Delta Objects as Half-Density Kernels: Identity, Stationary-Set Concentration, and Point Interactions

Abstract

Three seemingly different uses of the Dirac delta share one geometric meaning when amplitudes are treated as **half-densities**: 1. the delta as the Schwartz kernel of the identity operator, 2. the delta as a density supported on stationary points ($\delta(\nabla f)$), 3. the delta as a rank-one kernel defining a point interaction ($g|0\rangle\langle 0|$).

In each case, the amplitude-level object carries **square-root Jacobian** weights (half-density weights), while the corresponding "probability"/density-level object carries the unsquared Jacobians. This note collects the finite-dimensional identities and scaling computations that make this pattern explicit, and isolates where a physical length scale may enter when one insists on scalar representatives.

This note is a companion to `paper/main.md`. Statements are kept finite-dimensional unless explicitly labeled heuristic.

1. Half-densities and kernels (coordinate free)

Let M be a d-dimensional manifold and $|\Omega|^{1/2}$ the half-density bundle. An operator $K : \Gamma_c(|\Omega|^{1/2}) \to \Gamma(|\Omega|^{1/2})$ has a natural Schwartz kernel

$$\mathsf{K} \in \mathcal{D}'(M \times M;\ |\Omega|^{1/2} \boxtimes |\Omega|^{1/2}),$$

so that

$$(K\psi)(x) = \int_M \mathsf{K}(x,y)\,\psi(y),$$

is coordinate invariant: $\mathsf{K}(x,y)\psi(y)$ is a density in y valued in a half-density at x.

Scalarizing kernels (writing $\int dy$ with a scalar integrand) implicitly chooses a reference density/half-density; the half-density formalism keeps this choice explicit.

2. Delta as the identity kernel (and near-diagonal scaling)

The identity operator on half-densities has Schwartz kernel

$$\mathsf{K}_{\mathrm{Id}}(x,y) = \delta^{(d)}(x-y)\,|dx|^{1/2}|dy|^{1/2}.$$

Worked scaling computation (the $d/2$ exponent)

Introduce near-diagonal coordinates $y = x + \varepsilon v$. Then $\delta^{(d)}(x-y) = \delta^{(d)}(\varepsilon v) = \varepsilon^{-d}\delta^{(d)}(v)$ and $|dy|^{1/2} = \varepsilon^{d/2}|dv|^{1/2}$, so

$$\mathsf{K}_{\mathrm{Id}}(x, x+\varepsilon v) = \varepsilon^{-d/2}\,\delta^{(d)}(v)\,|dx|^{1/2}|dv|^{1/2}.$$

Thus the universal $\varepsilon^{-d/2}$ normalization exponent is already present in the identity delta kernel, once kernels are treated as half-densities.

3. Delta on the stationary set: ̆03b4(̆2207f) and determinant weights

3.1 One-dimensional identity (̆03b4(f'))

Let $f : \mathbb{R} \to \mathbb{R}$ have finitely many nondegenerate critical points x_i (so $f'(x_i) = 0$, $f''(x_i) \neq 0$). Then, as distributions,

$$\delta(f'(x)) = \sum_i \frac{\delta(x-x_i)}{|f''(x_i)|}.$$

So $\delta(f')\,dx$ is a density supported at stationary points with weights $1/|f''|$.

3.1a ̆03b4(f') versus ̆03b4': delta of a derivative vs derivative of delta

The notation $\delta(f')$ above means: apply the Dirac delta distribution $\delta(\cdot)$ to the **function** $f'(x)$, thereby localizing to the stationary set $f'(x) = 0$. It should not be confused with δ', the **distributional derivative** of δ, defined by duality:

$$\langle \delta', \varphi \rangle := -\langle \delta, \varphi' \rangle = -\varphi'(0).$$

So δ' is the distribution that probes derivatives of test functions at a point ("value of the derivative at zero", up to sign), whereas $\delta(f')$ is a stationary-set localization distribution.

3.1b ŏ3b4' from point splitting (difference quotient of shifted deltas)

The distribution δ' can be realized as a regulated point-splitting limit. Let $\varepsilon \to 0$ and consider the shifted delta $\delta(x+\varepsilon)$. For any test function φ,

$$\left\langle \frac{\delta(\,\cdot+\varepsilon)-\delta}{\varepsilon}, \varphi \right\rangle = \frac{\varphi(-\varepsilon)-\varphi(0)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} -\varphi'(0) = \langle \delta', \varphi \rangle.$$

Hence, in the sense of distributions,

$$\frac{\delta(x+\varepsilon)-\delta(x)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} \delta'(x).$$

This gives a clean dictionary item for "probing the derivative at a point":

$$f'(0) = \langle -\delta', f \rangle.$$

For the parallel smooth-function toy model ("difference quotient as divergence + subtraction") and further remarks, see `blackboards/2026-02-10-difference-quotients-counterterms-and-`

3.2 Multi-dimensional identity (ŏ3b4(ž207f))

Let $f : \mathbb{R}^n \to \mathbb{R}$ have finitely many nondegenerate critical points x_i (so $\nabla f(x_i) = 0$ and $\det(\operatorname{Hess} f)(x_i) \neq 0$). Then

$$\delta^{(n)}(\nabla f(x)) = \sum_i \frac{\delta^{(n)}(x-x_i)}{|\det(\operatorname{Hess} f)(x_i)|}.$$

3.3 Stationary phase and square-root weights (amplitudes vs densities)

For the oscillatory integral

$$I(\hbar) = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} f(x)}\, a(x)\, dx, \qquad \hbar \to 0^+,$$

stationary phase gives amplitude contributions weighted by

$$\frac{1}{\sqrt{|\det(\operatorname{Hess} f)(x_i)|}},$$

up to a universal $\hbar$-dependent factor and a signature phase. Squaring amplitude weights produces the density weights in $\delta^{(n)}(\nabla f)$. This is the finite-dimensional prototype of the slogan: **amplitudes are half-densities; probabilities are densities.**

3.4 Extremals in weak form: where ̆03b4 and ̆03b4’ appear in Euler̆2013Lagrange

For an action $S[q] = \int L(q, \dot{q}, t)\, dt$, the extremal condition is naturally distributional: for test variations $\eta(t)$ of compact support,

$$\delta S[q;\eta] = \int \Big(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \Big) \eta(t)\, dt.$$

If $\delta S[q;\eta] = 0$ for all η, then the Euler̆2013Lagrange expression vanishes as a distribution. Approximating η by bump functions converging to $\delta(t-t_*)$ localizes the equation at t_* under regularity.

When $\partial L/\partial \dot{q}$ has jumps (corners/impulses), the distributional derivative produces delta terms automatically; more generally, point-supported singularities are encoded by delta kernels and their derivatives $(\delta, \delta', \ldots)$, depending on distributional order. For a short dictionary, see `blackboards/2026-02-10-distribution-theory-for-extremals.md`.

4. Delta at a point: point interactions as rank-one kernels

A point interaction is naturally the rank-one operator

$$V = g\, |0\rangle\langle 0|.$$

In the half-density kernel calculus this is written as the bi-half-density distribution supported at $(0,0)$:

$$\mathsf{K}_V(x,y) = g\ \delta^{(d)}(x)\, \delta^{(d)}(y)\, |dx|^{1/2} |dy|^{1/2}.$$

This is the “projector-like delta” object underlying contact interactions.

5. Where scales enter upon scalarization (and why RG invariants are natural candidates)

Half-density kernels are canonical; scalar representatives are not. Choosing a reference half-density σ_* identifies any half-density ψ with a scalar f via $\psi = f\, \sigma_*$. If one insists that scalar representatives be dimensionless, then σ_* must carry a length$^{d/2}$ constant.

In marginal cases (notably the 2D point interaction), renormalization generates an RG-invariant scale κ_* (dimensional transmutation). This suggests a conditional identification: if one adds a universality hypothesis that scalarization scales must be built from physical invariants, then RG-invariant scales are natural candidates to supply the missing length$^{d/2}$ factors required by scalarization.

This note treats that identification as an organizing perspective, not as a theorem.

6. Outlook

1. Relate the determinant weights in $\delta(\nabla f)$ to the mixed Hessian determinants (Van Vleck type) that appear after eliminating intermediate variables in time slicing (Schur complement template).
2. Clarify which parts of the "functional delta $\delta(\delta S)$" story survive rigorous regularization and which remain heuristic.