Planck Area from Half-Density Normalization

Abstract

Half-densities are the natural "coordinate-free integrands" for composing kernels without choosing a background measure. But choosing a *universal* convention for turning half-density objects into dimensionless numerical amplitudes introduces a length$^{d/2}$ scale. In $d = 4$, this is an *area*. This note sharpens the hypothesis ladder needed for the claim "half-density normalization selects a universal area scale", and isolates a simple dimension-matching condition under which the Planck area appears without fractional powers of couplings. A gravitational anchor based on a minimal-areal-speed principle is recorded as a separate heuristic thread [RiveroAreal] [RiveroSimple].

Minimal claim (proved under stated hypotheses): half-density composition is intrinsic, but any scalar "amplitude function" representation requires extra scalarization structure, and if that scalar representative is demanded dimensionless then a length$^{d/2}$ constant is unavoidable. Conjectural claim (additional universality hypotheses): the scalarization constant is fixed by universal couplings and becomes a universal area scale in $D = 4$.

1. Purpose and Scope

This is a companion to `paper/main.md`. Its goal is to isolate one technical point that is only implicit in the main manuscript: the role of half-densities (and their scaling) in making composition laws coordinate-invariant *and* dimensionally well-defined.

Claims below are labeled as `Proposition` (math-precise under hypotheses) or `Heuristic` (programmatic bridge).

Notation (dimensions). The half-density weight length$^{d/2}$ uses the dimension of the manifold being integrated over in the composition law (the "composition variable" dimension). In nonrelativistic time-slicing this is typically spatial dimension; in covariant/field-theory settings it is spacetime dimension. In the gravity-only sieve (Derivation D1.3), read d as the spacetime dimension D if that avoids confusion: the "$D = 4 \Rightarrow$ area" conclusion is about spacetime $D = 4$, not a claim that *spatial* $d = 4$ is privileged.

2. Half-Densities and Composition Kernels

Let M be a d-dimensional manifold. A (positive) density is a section of $|\Lambda^d T^*M|$, and a half-density is a section of $|\Lambda^d T^*M|^{1/2}$.

The key operational point is: when a kernel is a half-density in its integration variable, composition of kernels does not depend on an arbitrary choice of coordinate measure.

`Heuristic H1.1 (Why half-densities).` If $K_1(x,y)$ and $K_2(y,z)$ are chosen so that their product in the intermediate variable y is a density, then $\int_M K_1(x,y)K_2(y,z)$ is coordinate-invariant without fixing a preferred dy. This matches the structural role of kernel composition used in `paper/main.md` (Section 6).

`Derivation D1.1 (Coordinate invariance of half-density pairing and composition).` In a local chart $y = (y^1, \dots, y^d)$, write a half-density as $\psi(y) = \varphi(y)\,|dy|^{1/2}$. Under a change of variables $y = y(y')$, one has $|dy|^{1/2} = |\det(\partial y/\partial y')|^{1/2}|dy'|^{1/2}$, so the coefficient transforms as $\varphi'(y') = \varphi(y(y'))\,|\det(\partial y/\partial y')|^{1/2}$.

Hence the product of two half-densities is a density: $\psi_1\psi_2 = (\varphi_1\varphi_2)\,|dy|$, and its integral is chart-independent: $\int_M \psi_1\psi_2$ is well-defined without choosing a background measure beyond the density bundle itself.

Kernel composition is the same mechanism: if $K_1(x,y)$ and $K_2(y,z)$ are half-densities in y, then K_1K_2 is a density in y and $\int_M K_1K_2$ is coordinate invariant.

3. Dimensional Analysis: Normalizing a Half-Density Requires a Scale

A density on M carries the units of length^d once physical units are assigned to coordinates. A half-density therefore carries units length$^{d/2}$.

`Proposition P1.1 (No canonical "half-density = function" identification).` There is no canonical identification of a half-density $\psi \in |\Lambda^d T^*M|^{1/2}$ with an ordinary scalar function f on M. Choosing such an identification is equivalent to choosing a nowhere-vanishing reference half-density σ_* (equivalently a positive density $\rho_* = \sigma_*^2$) and writing $\psi = f\,\sigma_*$.

`Derivation D1.2 (Dilation makes the \(\text{length}^{d/2}\) weight explicit).` On $\mathbb{R}^d$, consider a dilation $y \mapsto y' = ay$ with $a > 0$. Then $|dy'| = a^d|dy|$, so $|dy'|^{1/2} = a^{d/2}|dy|^{1/2}$. Thus even in flat space, half-densities carry an inherent length$^{d/2}$ scaling weight.

`Derivation D1.2a (Near-diagonal scaling forces the square-root Jacobian \(\varepsilon^{-d/2}\)).` On $M = \mathbb{R}^d$, introduce near-diagonal coordinates $y = x + \varepsilon v$ with $\varepsilon > 0$. Then $dy = \varepsilon^d\,dv$, hence $|dy|^{1/2} = \varepsilon^{d/2}|dv|^{1/2}$.

For a bi-half-density kernel written locally as

$$K_\varepsilon(x,y) = k_\varepsilon(x,y)\,|dx|^{1/2}|dy|^{1/2},$$

its pullback to (x,v) variables becomes

$$K_\varepsilon(x,x+\varepsilon v) = \big(\varepsilon^{d/2} k_\varepsilon(x,x+\varepsilon v)\big)\,|dx|^{1/2}|dv|^{1/2}.$$

Thus, any attempt to define a nontrivial "$\varepsilon \to 0$" near-diagonal limit of kernels (the scaling step that tangent-groupoid quantization packages) inevitably produces an $\varepsilon^{d/2}$ factor from the half-density Jacobian, and the corresponding scalar representative must be renormalized by $\varepsilon^{-d/2}$ to stay finite. This is the same exponent as in the finite-dimensional "square-root delta" normalization: the half-density is the square root of the density Jacobian. For a canonical distributional witness carrying the same exponent, see Derivation D1.2b.

`Derivation D1.2b (Identity kernel: the delta bi-half-density carries the same exponent).` The identity operator on half-densities has Schwartz kernel

$$K_{\mathrm{Id}}(x,y) = \delta^{(d)}(x-y)\,|dx|^{1/2}|dy|^{1/2}.$$

Under the same near-diagonal change of variables $y = x+\varepsilon v$, one has $\delta^{(d)}(x-y) = \varepsilon^{-d}\delta^{(d)}(v)$ and $|dy|^{1/2} = \varepsilon^{d/2}|dv|^{1/2}$, hence

$$K_{\mathrm{Id}}(x,x+\varepsilon v) = \varepsilon^{-d/2}\,\delta^{(d)}(v)\,|dx|^{1/2}|dv|^{1/2}.$$

So the universal $\varepsilon^{-d/2}$ exponent is already encoded in the half-density identity kernel. If one scalarizes by choosing a dimensionless reference half-density $\sigma_* := L_*^{-d/2}|dx|^{1/2}$ (for a constant length scale L_*), then the scalar representative of the identity kernel is the **dimensionless** distribution $k_{\mathrm{Id}}(x,y) = L_*^d\delta^{(d)}(x-y)$.

`Proposition P1.2 (Universal *dimensionless* amplitudes force a \(\text{length}^{d/2}\) constant).` If one imposes the extra requirement that the scalar representative f in $\psi = f\,\sigma_*$ be dimensionless in physical units, then the reference half-density σ_* must carry all of the length$^{d/2}$ dimension. In particular, a *constant* (field-independent) choice of σ_* is equivalent to choosing a universal length$^{d/2}$ scale.

In $d = 4$, this universal length$^{d/2}$ scale is a universal *area* scale.

`Heuristic H1.2 (Reciprocity claim).` Half-densities alone do not force a particular scale: the forced fact is that converting half-density objects into scalar numerical amplitudes requires extra structure (a reference half-density). The "universal area scale" claim begins only after adding two further hypotheses: 1. the reference σ_* is taken to be *constant* (no dependence on background metric/fields), and 2. the constant is required to be fixed by universal constants/couplings of the theory.

Under these hypotheses, $d = 4$ is the unique dimension in which the needed length$^{d/2}$ constant can be supplied by the gravitational coupling without fractional powers (Derivation D1.3).

`Derivation D1.3 (Gravity-only sieve: why \(d=4\) is singled out if only \(G_d\) is used).` In d spacetime dimensions, the Einstein–Hilbert action $\frac{1}{16\pi G_d}\int d^d x\,\sqrt{|g|}\,R$ shows that (in $c = \hbar = 1$ units) Newton's constant has dimension $[G_d] = \text{length}^{d-2}$. Assume the only available dimensionful coupling used to build the universal normalization constant is G_d itself (no cosmological constant, no additional dimensionful scales), and impose H2.5 in the literal "no fractional powers of G_d" sense. Then the normalization constant has dimension length$^{k(d-2)}$ for some integer k. Matching length$^{d/2}$ forces length$^{d/2}$ = length^{d-2}, which holds if and only if $d = 4$. In that case G_4 itself has dimension of area, and the corresponding area scale is the Planck area $L_P^2 \sim \hbar G_4/c^3$.

3.1 Hypotheses as Separate Knobs (What Is Forced vs Chosen)

The discussion above mixes three different kinds of statements: 1. **Geometric facts** (what half-densities are, how they compose, how they scale), 2. **Representational choices** (how one turns half-density objects into scalar numbers), 3. **Universality/selection principles** (what choices are allowed if we demand "background-free" and "built from couplings").

To study these separately, it is useful to keep the hypotheses explicit.

`Hypothesis H2.1 (Half-density formulation).` Quantum kernels are treated as bi-half-densities so that composition in intermediate variables is coordinate invariant (Section 2 and Derivation D1.4).

`Hypothesis H2.2 (Scalarization by a reference half-density).` To interpret half-density amplitudes as scalar numerical functions, we pick a nowhere-vanishing reference half-density σ_* and write $\psi = f\,\sigma_*$ (Proposition P1.1).

`Hypothesis H2.3 (Dimensionless scalar representative).` The scalar representative f is required to be dimensionless in physical units (Proposition P1.2). This forces σ_* to carry the full length$^{d/2}$ weight.

`Hypothesis H2.4 (Background-free constancy).` The reference σ_* is taken to be constant/field-independent, rather than determined by background geometry (e.g. a Riemannian volume $|g|^{1/4}|dx|^{1/2}$) or by dynamical fields (e.g. a dilaton-like factor). This is the first point where a *universal constant* enters.

`Hypothesis H2.5 (Analyticity / no fractional powers of couplings).` If the universal constant is required to be built from the theory's couplings without fractional powers, then dimensional analysis becomes a *dimension sieve* rather than a tautology. This hypothesis has at least two distinct readings: 1. **Integrality (integer-exponent) reading:** the constant is a monomial in the

available couplings with integer exponents (possibly allowing negative powers). Equivalently, dimension-matching becomes an integer (Diophantine) constraint on the exponents. 2. **Perturbative analyticity reading (stronger):** the constant admits a Taylor expansion around a specified weak-coupling limit, so only nonnegative integer powers appear.

To keep the branches explicit, we will refer to these as: - `Hypothesis H2.5a (Integrality / monomial sieve).` The normalization constant is a monomial in admitted couplings with integer exponents. - `Hypothesis H2.5b (Perturbative analyticity sieve).` The dependence is analytic at a specified weak-coupling point (hence forbids negative powers and other non-analytic dependence).

`Heuristic H2.5b1 (Analyticity for dimensionful couplings needs a reference scale).` If a coupling g_i is dimensionful, "weak coupling" is only meaningful after choosing a reference scale μ and forming a dimensionless parameter $\hat{g}_i(\mu) := \mu^{-a_i} g_i$ (where $[g_i] = \text{length}^{a_i}$). In that sense, perturbative analyticity is naturally analyticity in $\hat{g}_i(\mu)$ near $\hat{g}_i = 0$. Demanding at the same time that the scalarization constant be a μ-independent universal constant pushes one back to either: 1. an engineering-dimension monomial in admitted dimensionful couplings with nonnegative integer exponents (the sieve branch), or 2. a non-analytic RG-invariant transmutation scale (outside H2.5b).

Separately, the RG "dimensional transmutation" mechanism (Heuristic H2.13 / Derivation D1.6a) supplies a scale *outside* both H2.5a and H2.5b: it is typically non-analytic in the coupling.

Derivation D1.3 is the simplest gravity-only instance under the integrality reading: "use G_d without fractional powers" singles out $d = 4$.

`Heuristic H2.6 (Where "special dimensions" can appear).` Special dimensions do not come from half-densities alone (Hypothesis H2.1). They appear only after adding a selection principle like H2.4–H2.5: the requirement that the scalarization choice be universal, background-free, and coupling-built in a restricted (e.g. analytic) way.

`Heuristic H2.6a (Independent D=4 filter: conformal scalarization-gauge simplicity).` The "dimension sieve" (H2.5) is not the only way a special dimension can appear once one insists on scalar representatives. A different kind of filter comes from asking for *simplicity of how kinetic operators depend on the scalarization gauge.*

In the QFT-facing half-density calculus, a background metric supplies a reference half-density $|g|^{1/4}|dx|^{1/2}$, and the scalar Laplacian Δ_g conjugates to an operator on half-densities

$$\widetilde{\Delta} := |g|^{1/4} \Delta_g |g|^{-1/4}.$$

In a conformal background $g_{\mu\nu} = e^{2\sigma(x)}\delta_{\mu\nu}$ in spacetime dimension D, one finds (Derivation D1.3 in `papers/half-density-qft/main.md`, and the deriva-

tion/check notes in `blackboards/2026-02-10-conformal-rescaling-and-half-density-laplacian-family`

$$\widetilde{\Delta}\psi = e^{-2\sigma}\Big(\partial^2\psi - 2\,\partial\sigma\cdot\partial\psi - \frac{D}{2}(\partial^2\sigma)\,\psi + \frac{D(4-D)}{4}(\partial\sigma)^2\,\psi\Big).$$

Thus the quadratic-gradient term $(\partial\sigma)^2\psi$ cancels at $D = 4$ (within the conformal ansatz). If one adopts the extra criterion that scalarization-gauge changes should not generate such quadratic-gradient "potentials" in the half-density kinetic operator, then $D = 4$ is singled out by *operator simplicity* rather than by coupling-dimension matching.

This filter is independent of H2.5 and, by itself, does not supply a length scale; it is recorded only as an additional "special dimension" candidate knob to compare against the scale-sieve hypotheses.

3.2 What Changes When a Hypothesis Is Relaxed?

This subsection records the main "branches" that need separate study.

1. **Drop H2.3 (allow dimensionful f).** Then no universal length$^{d/2}$ constant is forced; the dimensional weight can be carried by the scalar representative itself (as in the usual statement "wavefunctions have dimension length$^{-d/2}$").
2. **Drop H2.4 (allow background geometry).** Then σ_* can be chosen from a metric (or other structure), and the "universal constant" is replaced by background-dependent normalization.
3. **Drop H2.5a/H2.5b (allow fractional powers or other non-analytic dependence).** Then in any $d > 2$ one can build a length$^{d/2}$ constant from gravity via $G_d^{d/(2(d-2))}$ (in $c = \hbar = 1$ units), so $d = 4$ is no longer singled out; instead, $d = 4$ is simply the unique case where the exponent is an integer.
4. **Change "which coupling supplies the scale".** Using other dimensionful couplings (cosmological constant, string tension, gauge couplings in various dimensions, etc.) yields different "special-dimension" sieves. This is conceptually aligned with the observation that some dimensions are singled out by other structures (division algebras, special holonomy, supersymmetry), but those filters are separate from the half-density story and should not be conflated.
5. **Replace H2.5 by transmutation (RG as the scale supplier).** If one allows RG-invariant scales generated by compatibility (dimensional transmutation), then even dimensionless couplings can supply a physical length scale (Heuristic H2.13 / Derivation D1.6a). This branch does not "sieve dimensions" in the same way as H2.5a/H2.5b; it supplies a scale by non-analytic RG invariants rather than by monomials in couplings.

Branch summary (keep the knobs separate). There are three distinct "scale supplier" mechanisms in play: 1. **Monomial sieve** (H2.5a/H2.5b): build

the needed length$^{d/2}$ constant from admitted couplings using restricted (e.g. analytic) dependence. 2. **Fractional/non-analytic coupling dependence:** allow fractional powers or other non-analytic dependence directly in couplings (dimensional analysis becomes permissive; the $d = 4$ sieve disappears). 3. **Dimensional transmutation:** generate a scale from RG compatibility (Derivation D1.6a / Example E5), which is typically non-analytic in naive coupling expansions.

3.3 Starting with H2.5a: Integrality as a Dimension Sieve

The point of H2.5a/H2.5b is not that dimensional analysis alone selects a unique scale (it does not), but that *restricting allowed functional dependence on couplings* can turn dimensional analysis into a selection principle.

`Derivation D1.6 (H2.5a: integer-exponent form of "no fractional powers").` Work in $c = \hbar = 1$ units for dimension counting. Let the available couplings $\{g_i\}$ have length dimensions $[g_i] = \text{length}^{a_i}$. Under the integrality reading of H2.5, the universal normalization constant is a monomial $C = \prod_i g_i^{n_i}$ with integers n_i. Its length dimension is $[C] = \text{length}^{\sum_i n_i a_i}$. Requiring $[C] = \text{length}^{d/2}$ is therefore the integer-exponent (Diophantine) condition

$$\sum_i n_i a_i = \frac{d}{2}.$$

Existence (and non-uniqueness) of solutions depends on: 1. which couplings are admitted as "universal" inputs, and 2. whether one allows negative exponents (non-analytic at zero coupling) or insists on perturbative analyticity (nonnegative exponents).

`Heuristic H2.7 (Why H2.5 needs a "what counts as a coupling" rule).` If one allows arbitrary redefinitions of couplings (e.g. adjoining a new symbol $\tilde{G} = G_d^{1/(d-2)}$), then "no fractional powers" becomes vacuous: the forbidden root has simply been renamed as an allowed coupling. H2.5 is meaningful only together with a prior criterion for admissible coupling dependence (e.g. perturbative analyticity around a distinguished limit such as $G_d \to 0$).

`Heuristic H2.7a (Admissible couplings: exclude scheme parameters and non-analytic reparametrizations).` To keep H2.5 from collapsing into a coordinate artifact, we implicitly adopt the following convention: 1. **Admitted couplings** are the independent parameters that appear as coefficients of local operators in the UV action after fixing a canonical normalization convention for fields (so that field-rescaling freedom is not used to hide roots/powers). 2. We allow only **analytic** reparametrizations near a chosen base point ("weak coupling"), so adjoining $\tilde{g} = g^{1/2}$ is disallowed when it is non-analytic at that base point. 3. We explicitly exclude scheme/scale conventions from the coupling set: the renormalization scale μ, regulator cutoffs Λ, and finite-subtraction

constants; and we also exclude the scalarization gauge scale L_* itself (since constraining L_* is the point of the ladder).

Under this convention, H2.5a is best viewed as a computational proxy once coupling coordinates are fixed, while H2.5b (analyticity at the base point) is the more invariant "no roots" statement. The phrase "canonical normalization of fields" in (1) is itself a convention choice; the point is that fixing such a convention makes the admissibility rule a controlled knob rather than an implicit loophole.

`Example E1 (Gravity-only).` With only G_d available, $a_1 = d - 2$ and the condition becomes $n(d-2) = d/2$. For integer $d \geq 3$, this has a solution only at $d = 4$ with $n = 1$, reproducing Derivation D1.3.

`Example E2 (Gravity + cosmological constant).` If one also allows the cosmological constant Λ_d with $[\Lambda_d] = \text{length}^{-2}$, then the condition becomes $n(d-2) - 2m = d/2$ for integers n, m. A simple family of solutions exists for d divisible by 4: take $n = 1$ and $m = d/4 - 1$, so

$$C \sim G_d \, \Lambda_d^{d/4-1},$$

has dimension $\text{length}^{d/2}$. Thus, even under H2.5, $d = 4$ is not automatically unique once additional dimensionful couplings are admitted; what is special about $d = 4$ in this family is that it is the only case with $m = 0$ (no need to involve Λ_d).

`Example E3 (Yang--Mills coupling as an alternative sieve).` In d spacetime dimensions, the Yang–Mills action is typically written as $\frac{1}{4g_d^2} \int d^d x \, F_{\mu\nu} F^{\mu\nu}$, so $[g_d^2] = \text{length}^{d-4}$ (equivalently $[g_d] = \text{length}^{(d-4)/2}$). If we (hypothetically) allow the half-density normalization constant to be a pure monomial in g_d, $C \sim g_d^p$ with integer $p \geq 0$, then

$$[C] = \text{length}^{p(d-4)/2}.$$

Matching $[C] = \text{length}^{d/2}$ gives the integer-exponent condition

$$p(d-4) = d \quad \Longrightarrow \quad d = \frac{4p}{p-1} = 4 + \frac{4}{p-1}.$$

Thus integer solutions occur only when $p - 1 \mid 4$, i.e. $p \in \{2, 3, 5\}$, giving $d \in \{8, 6, 5\}$ respectively.

In particular, in $d = 4$ the gauge coupling is dimensionless and cannot by itself supply the $\text{length}^{d/2}$ factor needed for half-density scalarization; in that case the scale must come from another dimensionful coupling (e.g. gravity) or from a non-analytic mechanism (dimensional transmutation).

`Example E4 (A universal area parameter \(\alpha_\ast\) as a scale supplier).` Suppose the UV theory admits a universal area parameter α_* with

dimension $[\alpha_*] = \text{length}^2$ (for example, in perturbative string theory α_* is the familiar $\alpha' = l_s^2$, with string tension $T \sim 1/\alpha'$). If one allows the half-density normalization constant to be built from α_* alone as a monomial $C \sim (\alpha_*)^n$ with integer n, then

$$[C] = \text{length}^{2n},$$

and matching $[C] = \text{length}^{d/2}$ forces $2n = d/2$, i.e. $d = 4n$. So α_* provides a "background-free" source of scale but does not single out $d = 4$ on its own; it selects dimensions divisible by 4 under the strict integrality reading of H2.5. In $d = 4$ it yields directly an area scale $C \sim \alpha_*$ (e.g. $C \sim \alpha'$ in the string example).

`Remark E4a (Emergent string tension is a transmutation-scale instance, not a UV parameter).` In confining phases one often defines an effective string tension σ with $[\sigma] = \text{length}^{-2}$, so σ^{-1} supplies an area scale. Operationally, for a large rectangular Wilson loop of spatial size R and Euclidean time extent T, one defines $V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln\langle W(R \times T)\rangle$, and an area law $\langle W(R \times T)\rangle \sim e^{-\sigma RT}$ implies $V(R) \sim \sigma R$ and identifies σ [Greensite2003Confinement]. In $d = 4$ gauge theories with dimensionless couplings, such a scale is expected to arise (when it exists) as an RG-invariant transmutation scale rather than as a new analytic monomial in the couplings. Logically, this places "string tension as area supplier" in the H2.13 branch unless one is explicitly assuming a fundamental UV area parameter α_*.

`Heuristic H2.12 (Link to gravity/Planck length in a UV completion).` The gravity-only sieve (Derivation D1.3) uses G_d as the unique universal coupling supplying dimension. In a UV completion where gravity is emergent from a sector with a universal area parameter α_* and a dimensionless coupling g, dimensional analysis suggests

$$G_d \propto g^2\,(\alpha_*)^{(d-2)/2} \times (\text{volume factors}),$$

so the Planck length/area is *derived* from α_* and g rather than fundamental. In perturbative string theory, one has $\alpha_* = \alpha'$ and $g = g_s$ (up to compactification-volume factors). In that framing the half-density "universal area scale" could naturally be α_* (or a simple function of it), while the Planck area is recovered as a consequence of how gravity emerges.

`Heuristic H2.8 (What H2.5 is really buying).` The value of H2.5 is comparative: it distinguishes dimensions in which the needed $\text{length}^{d/2}$ factor can be supplied by *simple* coupling dependence (integer powers of the already-present couplings), versus dimensions in which any such factor requires either (i) introducing extra scales/couplings, (ii) taking fractional powers, or (iii) invoking non-analytic mechanisms (dimensional transmutation).

`Heuristic H2.13 (Dimensional transmutation as a scale-supplier).` If one relaxes H2.5's "analytic monomial in couplings" requirement, then even a theory with only *dimensionless* couplings can generate a physical length scale through RG invariance: a running coupling $g(\mu)$ can be traded for an RG-invariant scale κ_* (or Λ), typically of the form $\kappa_* \sim \mu \exp(-\text{const}/g(\mu)^2)$

or $\kappa_* \sim \mu \exp(-\text{const}/g(\mu))$. In that branch, the half-density scalarization scale required by H2.4 can be supplied by $\kappa_*^{-d/2}$ (or its square in $d = 4$ as an area scale), but the scale is no longer an analytic monomial in the couplings: it is emergent and non-perturbative in the naive coupling expansion.

`Heuristic H2.14 (Bookkeeping: what "\(d\)" means in \(\text{length}^{d/2}\)).` The half-density weight $\text{length}^{d/2}$ refers to the dimension of the manifold whose coordinates are integrated over in the composition law (the intermediate-variable space). In a nonrelativistic time-sliced kernel this is typically the *spatial* dimension, while for covariant/proper-time kernels one may compose over *spacetime* points. The dimension-sieve discussion using G_d treats d as the **spacetime** dimension, so any $d = 4 \Rightarrow$ "area scale" conclusion should be read in that covariant sense unless stated otherwise.

`Derivation D1.6a (RG-invariant scale from a beta function).` Let a (dimensionless) running coupling $g(\mu)$ satisfy an RG equation $\mu\, dg/d\mu = \beta(g)$ with $\beta(g) \neq 0$ in the range of interest. Then the combination

$$\Lambda_* \equiv \mu \exp\left(- \int^{g(\mu)} \frac{dg'}{\beta(g')} \right)$$

is RG-invariant (independent of the subtraction scale μ), up to a finite multiplicative constant corresponding to a choice of scheme/normalization of the integral. In one-loop form $\beta(g) = -bg^2 + O(g^3)$, one obtains the familiar transmutation scale $\Lambda_* \sim \mu\, e^{-1/(bg(\mu))} \times$ (scheme factor).

If H2.3–H2.4 demand a universal scalarization constant C with $[C] = \text{length}^{d/2}$, then any RG-invariant inverse length Λ_* supplies one by $C \sim \Lambda_*^{-d/2}$. In particular, for $d = 4$ this produces a universal **area** scale $C \sim \Lambda_*^{-2}$, without requiring the scale to be an analytic monomial in couplings (so this branch sits outside H2.5).

`Example E5 (2D delta: transmutation yields a length scale).` In the 2D delta interaction, the contact coupling is marginal and the renormalized theory is naturally parameterized by an RG-invariant inverse length κ_* rather than by the bare coupling. Concretely, one finds (up to conventions) a running coupling $g_R(\mu)$ with beta function $\beta(g_R) \propto g_R^2$, and the RG invariant

$$\kappa_*^2 \equiv \mu^2 \exp\left(\frac{2\pi\hbar^2}{m} \frac{1}{g_R(\mu)} \right),$$

so κ_* is independent of the subtraction scale μ and sets a bound-state/scattering scale [ManuelTarrach1994PertRenQM]. This is a minimal witness that "a scale is forced by compatibility" can occur even without a dimensionful coupling, via renormalization rather than via analytic monomials.

3.4 Running H2.3: Is "Dimensionless f" Physics or Convention?

The half-density formalism (H2.1) gives a canonical pairing $\int \bar{\psi}\,\psi$ that does not require choosing a background measure. But when we write $\psi = f\,\sigma_*$ (H2.2), we are choosing a *representation* of the same object as a scalar function with respect to a chosen positive density $\rho_* = \sigma_*^2$.

`Proposition P1.3 (Scalarization is a choice of measure, not new physics).` Choosing a reference half-density σ_* identifies the canonical Hilbert space of L^2 half-densities on M with the scalar Hilbert space $L^2(M, \rho_*)$, where $\rho_* = \sigma_*^2$. Different choices σ_* yield unitarily equivalent scalar representations.

`Derivation D1.7 (Change of reference half-density acts by multiplication).` Let σ_1, σ_2 be nowhere-vanishing half-densities on M and set $r := \sigma_2/\sigma_1$, a positive scalar function. Writing the same half-density state ψ as $\psi = f_1\,\sigma_1 = f_2\,\sigma_2$ gives $f_2 = r^{-1} f_1$. Moreover,

$$\int_M \bar{\psi}\,\psi = \int_M |f_1|^2\,\sigma_1^2 = \int_M |f_2|^2\,\sigma_2^2 = \int_M |f_2|^2\,r^2\,\sigma_1^2,$$

so the two scalar pictures differ by a compensating change of measure and pointwise multiplication. In particular, if $\sigma_2 = c\,\sigma_1$ is a constant rescaling, then $f_2 = c^{-1} f_1$ is the familiar global wavefunction normalization freedom.

`Heuristic H2.9 (How H2.3 creates a scale).` In the usual "scalar wavefunction" presentation on $\mathbb{R}^d$, one implicitly chooses $\sigma_* = |dx|^{1/2}$ and allows the scalar representative to carry dimension $\text{length}^{-d/2}$ so that $|\psi|^2\,d^dx$ is dimensionless probability. Requiring instead that the scalar representative f be dimensionless (H2.3) shifts the $\text{length}^{-d/2}$ factor into the reference half-density:

$$\sigma_* \sim L_*^{-d/2} |dx|^{1/2},$$

so "dimensionless f" is a convention unless the scale $L_*^{d/2}$ is fixed by an additional universality principle (H2.4–H2.5).

3.5 Running H2.4: What Does "Background-Free Constancy" Mean?

From Derivation D1.7, changing the reference half-density σ_* by a positive function $r(x)$ changes the scalar representative by $f \mapsto r^{-1} f$ and changes the scalar measure by $\rho_* \mapsto r^2 \rho_*$. So the raw half-density formulation has a large "scalarization gauge freedom".

`Heuristic H2.10 (Constancy = no extra background function).` If we take "background-free" in the strong sense "no additional structure beyond the manifold and the theory's couplings", then allowing an arbitrary non-constant $r(x)$ would amount to introducing a new background field/function by hand. In that strong sense, the only admissible changes are constant rescalings, and

choosing σ_* becomes a choice of a single global scale (fixed or not fixed by couplings depending on H2.5).

`Derivation D1.8 (Three natural families of \(\sigma_\ast\) and what they mean).` On a configuration space M, the common ways to choose a reference half-density are: 1. **Flat/affine choice (when available):** on $\mathbb{R}^d$ with its affine structure, translation invariance picks $|dx|^{1/2}$ uniquely up to a constant factor. This is "constant" in the sense of being homogeneous under translations. 2. **Metric-derived choice:** given a Riemannian/Lorentzian metric g, one can take $\sigma_g := |g|^{1/4}|dx|^{1/2}$, so that $\rho_g = \sigma_g^2 = \sqrt{|g|}\,|dx|$ is the familiar invariant volume density. This makes the scalar representative f a genuine scalar field but makes the scalarization depend on background geometry. 3. **Field-derived (dilaton-like) choice:** given a scalar field Φ (background or dynamical), one can take $\sigma_\Phi := e^{-\Phi}\sigma_g$. In the scalar picture this is a local rescaling of the measure, and it is the natural way to encode "local units" or Weyl factors.

H2.4 asserts that the theory supplies (or selects) a choice of type (1) with no x-dependent factor: a fixed reference σ_* whose only remaining ambiguity is an overall constant scale.

`Heuristic H2.11 (RG as scale dependence of scalarization).` If refinement/coarse-graining forces an x-independent but scale-dependent choice $\sigma_*(\mu)$ (equivalently a scale-dependent constant $L_*(\mu)$), then H2.4 is replaced by an RG statement: the scalarization convention becomes part of the renormalization scheme (a "wavefunction renormalization" for the scalar representative). In that case, a universal area/length scale can still appear, but typically as an RG invariant (dimensional transmutation scale) rather than as a fixed analytic monomial in couplings.

`Derivation D1.8a (Running scalarization \(\sigma_\ast(\mu)\) is a \(Z(\mu)\) factor on scalar representatives).` Assume the intrinsic (half-density) state ψ is fixed, and consider two scalarizations related by a μ-dependent *constant* rescaling, $\sigma_*(\mu) = c(\mu)\,\sigma_0$ with $c(\mu) > 0$. Writing $\psi = f(\mu)\,\sigma_*(\mu)$ gives $f(\mu) = c(\mu)^{-1} f_0$, where f_0 is the scalar representative in the σ_0 convention. Defining $Z(\mu) := c(\mu)^2$, this becomes the familiar multiplicative form

$$f(\mu) = Z(\mu)^{-1/2}\, f_0.$$

So an x-independent running of scalarization is *formally equivalent* to a wavefunction renormalization factor on scalar representatives. This is bookkeeping: $\sigma_*(\mu)$ is a convention/scheme choice; only RG-invariant combinations (e.g. transmutation scales) are candidates for physical statements.

`Heuristic H2.11a (Guardrail: geometric weight \(\neq\) anomalous dimension).` The half-density square-root Jacobian is a geometric transformation law under coordinate changes; anomalous dimensions are RG scaling data in interacting theories. They should not be conflated: allowing $\sigma_*(\mu)$ to run is a

representation convenience, not a claim that geometry "produces" anomalous scaling.

4. Stationary Phase Produces Half-Density Prefactors (Short-Time Kernel)

The main manuscript uses stationary phase to explain why classical extremals dominate refinement limits. Here we add the complementary kernel-level fact: stationary phase does not only pick the extremal; it also produces a determinant prefactor that transforms as a half-density, i.e. the object needed for coordinate-free kernel composition.

`Derivation D1.4 (Van Vleck prefactor is a bi-half-density).` Let $S_{\mathrm{cl}}(x, z; t)$ be the classical action as a function of endpoints and time, treated as a generating function. The standard short-time/stationary-phase approximation to the propagator has the form

$$K(x, z; t) \approx \frac{1}{(2\pi i\hbar)^{d/2}} \left|\det\left(-\frac{\partial^2 S_{\mathrm{cl}}}{\partial x\, \partial z}\right)\right|^{1/2} \exp\left(\frac{i}{\hbar} S_{\mathrm{cl}}(x, z; t)\right).$$

Under a change of coordinates $x = x(x')$, $z = z(z')$, the mixed Hessian transforms by the chain rule, and its determinant acquires Jacobian factors:

$$\det\left(-\frac{\partial^2 S_{\mathrm{cl}}}{\partial x'\, \partial z'}\right) = \det\left(\frac{\partial x}{\partial x'}\right) \det\left(\frac{\partial z}{\partial z'}\right) \det\left(-\frac{\partial^2 S_{\mathrm{cl}}}{\partial x\, \partial z}\right).$$

Taking square roots shows that the prefactor transforms with $|\det(\partial x/\partial x')|^{1/2} |\det(\partial z/\partial z')|^{1/2}$, i.e. exactly as a half-density factor at each endpoint. Thus the stationary-phase prefactor is naturally interpreted as making K a half-density in each variable, so that kernel composition does not depend on a background measure choice. This is the standard "Van Vleck type" semiclassical prefactor in the correspondence/semiclassical tradition [VanVleck1928Correspondence].

`Derivation D1.9 (Square-root delta normalization has half-density weight).` In finite dimension, the "localize on critical points" distribution is $\delta(\nabla f)$, supported on $\mathrm{Crit}(f)$. A concrete way it appears is via a "halved" oscillatory integral with a normalization exponent fixed by dimension.

Let $f : \mathbb{R}^N \to \mathbb{R}$ be smooth and define, for $\varepsilon > 0$,

$$A_\varepsilon(O) := \varepsilon^{-N/2} \int_{\mathbb{R}^N} e^{\frac{i}{\varepsilon} f(x)}\, O(x)\, dx.$$

Then

$$|A_\varepsilon(O)|^2 = \varepsilon^{-N} \iint e^{\frac{i}{\varepsilon}(f(x) - f(y))}\, O(x)\, \overline{O(y)}\, dx\, dy.$$

Applying the near-diagonal scaling $y = x + \varepsilon z$ (so $dy = \varepsilon^N dz$) gives

$$|A_\varepsilon(O)|^2 = \iint e^{-\frac{i}{\varepsilon}(f(x+\varepsilon z)-f(x))}\, O(x)\, \overline{O(x+\varepsilon z)}\, dz\, dx.$$

Formally letting $\varepsilon \to 0$ yields

$$|A_\varepsilon(O)|^2 \;\to\; \iint e^{-iz\cdot\nabla f(x)}\, |O(x)|^2\, dz\, dx = (2\pi)^N \int \delta(\nabla f(x))\, |O(x)|^2\, dx.$$

The exponent $N/2$ in the prefactor is exactly the half-density scaling: it cancels the Jacobian $dy = \varepsilon^N dz$ under near-diagonal rescaling, and it is the "square root" of the density normalization that produces $\delta(\nabla f)$.

`Heuristic H1.4 (Where Planck area can enter, minimally).` Derivation D1.3 isolates one minimal route by which a Planck-scale quantity can enter: if the theory supplies a single universal coupling with dimension of length (Newton's constant) and one demands that the half-density normalization constant be built from that coupling *without fractional powers*, then $d = 4$ is singled out and the resulting constant has the dimension of an area, naturally identified with the Planck area $L_P^2 \sim \hbar G_4/c^3$.

5. A Gravitational Anchor: Minimal Areal Speed and the $D = 4$ Cancellation

Rivero's "Planck areal speed" observation gives a concrete route by which Planck-scale discreteness reappears at Compton scales in inverse-square gravity [RiveroAreal] [RiveroSimple].

`Heuristic H1.3 (Areal-speed selection).` In $3 + 1$ Newtonian gravity (inverse-square), imposing a discrete areal-speed/area-time condition at a Planck scale can yield characteristic radii proportional to a reduced Compton length, with Newton's constant canceling when expressed in Planck units. This is a non-trivial indication that "a universal area scale" can be operationally meaningful at low energies in $D = 4$.

`Derivation D1.5 (Inverse-square circular orbit + Planck areal speed \(\Rightarrow\) Compton radius).` For a circular orbit under an inverse-square central force $F(r) = K/r^2$ (with coupling $K > 0$), the centripetal balance is $mv^2/r = K/r^2$. The areal speed is $\dot{A} = \frac{1}{2}rv$, so $v = 2\dot{A}/r$. Substituting into the force balance gives

$$m\left(\frac{2\dot{A}}{r}\right)^2 = \frac{K}{r} \quad\Longrightarrow\quad r = \frac{4m\,\dot{A}^2}{K}.$$

For Newtonian gravity between a source mass M and test mass m, $K = GMm$, hence

$$r = \frac{4\dot{A}^2}{GM},$$

independent of the test mass m. If one now imposes $\dot{A} = k\,\dot{A}_P$, where Rivero's Planck areal speed is $\dot{A}_P = cL_P$ [RiveroAreal], then using $L_P^2 = G\hbar/c^3$ yields

$$r = \frac{4k^2(cL_P)^2}{GM} = \frac{4k^2(G\hbar/c)}{GM} = 4k^2\,\frac{\hbar}{cM}.$$

Thus r becomes a multiple of the reduced Compton length $L_M = \hbar/(cM)$, with Newton's constant canceled out. In particular, $k = \frac{1}{2}$ gives $r = L_M$. This is the "Planck area per Planck time ⇒ Compton scale" cancellation highlighted in [RiveroAreal] and summarized in [RiveroSimple].

`Remark D1.5a (Generic \(F=K/r^q\): only \(q=2\) yields linear Compton scaling; \(q\) links to dimension).` For a power-law central force $F(r) = K/r^q$ with $K > 0$, circular balance gives $mv^2/r = K/r^q$, i.e. $mv^2 = Kr^{1-q}$. Using the circular areal speed $\dot{A} = \frac{1}{2}rv$ (so $v = 2\dot{A}/r$) yields

$$4m\dot{A}^2 = K\,r^{3-q}.$$

Hence, for $q \neq 3$,

$$r = \left(\frac{4m\dot{A}^2}{K}\right)^{\frac{1}{3-q}},$$

while for $q = 3$ the radius drops out and $4m\dot{A}^2 = K$.

In the gravitational specialization $K = GMm$, the test mass cancels as before and

$$r = \left(\frac{4\dot{A}^2}{GM}\right)^{\frac{1}{3-q}}.$$

If one imposes $\dot{A} = cL_P$ and uses the $D = 4$ identity $L_P^2 = G\hbar/c^3$, then

$$r = \left(\frac{4\hbar}{cM}\right)^{\frac{1}{3-q}}.$$

Thus the Planck-areal-speed substitution produces *linear* reduced-Compton scaling $r \propto \hbar/(cM)$ only for $q = 2$ (inverse-square). For Newtonian long-range fields in n spatial dimensions, the Laplacian Green function gives $\Phi(r) \propto r^{2-n}$ (for $n > 2$), so $F \sim |\nabla\Phi| \propto r^{1-n}$, i.e. $q = n - 1$ (with the $n = 2$ logarithmic exception) [Tanaka2021KernelQuadrature]. In this sense the $q = 2$ special case corresponds to $n = 3$ spatial dimensions (spacetime $D = 4$). Equivalently, substituting $q = n - 1$ into the mass scaling gives $r \propto M^{-1/(4-n)}$ (for $n \neq 4$), so the linear Compton scaling (and the G-cancellation in the $D = 4$ identity $L_P^2 = G\hbar/c^3$) is uniquely $n = 3$; the $n = 4$ case is the degenerate $q = 3$ condition where the radius drops out.

Remark D1.5b (SR continuation of the inverse-square witness: Compton branch \(\to\) Planck floor). Inside the same mechanical SR model used in papers/relativistic-central-orbits/main.md (external inverse-square force, no GR field dynamics), one can keep the fixed coordinate-time areal speed $\dot{A}_0 = dA/dt$ and continue D1.5 exactly.

For inverse-square forces, SR circular motion gives $v = K/L$, while

$$L = \gamma m r v = 2\gamma m \dot{A}_0, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Hence

$$\gamma v = \frac{K}{2m\dot{A}_0},$$

so

$$v = \frac{\frac{K}{2m\dot{A}_0}}{\sqrt{1 + \left(\frac{K}{2m\dot{A}_0 c}\right)^2}}, \qquad r = \frac{2\dot{A}_0}{v} = \frac{4m\dot{A}_0^2}{K}\sqrt{1 + \left(\frac{K}{2m\dot{A}_0 c}\right)^2}.$$

For gravity $K = GMm$, m cancels:

$$r(M, \dot{A}_0) = \frac{4\dot{A}_0^2}{GM}\sqrt{1 + \left(\frac{GM}{2\dot{A}_0 c}\right)^2}.$$

With $\dot{A}_0 = cL_P$, using $L_P^2 = \hbar G/c^3$ and $M_P^2 = \hbar c/G$,

$$r(M) = \frac{4\hbar}{Mc}\sqrt{1 + \frac{M^2}{4M_P^2}}.$$

Therefore $M \ll M_P$ reproduces the D1.5 Compton-like branch $r \approx 4\hbar/(Mc)$, while $M \gg M_P$ saturates at

$$r \to 2L_P.$$

So in this SR continuation, the inverse-square Planck-areal-speed witness is not destroyed; it is regularized into a bounded interpolation.

Remark D1.5c (Clock-choice sensitivity: fixed \(dA/dt\) vs fixed \(dA/d\tau\)). The $2L_P$ high-mass saturation in D1.5b is tied to fixing the coordinate-time areal speed $\dot{A}_t = dA/dt$. If one instead fixes proper-time areal speed $\dot{A}_\tau = dA/d\tau$ in the same inverse-square SR model, then

$$r(M, \dot{A}_\tau) = \frac{4\dot{A}_\tau^2}{GM}\sqrt{1 - \left(\frac{GM}{2\dot{A}_\tau c}\right)^2},$$

which is defined only for $GM < 2\dot{A}_\tau c$. With $\dot{A}_\tau = cL_P$,

$$r(M) = \frac{4\hbar}{Mc}\sqrt{1 - \frac{M^2}{4M_P^2}},$$

so $M \to 2M_P^- \Rightarrow r \to 0$, not $2L_P$. Therefore the low-mass Compton-like branch is robust, while the high-mass asymptotic is clock-convention dependent in this mechanical SR setting.

`Remark D1.5d (Invariant-candidate reformulation via specific angular momentum).` A clock-independent candidate in the same central-source setup is the specific angular momentum scalar

$$\ell := \frac{1}{m}\sqrt{\frac{1}{2}\, L_{\mu\nu}L^{\mu\nu}},$$

where $L^{\mu\nu}$ is the source-rest-space projection of orbital $J^{\mu\nu}$ (using source 4-velocity U^μ). In the source rest frame this reduces to

$$\ell = \gamma r v, \qquad \dot{A}_t = \frac{\ell}{2\gamma}, \qquad \dot{A}_\tau = \frac{\ell}{2}.$$

So fixing ℓ selects the proper-time branch rather than the coordinate-time branch. For inverse-square gravity,

$$r(M,\ell) = \frac{\ell^2}{GM}\sqrt{1 - \left(\frac{GM}{\ell c}\right)^2}, \qquad \ell > \frac{GM}{c},$$

and $\ell = 2cL_P$ reproduces D1.5c. This does not yet prove universality, but it provides a structurally covariant way to encode the postulate without choosing a clock variable directly.

`Remark D1.5e (Why simple frame-free bivector invariants are insufficient here).` One might try to avoid the source-velocity projection entirely and use only Lorentz invariants of

$$M^{\mu\nu} = R^\mu p^\nu - R^\nu p^\mu,$$

namely

$$I_1 = \frac{1}{2}M_{\mu\nu}M^{\mu\nu} = R^2p^2 - (R\cdot p)^2, \qquad I_2 = \frac{1}{2}M_{\mu\nu}{}^\star M^{\mu\nu}.$$

But for the circular central branch ($R\cdot p = 0$, $R^2 = r^2$, $p^2 = -m^2c^2$), this gives

$$I_1 = -m^2c^2r^2,$$

independent of orbital speed, while I_2 vanishes in the planar case. So these simple frame-free invariants do not encode the areal-rate branch parameter. In this setup, a timelike direction (e.g. source U^μ) appears to be minimal extra structure for a useful covariant postulate.

`Remark D1.5f (Minimal timelike-structure rule for this branch).` For the present central-source inverse-square model, a practical "minimal structure" rule is: 1. use the source worldline 4-velocity U^μ as the distinguished

timelike direction (already part of the model input), and 2. formulate the postulate on

$$\ell_{(U)} := \frac{1}{m}\sqrt{\frac{1}{2}\, M^{(U)}_{\mu\nu} M^{\mu\nu}_{(U)}}, \qquad M^{(U)\mu\nu} = h(U)^{\mu}{}_{\alpha} h(U)^{\nu}{}_{\beta} M^{\alpha\beta},$$

rather than directly on coordinate-time areal rate. In the source rest frame this is equivalent to fixing $\dot{A}_{\tau,U} = \ell_{(U)}/2$, while coordinate-time rates are derived via the corresponding lapse factor. This keeps the branch covariant-with-source and avoids introducing an additional arbitrary observer field $u^{\mu}(x)$.

`Remark D1.5g (Non-circular planar extension is kinematic).` The ℓ-based rule is not restricted to circular trajectories. For general planar motion with tangential component $v_{\perp} = r\dot{\phi}$,

$$\ell = \gamma r v_{\perp} = \gamma r^2 \dot{\phi}, \qquad \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{\ell}{2\gamma}, \qquad \frac{dA}{d\tau} = \frac{\ell}{2}.$$

Thus "fix ℓ" remains equivalent to fixing proper-time areal rate in the source frame even away from circular orbits; this part is kinematic and does not depend on the specific force law.

`Remark D1.5h (Non-planar caution: vector area rate is fundamental).` For general 3D motion the natural identity is vector-valued:

$$\boldsymbol{\ell} = \frac{1}{m}(\mathbf{r} \times \mathbf{p}) = \gamma(\mathbf{r} \times \mathbf{v}), \qquad \frac{d\mathbf{A}}{d\tau} = \frac{\boldsymbol{\ell}}{2}.$$

So the ℓ-rule still survives kinematically, but scalar areal rates require a chosen normal $\mathbf{n}$:

$$\frac{dA_{\mathbf{n}}}{d\tau} = \frac{\boldsymbol{\ell} \cdot \mathbf{n}}{2}.$$

In non-planar perturbations, interpreting a scalar "areal speed postulate" without specifying this projection is ambiguous; the projection choice is part of the model specification.

`Remark D1.5i (Observability criterion for projected areal-rate postulates).` In perturbed-orbit settings, a projected areal-rate claim is empirically meaningful only after specifying: 1. projection normal $\mathbf{n}$, 2. clock convention (t-based or τ-based), 3. reconstruction map for $(\mathbf{r}, \mathbf{v})$ in the source frame. With those choices fixed, the observable is

$$\dot{A}_{\mathbf{n}}(t) = \frac{1}{2}\, \mathbf{n} \cdot (\mathbf{r} \times \mathbf{v}), \qquad \frac{dA_{\mathbf{n}}}{d\tau} = \frac{1}{2}\, \mathbf{n} \cdot \boldsymbol{\ell},$$

and departures from constant projected areal rate are governed by projected torque

$$\frac{d}{dt}\big(\mathbf{n} \cdot \boldsymbol{\ell}\big) = \frac{1}{m}\, \mathbf{n} \cdot (\mathbf{r} \times \mathbf{F}).$$

So the postulate becomes falsifiable precisely when projection, clock, and reconstruction are part of the model declaration.

`Remark D1.5j (Minimal implementation pipeline).` A compact data-to-test pipeline is: 1. reconstruct object state in observer frame from direction $\hat{\mathbf{n}}(t)$, range $\rho(t)$, and line-of-sight velocity $\dot{\rho}(t)$:

$$\mathbf{r}_{\mathrm{obj}} = \rho\hat{\mathbf{n}}, \qquad \mathbf{v}_{\mathrm{obj}} = \dot{\rho}\,\hat{\mathbf{n}} + \rho\,\dot{\hat{\mathbf{n}}},$$

2. subtract source ephemeris to obtain source-frame relative state $(\mathbf{r}, \mathbf{v})$, 3. evaluate

$$\dot{A}_{\mathbf{n}} = \frac{1}{2}\,\mathbf{n}\cdot(\mathbf{r}\times\mathbf{v}), \quad \frac{dA_{\mathbf{n}}}{d\tau} = \frac{1}{2}\,\mathbf{n}\cdot\boldsymbol{\ell},$$

and, if a force model is supplied, the projected-torque residual

$$\mathcal{T}_{\mathbf{n}} := \frac{d}{dt}\big(\mathbf{n}\cdot\boldsymbol{\ell}\big) - \frac{1}{m}\,\mathbf{n}\cdot(\mathbf{r}\times\mathbf{F}).$$

This keeps the postulate test tied to explicit reconstruction and uncertainty handling rather than to abstract kinematic statements alone.

`Remark D1.5k (Minimal uncertainty scaffold).` At first order, uncertainty in projected observables can be propagated by Jacobians:

$$q := \dot{A}_{\mathbf{n}} \quad \Rightarrow \quad \sigma_q^2 \approx J_q\,\Sigma_x\,J_q^{\top},$$

for reconstructed state vector x and covariance Σ_x, and

$$\sigma_{\mathcal{T}}^2 \approx J_{\mathcal{T}}\,\Sigma_z\,J_{\mathcal{T}}^{\top}$$

for residual $\mathcal{T}_{\mathbf{n}}$ with augmented state z. This linear scaffold is a baseline; in strongly nonlinear regimes the same quantities should be cross-checked with nonlinear propagation (e.g. Monte Carlo) before interpretation.

`Remark D1.5l (Practical nonlinear-validation trigger).` A lightweight policy is to run a pilot nonlinear propagation check and compare against the linear σ estimate; if the discrepancy is at the few-percent level (or larger), treat linearized errors as insufficient and switch to nonlinear uncertainty propagation for reporting.

`Remark D1.5m (Regime-dependent trigger calibration).` The trigger in D1.5l should be calibrated by uncertainty regime, not treated as universal. A practical diagnostic pair is

$$\epsilon_{\mathrm{nl}} := \frac{|\sigma_{\mathrm{MC}} - \sigma_{\mathrm{lin}}|}{\sigma_{\mathrm{lin}}}, \qquad \chi := \max\left(\frac{\sqrt{\mathrm{tr}\,\Sigma_r}}{\|\mathbf{r}\|}, \frac{\sqrt{\mathrm{tr}\,\Sigma_v}}{\|\mathbf{v}\|}\right),$$

with Σ_r, Σ_v the position/velocity covariance blocks in the chosen reconstruction model. Pilot scans in correlated-noise families can then map $\epsilon_{\mathrm{nl}}(\chi)$ for the instrument/model pair; the "few-percent" policy corresponds to selecting an operational ϵ_{nl} band after this calibration, rather than imposing a context-free constant.

6. Interface with the Main Paper

The main manuscript argues that: 1. classical dynamics are recovered from quantum composition by stationary-phase concentration, and 2. refinement across scales forces RG-style consistency conditions when naive limits diverge.

This note adds a complementary ingredient: the kernel side is most naturally formulated in half-density language, and stationary phase produces the bi-half-density prefactor directly. A universal convention for turning those half-densities into scalar amplitudes then requires a length$^{d/2}$ scale; in $d = 4$ this is an area scale.

7. Open Problems and Outlook

1. Make the half-density normalization argument precise for a concrete groupoid or kernel model (tangent-groupoid or short-time propagator model).
2. Show how the area scale enters stationary-phase prefactors and how this interacts with RG scaling.
3. General-dimension analysis: clarify what replaces "area" in odd dimensions and whether a universal normalization is still defensible.
4. Identify minimal hypotheses under which "need of half-density scale ⇒ Planck area" is more than dimensional bookkeeping.
5. Track minimal-length/GUP scenarios as a comparison branch: do they implement the "needed scale" at the level of kinematics (modified commutators/dispersion) or can they be reframed as a refinement-compatibility condition? Use [Hossenfelder2013MinimalLength] as an OA entry point.

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