

(1) Derive DC Gain of  $P(s)$  in terms of  $a, b, T$

$$P(s) = \frac{bT}{s + aT}, a, b > 0$$

$$T \in \{10, 100\}$$

$$P(0) = \frac{bT}{aT} = \frac{b}{a}$$

$$P(s) = \frac{K}{\tau s + 1}, K \text{ is DC Gain}$$

$$\begin{aligned} P(s) &= \frac{bT}{s + aT} = \frac{\frac{bT}{aT}}{\frac{1}{aT}s + 1} \\ &= \frac{b/a}{\frac{1}{aT}s + 1} \Rightarrow K = \frac{b}{a}, \tau = \frac{1}{aT} \end{aligned}$$

(2) Derive formula for b.w. of  $P(s)$   
in terms of  $a, b, T$

$$\frac{\|P(j\omega)\|}{\|P(0)\|} = \frac{1}{\sqrt{2}} \Rightarrow \left\| \frac{bT}{j\omega + aT} \right\| \cdot \frac{a}{b} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{bT}{\sqrt{a^2T^2 + \omega^2}} \cdot \frac{a}{b} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow a^2T^2 + \omega^2 = 2a^2T^2$$

$$\Rightarrow \omega^2 = 2a^2T^2 - a^2T^2 = a^2T^2$$

$$\Rightarrow \omega = aT$$

(3) Estimate  $a, b, T$  from measured bandwidth & DC Gain in Procedures 1.1 and 1.2 using  $\omega_{BW} = aT$  and  $K = \frac{b}{a}$  ( $\omega_{BW}$  := bandwidth freq.,  $K$  := DC Gain)

From Procedure 1.1,  $K = 0.9127$

From Procedure 1.2,  $f = 35.6 \text{ Hz}$

$$\Rightarrow \omega_{BW} \doteq 223.68 \text{ rad/s}$$

$$\Rightarrow T = 100 \Rightarrow a \doteq 2.2368, b \doteq 2.0415$$

(4) Compare these estimates with actual simulation params.

Actual Params.:

$$a_{sim} \doteq 2.0646, b_{sim} \doteq 1.8846, T_{sim} = 100$$

$$\Rightarrow a_{err} = \frac{|a - a_{sim}|}{a_{sim}} \doteq 8.34\% \quad (\text{absolute relative true error})$$

$$\Rightarrow b_{err} = \frac{|b - b_{sim}|}{b_{sim}} \doteq 8.33\% \quad (\text{absolute relative true error})$$

$$\Rightarrow T_{err} = 0\%$$

(5) Find formula for  $\tau$  in terms of  $a, b, T$

$$P(s) = \frac{K}{\tau s + 1}, K \text{ is DC Gain}$$

$$\begin{aligned} P(s) &= \frac{bT}{s + aT} = \frac{\frac{bT}{aT}}{\frac{1}{aT}s + 1} \\ &= \frac{b/a}{\frac{1}{aT} \cdot s + 1} \Rightarrow K = \frac{b}{a}, \quad \boxed{\tau = \frac{1}{aT}} \end{aligned}$$

(6) Find formula for 2% settling time of  $P(s)$  in terms of  $a, b, T$

$$Y(s) = P(s)U(s) = \frac{bT}{s(s+aT)} = \frac{b}{as} - \frac{b}{a(s+aT)}$$

$$\Rightarrow y(t) = \frac{b}{a} 1(t) - \frac{b}{a} e^{-aTt}$$

$$y(t) = 0.98 \frac{b}{a} \Rightarrow 0.98 = 1(t) - e^{-aTt}$$

$$\Rightarrow 0.98 = 1 - e^{-aTt}, t > 0$$

$$\Rightarrow \ln(0.02) = -aTt \Rightarrow t = -\frac{\ln(0.02)}{aT}$$

$$\doteq \frac{3.91}{aT}$$

(7) Estimate  $a, b, T$  using  $t_{2\%}, \tau = \frac{1}{aT}$   
and results from Procedure 1.4

$$t_{2\% \text{ sim}} = 2.230355 \times 10^{-2} [\text{s}]$$

$$\tau_{\text{sim}} = 4.8159299 \times 10^{-3} [\text{s}]$$

$$T = 100,$$

$$\Rightarrow a = \frac{1}{\tau_{\text{sim}} T} \doteq 2.0764$$

$$\Rightarrow b = ka = 0.9128 \cdot 2.0764 \doteq 1.8953$$

(8) Compare these estimates to actual params.  
from Simulation

Actual Params.:

$$a_{\text{sim}} \doteq 2.0646, b_{\text{sim}} \doteq 1.8846, T_{\text{sim}} = 100$$

$$\Rightarrow a_{\text{err}} = \frac{|a - a_{\text{sim}}|}{a_{\text{sim}}} \doteq 0.571\% \quad (\text{absolute relative true error})$$

$$\Rightarrow b_{\text{err}} = \frac{|b - b_{\text{sim}}|}{b_{\text{sim}}} \doteq 0.568 \% \quad (\text{absolute relative true error})$$

$$\Rightarrow T_{\text{err}} = 0 \%.$$

(9) Find transfer func. from input to output for closed-loop system.

$$U(s) = K_p(R(s) - Y(s))$$

$$R(s) = U(s) - Y(s)$$

$$Y(s) = P(s)U(s)$$

$$\Rightarrow Y(s) = P(s)K_p(R(s) - Y(s))$$

$$\Rightarrow Y(s) = P(s)K_p R(s) - P(s)K_p Y(s)$$

$$\Rightarrow Y(s) = \frac{P(s)K_p R(s)}{1 + P(s)K_p}$$

let  $G(s) = \frac{K_p P(s)}{1 + K_p P(s)} \Rightarrow Y(s) = G(s)R(s)$

$$P(s) = \frac{bT}{s+aT} \Rightarrow G(s) = \frac{\frac{bT}{s+aT} \cdot K_p}{1 + \frac{bT}{s+aT} \cdot K_p}$$

$$\Rightarrow G(s) = \frac{bT K_p}{s+aT + bT K_p}, \text{ let } \tau = \frac{1}{T(a+bK_p)}$$

$$\Rightarrow G(s) = \frac{\tau b T K_p}{\tau s + 1}, \quad \left( \text{or } G(s) = \frac{\frac{b K_p}{a+b K_p}}{\frac{s}{\tau(a+b K_p)} + 1} \right)$$

where  $\tau = T^{-1}(a+bK_p)^{-1}$  is time const. and

$K = \tau b T K_p$  is DC Gain

(10) How does closing the loop in Procedure 1.3 affect DC Gain and bandwidth?

Open-loop DC Gain =  $\frac{b}{a}$ ,  $a, b > 0$ ,  $a > b$  in sim.params.

Closed-loop DC Gain =  $\frac{b|K_p|}{a+b|K_p|}$ ,  $a, b > 0$ ,  $K_p \in \mathbb{R}$ ,  $a > b$  in sim.params.

Closing the loop decreases the magnitude of the DC Gain. (i.e.  $\frac{b|K_p|}{a+b|K_p|} < \frac{b}{a}$ )

Proof by contradiction. Assume  $\frac{b|K_p|}{a+b|K_p|} > \frac{b}{a}$

$$\frac{b|K_p|}{a+b|K_p|} > \frac{b}{a} \Rightarrow ab|K_p| > ab + b^2|K_p|$$

$$\Rightarrow |K_p| > 1 + \frac{b}{a}|K_p| \Rightarrow |K_p| < \frac{1}{\frac{b}{a}-1}$$

From simulation params.,  $a > b$

$$\Rightarrow \frac{b}{a}-1 < 0 \Rightarrow \frac{1}{\frac{b}{a}-1} < 0$$

$\Rightarrow \left( |K_p| < \frac{1}{\frac{b}{a}-1} \right) \Rightarrow |K_p| < 0$ , which is a contradiction

$$\Rightarrow \frac{b|K_p|}{a+b|K_p|} < \frac{b}{a} \quad \text{for } a > b > 0$$



$$\begin{aligned} \frac{\|G(j\omega)\|}{\|G(0)\|} = \frac{1}{\sqrt{2}} &\Rightarrow \frac{\left| \frac{\tau b T K_p}{\tau j\omega + 1} \right|}{\left| \frac{b K_p}{a + b K_p} \right|} = \frac{1}{\sqrt{2}} \\ \Rightarrow \left| \frac{a + b K_p}{b K_p} \right| \cdot \frac{\left| \tau b T K_p \right|}{\sqrt{1 + \tau^2 \omega^2}} &= \frac{1}{\sqrt{2}} \\ \Rightarrow \frac{|b K_p| \cdot \sqrt{1 + \tau^2 \omega^2}}{|\tau b T K_p| \cdot |a + b K_p|} &= \sqrt{2} \\ \Rightarrow \frac{\cancel{b^2 K_p^2} (1 + \tau^2 \omega^2)}{\cancel{\tau^2 b^2 T^2 K_p^2} (a + b K_p)^2} &= 2 \Rightarrow \frac{1 + \tau^2 \omega^2}{\tau^2 T^2 (a + b K_p)^2} = 2 \\ \Rightarrow 1 + \tau^2 \omega^2 &= 2 \tau^2 T^2 (a + b K_p)^2 \\ \Rightarrow \omega &= \sqrt{\frac{2 \tau^2 T^2 (a + b K_p)^2 - 1}{\tau}}, \quad \tau = \frac{1}{T(a + b K_p)} \\ \rightarrow \omega &= \frac{2 T^2 \tau^{-2} (a + b K_p)^2 (a + b K_p)^2 - 1}{T^{-1} (a + b K_p)^{-1}} \\ \Rightarrow \omega &= T(a + b K_p) \end{aligned}$$

Open-loop b.w. freq.:  $\omega_o = aT$ ,  $a > 0$

Closed-loop b.w. freq.:  $\omega_c = aT + bK_p T$ ,  $a, b > 0$ ,  $K_p \in \mathbb{R}$ ,  $a > b$  in sim. params

$$\Rightarrow \frac{\omega_c}{\omega_o} = 1 + \frac{b}{a} K_p \Rightarrow \omega_c > \omega_o \text{ if } K_p > 0$$

$$\omega_c < \omega_o \text{ if } K_p < 0$$

(11) How does closing the loop in Procedure 1.5 affect settling time and time-const.?

$$\text{Open-loop time const. : } \tau_o = \frac{1}{aT}$$

$$\text{Closed-loop time const. : } \tau_c = \frac{1}{T(a+bk_p)}$$

$$\Rightarrow \frac{\tau_c}{\tau_o} = \frac{a}{a+bk_p} \Rightarrow \begin{cases} \tau_c < \tau_o & \text{if } k_p > 0 \\ \tau_c > \tau_o & \text{if } k_p < 0 \end{cases}$$

$$Y(s) = G(s)R(s) = G(s)U(s) = \frac{\tau b T k_p}{s(\tau s + 1)},$$

$$\Rightarrow = \frac{\tau b k_p T}{s} - \frac{\tau b k_p T}{s + \tau^{-1}}$$

$$\Rightarrow y(t) = \tau b k_p T (1(t) - e^{-\frac{t}{\tau}}), t > 0$$

$$y(t) = \frac{b k_p}{a+bk_p} (0.98) \Rightarrow \frac{0.98}{a+bk_p} = \tau T (1 - e^{-\frac{t}{\tau}})$$

$$\Rightarrow \frac{0.98}{\tau T (a+bk_p)} = 1 - e^{-t/\tau}$$

$$\Rightarrow 0.02 = e^{-\frac{t}{\tau}} \Rightarrow t = -\tau \ln(0.02)$$

$$\Rightarrow t = \frac{-\ln(0.02)}{\tau(a+bk_p)}$$

Open-loop settling time :  $t_o = -\frac{\ln(0.02)}{aT}$

Closed-loop settling time:  $t_c = \frac{-\ln(0.02)}{T(a+bk_p)}$

$$\Rightarrow \frac{t_c}{t_o} = \frac{a}{a+bk_p} = \frac{1}{1 + \frac{b}{a}k_p}$$

$\Rightarrow t_c < t_o$  if  $k_p > 0$

$t_c > t_o$  if  $k_p < 0$