

# Half-Densities in QFT: Propagators as Bi-Half-Density Kernels

## Abstract

In QFT, the basic free object is the inverse of a kinetic operator, i.e. a propagator/Green kernel. On a manifold, writing “ $P_x G(x, y) = \delta(x, y)$ ” hides conventions: which volume form defines the adjoint, and which delta normalization realizes the identity. This note adopts a single organizing choice: treat fields (or kernels) as **half-densities**, so the identity kernel is canonical and kernel composition is coordinate-invariant without choosing a background measure. A worked computation shows how a scalar field on  $(M, g)$  becomes a half-density  $\psi = |g|^{1/4} \phi$ , with kinetic operator  $\tilde{P} = |g|^{1/4} P |g|^{-1/4}$  symmetric in the coordinate pairing. We also record a kernel-level remark: local counterterms/contact terms appear as distributions supported on the diagonal ( $x = y$ ) (delta kernels and their derivatives), which are most naturally expressed using the canonical bi-half-density delta.

This note is written to be readable on its own; it also connects to broader themes (scalarization scales and RG as compatibility) developed elsewhere.

## 1. Purpose and Scope

This note is intentionally narrow: 1. establish the “kernel as bi-half-density” semantics for spacetime propagators in QFT, 2. isolate what is **canonical** (half-density kernels, identity delta kernel) versus what is **a convention** (scalarization choices such as  $\sqrt{|g|}$ ), 3. give one explicit computation that can later be promoted (densitized scalar field).

BV/BRST/field-space half-densities are only flagged as outlook here; a full treatment would require additional dedicated sources and is beyond scope.

## 2. Kernels on a Manifold: half-densities make the identity canonical

Let  $M$  be a  $D$ -dimensional manifold. A half-density is a section of  $|\Lambda^D T^* M|^{1/2}$ .

The key operational point (as in the main paper's kernel-composition spine) is: if an operator acts on half-densities, then its Schwartz kernel is naturally a **bi-half-density**

$$K_A(x, y) = a(x, y) |dx|^{1/2} |dy|^{1/2},$$

and composition is intrinsic:

$$(A \circ B)(x, z) = \int_M K_A(x, y) K_B(y, z),$$

because the product in the intermediate variable  $y$  is a density.

**Derivation D1.1 (Identity kernel).** The identity operator on half-densities has kernel

$$K_{\text{Id}}(x, y) = \delta^{(D)}(x - y) |dx|^{1/2} |dy|^{1/2},$$

which is canonical: it does not require choosing a background density/volume form.

**Derivation D1.1a (Normalization witness: why  $\varepsilon^{-D/2}$  appears).** In local coordinates on  $\mathbb{R}^D$ , a standard approximate identity is the Gaussian family

$$\rho_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{D/2}} \exp\left(-\frac{|x|^2}{2\varepsilon}\right), \quad \varepsilon > 0.$$

The exponent  $D/2$  is forced by normalization: by the change of variables  $x = \sqrt{\varepsilon} u$ ,

$$\int_{\mathbb{R}^D} \rho_\varepsilon(x) d^D x = (2\pi\varepsilon)^{-D/2} \varepsilon^{D/2} \int_{\mathbb{R}^D} e^{-|u|^2/2} d^D u = 1.$$

Thus  $\rho_\varepsilon \rightharpoonup \delta^{(D)}$  as  $\varepsilon \rightarrow 0^+$ , and the diagonal delta kernel is the distributional limit of families whose normalization scales as  $\varepsilon^{-D/2}$ .

### 3. Worked computation: densitized scalar field

$$\psi = |g|^{1/4} \phi$$

Consider a real scalar field on a fixed Lorentzian/Euclidean background  $(M, g)$  with quadratic action

$$S[\phi] = \frac{1}{2} \int_M d^D x \sqrt{|g|} \phi P \phi, \quad P = -\nabla^2 + m^2 + \xi R \quad (\text{example}).$$

The pairing for which  $P$  is symmetric is

$$(\phi_1, \phi_2)_g = \int d^D x \sqrt{|g|} \phi_1 \phi_2.$$

Define the densitized field (a half-density in coordinates)

$$\psi := |g|^{1/4}\phi, \quad \text{so} \quad \phi = |g|^{-1/4}\psi.$$

Then the action becomes

$$S[\phi] = \frac{1}{2} \int d^D x \, \psi \, \tilde{P} \, \psi, \quad \tilde{P} := |g|^{1/4} P |g|^{-1/4},$$

so the pairing is now just the coordinate density  $d^D x$ .

**Derivation D1.2 (Explicit form of the conjugated kinetic operator).**

For  $P_{\text{kin}} = -\nabla^2$  one has in coordinates

$$\nabla^2 \phi = |g|^{-1/2} \partial_\mu \left( \sqrt{|g|} g^{\mu\nu} \partial_\nu \phi \right),$$

hence

$$\tilde{P}_{\text{kin}} = -|g|^{1/4} \nabla^2 |g|^{-1/4} = -|g|^{-1/4} \partial_\mu \left( \sqrt{|g|} g^{\mu\nu} \partial_\nu (|g|^{-1/4} \cdot) \right).$$

Assuming compact support (or boundary conditions killing boundary terms),

$$\int d^D x \, \psi_1 \tilde{P}_{\text{kin}} \psi_2 = \int d^D x \, \sqrt{|g|} g^{\mu\nu} \partial_\mu (|g|^{-1/4} \psi_1) \partial_\nu (|g|^{-1/4} \psi_2),$$

which is manifestly symmetric under  $(1 \leftrightarrow 2)$ .

**Derivation D1.3 (Conformal metric expansion; D=4 simplification in the conformal class).** As a worked expansion, take a conformally flat background  $g_{\mu\nu} = e^{2\sigma(x)} \delta_{\mu\nu}$  (Euclidean for simplicity). Then  $\sqrt{|g|} = e^{D\sigma}$ ,  $|g|^{1/4} = e^{D\sigma/2}$ ,  $g^{\mu\nu} = e^{-2\sigma} \delta^{\mu\nu}$ , and one finds

$$\Delta_g f = |g|^{-1/2} \partial_\mu \left( \sqrt{|g|} g^{\mu\nu} \partial_\nu f \right) = e^{-2\sigma} (\partial^2 f + (D-2) \partial\sigma \cdot \partial f).$$

Setting  $\phi = |g|^{-1/4} \psi = e^{-D\sigma/2} \psi$  and expanding derivatives gives the conjugated operator

$$\tilde{\Delta} \psi := |g|^{1/4} \Delta_g |g|^{-1/4} \psi = e^{-2\sigma} \left( \partial^2 \psi - 2 \partial\sigma \cdot \partial \psi - \frac{D}{2} (\partial^2 \sigma) \psi + \frac{D(4-D)}{4} (\partial\sigma)^2 \psi \right),$$

so the kinetic operator  $\tilde{P}_{\text{kin}} = -\tilde{\Delta}$  contains a term proportional to  $D(4-D)(\partial\sigma)^2$ , which cancels at  $D = 4$  in this conformal ansatz.

Scope disclaimer: this is recorded only as a checked simplification for  $\tilde{\Delta}$  in this metric class; it is not, by itself, a dimension-selection claim or a conformal-invariance statement. A symbolic coefficient/sign check (SymPy) confirms the expansion.

Interpretation: - the metric half-density  $|g|^{1/4} |dx|^{1/2}$  is a **scalarization gauge** (a choice of reference half-density) on a fixed background, - writing the field as  $\psi$  makes the “half-density priority” viewpoint explicit: both the field and the kernels live naturally as half-density objects.

## 4. Propagators/Green functions as bi-half-density kernels

Let  $\tilde{G}$  be the inverse kernel of  $\tilde{P}$  with respect to the coordinate pairing:

$$(\tilde{P}^{-1}f)(x) = \int \tilde{G}(x, y) f(y) d^D y, \quad \tilde{P}_x \tilde{G}(x, y) = \delta^{(D)}(x - y).$$

Then the corresponding canonical bi-half-density kernel is

$$K_{\tilde{P}^{-1}}(x, y) = \tilde{G}(x, y) |dx|^{1/2} |dy|^{1/2}.$$

Equivalently, if  $G_g(x, y)$  denotes the usual **scalar** Green function for  $P$  defined with respect to the metric pairing  $\int \sqrt{|g|} d^D y$  (so  $(P^{-1}J)(x) = \int G_g(x, y) J(y) \sqrt{|g(y)|} d^D y$ ), then the kernels are related by

$$\tilde{G}(x, y) = |g(x)|^{1/4} G_g(x, y) |g(y)|^{1/4}.$$

This is exactly the same “kernel as bi-half-density” structure used for QM propagators in the main manuscript, now applied to spacetime Green functions in QFT.

**Remark D4.1 (Doubling: densities live on  $(M \times M)$ ).** Half-density kernels also make the amplitude-vs-density doubling completely explicit. Let  $U_t$  be a (unitary) evolution operator on half-densities with kernel  $K_t(x, y)$ . Then a density operator  $\rho_t = U_t \rho_0 U_t^{-1}$  has a kernel satisfying

$$K_{\rho_t}(x, y) = \int_{M \times M} K_t(x, x') K_{\rho_0}(x', y') \overline{K_t(y, y')}.$$

So densities naturally propagate by a kernel on the doubled space  $M \times M$ , built from the forward kernel and its conjugate. This is the kernel-level origin of bra/ket (forward/backward) doubling in real expectation values; a fuller discussion is beyond this note’s scope.

## 5. Contact terms and counterterms as diagonal delta kernels (kernel-level remark)

In QFT, divergences are removed by adding local counterterms to the action, e.g.  $\delta m^2 \phi^2$ ,  $\delta Z (\partial\phi)^2$ , curvature couplings, etc. At the operator/kernel level this means: the kinetic operator  $P$  is modified by local (differential) operators, and therefore its kernel acquires **distributions supported on the diagonal**  $x = y$ .

In the half-density kernel language the diagonal object is canonical:

$$K_{\text{Id}}(x, y) = \delta^{(D)}(x - y) |dx|^{1/2} |dy|^{1/2}.$$

Multiplication counterterms correspond to  $c(x) K_{\text{Id}}(x, y)$ , and derivative counterterms correspond to derivatives acting on the delta kernel (e.g.  $\partial_x \delta^{(D)}(x - y)$  and higher).

**Remark D5.1 (Derivative of the diagonal delta kernel; coordinate-free identity).** The slogan “ $\partial_x \delta(x - y) = -\partial_y \delta(x - y)$ ” has a clean, connection-free formulation in the half-density kernel calculus. The identity kernel  $K_{\text{Id}}$  is invariant under diagonal diffeomorphisms  $(\Phi \times \Phi)$ , so for any vector field  $V$  on  $M$  one has

$$(\mathcal{L}_{V_x} + \mathcal{L}_{V_y}) K_{\text{Id}} = 0, \quad \text{hence} \quad \mathcal{L}_{V_x} K_{\text{Id}} = -\mathcal{L}_{V_y} K_{\text{Id}},$$

where  $\mathcal{L}$  is the Lie derivative acting on half-densities. In local coordinates, taking  $V = \partial/\partial x^\mu$  recovers  $\partial_{x^\mu} \delta^{(D)}(x - y) = -\partial_{y^\mu} \delta^{(D)}(x - y)$ . This is the kernel-level mechanism behind “moving derivatives between slots” in integration by parts and in contact-term identities.

**Remark D5.2 (Point splitting produces ‘contact terms’).** Point splitting makes the simplest derivative contact term explicit: in one dimension,

$$\frac{\delta(x + \varepsilon) - \delta(x)}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \delta'(x), \quad \langle \delta', \varphi \rangle = -\varphi'(0).$$

This limit is the distributional companion of the “difference quotient as divergence + subtraction” toy model.

This framing is useful for two reasons: 1. it makes “locality = diagonal support” literal at the kernel level, 2. it separates the canonical distributional objects from scheme-dependent scalarizations and finite-subtraction conventions.

## 6. Link to the half-density scale program (where Planck-area enters conditionally)

On a fixed background  $(M, g)$ , the metric provides a natural reference half-density  $|g|^{1/4} |dx|^{1/2}$ . The Planck-area program begins only when we ask for a **background-free** scalarization convention that turns half-density amplitudes into universal dimensionless numbers. A separate note develops that stronger hypothesis ladder; it is not needed for the present kernel/propagator semantics.

This paper’s role is only to show that half-densities are not a QM quirk: the same kernel semantics is already present in standard QFT propagator definitions, once the hidden measure conventions are made explicit.

## 7. Outlook: BV half-densities

Gauge theories suggest a second, deeper appearance of half-densities: the BV formalism treats the integrand as a (half-)density on an (odd) symplectic space

of fields/antifields, and the quantum master equation expresses independence of gauge-fixing choices. This note does not develop BV beyond this remark; doing so responsibly would require additional authoritative sources and a separate dedicated treatment.