Fermionic Mediators, Static Potentials, and Contact/Boundary-Condition Limits

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Abstract

The textbook derivation of a static potential from "field exchange" uses a bosonic mediator linearly sourced by a commuting classical density, yielding an effective action quadratic in the source and (in a static limit) a central Yukawa/Coulomb potential. This derivation does not transplant verbatim to fermionic fields: the linear source terms for fermions require Grassmann-valued sources, so there is no ordinary commuting classical source whose elimination produces a classical potential in the same way. This note isolates the precise obstruction and records the robust infrared replacement: when a microscopic description reduces to local operators at low resolution, the effective interaction is encoded by contact terms (delta kernels and their derivatives) or, equivalently, boundary-condition/self-adjoint-extension data, with renormalization-group running when the contact limit is singular.

This is a dependent note aligned with the broader refinement-compatibility program: contact terms are diagonal-support kernels, and their scale dependence is a compatibility condition rather than an afterthought.

1. Purpose and scope

This note answers a narrowly phrased question: what can it mean for a **fermionic** field to "generate a (central) potential" in the same sense that a massive bosonic field generates a Yukawa potential?

We keep the scope bounded: 1. state the bosonic sourcing ⇒ potential mechanism (derivation-first, brief), 2. state the fermionic obstruction precisely (Grassmann sources), 3. give one explicit IR matching witness: **local operators ⇒ contact/derivative-contact kernels**, 4. connect contact kernels to related point-interaction/RG witnesses.

We do **not** claim that fermions cannot affect forces; we only isolate which parts of the "classical source ⇒ potential" story fail, and what the correct replacement statement is at low resolution.

2. What "a field generates a potential" means in the bosonic source story

The archetypal construction is a bosonic mediator φ linearly coupled to a commuting source $J(x)$:

$$S[\varphi; J] = \int d^D x \left(\frac{1}{2} \varphi \, K \, \varphi + J \, \varphi \right), \qquad K = (\Box + m^2) \text{ (example).}$$

Integrating out φ (Gaussian elimination) yields an effective action quadratic in the source,

$$S_{\text{eff}}[J] = -\frac{1}{2} \int d^D x \, d^D y \; J(x) \, K^{-1}(x, y) \, J(y),$$

so the static, nonrelativistic limit of K^{-1} produces a central potential (Yukawa for $m \neq 0$, Coulomb-type when $m = 0$).

The key structural ingredient is that the source is an ordinary commuting function (a classical background density).

3. Fermionic fields: linear sources are Grassmann, so the classical-source story does not transplant

For a Dirac fermion Ψ, the generating functional with sources is written with **Grassmann-valued** sources $\eta, \bar{\eta}$:

$$Z[\bar{\eta}, \eta] = \int D\bar{\Psi} \, D\Psi \; \exp\!\left(i \int d^D x \; \bar{\Psi} \, (i\gamma^\mu \partial_\mu - m) \Psi + i \int d^D x \; (\bar{\eta} \, \Psi + \bar{\Psi} \, \eta) \right).$$

An explicit statement of this form, including that $\eta, \bar{\eta}$ are Grassmann-valued, is recorded in [Floerchinger2024QFT1Lecture21].

Two immediate consequences follow.

`Remark 3.1 (Obstruction statement).` The bosonic derivation "choose a commuting classical source J, integrate out the field, and read off a classical potential" does not directly apply to fermions, because the linear source terms that couple to Ψ require Grassmann sources rather than commuting c-number densities. Therefore, "fermion exchange generates a classical potential between commuting sources" is not a well-posed transplant of the bosonic story.

This does **not** mean fermions are irrelevant: fermions can and do affect effective interactions through loop effects, through bosonic composite modes (bilinears), and through low-energy EFT operators. The point is that the meaning of "generates a potential" must be stated through one of these controlled mechanisms.

3.1 The controlled alternative: fermion loops modify bosonic propagators

The standard example is vacuum polarization in quantum electrodynamics. A closed electron–positron loop inserted into the photon propagator gives a momentum-dependent correction to the effective electromagnetic coupling,

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha}{1-\Pi(q^2)}, \qquad \Pi(q^2) = -\frac{\alpha}{3\pi}\ln\left(\frac{q^2}{\mu^2}\right) + \cdots,$$

where $\Pi(q^2)$ is the vacuum polarization function (the photon self-energy from a fermion one-loop diagram). At low momentum transfer ($|q| \ll m_e$), the loop correction is analytic in q^2 and generates precisely the local operators $C_0 + C_2 q^2 + \cdots$ discussed in Section 4 below.

The structural point: fermions affect forces, but the path from "fermion field" to "effective interaction" runs through a quantum loop (not through a tree-level Gaussian elimination of a classical source), and the low-energy residue takes the form of local/contact operators.

4. IR replacement: local operators ⇒ contact kernels / boundary-condition data

At low resolution, integrating out heavy degrees of freedom typically produces local operators. In a two-body, nonrelativistic sector, this appears as an amplitude expansion analytic in momentum transfer q:

$$\mathcal{A}(q) = C_0 + C_2 q^2 + O(q^4).$$

The coordinate-space interaction associated to such an analytic expansion is distributional and diagonal-supported. The invariant core is a Fourier-transform identity:

$$\int \frac{d^d q}{(2\pi)^d}\, e^{iq\cdot r} = \delta^{(d)}(r), \qquad \int \frac{d^d q}{(2\pi)^d}\, q^2\, e^{iq\cdot r} = -\nabla^2 \delta^{(d)}(r).$$

`Derivation 4.1 (Contact expansion gives \(\delta\) and derivative contacts).` Interpreting the low-energy interaction kernel as the inverse Fourier transform of $\mathcal{A}(q)$ (Born-level language, up to overall convention-dependent factors), we obtain

$$V(r) \;\propto\; \int \frac{d^d q}{(2\pi)^d}\, e^{iq\cdot r}\, \mathcal{A}(q) \;\propto\; C_0\, \delta^{(d)}(r) \;-\; C_2\, \nabla^2 \delta^{(d)}(r) \;+\; \cdots .$$

Thus locality at low energy naturally becomes **contact data**: delta kernels and their derivatives supported at a point (or on the diagonal, in kernel language).

In singular cases (notably δ interactions in $d \geq 2$ in various channels), this contact data is not simply a fixed number: it is defined by a renormalization condition and can generate RG-invariant scales and bound states [Jackiw1991DeltaPotentials] [ManuelTarrach1994PertRenQM].

5. Boundary-condition reading (point interactions)

Point-supported interactions can be encoded as self-adjoint extension / boundary-condition data rather than as ordinary functions $V(r)$. This is the natural operator-theoretic counterpart of "diagonal-support kernels." For standard references and pedagogical framing, see [BonneauFarautValent2001SAE] and the delta-potential discussion in [Jackiw1991DeltaPotentials].

This viewpoint matches the controlled-refinement perspective: when a continuum description is defined as a refinement limit, UV data can survive in the limit precisely as boundary-condition parameters (contact terms), with RG flow expressing compatibility across resolutions.

6. Outlook (kept minimal)

Longer-range effects associated to fermionic degrees of freedom can arise through loop-induced mechanisms or through emergent bosonic composite modes. Treating those responsibly would require a separate bibliography-hardening pass and is outside this note's scope.

References

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3. [Jackiw1991DeltaPotentials] R. Jackiw, "Delta-function potentials in two- and three-dimensional quantum mechanics," MIT-CTP-1937 (Jan 1991). Reprinted in *M.A.B. Bég Memorial Volume* (World Scientific, 1991), pp. 25–42. OA mirror: https://www.physics.smu.edu/scalise/P6335fa21/notes/Jackiw.pdf.
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generating functional $Z[\bar{\eta}, \eta]$ with Grassmann-valued sources.) OA: lecture webpage.