1) Determine $\hat{K}$, ω_n, ζ from procedure 2.1.4

$$\lim_{t\to\infty} y(t) = y_{ss}(t) = \boxed{\hat{K} \approx 0.557}$$

$$\%\,O.S. = 100\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \approx 7.31\,\%$$

$$\Rightarrow \frac{-\zeta^2\pi^2}{1-\zeta^2} = 2\ln\left(\frac{\%\,O.S.}{100}\right)$$

$$\Rightarrow -\zeta^2\pi^2 = 2\ln(\|\%\,O.S.\|) - 2\zeta^2\ln(\|\%\,O.S.\|)$$

$$\Rightarrow -\zeta^2\left(\pi^2 - 2\ln(\|\%\,O.S.\|)\right) = 2\ln(\|\%\,O.S.\|)$$

$$\Rightarrow \boxed{\zeta} = \sqrt{\frac{-2\ln(\|\%\,O.S.\|)}{\pi^2 - 2\ln(\|\%\,O.S.\|)}} \; \boxed{\approx 0.589}$$

$$T_{2\%} \approx \frac{4}{\zeta\omega_n} \approx 0.227\ s$$

$$\Rightarrow \boxed{\omega_n} \approx \frac{4}{\zeta T_{2\%}} \; \boxed{\approx 29.9\ \frac{rad}{s}}$$

2) Estimate ω_n using time-to-peak.
Which is a better estimator for ω_n, using time-to-peak or 2% settling time?

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta}} \Rightarrow \omega_n = \frac{\pi}{T_p\sqrt{1-\zeta}} \approx 21.577$$

Would rather use time-to-peak to estimate ω_n since the relationship between 2% settling time and ω_n for an underdamped, standard 2nd order system is approximate whereas the relationship between time-to-peak and ω_n isn't.

3) Compute closed-loop transfer func. for Fig. 2.1.

Compute damping ratio, DC Gain and natural freq. in terms of plant's $\hat{k}$, ω_n and ζ and gain k_p

$$E(s) = R(s) - Y(s)$$

$$Y(s) = D(s) + P(s)E(s)k_p$$

$$\Rightarrow Y(s) = D(s) + k_p(R(s) - Y(s))P(s)$$

$$\Rightarrow Y(s) = D(s) + k_p P(s) R(s) - k_p P(s) Y(s), \quad D(s) = 0$$

$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{k_p P(s)}{1 + k_p P(s)}$$

$$= \frac{k_p \hat{k} \omega_n^2 (s^2 + 2\zeta\omega_n s + \omega_n^2)^{-1}}{1 + k_p \hat{k} \omega_n^2 (s^2 + 2\zeta\omega_n s + \omega_n^2)^{-1}}$$

$$= \frac{k_p \hat{k} \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2 + k_p \hat{k} \omega_n^2}$$

$$= \frac{k_p \hat{k} \omega_n^2 \cancel{(s^2 + 2\zeta\omega_n s + \omega_n^2)}}{(s^2 + 2\zeta\omega_n s + \omega_n^2)^{\cancel{2}\,1} + \cancel{(s^2 + 2\zeta\omega_n s + \omega_n^2)}(k_p \hat{k} \omega_n^2)}$$

$$= \frac{k_p \hat{k} \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2 + k_p \hat{k} \omega_n^2} = \frac{\hat{k}' \omega_n'^2}{s^2 + 2\zeta'\omega_n' s + \omega_n'^2}$$

$\hat{K}$, ω_n and ζ from part 1 & 2

$$\Rightarrow \omega_n'^2 = \omega_n^2(1 + K_p\hat{K})$$

$\Rightarrow$ $\boxed{\text{natural freq. } \omega_n' = \omega_n\sqrt{1 + K_p\hat{K}}}$

$$\Rightarrow \cancel{2}\zeta\omega_n\cancel{s} = \cancel{2}\zeta'\omega_n'\cancel{s}$$

$\Rightarrow$ $\boxed{\text{damping ratio } \zeta' = \dfrac{\zeta}{\sqrt{1 + K_p\hat{K}}}}$

$\Rightarrow$ $\boxed{\text{DC Gain } \hat{K}' = \dfrac{K_p\hat{K}}{1 + K_p\hat{K}}}$

8) Derive closed-loop transfer func. from d to y.

What happens when k_p changes?

$$Y(s) = D(s) + k_p P(s) R(s) - k_p P(s) Y(s), \quad R(s) = 0$$

$$\Rightarrow Y(s) = D(s) - k_p P(s) Y(s)$$

$$\Rightarrow \frac{Y(s)}{D(s)} = \frac{1}{1 + k_p P(s)} = \frac{1}{1 + \frac{k_p \hat{k}_c \omega_{n_c}^2}{s^2 + 2\zeta_c \omega_{n_c} s + \omega_{n_c}^2}}$$

$$= \frac{s^2 + 2\zeta_c \omega_{n_c} s + \omega_{n_c}^2}{s^2 + 2\zeta_c \omega_{n_c} s + \omega_{n_c}^2 (1 + k_p \hat{k}_c)}$$

$$= (s^2 + 2\zeta_c \omega_{n_c} s + \omega_{n_c}^2) \cdot \boxed{\frac{1}{\omega_{n_c}^2 (1 + k_p \hat{k}_c)} \cdot \frac{\omega_{n_c}^2 (1 + k_p \hat{k}_c)}{s^2 + 2\zeta_c \omega_{n_c} s + \omega_{n_c}^2 (1 + k_p \hat{k}_c)}}$$

Only latter half of transfer func. depends on k_p

$$\Rightarrow \dot{\omega}_{n_c}^2 = \omega_{n_c}^4 (1 + k_p \hat{k}_c)^2 \Rightarrow \boxed{\dot{\omega}_{n_c} = \omega_{n_c}^2 (1 + k_p \hat{k}_c)}$$

$$\cancel{2} \zeta_c \omega_{n_c}^3 (1 + k_p \hat{k}_c) = \cancel{2} \dot{\zeta}_c \dot{\omega}_{n_c}$$

$$\Rightarrow \boxed{\dot{\zeta}_c} = \frac{\zeta_c \omega_{n_c}^{\cancel{3}} \cancel{(1 + k_p \hat{k}_c)}}{\cancel{\omega_{n_c}^2} \cancel{(1 + k_p \hat{k}_c)}} = \boxed{\zeta_c \omega_{n_c}}$$

$$\hat{K}_c \dot{\omega}_{n_c}^2 = \omega_{n_c}^2 (1 + k_p \hat{K}_c) \Rightarrow \hat{K}_c = \frac{\cancel{\omega_{n_c}^2 (1 + k_p \hat{K}_c)}}{\omega_{n_c}^{\cancel{2}\,4} (1 + k_p \hat{K}_c)^{\cancel{2}}}$$

$$\Rightarrow \boxed{\dot{\hat{K}}_c = \frac{1}{\omega_{n_c}^2 (1 + k_p \hat{K}_c)}}$$

$\Rightarrow$ as k_p increases, natural frequency increases

$\Rightarrow$ damping ratio is unaffected by k_p

$\Rightarrow$ DC Gain decreases as k_p increases.

DC Gain $= \omega_{n_c}^{-2}$ when $k_p = 0$

Increasing $k_p \Rightarrow$ decrease in magnitude of transfer function $\Rightarrow$ better disturbance rejection (since $20 \log |G(s)| \to -\infty$ as $|G(s)| \to 0$)

When k_p increases, disturbance rejection properties of system improve.