

Half-Densities in QFT: Propagators as Bi-Half-Density Kernels

Abstract

In QFT, the basic free object is the inverse of a kinetic operator, i.e. a propagator/Green kernel. On a manifold, writing “ $P_x G(x, y) = \delta(x, y)$ ” hides conventions: which volume form defines the adjoint, and which delta normalization realizes the identity. This note adopts a single organizing choice: treat fields (or kernels) as **half-densities**, so the identity kernel is canonical and kernel composition is coordinate-invariant without choosing a background measure. A worked computation shows how a scalar field on (M, g) becomes a half-density $\psi = |g|^{1/4} \phi$, with kinetic operator $\tilde{P} = |g|^{1/4} P |g|^{-1/4}$ symmetric in the coordinate pairing. We also record a kernel-level remark: local counterterms/contact terms appear as distributions supported on the diagonal ($x = y$) (delta kernels and their derivatives), which are most naturally expressed using the canonical bi-half-density delta.

This note is written to be readable on its own; it also connects to broader themes (scalarization scales and RG as compatibility) developed elsewhere.

1. Purpose and Scope

This note is intentionally narrow: 1. establish the “kernel as bi-half-density” semantics for spacetime propagators in QFT, 2. isolate what is **canonical** (half-density kernels, identity delta kernel) versus what is **a convention** (scalarization choices such as $\sqrt{|g|}$), 3. give one explicit computation that can later be promoted (densitized scalar field).

BV/BRST/field-space half-densities are only flagged as outlook here; a full treatment would require additional dedicated sources and is beyond scope.

2. Kernels on a Manifold: half-densities make the identity canonical

Let M be a D -dimensional manifold. A half-density is a section of $|\Lambda^D T^* M|^{1/2}$.

The key operational point (as in the main paper's kernel-composition spine) is: if an operator acts on half-densities, then its Schwartz kernel is naturally a **bi-half-density** [BatesWeinstein1997] [Hormander2003]

$$K_A(x, y) = a(x, y) |dx|^{1/2} |dy|^{1/2},$$

and composition is intrinsic:

$$(A \circ B)(x, z) = \int_M K_A(x, y) K_B(y, z),$$

because the product in the intermediate variable y is a density.

Derivation HD-D1.1 (Identity kernel). The identity operator on half-densities has kernel

$$K_{\text{Id}}(x, y) = \delta^{(D)}(x - y) |dx|^{1/2} |dy|^{1/2},$$

which is canonical: it does not require choosing a background density/volume form.

Derivation HD-D1.1a (Normalization witness: why $\backslash(\backslash varepsilon^{-D/2}\backslash)$ appears). In local coordinates on \mathbb{R}^D , a standard approximate identity is the Gaussian family

$$\rho_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{D/2}} \exp\left(-\frac{|x|^2}{2\varepsilon}\right), \quad \varepsilon > 0.$$

The exponent $D/2$ is forced by normalization: by the change of variables $x = \sqrt{\varepsilon} u$,

$$\int_{\mathbb{R}^D} \rho_\varepsilon(x) d^D x = (2\pi\varepsilon)^{-D/2} \varepsilon^{D/2} \int_{\mathbb{R}^D} e^{-|u|^2/2} d^D u = 1.$$

Thus $\rho_\varepsilon \rightharpoonup \delta^{(D)}$ as $\varepsilon \rightarrow 0^+$, and the diagonal delta kernel is the distributional limit of families whose normalization scales as $\varepsilon^{-D/2}$.

3. Worked computation: densitized scalar field

$$\psi = |g|^{1/4} \phi$$

Consider a real scalar field on a fixed Lorentzian/Euclidean background (M, g) [Wald1984] with quadratic action

$$S[\phi] = \frac{1}{2} \int_M d^D x \sqrt{|g|} \phi P \phi, \quad P = -\nabla^2 + m^2 + \xi R \text{ (example).}$$

The pairing for which P is symmetric is

$$(\phi_1, \phi_2)_g = \int d^D x \sqrt{|g|} \phi_1 \phi_2.$$

Define the densitized field (a half-density in coordinates)

$$\psi := |g|^{1/4}\phi, \quad \text{so} \quad \phi = |g|^{-1/4}\psi.$$

Then the action becomes

$$S[\phi] = \frac{1}{2} \int d^Dx \psi \tilde{P} \psi, \quad \tilde{P} := |g|^{1/4} P |g|^{-1/4},$$

so the pairing is now just the coordinate density d^Dx .

Derivation HD-D1.2 (Explicit form of the conjugated kinetic operator). For $P_{\text{kin}} = -\nabla^2$ one has in coordinates

$$\nabla^2\phi = |g|^{-1/2}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right),$$

hence

$$\tilde{P}_{\text{kin}} = -|g|^{1/4}\nabla^2|g|^{-1/4} = -|g|^{-1/4}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu(|g|^{-1/4}\cdot)\right).$$

Assuming compact support (or boundary conditions killing boundary terms),

$$\int d^Dx \psi_1 \tilde{P}_{\text{kin}} \psi_2 = \int d^Dx \sqrt{|g|} g^{\mu\nu} \partial_\mu(|g|^{-1/4}\psi_1) \partial_\nu(|g|^{-1/4}\psi_2),$$

which is manifestly symmetric under $(1 \leftrightarrow 2)$.

Derivation HD-D1.3 (Conformal metric expansion; D=4 simplification in the conformal class). As a worked expansion, take a conformally flat background $g_{\mu\nu} = e^{2\sigma(x)}\delta_{\mu\nu}$ (Euclidean for simplicity). Then $\sqrt{|g|} = e^{D\sigma}$, $|g|^{1/4} = e^{D\sigma/2}$, $g^{\mu\nu} = e^{-2\sigma}\delta^{\mu\nu}$, and one finds

$$\Delta_g f = |g|^{-1/2}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu f) = e^{-2\sigma}(\partial^2 f + (D-2)\partial\sigma \cdot \partial f).$$

Setting $\phi = |g|^{-1/4}\psi = e^{-D\sigma/2}\psi$ and expanding derivatives gives the conjugated operator

$$\tilde{\Delta}\psi := |g|^{1/4}\Delta_g|g|^{-1/4}\psi = e^{-2\sigma}\left(\partial^2\psi - 2\partial\sigma \cdot \partial\psi - \frac{D}{2}(\partial^2\sigma)\psi + \frac{D(4-D)}{4}(\partial\sigma)^2\psi\right),$$

so the kinetic operator $\tilde{P}_{\text{kin}} = -\tilde{\Delta}$ contains a term proportional to $D(4-D)(\partial\sigma)^2$, which cancels at $D = 4$ in this conformal ansatz.

Scope disclaimer: this is recorded only as a checked simplification for $\tilde{\Delta}$ in this metric class; it is not, by itself, a dimension-selection claim or a conformal-invariance statement. A symbolic coefficient/sign check (SymPy) confirms the expansion.

Interpretation: - the metric half-density $|g|^{1/4}|dx|^{1/2}$ is a **scalarization gauge** (a choice of reference half-density) on a fixed background, - writing the field as

ψ makes the “half-density priority” viewpoint explicit: both the field and the kernels live naturally as half-density objects.

Remark HD-D1.3a (Half-density conjugation is distinct from conformal coupling). The conjugated Laplacian $\tilde{\Delta} = |g|^{1/4} \Delta_g |g|^{-1/4}$ acts on sections of the half-density bundle, which carry conformal weight $D/2$. The Yamabe operator $Y = -\Delta_g + \frac{D-2}{4(D-1)} R$ acts on conformal weight $(D-2)/2$. These are different operators: their first-order, Laplacian, and potential terms all differ.

In particular, the zeroth-order (“potential”) terms generated by the half-density conjugation in the conformal class cannot be written as ξR for any single value of ξ . Matching the $\partial^2 \sigma$ coefficient gives $\xi = D/(4(D-1))$, but the $(\partial\sigma)^2$ coefficient then has a residual mismatch of $D/2$. Thus the $D=4$ cancellation of the $(\partial\sigma)^2$ term in $\tilde{\Delta}$ is a property of the half-density weight, not of conformal invariance or conformal coupling (which corresponds to $\xi_{\text{conf}} = (D-2)/(4(D-1))$).

This cancellation is also conformal-class-specific: for a general metric, the conjugation potential is $V = |\nabla\omega|_g^2 - \Delta_g \omega$ with $\omega = \frac{1}{4} \ln |g|$, and the partial cancellation at $D=4$ relies on the conformal structure (contracted Christoffel symbols proportional to $\partial\sigma$).

4. Propagators/Green functions as bi-half-density kernels

Let \tilde{G} be the inverse kernel of \tilde{P} with respect to the coordinate pairing:

$$(\tilde{P}^{-1} f)(x) = \int \tilde{G}(x, y) f(y) d^D y, \quad \tilde{P}_x \tilde{G}(x, y) = \delta^{(D)}(x - y).$$

Then the corresponding canonical bi-half-density kernel is

$$K_{\tilde{P}^{-1}}(x, y) = \tilde{G}(x, y) |dx|^{1/2} |dy|^{1/2}.$$

Equivalently, if $G_g(x, y)$ denotes the usual **scalar** Green function for P defined with respect to the metric pairing $\int \sqrt{|g|} d^D y$ (so $(P^{-1} J)(x) = \int G_g(x, y) J(y) \sqrt{|g(y)|} d^D y$), then the kernels are related by

$$\tilde{G}(x, y) = |g(x)|^{1/4} G_g(x, y) |g(y)|^{1/4}.$$

This is exactly the same “kernel as bi-half-density” structure used for QM propagators [BatesWeinstein1997] [deGosson2018ShortTimePropagators], now applied to spacetime Green functions in QFT.

Remark HD-D4.1 (Doubling: densities live on $\backslash(M \times M)$). Half-density kernels also make the amplitude-vs-density doubling completely explicit.

Let U_t be a (unitary) evolution operator on half-densities with kernel $K_t(x, y)$. Then a density operator $\rho_t = U_t \rho_0 U_t^{-1}$ has a kernel satisfying

$$K_{\rho_t}(x, y) = \int_{M \times M} K_t(x, x') K_{\rho_0}(x', y') \overline{K_t(y, y')}.$$

So densities naturally propagate by a kernel on the doubled space $M \times M$, built from the forward kernel and its conjugate. This is the kernel-level origin of bra/ket (forward/backward) doubling in real expectation values; a fuller discussion is beyond this note's scope.

5. Contact terms and counterterms as diagonal delta kernels (kernel-level remark)

In QFT, divergences are removed by adding local counterterms to the action, e.g. $\delta m^2 \phi^2$, $\delta Z (\partial\phi)^2$, curvature couplings, etc. At the operator/kernel level this means: the kinetic operator P is modified by local (differential) operators, and therefore its kernel acquires **distributions supported on the diagonal** $x = y$.

In the half-density kernel language the diagonal object is canonical:

$$K_{\text{Id}}(x, y) = \delta^{(D)}(x - y) |dx|^{1/2} |dy|^{1/2}.$$

Multiplication counterterms correspond to $c(x) K_{\text{Id}}(x, y)$, and derivative counterterms correspond to derivatives acting on the delta kernel (e.g. $\partial_x \delta^{(D)}(x - y)$ and higher).

Remark HD-D5.1 (Derivative of the diagonal delta kernel; coordinate-free identity). The slogan “ $\partial_x \delta(x - y) = -\partial_y \delta(x - y)$ ” has a clean, connection-free formulation in the half-density kernel calculus. The identity kernel K_{Id} is invariant under diagonal diffeomorphisms $(\Phi \times \Phi)$, so for any vector field V on M one has

$$(\mathcal{L}_{V_x} + \mathcal{L}_{V_y}) K_{\text{Id}} = 0, \quad \text{hence} \quad \mathcal{L}_{V_x} K_{\text{Id}} = -\mathcal{L}_{V_y} K_{\text{Id}},$$

where \mathcal{L} is the Lie derivative acting on half-densities. In local coordinates, taking $V = \partial/\partial x^\mu$ recovers $\partial_{x^\mu} \delta^{(D)}(x - y) = -\partial_{y^\mu} \delta^{(D)}(x - y)$. This is the kernel-level mechanism behind “moving derivatives between slots” in integration by parts and in contact-term identities.

Remark HD-D5.2 (Point splitting produces ‘contact terms’). Point splitting makes the simplest derivative contact term explicit: in one dimension,

$$\frac{\delta(x + \varepsilon) - \delta(x)}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \delta'(x), \quad \langle \delta', \varphi \rangle = -\varphi'(0).$$

This limit is the distributional companion of the “difference quotient as divergence + subtraction” toy model.

This framing is useful for two reasons: 1. it makes “locality = diagonal support” literal at the kernel level, 2. it separates the canonical distributional objects from scheme-dependent scalarizations and finite-subtraction conventions.

6. Link to the half-density scale program (where Planck-area enters conditionally)

On a fixed background (M, g) , the metric provides a natural reference half-density $|g|^{1/4}|dx|^{1/2}$. The Planck-area program begins only when we ask for a **background-free** scalarization convention that turns half-density amplitudes into universal dimensionless numbers. A separate note develops that stronger hypothesis ladder; it is not needed for the present kernel/propagator semantics.

This paper’s role is only to show that half-densities are not a QM quirk: the same kernel semantics is already present in standard QFT propagator definitions, once the hidden measure conventions are made explicit.

7. Outlook: BV half-densities

Gauge theories suggest a second, deeper appearance of half-densities: the BV formalism treats the integrand as a (half-)density on an (odd) symplectic space of fields/antifields, and the quantum master equation expresses independence of gauge-fixing choices [Costello2011]. This note does not develop BV beyond this remark; doing so responsibly would require additional authoritative sources and a separate dedicated treatment.

References

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