Renormalization Group as a Fundamental Compatibility Principle (Draft)

Abstract

This note develops the viewpoint that the renormalization group (RG) is not an optional add-on to quantization, but the *compatibility condition* that makes refinement-based composition meaningful when naive continuum limits diverge. The core claim is structural: once a theory is defined by composing local pieces across refinement steps, regulator dependence is unavoidable in intermediate objects, and RG invariance is the demand that the composed predictions remain stable. A simple calculus analogy makes the point vivid: the derivative can be viewed as a renormalized limit obtained by subtracting a divergent counterterm from two individually divergent quantities [RiveroOde2002]. We connect this perspective to rooted-tree bookkeeping (Butcher group) and the Hopf-algebra formulation of perturbative renormalization [Brouder1999] [ConnesKreimer2000] [McLachlan2017].

1. Scope

Dependent follow-up to `paper/main.md`, focusing on: 1. RG as "scale-compatibility" within the Refinement Compatibility Principle (RCP) perspective. 2. Why RG is semigroup-like (information loss under coarse-graining). 3. A minimal worked analogy (difference quotients) showing "counterterm subtraction = taking a derivative".

2. RG as Composition Consistency

`Proposition P1.1 (Semigroup compatibility).` Let $W_{\Lambda\to\mu}$ map effective data at cutoff Λ to effective data at scale μ. If refinement/coarse-graining is composable, $W_{\kappa\to\mu}\circ W_{\Lambda\to\kappa} = W_{\Lambda\to\mu}$, then an infinitesimal generator exists under differentiability assumptions, yielding beta functions as the differential form of scale-compatibility.

This is the main manuscript's Section 8 claim restated as a "fundamental" postulate: a theory is well-defined only if it survives composed changes of scale.

3. A Calculus Micro-Model: Derivative as Counterterm Subtraction

Let f be smooth and consider $\varepsilon \to 0^+$. The quantities $f(x+\varepsilon)/\varepsilon$ and $f(x)/\varepsilon$ individually diverge as $1/\varepsilon$, but their difference is finite:

$$\frac{f(x+\varepsilon)}{\varepsilon} - \frac{f(x)}{\varepsilon} = \frac{f(x+\varepsilon)-f(x)}{\varepsilon} \xrightarrow{\varepsilon\to 0} f'(x).$$

`Heuristic H1.1 (Interpretation as renormalization).` View $f(x)/\varepsilon$ as a local counterterm subtracting the divergence of $f(x+\varepsilon)/\varepsilon$. The renormalized limit is the derivative $f'(x)$. This is the simplest instance of the general pattern: regulated intermediate objects diverge, counterterms remove local divergences, and the renormalized quantity is the cutoff-stable remainder [RiveroOde2002].

4. Rooted Trees: Bookkeeping for Composed Refinements

Runge–Kutta composition and perturbative renormalization share rooted-tree combinatorics. Butcher's group organizes method composition, while the Hopf-algebra approach organizes subtraction/recursion patterns in renormalization [Brouder1999] [McLachlan2017] [ConnesKreimer2000].

`Heuristic H1.2 (One compatibility story, two domains).` The shared tree bookkeeping suggests that "RG is fundamental" can be understood as: whenever you define a theory by iterating/composing local approximations, you must control the discrepancy between "two steps" and "one step". That discrepancy is the counterterm/correction data, and the RG is the law governing how those corrections change with scale.

5. Worked Singular QM Model: The 2D Delta Potential

The cleanest "RG appears before QFT" example is the contact (delta) interaction in two spatial dimensions. The point is not that this is the most physical model; it is that: 1. the interaction is Dirac-supported (singular), 2. the naive continuum limit is ill-defined, 3. a renormalization prescription plus RG invariance *defines* the theory.

For a compact treatment that explicitly parallels QFT-style renormalization in a two-dimensional delta interaction (and also treats the Aharonov–Bohm case), see [ManuelTarrach1994PertRenQM]. The derivation below matches the same logarithmic short-distance divergence and dimensional transmutation scale, up to conventions for normalization and for where one places scheme-dependent constants. For a complementary Wilson–Kogut-style RG study of contact interactions in 1D quantum mechanics (including a scaling analysis on the full line), see [BoyaRivero1994Contact]. The 2D delta model below is chosen because it is the simplest setting where the contact coupling is marginal and the renormalization produces genuine logarithmic running and dimensional transmutation.

5.1 Setup

Consider $H = -\frac{\hbar^2}{2m}\Delta + g\,\delta^{(2)}(x)$ on $\mathbb{R}^2$. At the level of formal perturbation theory, the contact interaction generates momentum integrals that diverge logarithmically.

The Lippmann–Schwinger equation for the T-matrix reads

$$T(E) = g + g\, I(E)\, T(E),$$

where the divergent loop integral is the free resolvent at the origin,

$$I(E) = \langle 0|(E - H_0 + i0)^{-1}|0\rangle = \int \frac{d^2q}{(2\pi)^2}\, \frac{1}{E - \frac{\hbar^2 q^2}{2m} + i0}.$$

`Derivation D1.1 (Cutoff evaluation of the loop integral).` Introduce a wavevector cutoff $|q| < \Lambda$ and write $E = \hbar^2 k^2/(2m)$ for $E > 0$. Then

$$I(E;\Lambda) = \frac{1}{2\pi} \int_0^\Lambda \frac{q\, dq}{E - \frac{\hbar^2 q^2}{2m} + i0} = -\frac{m}{2\pi\hbar^2} \left[\ln\!\left(\frac{\Lambda^2}{k^2}\right) + i\pi \right] + O\!\left(\frac{k^2}{\Lambda^2}\right).$$

The key feature is the logarithmic divergence $\sim -\frac{m}{2\pi\hbar^2} \ln \Lambda^2$.

5.2 Renormalized Coupling and RG Equation

For a contact interaction the T-matrix is algebraic:

$$T(E;\Lambda) = \frac{1}{g_B(\Lambda)^{-1} - I(E;\Lambda)}.$$

Define the renormalized coupling at scale μ by subtracting the logarithmic divergence:

$$\frac{1}{g_R(\mu)} \equiv \frac{1}{g_B(\Lambda)} + \frac{m}{2\pi\hbar^2} \ln\!\left(\frac{\Lambda^2}{\mu^2}\right).$$

`Derivation D1.2 (Finite \(T\)-matrix and beta function).` Substituting the definition of $g_R(\mu)$ into $T(E;\Lambda)$ and using the expression for $I(E;\Lambda)$ gives a cutoff-independent amplitude:

$$T(E) = \frac{1}{\frac{1}{g_R(\mu)} - \frac{m}{2\pi\hbar^2} \ln\!\left(\frac{k^2}{\mu^2}\right) + i\,\frac{m}{2\hbar^2}}.$$

Physical quantities cannot depend on the arbitrary subtraction scale μ, so impose $dT/d\ln\mu = 0$. Since $\frac{d}{d\ln\mu} \ln(k^2/\mu^2) = -2$, this yields the RG equation in the unambiguous form

$$\mu \frac{d}{d\mu} \left(\frac{1}{g_R(\mu)} \right) = -\frac{m}{\pi\hbar^2}.$$

Equivalently,

$$\beta(g_R) \equiv \mu \frac{dg_R}{d\mu} = \frac{m}{\pi\hbar^2}\, g_R^2.$$

The semigroup/composition property is explicit in the integrated flow:

$$\frac{1}{g_R(\mu_2)} = \frac{1}{g_R(\mu_1)} - \frac{m}{\pi\hbar^2} \ln\!\left(\frac{\mu_2}{\mu_1}\right).$$

5.3 Dimensional Transmutation via the Bound State

For $E < 0$ write $E = -\hbar^2\kappa^2/(2m)$ with $\kappa > 0$. Analytic continuation removes the $i\pi$ term and the pole condition $T(E)^{-1} = 0$ becomes

$$\frac{1}{g_R(\mu)} + \frac{m}{2\pi\hbar^2} \ln\!\left(\frac{\mu^2}{\kappa^2}\right) = 0.$$

`Proposition P1.2 (RG-invariant scale from the delta interaction).` Define

$$\kappa_*^2 \equiv \mu^2 \exp\!\left(\frac{2\pi\hbar^2}{m} \frac{1}{g_R(\mu)}\right).$$

Using the RG equation for $1/g_R(\mu)$, the quantity κ_* is independent of μ. The scattering amplitude can be rewritten purely in terms of κ_*:

$$T(E) = \frac{2\pi\hbar^2/m}{\ln(\kappa_*^2/k^2) + i\pi}.$$

Thus the renormalized delta interaction trades the bare coupling for a physical scale κ_* (equivalently a bound-state energy $E_B = \hbar^2\kappa_*^2/(2m)$).

5.4 Scheme Dependence: What Changes vs What Is Invariant

The subtraction condition defining $g_R(\mu)$ is not unique: one may add a finite constant to the definition. For example, define an alternative coupling $g_R^{(C)}$ by

$$\frac{1}{g_R^{(C)}(\mu)} \equiv \frac{1}{g_B(\Lambda)} + \frac{m}{2\pi\hbar^2}\left[\ln\!\left(\frac{\Lambda^2}{\mu^2}\right) + C\right] = \frac{1}{g_R(\mu)} + \frac{m}{2\pi\hbar^2} C,$$

with fixed dimensionless C.

`Derivation D1.3 (Scheme dependence as coupling reparametrization).` Differentiating with respect to $\ln\mu$ removes the constant C, so the RG equation

for $1/g_R(\mu)$ and the beta function $\beta(g_R)$ are unchanged by this finite subtraction shift.

The mapping between the running coupling and the RG-invariant scale does change:

$$\kappa_*^{(C)\,2} \equiv \mu^2 \exp\left(\frac{2\pi\hbar^2}{m}\frac{1}{g_R^{(C)}(\mu)}\right) = e^C\,\kappa_*^2.$$

Thus, in this one-scale model, scheme dependence is exactly a rescaling of the dimensional transmutation scale. Once one physical datum is used to fix κ_* (equivalently E_B), all predictions are scheme-independent.

5.5 Bubble Chains as Ladder Trees (Explicit Recursion)

The contact interaction has a particularly transparent perturbative structure:

$$T(E;\Lambda) = \frac{g_B(\Lambda)}{1 - g_B(\Lambda)\,I(E;\Lambda)} = \sum_{n\geq 0} g_B(\Lambda)^{n+1}\,I(E;\Lambda)^n.$$

Each term corresponds to a "bubble-chain" diagram. Its divergence structure is the simplest possible: repeated insertions of the same logarithmically divergent loop.

`Derivation D1.4 (Perturbative counterterm cancellation through order \(g_R^2\)).` Write $I(E;\Lambda) = I_{\rm div}(\Lambda) + I_{\rm fin}(E;\mu)$ with $I_{\rm div}(\Lambda) = -\frac{m}{2\pi\hbar^2}\ln(\Lambda^2/\mu^2)$ and $I_{\rm fin}(E;\mu) = -\frac{m}{2\pi\hbar^2}\ln(\mu^2/k^2) - i\frac{m}{2\hbar^2}$. From the renormalization condition, $g_B = g_R + g_R^2 I_{\rm div} + O(g_R^3)$. Then, expanding $T = g_B + g_B^2 I + O(g_B^3)$ gives

$$T = g_R + g_R^2\big(I_{\rm div} + I_{\rm fin}\big) + g_R^2(-I_{\rm div}) + O(g_R^3) = g_R + g_R^2 I_{\rm fin} + O(g_R^3),$$

so the cutoff divergence cancels explicitly at this order. Higher orders cancel similarly because the series is geometric.

`Heuristic H1.3 (Rooted-tree/Hopf-algebra view, toy-level).` In the Connes–Kreimer Hopf algebra of rooted trees, ladder (linear) trees model iterated insertions of a single primitive divergence. For the ladder tree L_n with n vertices, admissible cuts yield a deconcatenation coproduct

$$\Delta L_n = L_n \otimes 1 + 1 \otimes L_n + \sum_{k=1}^{n-1} L_{n-k} \otimes L_k,$$

and the antipode recursion generates the alternating subtraction pattern (e.g. $S(L_1) = -L_1$, $S(L_2) = -L_2 + L_1^2$, $S(L_3) = -L_3 + 2L_1L_2 - L_1^3$). This

is the algebraic skeleton underlying the Bogoliubov counterterm recursion in perturbative renormalization [ConnesKreimer2000] and the rooted-tree bookkeeping emphasized in the Runge–Kutta/RG bridge literature [Brouder1999] [McLachlan2017].

5.6 Why This Supports "RG Is Definitional"

This model is the precise analog of the calculus micro-model in Section 3: 1. The regulated object $I(E;\Lambda)$ diverges. 2. A local counterterm (here the subtraction defining $g_R(\mu)$) removes the divergence. 3. RG invariance (μ-independence) enforces a flow law and produces a cutoff-free theory.

In short: in a singular interaction, "the theory" is the triple (regularization, subtraction condition, RG invariance). The RG is not an afterthought; it is the compatibility condition that makes the refinement limit meaningful.

5.7 Point Interactions as Short-Distance Boundary Conditions (Where the Scale Lives)

The renormalized 2D contact interaction can be understood without momentum cutoffs: it can be formulated as a boundary condition at the removed point $r = 0$ (a self-adjoint extension parameter). This makes the "scale from a point" intuition explicit. For a perturbative-QFT-style presentation of this mechanism in the delta model, including the explicit short-distance logarithm, see [ManuelTarrach1994PertRenQM].

`Proposition P1.4 (Point interaction parameterizes a length scale).` For the free Hamiltonian on $\mathbb{R}^2 \setminus \{0\}$, admissible s-wave states can be characterized near $r = 0$ by an asymptotic expansion $\psi(r) = A \ln r + B + o(1)$. A 2D point interaction is (equivalently) the choice of a boundary condition relating A and B, e.g. $B = A \ln R$, or $\psi(r) = A \ln(r/R) + o(1)$ for some length scale $R > 0$.

`Derivation D1.5 (Bound state scale from the \(K_0\) short-distance expansion).` Consider a bound state $E = -\hbar^2\kappa^2/(2m)$ with $\kappa > 0$. For $r > 0$ the equation is free: $(\Delta - \kappa^2)\psi = 0$. The s-wave solution decaying at infinity is $\psi(r) = A\, K_0(\kappa r)$. As $r \to 0$, $K_0(z) = -\ln(z/2) - \gamma + O(z^2)$, so

$$\psi(r) = A\Big[-\ln r - \ln(\kappa/2) - \gamma\Big] + o(1) = A \ln\Big(\frac{R}{r}\Big) + o(1), \qquad R \equiv \frac{2e^{-\gamma}}{\kappa}.$$

Thus the boundary-condition parameter R is equivalent to the bound-state scale κ, i.e. the physical transmutation scale κ_* in Section 5.3 can be taken as $\kappa_* = 2e^{-\gamma}/R$ up to conventions.

`Heuristic H1.6 (Scheme dependence is the constant in "\(\ln r\)").` The only ambiguity in the short-distance matching is the additive constant accompanying $\ln r$ (here $-\ln(\kappa/2) - \gamma$). Changing that finite constant is exactly the finite-subtraction freedom of Section 5.4, and it rescales the physical length/energy scale (R, κ_*) without changing the beta function.

5.8 Contact Interactions Across Dimensions: 1D vs 2D vs 3D

The 2D delta is singled out above because its coupling is marginal and produces a clean log-running and transmutation scale. But contact interactions exist in all dimensions, and comparing 1D/2D/3D is a good sanity-check on the "RG = compatibility" thesis: the dimension controls which kind of divergence (or boundary-condition flow) appears.

`Proposition P1.6 (Canonical dimension of the delta coupling).` In d spatial dimensions, the contact potential $V(x) = g\,\delta^{(d)}(x)$ requires $[g] = \text{length}^{d-2}$ (in $\hbar = c = 1$ units), because $[\delta^{(d)}] = \text{length}^{-d}$ and the kinetic term sets $[E] = \text{length}^{-2}$. Thus: 1. $d = 1$: $[g] = \text{length}^{-1}$ (irrelevant in naive power counting), 2. $d = 2$: $[g] = \text{length}^{0}$ (marginal; logarithms), 3. $d = 3$: $[g] = \text{length}^{+1}$ (relevant; power divergences).

5.8.1 The 3D delta: scattering length and (Wilsonian) fixed points

In 3D, the loop integral $I(E)$ diverges linearly with a momentum cutoff. With $E = \hbar^2 k^2/(2m)$ and $|q| < \Lambda$,

$$I_{3D}(E;\Lambda) = \int \frac{d^3q}{(2\pi)^3}\,\frac{1}{E - \frac{\hbar^2 q^2}{2m} + i0} = -\frac{m}{\pi^2\hbar^2}\Lambda i\,\frac{m}{2\pi\hbar^2}kO\!\left(\frac{k^2}{\Lambda}\right).$$

So

$$T(E;\Lambda) = \frac{1}{g_B(\Lambda)^{-1} - I_{3D}(E;\Lambda)}$$

is cutoff-dependent unless one trades $g_B(\Lambda)$ for a physical parameter. A standard choice is to define a renormalized coupling at threshold (equivalently a scattering length a) by fixing $T(0)$:

$$\frac{1}{T(0)} \equiv \frac{m}{2\pi\hbar^2}\,\frac{1}{a} \quad\Longleftrightarrow\quad \frac{1}{g_B(\Lambda)} + \frac{m}{\pi^2\hbar^2}\Lambda = \frac{m}{2\pi\hbar^2}\,\frac{1}{a}.$$

Then the renormalized amplitude takes the universal one-parameter form

$$T(E) = \frac{2\pi\hbar^2/m}{\frac{1}{a} - ik},$$

up to conventions for the overall normalization of T.

`Heuristic H1.9 (3D "no log" does not mean "no RG").` Because the divergence is power-like, a minimal-subtraction style scheme can hide running. But in a Wilsonian parameterization with a dimensionless coupling (roughly $\hat{g}(\Lambda) \propto \Lambda g_B(\Lambda)$), the flow can exhibit fixed points corresponding to $a = 0$ (free/transparent) and $a = \infty$ (unitarity). The point for this note is structural: even when running is not logarithmic, a compatibility condition still controls how the effective description must change with scale.

5.8.2 The 1D line: U(2) contact family and richer fixed-point set In 1D, the pure $\delta(x)$ interaction is already a well-defined self-adjoint Hamiltonian and does not require a UV subtraction to make scattering finite. However, the *space of all* contact interactions on the full line is richer: it is naturally parameterized by boundary conditions at the origin (a self-adjoint extension of the free Hamiltonian on $\mathbb{R} \setminus \{0\}$), and scaling acts nontrivially on that parameter space.

One convenient description is in terms of the (dimensionless) cutoff-a scattering matrix $\tilde{S}_{\tilde{k},a}$ as a function of $\tilde{k} = ak$. A Wilson–Kogut style scaling transformation acts by

$$\tilde{S}^{(t)}_{\tilde{k}} = T^t[\tilde{S}]_{\tilde{k}} = \tilde{S}_{e^{-t}\tilde{k}},$$

so fixed points are exactly **constant** unitary matrices satisfying the usual 1D constraints (unitarity and $S_{-k} = QS_k^\dagger Q$ with Q swapping left/right channels). Boya–Rivero show that this yields a nontrivial fixed-point set (including a circle of fixed points in the time-reversal-invariant sector) and that familiar contact interactions (δ, δ', decoupled half-lines) sit on trajectories connecting these fixed points [BoyaRivero1994Contact].

`Heuristic H1.10 (Why 1D looks "more complicated").` The lesson is not that 1D is "more renormalizable"; it is that the relevant notion of RG data for point interactions is the boundary-condition parameter space. In 2D the same "scale from a point" reappears as a single transmutation scale. In 1D, because the full contact family is larger (U(2) rather than a one-parameter subset), the scaling flow can have a correspondingly richer fixed-point structure.

`Heuristic H1.11 (Fixed points as quantization rules in finite volume).` In one dimension, a contact interaction is boundary data at a point, and RG fixed points are precisely the **scale-invariant** boundary conditions (energy-independent S-matrices). When the system is placed in a finite box (or on a circle), boundary/scattering data become **quantization conditions** for the allowed momenta. Thus fixed points naturally correspond to simple "quantization rules" (linear spectra up to constant phase shifts), while departures from the fixed point appear as scale-dependent phase shifts.

`Derivation D1.6 (Robin boundary condition gives a quantization condition).` Consider a free particle on the interval $[0, L]$ with a Robin boundary condition at the left endpoint, $\psi'(0) = \lambda\,\psi(0)$, and (for simplicity) a Dirichlet boundary at the right endpoint, $\psi(L) = 0$. For energy $E = \hbar^2k^2/(2m)$, the general solution is $\psi(x) = A\sin(kx) + B\cos(kx)$. The Dirichlet condition gives $A\sin(kL) + B\cos(kL) = 0$, hence $B = -A\tan(kL)$. The Robin condition gives $Ak = \lambda B$, so

$$k = -\lambda\,\tan(kL), \qquad \text{equivalently} \qquad \tan(kL) = -\frac{k}{\lambda}.$$

This is a quantization rule: the allowed k are the roots of the displayed equation. At the scale-invariant endpoints $\lambda = \infty$ (Dirichlet at 0) and $\lambda = 0$ (Neumann at

0), one recovers the familiar fixed-point spectra

$$\lambda = \infty : \; k_n = \frac{n\pi}{L}, \qquad \lambda = 0 : \; k_n = \frac{(n + \frac{1}{2})\pi}{L},$$

up to the usual $n \in \mathbb{N}$ conventions. More general point interactions (including the full U(2) family on the line) act similarly: they change the phase relation at the defect, hence shift the finite-volume quantization condition by a phase.

6. Semigroup vs Group: What Is (Not) Invertible

The phrase "renormalization group" hides two different notions: 1. an information-losing *coarse-graining map* on descriptions of the system, and 2. the induced *flow* of a chosen parameterization (couplings) needed to keep predictions stable.

`Proposition P1.3 (Coarse-graining is a semigroup).` Let $W_{\Lambda\to\mu}$ be the operation "integrate out modes between μ and Λ" acting on the space of effective descriptions (e.g. effective actions). Then: 1. $W_{\Lambda\to\Lambda} = \mathrm{id}$, 2. $W_{\kappa\to\mu} \circ W_{\Lambda\to\kappa} = W_{\Lambda\to\mu}$ for $\Lambda \geq \kappa \geq \mu$.

So W is a semigroup under composition.

`Heuristic H1.4 (Why it is not a group).` In general there is no inverse $W_{\mu\to\Lambda}$ that reconstructs the UV degrees of freedom that were discarded in coarse-graining. Many distinct UV theories can flow to the same IR effective description. This non-invertibility is the structural reason semigroup language is more accurate than group language at the level of coarse-graining maps.

`Heuristic H1.5 (Why coupling "running" can look invertible anyway).` If one restricts attention to a finite-dimensional ansatz for the effective description (a truncation to finitely many couplings), then the RG flow becomes an ordinary differential equation for those couplings. Such flows can often be integrated "up" or "down" in scale given initial data, but this mathematical reversibility should not be confused with physical invertibility of coarse-graining. It is an artifact of restricting to (and assuming closure of) a low-dimensional manifold of descriptions.

6.1 A Fully Explicit Coarse-Graining Map: Gaussian Integration (Schur Complement)

This toy model is deliberately "too simple" in the same way the difference quotient is "too simple": it makes the semigroup structure and non-invertibility *visible*.

`Derivation D1.6 (Exact coarse-graining in a quadratic model).` Let x be "IR" and y be "UV", and consider the quadratic action

$$S(x, y) = \frac{1}{2}\left(a\,x^2 + 2b\,xy + c\,y^2\right), \qquad c > 0.$$

Define coarse-graining by integrating out y:

$$e^{-S_{\text{eff}}(x)/\hbar} \equiv \int_{\mathbb{R}} dy\, e^{-S(x,y)/\hbar}.$$

Completing the square gives

$$S(x,y) = \frac{1}{2}\left(a - \frac{b^2}{c}\right)x^2 + \frac{c}{2}\left(y + \frac{b}{c}x\right)^2,$$

so

$$S_{\text{eff}}(x) = \frac{1}{2}\left(a - \frac{b^2}{c}\right)x^2 + \frac{\hbar}{2}\ln c + (\text{constant}).$$

The flow of the quadratic coupling is therefore $a \mapsto a_{\text{eff}} = a - b^2/c$: the Schur complement of the UV block.

`Proposition P1.5 (Semigroup property as associativity of integration).` For a positive definite quadratic form on (x, y, z), define the coarse-graining map W by integrating out a chosen subset of variables. Then

$$W_{(y,z)\to x} = W_{y\to x} \circ W_{z\to(x,y)},$$

because the integrals are iterated applications of Fubini's theorem (and, in the quadratic case, correspond algebraically to nested Schur complements of the Hessian matrix).

`Heuristic H1.7 (Non-invertibility is concrete: many UV completions, one IR).` The effective parameter $a_{\text{eff}} = a - b^2/c$ does not determine the UV data (a, b, c). Infinitely many triples (a, b, c) give the same a_{eff}. This is the simplest explicit witness that coarse-graining is generally not invertible.

`Heuristic H1.8 (Stationary phase link: coarse-graining = extremize + fluctuations).` In this quadratic model, integrating out y is equivalent to: 1. solving the UV Euler–Lagrange equation $\partial_y S = bx + cy = 0$, giving $y_* = -(b/c)x$; 2. substituting y_* into S, yielding the classical effective action $\frac{1}{2}(a - b^2/c)x^2$; 3. multiplying by the fluctuation determinant factor $\sim c^{-1/2}$ (the $\frac{\hbar}{2}\ln c$ term).

Thus even the simplest exact coarse-graining step already splits into "extremal selection" plus a one-loop prefactor, matching the main manuscript's stationary-phase narrative.

7. What This Paper Must Still Do

1. Identify/ingest an independent standard reference for the 2D delta renormalization (OA if possible; otherwise mark `PENDING`) and align conventions.
2. Identify/ingest the intended "Cristina 2D delta" reference and match key formulae explicitly.