"Uncuttable" as Controlled Refinement

Alejandro Rivero

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Abstract

This note fixes a meaning of "uncuttable" aligned with the refinement-compatibility thesis of the cornerstone manuscript. Here **uncuttable** does not mean "indivisible." It means: the quantity of interest is not determined by any *finite* dissection alone; it is a **limit object** whose definition requires a refinement rule and a comparison structure across refinements.

The point is structural and mathematical: once a theory is built from composable local pieces, the continuum theory is the stable target of a refinement limit, and extra control data may be required for that limit to exist or be unique.

1. Definition

Call a quantity Q **uncuttable** (in this note's sense) if: 1. there exists a family of finite approximants Q_N produced by a finite dissection/refinement scheme of depth N, but 2. the value of interest is not any finite Q_N; it is a controlled limit $Q = \lim_{N\to\infty} Q_N$, and 3. specifying the *rule of refinement* and the *comparison across refinements* is part of the definition of Q.

The historical resonance is deliberate. The Greek $\acute{\alpha}\tau o\mu o\varsigma$ ("a-tomos," uncuttable) was coined by Leucippus and Democritus to denote indivisible substance — matter that cannot be divided further. The shift proposed here is from ontology to procedure: what is "uncuttable" is not a smallest piece of stuff, but a limit object that no single finite dissection captures. The indivisibility is not in the substance but in the definition: you cannot "cut" the limit into finitely many pieces and recover it without specifying how the pieces are to be reassembled and refined.

This is the ordinary situation in analysis: finite partitions approximate, but the object is defined by a limiting procedure together with hypotheses that ensure convergence/uniqueness.

2. Toy model: an integral is already a refinement limit

Let $f : [a, b] \to \mathbb{R}$. A prototypical refinement family is a partition $P_N = \{a = t_0 < \cdots < t_N = b\}$ with mesh $\|P_N\| := \max_k (t_{k+1} - t_k) \to 0$. Define the Riemann-sum approximants

$$Q_N := \sum_{k=0}^{N-1} f(\xi_k)\,(t_{k+1} - t_k), \qquad \xi_k \in [t_k, t_{k+1}].$$

In good cases, $Q_N \to \int_a^b f(t)\,dt$ as $\|P_N\| \to 0$, and the limit is independent of the tags ξ_k. But this is not a tautology: the limit can fail to exist or can depend on the refinement rule unless hypotheses are stated.

A concrete failure case appears already in calculus. Consider the difference quotient $f(x + \varepsilon)/\varepsilon$: for each finite ε, the quantity diverges as $\varepsilon \to 0$. The "refinement limit" exists only after subtracting a counterterm $f(x)/\varepsilon$, yielding the derivative $f'(x)$. The subtraction is part of the definition of the limit — without it, the refinement procedure does not converge. This is the simplest model of renormalization: a "cut" at finite ε is not the answer; the answer requires a controlled $\varepsilon \to 0$ limit with explicit subtraction rules.

In the present program, this is the basic moral: finite cuts approximate, but the value is defined by **controlled refinement**.

2.5 Worked example: the derivative as a renormalized refinement limit

The paragraph above describes the difference-quotient subtraction informally. We now make it explicit as a minimal model of renormalization.

`Example 2.1 (Derivative as counterterm-subtracted limit).` Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at x. The naive "refinement" approximant at scale ε is

$$R(\varepsilon) := \frac{f(x + \varepsilon)}{\varepsilon}.$$

This diverges as $\varepsilon \to 0$ whenever $f(x) \neq 0$. To extract a finite limit, subtract a counterterm:

$$R_{\mathrm{ren}}(\varepsilon) := \frac{f(x + \varepsilon) - f(x)}{\varepsilon}.$$

Now $\lim_{\varepsilon \to 0} R_{\mathrm{ren}}(\varepsilon) = f'(x)$, which is finite and independent of the "regulator" ε. The structure is:

1. **Regulated quantity**: $R(\varepsilon)$, finite for each $\varepsilon > 0$ but divergent as $\varepsilon \to 0$.
2. **Counterterm**: $f(x)/\varepsilon$, which absorbs the divergence.
3. **Renormalized observable**: $f'(x)$, the ε-independent limit.

This is formally identical to the pattern in perturbative renormalization: a bare quantity diverges as the regulator is removed, but a systematic subtraction yields a finite, physically meaningful result. The subtraction rule is not ad hoc; it is forced by the requirement that the result depend smoothly on the data and be independent of the regulator.

`Remark 2.2 (Higher-order counterterms).` If f is C^n, the Taylor expansion

$$f(x+\varepsilon) = f(x) + f'(x)\varepsilon + \tfrac{1}{2}f''(x)\varepsilon^2 + \cdots + \tfrac{1}{n!}f^{(n)}(x)\varepsilon^n + o(\varepsilon^n)$$

provides a systematic tower of subtractions, one per order. Removing terms through order $k-1$ and dividing by ε^k gives the k-th Taylor coefficient $f^{(k)}(x)/k!$ plus vanishing corrections — or equivalently, dividing by $\varepsilon^k/k!$ recovers the derivative $f^{(k)}(x)$ itself. Each order is a "counterterm-subtracted refinement limit" — the analog of loop-by-loop renormalization in QFT.

`Remark 2.3 (Euler-Maclaurin: correction tower for the Riemann sum).` The same pattern applies to the integral approximation itself. The Euler–Maclaurin formula expresses the error of the left-point Riemann sum as a systematic expansion in the mesh size $h = (b-a)/N$:

$$\sum_{k=0}^{N-1} f(a+kh)\,h = \int_a^b f(t)\,dt + \sum_{j=1}^{p} \frac{B_j}{j!}\,h^j\,\left[f^{(j-1)}(b) - f^{(j-1)}(a)\right] + O(h^{p+1}),$$

where B_j are Bernoulli numbers ($B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0, \ldots$). Each correction involves a power of the mesh (the "regulator"), multiplied by endpoint derivatives (boundary data). The leading correction $j = 1$ converts the left-point sum into the trapezoidal rule; the $j = 2$ term adds an endpoint-derivative correction that raises the order to $O(h^4)$. This is the interval counterpart of Remark 2.2's single-point tower: there the subtractions were local (Taylor coefficients at x); here they are boundary-localized (endpoint derivatives of f).

`Remark 2.4 (Richardson extrapolation: subtracting leading errors across refinement levels).` Given an approximation family $Q(h) = Q + c_1 h^p + O(h^{p+1})$, the combination $R(h) = (2^p\,Q(h/2) - Q(h))/(2^p - 1)$ eliminates the leading error term without knowledge of c_1 — only the exponent p is needed. Iterating at mesh sizes $h, h/2, h/4, \ldots$ produces a triangular tableau (Romberg) that peels off one order of correction per column, mirroring the counterterm tower of Remark 2.2. Applied to the trapezoidal rule (whose even-power error expansion follows from Remark 2.3 with $B_{2k+1} = 0$ for $k \geq 1$), the first Richardson step ($p = 2$) yields Simpson's rule; the second ($p = 4$) yields Boole's rule — each time eliminating the next Euler–Maclaurin correction without computing endpoint derivatives. The structural parallel to minimal subtraction in dimensional regularization is precise: the exponent p plays the role of the pole order, and the subtraction procedure is universal — independent of the specific integrand.

3. Dynamics: action, stationarity, and the need for control data

The cornerstone manuscript uses the same template in mechanics. Given a partition of time, the discrete action

$$S_N[q] = \sum_k \mathcal{L}\left(q_k, \frac{q_{k+1} - q_k}{\Delta t_k}, t_k\right) \Delta t_k$$

is a finite refinement approximant. The continuum action $S[q] = \int \mathcal{L}\, dt$ is a refinement limit.

Two "uncuttable" features appear immediately when one pushes beyond smooth classical paths: 1. **Singular probes and corners:** stationarity must be interpreted in weak/distributional form; point-supported variations require mollification. 2. **Non-uniqueness of refinement schemes:** different discretization conventions (even if classically equivalent) can produce distinct refined objects unless an equivalence or control map is specified.

These are exactly the obstructions discussed in the cornerstone manuscript: the point is not indivisible atoms, but limit control.

In the quantum setting, the "uncuttable" character becomes sharper. The path-integral amplitude

$$K(q_f, t_f; q_i, t_i) = \int \prod_{k=1}^{N-1} dq_k \prod_{k=0}^{N-1} K_\Delta(q_{k+1}, q_k; t_k)$$

is a product of short-time kernels composed over a time partition of depth N. No finite N gives the exact propagator; the propagator is the $N \to \infty$ refinement limit. Crucially, the control parameter $\hbar$ enters the short-time kernels as $\exp(iS_\Delta/\hbar)$, and different discretization conventions (left-point, midpoint, symmetric) can produce distinct $O(\hbar)$ corrections even though they share the same classical $\hbar \to 0$ limit [FeynmanHibbs1965]. Thus the quantum amplitude is doubly "uncuttable": it requires both a refinement rule (time-slicing prescription) and a comparison/equivalence structure (ordering convention or half-density normalization) before the limit is well-defined.

`Example 3.1 (Non-uniqueness of refinement: \(\alpha\)-ordering).` For the classical Hamiltonian $H(q,p) = qp$, different time-slicing prescriptions — evaluating position at $q_\alpha = (1-\alpha)q_k + \alpha q_{k+1}$ in each slice — produce different quantum operators:

$$\hat{H}_\alpha = \alpha \hat{q}\hat{p} + (1-\alpha)\hat{p}\hat{q} = \hat{p}\hat{q} + \alpha\, i\hbar.$$

At $\alpha = 0$ (prepoint): $\hat{p}\hat{q}$. At $\alpha = 1/2$ (midpoint/Weyl): $\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$. At $\alpha = 1$ (postpoint): $\hat{q}\hat{p}$. All three share the classical limit $H = qp$ as $\hbar \to 0$, but they are distinct quantum objects. The "uncuttable" message: the continuum

quantum Hamiltonian is not determined by any single finite time-slicing; it requires specifying the refinement convention α as part of the definition.

`Remark 3.2 (Stochastic counterpart: Itô versus Stratonovich).` The same refinement non-uniqueness appears in stochastic calculus. For a Wiener process W_t, the stochastic integral $\int f(W_t)\,dW_t$ can be defined using left-point (Itô) or midpoint (Stratonovich) evaluation in the Riemann sums:

$$\text{Itô: } \sum_k f(W_{t_k})\,\Delta W_k, \qquad \text{Stratonovich: } \sum_k f\Big(\tfrac{W_{t_k}+W_{t_{k+1}}}{2}\Big)\,\Delta W_k.$$

Both converge as mesh $\to 0$ (for $f \in C^2$), but to different limits related by $\int f(W)\circ dW = \int f(W)\,dW + \frac{1}{2}\int f'(W)\,dt$ [Oksendal2003]. The correction $\frac{1}{2}f'\,dt$ arises because Brownian paths have non-zero quadratic variation ($\sum(\Delta W_k)^2 \to T \neq 0$); for paths of bounded variation, the quadratic variation vanishes and all evaluation-point prescriptions agree. This parallels Example 3.1: the ordering correction $\alpha\, i\hbar$ vanishes when $\hbar \to 0$ (smooth classical paths), but is unavoidable at finite $\hbar$. Both cases instantiate the "uncuttable" pattern: when paths are rough enough, the refinement prescription becomes part of the definition.

`Remark 3.3 (Trotter product formula as a refinement theorem).` The mathematical backbone of the path-integral refinement limit is the Trotter product formula: for self-adjoint operators A and B with $A + B$ essentially self-adjoint on a common dense domain,

$$e^{t(A+B)} = \lim_{N\to\infty} \left(e^{tA/N}\, e^{tB/N}\right)^N.$$

In the path-integral context, $A = -i\hat{T}/\hbar$ (kinetic) and $B = -i\hat{V}/\hbar$ (potential), so each factor is a free propagation or a potential phase-kick at one time slice. The formula states that the exact propagator is a refinement limit of N-fold compositions — and that this limit converges, with total error $O(1/N)$. The symmetric (Suzuki–Trotter) splitting $e^{tA/(2N)}e^{tB/N}e^{tA/(2N)}$ reduces the total error to $O(1/N^2)$ by canceling the leading Baker–Campbell–Hausdorff commutator $[A, B]$ contribution at each step. This is the "uncuttable" pattern in operator-algebraic form: no finite product equals $e^{t(A+B)}$, and the rate of convergence depends on controlling the non-commutativity of the pieces [FeynmanHibbs1965].

`Remark 3.4 (Symplectic integrators: structural compatibility at finite resolution).` The symmetric Trotter splitting is the operator form of the Störmer–Verlet (leapfrog) integrator. The discrete map is *exactly* symplectic at every finite N: it preserves the Poisson brackets (or, quantum-mechanically, unitarity) not just asymptotically but at each approximation level. Backward error analysis shows that the numerical flow is the exact flow of a nearby "shadow Hamiltonian" $\tilde{H} = H + O(\Delta t^2)$, which is conserved exactly along the discrete orbit; the original H oscillates with amplitude $O(\Delta t^2)$ without secular drift. Higher-order symplectic splittings (Yoshida, Forest–Ruth) systematically cancel

further Baker–Campbell–Hausdorff commutators, paralleling the counterterm tower of Remark 2.2. This is refinement compatibility made precise: the approximants at every level carry a structural invariant (symplecticity) that the continuum limit inherits.

`Remark 3.5 (Adiabatic limit: Berry phase as geometric refinement correction).` The adiabatic theorem is another instance of the same refinement pattern. A slowly varying Hamiltonian $H(t/T)$ with $T \to \infty$ has instantaneous eigenstates as its finite approximants; the exact evolution is the refinement limit, controlled by the gap condition (transition frequencies bounded away from zero). Diabatic transitions are exponentially suppressed as $T \to \infty$, but a geometric correction survives exactly: the Berry phase $\gamma_n = i \oint \langle n|\nabla_R n\rangle \cdot dR$, which is the holonomy of a U(1) connection on the eigenstate bundle over parameter space. This is the "extra structure" of Section 4 in geometric dress — a connection datum that no local approximant captures, yet persists through the refinement limit and can be topologically quantized.

4. Outlook: refinement compatibility as "the extra structure"

In the companion papers, the "extra structure" used to control refinement limits is made explicit: - half-densities make kernel composition coordinate-free without hidden measure choices, - control maps τ encode how parameters must flow under refinement to maintain stability, - renormalization is the compatibility rule when naive refinement limits diverge.

This note is therefore a small conceptual bridge: it isolates an early, analysis-level instance of the same meta-problem that reappears in quantization and in QFT.

References

1. [FeynmanHibbs1965] Richard P. Feynman and Albert R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, 1965. (Path integral as refinement limit of time-sliced amplitudes; foundational treatment.)
2. [Oksendal2003] Bernt Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 6th ed., Springer, 2003. ISBN `978-3-540-04758-2`. DOI `10.1007/978-3-642-14394-6`. (Standard textbook on Itô vs Stratonovich integrals and their relationship; used in uncuttable satellite for Remark 3.2.)