Planck Area from Half-Density Normalization (Draft)

Abstract

Half-densities are the natural "coordinate-free integrands" for composing kernels without choosing a background measure. But choosing a *universal* convention for turning half-density objects into dimensionless numerical amplitudes introduces a length$^{d/2}$ scale. In $d = 4$, this is an *area*. This note sharpens the hypothesis ladder needed for the claim "half-density normalization selects a universal area scale", and isolates a simple dimension-matching condition under which the Planck area appears without fractional powers of couplings. A gravitational anchor based on a minimal-areal-speed principle is recorded as a separate heuristic thread [RiveroAreal] [RiveroSimple].

1. Purpose and Status

This is a dependent follow-up to `paper/main.md`. It is not yet a finished paper; its goal is to isolate one technical point that is only implicit in the main manuscript: the role of half-densities (and their scaling) in making composition laws coordinate-invariant *and* dimensionally well-defined.

Claims below are labeled as `Proposition` (math-precise under hypotheses) or `Heuristic` (programmatic bridge).

2. Half-Densities and Composition Kernels

Let M be a d-dimensional manifold. A (positive) density is a section of $|\Lambda^d T^*M|$, and a half-density is a section of $|\Lambda^d T^*M|^{1/2}$.

The key operational point is: when a kernel is a half-density in its integration variable, composition of kernels does not depend on an arbitrary choice of coordinate measure.

`Heuristic H1.1 (Why half-densities).` If $K_1(x,y)$ and $K_2(y,z)$ are chosen so that their product in the intermediate variable y is a density, then $\int_M K_1(x,y)K_2(y,z)$ is coordinate-invariant without fixing a preferred dy. This matches the structural role of kernel composition used in `paper/main.md` (Section 6).

`Derivation D1.1 (Coordinate invariance of half-density pairing and composition).` In a local chart $y = (y^1, \ldots, y^d)$, write a half-density as $\psi(y) = \varphi(y)\,|dy|^{1/2}$. Under a change of variables $y = y(y')$, one has $|dy|^{1/2} = |\det(\partial y/\partial y')|^{1/2}|dy'|^{1/2}$, so the coefficient transforms as $\varphi'(y') = \varphi(y(y'))\,|\det(\partial y/\partial y')|^{1/2}$.

Hence the product of two half-densities is a density: $\psi_1\psi_2 = (\varphi_1\varphi_2)\,|dy|$, and its integral is chart-independent: $\int_M \psi_1\psi_2$ is well-defined without choosing a background measure beyond the density bundle itself.

Kernel composition is the same mechanism: if $K_1(x, y)$ and $K_2(y, z)$ are half-densities in y, then K_1K_2 is a density in y and $\int_M K_1K_2$ is coordinate invariant.

3. Dimensional Analysis: Normalizing a Half-Density Requires a Scale

A density on M carries the units of length^d once physical units are assigned to coordinates. A half-density therefore carries units length$^{d/2}$.

`Proposition P1.1 (No canonical "half-density = function" identification).` There is no canonical identification of a half-density $\psi \in |\Lambda^d T^* M|^{1/2}$ with an ordinary scalar function f on M. Choosing such an identification is equivalent to choosing a nowhere-vanishing reference half-density σ_* (equivalently a positive density $\rho_* = \sigma_*^2$) and writing $\psi = f\,\sigma_*$.

`Derivation D1.2 (Dilation makes the \(\text{length}^{d/2}\) weight explicit).` On $\mathbb{R}^d$, consider a dilation $y \mapsto y' = ay$ with $a > 0$. Then $|dy'| = a^d|dy|$, so $|dy'|^{1/2} = a^{d/2}|dy|^{1/2}$. Thus even in flat space, half-densities carry an inherent length$^{d/2}$ scaling weight.

`Proposition P1.2 (Universal *dimensionless* amplitudes force a \(\text{length}^{d/2}\) constant).` If one imposes the extra requirement that the scalar representative f in $\psi = f\,\sigma_*$ be dimensionless in physical units, then the reference half-density σ_* must carry all of the length$^{d/2}$ dimension. In particular, a *constant* (field-independent) choice of σ_* is equivalent to choosing a universal length$^{d/2}$ scale.

In $d = 4$, this universal length$^{d/2}$ scale is a universal *area* scale.

`Heuristic H1.2 (Reciprocity claim).` Half-densities alone do not force a particular scale: the forced fact is that converting half-density objects into scalar numerical amplitudes requires extra structure (a reference half-density). The "universal area scale" claim begins only after adding two further hypotheses: 1. the reference σ_* is taken to be *constant* (no dependence on background metric/fields), and 2. the constant is required to be fixed by universal constants/couplings of the theory.

Under these hypotheses, $d = 4$ is the unique dimension in which the needed length$^{d/2}$ constant can be supplied by the gravitational coupling without fractional powers (Derivation D1.3).

`Derivation D1.3 (Gravity-only sieve: why \(d=4\) is singled out if only \(G_d\) is used).` In d spacetime dimensions, the Einstein–Hilbert action $\frac{1}{16\pi G_d}\int d^dx\,\sqrt{|g|}\,R$ shows that (in $c = \hbar = 1$ units) Newton's constant has dimension $[G_d] = \text{length}^{d-2}$. Assume the only available dimensionful coupling used to build the universal normalization constant is G_d itself (no cosmological constant, no additional dimensionful scales), and impose H2.5

in the literal "no fractional powers of G_d" sense. Then the normalization constant has dimension length$^{k(d-2)}$ for some integer k. Matching length$^{d/2}$ forces length$^{d/2}$ = length^{d-2}, which holds if and only if $d = 4$. In that case G_4 itself has dimension of area, and the corresponding area scale is the Planck area $L_P^2 \sim \hbar G_4/c^3$.

3.1 Hypotheses as Separate Knobs (What Is Forced vs Chosen)

The discussion above mixes three different kinds of statements: 1. **Geometric facts** (what half-densities are, how they compose, how they scale), 2. **Representational choices** (how one turns half-density objects into scalar numbers), 3. **Universality/selection principles** (what choices are allowed if we demand "background-free" and "built from couplings").

To study these separately, it is useful to keep the hypotheses explicit.

`Hypothesis H2.1 (Half-density formulation).` Quantum kernels are treated as bi-half-densities so that composition in intermediate variables is coordinate invariant (Section 2 and Derivation D1.4).

`Hypothesis H2.2 (Scalarization by a reference half-density).` To interpret half-density amplitudes as scalar numerical functions, we pick a nowhere-vanishing reference half-density σ_* and write $\psi = f\,\sigma_*$ (Proposition P1.1).

`Hypothesis H2.3 (Dimensionless scalar representative).` The scalar representative f is required to be dimensionless in physical units (Proposition P1.2). This forces σ_* to carry the full length$^{d/2}$ weight.

`Hypothesis H2.4 (Background-free constancy).` The reference σ_* is taken to be constant/field-independent, rather than determined by background geometry (e.g. a Riemannian volume $|g|^{1/4}|dx|^{1/2}$) or by dynamical fields (e.g. a dilaton-like factor). This is the first point where a *universal constant* enters.

`Hypothesis H2.5 (Analyticity / no fractional powers of couplings).` If the universal constant is required to be built from the theory's couplings without fractional powers, then dimensional analysis becomes a *dimension sieve* rather than a tautology. This hypothesis has at least two distinct readings: 1. **Integrality (Diophantine) reading:** the constant is a monomial in the available couplings with integer exponents (possibly allowing negative powers). 2. **Perturbative analyticity reading (stronger):** the constant admits a Taylor expansion around zero couplings, so only nonnegative integer powers appear.

Derivation D1.3 is the simplest gravity-only instance under the integrality reading: "use G_d without fractional powers" singles out $d = 4$.

`Heuristic H2.6 (Where "special dimensions" can appear).` Special dimensions do not come from half-densities alone (Hypothesis H2.1). They appear only after adding a selection principle like H2.4–H2.5: the requirement that the scalarization choice be universal, background-free, and coupling-built in a restricted (e.g. analytic) way.

3.2 What Changes When a Hypothesis Is Relaxed?

This subsection records the main "branches" that need separate study.

1. **Drop H2.3 (allow dimensionful f).** Then no universal length$^{d/2}$ constant is forced; the dimensional weight can be carried by the scalar representative itself (as in the usual statement "wavefunctions have dimension length$^{-d/2}$").
2. **Drop H2.4 (allow background geometry).** Then σ_* can be chosen from a metric (or other structure), and the "universal constant" is replaced by background-dependent normalization.
3. **Drop H2.5 (allow fractional powers).** Then in any $d > 2$ one can build a length$^{d/2}$ constant from gravity via $G_d^{d/(2(d-2))}$ (in $c = \hbar = 1$ units), so $d = 4$ is no longer singled out; instead, $d = 4$ is simply the unique case where the exponent is an integer.
4. **Change "which coupling supplies the scale".** Using other dimensionful couplings (cosmological constant, string tension, gauge couplings in various dimensions, etc.) yields different "special-dimension" sieves. This is conceptually aligned with the observation that some dimensions are singled out by other structures (division algebras, special holonomy, supersymmetry), but those filters are separate from the half-density story and should not be conflated.

3.3 Starting with H2.5: Integrality as a Dimension Sieve

The point of H2.5 is not that dimensional analysis alone selects a unique scale (it does not), but that *restricting allowed functional dependence on couplings* can turn dimensional analysis into a selection principle.

`Derivation D1.6 (Diophantine form of "no fractional powers").`
Work in $c = \hbar = 1$ units for dimension counting. Let the available couplings $\{g_i\}$ have length dimensions $[g_i] = \text{length}^{a_i}$. Under the integrality reading of H2.5, the universal normalization constant is a monomial $C = \prod_i g_i^{n_i}$ with integers n_i. Its length dimension is $[C] = \text{length}^{\sum_i n_i a_i}$. Requiring $[C] = \text{length}^{d/2}$ is therefore the Diophantine condition

$$\sum_i n_i a_i = \frac{d}{2}.$$

Existence (and non-uniqueness) of solutions depends on: 1. which couplings are admitted as "universal" inputs, and 2. whether one allows negative exponents (non-analytic at zero coupling) or insists on perturbative analyticity (nonnegative exponents).

`Heuristic H2.7 (Why H2.5 needs a "what counts as a coupling" rule).` If one allows arbitrary redefinitions of couplings (e.g. adjoining a new symbol $\tilde{G} = G_d^{1/(d-2)}$), then "no fractional powers" becomes vacuous: the forbidden root has simply been renamed as an allowed coupling. H2.5 is meaningful only together with a prior criterion for admissible coupling dependence (e.g. perturbative analyticity around a distinguished limit such as $G_d \to 0$).

`Example E1 (Gravity-only).` With only G_d available, $a_1 = d - 2$ and the condition becomes $n(d-2) = d/2$. For integer $d \geq 3$, this has a solution only at $d = 4$ with $n = 1$, reproducing Derivation D1.3.

`Example E2 (Gravity + cosmological constant).` If one also allows the cosmological constant Λ_d with $[\Lambda_d] = \text{length}^{-2}$, then the condition becomes $n(d-2) - 2m = d/2$ for integers n, m. A simple family of solutions exists for d divisible by 4: take $n = 1$ and $m = d/4 - 1$, so

$$C \sim G_d \, \Lambda_d^{d/4-1},$$

has dimension length$^{d/2}$. Thus, even under H2.5, $d = 4$ is not automatically unique once additional dimensionful couplings are admitted; what is special about $d = 4$ in this family is that it is the only case with $m = 0$ (no need to involve Λ_d).

`Heuristic H2.8 (What H2.5 is really buying).` The value of H2.5 is comparative: it distinguishes dimensions in which the needed length$^{d/2}$ factor can be supplied by *simple* coupling dependence (integer powers of the already-present couplings), versus dimensions in which any such factor requires either (i) introducing extra scales/couplings, (ii) taking fractional powers, or (iii) invoking non-analytic mechanisms (dimensional transmutation).

4. Stationary Phase Produces Half-Density Prefactors (Short-Time Kernel)

The main manuscript uses stationary phase to explain why classical extremals dominate refinement limits. Here we add the complementary kernel-level fact: stationary phase does not only pick the extremal; it also produces a determinant prefactor that transforms as a half-density, i.e. the object needed for coordinate-free kernel composition.

`Derivation D1.4 (Van Vleck prefactor is a bi-half-density).` Let $S_{\text{cl}}(x, z; t)$ be the classical action as a function of endpoints and time, treated as a generating function. The standard short-time/stationary-phase approximation to the propagator has the form

$$K(x, z; t) \approx \frac{1}{(2\pi i\hbar)^{d/2}} \left| \det\left(-\frac{\partial^2 S_{\text{cl}}}{\partial x \, \partial z} \right) \right|^{1/2} \exp\left(\frac{i}{\hbar} S_{\text{cl}}(x, z; t) \right).$$

Under a change of coordinates $x = x(x')$, $z = z(z')$, the mixed Hessian transforms by the chain rule, and its determinant acquires Jacobian factors:

$$\det\left(-\frac{\partial^2 S_{\mathrm{cl}}}{\partial x'\,\partial z'}\right) = \det\left(\frac{\partial x}{\partial x'}\right)\det\left(\frac{\partial z}{\partial z'}\right)\det\left(-\frac{\partial^2 S_{\mathrm{cl}}}{\partial x\,\partial z}\right).$$

Taking square roots shows that the prefactor transforms with $|\det(\partial x/\partial x')|^{1/2}|\det(\partial z/\partial z')|^{1/2}$, i.e. exactly as a half-density factor at each endpoint. Thus the stationary-phase prefactor is naturally interpreted as making K a half-density in each variable, so that kernel composition does not depend on a background measure choice.

`Heuristic H1.4 (Where Planck area can enter, minimally).` Derivation D1.3 isolates one minimal route by which a Planck-scale quantity can enter: if the theory supplies a single universal coupling with dimension of length (Newton's constant) and one demands that the half-density normalization constant be built from that coupling *without fractional powers*, then $d = 4$ is singled out and the resulting constant has the dimension of an area, naturally identified with the Planck area $L_P^2 \sim \hbar G_4/c^3$.

5. A Gravitational Anchor: Minimal Areal Speed and the $D = 4$ Cancellation

Rivero's "Planck areal speed" observation gives a concrete route by which Planck-scale discreteness reappears at Compton scales in inverse-square gravity [RiveroAreal] [RiveroSimple].

`Heuristic H1.3 (Areal-speed selection).` In $3 + 1$ Newtonian gravity (inverse-square), imposing a discrete areal-speed/area-time condition at a Planck scale can yield characteristic radii proportional to a reduced Compton length, with Newton's constant canceling when expressed in Planck units. This is a nontrivial indication that "a universal area scale" can be operationally meaningful at low energies in $D = 4$.

`Derivation D1.5 (Inverse-square circular orbit + Planck areal speed \(\Rightarrow\) Compton radius).` For a circular orbit under an inverse-square central force $F(r) = K/r^2$ (with coupling $K > 0$), the centripetal balance is $mv^2/r = K/r^2$. The areal speed is $\dot{A} = \frac{1}{2}rv$, so $v = 2\dot{A}/r$. Substituting into the force balance gives

$$m\left(\frac{2\dot{A}}{r}\right)^2 = \frac{K}{r} \quad\Longrightarrow\quad r = \frac{4m\,\dot{A}^2}{K}.$$

For Newtonian gravity between a source mass M and test mass m, $K = GMm$, hence

$$r = \frac{4\dot{A}^2}{GM},$$

independent of the test mass m. If one now imposes $\dot{A} = k\,\dot{A}_P$, where Rivero's Planck areal speed is $\dot{A}_P = cL_P$ [RiveroAreal], then using $L_P^2 = G\hbar/c^3$ yields

$$r = \frac{4k^2(cL_P)^2}{GM} = \frac{4k^2(G\hbar/c)}{GM} = 4k^2\,\frac{\hbar}{cM}.$$

Thus r becomes a multiple of the reduced Compton length $L_M = \hbar/(cM)$, with Newton's constant canceled out. In particular, $k = \frac{1}{2}$ gives $r = L_M$. This is the "Planck area per Planck time $\Rightarrow$ Compton scale" cancellation highlighted in [RiveroAreal] and summarized in [RiveroSimple].

6. Interface with the Main Paper

The main manuscript argues that: 1. classical dynamics are recovered from quantum composition by stationary-phase concentration, and 2. refinement across scales forces RG-style consistency conditions when naive limits diverge.

This draft adds a complementary ingredient: the kernel side is most naturally formulated in half-density language, and stationary phase produces the bi-half-density prefactor directly. A universal convention for turning those half-densities into scalar amplitudes then requires a length$^{d/2}$ scale; in $d = 4$ this is an area scale.

7. Open Problems (Needed for a Real Paper)

1. Make the half-density normalization argument precise for a concrete groupoid or kernel model (tangent-groupoid or short-time propagator model).
2. Show how the area scale enters stationary-phase prefactors and how this interacts with RG scaling.
3. General-dimension analysis: clarify what replaces "area" in odd dimensions and whether a universal normalization is still defensible.
4. Identify minimal hypotheses under which "need of half-density scale $\Rightarrow$ Planck area" is more than dimensional bookkeeping.