Planck Area from Half-Density Normalization

Abstract

Half-densities are the natural "coordinate-free integrands" for composing kernels without choosing a background measure. But choosing a *universal* convention for turning half-density objects into dimensionless numerical amplitudes introduces a length$^{d/2}$ scale. In $d = 4$, this is an *area*. This note sharpens the hypothesis ladder needed for the claim "half-density normalization selects a universal area scale", and isolates a simple dimension-matching condition under which the Planck area appears without fractional powers of couplings. A gravitational anchor based on a minimal-areal-speed principle is recorded as a separate heuristic thread [RiveroAreal] [RiveroSimple].

Minimal claim (proved under stated hypotheses): half-density composition is intrinsic, but any scalar "amplitude function" representation requires extra scalarization structure, and if that scalar representative is demanded dimensionless then a length$^{d/2}$ constant is unavoidable. Conjectural claim (additional universality hypotheses): the scalarization constant is fixed by universal couplings and becomes a universal area scale in $D = 4$.

1. Purpose and Scope

This note isolates one technical point that is often implicit in treatments of path integrals and quantum composition: the role of half-densities (and their scaling) in making composition laws coordinate-invariant *and* dimensionally well-defined.

Claims below are labeled as `Proposition` (math-precise under hypotheses) or `Heuristic` (programmatic bridge).

Notation (dimensions). The half-density weight length$^{d/2}$ uses the dimension of the manifold being integrated over in the composition law (the "composition variable" dimension). In nonrelativistic time-slicing this is typically spatial dimension; in covariant/field-theory settings it is spacetime dimension. In the gravity-only sieve (Derivation PA-D1.3), read d as the spacetime dimension D if that avoids confusion: the "$D = 4 \Rightarrow$ area" conclusion is about spacetime $D = 4$, not a claim that *spatial* $d = 4$ is privileged.

2. Half-Densities and Composition Kernels

Let M be a d-dimensional manifold. A (positive) density is a section of $|\Lambda^d T^*M|$, and a half-density is a section of $|\Lambda^d T^*M|^{1/2}$.

The key operational point is: when a kernel is a half-density in its integration variable, composition of kernels does not depend on an arbitrary choice of coordinate measure.

`Heuristic PA-H1.1 (Why half-densities).` If $K_1(x, y)$ and $K_2(y, z)$ are chosen so that their product in the intermediate variable y is a density, then $\int_M K_1(x, y)K_2(y, z)$ is coordinate-invariant without fixing a preferred dy. This is the structural reason why kernel composition in the path-integral formalism is coordinate-invariant.

`Derivation PA-D1.1 (Coordinate invariance of half-density pairing and composition).` In a local chart $y = (y^1, \dots, y^d)$, write a half-density as $\psi(y) = \varphi(y)\,|dy|^{1/2}$. Under a change of variables $y = y(y')$, one has $|dy|^{1/2} = |\det(\partial y/\partial y')|^{1/2}|dy'|^{1/2}$, so the coefficient transforms as $\varphi'(y') = \varphi(y(y'))\,|\det(\partial y/\partial y')|^{1/2}$.

Hence the product of two half-densities is a density: $\psi_1\psi_2 = (\varphi_1\varphi_2)\,|dy|$, and its integral is chart-independent: $\int_M \psi_1\psi_2$ is well-defined without choosing a background measure beyond the density bundle itself.

Kernel composition is the same mechanism: if $K_1(x, y)$ and $K_2(y, z)$ are half-densities in y, then K_1K_2 is a density in y and $\int_M K_1K_2$ is coordinate invariant.

3. Dimensional Analysis: Normalizing a Half-Density Requires a Scale

A density on M carries the units of length^d once physical units are assigned to coordinates. A half-density therefore carries units length$^{d/2}$.

`Proposition PA-P1.1 (No canonical "half-density = function" identification).` There is no canonical identification of a half-density $\psi \in |\Lambda^d T^*M|^{1/2}$ with an ordinary scalar function f on M. Choosing such an identification is equivalent to choosing a nowhere-vanishing reference half-density σ_* (equivalently a positive density $\rho_* = \sigma_*^2$) and writing $\psi = f\,\sigma_*$.

`Derivation PA-D1.2 (Dilation makes the \(\text{length}^{d/2}\) weight explicit).` On $\mathbb{R}^d$, consider a dilation $y \mapsto y' = ay$ with $a > 0$. Then $|dy'| = a^d|dy|$, so $|dy'|^{1/2} = a^{d/2}|dy|^{1/2}$. Thus even in flat space, half-densities carry an inherent length$^{d/2}$ scaling weight.

`Derivation PA-D1.2a (Near-diagonal scaling forces the square-root Jacobian \(\varepsilon^{-d/2}\)).` On $M = \mathbb{R}^d$, introduce near-diagonal coordinates $y = x + \varepsilon v$ with $\varepsilon > 0$. Then $dy = \varepsilon^d\,dv$, hence $|dy|^{1/2} = \varepsilon^{d/2}|dv|^{1/2}$.

For a bi-half-density kernel written locally as

$$K_\varepsilon(x,y) = k_\varepsilon(x,y)\,|dx|^{1/2}|dy|^{1/2},$$

its pullback to (x,v) variables becomes

$$K_\varepsilon(x, x+\varepsilon v) = \left(\varepsilon^{d/2} k_\varepsilon(x, x+\varepsilon v)\right)|dx|^{1/2}|dv|^{1/2}.$$

Thus, any attempt to define a nontrivial "$\varepsilon \to 0$" near-diagonal limit of kernels (the scaling step that tangent-groupoid quantization packages) inevitably produces an $\varepsilon^{d/2}$ factor from the half-density Jacobian, and the corresponding scalar representative must be renormalized by $\varepsilon^{-d/2}$ to stay finite. This is the same exponent as in the finite-dimensional "square-root delta" normalization: the half-density is the square root of the density Jacobian. For a canonical distributional witness carrying the same exponent, see Derivation PA-D1.2b.

`Derivation PA-D1.2b (Identity kernel: the delta bi-half-density carries the same exponent).` The identity operator on half-densities has Schwartz kernel

$$K_{\mathrm{Id}}(x,y) = \delta^{(d)}(x-y)\,|dx|^{1/2}|dy|^{1/2}.$$

Under the same near-diagonal change of variables $y = x+\varepsilon v$, one has $\delta^{(d)}(x-y) = \varepsilon^{-d}\delta^{(d)}(v)$ and $|dy|^{1/2} = \varepsilon^{d/2}|dv|^{1/2}$, hence

$$K_{\mathrm{Id}}(x, x+\varepsilon v) = \varepsilon^{-d/2}\,\delta^{(d)}(v)\,|dx|^{1/2}|dv|^{1/2}.$$

So the universal $\varepsilon^{-d/2}$ exponent is already encoded in the half-density identity kernel. If one scalarizes by choosing a dimensionless reference half-density $\sigma_* := L_*^{-d/2}|dx|^{1/2}$ (for a constant length scale L_*), then the scalar representative of the identity kernel is the **dimensionless** distribution $k_{\mathrm{Id}}(x,y) = L_*^d\delta^{(d)}(x-y)$.

`Proposition PA-P1.2 (Universal *dimensionless* amplitudes force a \(\text{length}^{d/2}\) constant).` If one imposes the extra requirement that the scalar representative f in $\psi = f\,\sigma_*$ be dimensionless in physical units, then the reference half-density σ_* must carry all of the length$^{d/2}$ dimension. In particular, a *constant* (field-independent) choice of σ_* is equivalent to choosing a universal length$^{d/2}$ scale.

In $d = 4$, this universal length$^{d/2}$ scale is a universal *area* scale.

`Heuristic PA-H1.2 (Reciprocity claim).` Half-densities alone do not force a particular scale: the forced fact is that converting half-density objects into scalar numerical amplitudes requires extra structure (a reference half-density). The "universal area scale" claim begins only after adding two further hypotheses: 1. the reference σ_* is taken to be *constant* (no dependence on background metric/fields), and 2. the constant is required to be fixed by universal constants/couplings of the theory.

Under these hypotheses, $d = 4$ is the unique dimension in which the needed length$^{d/2}$ constant can be supplied by the gravitational coupling without fractional powers (Derivation PA-D1.3).

`Derivation PA-D1.3 (Gravity-only sieve: why \(d=4\) is singled out if only \(G_d\) is used).` In d spacetime dimensions, the Einstein–Hilbert action $\frac{1}{16\pi G_d}\int d^dx\,\sqrt{|g|}\,R$ shows that (in $c=\hbar=1$ units) Newton's constant has dimension $[G_d]=\mathrm{length}^{d-2}$. Assume the only available dimensionful coupling used to build the universal normalization constant is G_d itself (no cosmological constant, no additional dimensionful scales), and impose PA-H2.5 in the literal "no fractional powers of G_d" sense. Then the normalization constant has dimension $\mathrm{length}^{k(d-2)}$ for some integer k. Matching $\mathrm{length}^{d/2}$ forces $\mathrm{length}^{d/2}=\mathrm{length}^{d-2}$, which holds if and only if $d=4$. In that case G_4 itself has dimension of area, and the corresponding area scale is the Planck area $L_P^2\sim \hbar G_4/c^3$.

`Remark PA-D1.3a (Three obstruction mechanisms: why \(d=4\) is uniquely un-obstructed).` The failure of the gravity-only sieve away from $d=4$ has three qualitatively distinct causes. For odd d, the target exponent $d/2$ is a half-integer, while any monomial in couplings with integer length dimensions has integer total dimension — a *parity* obstruction that is categorical regardless of which integer-dimensional couplings are admitted (including Λ_d). For even $d\geq 6$, the target $d/2$ is an integer but $k=d/(2(d-2))<1$, so no positive-integer power of G_d alone suffices — a *magnitude* obstruction. For $d=2$, G_2 is dimensionless and gravity provides no scale. Thus $d=4$ is not merely the solution of a single Diophantine equation; it is the unique integer dimension that evades all three obstructions simultaneously.

3.1 Hypotheses as Separate Knobs (What Is Forced vs Chosen)

The discussion above mixes three different kinds of statements: 1. **Geometric facts** (what half-densities are, how they compose, how they scale), 2. **Representational choices** (how one turns half-density objects into scalar numbers), 3. **Universality/selection principles** (what choices are allowed if we demand "background-free" and "built from couplings").

To study these separately, it is useful to keep the hypotheses explicit.

`Hypothesis PA-H2.1 (Half-density formulation).` Quantum kernels are treated as bi-half-densities so that composition in intermediate variables is coordinate invariant (Section 2 and Derivation PA-D1.4).

`Hypothesis PA-H2.2 (Scalarization by a reference half-density).` To interpret half-density amplitudes as scalar numerical functions, we pick a nowhere-vanishing reference half-density σ_* and write $\psi=f\,\sigma_*$ (Proposition PA-P1.1).

`Hypothesis PA-H2.3 (Dimensionless scalar representative).` The scalar representative f is required to be dimensionless in physical units (Proposition PA-P1.2). This forces σ_* to carry the full $\mathrm{length}^{d/2}$ weight.

`Hypothesis PA-H2.4 (Background-free constancy).` The reference σ_* is taken to be constant/field-independent, rather than determined by background geometry (e.g. a Riemannian volume $|g|^{1/4}|dx|^{1/2}$) or by dynamical fields (e.g. a dilaton-like factor). This is the first point where a *universal constant* enters.

`Hypothesis PA-H2.5 (Analyticity / no fractional powers of couplings).` If the universal constant is required to be built from the theory's couplings without fractional powers, then dimensional analysis becomes a *dimension sieve* rather than a tautology. This hypothesis has at least two distinct readings: 1. **Integrality (integer-exponent) reading:** the constant is a monomial in the available couplings with integer exponents (possibly allowing negative powers). Equivalently, dimension-matching becomes an integer (Diophantine) constraint on the exponents. 2. **Perturbative analyticity reading (stronger):** the constant admits a Taylor expansion around a specified weak-coupling limit, so only nonnegative integer powers appear.

To keep the branches explicit, we will refer to these as: `- Hypothesis PA-H2.5a (Integrality / monomial sieve).` The normalization constant is a monomial in admitted couplings with integer exponents. `- Hypothesis PA-H2.5b (Perturbative analyticity sieve).` The dependence is analytic at a specified weak-coupling point (hence forbids negative powers and other non-analytic dependence).

`Heuristic PA-H2.5b1 (Analyticity for dimensionful couplings needs a reference scale).` If a coupling g_i is dimensionful, "weak coupling" is only meaningful after choosing a reference scale μ and forming a dimensionless parameter $\hat{g}_i(\mu) := \mu^{-a_i} g_i$ (where $[g_i] = \text{length}^{a_i}$). In that sense, perturbative analyticity is naturally analyticity in $\hat{g}_i(\mu)$ near $\hat{g}_i = 0$. Demanding at the same time that the scalarization constant be a μ-independent universal constant pushes one back to either: 1. an engineering-dimension monomial in admitted dimensionful couplings with nonnegative integer exponents (the sieve branch), or 2. a non-analytic RG-invariant transmutation scale (outside PA-H2.5b).

Separately, the RG "dimensional transmutation" mechanism (Heuristic PA-H2.13 / Derivation PA-D1.6a) supplies a scale *outside* both PA-H2.5a and PA-H2.5b: it is typically non-analytic in the coupling.

Derivation PA-D1.3 is the simplest gravity-only instance under the integrality reading: "use G_d without fractional powers" singles out $d = 4$.

`Heuristic PA-H2.6 (Where "special dimensions" can appear).` Special dimensions do not come from half-densities alone (Hypothesis PA-H2.1). They appear only after adding a selection principle like PA-H2.4–PA-H2.5: the requirement that the scalarization choice be universal, background-free, and coupling-built in a restricted (e.g. analytic) way.

`Heuristic PA-H2.6a (Independent D=4 filter: conformal scalarization-gauge simplicity).` The "dimension sieve" (PA-H2.5) is not the only way a special dimension can appear once one insists on scalar representatives. A different kind of filter comes from asking for *simplicity of how kinetic operators depend on the scalarization gauge.*

In the QFT-facing half-density calculus, a background metric supplies a reference half-density $|g|^{1/4}|dx|^{1/2}$, and the scalar Laplacian Δ_g conjugates to an operator on half-densities

$$\widetilde{\Delta} := |g|^{1/4}\Delta_g|g|^{-1/4}.$$

In a conformal background $g_{\mu\nu} = e^{2\sigma(x)}\delta_{\mu\nu}$ in spacetime dimension D, a direct computation gives

$$\widetilde{\Delta}\psi = e^{-2\sigma}\Big(\partial^2\psi - 2\,\partial\sigma\cdot\partial\psi - \frac{D}{2}(\partial^2\sigma)\,\psi + \frac{D(4-D)}{4}(\partial\sigma)^2\,\psi\Big).$$

Thus the quadratic-gradient term $(\partial\sigma)^2\psi$ cancels at $D = 4$ (within the conformal ansatz). If one adopts the extra criterion that scalarization-gauge changes should not generate such quadratic-gradient "potentials" in the half-density kinetic operator, then $D = 4$ is singled out by *operator simplicity* rather than by coupling-dimension matching.

This filter is independent of PA-H2.5 and, by itself, does not supply a length scale; it is recorded only as an additional "special dimension" candidate knob to compare against the scale-sieve hypotheses.

3.2 What Changes When a Hypothesis Is Relaxed?

This subsection records the main "branches" that need separate study.

1. **Drop PA-H2.3 (allow dimensionful f).** Then no universal length$^{d/2}$ constant is forced; the dimensional weight can be carried by the scalar representative itself (as in the usual statement "wavefunctions have dimension length$^{-d/2}$").
2. **Drop PA-H2.4 (allow background geometry).** Then σ_* can be chosen from a metric (or other structure), and the "universal constant" is replaced by background-dependent normalization.
3. **Drop PA-H2.5a/PA-H2.5b (allow fractional powers or other non-analytic dependence).** Then in any $d > 2$ one can build a length$^{d/2}$ constant from gravity via $G_d^{d/(2(d-2))}$ (in $c = \hbar = 1$ units), so $d = 4$ is no longer singled out; instead, $d = 4$ is simply the unique case where the exponent is an integer.
4. **Change "which coupling supplies the scale".** Using other dimensionful couplings (cosmological constant, string tension, gauge couplings in various dimensions, etc.) yields different "special-dimension" sieves. This is conceptually aligned with the observation that some dimensions are singled out by other structures (division algebras, special holonomy, supersymmetry), but those filters are separate from the half-density story and should not be conflated.

5. **Replace PA-H2.5 by transmutation (RG as the scale supplier).** If one allows RG-invariant scales generated by compatibility (dimensional transmutation), then even dimensionless couplings can supply a physical length scale (Heuristic PA-H2.13 / Derivation PA-D1.6a). This branch does not "sieve dimensions" in the same way as PA-H2.5a/PA-H2.5b; it supplies a scale by non-analytic RG invariants rather than by monomials in couplings.

Branch summary (keep the knobs separate). There are three distinct "scale supplier" mechanisms in play: 1. **Monomial sieve** (PA-H2.5a/PA-H2.5b): build the needed length$^{d/2}$ constant from admitted couplings using restricted (e.g. analytic) dependence. 2. **Fractional/non-analytic coupling dependence:** allow fractional powers or other non-analytic dependence directly in couplings (dimensional analysis becomes permissive; the $d = 4$ sieve disappears). 3. **Dimensional transmutation:** generate a scale from RG compatibility (Derivation PA-D1.6a / Example PA-E5), which is typically non-analytic in naive coupling expansions.

3.3 Starting with PA-H2.5a: Integrality as a Dimension Sieve

The point of PA-H2.5a/PA-H2.5b is not that dimensional analysis alone selects a unique scale (it does not), but that *restricting allowed functional dependence on couplings* can turn dimensional analysis into a selection principle.

`Derivation PA-D1.6 (PA-H2.5a: integer-exponent form of "no fractional powers").` Work in $c = \hbar = 1$ units for dimension counting. Let the available couplings $\{g_i\}$ have length dimensions $[g_i] = \text{length}^{a_i}$. Under the integrality reading of PA-H2.5, the universal normalization constant is a monomial $C = \prod_i g_i^{n_i}$ with integers n_i. Its length dimension is $[C] = \text{length}^{\sum_i n_i a_i}$. Requiring $[C] = \text{length}^{d/2}$ is therefore the integer-exponent (Diophantine) condition

$$\sum_i n_i a_i = \frac{d}{2}.$$

Existence (and non-uniqueness) of solutions depends on: 1. which couplings are admitted as "universal" inputs, and 2. whether one allows negative exponents (non-analytic at zero coupling) or insists on perturbative analyticity (nonnegative exponents).

`Heuristic PA-H2.7 (Why PA-H2.5 needs a "what counts as a coupling" rule).` If one allows arbitrary redefinitions of couplings (e.g. adjoining a new symbol $\tilde{G} = G_d^{1/(d-2)}$), then "no fractional powers" becomes vacuous: the forbidden root has simply been renamed as an allowed coupling. PA-H2.5 is meaningful only together with a prior criterion for admissible coupling dependence (e.g. perturbative analyticity around a distinguished limit such as $G_d \to 0$).

`Heuristic PA-H2.7a (Admissible couplings: exclude scheme parameters and non-analytic reparametrizations).` To keep PA-H2.5 from collapsing into a coordinate artifact, we implicitly adopt the following convention: 1. **Admitted couplings** are the independent parameters that appear as coefficients of local operators in the UV action after fixing a canonical normalization convention for fields (so that field-rescaling freedom is not used to hide roots/powers). 2. We allow only **analytic** reparametrizations near a chosen base point ("weak coupling"), so adjoining $\tilde{g} = g^{1/2}$ is disallowed when it is non-analytic at that base point. 3. We explicitly exclude scheme/scale conventions from the coupling set: the renormalization scale μ, regulator cutoffs Λ, and finite-subtraction constants; and we also exclude the scalarization gauge scale L_* itself (since constraining L_* is the point of the ladder).

Under this convention, PA-H2.5a is best viewed as a computational proxy once coupling coordinates are fixed, while PA-H2.5b (analyticity at the base point) is the more invariant "no roots" statement. The phrase "canonical normalization of fields" in (1) is itself a convention choice; the point is that fixing such a convention makes the admissibility rule a controlled knob rather than an implicit loophole.

`Example PA-E1 (Gravity-only).` With only G_d available, $a_1 = d - 2$ and the condition becomes $n(d-2) = d/2$. For integer $d \geq 3$, this has a solution only at $d = 4$ with $n = 1$, reproducing Derivation PA-D1.3.

`Example PA-E2 (Gravity + cosmological constant).` If one also allows the cosmological constant Λ_d with $[\Lambda_d] = \text{length}^{-2}$, then the condition becomes $n(d-2) - 2m = d/2$ for integers n, m. A simple family of solutions exists for d divisible by 4: take $n = 1$ and $m = d/4 - 1$, so

$$C \sim G_d \, \Lambda_d^{d/4-1},$$

has dimension length$^{d/2}$. Thus, even under PA-H2.5, $d = 4$ is not automatically unique once additional dimensionful couplings are admitted; what is special about $d = 4$ in this family is that it is the only case with $m = 0$ (no need to involve Λ_d).

`Example PA-E3 (Yang--Mills coupling as an alternative sieve).` In d spacetime dimensions, the Yang–Mills action is typically written as $\frac{1}{4g_d^2} \int d^d x \, F_{\mu\nu} F^{\mu\nu}$, so $[g_d^2] = \text{length}^{d-4}$ (equivalently $[g_d] = \text{length}^{(d-4)/2}$). If we (hypothetically) allow the half-density normalization constant to be a pure monomial in g_d, $C \sim g_d^p$ with integer $p \geq 0$, then

$$[C] = \text{length}^{p(d-4)/2}.$$

Matching $[C] = \text{length}^{d/2}$ gives the integer-exponent condition

$$p(d-4) = d \quad \Longrightarrow \quad d = \frac{4p}{p-1} = 4 + \frac{4}{p-1}.$$

Thus integer solutions occur only when $p-1 \mid 4$, i.e. $p \in \{2,3,5\}$, giving $d \in \{8,6,5\}$ respectively.

In particular, in $d=4$ the gauge coupling is dimensionless and cannot by itself supply the $\text{length}^{d/2}$ factor needed for half-density scalarization; in that case the scale must come from another dimensionful coupling (e.g. gravity) or from a non-analytic mechanism (dimensional transmutation).

`Example PA-E4 (A universal area parameter \(\alpha_\ast\) as a scale supplier).` Suppose the UV theory admits a universal area parameter α_* with dimension $[\alpha_*] = \text{length}^2$ (for example, in perturbative string theory α_* is the familiar $\alpha' = l_s^2$, with string tension $T \sim 1/\alpha'$). If one allows the half-density normalization constant to be built from α_* alone as a monomial $C \sim (\alpha_*)^n$ with integer n, then

$$[C] = \text{length}^{2n},$$

and matching $[C] = \text{length}^{d/2}$ forces $2n = d/2$, i.e. $d = 4n$. So α_* provides a "background-free" source of scale but does not single out $d=4$ on its own; it selects dimensions divisible by 4 under the strict integrality reading of PA-H2.5. In $d=4$ it yields directly an area scale $C \sim \alpha_*$ (e.g. $C \sim \alpha'$ in the string example).

`Remark PA-E4a (Emergent string tension is a transmutation-scale instance, not a UV parameter).` In confining phases one often defines an effective string tension σ with $[\sigma] = \text{length}^{-2}$, so σ^{-1} supplies an area scale. Operationally, for a large rectangular Wilson loop of spatial size R and Euclidean time extent T, one defines $V(R) = -\lim_{T\to\infty} \frac{1}{T}\ln\langle W(R\times T)\rangle$, and an area law $\langle W(R\times T)\rangle \sim e^{-\sigma RT}$ implies $V(R) \sim \sigma R$ and identifies σ [Greensite2003Confinement]. In $d=4$ gauge theories with dimensionless couplings, such a scale is expected to arise (when it exists) as an RG-invariant transmutation scale rather than as a new analytic monomial in the couplings. Logically, this places "string tension as area supplier" in the PA-H2.13 branch unless one is explicitly assuming a fundamental UV area parameter α_*.

`Heuristic PA-H2.12 (Link to gravity/Planck length in a UV completion).` The gravity-only sieve (Derivation PA-D1.3) uses G_d as the unique universal coupling supplying dimension. In a UV completion where gravity is emergent from a sector with a universal area parameter α_* and a dimensionless coupling g, dimensional analysis suggests

$$G_d \propto g^2\,(\alpha_*)^{(d-2)/2} \times \text{(volume factors)},$$

so the Planck length/area is *derived* from α_* and g rather than fundamental. In perturbative string theory, one has $\alpha_* = \alpha'$ and $g = g_s$ (up to compactification-volume factors). In that framing the half-density "universal area scale" could naturally be α_* (or a simple function of it), while the Planck area is recovered as a consequence of how gravity emerges.

`Heuristic PA-H2.8 (What PA-H2.5 is really buying).` The value of PA-H2.5 is comparative: it distinguishes dimensions in which the needed length$^{d/2}$ factor can be supplied by *simple* coupling dependence (integer powers of the already-present couplings), versus dimensions in which any such factor requires either (i) introducing extra scales/couplings, (ii) taking fractional powers, or (iii) invoking non-analytic mechanisms (dimensional transmutation).

`Heuristic PA-H2.13 (Dimensional transmutation as a scale-supplier).` If one relaxes PA-H2.5's "analytic monomial in couplings" requirement, then even a theory with only *dimensionless* couplings can generate a physical length scale through RG invariance: a running coupling $g(\mu)$ can be traded for an RG-invariant scale κ_* (or Λ), typically of the form $\kappa_* \sim \mu\, \exp(-\text{const}/g(\mu)^2)$ or $\kappa_* \sim \mu\, \exp(-\text{const}/g(\mu))$. In that branch, the half-density scalarization scale required by PA-H2.4 can be supplied by $\kappa_*^{-d/2}$ (or its square in $d = 4$ as an area scale), but the scale is no longer an analytic monomial in the couplings: it is emergent and non-perturbative in the naive coupling expansion.

`Heuristic PA-H2.14 (Bookkeeping: what "\(d\)" means in \(\text{length}^{d/2}\)).` The half-density weight length$^{d/2}$ refers to the dimension of the manifold whose coordinates are integrated over in the composition law (the intermediate-variable space). In a nonrelativistic time-sliced kernel this is typically the *spatial* dimension, while for covariant/proper-time kernels one may compose over *spacetime* points. The dimension-sieve discussion using G_d treats d as the **spacetime** dimension, so any $d = 4 \Rightarrow$ "area scale" conclusion should be read in that covariant sense unless stated otherwise.

`Derivation PA-D1.6a (RG-invariant scale from a beta function).` Let a (dimensionless) running coupling $g(\mu)$ satisfy an RG equation $\mu\, dg/d\mu = \beta(g)$ with $\beta(g) \neq 0$ in the range of interest. Then the combination

$$\Lambda_* \equiv \mu \exp\left(- \int^{g(\mu)} \frac{dg'}{\beta(g')} \right)$$

is RG-invariant (independent of the subtraction scale μ), up to a finite multiplicative constant corresponding to a choice of scheme/normalization of the integral. In one-loop form $\beta(g) = -bg^2 + O(g^3)$, one obtains the familiar transmutation scale $\Lambda_* \sim \mu\, e^{-1/(bg(\mu))} \times$ (scheme factor).

If PA-H2.3–PA-H2.4 demand a universal scalarization constant C with $[C] =$ length$^{d/2}$, then any RG-invariant inverse length Λ_* supplies one by $C \sim \Lambda_*^{-d/2}$. In particular, for $d = 4$ this produces a universal **area** scale $C \sim \Lambda_*^{-2}$, without requiring the scale to be an analytic monomial in couplings (so this branch sits outside PA-H2.5).

`Example PA-E5 (2D delta: transmutation yields a length scale).` In the 2D delta interaction, the contact coupling is marginal and the renormalized theory is naturally parameterized by an RG-invariant inverse length κ_* rather than by the bare coupling. Concretely, one finds (up to conventions) a running coupling $g_R(\mu)$ with beta function $\beta(g_R) \propto g_R^2$, and the RG invariant

$$\kappa_*^2 \equiv \mu^2 \exp\left(\frac{2\pi\hbar^2}{m}\frac{1}{g_R(\mu)}\right),$$

so κ_* is independent of the subtraction scale μ and sets a bound-state/scattering scale [ManuelTarrach1994PertRenQM]. This is a minimal witness that "a scale is forced by compatibility" can occur even without a dimensionful coupling, via renormalization rather than via analytic monomials.

`Remark PA-E5a (Half-density match and three-level RG hierarchy).` In the 2D delta, the transmutation scale κ_*^{-1} has dimension length^1 = length$^{d/2}$ for $d = 2$, exactly the half-density scalarization weight required by PA-H2.3–PA-H2.4. The beta function $\beta(g_R) \propto g_R^2$ vanishes to order 2 at the Gaussian fixed point; transmutation requires this nonlinearity — a linear beta function produces only algebraic (power-law) RG invariants without generating a new scale. Three levels of the RG hierarchy are now witnessed by explicit computations: 1. *Semigroup structure*: shared by all refinement flows, including linear beta functions (any toy ODE with a linear beta, e.g. $\beta(a) = \frac{1}{2} - a$, exhibits the semigroup property without transmutation). 2. *Transmutation*: requires β of order ≥ 2 at the fixed point, producing a non-analytic RG-invariant scale. Witness: this example (2D delta, $\beta \propto g_R^2$). 3. *Dimension sieve* (PA-H2.5): demands the scale be an analytic monomial in couplings, selecting $d = 4$ under the gravity-only hypothesis. Witness: Derivation PA-D1.3. Transmutation (level 2) supplies a scale in any d where a marginal coupling exists, so it does not sieve dimensions. The half-density weight length$^{d/2}$ correctly tracks the geometric type of the resulting scale: a length in $d = 2$, an area in $d = 4$.

`Remark PA-E5b (Where in the kernel the transmutation scale acts).` The full resolvent of the 2D delta interaction factorizes via the Lippmann–Schwinger identity as $G = G_0 + G_0\, T\, G_0$, where $G_0(x, z; E)$ is the free resolvent and $T(E)$ is the scalar T-matrix at the contact vertex. Each G_0 factor carries the Van Vleck half-density weight at its endpoint (Derivation PA-D1.4), while $T(E)$ is a scalar amplitude containing the transmutation scale κ_*. Thus the two ingredients play complementary roles: the Van Vleck prefactor implements the half-density *transformation law* (PA-H2.1), and the transmutation scale supplies the *scalarization constant* needed to extract dimensionless amplitudes (PA-H2.2–PA-H2.4). They are structurally independent and combine multiplicatively in the kernel; the geometric weight comes from free propagation between vertices, while the dynamical scale comes from RG invariance at the vertex.

3.4 Running PA-H2.3: Is "Dimensionless f" Physics or Convention?

The half-density formalism (PA-H2.1) gives a canonical pairing $\int \bar{\psi}\, \psi$ that does not require choosing a background measure. But when we write $\psi = f\, \sigma_*$ (PA-

H2.2), we are choosing a *representation* of the same object as a scalar function with respect to a chosen positive density $\rho_* = \sigma_*^2$.

`Proposition PA-P1.3 (Scalarization is a choice of measure, not new physics).` Choosing a reference half-density σ_* identifies the canonical Hilbert space of L^2 half-densities on M with the scalar Hilbert space $L^2(M, \rho_*)$, where $\rho_* = \sigma_*^2$. Different choices σ_* yield unitarily equivalent scalar representations.

`Derivation PA-D1.7 (Change of reference half-density acts by multiplication).` Let σ_1, σ_2 be nowhere-vanishing half-densities on M and set $r := \sigma_2/\sigma_1$, a positive scalar function. Writing the same half-density state ψ as $\psi = f_1\,\sigma_1 = f_2\,\sigma_2$ gives $f_2 = r^{-1} f_1$. Moreover,

$$\int_M \bar{\psi}\,\psi = \int_M |f_1|^2\,\sigma_1^2 = \int_M |f_2|^2\,\sigma_2^2 = \int_M |f_2|^2\,r^2\,\sigma_1^2,$$

so the two scalar pictures differ by a compensating change of measure and pointwise multiplication. In particular, if $\sigma_2 = c\,\sigma_1$ is a constant rescaling, then $f_2 = c^{-1} f_1$ is the familiar global wavefunction normalization freedom.

`Heuristic PA-H2.9 (How PA-H2.3 creates a scale).` In the usual "scalar wavefunction" presentation on $\mathbb{R}^d$, one implicitly chooses $\sigma_* = |dx|^{1/2}$ and allows the scalar representative to carry dimension length$^{-d/2}$ so that $|\psi|^2\,d^d x$ is dimensionless probability. Requiring instead that the scalar representative f be dimensionless (PA-H2.3) shifts the length$^{-d/2}$ factor into the reference half-density:

$$\sigma_* \sim L_*^{-d/2} |dx|^{1/2},$$

so "dimensionless f" is a convention unless the scale $L_*^{d/2}$ is fixed by an additional universality principle (PA-H2.4–PA-H2.5).

3.5 Running PA-H2.4: What Does "Background-Free Constancy" Mean?

From Derivation PA-D1.7, changing the reference half-density σ_* by a positive function $r(x)$ changes the scalar representative by $f \mapsto r^{-1} f$ and changes the scalar measure by $\rho_* \mapsto r^2 \rho_*$. So the raw half-density formulation has a large "scalarization gauge freedom".

`Heuristic PA-H2.10 (Constancy = no extra background function).`
If we take "background-free" in the strong sense "no additional structure beyond the manifold and the theory's couplings", then allowing an arbitrary non-constant $r(x)$ would amount to introducing a new background field/function by hand. In that strong sense, the only admissible changes are constant rescalings, and choosing σ_* becomes a choice of a single global scale (fixed or not fixed by couplings depending on PA-H2.5).

`Derivation PA-D1.8 (Three natural families of \(\sigma_\ast\) and what they mean).` On a configuration space M, the common ways to choose a reference half-density are: 1. **Flat/affine choice (when available):** on $\mathbb{R}^d$ with its affine structure, translation invariance picks $|dx|^{1/2}$ uniquely up to a constant factor. This is "constant" in the sense of being homogeneous under translations. 2. **Metric-derived choice:** given a Riemannian/Lorentzian metric g, one can take $\sigma_g := |g|^{1/4}|dx|^{1/2}$, so that $\rho_g = \sigma_g^2 = \sqrt{|g|}\,|dx|$ is the familiar invariant volume density. This makes the scalar representative f a genuine scalar field but makes the scalarization depend on background geometry. 3. **Field-derived (dilaton-like) choice:** given a scalar field Φ (background or dynamical), one can take $\sigma_\Phi := e^{-\Phi}\sigma_g$. In the scalar picture this is a local rescaling of the measure, and it is the natural way to encode "local units" or Weyl factors.

PA-H2.4 asserts that the theory supplies (or selects) a choice of type (1) with no x-dependent factor: a fixed reference σ_* whose only remaining ambiguity is an overall constant scale.

`Heuristic PA-H2.11 (RG as scale dependence of scalarization).` If refinement/coarse-graining forces an x-independent but scale-dependent choice $\sigma_*(\mu)$ (equivalently a scale-dependent constant $L_*(\mu)$), then PA-H2.4 is replaced by an RG statement: the scalarization convention becomes part of the renormalization scheme (a "wavefunction renormalization" for the scalar representative). In that case, a universal area/length scale can still appear, but typically as an RG invariant (dimensional transmutation scale) rather than as a fixed analytic monomial in couplings.

`Derivation PA-D1.8a (Running scalarization \(\sigma_\ast(\mu)\) is a \(Z(\mu)\) factor on scalar representatives).` Assume the intrinsic (half-density) state ψ is fixed, and consider two scalarizations related by a μ-dependent *constant* rescaling, $\sigma_*(\mu) = c(\mu)\,\sigma_0$ with $c(\mu) > 0$. Writing $\psi = f(\mu)\,\sigma_*(\mu)$ gives $f(\mu) = c(\mu)^{-1}f_0$, where f_0 is the scalar representative in the σ_0 convention. Defining $Z(\mu) := c(\mu)^2$, this becomes the familiar multiplicative form

$$f(\mu) = Z(\mu)^{-1/2}\,f_0.$$

So an x-independent running of scalarization is *formally equivalent* to a wavefunction renormalization factor on scalar representatives. This is bookkeeping: $\sigma_*(\mu)$ is a convention/scheme choice; only RG-invariant combinations (e.g. transmutation scales) are candidates for physical statements.

`Heuristic PA-H2.11a (Guardrail: geometric weight \(\neq\) anomalous dimension).` The half-density square-root Jacobian is a geometric transformation law under coordinate changes; anomalous dimensions are RG scaling data in interacting theories. They should not be conflated: allowing $\sigma_*(\mu)$ to run is a representation convenience, not a claim that geometry "produces" anomalous scaling.

4. Stationary Phase Produces Half-Density Prefactors (Short-Time Kernel)

In the standard path-integral formalism, stationary phase explains why classical extremals dominate refinement limits. Here we add the complementary kernel-level fact: stationary phase does not only pick the extremal; it also produces a determinant prefactor that transforms as a half-density, i.e. the object needed for coordinate-free kernel composition.

`Derivation PA-D1.4 (Van Vleck prefactor is a bi-half-density).` Let $S_{\text{cl}}(x, z; t)$ be the classical action as a function of endpoints and time, treated as a generating function. The standard short-time/stationary-phase approximation to the propagator has the form

$$K(x, z; t) \approx \frac{1}{(2\pi i\hbar)^{d/2}} \left| \det\left(-\frac{\partial^2 S_{\text{cl}}}{\partial x\, \partial z} \right) \right|^{1/2} \exp\left(\frac{i}{\hbar} S_{\text{cl}}(x, z; t) \right).$$

Under a change of coordinates $x = x(x')$, $z = z(z')$, the mixed Hessian transforms by the chain rule, and its determinant acquires Jacobian factors:

$$\det\left(-\frac{\partial^2 S_{\text{cl}}}{\partial x'\, \partial z'} \right) = \det\left(\frac{\partial x}{\partial x'} \right) \det\left(\frac{\partial z}{\partial z'} \right) \det\left(-\frac{\partial^2 S_{\text{cl}}}{\partial x\, \partial z} \right).$$

Taking square roots shows that the prefactor transforms with $|\det(\partial x/\partial x')|^{1/2} |\det(\partial z/\partial z')|^{1/2}$, i.e. exactly as a half-density factor at each endpoint. Thus the stationary-phase prefactor is naturally interpreted as making K a half-density in each variable, so that kernel composition does not depend on a background measure choice. This is the standard "Van Vleck type" semiclassical prefactor in the correspondence/semiclassical tradition [VanVleck1928Correspondence].

`Derivation PA-D1.9 (Square-root delta normalization has half-density weight).` In finite dimension, the "localize on critical points" distribution is $\delta(\nabla f)$, supported on $\mathrm{Crit}(f)$. A concrete way it appears is via a "halved" oscillatory integral with a normalization exponent fixed by dimension.

Let $f : \mathbb{R}^N \to \mathbb{R}$ be smooth and define, for $\varepsilon > 0$,

$$A_\varepsilon(O) := \varepsilon^{-N/2} \int_{\mathbb{R}^N} e^{\frac{i}{\varepsilon} f(x)}\, O(x)\, dx.$$

Then

$$|A_\varepsilon(O)|^2 = \varepsilon^{-N} \iint e^{\frac{i}{\varepsilon}(f(x) - f(y))}\, O(x)\, \overline{O(y)}\, dx\, dy.$$

Applying the near-diagonal scaling $y = x + \varepsilon z$ (so $dy = \varepsilon^N dz$) gives

$$|A_\varepsilon(O)|^2 = \iint e^{-\frac{i}{\varepsilon}(f(x+\varepsilon z) - f(x))}\, O(x)\, \overline{O(x + \varepsilon z)}\, dz\, dx.$$

Formally letting $\varepsilon \to 0$ yields

$$|A_\varepsilon(O)|^2 \;\to\; \iint e^{-iz\cdot\nabla f(x)}\,|O(x)|^2\,dz\,dx = (2\pi)^N \int \delta(\nabla f(x))\,|O(x)|^2\,dx.$$

The exponent $N/2$ in the prefactor is exactly the half-density scaling: it cancels the Jacobian $dy = \varepsilon^N dz$ under near-diagonal rescaling, and it is the "square root" of the density normalization that produces $\delta(\nabla f)$.

`Heuristic PA-H1.4 (Where Planck area can enter, minimally).` Derivation PA-D1.3 isolates one minimal route by which a Planck-scale quantity can enter: if the theory supplies a single universal coupling with dimension of length (Newton's constant) and one demands that the half-density normalization constant be built from that coupling *without fractional powers*, then $d = 4$ is singled out and the resulting constant has the dimension of an area, naturally identified with the Planck area $L_P^2 \sim \hbar G_4/c^3$.

5. A Gravitational Anchor: Minimal Areal Speed and the $D = 4$ Cancellation

Rivero's "Planck areal speed" observation gives a concrete route by which Planck-scale discreteness reappears at Compton scales in inverse-square gravity [RiveroAreal] [RiveroSimple].

`Heuristic PA-H1.3 (Areal-speed selection).` In $3+1$ Newtonian gravity (inverse-square), imposing a discrete areal-speed/area-time condition at a Planck scale can yield characteristic radii proportional to a reduced Compton length, with Newton's constant canceling when expressed in Planck units. This is a nontrivial indication that "a universal area scale" can be operationally meaningful at low energies in $D = 4$.

`Derivation PA-D1.5 (Inverse-square circular orbit + Planck areal speed \(\Rightarrow\) Compton radius).` For a circular orbit under an inverse-square central force $F(r) = K/r^2$ (with coupling $K > 0$), the centripetal balance is $mv^2/r = K/r^2$. The areal speed is $\dot{A} = \frac{1}{2}rv$, so $v = 2\dot{A}/r$. Substituting into the force balance gives

$$m\left(\frac{2\dot{A}}{r}\right)^2 = \frac{K}{r} \quad\Longrightarrow\quad r = \frac{4m\,\dot{A}^2}{K}.$$

For Newtonian gravity between a source mass M and test mass m, $K = GMm$, hence

$$r = \frac{4\dot{A}^2}{GM},$$

independent of the test mass m. If one now imposes $\dot{A} = k\,\dot{A}_P$, where Rivero's Planck areal speed is $\dot{A}_P = cL_P$ [RiveroAreal], then using $L_P^2 = G\hbar/c^3$ yields

$$r = \frac{4k^2(cL_P)^2}{GM} = \frac{4k^2(G\hbar/c)}{GM} = 4k^2\,\frac{\hbar}{cM}.$$

Thus r becomes a multiple of the reduced Compton length $L_M = \hbar/(cM)$, with Newton's constant canceled out. In particular, $k = \frac{1}{2}$ gives $r = L_M$. This is the "Planck area per Planck time $\Rightarrow$ Compton scale" cancellation highlighted in [RiveroAreal] and summarized in [RiveroSimple].

`Remark PA-D1.5a (Generic \(F=K/r^q\): only \(q=2\) yields linear Compton scaling; \(q\) links to dimension).` For a power-law central force $F(r) = K/r^q$ with $K > 0$, circular balance gives $mv^2/r = K/r^q$, i.e. $mv^2 = Kr^{1-q}$. Using the circular areal speed $\dot{A} = \frac{1}{2}rv$ (so $v = 2\dot{A}/r$) yields

$$4m\dot{A}^2 = K\,r^{3-q}.$$

Hence, for $q \neq 3$,

$$r = \left(\frac{4m\dot{A}^2}{K}\right)^{\frac{1}{3-q}},$$

while for $q = 3$ the radius drops out and $4m\dot{A}^2 = K$.

In the gravitational specialization $K = GMm$, the test mass cancels as before and

$$r = \left(\frac{4\dot{A}^2}{GM}\right)^{\frac{1}{3-q}}.$$

If one imposes $\dot{A} = cL_P$ and uses the $D = 4$ identity $L_P^2 = G\hbar/c^3$, then

$$r = \left(\frac{4\hbar}{cM}\right)^{\frac{1}{3-q}}.$$

Thus the Planck-areal-speed substitution produces *linear* reduced-Compton scaling $r \propto \hbar/(cM)$ only for $q = 2$ (inverse-square). For Newtonian long-range fields in n spatial dimensions, the Laplacian Green function gives $\Phi(r) \propto r^{2-n}$ (for $n > 2$), so $F \sim |\nabla\Phi| \propto r^{1-n}$, i.e. $q = n - 1$ (with the $n = 2$ logarithmic exception) [Tanaka2021KernelQuadrature]. In this sense the $q = 2$ special case corresponds to $n = 3$ spatial dimensions (spacetime $D = 4$). Equivalently, substituting $q = n - 1$ into the mass scaling gives $r \propto M^{-1/(4-n)}$ (for $n \neq 4$), so the linear Compton scaling (and the G-cancellation in the $D = 4$ identity $L_P^2 = G\hbar/c^3$) is uniquely $n = 3$; the $n = 4$ case is the degenerate $q = 3$ condition where the radius drops out.

`Remark PA-D1.5b (SR continuation of the inverse-square witness: Compton branch \(\to\) Planck floor).` Inside a mechanical special-relativistic model with an external inverse-square force (no GR field dynamics), one can keep the fixed coordinate-time areal speed $\dot{A}_0 = dA/dt$ and continue PA-D1.5 exactly.

For inverse-square forces, SR circular motion gives $v = K/L$, while

$$L = \gamma m r v = 2\gamma m \dot{A}_0, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Hence

$$\gamma v = \frac{K}{2m\dot{A}_0},$$

so

$$v = \frac{\frac{K}{2m\dot{A}_0}}{\sqrt{1 + \left(\frac{K}{2m\dot{A}_0 c}\right)^2}}, \qquad r = \frac{2\dot{A}_0}{v} = \frac{4m\dot{A}_0^2}{K}\sqrt{1 + \left(\frac{K}{2m\dot{A}_0 c}\right)^2}.$$

For gravity $K = GMm$, m cancels:

$$r(M, \dot{A}_0) = \frac{4\dot{A}_0^2}{GM}\sqrt{1 + \left(\frac{GM}{2\dot{A}_0 c}\right)^2}.$$

With $\dot{A}_0 = cL_P$, using $L_P^2 = \hbar G/c^3$ and $M_P^2 = \hbar c/G$,

$$r(M) = \frac{4\hbar}{Mc}\sqrt{1 + \frac{M^2}{4M_P^2}}.$$

Therefore $M \ll M_P$ reproduces the PA-D1.5 Compton-like branch $r \approx 4\hbar/(Mc)$, while $M \gg M_P$ saturates at

$$r \to 2L_P.$$

So in this SR continuation, the inverse-square Planck-areal-speed witness is not destroyed; it is regularized into a bounded interpolation.

`Remark PA-D1.5c (Clock-choice sensitivity: fixed \(dA/dt\) vs fixed \(dA/d\tau\)).` The $2L_P$ high-mass saturation in PA-D1.5b is tied to fixing the coordinate-time areal speed $\dot{A}_t = dA/dt$. If one instead fixes proper-time areal speed $\dot{A}_\tau = dA/d\tau$ in the same inverse-square SR model, then

$$r(M, \dot{A}_\tau) = \frac{4\dot{A}_\tau^2}{GM}\sqrt{1 - \left(\frac{GM}{2\dot{A}_\tau c}\right)^2},$$

which is defined only for $GM < 2\dot{A}_\tau c$. With $\dot{A}_\tau = cL_P$,

$$r(M) = \frac{4\hbar}{Mc}\sqrt{1 - \frac{M^2}{4M_P^2}},$$

so $M \to 2M_P^- \Rightarrow r \to 0$, not $2L_P$. Therefore the low-mass Compton-like branch is robust, while the high-mass asymptotic is clock-convention dependent in this mechanical SR setting.

`Remark PA-D1.5d (Invariant-candidate reformulation via specific angular momentum).` A clock-independent candidate in the same central-source setup is the specific angular momentum scalar

$$\ell := \frac{1}{m}\sqrt{\frac{1}{2}\,L_{\mu\nu}L^{\mu\nu}},$$

where $L^{\mu\nu}$ is the source-rest-space projection of orbital $J^{\mu\nu}$ (using source 4-velocity U^μ). In the source rest frame this reduces to

$$\ell = \gamma r v, \qquad \dot{A}_t = \frac{\ell}{2\gamma}, \qquad \dot{A}_\tau = \frac{\ell}{2}.$$

So fixing ℓ selects the proper-time branch rather than the coordinate-time branch. For inverse-square gravity,

$$r(M,\ell) = \frac{\ell^2}{GM}\sqrt{1-\left(\frac{GM}{\ell c}\right)^2}, \qquad \ell > \frac{GM}{c},$$

and $\ell = 2cL_P$ reproduces PA-D1.5c. This does not yet prove universality, but it provides a structurally covariant way to encode the postulate without choosing a clock variable directly.

`Remark PA-D1.5e (Why simple frame-free bivector invariants are insufficient here).` One might try to avoid the source-velocity projection entirely and use only Lorentz invariants of

$$M^{\mu\nu} = R^\mu p^\nu - R^\nu p^\mu,$$

namely

$$I_1 = \frac{1}{2}M_{\mu\nu}M^{\mu\nu} = R^2p^2 - (R\cdot p)^2, \qquad I_2 = \frac{1}{2}M_{\mu\nu}{}^\star M^{\mu\nu}.$$

But for the circular central branch ($R\cdot p = 0$, $R^2 = r^2$, $p^2 = -m^2c^2$), this gives

$$I_1 = -m^2c^2r^2,$$

independent of orbital speed, while I_2 vanishes in the planar case. So these simple frame-free invariants do not encode the areal-rate branch parameter. In this setup, a timelike direction (e.g. source U^μ) appears to be minimal extra structure for a useful covariant postulate.

`Remark PA-D1.5f (Minimal timelike-structure rule for this branch).` For the present central-source inverse-square model, a practical "minimal structure" rule is: 1. use the source worldline 4-velocity U^μ as the distinguished timelike direction (already part of the model input), and 2. formulate the postulate on

$$\ell_{(U)} := \frac{1}{m}\sqrt{\frac{1}{2}\,M^{(U)}_{\mu\nu}M^{\mu\nu}_{(U)}}, \qquad M^{(U)\mu\nu} = h(U)^{\mu}{}_{\alpha}h(U)^{\nu}{}_{\beta}M^{\alpha\beta},$$

rather than directly on coordinate-time areal rate. In the source rest frame this is equivalent to fixing $\dot{A}_{\tau,U} = \ell_{(U)}/2$, while coordinate-time rates are derived via the corresponding lapse factor. This keeps the branch covariant-with-source and avoids introducing an additional arbitrary observer field $u^{\mu}(x)$.

`Remark PA-D1.5g (Non-circular planar extension is kinematic).` The ℓ-based rule is not restricted to circular trajectories. For general planar motion with tangential component $v_{\perp} = r\dot{\phi}$,

$$\ell = \gamma r v_{\perp} = \gamma r^2\dot{\phi}, \qquad \frac{dA}{dt} = \frac{1}{2}r^2\dot{\phi} = \frac{\ell}{2\gamma}, \qquad \frac{dA}{d\tau} = \frac{\ell}{2}.$$

Thus "fix ℓ" remains equivalent to fixing proper-time areal rate in the source frame even away from circular orbits; this part is kinematic and does not depend on the specific force law.

`Remark PA-D1.5h (Non-planar caution: vector area rate is fundamental).` For general 3D motion the natural identity is vector-valued:

$$\boldsymbol{\ell} = \frac{1}{m}(\mathbf{r}\times\mathbf{p}) = \gamma(\mathbf{r}\times\mathbf{v}), \qquad \frac{d\mathbf{A}}{d\tau} = \frac{\boldsymbol{\ell}}{2}.$$

So the ℓ-rule still survives kinematically, but scalar areal rates require a chosen normal $\mathbf{n}$:

$$\frac{dA_{\mathbf{n}}}{d\tau} = \frac{\boldsymbol{\ell}\cdot\mathbf{n}}{2}.$$

In non-planar perturbations, interpreting a scalar "areal speed postulate" without specifying this projection is ambiguous; the projection choice is part of the model specification.

`Remark PA-D1.5i (Observability criterion for projected areal-rate postulates).` In perturbed-orbit settings, a projected areal-rate claim is empirically meaningful only after specifying: 1. projection normal $\mathbf{n}$, 2. clock convention (t-based or τ-based), 3. reconstruction map for $(\mathbf{r},\mathbf{v})$ in the source frame. With those choices fixed, the observable is

$$\dot{A}_{\mathbf{n}}(t) = \frac{1}{2}\,\mathbf{n}\cdot(\mathbf{r}\times\mathbf{v}), \qquad \frac{dA_{\mathbf{n}}}{d\tau} = \frac{1}{2}\,\mathbf{n}\cdot\boldsymbol{\ell},$$

and departures from constant projected areal rate are governed by projected torque

$$\frac{d}{dt}\big(\mathbf{n}\cdot\boldsymbol{\ell}\big) = \frac{1}{m}\,\mathbf{n}\cdot(\mathbf{r}\times\mathbf{F}).$$

So the postulate becomes falsifiable precisely when projection, clock, and reconstruction are part of the model declaration.

`Remark PA-D1.5j (Minimal implementation pipeline).` A compact data-to-test pipeline is: 1. reconstruct object state in observer frame from direction $\hat{\mathbf{n}}(t)$, range $\rho(t)$, and line-of-sight velocity $\dot{\rho}(t)$:

$$\mathbf{r}_{\text{obj}} = \rho\hat{\mathbf{n}}, \qquad \mathbf{v}_{\text{obj}} = \dot{\rho}\,\hat{\mathbf{n}} + \rho\,\dot{\hat{\mathbf{n}}},$$

2. subtract source ephemeris to obtain source-frame relative state $(\mathbf{r}, \mathbf{v})$, 3. evaluate

$$\dot{A}_{\mathbf{n}} = \frac{1}{2}\,\mathbf{n}\cdot(\mathbf{r}\times\mathbf{v}), \quad \frac{dA_{\mathbf{n}}}{d\tau} = \frac{1}{2}\,\mathbf{n}\cdot\boldsymbol{\ell},$$

and, if a force model is supplied, the projected-torque residual

$$\mathcal{T}_{\mathbf{n}} := \frac{d}{dt}\big(\mathbf{n}\cdot\boldsymbol{\ell}\big) - \frac{1}{m}\,\mathbf{n}\cdot(\mathbf{r}\times\mathbf{F}).$$

This keeps the postulate test tied to explicit reconstruction and uncertainty handling rather than to abstract kinematic statements alone.

`Remark PA-D1.5k (Minimal uncertainty scaffold).` At first order, uncertainty in projected observables can be propagated by Jacobians:

$$q := \dot{A}_{\mathbf{n}} \quad \Rightarrow \quad \sigma_q^2 \approx J_q\,\Sigma_x\,J_q^{\top},$$

for reconstructed state vector x and covariance Σ_x, and

$$\sigma_{\mathcal{T}}^2 \approx J_{\mathcal{T}}\,\Sigma_z\,J_{\mathcal{T}}^{\top}$$

for residual $\mathcal{T}_{\mathbf{n}}$ with augmented state z. This linear scaffold is a baseline; in strongly nonlinear regimes the same quantities should be cross-checked with nonlinear propagation (e.g. Monte Carlo) before interpretation.

`Remark PA-D1.5l (Practical nonlinear-validation trigger).` A lightweight policy is to run a pilot nonlinear propagation check and compare against the linear σ estimate; if the discrepancy is at the few-percent level (or larger), treat linearized errors as insufficient and switch to nonlinear uncertainty propagation for reporting.

`Remark PA-D1.5m (Regime-dependent trigger calibration).` The trigger in PA-D1.5l should be calibrated by uncertainty regime, not treated as universal. A practical diagnostic pair is

$$\epsilon_{\text{nl}} := \frac{|\sigma_{\text{MC}} - \sigma_{\text{lin}}|}{\sigma_{\text{lin}}}, \qquad \chi := \max\!\left(\frac{\sqrt{\operatorname{tr}\Sigma_r}}{\|\mathbf{r}\|}, \frac{\sqrt{\operatorname{tr}\Sigma_v}}{\|\mathbf{v}\|}\right),$$

with Σ_r, Σ_v the position/velocity covariance blocks in the chosen reconstruction model. Pilot scans in correlated-noise families can then map $\epsilon_{\text{nl}}(\chi)$ for the instrument/model pair; the "few-percent" policy corresponds to selecting an operational ϵ_{nl} band after this calibration, rather than imposing a context-free constant.

6. Connection to the Refinement-Composition Framework

The broader program in which this note sits argues that: 1. classical dynamics are recovered from quantum composition by stationary-phase concentration, and 2. refinement across scales forces RG-style consistency conditions when naive limits diverge.

This note adds a complementary ingredient: the kernel side is most naturally formulated in half-density language, and stationary phase produces the bi-half-density prefactor directly. A universal convention for turning those half-densities into scalar amplitudes then requires a length$^{d/2}$ scale; in $d = 4$ this is an area scale.

7. Open Problems and Outlook

1. Make the half-density normalization argument precise for a concrete groupoid or kernel model (tangent-groupoid or short-time propagator model).
2. Show how the area scale enters stationary-phase prefactors and how this interacts with RG scaling.
3. General-dimension analysis: clarify what replaces "area" in odd dimensions and whether a universal normalization is still defensible.
4. Identify minimal hypotheses under which "need of half-density scale $\Rightarrow$ Planck area" is more than dimensional bookkeeping.
5. Track minimal-length/GUP scenarios as a comparison branch: do they implement the "needed scale" at the level of kinematics (modified commutators/dispersion) or can they be reframed as a refinement-compatibility condition? Use [Hossenfelder2013MinimalLength] as an OA entry point.

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