(10) How does closing the loop in Procedure 1.3 affect DC Gain and bandwidth?

Open-loop DC Gain $= \frac{b}{a}$, $a, b > 0$, $a > b$ in sim. params

Closed-loop DC Gain $= \frac{bK_p}{a + bK_p}$, $a, b > 0$, $K_p \in \mathbb{R}$, $a > b$ in sim. params

Closing the loop <u>decreases</u> the magnitude of the DC Gain. (i.e. $\frac{b|K_p|}{a+b|K_p|} < \frac{b}{a}$)

Proof by contradiction. Assume $\frac{b|K_p|}{a+b|K_p|} > \frac{b}{a}$

$$\frac{b|K_p|}{a+b|K_p|} > \frac{b}{a} \Rightarrow ab|K_p| > ab + b^2|K_p|$$

$$\Rightarrow |K_p| > 1 + \frac{b}{a}|K_p| \Rightarrow |K_p| < \frac{1}{\frac{b}{a} - 1}$$

From simulation params., $a > b$

$$\Rightarrow \frac{b}{a} - 1 < 0 \Rightarrow \frac{1}{\frac{b}{a} - 1} < 0$$

$$\Rightarrow \left(|K_p| < \frac{1}{\frac{b}{a} - 1}\right) \Rightarrow |K_p| < 0,$$ which is a contradiction

$$\Rightarrow \boxed{\frac{bK_p}{a+bK_p} < \frac{b}{a} \quad \text{for } a > b > 0}$$ ∎

$$\frac{\|G(j\omega)\|}{\|G(0)\|} = \frac{1}{\sqrt{2}} \Rightarrow \frac{\left|\frac{\tau b T k_p}{\tau j\omega + 1}\right|}{\left|\frac{b k_p}{a + b k_p}\right|} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \left|\frac{a + b k_p}{b k_p}\right| \cdot \frac{|\tau b T k_p|}{\sqrt{1 + \tau^2 \omega^2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{|b k_p| \cdot \sqrt{1 + \tau^2 \omega^2}}{|\tau b T k_p| \cdot |a + b k_p|} = \sqrt{2}$$

$$\Rightarrow \frac{\cancel{b^2} \cancel{k_p^2} (1 + \tau^2 \omega^2)}{\tau^2 \cancel{b^2} T^2 \cancel{k_p^2} (a + b k_p)^2} = 2 \Rightarrow \frac{1 + \tau^2 \omega^2}{\tau^2 T^2 (a + b k_p)^2} = 2$$

$$\Rightarrow 1 + \tau^2 \omega^2 = 2 \tau^2 T^2 (a + b k_p)^2$$

$$\Rightarrow \omega = \frac{\sqrt{2 \tau^2 T^2 (a + b k_p)^2 - 1}}{\tau}, \quad \tau = \frac{1}{T(a + b k_p)}$$

$$\rightarrow \omega = \frac{2 \cancel{T^2} \cancel{T^{-2}} \cancel{(a + b k_p)^{-2}} \cancel{(a + b k_p)^2} - 1}{T^{-1} (a + b k_p)^{-1}}$$

$$\rightarrow \boxed{\omega = T(a + b k_p)}$$

Open-loop b.w. freq.: $\omega_o = aT, \; a > 0$

Closed-loop b.w. freq.: $\omega_c = aT + b k_p T, \; a, b > 0, \; k_p \in \mathbb{R}$, $a > b$ in sim. params

$$\Rightarrow \boxed{\frac{\omega_c}{\omega_o} = 1 + \frac{b}{a} k_p \Rightarrow \begin{array}{l} \omega_c > \omega_o \text{ if } k_p > 0 \\ \omega_c < \omega_o \text{ if } k_p < 0 \end{array}}$$