Fermionic Mediators, Static Potentials, and Contact/Boundary-Condition Limits

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2026

Abstract

The textbook derivation of a static potential from "field exchange" uses a bosonic mediator linearly sourced by a commuting classical density, yielding an effective action quadratic in the source and (in a static limit) a central Yukawa/Coulomb potential. This derivation does not transplant verbatim to fermionic fields: the linear source terms for fermions require Grassmann-valued sources, so there is no ordinary commuting classical source whose elimination produces a classical potential in the same way. This note isolates the precise obstruction and records the robust infrared replacement: when a microscopic description reduces to local operators at low resolution, the effective interaction is encoded by contact terms (delta kernels and their derivatives) or, equivalently, boundary-condition/self-adjoint-extension data, with renormalization-group running when the contact limit is singular.

This is a dependent note aligned with the broader refinement-compatibility program: contact terms are diagonal-support kernels, and their scale dependence is a compatibility condition rather than an afterthought.

1. Purpose and scope

This note answers a narrowly phrased question: what can it mean for a **fermionic** field to "generate a (central) potential" in the same sense that a massive bosonic field generates a Yukawa potential?

We keep the scope bounded: 1. state the bosonic sourcing $\Rightarrow$ potential mechanism (derivation-first, brief), 2. state the fermionic obstruction precisely (Grassmann sources), 3. give one explicit IR matching witness: **local operators** $\Rightarrow$ **contact/derivative-contact kernels**, 4. connect contact kernels to related point-interaction/RG witnesses.

We do **not** claim that fermions cannot affect forces; we only isolate which parts of the "classical source $\Rightarrow$ potential" story fail, and what the correct replacement statement is at low resolution.

2. What "a field generates a potential" means in the bosonic source story

The archetypal construction is a bosonic mediator φ linearly coupled to a commuting source $J(x)$:

$$S[\varphi; J] = \int d^D x \left(\frac{1}{2} \varphi \, K \, \varphi + J \, \varphi \right), \qquad K = (\Box + m^2) \text{ (example)}.$$

Integrating out φ (Gaussian elimination) yields an effective action quadratic in the source,

$$S_{\text{eff}}[J] = -\frac{1}{2} \int d^D x \, d^D y \; J(x) \, K^{-1}(x, y) \, J(y),$$

so the static, nonrelativistic limit of K^{-1} produces a central potential (Yukawa for $m \neq 0$, Coulomb-type when $m = 0$).

The key structural ingredient is that the source is an ordinary commuting function (a classical background density).

3. Fermionic fields: linear sources are Grassmann, so the classical-source story does not transplant

For a Dirac fermion Ψ, the generating functional with sources is written with **Grassmann-valued** sources $\eta, \bar{\eta}$:

$$Z[\bar{\eta}, \eta] = \int D\bar{\Psi} \, D\Psi \; \exp\!\left(i \int d^D x \; \bar{\Psi} \, (i\gamma^\mu \partial_\mu - m) \Psi + i \int d^D x \; (\bar{\eta} \, \Psi + \bar{\Psi} \, \eta) \right).$$

An explicit statement of this form, including that $\eta, \bar{\eta}$ are Grassmann-valued, is recorded in [Floerchinger2024QFT1Lecture21].

Two immediate consequences follow.

`Remark 3.1 (Obstruction statement).` The bosonic derivation "choose a commuting classical source J, integrate out the field, and read off a classical potential" does not directly apply to fermions, because the linear source terms that couple to Ψ require Grassmann sources rather than commuting c-number densities. Therefore, "fermion exchange generates a classical potential between commuting sources" is not a well-posed transplant of the bosonic story.

This does **not** mean fermions are irrelevant: fermions can and do affect effective interactions through loop effects, through bosonic composite modes (bilinears), and through low-energy EFT operators. The point is that the meaning of "generates a potential" must be stated through one of these controlled mechanisms.

3.1 The controlled alternative: fermion loops modify bosonic propagators

The standard example is vacuum polarization in quantum electrodynamics. A closed electron–positron loop inserted into the photon propagator gives a momentum-dependent correction to the effective electromagnetic coupling,

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha}{1 - \Pi(q^2)}, \qquad \Pi(q^2) = -\frac{\alpha}{3\pi} \ln\left(\frac{q^2}{\mu^2}\right) + \cdots ,$$

where $\Pi(q^2)$ is the vacuum polarization function (the photon self-energy from a fermion one-loop diagram). At low momentum transfer ($|q| \ll m_e$), the loop correction is analytic in q^2 and generates precisely the local operators $C_0 + C_2 q^2 + \cdots$ discussed in Section 4 below.

The structural point: fermions affect forces, but the path from "fermion field" to "effective interaction" runs through a quantum loop (not through a tree-level Gaussian elimination of a classical source), and the low-energy residue takes the form of local/contact operators.

4. IR replacement: local operators $\Rightarrow$ contact kernels / boundary-condition data

At low resolution, integrating out heavy degrees of freedom typically produces local operators. In a two-body, nonrelativistic sector, this appears as an amplitude expansion analytic in momentum transfer q:

$$\mathcal{A}(q) = C_0 + C_2 q^2 + O(q^4).$$

The coordinate-space interaction associated to such an analytic expansion is distributional and diagonal-supported. The invariant core is a Fourier-transform identity:

$$\int \frac{d^d q}{(2\pi)^d}\, e^{iq\cdot r} = \delta^{(d)}(r), \qquad \int \frac{d^d q}{(2\pi)^d}\, q^2\, e^{iq\cdot r} = -\nabla^2 \delta^{(d)}(r).$$

`Derivation 4.1 (Contact expansion gives \(\delta\) and derivative contacts).` Interpreting the low-energy interaction kernel as the inverse Fourier transform of $\mathcal{A}(q)$ (Born-level language, up to overall convention-dependent factors), we obtain

$$V(r) \;\propto\; \int \frac{d^d q}{(2\pi)^d}\, e^{iq\cdot r}\, \mathcal{A}(q) \;\propto\; C_0\, \delta^{(d)}(r) \;-\; C_2\, \nabla^2 \delta^{(d)}(r) \;+\; \cdots .$$

Thus locality at low energy naturally becomes **contact data**: delta kernels and their derivatives supported at a point (or on the diagonal, in kernel language).

In singular cases (notably δ interactions in $d \geq 2$ in various channels), this contact data is not simply a fixed number: it is defined by a renormalization condition and can generate RG-invariant scales and bound states [Jackiw1991DeltaPotentials] [ManuelTarrach1994PertRenQM].

`Remark 4.2 (Connection to the effective range expansion).` In scattering theory the s-wave amplitude is parametrized by the effective range expansion (ERE) $k \cot \delta_0(k) = -1/a + r_0 k^2/2 + O(k^4)$, where a is the scattering length and r_0 the effective range. The contact expansion of Derivation 4.1 is the momentum-space counterpart: at Born level, C_0 determines a, C_2 determines r_0, and each higher C_{2n} maps to a shape parameter. Examples 5.1 and 5.2 below are the leading-order case $C_2 = 0$ (zero effective range, $r_0 = 0$), for which the full amplitude $f_0(k) = -a/(1 + ika)$ depends on a single parameter — the scattering length.

5. Boundary-condition reading (point interactions)

Point-supported interactions can be encoded as self-adjoint extension / boundary-condition data rather than as ordinary functions $V(r)$. This is the natural operator-theoretic counterpart of "diagonal-support kernels." For standard references and pedagogical framing, see [BonneauFarautValent2001SAE] and the delta-potential discussion in [Jackiw1991DeltaPotentials].

This viewpoint matches the controlled-refinement perspective: when a continuum description is defined as a refinement limit, UV data can survive in the limit precisely as boundary-condition parameters (contact terms), with RG flow expressing compatibility across resolutions.

`Example 5.1 (Contact coupling generates a scale: 2D delta potential).` In two spatial dimensions, a contact interaction $V(r) = g_0\, \delta^{(2)}(r)$ with bare coupling g_0 and UV cutoff Λ leads, after a standard loop integral, to a T-matrix with the structure

$$T(k)^{-1} = \frac{1}{g_0} + \frac{m}{\pi \hbar^2} \ln\left(\frac{\Lambda}{k}\right),$$

which diverges as $\Lambda \to \infty$ unless g_0 is tuned. Define a renormalized coupling at reference scale μ by absorbing the $\ln \Lambda$ divergence; cutoff independence then gives the beta function $\beta(g_R) = \mu\, dg_R/d\mu = (m/\pi\hbar^2)\, g_R^2$. This is a quadratic beta function of the same form as the toy logarithmic model in the cornerstone (Section 8.3), with solution producing a dynamically generated scale $\mu_* = \mu\, e^{\pi\hbar^2/(m g_R)}$. For attractive coupling ($g_R < 0$) this scale is below the reference scale and sets the bound-state energy: $E = -\hbar^2 \mu_*^2/(2m)$.

The structural lesson: the "contact" limit of the effective interaction is not a number (coupling constant) but a flow — a scale-dependent parameter whose RG trajectory is part of the definition. This is "uncuttable" in the sense of the companion note: the continuum theory requires the refinement rule (cutoff removal + beta function) and not merely a single-cutoff value.

`Example 5.2 (3D contact interaction: scattering length).` In three spatial dimensions, the same contact interaction $V(r) = g_0\,\delta^{(3)}(r)$ with UV cutoff Λ produces a linearly divergent loop integral (compared to the logarithmic divergence in $d = 2$). After resummation, the s-wave scattering amplitude takes the standard effective-range form with zero effective range:

$$f_0(k) = \frac{-a}{1 + ika},$$

where the scattering length a is defined by absorbing the Λ-dependent bare coupling into a single physical parameter via a renormalization condition of the form $1/g_0 \propto \Lambda +$ (finite part depending on a). When $a > 0$, a pole at $k = i/a$ gives a bound state with energy $E = -\hbar^2/(2ma^2)$ [AlbeverioGesztesyHoeghKrohn-Holden2005]. The comparison with Example 5.1 highlights how the divergence character changes with dimension — logarithmic ($d = 2$) versus linear ($d = 3$) — while the structural lesson is identical: the "coupling constant" of a contact interaction is not a bare number but a renormalization-group datum, defined only through a refinement rule (cutoff removal + physical matching condition).

6. Outlook (kept minimal)

Longer-range effects associated to fermionic degrees of freedom can arise through loop-induced mechanisms or through emergent bosonic composite modes. Treating those responsibly would require a separate bibliography-hardening pass and is outside this note's scope.

References

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