

# Action–Angle Indeterminacy in Central Potentials: A Referee-Safe Witness

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## Abstract

“Action–angle indeterminacy” should not be read as a force-range heuristic (in the style of energy–time slogans), but as a clean conjugacy statement: sharpening an action variable broadens the conjugate angle variable. For central potentials the safest, most explicit instance is the azimuthal pair  $(\phi, L_z)$ : an  $L_z$  eigenstate has  $\phi$ -dependence  $e^{im\phi}$ , hence a uniform azimuthal probability distribution; conversely, any state localized in  $\phi$  must involve a broad superposition of angular-momentum modes (Fourier on the circle). This note records that witness and explains its foundations-level message: classical orbit-phase/orientation pictures correspond to semiclassical packets/superpositions rather than single stationary eigenstates.

## 1. Purpose and scope

This dependent note isolates one specific “action–angle indeterminacy” statement that is both explicit and referee-safe in a central potential:  **$\phi$  is delocalized in an  $L_z$  eigenstate, and conversely localizing  $\phi$  requires a superposition over many  $m$  modes.**

We deliberately keep the scope bounded. We do **not** enter the self-adjoint “angle operator” debate; instead we use the standard circle/Fourier structure and the unitary phase variable  $e^{i\phi}$ . We also do **not** make any claims about the range of forces or potentials; the point here is about **which variables can be simultaneously sharp** in stationary states.

## 2. The safe conjugate pair on the circle: $\phi$ and $L_z$

In spherical coordinates the azimuthal angle is periodic,  $\phi \sim \phi + 2\pi$ . The generator of rotations about the  $z$ -axis is

$$L_z = -i\hbar \frac{\partial}{\partial \phi}.$$

The periodicity makes the naive commutator  $[\phi, L_z] = i\hbar$  subtle if one insists on an everywhere-defined self-adjoint  $\phi$  operator. A standard way to stay on safe ground is to use the unitary “phase” variable

$$E := e^{i\phi}.$$

Acting on  $2\pi$ -periodic wavefunctions,  $E$  is well-defined and satisfies the canonical shift relation

$$[L_z, E] = \hbar E,$$

which already captures the operational content: sharp  $L_z$  implies maximal delocalization in the conjugate angle.

### 3. Central potentials: $L_z$ eigenstates have uniform $\phi$ distribution

For a central potential (or any Hamiltonian commuting with  $L_z$ ), one may choose simultaneous eigenstates of  $L_z$ . In the standard separation of variables, the azimuthal dependence of an angular-momentum eigenstate is the Fourier mode  $e^{im\phi}$  with integer  $m$  (for example in the spherical-harmonic factor  $Y_{\ell m}(\theta, \phi) \propto P_{\ell m}(\cos \theta) e^{im\phi}$ ) [TongQMlectures].

Thus an  $L_z$  eigenstate may be written as

$$\psi(r, \theta, \phi) = F(r, \theta) e^{im\phi}, \quad m \in \mathbb{Z},$$

and therefore

$$|\psi(r, \theta, \phi)|^2 = |F(r, \theta)|^2,$$

independent of  $\phi$ . In particular, the marginal distribution of  $\phi$  is uniform on  $[0, 2\pi)$ . This is the minimal “angle indeterminacy” witness for central potentials.

### 4. Fourier tradeoff: localizing $\phi$ forces a broad $m$ -superposition

Any square-integrable  $2\pi$ -periodic function admits a Fourier series

$$\psi(\phi) = \sum_{m \in \mathbb{Z}} c_m e^{im\phi}, \quad \sum_{m \in \mathbb{Z}} |c_m|^2 < \infty.$$

If only one Fourier mode is present (sharp  $m$ , hence sharp  $L_z$ ), then  $|\psi(\phi)|^2$  is constant; conversely, a state that is peaked in  $\phi$  necessarily uses many Fourier modes (broad  $m$ -support).

**Example 4.1 (Dirichlet-kernel packet).** The normalized superposition of modes  $-M \leq m \leq M$ ,

$$\psi_M(\phi) = \frac{1}{\sqrt{2\pi(2M+1)}} \sum_{m=-M}^M e^{im\phi},$$

is peaked near  $\phi = 0$  with an angular width that scales like  $1/M$ , while its  $m$ -distribution is spread across  $\{-M, \dots, M\}$ . This makes the “sharpening  $\phi \Rightarrow$  broadening  $L_z$ ” tradeoff completely explicit without invoking any disputed angle-operator formalism.

The Fourier tradeoff above can be made into a sharp quantitative bound using only the self-adjoint observables  $\cos \phi$  and  $\sin \phi$ :

**Proposition 4.2 (Circular uncertainty relation).** For any state on the circle, define the circular concentration  $R = |\langle e^{i\phi} \rangle| \in [0, 1]$ . Adding the Robertson inequalities for the two self-adjoint pairs  $(L_z, \cos \phi)$  and  $(L_z, \sin \phi)$  — using  $[L_z, \cos \phi] = i\hbar \sin \phi$  and  $[L_z, \sin \phi] = -i\hbar \cos \phi$  — and the identity  $\text{Var}(\cos \phi) + \text{Var}(\sin \phi) = 1 - R^2$ , gives

$$\text{Var}(L_z) \cdot (1 - R^2) \geq \frac{\hbar^2}{4} R^2.$$

When  $R = 0$  (uniform distribution, as in an  $L_z$  eigenstate) the bound is trivial. As  $R \rightarrow 1$  (sharply localized angle) the bound forces  $\text{Var}(L_z) \rightarrow \infty$ : angular localization requires spreading across many  $m$ -modes. This quantifies the Fourier tradeoff above without invoking a self-adjoint angle operator.

## 5. Foundations message: orbit pictures require packets/superpositions

This witness supports a simple interpretive guardrail for central-force intuition: a single stationary eigenstate (even when it carries classical-sounding quantum numbers) is typically **not** a localized classical orbit with a definite phase/orientation. Variables like the azimuthal phase  $\phi$  (and, in more structured integrable cases, other angle variables on the invariant torus) become localized only in **coherent superpositions** of many stationary modes.

In other words, “classical orbit pictures” correspond to semiclassical packets and stationary-phase concentration, not to exact eigenstates that are sharp in the conserved actions.

## 6. A second witness: the harmonic oscillator

The same structure appears in the simplest one-dimensional integrable system.

**Example 6.1 (Harmonic oscillator: Fock states vs coherent states).** For a harmonic oscillator of frequency  $\omega$ , define the classical action variable  $J = E/\omega$ . The quantum Fock states  $|n\rangle$  are the action eigenstates ( $J_n = (n + \frac{1}{2})\hbar$ ), and their phase-space (Husimi) distribution is a ring centered at the origin — the orbit phase  $\theta$  is uniformly delocalized, exactly as  $\phi$  is

delocalized in an  $L_z$  eigenstate. Conversely, a coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \alpha = |\alpha| e^{i\theta_0},$$

is the closest quantum analog of a classical orbit with definite amplitude  $|\alpha|$  and phase  $\theta_0$ . Its Fock-state weights follow a Poisson distribution with mean  $\bar{n} = |\alpha|^2$ , so localizing the phase to width  $\Delta\theta \sim 1/|\alpha|$  requires spreading the action over  $\Delta n \sim |\alpha|$  modes. The tradeoff is the same as in Section 4: sharp action implies delocalized phase, and vice versa.

**Example 6.2 (Hydrogen atom: three action-angle pairs).** In the hydrogen atom, the  $n^2$ -fold degeneracy ( $E_n$  depending only on the principal quantum number  $n$ ) reflects an enhanced  $SO(4)$  symmetry [Sakurai2020]. Semiclassically, the bound orbits lie on a three-torus with action integrals quantized by  $(n, \ell, m)$ . A stationary eigenstate  $|n, \ell, m\rangle$  is sharp in all three actions and therefore delocalized in all three conjugate angles: the azimuthal phase  $\phi$  is uniform (Section 3), the in-plane orbit orientation has no preferred direction (the Runge–Lenz vector has vanishing expectation value, since it connects states of different  $\ell$ ), and the radial probability  $|R_{n\ell}(r)|^2$  is time-independent — the sharp radial action leaves the conjugate radial phase uniformly delocalized. A classical Keplerian ellipse with definite eccentricity, orientation, and timing requires a coherent superposition over ranges of  $(n, \ell, m)$ , just as a coherent state in Example 6.1 requires superposing many Fock states.

## 7. Outlook (kept minimal)

The preceding witnesses illustrate the action–angle tradeoff in systems with one, two, and three action–angle pairs. More generally, any integrable system with  $d$  degrees of freedom admits  $d$  action–angle pairs on an invariant torus, and EBK/WKB quantization makes the same structural point: the more sharply the actions are specified, the less information remains in the conjugate phases.

## References

1. [TongQMlectures] David Tong, “Quantum Mechanics” (lecture notes, no DOI). OA: lecture-note PDF. (Contains  $Y_{l,m}(\theta, \phi) = P_{l,m}(\cos \theta) e^{im\phi}$  as simultaneous eigenstates of  $L^2$  and  $L_z$ .)
2. [Sakurai2020] J. J. Sakurai and Jim Napolitano, *Modern Quantum Mechanics*, 3rd ed., Cambridge University Press, 2020. ISBN 978-1-108-47322-4. (Standard treatment of angular momentum, spherical harmonics, and quantum measurement.)