“Uncuttable” as Controlled Refinement

Abstract

This note fixes a project-internal meaning of “uncuttable” aligned with the refinement-compatibility thesis of `paper/main.md`. Here **uncuttable** does not mean “indivisible.” It means: the quantity of interest is not determined by any *finite* dissection alone; it is a **limit object** whose definition requires a refinement rule and a comparison structure across refinements.

The point is structural and mathematical: once a theory is built from composable local pieces, the continuum theory is the stable target of a refinement limit, and extra control data may be required for that limit to exist or be unique.

1. Definition

Call a quantity Q **uncuttable** (in this note’s sense) if: 1. there exists a family of finite approximants Q_N produced by a finite dissection/refinement scheme of depth N, but 2. the value of interest is not any finite Q_N; it is a controlled limit $Q = \lim_{N\to\infty} Q_N$, and 3. specifying the *rule of refinement* and the *comparison across refinements* is part of the definition of Q.

This is the ordinary situation in analysis: finite partitions approximate, but the object is defined by a limiting procedure together with hypotheses that ensure convergence/uniqueness.

2. Toy model: an integral is already a refinement limit

Let $f : [a,b] \to \mathbb{R}$. A prototypical refinement family is a partition $P_N = \{a = t_0 < \cdots < t_N = b\}$ with mesh $\|P_N\| := \max_k (t_{k+1} - t_k) \to 0$. Define the Riemann-sum approximants

$$Q_N := \sum_{k=0}^{N-1} f(\xi_k)\,(t_{k+1} - t_k), \qquad \xi_k \in [t_k, t_{k+1}].$$

In good cases, $Q_N \to \int_a^b f(t)\,dt$ as $\|P_N\| \to 0$, and the limit is independent of the tags ξ_k. But this is not a tautology: the limit can fail to exist or can depend on the refinement rule unless hypotheses are stated.

In the present program, this is the basic moral: finite cuts approximate, but the value is defined by **controlled refinement**.

3. Dynamics: action, stationarity, and the need for control data

The cornerstone manuscript uses the same template in mechanics. Given a partition of time, the discrete action

$$S_N[q] = \sum_k \mathcal{L}\left(q_k, \frac{q_{k+1} - q_k}{\Delta t_k}, t_k\right) \Delta t_k$$

is a finite refinement approximant. The continuum action $S[q] = \int \mathcal{L}\, dt$ is a refinement limit.

Two "uncuttable" features appear immediately when one pushes beyond smooth classical paths: 1. **Singular probes and corners:** stationarity must be interpreted in weak/distributional form; point-supported variations require mollification. 2. **Non-uniqueness of refinement schemes:** different discretization conventions (even if classically equivalent) can produce distinct refined objects unless an equivalence or control map is specified.

These are exactly the obstructions enumerated in `paper/main.md` (`Heuristic H0.2`): the point is not indivisible atoms, but limit control.

4. Outlook: refinement compatibility as "the extra structure"

In this repo, the "extra structure" used to control refinement limits is made explicit: - half-densities make kernel composition coordinate-free without hidden measure choices, - control maps τ encode how parameters must flow under refinement to maintain stability, - renormalization is the compatibility rule when naive refinement limits diverge.

This note is therefore a small conceptual bridge: it isolates an early, analysis-level instance of the same meta-problem that reappears in quantization and in QFT.