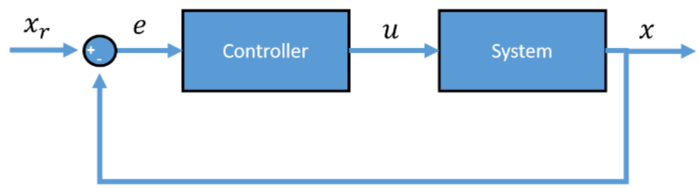
RBE 500 Homework #5

Arjan Gupta

Problem 1

Consider the following block diagram:



The dynamics of the system is given in the following differential equation

$$m\ddot{x} + b\dot{x} = u$$

The controller is designed as

$$k_p e + k_d \dot{e} = u$$

Convert the system model and the controller to the Laplace domain.

Solution

For the system model, take the Laplacian on both sides,

$$\begin{aligned} \mathcal{L}\{m\ddot{x} + b\dot{x}\} &= \mathcal{L}\{u\} \\ m\mathcal{L}\{\ddot{x}\} + b\mathcal{L}\{\dot{x}\} &= U(s) \\ \boxed{ms^2X(s) + bsX(s) = U(s)} \end{aligned}$$

Similarly, take the Laplacian on both sides of the controller model,

$$\begin{aligned} \mathcal{L}\{k_p e + k_d \dot{e}\} &= \mathcal{L}\{u\} \\ k_p\mathcal{L}\{e\} + k_d\mathcal{L}\{\dot{e}\} &= U(s) \\ \boxed{k_pE(s) + k_dsE(s) = U(s)} \end{aligned}$$

Problem 2

Find the transfer functions for $\frac{U(s)}{E(s)}$, $\frac{X(s)}{U(s)}$, and $\frac{X(s)}{E(s)}$.

Solution

Re-arranging the Laplace-domain controller model from Problem 1,

$$E(s)\,(k_p + k_d s) = U(s) \tag{1}$$

$$\boxed{\frac{U(s)}{E(s)} = (k_p + k_d s)}$$

Re-arranging the Laplace-domain system model from Problem 1,

$$X(s)\left(ms^2 + bs\right) = U(s) \tag{2}$$

$$\boxed{\frac{X(s)}{U(s)} = \frac{1}{(ms^2 + bs)}}$$

Equating equations (1) and (2) as we have found above, we get

$$E(s)\,(k_p + k_d s) = X(s)\left(ms^2 + bs\right)$$

$$\frac{(k_p + k_d s)}{(ms^2 + bs)} = \frac{X(s)}{E(s)}$$

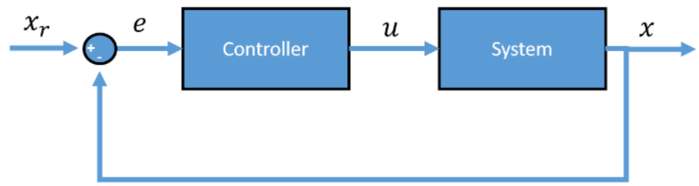
$$\boxed{\frac{X(s)}{E(s)} = \frac{(k_p + k_d s)}{(ms^2 + bs)}} \tag{3}$$

Problem 3

Find the closed loop transfer function.

Solution

The closed loop transfer function is given by $\dfrac{X}{X_r}$.



As shown in the figure,

$$E(s) = X_r - X \tag{4}$$

And re-arranging equation (3) from Problem 2,

$$X(s) = \frac{(k_p + k_d s)}{(ms^2 + bs)} E(s)$$

Plug in equation (4),

$$\begin{aligned}
X(s) &= \frac{(k_p + k_d s)}{(ms^2 + bs)} \left(X_r(s) - X(s)\right) \\
X(s)\frac{(ms^2 + bs)}{(k_p + k_d s)} &= X_r(s) - X(s) \\
X(s)\frac{(ms^2 + bs)}{(k_p + k_d s)} + X(s) &= X_r(s) \\
X(s)\left(\frac{ms^2 + bs}{k_p + k_d s} + 1\right) &= X_r(s) \\
X(s)\left(\frac{ms^2 + bs + k_p + k_d s}{k_p + k_d s}\right) &= X_r(s) \\
\frac{ms^2 + bs + k_p + k_d s}{k_p + k_d s} &= \frac{X_r}{X} \\
\frac{k_p + k_d s}{ms^2 + bs + k_p + k_d s} &= \frac{X}{X_r}
\end{aligned}$$

So, the closed loop transfer function is

$$\boxed{\frac{X}{X_r} = \frac{k_p + k_d s}{ms^2 + bs + k_p + k_d s}}$$