

# “Uncuttable” as Controlled Refinement

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## Abstract

This note fixes a meaning of “uncuttable” aligned with the refinement-compatibility thesis of the cornerstone manuscript. Here **uncuttable** does not mean “indivisible.” It means: the quantity of interest is not determined by any *finite* dissection alone; it is a **limit object** whose definition requires a refinement rule and a comparison structure across refinements.

The point is structural and mathematical: once a theory is built from composable local pieces, the continuum theory is the stable target of a refinement limit, and extra control data may be required for that limit to exist or be unique.

## 1. Definition

Call a quantity  $Q$  **uncuttable** (in this note’s sense) if: 1. there exists a family of finite approximants  $Q_N$  produced by a finite dissection/refinement scheme of depth  $N$ , but 2. the value of interest is not any finite  $Q_N$ ; it is a controlled limit  $Q = \lim_{N \rightarrow \infty} Q_N$ , and 3. specifying the *rule of refinement* and the *comparison across refinements* is part of the definition of  $Q$ .

The historical resonance is deliberate. The Greek *ἄτομος* (“a-tomos,” uncuttable) was coined by Leucippus and Democritus to denote indivisible substance — matter that cannot be divided further. The shift proposed here is from ontology to procedure: what is “uncuttable” is not a smallest piece of stuff, but a limit object that no single finite dissection captures. The indivisibility is not in the substance but in the definition: you cannot “cut” the limit into finitely many pieces and recover it without specifying how the pieces are to be reassembled and refined.

This is the ordinary situation in analysis: finite partitions approximate, but the object is defined by a limiting procedure together with hypotheses that ensure convergence/uniqueness.

## 2. Toy model: an integral is already a refinement limit

Let  $f : [a, b] \rightarrow \mathbb{R}$ . A prototypical refinement family is a partition  $P_N = \{a = t_0 < \dots < t_N = b\}$  with mesh  $\|P_N\| := \max_k(t_{k+1} - t_k) \rightarrow 0$ . Define the Riemann-sum approximants

$$Q_N := \sum_{k=0}^{N-1} f(\xi_k) (t_{k+1} - t_k), \quad \xi_k \in [t_k, t_{k+1}].$$

In good cases,  $Q_N \rightarrow \int_a^b f(t) dt$  as  $\|P_N\| \rightarrow 0$ , and the limit is independent of the tags  $\xi_k$ . But this is not a tautology: the limit can fail to exist or can depend on the refinement rule unless hypotheses are stated.

A concrete failure case appears already in calculus. Consider the difference quotient  $f(x + \varepsilon)/\varepsilon$ : for each finite  $\varepsilon$ , the quantity diverges as  $\varepsilon \rightarrow 0$ . The “refinement limit” exists only after subtracting a counterterm  $f(x)/\varepsilon$ , yielding the derivative  $f'(x)$ . The subtraction is part of the definition of the limit — without it, the refinement procedure does not converge. This is the simplest model of renormalization: a “cut” at finite  $\varepsilon$  is not the answer; the answer requires a controlled  $\varepsilon \rightarrow 0$  limit with explicit subtraction rules.

In the present program, this is the basic moral: finite cuts approximate, but the value is defined by **controlled refinement**.

### 2.5 Worked example: the derivative as a renormalized refinement limit

The paragraph above describes the difference-quotient subtraction informally. We now make it explicit as a minimal model of renormalization.

**Example 2.1 (Derivative as counterterm-subtracted limit).** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable at  $x$ . The naive “refinement” approximant at scale  $\varepsilon$  is

$$R(\varepsilon) := \frac{f(x + \varepsilon)}{\varepsilon}.$$

This diverges as  $\varepsilon \rightarrow 0$  whenever  $f'(x) \neq 0$ . To extract a finite limit, subtract a counterterm:

$$R_{\text{ren}}(\varepsilon) := \frac{f(x + \varepsilon) - f(x)}{\varepsilon}.$$

Now  $\lim_{\varepsilon \rightarrow 0} R_{\text{ren}}(\varepsilon) = f'(x)$ , which is finite and independent of the “regulator”  $\varepsilon$ . The structure is:

1. **Regulated quantity:**  $R(\varepsilon)$ , finite for each  $\varepsilon > 0$  but divergent as  $\varepsilon \rightarrow 0$ .
2. **Counterterm:**  $f(x)/\varepsilon$ , which absorbs the divergence.
3. **Renormalized observable:**  $f'(x)$ , the  $\varepsilon$ -independent limit.

This is formally identical to the pattern in perturbative renormalization: a bare quantity diverges as the regulator is removed, but a systematic subtraction yields a finite, physically meaningful result. The subtraction rule is not ad hoc; it is forced by the requirement that the result depend smoothly on the data and be independent of the regulator.

**Remark 2.2 (Higher-order counterterms).** If  $f$  is  $C^n$ , the Taylor expansion

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon + \frac{1}{2}f''(x)\varepsilon^2 + \cdots + \frac{1}{n!}f^{(n)}(x)\varepsilon^n + o(\varepsilon^n)$$

provides a systematic tower of subtractions, one per order. Removing terms through order  $k - 1$  and dividing by  $\varepsilon^k$  gives the  $k$ -th Taylor coefficient  $f^{(k)}(x)/k!$  plus vanishing corrections — or equivalently, dividing by  $\varepsilon^k/k!$  recovers the derivative  $f^{(k)}(x)$  itself. Each order is a “counterterm-subtracted refinement limit” — the analog of loop-by-loop renormalization in QFT.

**Remark 2.3 (Euler–Maclaurin: correction tower for the Riemann sum).** The same pattern applies to the integral approximation itself. The Euler–Maclaurin formula expresses the error of the left-point Riemann sum as a systematic expansion in the mesh size  $h = (b - a)/N$ :

$$\sum_{k=0}^{N-1} f(a + kh) h = \int_a^b f(t) dt + \sum_{j=1}^p \frac{B_j}{j!} h^j [f^{(j-1)}(b) - f^{(j-1)}(a)] + O(h^{p+1}),$$

where  $B_j$  are Bernoulli numbers ( $B_1 = -\frac{1}{2}$ ,  $B_2 = \frac{1}{6}$ ,  $B_3 = 0, \dots$ ). Each correction involves a power of the mesh (the “regulator”), multiplied by endpoint derivatives (boundary data). The leading correction  $j = 1$  converts the left-point sum into the trapezoidal rule; the  $j = 2$  term adds an endpoint-derivative correction that raises the order to  $O(h^4)$ . This is the interval counterpart of Remark 2.2’s single-point tower: there the subtractions were local (Taylor coefficients at  $x$ ); here they are boundary-localized (endpoint derivatives of  $f$ ).

### 3. Dynamics: action, stationarity, and the need for control data

The cornerstone manuscript uses the same template in mechanics. Given a partition of time, the discrete action

$$S_N[q] = \sum_k \mathcal{L}\left(q_k, \frac{q_{k+1} - q_k}{\Delta t_k}, t_k\right) \Delta t_k$$

is a finite refinement approximant. The continuum action  $S[q] = \int \mathcal{L} dt$  is a refinement limit.

Two “uncuttable” features appear immediately when one pushes beyond smooth classical paths: 1. **Singular probes and corners:** stationarity must be

interpreted in weak/distributional form; point-supported variations require mollification.

**2. Non-uniqueness of refinement schemes:** different discretization conventions (even if classically equivalent) can produce distinct refined objects unless an equivalence or control map is specified.

These are exactly the obstructions discussed in the cornerstone manuscript: the point is not indivisible atoms, but limit control.

In the quantum setting, the “uncuttable” character becomes sharper. The path-integral amplitude

$$K(q_f, t_f; q_i, t_i) = \int \prod_{k=1}^{N-1} dq_k \prod_{k=0}^{N-1} K_\Delta(q_{k+1}, q_k; t_k)$$

is a product of short-time kernels composed over a time partition of depth  $N$ . No finite  $N$  gives the exact propagator; the propagator is the  $N \rightarrow \infty$  refinement limit. Crucially, the control parameter  $\hbar$  enters the short-time kernels as  $\exp(iS_\Delta/\hbar)$ , and different discretization conventions (left-point, midpoint, symmetric) can produce distinct  $O(\hbar)$  corrections even though they share the same classical  $\hbar \rightarrow 0$  limit [FeynmanHibbs1965]. Thus the quantum amplitude is doubly “uncuttable”: it requires both a refinement rule (time-slicing prescription) and a comparison/equivalence structure (ordering convention or half-density normalization) before the limit is well-defined.

**Example 3.1 (Non-uniqueness of refinement:  $\backslash(\alpha\backslash)$ -ordering).** For the classical Hamiltonian  $H(q, p) = qp$ , different time-slicing prescriptions — evaluating position at  $q_\alpha = (1 - \alpha)q_k + \alpha q_{k+1}$  in each slice — produce different quantum operators:

$$\hat{H}_\alpha = \alpha \hat{q}\hat{p} + (1 - \alpha)\hat{p}\hat{q} = \hat{p}\hat{q} + \alpha i\hbar.$$

At  $\alpha = 0$  (prepoint):  $\hat{p}\hat{q}$ . At  $\alpha = 1/2$  (midpoint/Weyl):  $\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$ . At  $\alpha = 1$  (postpoint):  $\hat{q}\hat{p}$ . All three share the classical limit  $H = qp$  as  $\hbar \rightarrow 0$ , but they are distinct quantum objects. The “uncuttable” message: the continuum quantum Hamiltonian is not determined by any single finite time-slicing; it requires specifying the refinement convention  $\alpha$  as part of the definition.

**Remark 3.2 (Stochastic counterpart: Itô versus Stratonovich).** The same refinement non-uniqueness appears in stochastic calculus. For a Wiener process  $W_t$ , the stochastic integral  $\int f(W_t) dW_t$  can be defined using left-point (Itô) or midpoint (Stratonovich) evaluation in the Riemann sums:

$$\text{Itô: } \sum_k f(W_{t_k}) \Delta W_k, \quad \text{Stratonovich: } \sum_k f\left(\frac{W_{t_k} + W_{t_{k+1}}}{2}\right) \Delta W_k.$$

Both converge as mesh  $\rightarrow 0$  (for  $f \in C^2$ ), but to different limits related by  $\int f(W) \circ dW = \int f(W) dW + \frac{1}{2} \int f'(W) dt$  [Oksendal2003]. The correction  $\frac{1}{2}f' dt$  arises because Brownian paths have non-zero quadratic variation ( $\sum (\Delta W_k)^2 \rightarrow T \neq 0$ ); for paths of bounded variation, the quadratic variation vanishes and all

evaluation-point prescriptions agree. This parallels Example 3.1: the ordering correction  $\alpha i\hbar$  vanishes when  $\hbar \rightarrow 0$  (smooth classical paths), but is unavoidable at finite  $\hbar$ . Both cases instantiate the “uncuttable” pattern: when paths are rough enough, the refinement prescription becomes part of the definition.

**Remark 3.3 (Trotter product formula as a refinement theorem).** The mathematical backbone of the path-integral refinement limit is the Trotter product formula: for self-adjoint operators  $A$  and  $B$  with  $A + B$  essentially self-adjoint on a common dense domain,

$$e^{t(A+B)} = \lim_{N \rightarrow \infty} \left( e^{tA/N} e^{tB/N} \right)^N.$$

In the path-integral context,  $A = -i\hat{T}/\hbar$  (kinetic) and  $B = -i\hat{V}/\hbar$  (potential), so each factor is a free propagation or a potential phase-kick at one time slice. The formula states that the exact propagator is a refinement limit of  $N$ -fold compositions — and that this limit converges, with total error  $O(1/N)$ . The symmetric (Suzuki–Trotter) splitting  $e^{tA/(2N)} e^{tB/N} e^{tA/(2N)}$  reduces the total error to  $O(1/N^2)$  by canceling the leading Baker–Campbell–Hausdorff commutator  $[A, B]$  contribution at each step. This is the “uncuttable” pattern in operator-algebraic form: no finite product equals  $e^{t(A+B)}$ , and the rate of convergence depends on controlling the non-commutativity of the pieces [FeynmanHibbs1965].

**Remark 3.4 (Symplectic integrators: structural compatibility at finite resolution).** The symmetric Trotter splitting is the operator form of the Störmer–Verlet (leapfrog) integrator. The discrete map is *exactly* symplectic at every finite  $N$ : it preserves the Poisson brackets (or, quantum-mechanically, unitarity) not just asymptotically but at each approximation level. Backward error analysis shows that the numerical flow is the exact flow of a nearby “shadow Hamiltonian”  $\tilde{H} = H + O(\Delta t^2)$ , which is conserved exactly along the discrete orbit; the original  $H$  oscillates with amplitude  $O(\Delta t^2)$  without secular drift. Higher-order symplectic splittings (Yoshida, Forest–Ruth) systematically cancel further Baker–Campbell–Hausdorff commutators, paralleling the counterterm tower of Remark 2.2. This is refinement compatibility made precise: the approximants at every level carry a structural invariant (symplecticity) that the continuum limit inherits.

## 4. Outlook: refinement compatibility as “the extra structure”

In the companion papers, the “extra structure” used to control refinement limits is made explicit: - half-densities make kernel composition coordinate-free without hidden measure choices, - control maps  $\tau$  encode how parameters must flow under refinement to maintain stability, - renormalization is the compatibility rule when naive refinement limits diverge.

This note is therefore a small conceptual bridge: it isolates an early, analysis-level instance of the same meta-problem that reappears in quantization and in QFT.

## References

1. [FeynmanHibbs1965] Richard P. Feynman and Albert R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, 1965. (Path integral as refinement limit of time-sliced amplitudes; foundational treatment.)
2. [Øksendal2003] Bernt Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 6th ed., Springer, 2003. ISBN 978-3-540-04758-2. DOI 10.1007/978-3-642-14394-6. (Standard textbook on Itô vs Stratonovich integrals and their relationship; used in uncuttable satellite for Remark 3.2.)