

Planck Area from Half-Density Normalization (Draft)

Abstract

Half-densities are the natural “coordinate-free integrands” for composing kernels without choosing a background measure. But choosing a *universal* convention for turning half-density objects into dimensionless numerical amplitudes introduces a length $^{d/2}$ scale. In $d = 4$, this is an *area*. This note sharpens the hypothesis ladder needed for the claim “half-density normalization selects a universal area scale”, and isolates a simple dimension-matching condition under which the Planck area appears without fractional powers of couplings. A gravitational anchor based on a minimal-areal-speed principle is recorded as a separate heuristic thread [RiveroAreal] [RiveroSimple].

1. Purpose and Status

This is a dependent follow-up to `paper/main.md`. It is not yet a finished paper; its goal is to isolate one technical point that is only implicit in the main manuscript: the role of half-densities (and their scaling) in making composition laws coordinate-invariant *and* dimensionally well-defined.

Claims below are labeled as **Proposition** (math-precise under hypotheses) or **Heuristic** (programmatic bridge).

2. Half-Densities and Composition Kernels

Let M be a d -dimensional manifold. A (positive) density is a section of $|\Lambda^d T^* M|$, and a half-density is a section of $|\Lambda^d T^* M|^{1/2}$.

The key operational point is: when a kernel is a half-density in its integration variable, composition of kernels does not depend on an arbitrary choice of coordinate measure.

Heuristic H1.1 (Why half-densities). If $K_1(x, y)$ and $K_2(y, z)$ are chosen so that their product in the intermediate variable y is a density, then $\int_M K_1(x, y)K_2(y, z)$ is coordinate-invariant without fixing a preferred dy . This matches the structural role of kernel composition used in `paper/main.md` (Section 6).

Derivation D1.1 (Coordinate invariance of half-density pairing and composition). In a local chart $y = (y^1, \dots, y^d)$, write a half-density as $\psi(y) = \varphi(y)|dy|^{1/2}$. Under a change of variables $y = y(y')$, one has $|dy|^{1/2} = |\det(\partial y / \partial y')|^{1/2}|dy'|^{1/2}$, so the coefficient transforms as $\varphi'(y') = \varphi(y(y'))|\det(\partial y / \partial y')|^{1/2}$.

Hence the product of two half-densities is a density: $\psi_1\psi_2 = (\varphi_1\varphi_2)|dy|$, and its integral is chart-independent: $\int_M \psi_1\psi_2$ is well-defined without choosing a

background measure beyond the density bundle itself.

Kernel composition is the same mechanism: if $K_1(x, y)$ and $K_2(y, z)$ are half-densities in y , then $K_1 K_2$ is a density in y and $\int_M K_1 K_2$ is coordinate invariant.

3. Dimensional Analysis: Normalizing a Half-Density Requires a Scale

A density on M carries the units of length d once physical units are assigned to coordinates. A half-density therefore carries units length $^{d/2}$.

Proposition P1.1 (No canonical "half-density = function" identification). There is no canonical identification of a half-density $\psi \in |\Lambda^d T^* M|^{1/2}$ with an ordinary scalar function f on M . Choosing such an identification is equivalent to choosing a nowhere-vanishing reference half-density σ_* (equivalently a positive density $\rho_* = \sigma_*^2$) and writing $\psi = f \sigma_*$.

Derivation D1.2 (Dilation makes the $\text{length}^{d/2}$ weight explicit). On \mathbb{R}^d , consider a dilation $y \mapsto y' = ay$ with $a > 0$. Then $|dy'| = a^d |dy|$, so $|dy'|^{1/2} = a^{d/2} |dy|^{1/2}$. Thus even in flat space, half-densities carry an inherent length $^{d/2}$ scaling weight.

Proposition P1.2 (Universal *dimensionless* amplitudes force a $\text{length}^{d/2}$ constant). If one imposes the extra requirement that the scalar representative f in $\psi = f \sigma_*$ be dimensionless in physical units, then the reference half-density σ_* must carry all of the length $^{d/2}$ dimension. In particular, a *constant* (field-independent) choice of σ_* is equivalent to choosing a universal length $^{d/2}$ scale.

In $d = 4$, this universal length $^{d/2}$ scale is a universal *area* scale.

Heuristic H1.2 (Reciprocity claim). Half-densities alone do not force a particular scale: the forced fact is that converting half-density objects into scalar numerical amplitudes requires extra structure (a reference half-density). The “universal area scale” claim begins only after adding two further hypotheses: 1. the reference σ_* is taken to be *constant* (no dependence on background metric/fields), and 2. the constant is required to be fixed by universal constants/couplings of the theory.

Under these hypotheses, $d = 4$ is the unique dimension in which the needed length $^{d/2}$ constant can be supplied by the gravitational coupling without fractional powers (Derivation D1.3).

Derivation D1.3 (Gravity-only sieve: why $(d=4)$ is singled out if only (G_d) is used). In d spacetime dimensions, the Einstein–Hilbert action $\frac{1}{16\pi G_d} \int d^d x \sqrt{|g|} R$ shows that (in $c = \hbar = 1$ units) Newton’s constant has dimension $[G_d] = \text{length}^{d-2}$. Assume the only available dimensionful coupling used to build the universal normalization constant is G_d itself (no cosmological constant, no additional dimensionful scales), and impose H2.5

in the literal “no fractional powers of G_d ” sense. Then the normalization constant has dimension $\text{length}^{k(d-2)}$ for some integer k . Matching $\text{length}^{d/2}$ forces $\text{length}^{d/2} = \text{length}^{d-2}$, which holds if and only if $d = 4$. In that case G_4 itself has dimension of area, and the corresponding area scale is the Planck area $L_P^2 \sim \hbar G_4/c^3$.

3.1 Hypotheses as Separate Knobs (What Is Forced vs Chosen)

The discussion above mixes three different kinds of statements: 1. **Geometric facts** (what half-densities are, how they compose, how they scale), 2. **Representational choices** (how one turns half-density objects into scalar numbers), 3. **Universality/selection principles** (what choices are allowed if we demand “background-free” and “built from couplings”).

To study these separately, it is useful to keep the hypotheses explicit.

Hypothesis H2.1 (Half-density formulation). Quantum kernels are treated as bi-half-densities so that composition in intermediate variables is coordinate invariant (Section 2 and Derivation D1.4).

Hypothesis H2.2 (Scalarization by a reference half-density). To interpret half-density amplitudes as scalar numerical functions, we pick a nowhere-vanishing reference half-density σ_* and write $\psi = f \sigma_*$ (Proposition P1.1).

Hypothesis H2.3 (Dimensionless scalar representative). The scalar representative f is required to be dimensionless in physical units (Proposition P1.2). This forces σ_* to carry the full $\text{length}^{d/2}$ weight.

Hypothesis H2.4 (Background-free constancy). The reference σ_* is taken to be constant/field-independent, rather than determined by background geometry (e.g. a Riemannian volume $|g|^{1/4}|dx|^{1/2}$) or by dynamical fields (e.g. a dilaton-like factor). This is the first point where a *universal constant* enters.

Hypothesis H2.5 (Analyticity / no fractional powers of couplings). If the universal constant is required to be built from the theory’s couplings without fractional powers, then dimensional analysis becomes a *dimension sieve* rather than a tautology. This hypothesis has at least two distinct readings:

1. **Integrality (Diophantine) reading:** the constant is a monomial in the available couplings with integer exponents (possibly allowing negative powers).
2. **Perturbative analyticity reading (stronger):** the constant admits a Taylor expansion around zero couplings, so only nonnegative integer powers appear.

Derivation D1.3 is the simplest gravity-only instance under the integrality reading: “use G_d without fractional powers” singles out $d = 4$.

Heuristic H2.6 (Where “special dimensions” can appear). Special dimensions do not come from half-densities alone (Hypothesis H2.1). They appear only after adding a selection principle like H2.4–H2.5: the requirement that

the scalarization choice be universal, background-free, and coupling-built in a restricted (e.g. analytic) way.

3.2 What Changes When a Hypothesis Is Relaxed?

This subsection records the main “branches” that need separate study.

1. **Drop H2.3 (allow dimensionful f).** Then no universal length $^{d/2}$ constant is forced; the dimensional weight can be carried by the scalar representative itself (as in the usual statement “wavefunctions have dimension length $^{-d/2}$ ”).
2. **Drop H2.4 (allow background geometry).** Then σ_* can be chosen from a metric (or other structure), and the “universal constant” is replaced by background-dependent normalization.
3. **Drop H2.5 (allow fractional powers).** Then in any $d > 2$ one can build a length $^{d/2}$ constant from gravity via $G_d^{d/(2(d-2))}$ (in $c = \hbar = 1$ units), so $d = 4$ is no longer singled out; instead, $d = 4$ is simply the unique case where the exponent is an integer.
4. **Change “which coupling supplies the scale”.** Using other dimensionful couplings (cosmological constant, string tension, gauge couplings in various dimensions, etc.) yields different “special-dimension” sieves. This is conceptually aligned with the observation that some dimensions are singled out by other structures (division algebras, special holonomy, supersymmetry), but those filters are separate from the half-density story and should not be conflated.

3.3 Starting with H2.5: Integrality as a Dimension Sieve

The point of H2.5 is not that dimensional analysis alone selects a unique scale (it does not), but that *restricting allowed functional dependence on couplings* can turn dimensional analysis into a selection principle.

Derivation D1.6 (Diophantine form of "no fractional powers").

Work in $c = \hbar = 1$ units for dimension counting. Let the available couplings $\{g_i\}$ have length dimensions $[g_i] = \text{length}^{a_i}$. Under the integrality reading of H2.5, the universal normalization constant is a monomial $C = \prod_i g_i^{n_i}$ with integers n_i . Its length dimension is $[C] = \text{length}^{\sum_i n_i a_i}$. Requiring $[C] = \text{length}^{d/2}$ is therefore the Diophantine condition

$$\sum_i n_i a_i = \frac{d}{2}.$$

Existence (and non-uniqueness) of solutions depends on: 1. which couplings are admitted as “universal” inputs, and 2. whether one allows negative exponents (non-analytic at zero coupling) or insists on perturbative analyticity (nonnegative exponents).

Heuristic H2.7 (Why H2.5 needs a "what counts as a coupling" rule). If one allows arbitrary redefinitions of couplings (e.g. adjoining a new symbol $\tilde{G} = G_d^{1/(d-2)}$), then "no fractional powers" becomes vacuous: the forbidden root has simply been renamed as an allowed coupling. H2.5 is meaningful only together with a prior criterion for admissible coupling dependence (e.g. perturbative analyticity around a distinguished limit such as $G_d \rightarrow 0$).

Example E1 (Gravity-only). With only G_d available, $a_1 = d - 2$ and the condition becomes $n(d - 2) = d/2$. For integer $d \geq 3$, this has a solution only at $d = 4$ with $n = 1$, reproducing Derivation D1.3.

Example E2 (Gravity + cosmological constant). If one also allows the cosmological constant Λ_d with $[\Lambda_d] = \text{length}^{-2}$, then the condition becomes $n(d - 2) - 2m = d/2$ for integers n, m . A simple family of solutions exists for d divisible by 4: take $n = 1$ and $m = d/4 - 1$, so

$$C \sim G_d \Lambda_d^{d/4-1},$$

has dimension $\text{length}^{d/2}$. Thus, even under H2.5, $d = 4$ is not automatically unique once additional dimensionful couplings are admitted; what is special about $d = 4$ in this family is that it is the only case with $m = 0$ (no need to involve Λ_d).

Heuristic H2.8 (What H2.5 is really buying). The value of H2.5 is comparative: it distinguishes dimensions in which the needed $\text{length}^{d/2}$ factor can be supplied by *simple* coupling dependence (integer powers of the already-present couplings), versus dimensions in which any such factor requires either (i) introducing extra scales/couplings, (ii) taking fractional powers, or (iii) invoking non-analytic mechanisms (dimensional transmutation).

3.4 Running H2.3: Is “Dimensionless f ” Physics or Convention?

The half-density formalism (H2.1) gives a canonical pairing $\int \bar{\psi} \psi$ that does not require choosing a background measure. But when we write $\psi = f \sigma_*$ (H2.2), we are choosing a *representation* of the same object as a scalar function with respect to a chosen positive density $\rho_* = \sigma_*^2$.

Proposition P1.3 (Scalarization is a choice of measure, not new physics). Choosing a reference half-density σ_* identifies the canonical Hilbert space of L^2 half-densities on M with the scalar Hilbert space $L^2(M, \rho_*)$, where $\rho_* = \sigma_*^2$. Different choices σ_* yield unitarily equivalent scalar representations.

Derivation D1.7 (Change of reference half-density acts by multiplication). Let σ_1, σ_2 be nowhere-vanishing half-densities on M and set $r := \sigma_2/\sigma_1$, a positive scalar function. Writing the same half-density state ψ as $\psi = f_1 \sigma_1 = f_2 \sigma_2$

gives $f_2 = r^{-1}f_1$. Moreover,

$$\int_M \bar{\psi} \psi = \int_M |f_1|^2 \sigma_1^2 = \int_M |f_2|^2 \sigma_2^2 = \int_M |f_2|^2 r^2 \sigma_1^2,$$

so the two scalar pictures differ by a compensating change of measure and pointwise multiplication. In particular, if $\sigma_2 = c\sigma_1$ is a constant rescaling, then $f_2 = c^{-1}f_1$ is the familiar global wavefunction normalization freedom.

Heuristic H2.9 (How H2.3 creates a scale). In the usual “scalar wavefunction” presentation on \mathbb{R}^d , one implicitly chooses $\sigma_* = |dx|^{1/2}$ and allows the scalar representative to carry dimension length $^{-d/2}$ so that $|\psi|^2 d^d x$ is dimensionless probability. Requiring instead that the scalar representative f be dimensionless (H2.3) shifts the length $^{-d/2}$ factor into the reference half-density:

$$\sigma_* \sim L_*^{-d/2} |dx|^{1/2},$$

so “dimensionless f ” is a convention unless the scale $L_*^{d/2}$ is fixed by an additional universality principle (H2.4–H2.5).

3.5 Running H2.4: What Does “Background-Free Constancy” Mean?

From Derivation D1.7, changing the reference half-density σ_* by a positive function $r(x)$ changes the scalar representative by $f \mapsto r^{-1}f$ and changes the scalar measure by $\rho_* \mapsto r^2 \rho_*$. So the raw half-density formulation has a large “scalarization gauge freedom”.

Heuristic H2.10 (Constancy = no extra background function). If we take “background-free” in the strong sense “no additional structure beyond the manifold and the theory’s couplings”, then allowing an arbitrary non-constant $r(x)$ would amount to introducing a new background field/function by hand. In that strong sense, the only admissible changes are constant rescalings, and choosing σ_* becomes a choice of a single global scale (fixed or not fixed by couplings depending on H2.5).

Derivation D1.8 (Three natural families of σ_* and what they mean). On a configuration space M , the common ways to choose a reference half-density are:

- Flat/affine choice (when available):** on \mathbb{R}^d with its affine structure, translation invariance picks $|dx|^{1/2}$ uniquely up to a constant factor. This is “constant” in the sense of being homogeneous under translations.
- Metric-derived choice:** given a Riemannian/Lorentzian metric g , one can take $\sigma_g := |g|^{1/4} |dx|^{1/2}$, so that $\rho_g = \sigma_g^2 = \sqrt{|g|} |dx|$ is the familiar invariant volume density. This makes the scalar representative f a genuine scalar field but makes the scalarization depend on background geometry.
- Field-derived (dilaton-like) choice:** given a scalar field Φ (background or dynamical), one can take $\sigma_\Phi := e^{-\Phi} \sigma_g$. In the scalar picture this is a local rescaling of the measure, and it is the natural way to encode “local units” or Weyl factors.

H2.4 asserts that the theory supplies (or selects) a choice of type (1) with no x -dependent factor: a fixed reference σ_* whose only remaining ambiguity is an overall constant scale.

Heuristic H2.11 (RG as scale dependence of scalarization). If refinement/coarse-graining forces an x -independent but scale-dependent choice $\sigma_*(\mu)$ (equivalently a scale-dependent constant $L_*(\mu)$), then H2.4 is replaced by an RG statement: the scalarization convention becomes part of the renormalization scheme (a “wavefunction renormalization” for the scalar representative). In that case, a universal area/length scale can still appear, but typically as an RG invariant (dimensional transmutation scale) rather than as a fixed analytic monomial in couplings.

4. Stationary Phase Produces Half-Density Prefactors (Short-Time Kernel)

The main manuscript uses stationary phase to explain why classical extremals dominate refinement limits. Here we add the complementary kernel-level fact: stationary phase does not only pick the extremal; it also produces a determinant prefactor that transforms as a half-density, i.e. the object needed for coordinate-free kernel composition.

Derivation D1.4 (Van Vleck prefactor is a bi-half-density). Let $S_{\text{cl}}(x, z; t)$ be the classical action as a function of endpoints and time, treated as a generating function. The standard short-time/stationary-phase approximation to the propagator has the form

$$K(x, z; t) \approx \frac{1}{(2\pi i\hbar)^{d/2}} \left| \det \left(-\frac{\partial^2 S_{\text{cl}}}{\partial x \partial z} \right) \right|^{1/2} \exp \left(\frac{i}{\hbar} S_{\text{cl}}(x, z; t) \right).$$

Under a change of coordinates $x = x(x')$, $z = z(z')$, the mixed Hessian transforms by the chain rule, and its determinant acquires Jacobian factors:

$$\det \left(-\frac{\partial^2 S_{\text{cl}}}{\partial x' \partial z'} \right) = \det \left(\frac{\partial x}{\partial x'} \right) \det \left(\frac{\partial z}{\partial z'} \right) \det \left(-\frac{\partial^2 S_{\text{cl}}}{\partial x \partial z} \right).$$

Taking square roots shows that the prefactor transforms with $|\det(\partial x/\partial x')|^{1/2} |\det(\partial z/\partial z')|^{1/2}$, i.e. exactly as a half-density factor at each endpoint. Thus the stationary-phase prefactor is naturally interpreted as making K a half-density in each variable, so that kernel composition does not depend on a background measure choice.

Derivation D1.9 (Square-root delta normalization has half-density weight). In finite dimension, the “localize on critical points” distribution is $\delta(\nabla f)$, supported on $\text{Crit}(f)$. A concrete way it appears is via a “halved” oscillatory integral with a normalization exponent fixed by dimension.

Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be smooth and define, for $\varepsilon > 0$,

$$A_\varepsilon(O) := \varepsilon^{-N/2} \int_{\mathbb{R}^N} e^{\frac{i}{\varepsilon} f(x)} O(x) dx.$$

Then

$$|A_\varepsilon(O)|^2 = \varepsilon^{-N} \iint_{\mathbb{R}^N} e^{\frac{i}{\varepsilon}(f(x)-f(y))} O(x) \overline{O(y)} dx dy.$$

Applying the near-diagonal scaling $y = x + \varepsilon z$ (so $dy = \varepsilon^N dz$) gives

$$|A_\varepsilon(O)|^2 = \iint e^{-\frac{i}{\varepsilon}(f(x+\varepsilon z)-f(x))} O(x) \overline{O(x+\varepsilon z)} dz dx.$$

Formally letting $\varepsilon \rightarrow 0$ yields

$$|A_\varepsilon(O)|^2 \rightarrow \iint e^{-iz \cdot \nabla f(x)} |O(x)|^2 dz dx = (2\pi)^N \int \delta(\nabla f(x)) |O(x)|^2 dx.$$

The exponent $N/2$ in the prefactor is exactly the half-density scaling: it cancels the Jacobian $dy = \varepsilon^N dz$ under near-diagonal rescaling, and it is the “square root” of the density normalization that produces $\delta(\nabla f)$.

Heuristic H1.4 (Where Planck area can enter, minimally). Derivation D1.3 isolates one minimal route by which a Planck-scale quantity can enter: if the theory supplies a single universal coupling with dimension of length (Newton’s constant) and one demands that the half-density normalization constant be built from that coupling *without fractional powers*, then $d = 4$ is singled out and the resulting constant has the dimension of an area, naturally identified with the Planck area $L_P^2 \sim \hbar G_4/c^3$.

5. A Gravitational Anchor: Minimal Areal Speed and the $D = 4$ Cancellation

Rivero’s “Planck areal speed” observation gives a concrete route by which Planck-scale discreteness reappears at Compton scales in inverse-square gravity [RiveroAreal] [RiveroSimple].

Heuristic H1.3 (Areal-speed selection). In $3 + 1$ Newtonian gravity (inverse-square), imposing a discrete areal-speed/area-time condition at a Planck scale can yield characteristic radii proportional to a reduced Compton length, with Newton’s constant canceling when expressed in Planck units. This is a non-trivial indication that “a universal area scale” can be operationally meaningful at low energies in $D = 4$.

Derivation D1.5 (Inverse-square circular orbit + Planck areal speed \Rightarrow Compton radius). For a circular orbit under an inverse-square central force $F(r) = K/r^2$ (with coupling $K > 0$), the centripetal balance is $mv^2/r = K/r^2$. The areal speed is $\dot{A} = \frac{1}{2}rv$, so $v = 2\dot{A}/r$. Substituting into the force balance gives

$$m \left(\frac{2\dot{A}}{r} \right)^2 = \frac{K}{r} \implies r = \frac{4m\dot{A}^2}{K}.$$

For Newtonian gravity between a source mass M and test mass m , $K = GMm$, hence

$$r = \frac{4\dot{A}^2}{GM},$$

independent of the test mass m . If one now imposes $\dot{A} = k \dot{A}_P$, where Rivero's Planck areal speed is $\dot{A}_P = cL_P$ [RiveroAreal], then using $L_P^2 = G\hbar/c^3$ yields

$$r = \frac{4k^2(cL_P)^2}{GM} = \frac{4k^2(G\hbar/c)}{GM} = 4k^2 \frac{\hbar}{cM}.$$

Thus r becomes a multiple of the reduced Compton length $L_M = \hbar/(cM)$, with Newton's constant canceled out. In particular, $k = \frac{1}{2}$ gives $r = L_M$. This is the "Planck area per Planck time \Rightarrow Compton scale" cancellation highlighted in [RiveroAreal] and summarized in [RiveroSimple].

6. Interface with the Main Paper

The main manuscript argues that: 1. classical dynamics are recovered from quantum composition by stationary-phase concentration, and 2. refinement across scales forces RG-style consistency conditions when naive limits diverge.

This draft adds a complementary ingredient: the kernel side is most naturally formulated in half-density language, and stationary phase produces the bi-half-density prefactor directly. A universal convention for turning those half-densities into scalar amplitudes then requires a length $^{d/2}$ scale; in $d = 4$ this is an area scale.

7. Open Problems (Needed for a Real Paper)

1. Make the half-density normalization argument precise for a concrete groupoid or kernel model (tangent-groupoid or short-time propagator model).
2. Show how the area scale enters stationary-phase prefactors and how this interacts with RG scaling.
3. General-dimension analysis: clarify what replaces "area" in odd dimensions and whether a universal normalization is still defensible.
4. Identify minimal hypotheses under which "need of half-density scale \Rightarrow Planck area" is more than dimensional bookkeeping.