The Cornerstone:

Butcher Trees, Counterterm Subtraction, and the Forced Passage to Quantum Mechanics

Companion note to "From Newton to the Path Integral"

February 2026

Abstract

We develop the identification that the rooted-tree algebra organising Runge–Kutta order conditions is the *same* algebra that organises perturbative renormalisation, and that the passage from classical mechanics to quantum mechanics is forced by requiring this tree-organised counterterm subtraction to *compose*. Each edge of a Butcher tree encodes one derivative, and each derivative is one counterterm subtraction. The exact classical flow is the "fully renormalised" character of the Connes–Kreimer Hopf algebra. Quantisation extends the domain from one character to a weighted sum over characters; the weight requires $\hbar > 0$ for the sum to compose with an identity limit. Renormalisation handles divergences in this sum using the same coproduct that organised the original ODE order conditions.

We state this as a *Cornerstone Proposition* (Proposition 6.1), identify precisely what is a theorem and what remains conjectural, and trace the three-level tree hierarchy (Butcher $\to$ Frabetti $\to$ Hairer) that parallels the paper's refinement chain (ODE $\to$ QM $\to$ QFT).

Contents

1 The derivative as a single counterterm subtraction

1.1 The regulated difference quotient

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. For a "cutoff" $\varepsilon > 0$, define the regulated quantities

$$A_\varepsilon(x) := \frac{f(x+\varepsilon)}{\varepsilon}, \qquad C_\varepsilon(x) := \frac{f(x)}{\varepsilon}. \tag{1}$$

Both diverge as $\varepsilon \to 0$ whenever $f(x) \neq 0$. The **renormalised** quantity is their difference:

$$R_\varepsilon(x) := A_\varepsilon(x) - C_\varepsilon(x) = \frac{f(x+\varepsilon) - f(x)}{\varepsilon}. \tag{2}$$

If f is differentiable at x, the limit $\lim_{\varepsilon\to 0} R_\varepsilon(x) = f'(x)$ exists and is finite: the divergences cancel.

1.2 The Lipschitz condition as uniform renormalisability

Proposition 1.1. *f is Lipschitz with constant L if and only if the renormalised quantity is uniformly bounded at all scales:*

$$\sup_{x\in\mathbb{R},\, \varepsilon>0} |R_\varepsilon(x)| \leq L. \tag{3}$$

This is the **zeroth-order RG condition**: the single-vertex counterterm subtraction produces a bounded output at every scale $\varepsilon > 0$.

1.3 Graphical notation

We represent this single counterterm subtraction as a vertex with one incoming edge:

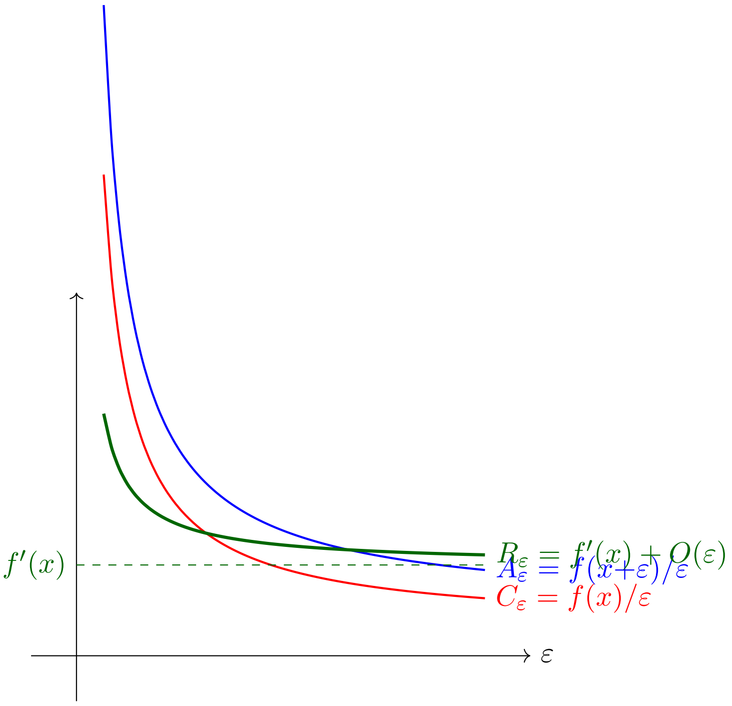
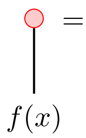


Figure 1: Both A_ε and C_ε diverge as $\varepsilon \to 0$, but their difference R_ε converges to $f'(x)$. The "counterterm" C_ε subtracts the local divergence, leaving a finite "renormalised" quantity.



$$= f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

The vertex (red circle) is the subtraction point.
The edge below it carries the input $f(x)$.
The output is the derivative $f'(x)$.

Figure 2: A single counterterm subtraction = one derivative.

2 Trees as iterated counterterm subtractions

2.1 The first Butcher trees

For the ODE $\dot{y} = f(y)$, the Taylor expansion of the exact flow $\Phi_h(y)$ is indexed by rooted trees. Each tree prescribes a pattern of nested derivatives of f. Since each derivative is a counterterm subtraction (Section 1), **each tree is a pattern of nested counterterm subtractions**.

2.2 The order-4 trees

At order 4 there are four distinct (unordered) rooted trees. They exhaust the ways to nest three counterterm subtractions:

2.3 The general count

Proposition 2.1. *For a rooted tree τ with $|\tau|$ vertices:*

(i) The number of edges is $|\tau| - 1$.

(ii) Each edge encodes one application of a derivative of f, which is one counterterm subtraction.

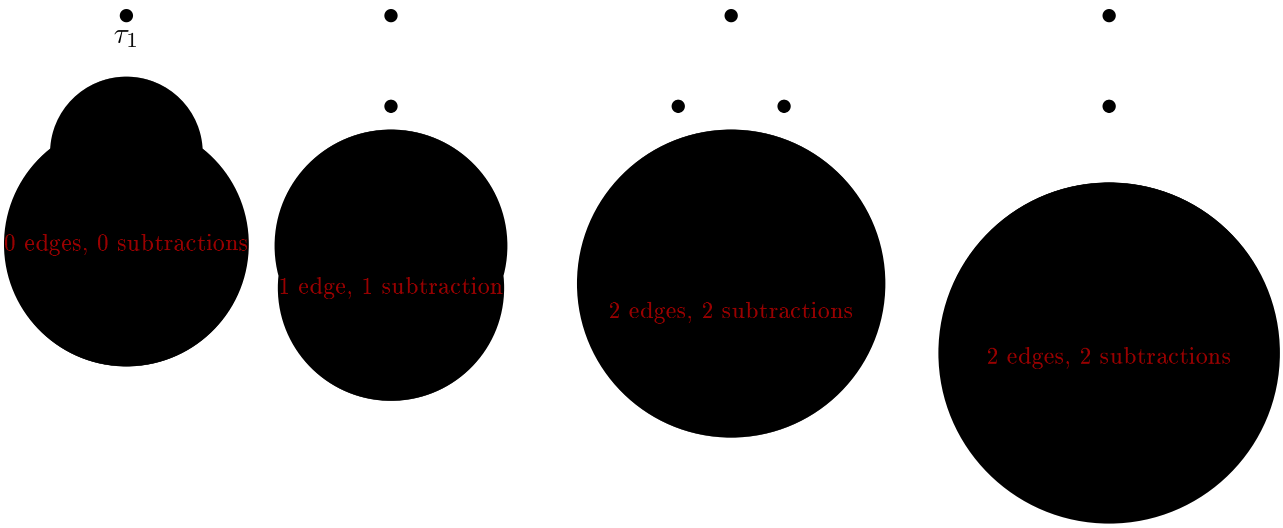


Figure 3: The first four Butcher trees (orders 1–3) with their elementary differentials. Each **edge** represents one application of a derivative of f, which is one counterterm subtraction. The number of edges equals the number of subtractions: $|\tau| - 1$.

(iii) The total "subtraction depth" of the tree is $\sum_{v \in \tau} k_v = |\tau| - 1$*, where* k_v *is the number of children of vertex* v*.*

Proof. Every tree on n vertices has exactly $n-1$ edges. Each edge connects a child to its parent; the parent vertex applies $f^{(k_v)}[\cdot, \dots, \cdot]$ where k_v is its number of children. Since $f^{(k)}$ involves k limiting subtractions (each partial derivative is a limit of a difference quotient), the total is $\sum_v k_v = |\tau| - 1$. □

3 The exact flow as the fully renormalised character

3.1 The Butcher B-series

The Taylor expansion of the exact flow $\Phi_h(y)$ of $\dot{y} = f(y)$ is the **B-series**:

$$\Phi_h(y) = y + \sum_{\tau \in \mathcal{T}} \frac{h^{|\tau|}}{\sigma(\tau)} F(\tau)(y), \tag{4}$$

where $\mathcal{T}$ is the set of rooted trees, $|\tau|$ is the order (number of vertices), $\sigma(\tau)$ is a symmetry factor ($\sigma(\tau_1) = 1$, $\sigma(\tau_2) = 2$, $\sigma(\tau_{3a}) = 3$, $\sigma(\tau_{3b}) = 6$, $\dots$), and $F(\tau)$ is the elementary differential (Figures 3–4).

3.2 Characters of the Hopf algebra

The Connes–Kreimer Hopf algebra $\mathcal{H}_{\mathrm{CK}}$ is the commutative polynomial algebra generated by rooted trees, with coproduct Δ defined by admissible cuts (Section 4).

Definition 3.1. A **character** of $\mathcal{H}_{\mathrm{CK}}$ is an algebra homomorphism $\phi : \mathcal{H}_{\mathrm{CK}} \to \mathbb{R}$. It assigns a coefficient $\phi(\tau) \in \mathbb{R}$ to each tree. The set of all characters forms a group $G(\mathcal{H}_{\mathrm{CK}})$ under the convolution product $\star$ (the **Butcher group**).

Definition 3.2. The **exact-flow character** is $\phi_{\mathrm{exact}}(\tau) := 1/\sigma(\tau)$.

Theorem 3.3 (Butcher 1972)**.** *A Runge–Kutta method with Butcher tableau* (A, b, c) *defines a character* ϕ_{RK} *via* $\phi_{\mathrm{RK}}(\tau) = \Phi_\tau(A, b, c)$ *(a polynomial in the tableau entries). The method has order* p *if and only if*

$$\phi_{\mathrm{RK}}(\tau) = \phi_{\mathrm{exact}}(\tau) \quad \textit{for all } |\tau| \le p. \tag{5}$$

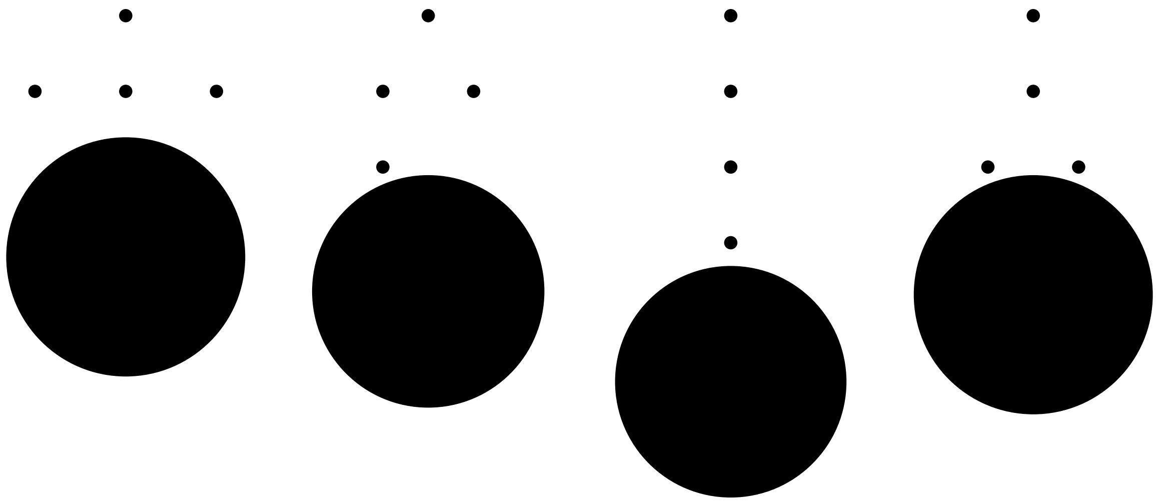


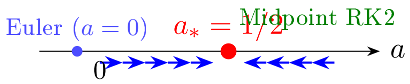
Figure 4: The four order-4 Butcher trees. Each has 3 edges = 3 counterterm subtractions. The topology determines *how* the subtractions are composed: all-parallel (τ_{4a}), mixed (τ_{4b}, τ_{4d}), or all-serial (τ_{4c}).

Remark 3.4 (The exact flow is "fully renormalised")**.** The exact-flow character ϕ_{exact} is the unique element of $G(\mathcal{H}_{\text{CK}})$ that gets *every* counterterm subtraction right at *every* order. A numerical method of order p gets them right up to order p; the discrepancy $\delta_\tau := \phi_{\text{RK}}(\tau) - 1/\sigma(\tau)$ at order $p+1$ is the **counterterm** needed to correct the method.

3.3 The control map as a counterterm

The main paper's Derivation D6.2a defines the step-halving comparison: compose two half-steps $\Phi_{h/2} \circ \Phi_{h/2}$ and compare with Φ_h. In the family $\Phi_h^{(a)}(y) = y + hf(y) + ah^2 f'(y)[f(y)] + O(h^3)$, this gives

$$\Phi_{h/2}^{(a)} \circ \Phi_{h/2}^{(a)} = \Phi_h^{(\tau_2(a))} + O(h^3), \qquad \tau_2(a) = \frac{a}{2} + \frac{1}{4}. \tag{6}$$



The beta function $\beta(a) = 1/2 - a$ drives any initial a to the fixed point $a_* = 1/2$ (the correct order-2 coefficient of the exact flow).

Figure 5: The control map $\tau_2(a) = a/2 + 1/4$ on the parameter space of order-2 coefficients. This is the counterterm for the single tree τ_2 = (chain of 2). The fixed point $a_* = 1/2$ is the exact-flow value. Repeated halving drives any initial method toward the exact flow.

In the Hopf-algebraic language: $\tau_2(a)$ is the $|\tau| = 2$ component of the Butcher group product $\phi_{h/2} \star \phi_{h/2}$ projected onto the one-parameter family. The full tree-level comparison would give a control map $\tau_b^{(\tau)}$ for *each* tree τ, with the semigroup property $\tau_b^{(\tau)} \circ \tau_c^{(\tau)} = \tau_{bc}^{(\tau)}$ inherited from the group law of $G(\mathcal{H}_{\text{CK}})$.

4 The coproduct: same structure for ODEs and QFT

4.1 Admissible cuts

The coproduct $\Delta : \mathcal{H}_{\text{CK}} \to \mathcal{H}_{\text{CK}} \otimes \mathcal{H}_{\text{CK}}$ is defined by **admissible cuts**: remove a subset of edges from the tree such that at most one edge on each root-to-leaf path is cut. The pieces below the cuts form the "pruned part" $P^c(\tau)$ (a forest of subtrees), and the piece containing the root is the "trunk" $R^c(\tau)$.

$$\Delta(\tau) = \tau \otimes \mathbf{1} + \mathbf{1} \otimes \tau + \sum_{\text{admissible cuts } c} P^c(\tau) \otimes R^c(\tau). \tag{7}$$

4.2 Worked example: $\Delta(\tau_2)$

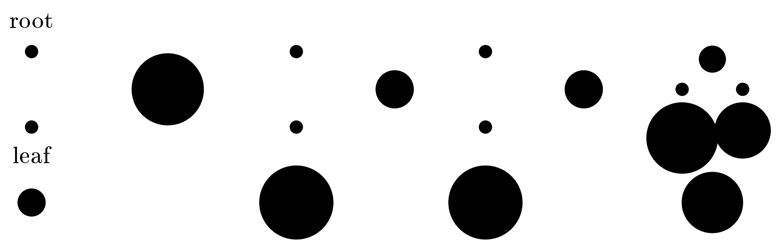


Figure 6: The coproduct of the chain tree τ_2. There is one non-trivial admissible cut (cutting the single edge), which separates the leaf (pruned: $\bullet = f$) from the root (trunk: $\bullet = f'[\cdot]$ waiting for input).

4.3 Worked example: $\Delta(\tau_{3b})$

The chain of three has two edges and hence more admissible cuts:

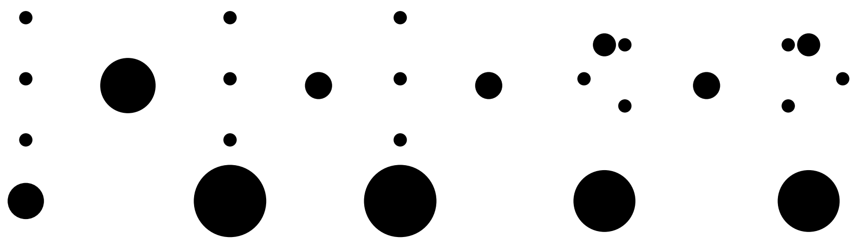


Figure 7: The coproduct of the chain τ_{3b}. Two admissible cuts yield $\bullet \otimes \tau_2$ (cut bottom edge) and $\tau_2 \otimes \bullet$ (cut top edge). Cutting both edges simultaneously is *not* admissible (it would cut two edges on the same root-to-leaf path).

4.4 The dual interpretation

ODE / Runge–Kutta	**QFT / Renormalisation**
Admissible cut separates a sub-pattern of derivatives	Admissible cut separates a divergent subdiagram
Pruned part = inner composition step	Pruned part = subdivergence
Trunk = outer composition step	Trunk = reduced diagram
$(\phi \star \psi)(\tau) = \sum \phi(P^c)\psi(R^c)$: compose two flows	Same formula: compose renormalisation maps
Order conditions: $\phi_{\mathrm{RK}}(\tau) = 1/\sigma(\tau)$	Renormalisation conditions: $\phi_+(\tau) =$ finite part

This is the content of Brouder's theorem (1999): the Butcher group and the Connes–Kreimer renormalisation group are the **same group** $G(\mathcal{H}_{\mathrm{CK}})$.

5 From one character to all: why $\hbar$ enters

5.1 The classical situation

Classically, the flow Φ_t is deterministic. Given y_0, there is one trajectory $y(t) = \Phi_t(y_0)$. The B-series (4) computes this single trajectory. All trees contribute, but to a **single** character ϕ_{exact}.

5.2 The compositional question

Now ask: can the flow be represented as a smooth **transition kernel** $K(x, y; t)$ satisfying

$$K(x, y;\, t_1{+}t_2) = \int K(x, z;\, t_1)\, K(z, y;\, t_2)\, dz \tag{8}$$

with the identity limit $K(x, y; t) \to \delta(x{-}y)$ as $t \to 0^+$?

The classical answer $K_{\mathrm{cl}}(x, y; t) = \delta(y - \Phi_t(x))$ satisfies (8) but is not smooth.

5.3 The smooth obstruction (P4.2)

If we require K to be a smooth function (not a distribution), then the main paper's Proposition P4.2 shows:

(a) Composition closure on smooth kernels forces the normalisation $K \propto t^{-d/2}$.

(b) The identity limit forces an exponential weight $K \propto \exp\bigl(-Q(x, y)/(\kappa t)\bigr)$ with $[\kappa] =$ [action].

(c) Setting $\kappa \to 0$ concentrates K on $\delta(y - \Phi_t(x))$, but the identity limit is lost for $t > 0$ (the flow Φ_t is not the identity for $t > 0$).

(d) Therefore $\kappa = \hbar > 0$ is forced.

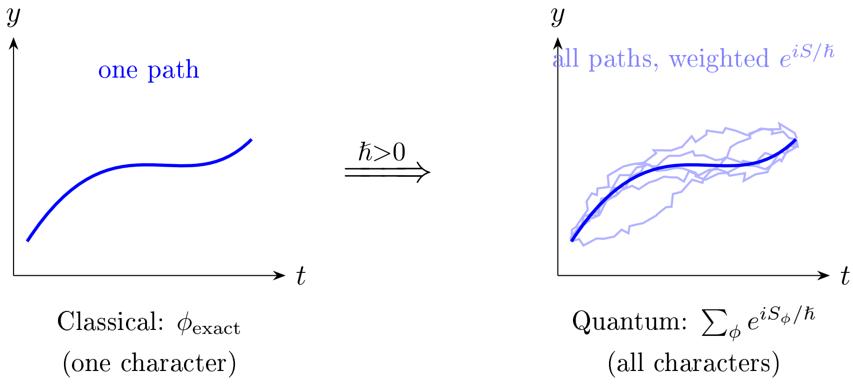


Figure 8: Classical mechanics uses one character (the extremal path). Quantum mechanics sums over all characters, weighted by $e^{iS/\hbar}$. The classical limit $\hbar \to 0$ projects back to the dominant (stationary-phase) character. Composition (8) forces $\hbar > 0$.

5.4 The tree-level reading

In the perturbative expansion, the quantum propagator decomposes as

$$K(x,y;t) \sim K_0(x,y;t)\left(1+\sum_{n=1}^{\infty}\hbar^n \sum_{|\tau|=n+1} c_\tau\, F(\tau)(x,y)\right), \tag{9}$$

where K_0 is the free (Gaussian) kernel and the c_τ are combinatorial coefficients. The trees that appear are the *same* Butcher trees, but now they index the perturbative corrections to the propagator rather than the Taylor corrections to the ODE flow.

	Classical (ODE)	**Quantum (path integral)**
Character	one: ϕ_{exact}	all: $\sum_\phi e^{iS_\phi/\hbar}$
Expansion parameter	h (step size)	$\hbar$ (action scale)
Trees index	Taylor corrections	perturbative corrections
$\hbar \to 0$ limit	—	stationary phase $\to \phi_{\text{exact}}$
Lipschitz condition	$\lVert f' \rVert \le L_0$ (well-posedness)	$V \in \mathcal{K}_d$ (Kato class)

6 The Cornerstone Proposition

Proposition 6.1 (Cornerstone)**.** *Let $\mathcal{H}_{\text{CK}}$ be the Connes–Kreimer Hopf algebra of rooted trees and $G(\mathcal{H}_{\text{CK}})$ its group of characters (the Butcher group).*

(i) ***Trees = organised subtractions.*** *The exact flow Φ_t of $\dot y = f(y)$ defines a character $\phi_{\text{exact}} \in G(\mathcal{H}_{\text{CK}})$ via $\phi_{\text{exact}}(\tau) = 1/\sigma(\tau)$. Each edge of τ encodes one derivative of f, which is one counterterm subtraction (D6.3). A Runge–Kutta method of order p is a character ϕ_{RK} agreeing with ϕ_{exact} on $|\tau| \le p$; the discrepancy δ_τ at order $p+1$ is a counterterm.*

(ii) ***Composition = Hopf product.*** *The composition $\Phi_s \circ \Phi_t$ corresponds to the convolution product $\phi_s \star \phi_t$ in $G(\mathcal{H}_{\text{CK}})$, defined via the coproduct Δ. The control map τ_b of D6.2a is the $|\tau| = 2$ truncation. The semigroup $\tau_b \circ \tau_c = \tau_{bc}$ is a shadow of the group law of $G(\mathcal{H}_{\text{CK}})$.*

(iii) ***Composition forces $\hbar$.*** *When the transition kernel $K(x,y;t)$ is required to be smooth, compose via (8), and satisfy $K \to \delta$ as $t \to 0^+$, then K must have the form $K \propto$*

$t^{-d/2}\exp(-Q/(\kappa t))$ *with* $\kappa = \hbar > 0$ *(P4.2). The smooth kernel is parametrised by a* family *of characters weighted by* $\exp(iS[path]/\hbar)$.

(iv) ***Renormalisation = the same subtraction, extended.*** *When the weighted sum over characters diverges, renormalisation is the Birkhoff decomposition* $\phi = \phi_-^{-1} \star \phi_+$ *in* $G(\mathcal{H}_{\mathrm{CK}})$ *— using the* same *coproduct* Δ *that organised the ODE order conditions. The counterterm* ϕ_- *subtracts divergent tree contributions, exactly as* δ_τ *corrected the numerical method.*

Remark 6.2 (Status)**.** Parts (i)–(ii) are theorems (Butcher 1972, Connes–Kreimer 1998, Brouder 1999). Part (iii) is conditional on P4.2 of the main paper. Part (iv) as stated is an interpretation: the algebraic isomorphism is proved, but the physical identification (which B-series tree = which Feynman diagram) is model-dependent. See Section 7 for a precise accounting.

6.1 The cornerstone diagram

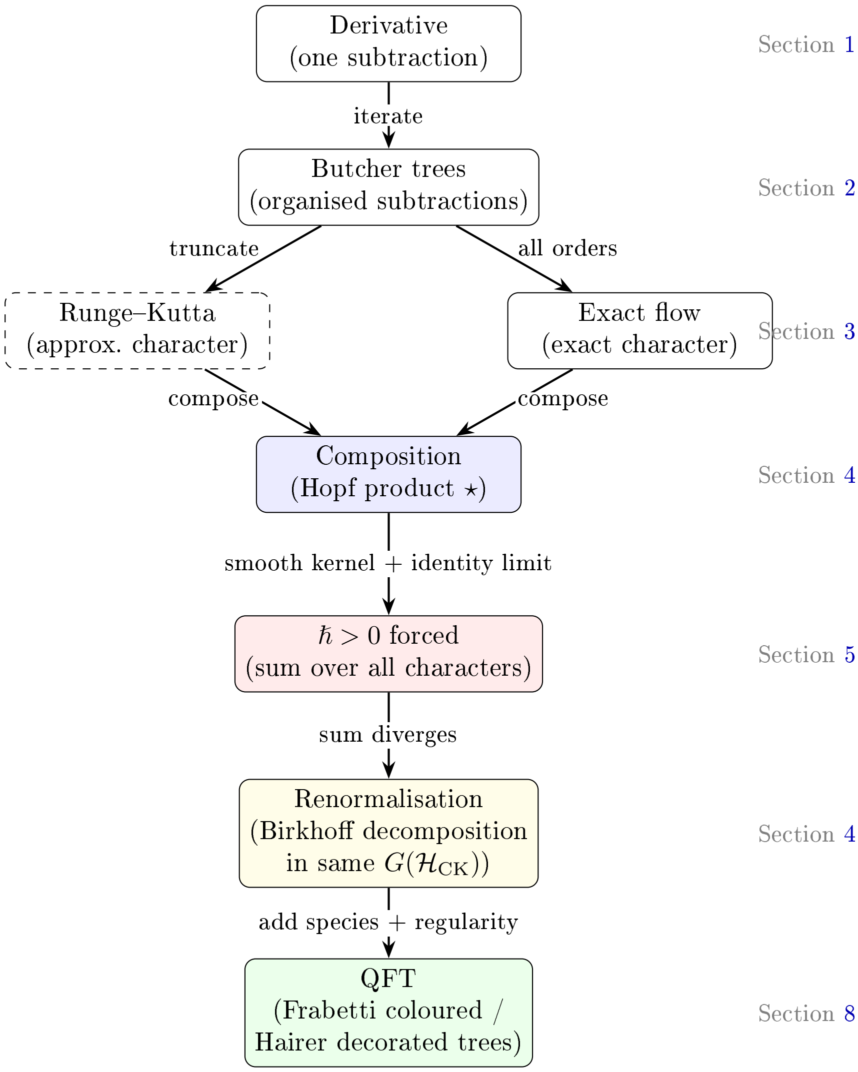


Figure 9: The cornerstone diagram. The entire chain — from the derivative as a single subtraction to QFT renormalisation — is organised by the same Hopf algebra $\mathcal{H}_{\mathrm{CK}}$. Each downward arrow adds compositional complexity; the algebraic structure (coproduct, characters, group law) remains the same throughout.

7 Honest gaps and open problems

7.1 The tree decoration problem

In the ODE/B-series setting, trees are decorated by elementary differentials $F(\tau)$ — patterns of derivatives of the vector field f. In the QFT/Feynman setting, trees are decorated by residues of Feynman diagrams — patterns of propagators and vertices.

Brouder's theorem establishes that the *undecorated* Hopf algebras are isomorphic. But the identification "which ODE tree $\leftrightarrow$ which Feynman diagram" depends on the specific theory.

Example 7.1 (Where the identification is explicit)**.** For the 1D anharmonic oscillator $\ddot{x} + x + \lambda x^3 = 0$, written as a system $\dot{y} = f(y)$ with $f(x, v) = (v, -x - \lambda x^3)$:

- The B-series trees of f map directly to the Feynman diagrams of ϕ^4 in 0+1 dimensions.
- The $|\tau| = 2$ tree gives the one-loop correction.
- The $|\tau| = 3$ trees give two-loop corrections.

In this case, the identification is canonical.

Honest Gap (General theories). For gauge theories (QED, QCD), the tree structure is more complex: coloured/planar trees (Frabetti), with non-commutative Hopf algebra. The identification with B-series is indirect.

7.2 Path roughness

Honest Gap (Non-perturbative paths). The path integral sums over Wiener paths, which are Hölder-1/2 (not C^∞). A generic path does not have a B-series expansion. The tree-algebraic description applies to the **perturbative** expansion of the propagator (loop expansion in powers of $\hbar$ or couplings), not to individual paths.

The honest statement is therefore:

> The perturbative quantum propagator and the perturbative ODE solution (B-series) are organised by the same Hopf algebra. Quantisation, at the perturbative level, is the passage from one character to a sum over characters.

Non-perturbative content (tunnelling, instantons, resurgence) lies beyond the tree algebra. The main paper's Remark H6.3 discusses partial recovery via Borel resummation.

7.3 The "quantisation = extending the domain" claim

A possible precise formulation:

> Quantisation is a functor from $G(\mathcal{H}_{\mathrm{CK}})$ (the Butcher group of single-character flows) to $\mathrm{Meas}(G(\mathcal{H}_{\mathrm{CK}}))$ (measures on the Butcher group, i.e., weighted sums of characters). The parameter $\hbar$ controls the concentration of these measures: $\hbar \to 0$ gives a delta-measure on ϕ_{exact}; finite $\hbar$ gives a spread measure.

This formulation has not been proved. It would require:

(a) A rigorous definition of "measure on $G(\mathcal{H}_{\mathrm{CK}})$" (the Butcher group is an infinite-dimensional pro-unipotent group).

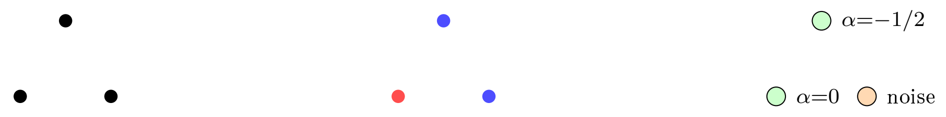
(b) A proof that the path-integral measure, restricted to its perturbative sector, defines such a measure.

(c) A characterisation of $\hbar$ as the variance parameter.

8 Extensions: coloured and decorated trees

8.1 Frabetti coloured trees (2000–2008)

For QED, which has two propagator types (electron, photon), Brouder and Frabetti constructed a **non-commutative** Hopf algebra on **planar binary trees**, with one tree species per propagator type.



Uncoloured (scalar ϕ^3)

Coloured (electron, photon)

Decorated (regularity structure)

Figure 10: Three levels of tree structure. **Left:** Butcher/Connes–Kreimer uncoloured trees (commutative Hopf algebra, scalar ODE/QFT). **Centre:** Frabetti coloured planar trees (non-commutative, gauge theories with multiple propagator types). **Right:** Hairer decorated trees (regularity structures, singular SPDEs; vertices carry regularity exponents, edges may carry noise type).

In the ODE analogy, coloured trees correspond to systems with multiple coupling types: $\dot{y}_i = f_i(y_1, \ldots, y_n)$ where the f_i depend on the y_j in structurally different ways.

Status: the algebraic structure is proved. The physical payoff was limited: Frabetti confirmed that Connes–Kreimer extends to gauge theories but did not simplify calculations.

8.2 Hairer decorated trees (2014)

For singular SPDEs (KPZ equation, Φ^4_3), Hairer's regularity structures use trees decorated with:

- **Regularity exponents** at vertices (how singular the distribution is).
- **Noise types** on certain edges (stochastic forcing terms).

Bruned–Hairer–Zambotti (2019) proved that Hairer's BPHZ-type renormalisation of regularity structures *is* a Hopf-algebraic renormalisation on decorated trees, with the same structural coproduct as Connes–Kreimer.

In the cornerstone language:

- **Butcher:** each vertex = one smooth derivative (counterterm subtraction of a smooth function).
- **Hairer:** each vertex = one distributional operation (counterterm subtraction of a distribution, which may not exist without renormalisation).

8.3 The three-level hierarchy

9 Implications for the main paper

9.1 The missing section

The main paper's Section 8 presents D6.2a (control map) and H6.2 (rooted-tree bookkeeping) as an *analogy* between Runge–Kutta and renormalisation. The cornerstone says it is an *identity*.

A new subsection (8.7 or equivalent) should state:

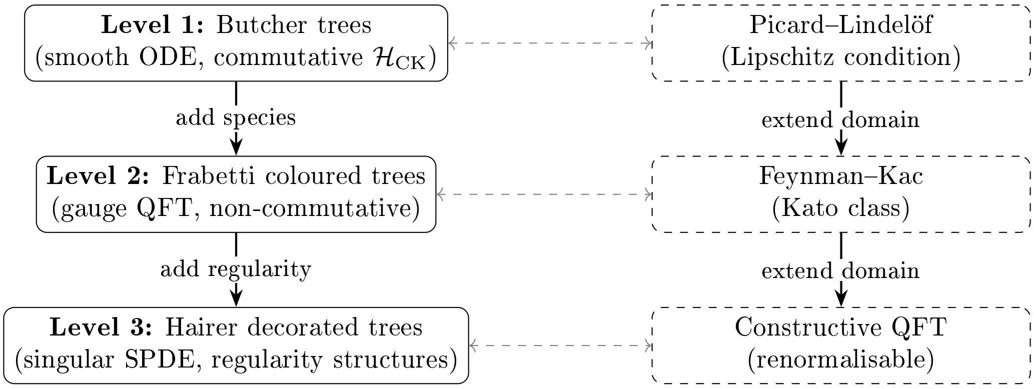


Figure 11: The three-level tree hierarchy (left) mirrors the paper's well-posedness hierarchy (right). At each level, the tree algebra becomes richer (coloured, then decorated) and the domain of well-posedness extends (Lipschitz $\subset$ Kato $\subset$ renormalisable).

1. The exact flow is the fully renormalised character of $\mathcal{H}_{\mathrm{CK}}$.
2. Numerical methods are approximate characters; order conditions = tree-by-tree matching.
3. The control map τ_b is the counterterm at $|\tau| = 2$.
4. Quantisation extends from one character to all characters.
5. $\hbar > 0$ is forced by composition + identity (P4.2).
6. Renormalisation handles divergences using the same coproduct.

9.2 What it would prove

If made fully precise (resolving the gaps of Section 7), this would establish:

> **Quantisation is the compositional completion of the counterterm-subtraction algebra.**

Classical mechanics uses one character. Quantum mechanics uses all characters. The passage is forced by requiring smooth composition with identity. Renormalisation controls the extension. Derivative, quantisation, and renormalisation are three instances of the same algebraic operation (counterterm subtraction organised by rooted trees) at increasing levels of compositional complexity.

This is the main paper's thesis — refinement-compatibility forces the chain Newton → quantum → QFT — stated in its most algebraic form.

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