RBE 500 Homework #4

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Problem 4.6

Given $R = R_{x,\theta} R_{y,\phi}$, compute $\frac{\partial R}{\partial \phi}$. Evaluate $\frac{\partial R}{\partial \phi}$ at $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$. First parametrically compute, then evaluate by plugging the values in.

Solution

$$\frac{\partial}{\partial \phi}\left(R_{x,\theta} R_{y,\phi}\right) = R_{x,\theta} \frac{\partial}{\partial \phi}\left(R_{y,\phi}\right)$$

Using the the fact that $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(R_{y,\theta}\right) = S(j) R_{y,\theta}$,

$$\begin{aligned} R_{x,\theta} \frac{\partial}{\partial \phi}\left(R_{y,\phi}\right) = R_{x,\theta} S(j) R_{y,\phi} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} -\sin(\phi) & 0 & \cos(\phi) \\ \cos(\phi)\sin(\theta) & 0 & \sin(\phi)\sin(\theta) \\ -\cos(\phi)\cos(\theta) & 0 & -\cos(\theta)\sin(\phi) \end{bmatrix} \end{aligned}$$

Now, plugging in the values $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$, we get

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We performed the above computations using the following MATLAB code.

```matlab
% Calculation code for problem 4.6 of the RBE500 textbook (HW 4)

clear; close all; clc;

syms theta phi

% Define the matrices in our problem
rotx_theta = [1 0 0; 0 cos(theta) -sin(theta); 0 sin(theta) cos(theta)];
Sj = [0 0 1; 0 0 0; -1 0 0];
roty_phi = [cos(phi) 0 sin(phi); 0 1 0; -sin(phi) 0 cos(phi)];

% Multiply the matrices
product = rotx_theta*Sj*roty_phi;

% Get latex output
latex(product)

phi_val = pi/2;
theta_val = pi/2;

% Now plug in values
product_val = [ -sin(phi_val), 0, cos(phi_val);
cos(phi_val)*sin(theta_val), 0, sin(phi_val)*sin(theta_val);
-cos(phi_val)*cos(theta_val), 0, -cos(theta_val)*sin(phi_val)]
```

Problem 4.10

Two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the homogeneous transformation

$$H = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A particle has velocity $v_1(t) = (3, 1, 0)$ relative to frame $o_1x_1y_1z_1$. What is the velocity of the particle in frame $o_0x_0y_0z_0$?

Solution

The given H is the homogeneous transformation H_1^0. Therefore, $R_1^0 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $o_1^0 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

At any given point in time, the position of the particle with respect to frame $o_1x_1y_1z_1$ is given as $p^1(t)$. We know that

$$p^0(t) = R_1^0 p^1(t) + o_1^0$$

Taking the derivative of both sides and using the product rule, we get

$$v^0(t) = \dot{R}_1^0 p^1(t) + R_1^0 v^1(t) + 0$$

But, $\dot{R}_1^0 = 0$ since R_1^0 is a constant in time. Therefore,

$$v^0(t) = R_1^0 v^1(t)$$

$$v^0(t) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$v^0(t) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

Which was calculated by the following MATLAB script

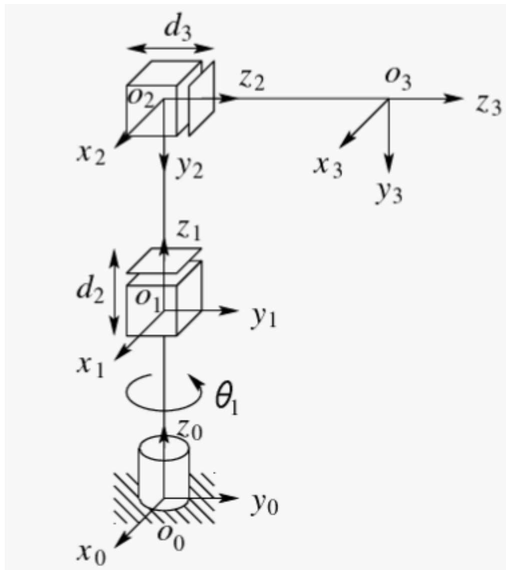
```
% Calculation code for problem 4.10 of the RBE500 textbook (HW 4)

clear; close all; clc;

R = [0 -1 0; 1 0 0; 0 0 1];
v1 = [3; 1; 0];

v0 = R*v1
```

Problem 4.15

Find the 6×3 Jacobian for the three links of the cylindrical manipulator of Figure 3.7. Find the singular configurations for this arm.

Solution

Figure 3.7 of our book is the following.



We can see that we have 3 joints, so $n = 3$. Let us also form the table given in the lecture videos:

	Linear component	Angular component
Revolute joint	$J_{v_i} = z_{i-1}^0 \times (o_n^0 - o_{i-1}^0)$	$J_{v_i} = z_{i-1}^0$
Prismatic joint	$J_{\omega_i} = z_{i-1}^0$	$J_{\omega_i} = 0$

Using this table, and the fact that the upper half of the Jacobian contains linear components while the bottom half contains angular components, we have

$$J = \begin{bmatrix} z_0 \times (o_3 - o_0) & z_1 & z_2 \\ z_0 & 0 & 0 \end{bmatrix}$$

In this Jacobian matrix, we know that $z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. To find z_1, z_2, and o_3, we need to find $T_n^0 = A_1 \ldots A_n$ for $n = 1, 2, 3$. Following the DH convention (which Figure 3.7 already abides by), we have the following table for quantities $\alpha_i, a_i, \theta_i, d_i$.

Link	α_i	a_i	θ_i	d_i
1	0	0	θ_1	d_1
2	-90°	0	0	d_2
3	0	0	0	d_3

Problem 4.15 continued on next page...

Which gives us the following A_i matrices.

$$A_1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we can compute our T matrices using the following MATLAB code,

```
% Calculation code for problem 4.15 (T matrices) of the RBE500 textbook (HW 4)

clear; close all; clc;

syms theta1 d1 d2 d3;

% Form the A matrices
A1 = [cos(theta1) -sin(theta1) 0 0; sin(theta1) cos(theta1) 0 0; 0 0 1 d1; 0 0 0 1];
A2 = [1 0 0 0; 0 0 1 0; 0 -1 0 d2; 0 0 0 1];
A3 = [1 0 0 0; 0 1 0 0; 0 0 1 d3; 0 0 0 1];

% Compute T matrices
T2 = A1*A2;
T3 = A1*A2*A3;

% Output to LaTex
latex(T2)
latex(T3)
```

$$T_1^0 = A_1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} \cos(\theta_1) & 0 & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & \cos(\theta_1) & 0 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} \cos(\theta_1) & 0 & -\sin(\theta_1) & -d_3 \sin(\theta_1) \\ \sin(\theta_1) & 0 & \cos(\theta_1) & d_3 \cos(\theta_1) \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From these T matrices, we get

$$z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, z_2 = \begin{bmatrix} -\sin(\theta_1) \\ \cos(\theta_1) \\ 0 \end{bmatrix}, o_3 = \begin{bmatrix} -d_3 \sin(\theta_1) \\ d_3 \cos(\theta_1) \\ d_1 + d_2 \end{bmatrix}$$

Now we are ready to compute our Jacobian matrix. We do this using the following MATLAB code.

```
% Calculation code for problem 4.15 (Jacobian) of the RBE500 textbook (HW 4)

clear; close all; clc;

syms theta1 a3 d1 d2 d3;

% Declare vectors
z0 = [0; 0; 1];
z1 = z0;
z2 = [-sin(theta1); cos(theta1); 0];
o0 = [0; 0; 0];
o3 = [-d3*sin(theta1); d3*cos(theta1); d1+d2];

% Compute Jacobian
J = [cross(z0, (o3 - o0)) z1 z2; z0 zeros(3,1) zeros(3,1)]

% Output LaTex
latex(J)
```

Which gives us the following Jacobian,

$$J = \begin{bmatrix} -d_3\cos(\theta_1) & 0 & -\sin(\theta_1) \\ -d_3\sin(\theta_1) & 0 & \cos(\theta_1) \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = J_P = \begin{bmatrix} J_{11} \\ \hline J_{21} \end{bmatrix}$$

Where J_{11} and J_{21} are 3×3 matrices. By setting $\det J_{11} = 0$ we can find the singular configurations for the arm portion of the manipulator (first three joints). We calculate the determinant using the following MATLAB code,

```
% Calculation code for problem 4.15 (Determinant) of the RBE500 textbook (HW 4)

clear; close all; clc;

% Declare symbolic variables
syms d3 theta1;

% Write J11 (first three rows of our jacobian)
J11 = [-d3*cos(theta1) 0 -sin(theta1); -d3*sin(theta1) 0 cos(theta1); 0 1 0];
% Compute the determinant
det_J11 = det(J11);

% Output to LaTex
latex(det_J11)
```

Which gives us,

$$J_{11} = d_3\cos^2(\theta_1) + d_3\sin^2(\theta_1) = d_3(\cos^2(\theta_1) + \sin^2(\theta_1)) = d_3$$

When we set $J_{11} = 0$ here, we get $d_3 = 0$. Therefore, the only singular configuration for this manipulator will occur when the third joint variable (which is a prismatic joint) goes to 0.