(1) Derive DC Gain of $P(s)$ in terms of a, b, T

$$P(s) = \frac{bT}{s + aT}, \quad a, b > 0$$

$$T \in \{10, 100\}$$

$$\boxed{P(0) = \frac{bT}{aT} = \frac{b}{a}}$$

$$P(s) = \frac{K}{\tau s + 1}, \quad K \text{ is DC Gain}$$

$$P(s) = \frac{bT}{s + aT} = \frac{\frac{b\cancel{T}}{a\cancel{T}}}{\frac{1}{aT}s + 1}$$

$$= \frac{b/a}{\frac{1}{aT}\cdot s + 1} \Rightarrow \boxed{K = \frac{b}{a}}, \quad \tau = \frac{1}{aT}$$

(2) Derive formula for b.w. of $P(s)$
in terms of a, b, T

$$\frac{\|P(j\omega)\|}{\|P(0)\|} = \frac{1}{\sqrt{2}} \Rightarrow \left\| \frac{bT}{j\omega + aT} \right\| \cdot \frac{a}{b} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{\cancel{b}T}{\sqrt{a^2T^2+\omega^2}} \cdot \frac{a}{\cancel{b}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow a^2T^2 + \omega^2 = 2a^2T^2$$

$$\Rightarrow \omega^2 = 2a^2T^2 - a^2T^2 = a^2T^2$$

$$\Rightarrow \boxed{\omega = aT}$$

(3) Estimate a, b, T from measured bandwidth & DC Gain in Procedures 1.1 and 1.2 using $\omega_{BW} = aT$ and $K = \frac{b}{a}$ (ω_{BW} := bandwidth freq., K := DC Gain)

From Procedure 1.1, $K = 0.9127$

From Procedure 1.2, $f = 35.6\ Hz$

$\Rightarrow \omega_{BW} \doteq 223.68\ rad/s$

$\Rightarrow T = 100 \Rightarrow a \doteq 2.2368,\ b \doteq 2.0415$

(4) Compare these estimates with actual simulation params.

Actual Params:

$a_{sim} \doteq 2.0646,\ b_{sim} \doteq 1.8846,\ T_{sim} = 100$

$$\Rightarrow a_{err} = \frac{|a - a_{sim}|}{a_{sim}} \doteq 8.34\%$$

(absolute relative true error)

$$\Rightarrow b_{err} = \frac{|b - b_{sim}|}{b_{sim}} \doteq 8.33\%$$

(absolute relative true error)

$$\Rightarrow T_{err} = 0\%$$

(5) Find formula for τ in terms of a, b, T

$$P(s) = \frac{K}{\tau s + 1} \quad , \quad K \text{ is DC Gain}$$

$$P(s) = \frac{bT}{s + aT} = \frac{\frac{b\cancel{T}}{a\cancel{T}}}{\frac{1}{aT}s + 1}$$

$$= \frac{b/a}{\frac{1}{aT} \cdot s + 1} \Rightarrow K = \frac{b}{a} \, , \quad \boxed{\tau = \frac{1}{aT}}$$

(6) Find formula for 2% settling time of P(s) in terms of a, b, T

$$Y(s) = P(s)U(s) = \frac{bT}{s(s+aT)} = \frac{b}{as} - \frac{b}{a(s+aT)}$$

$$\Rightarrow y(t) = \frac{b}{a} 1(t) - \frac{b}{a} e^{-aTt}$$

$$y(t) = 0.98 \frac{b}{a} \Rightarrow 0.98 = 1(t) - e^{-aTt}$$

$$\Rightarrow 0.98 = 1 - e^{-aTt} \quad , \quad t > 0$$

$$\Rightarrow \ln(0.02) = -aTt \Rightarrow \boxed{t = -\frac{\ln(0.02)}{aT}}$$

$$\doteq \frac{3.91}{aT}$$

(7) Estimate a, b, T using $t_{2\%}$, $\tau = \frac{1}{aT}$ and results from Procedure 1.4

$$t_{2\%\,sim} = 2.230355 \times 10^{-2}\,[s]$$

$$\tau_{sim} = 4.8159299 \times 10^{-3}\,[s]$$

$T = 100$,

$$\Rightarrow a = \frac{1}{\tau_{sim} T} \doteq 2.0764$$

$$\Rightarrow b = Ka = 0.9128 \cdot 2.0764 \doteq 1.8953$$

(8) Compare these estimates to actual params. from Simulation

Actual Params.:

$a_{sim} \doteq 2.0646$, $b_{sim} \doteq 1.8846$, $T_{sim} = 100$

$$\Rightarrow a_{err} = \frac{|a - a_{sim}|}{a_{sim}} \doteq 0.571\%$$

(absolute relative true error)

$$\Rightarrow b_{err} = \frac{|b - b_{sim}|}{b_{sim}} \doteq 0.568\ \%$$

(absolute relative true error)

$$\Rightarrow T_{err} = 0\ \%$$

(9) Find transfer func. from input to output for closed-loop system.

$$\mathcal{U}(s) = K_p\left(R(s) - Y(s)\right)$$
$$R(s) = \mathcal{U}(s) - Y(s)$$
$$Y(s) = P(s)\mathcal{U}(s)$$

$$\Rightarrow Y(s) = P(s)K_p\left(R(s) - Y(s)\right)$$

$$\Rightarrow Y(s) = P(s)K_p R(s) - P(s)K_p Y(s)$$

$$\Rightarrow Y(s) = \frac{P(s)K_p R(s)}{1 + P(s)K_p}$$

let $G(s) = \dfrac{K_p P(s)}{1 + K_p P(s)} \Rightarrow Y(s) = G(s)R(s)$

$$P(s) = \frac{bT}{s + aT} \Rightarrow G(s) = \frac{\frac{bT}{s+aT}\cdot K_p}{1 + \frac{bT}{s+aT}\cdot K_p}$$

$\Rightarrow G(s) = \dfrac{bTK_p}{s + aT + bTK_p}$, let $\tau = \dfrac{1}{T(a + bK_p)}$

$\Rightarrow G(s) = \dfrac{\tau bTK_p}{\tau s + 1}$, $\left(\text{or } G(s) = \dfrac{\frac{bK_p}{a+bK_p}}{\frac{s}{T(a+bK_p)} + 1}\right)$

where $\tau = T^{-1}(a + bK_p)^{-1}$ is time const. and $K = \tau bTK_p$ is DC Gain

(10) How does closing the loop in Procedure 1.3 affect DC Gain and bandwidth?

Open-loop DC Gain $= \frac{b}{a}$, $a, b > 0$, $a > b$ in sim. params

Closed-loop DC Gain $= \frac{bK_p}{a + bK_p}$, $a, b > 0$, $K_p \in \mathbb{R}$, $a > b$ in sim. params

$$\boxed{\text{Closing the loop } \underline{\text{decreases}} \text{ the magnitude of the DC Gain. (i.e. } \frac{b|K_p|}{a+b|K_p|} < \frac{b}{a}\text{)}}$$

Proof by contradiction. Assume $\frac{b|K_p|}{a+b|K_p|} > \frac{b}{a}$

$$\frac{b|K_p|}{a+b|K_p|} > \frac{b}{a} \Rightarrow ab|K_p| > ab + b^2|K_p|$$

$$\Rightarrow |K_p| > 1 + \frac{b}{a}|K_p| \Rightarrow |K_p| < \frac{1}{\frac{b}{a} - 1}$$

From simulation params., $a > b$

$$\Rightarrow \frac{b}{a} - 1 < 0 \Rightarrow \frac{1}{\frac{b}{a}-1} < 0$$

$$\Rightarrow \left(|K_p| < \frac{1}{\frac{b}{a}-1}\right) \Rightarrow |K_p| < 0,$$ which is a contradiction

$$\Rightarrow \boxed{\frac{bK_p}{a+bK_p} < \frac{b}{a} \quad \text{for } a > b > 0} \qquad \blacksquare$$

$$\frac{\|G(j\omega)\|}{\|G(0)\|} = \frac{1}{\sqrt{2}} \Rightarrow \frac{\left|\frac{\tau b T k_p}{\tau j\omega + 1}\right|}{\left|\frac{bk_p}{a+bk_p}\right|} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \left|\frac{a+bk_p}{bk_p}\right| \cdot \frac{|\tau b T k_p|}{\sqrt{1+\tau^2\omega^2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{|bk_p| \cdot \sqrt{1+\tau^2\omega^2}}{|\tau b T k_p| \cdot |a+bk_p|} = \sqrt{2}$$

$$\Rightarrow \frac{\cancel{b^2}\cancel{k_p^2}(1+\tau^2\omega^2)}{\tau^2 \cancel{b^2} T^2 \cancel{k_p^2}(a+bk_p)^2} = 2 \Rightarrow \frac{1+\tau^2\omega^2}{\tau^2 T^2 (a+bk_p)^2} = 2$$

$$\Rightarrow 1+\tau^2\omega^2 = 2\tau^2 T^2 (a+bk_p)^2$$

$$\Rightarrow \omega = \frac{\sqrt{2\tau^2 T^2 (a+bk_p)^2 - 1}}{\tau}, \quad \tau = \frac{1}{T(a+bk_p)}$$

$$\rightarrow \omega = \frac{2\cancel{T^2}\cancel{T^{-2}}\cancel{(a+bk_p)^{-2}}\cancel{(a+bk_p)^2} - 1}{T^{-1}(a+bk_p)^{-1}}$$

$$\rightarrow \boxed{\omega = T(a+bk_p)}$$

Open-loop b.w. freq.: $\omega_o = aT, \; a > 0$

Closed-loop b.w. freq.: $\omega_c = aT + bk_pT, \; a,b > 0, \; k_p \in \mathbb{R}, \; a > b$ in sim. params

$$\Rightarrow \boxed{\frac{\omega_c}{\omega_o} = 1 + \frac{b}{a}k_p \Rightarrow \begin{array}{l} \omega_c > \omega_o \text{ if } k_p > 0 \\ \omega_c < \omega_o \text{ if } k_p < 0 \end{array}}$$

(11) How does closing the loop in Procedure 1.5 affect settling time and time-const.?

Open-loop time const.: $\tau_o = \frac{1}{aT}$

Closed-loop time const.: $\tau_c = \frac{1}{T(a+bk_p)}$

$$\boxed{\Rightarrow \frac{\tau_c}{\tau_o} = \frac{a}{a+bk_p} \Rightarrow \begin{array}{ll} \tau_c < \tau_o & \text{if } k_p > 0 \\ \tau_c > \tau_o & \text{if } k_p < 0 \end{array}}$$

$$Y(s) = G(s)R(s) = G(s)U(s) = \frac{\tau b T k_p}{s(\tau s+1)},$$

$$\tau = T^{-1}(a+bk_p)^{-1}$$

$$\Rightarrow = \frac{\tau b k_p T}{s} - \frac{\tau b k_p T}{s+\tau^{-1}}$$

$$\Rightarrow y(t) = \tau b k_p T\left(1(t) - e^{-\frac{t}{\tau}}\right), t>0$$

$$y(t) = \frac{bk_p}{a+bk_p}(0.98) \Rightarrow \frac{0.98}{a+bk_p} = \tau T\left(1-e^{-\frac{t}{\tau}}\right)$$

$$\Rightarrow \frac{0.98}{\tau T(a+bk_p)} = 1 - e^{-t/\tau}$$

$$\Rightarrow 0.02 = e^{-\frac{t}{\tau}} \Rightarrow t = -\tau \ln(0.02)$$

$$\boxed{\Rightarrow t = \frac{-\ln(0.02)}{T(a+bk_p)}}$$

Open-loop settling time : $t_o = -\dfrac{\ln(0.02)}{aT}$

Closed-loop settling time: $t_c = \dfrac{-\ln(0.02)}{T(a+bk_p)}$

$$\boxed{\begin{aligned} &\Rightarrow \frac{t_c}{t_o} = \frac{a}{a+bk_p} = \frac{1}{1+\frac{b}{a}k_p} \\ &\qquad \Rightarrow t_c < t_o \quad \text{if} \quad k_p > 0 \\ &\qquad\qquad t_c > t_o \quad \text{if} \quad k_p < 0 \end{aligned}}$$