Relativistic Central Orbits as Refinement-Witnesses

Abstract

Central-force motion is a clean domain where "refinement" arguments can be made explicit: Newton's polygonal limit gives exact finite-step invariants, while relativistic kinematics introduces new admissibility constraints even before general relativity is invoked. This dependent note records one such constraint in special relativity for inverse-square forces: circular orbits obey $v = K/L$ and therefore require $L > K/c$. The goal is not to replace standard treatments, but to keep a minimal derivation-first record of what changes when the same geometric refinement story is pushed into relativistic regimes.

1. Purpose and Relation to the Cornerstone Paper

The cornerstone manuscript (`paper/main.md`) uses central-force refinement as a structural bridge (equal areas / angular momentum preservation $\leftrightarrow$ action additivity $\leftrightarrow$ composition).

This note is "outside scope" of the cornerstone paper but tracks a conversation branch: how relativistic kinematics modifies the simplest central-force circular-orbit conditions. The inverse-square case is singled out by an exact simplification already at the SR level.

2. SR Circular Motion Under a Power-Law Force

Assume a particle of rest mass m executes uniform circular motion of radius r and speed v. The Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

For uniform circular motion the acceleration is perpendicular to the velocity, so

$$F = \frac{dp}{dt} = \gamma m a_\perp = \gamma m \frac{v^2}{r}.$$

Assume an attractive central force magnitude

$$F(r) = \frac{K}{r^q}, \qquad K > 0.$$

Then the circular-orbit condition is

$$\frac{K}{r^q} = \gamma m \frac{v^2}{r} \quad \Longleftrightarrow \quad K = \gamma m v^2 r^{q-1}.$$

The angular momentum magnitude is

$$L = rp = \gamma m r v.$$

Eliminating v gives the SR circular-orbit condition

$$L^2 = K\,\gamma\,m\,r^{3-q}.$$

This reduces to the Newtonian relation when $\gamma \to 1$ (i.e. $c \to \infty$).

3. Inverse-Square ($q = 2$) Special Case: $v = K/L$ and the Bound $L > K/c$

For $q = 2$,

$$K = \gamma m v^2 r, \qquad L = \gamma m r v.$$

Dividing yields the exact identity

$$\boxed{v = \frac{K}{L}}.$$

Therefore a relativistic circular orbit requires $v < c$, hence the angular-momentum bound

$$\boxed{L > \frac{K}{c}}.$$

The radius follows from $r = L^2/(K\gamma m)$ with $\gamma = (1 - v^2/c^2)^{-1/2}$ and $v = K/L$:

$$r = \frac{L^2}{Km}\sqrt{1 - \frac{K^2}{c^2 L^2}} \xrightarrow[c\to\infty]{} \frac{L^2}{Km}.$$

This bound and the circular-solution branch appear in standard treatments of the relativistic Kepler problem (see e.g. [BoscagginDambrosioFeltrin2020RelKepler] for a dynamical-systems/variational analysis of the same SR equation).

4. GR Baseline: Schwarzschild Geodesics, Effective Potential, Photon Sphere

This section records the standard Schwarzschild baseline in a form parallel to the SR "effective 1D radial motion" viewpoint.

Conventions: set $G = c = 1$. Then the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

with horizon at $r = 2M$. Restrict to the equatorial plane $\theta = \pi/2$.

Using an affine parameter λ, define $\epsilon = 1$ (timelike, $\lambda = \tau$) and $\epsilon = 0$ (null). Energy and angular momentum from the Killing fields $\partial_t, \partial_\phi$ are

$$E = \left(1 - \frac{2M}{r}\right) \dot{t}, \qquad L = r^2 \dot{\phi},$$

so $\dot{t} = E/(1 - 2M/r)$, $\dot{\phi} = L/r^2$.

The normalization condition $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -\epsilon$ yields the radial equation

$$\dot{r}^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right),$$

i.e. $\dot{r}^2 + V_{\rm eff}(r) = E^2$ with

$$V_{\rm eff}(r) = \left(1 - \frac{2M}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right).$$

4.1 Null circular orbit (photon sphere)

For null geodesics $\epsilon = 0$,

$$V_{\rm eff}(r) = \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}.$$

Circular null orbits satisfy $\dot{r} = 0$ and $dV_{\rm eff}/dr = 0$, which gives the photon-sphere radius

$$\boxed{r_{\rm ph} = 3M}.$$

The impact parameter $b = L/E$ obeys $b^2 = r^2/(1 - 2M/r)$, so at $r = 3M$ one has $b = 3\sqrt{3}\, M$.

Baseline anchor for these standard results: [Carroll1997LectureGR].

4.2 Restoring units

Replace M by GM/c^2: $r_{\rm ph} = 3GM/c^2$, $r_h = 2GM/c^2$, and $b = 3\sqrt{3}\, GM/c^2$.

4.3 Timelike circular orbits and ISCO

For timelike geodesics $\epsilon = 1$,

$$V_{\rm eff}(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right).$$

Circular orbits satisfy $\dot r = 0$ and $dV_{\rm eff}/dr = 0$. Writing

$$V_{\rm eff}(r) = 1 - \frac{2M}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3},$$

one finds

$$\frac{dV_{\rm eff}}{dr} = \frac{2M}{r^2} - \frac{2L^2}{r^3} + \frac{6ML^2}{r^4}.$$

Thus the circular branch obeys

$$Mr^2 = L^2(r - 3M) \quad \Rightarrow \quad \boxed{L^2(r) = \frac{Mr^2}{r - 3M}}, \qquad r > 3M,$$

and using $E^2 = V_{\rm eff}(r)$ on the circular orbit gives

$$\boxed{E^2(r) = \frac{(r - 2M)^2}{r(r - 3M)}}.$$

Stability requires $V''_{\rm eff}(r) > 0$ at the circular orbit. Differentiating once more,

$$V''_{\rm eff}(r) = -\frac{4M}{r^3} + \frac{6L^2}{r^4} - \frac{24ML^2}{r^5},$$

and substituting the circular-orbit value $L^2 = Mr^2/(r - 3M)$ yields

$$V''_{\rm eff}(r) = \frac{2M(r - 6M)}{r^3(r - 3M)}.$$

So circular timelike orbits are stable iff $r > 6M$, and the innermost stable circular orbit is

$$\boxed{r_{\rm ISCO} = 6M}.$$

At the ISCO,

$$L = 2\sqrt{3}\,M, \qquad E = \frac{2\sqrt{2}}{3}.$$

5. SR Stability of Circular Orbits (Small Radial Perturbations)

This section stays within mechanical SR (a point particle in an external, time-independent central scalar potential $U(r)$). It is used as a kinematic witness: some restrictions already appear before GR or field-theoretic interactions enter.

Fix the (conserved) angular momentum magnitude L. For purely tangential motion ($p_r = 0$), define the fixed-L energy function

$$W_L(r) = \sqrt{m^2c^4 + \frac{L^2c^2}{r^2}} + U(r).$$

Assume the attractive power-law force $F(r) = K/r^q$ so that $U'(r) = K/r^q$.

5.1 Circularity and the effective-potential viewpoint

Circular orbits satisfy $W_L'(r_0) = 0$, i.e.

$$\frac{K}{r_0^q} = \frac{L^2c^2}{r_0^3\sqrt{m^2c^4 + L^2c^2/r_0^2}} \quad \Longleftrightarrow \quad L^2 = K\,\gamma\,m\,r_0^{3-q},$$

since $\sqrt{m^2c^4 + L^2c^2/r_0^2} = \gamma mc^2$. This recovers the SR circular-orbit condition used earlier.

5.2 Stability inequality

At fixed L, stability under small radial perturbations is the local-minimum condition

$$W_L''(r_0) > 0.$$

Writing the Lorentz factor at the orbit as

$$\gamma^2 = 1 + \frac{L^2}{m^2c^2r_0^2},$$

one finds the compact expression

$$\boxed{W_L''(r_0) = \frac{L^2}{mr_0^4\,\gamma^3}\Big(1 + (2-q)\gamma^2\Big),}$$

hence the stability criterion

$$\boxed{1 + (2-q)\gamma^2 > 0.}$$

In the Newtonian limit $\gamma \to 1$ this reduces to the standard condition $q < 3$. For $2 < q < 3$ SR adds a speed bound:

$$\gamma^2 < \frac{1}{q-2} \quad \Longleftrightarrow \quad \frac{v^2}{c^2} < 3 - q.$$

In particular, inverse-square forces ($q = 2$) are stable for all γ in this model, while $q \geq 3$ yields no stable circular orbits (beyond the Newtonian marginal case at $q = 3$).

6. Outlook

Two natural extensions are: 1. replace the "external potential" modeling assumption by an explicitly field-mediated interaction model, and compare which orbit admissibility/stability bounds survive that change; 2. connect the SR/GR orbit constraints more explicitly to the refinement-compatibility spine (what is preserved under refinement, and what new kinematic admissibility conditions appear when the refinement rules are Lorentz/GR-consistent).

References

1. [BoscagginDambrosioFeltrin2020RelKepler] Alberto Boscaggin, Walter Dambrosio, and Guglielmo Feltrin, "Periodic solutions to a perturbed relativistic Kepler problem," arXiv:`2003.03110` (v1, 6 Mar 2020). (Contains the unperturbed SR relativistic Kepler equation and discusses circular solutions/constraints.)
2. [Carroll1997LectureGR] Sean M. Carroll, "Lecture Notes on General Relativity," arXiv:`gr-qc/9712019` (v1, 3 Dec 1997). (Includes black holes/geodesic applications used as baseline GR anchors.)