

# Fermionic Mediators, Static Potentials, and Contact/Boundary-Condition Limits

Alejandro Rivero

2026

## Abstract

The textbook derivation of a static potential from “field exchange” uses a bosonic mediator linearly sourced by a commuting classical density, yielding an effective action quadratic in the source and (in a static limit) a central Yukawa/Coulomb potential. This derivation does not transplant verbatim to fermionic fields: the linear source terms for fermions require Grassmann-valued sources, so there is no ordinary commuting classical source whose elimination produces a classical potential in the same way. This note isolates the precise obstruction and records the robust infrared replacement: when a microscopic description reduces to local operators at low resolution, the effective interaction is encoded by contact terms (delta kernels and their derivatives) or, equivalently, boundary-condition/self-adjoint-extension data, with renormalization-group running when the contact limit is singular.

This is a dependent note aligned with the broader refinement-compatibility program: contact terms are diagonal-support kernels, and their scale dependence is a compatibility condition rather than an afterthought.

## 1. Purpose and scope

This note answers a narrowly phrased question: what can it mean for a **fermionic** field to “generate a (central) potential” in the same sense that a massive bosonic field generates a Yukawa potential?

We keep the scope bounded: 1. state the bosonic sourcing  $\Rightarrow$  potential mechanism (derivation-first, brief), 2. state the fermionic obstruction precisely (Grassmann sources), 3. give one explicit IR matching witness: **local operators  $\Rightarrow$  contact/derivative-contact kernels**, 4. connect contact kernels to related point-interaction/RG witnesses.

We do **not** claim that fermions cannot affect forces; we only isolate which parts of the “classical source  $\Rightarrow$  potential” story fail, and what the correct replacement statement is at low resolution.

## 2. What “a field generates a potential” means in the bosonic source story

The archetypal construction is a bosonic mediator  $\varphi$  linearly coupled to a commuting source  $J(x)$ :

$$S[\varphi; J] = \int d^D x \left( \frac{1}{2} \varphi K \varphi + J \varphi \right), \quad K = (\square + m^2) \text{ (example).}$$

Integrating out  $\varphi$  (Gaussian elimination) yields an effective action quadratic in the source,

$$S_{\text{eff}}[J] = -\frac{1}{2} \int d^D x d^D y J(x) K^{-1}(x, y) J(y),$$

so the static, nonrelativistic limit of  $K^{-1}$  produces a central potential (Yukawa for  $m \neq 0$ , Coulomb-type when  $m = 0$ ).

The key structural ingredient is that the source is an ordinary commuting function (a classical background density).

## 3. Fermionic fields: linear sources are Grassmann, so the classical-source story does not transplant

For a Dirac fermion  $\Psi$ , the generating functional with sources is written with **Grassmann-valued** sources  $\eta, \bar{\eta}$ :

$$Z[\bar{\eta}, \eta] = \int D\bar{\Psi} D\Psi \exp \left( i \int d^D x \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi + i \int d^D x (\bar{\eta} \Psi + \bar{\Psi} \eta) \right).$$

An explicit statement of this form, including that  $\eta, \bar{\eta}$  are Grassmann-valued, is recorded in [Floerchinger2024QFT1Lecture21].

Two immediate consequences follow.

**Remark 3.1 (Obstruction statement).** The bosonic derivation “choose a commuting classical source  $J$ , integrate out the field, and read off a classical potential” does not directly apply to fermions, because the linear source terms that couple to  $\Psi$  require Grassmann sources rather than commuting c-number densities. Therefore, “fermion exchange generates a classical potential between commuting sources” is not a well-posed transplant of the bosonic story.

This does **not** mean fermions are irrelevant: fermions can and do affect effective interactions through loop effects, through bosonic composite modes (bilinears), and through low-energy EFT operators. The point is that the meaning of “generates a potential” must be stated through one of these controlled mechanisms.

### 3.1 The controlled alternative: fermion loops modify bosonic propagators

The standard example is vacuum polarization in quantum electrodynamics. A closed electron–positron loop inserted into the photon propagator gives a momentum-dependent correction to the effective electromagnetic coupling,

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha}{1 - \Pi(q^2)}, \quad \Pi(q^2) = -\frac{\alpha}{3\pi} \ln\left(\frac{q^2}{\mu^2}\right) + \cdots,$$

where  $\Pi(q^2)$  is the vacuum polarization function (the photon self-energy from a fermion one-loop diagram). At low momentum transfer ( $|q| \ll m_e$ ), the loop correction is analytic in  $q^2$  and generates precisely the local operators  $C_0 + C_2 q^2 + \cdots$  discussed in Section 4 below.

The structural point: fermions affect forces, but the path from “fermion field” to “effective interaction” runs through a quantum loop (not through a tree-level Gaussian elimination of a classical source), and the low-energy residue takes the form of local/contact operators.

**Example 3.2 (Uehling potential: the coordinate-space face of vacuum polarization).** The momentum-dependent coupling above translates, via Fourier transform, into a coordinate-space correction to the Coulomb potential — the Uehling potential  $V_{\text{Uehl}}(r) = -(Z_1 Z_2 \alpha / r) \cdot (2\alpha / 3\pi) \int_1^\infty du \, (1 + 1/(2u^2)) \sqrt{1 - 1/u^2} u^{-1} e^{-2m_e r u}$ . At short distances ( $r \ll 1/m_e$ ) the integral yields a logarithmic correction  $\propto \ln(1/(m_e r))$ , reflecting the running coupling and matching the analytic  $C_0 + C_2 q^2 + \cdots$  expansion of Section 4; at long distances ( $r \gg 1/m_e$ ) it is exponentially suppressed  $\propto e^{-2m_e r}$ , confirming that the fermion decouples below its mass threshold. The dominant observable consequence is the vacuum-polarization contribution to the hydrogen Lamb shift ( $\approx 27$  MHz of the total  $\approx 1058$  MHz  $2S$ – $2P$  splitting), which probes the modified potential at nuclear distances where the  $S$ -wave wavefunction satisfies  $|\psi(0)|^2 \neq 0$ .

## 4. IR replacement: local operators $\Rightarrow$ contact kernels / boundary-condition data

At low resolution, integrating out heavy degrees of freedom typically produces local operators. In a two-body, nonrelativistic sector, this appears as an amplitude expansion analytic in momentum transfer  $q$ :

$$\mathcal{A}(q) = C_0 + C_2 q^2 + O(q^4).$$

The coordinate-space interaction associated to such an analytic expansion is distributional and diagonal-supported. The invariant core is a Fourier-transform

identity:

$$\int \frac{d^d q}{(2\pi)^d} e^{iq \cdot r} = \delta^{(d)}(r), \quad \int \frac{d^d q}{(2\pi)^d} q^2 e^{iq \cdot r} = -\nabla^2 \delta^{(d)}(r).$$

**Derivation 4.1 (Contact expansion gives  $\delta$  and derivative contacts).** Interpreting the low-energy interaction kernel as the inverse Fourier transform of  $\mathcal{A}(q)$  (Born-level language, up to overall convention-dependent factors), we obtain

$$V(r) \propto \int \frac{d^d q}{(2\pi)^d} e^{iq \cdot r} \mathcal{A}(q) \propto C_0 \delta^{(d)}(r) - C_2 \nabla^2 \delta^{(d)}(r) + \dots$$

Thus locality at low energy naturally becomes **contact data**: delta kernels and their derivatives supported at a point (or on the diagonal, in kernel language).

In singular cases (notably  $\delta$  interactions in  $d \geq 2$  in various channels), this contact data is not simply a fixed number: it is defined by a renormalization condition and can generate RG-invariant scales and bound states [Jackiw1991DeltaPotentials] [ManuelTarrach1994PertRenQM].

**Remark 4.2 (Connection to the effective range expansion).** In scattering theory the s-wave amplitude is parametrized by the effective range expansion (ERE)  $k \cot \delta_0(k) = -1/a + r_0 k^2/2 + O(k^4)$ , where  $a$  is the scattering length and  $r_0$  the effective range. The contact expansion of Derivation 4.1 is the momentum-space counterpart: at Born level,  $C_0$  determines  $a$ ,  $C_2$  determines  $r_0$ , and each higher  $C_{2n}$  maps to a shape parameter. Examples 5.1 and 5.2 below are the leading-order case  $C_2 = 0$  (zero effective range,  $r_0 = 0$ ), for which the full amplitude  $f_0(k) = -a/(1 + ika)$  depends on a single parameter — the scattering length.

## 5. Boundary-condition reading (point interactions)

Point-supported interactions can be encoded as self-adjoint extension / boundary-condition data rather than as ordinary functions  $V(r)$ . This is the natural operator-theoretic counterpart of “diagonal-support kernels.” For standard references and pedagogical framing, see [BonneauFarautValent2001SAE] and the delta-potential discussion in [Jackiw1991DeltaPotentials].

This viewpoint matches the controlled-refinement perspective: when a continuum description is defined as a refinement limit, UV data can survive in the limit precisely as boundary-condition parameters (contact terms), with RG flow expressing compatibility across resolutions.

**Example 5.1 (Contact coupling generates a scale: 2D delta potential).** In two spatial dimensions, a contact interaction  $V(r) = g_0 \delta^{(2)}(r)$  with bare

coupling  $g_0$  and UV cutoff  $\Lambda$  leads, after a standard loop integral, to a  $T$ -matrix with the structure

$$T(k)^{-1} = \frac{1}{g_0} + \frac{m}{\pi\hbar^2} \ln\left(\frac{\Lambda}{k}\right),$$

which diverges as  $\Lambda \rightarrow \infty$  unless  $g_0$  is tuned. Define a renormalized coupling at reference scale  $\mu$  by absorbing the  $\ln \Lambda$  divergence; cutoff independence then gives the beta function  $\beta(g_R) = \mu dg_R/d\mu = (m/\pi\hbar^2) g_R^2$ . This is a quadratic beta function of the same form as the toy logarithmic model in the cornerstone (Section 8.3), with solution producing a dynamically generated scale  $\mu_* = \mu e^{\pi\hbar^2/(mg_R)}$ . For attractive coupling ( $g_R < 0$ ) this scale is below the reference scale and sets the bound-state energy:  $E = -\hbar^2\mu_*^2/(2m)$ .

The structural lesson: the “contact” limit of the effective interaction is not a number (coupling constant) but a flow — a scale-dependent parameter whose RG trajectory is part of the definition. This is “uncuttable” in the sense of the companion note: the continuum theory requires the refinement rule (cutoff removal + beta function) and not merely a single-cutoff value.

**Example 5.2 (3D contact interaction: scattering length).** In three spatial dimensions, the same contact interaction  $V(r) = g_0 \delta^{(3)}(r)$  with UV cutoff  $\Lambda$  produces a linearly divergent loop integral (compared to the logarithmic divergence in  $d = 2$ ). After resummation, the s-wave scattering amplitude takes the standard effective-range form with zero effective range:

$$f_0(k) = \frac{-a}{1 + ika},$$

where the scattering length  $a$  is defined by absorbing the  $\Lambda$ -dependent bare coupling into a single physical parameter via a renormalization condition of the form  $1/g_0 \propto \Lambda + (\text{finite part depending on } a)$ . When  $a > 0$ , a pole at  $k = i/a$  gives a bound state with energy  $E = -\hbar^2/(2ma^2)$  [AlbeverioGesztesyHoeghKrohn- Holden2005]. The comparison with Example 5.1 highlights how the divergence character changes with dimension — logarithmic ( $d = 2$ ) versus linear ( $d = 3$ ) — while the structural lesson is identical: the “coupling constant” of a contact interaction is not a bare number but a renormalization-group datum, defined only through a refinement rule (cutoff removal + physical matching condition).

**Remark 5.3 (Unitarity limit: universality at the RG fixed point).** When the scattering length diverges ( $|a| \rightarrow \infty$ ), the contact coupling sits at a non-trivial RG fixed point and the theory becomes scale-invariant: no microscopic length survives beyond the interparticle spacing. Thermodynamic ratios become universal — for a spin- $\frac{1}{2}$  Fermi gas the ground-state energy is  $E = \xi E_{\text{FG}}$  with Bertsch parameter  $\xi \approx 0.37$ , independent of the short-range physics that produced the large scattering length. This fixed point controls the BEC–BCS crossover in cold atomic gases, where a magnetic Feshbach resonance tunes  $a$  through  $\pm\infty$ , providing a laboratory realization of the structural lesson in Examples 5.1–5.2: the contact coupling is not a number but a flow, and the fixed point of that flow generates universality.

## 6. Outlook (kept minimal)

Longer-range effects associated to fermionic degrees of freedom can arise through loop-induced mechanisms or through emergent bosonic composite modes. Treating those responsibly would require a separate bibliography-hardening pass and is outside this note's scope.

## References

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