Action–Angle Indeterminacy in Central Potentials: A Referee-Safe Witness

Abstract

"Action–angle indeterminacy" should not be read as a force-range heuristic (in the style of energy–time slogans), but as a clean conjugacy statement: sharpening an action variable broadens the conjugate angle variable. For central potentials the safest, most explicit instance is the azimuthal pair (ϕ, L_z): an L_z eigenstate has ϕ-dependence $e^{im\phi}$, hence a uniform azimuthal probability distribution; conversely, any state localized in ϕ must involve a broad superposition of angular-momentum modes (Fourier on the circle). This note records that witness and explains its foundations-level message: classical orbit-phase/orientation pictures correspond to semiclassical packets/superpositions rather than single stationary eigenstates.

1. Purpose and scope

This dependent note isolates one specific "action–angle indeterminacy" statement that is both explicit and referee-safe in a central potential: ϕ **is delocalized in an** L_z **eigenstate**, and conversely **localizing** ϕ **requires a superposition over many** m modes.

We deliberately keep the scope bounded. We do **not** enter the self-adjoint "angle operator" debate; instead we use the standard circle/Fourier structure and the unitary phase variable $e^{i\phi}$. We also do **not** make any claims about the range of forces or potentials; the point here is about **which variables can be simultaneously sharp** in stationary states.

2. The safe conjugate pair on the circle: ϕ and L_z

In spherical coordinates the azimuthal angle is periodic, $\phi \sim \phi + 2\pi$. The generator of rotations about the z-axis is

$$L_z = -i\hbar \frac{\partial}{\partial \phi}.$$

The periodicity makes the naive commutator $[\phi, L_z] = i\hbar$ subtle if one insists on an everywhere-defined self-adjoint ϕ operator. A standard way to stay on safe

ground is to use the unitary "phase" variable

$$E := e^{i\phi}.$$

Acting on 2π-periodic wavefunctions, E is well-defined and satisfies the canonical shift relation

$$[L_z, E] = \hbar E,$$

which already captures the operational content: sharp L_z implies maximal delocalization in the conjugate angle.

3. Central potentials: L_z eigenstates have uniform ϕ distribution

For a central potential (or any Hamiltonian commuting with L_z), one may choose simultaneous eigenstates of L_z. In the standard separation of variables, the azimuthal dependence of an angular-momentum eigenstate is the Fourier mode $e^{im\phi}$ with integer m (for example in the spherical-harmonic factor $Y_{\ell m}(\theta, \phi) \propto P_{\ell m}(\cos\theta)e^{im\phi}$) [TongQMLectures].

Thus an L_z eigenstate may be written as

$$\psi(r, \theta, \phi) = F(r, \theta)\, e^{im\phi}, \qquad m \in \mathbb{Z},$$

and therefore

$$|\psi(r, \theta, \phi)|^2 = |F(r, \theta)|^2,$$

independent of ϕ. In particular, the marginal distribution of ϕ is uniform on $[0, 2\pi)$. This is the minimal "angle indeterminacy" witness for central potentials.

4. Fourier tradeoff: localizing ϕ forces a broad m-superposition

Any square-integrable 2π-periodic function admits a Fourier series

$$\psi(\phi) = \sum_{m \in \mathbb{Z}} c_m e^{im\phi}, \qquad \sum_{m \in \mathbb{Z}} |c_m|^2 < \infty.$$

If only one Fourier mode is present (sharp m, hence sharp L_z), then $|\psi(\phi)|^2$ is constant; conversely, a state that is peaked in ϕ necessarily uses many Fourier modes (broad m-support).

Example 4.1 (Dirichlet-kernel packet). The normalized superposition of modes $-M \le m \le M$,

$$\psi_M(\phi) = \frac{1}{\sqrt{2\pi(2M+1)}} \sum_{m=-M}^{M} e^{im\phi},$$

is peaked near $\phi = 0$ with an angular width that scales like $1/M$, while its m-distribution is spread across $\{-M, \ldots, M\}$. This makes the "sharpening ϕ $\Rightarrow$ broadening L_z" tradeoff completely explicit without invoking any disputed angle-operator formalism.

5. Foundations message: orbit pictures require packets/superpositions

This witness supports a simple interpretive guardrail for central-force intuition: a single stationary eigenstate (even when it carries classical-sounding quantum numbers) is typically **not** a localized classical orbit with a definite phase/orientation. Variables like the azimuthal phase ϕ (and, in more structured integrable cases, other angle variables on the invariant torus) become localized only in **coherent superpositions** of many stationary modes.

In other words, "classical orbit pictures" correspond to semiclassical packets and stationary-phase concentration, not to exact eigenstates that are sharp in the conserved actions.

6. Outlook (kept minimal)

Beyond the (ϕ, L_z) sector, integrable central problems admit a fuller action–angle description (with a radial action and additional angle variables on the invariant torus). EBK/WKB quantization makes the same structural point: the more sharply the actions are specified, the less information remains in the conjugate phases. Hardening that broader story into a standalone foundations claim would require a separate study cycle to avoid conflating (i) action–angle existence/global issues with (ii) semiclassical quantization conditions.

References

1. [TongQMLectures] David Tong, "Quantum Mechanics" (lecture notes, no DOI). OA: lecture-note PDF. (Contains $Y_{l,m}(\theta, \phi) = P_{l,m}(\cos\theta)e^{im\phi}$ as simultaneous eigenstates of L^2 and L_z.)