend the half-density treatment to spacetime (Lorentzian) propagators and distributional kernels in field theory.

References

1. [VanVleck1928Correspondence] J. H. Van Vleck, "The Correspondence Principle in the Statistical Interpretation of Quantum Mechanics," *Proceedings of the National Academy of Sciences of the United States of America* 14(2) (1928), 178–188. DOI `10.1073/pnas.14.2.178`.
2. [BatesWeinstein1997] Sean Bates and Alan Weinstein, "Lectures on the Geometry of Quantization," Berkeley Mathematics Lecture Notes, vol. 8, AMS, 1997. ISBN `978-0-8218-0798-9`. OA: https://math.berkeley.edu/~alanw/GofQ.pdf. (Canonical reference for half-density formalism in geometric quantization; half-density kernels and composition.)
3. [Hormander2003] Lars Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, 2nd ed., Springer, 2003. DOI `10.1007/978-3-642-61497-2`. (Schwartz kernel theorem; distributional calculus for PDE Green functions.)
4. [AlbeverioGesztesyHoeghKrohnHolden2005] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed., AMS Chelsea Publishing, 2005. ISBN `978-0-8218-3624-4`. (Canonical reference for point interactions in quantum mechanics; self-adjoint extensions, delta potentials.)

5. [Morette1951] C. Morette, “On the Definition and Approximation of Feynman’s Path Integrals,” *Phys. Rev.* **81**, 848–852 (1951). DOI 10.1103/PhysRev.81.848. (Van Vleck determinant in path integral; semiclassical expansion weights.)