RBE 500 Homework #6

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Question 1

Consider the following robot joint model

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = u(t) + d(t)$$

where J is the inertia of the link, B is the effective damping on the link, θ is the joint angle, u is the actuator torque (input), and d is the disturbance acting on the system.

First, assume that disturbance is zero and take $J = 2$, $B = 0.5$. Design a PD controller such that the closed loop system is critically damped, and settling time is 2 second. Do not do this by tuning the gains; calculate the K_p and K_d gains using natural frequency and damping ratio.

Solution

Since $d(t) = 0$, $J = 2$, $B = 0.5$, we have

$$2\ddot{\theta}(t) + 0.5\dot{\theta}(t) = u(t)$$

Transform to Laplace domain,

$$2\Theta(s)s^2 + 0.5\Theta(s)s = U(s)$$

$$\Theta(s)[2s^2 + 0.5s] = U(s) \tag{1}$$

$$\frac{\Theta(s)}{U(s)} = \frac{1}{2s^2 + 0.5s}$$

Let our PD controller model be

$$K_p e + K_d \dot{e} = u$$

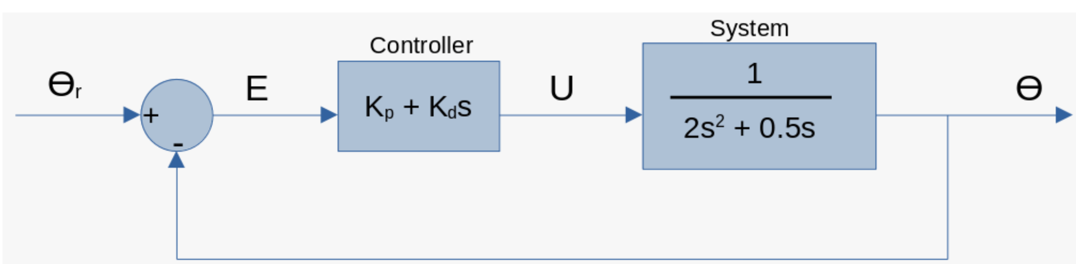
Transform to Laplace domain,

$$K_p E(s) + K_d E(s)s = U(s)$$
$$E(s)[K_p + K_d s] = U(s)$$

Therefore, the transfer function for the PD controller is

$$\frac{U(s)}{E(s)} = K_p + K_d s \tag{2}$$

Now we can draw the block diagram, as shown below.



From the block diagram, we can see that

$$E = \Theta_r - \Theta$$

Question 1 continued on next page. . .

Using equation 2,

$$\frac{U(s)}{K_p + K_d s} = \Theta_r - \Theta$$

Furthermore, using equation 1,

$$\begin{aligned}
\frac{\Theta(s)[2s^2 + 0.5s]}{K_p + K_d s} &= \Theta_r - \Theta \\
\frac{\Theta[2s^2 + 0.5s]}{K_p + K_d s} + \Theta &= \Theta_r \\
\Theta\left(\frac{2s^2 + 0.5s}{K_p + K_d s} + 1\right) &= \Theta \\
\Theta\left(\frac{2s^2 + 0.5s + K_p + K_d s}{K_p + K_d s}\right) &= \Theta_r
\end{aligned}$$

Therefore,

$$\frac{\Theta}{\Theta_r} = \frac{K_p + K_d s}{2s^2 + s(0.5 + K_d) + K_p}$$

So our charateristic equation is,

$$\begin{aligned}
2s^2 + s(0.5 + K_d) + K_p &= 0 \\
s^2 + s\frac{(0.5 + K_d)}{2} + \frac{K_p}{2} &= 0
\end{aligned}$$

The general form of the charateristic equation is

$$s^2 + (2\xi\omega_n)s + {\omega_n}^2 = 0$$

Where ξ is the damping ratio and ω_n is the natural frequency.

Hence, we have,

$${\omega_n}^2 = \frac{K_p}{2} \tag{3}$$

and

$$2\xi\omega_n = \frac{(0.5 + K_d)}{2} \tag{4}$$

Also, we know that the natural frequency and settling time T_s are related by

$$\xi\omega_n T_s = 4$$

Since we are solving for a critically damped system, we set $\xi = 1$. We also want settling time $T_s = 2$ seconds.

So,

$$\begin{aligned}
\xi\omega_n T_s &= 4 \\
1 \cdot \omega_n \cdot 2 &= 4 \\
\omega_n &= 2
\end{aligned}$$

Plugging this into equation 3, we have

$$
\begin{aligned}
(2)^2 &= \frac{K_p}{2} \\
4 &= \frac{K_p}{2} \\
\boxed{K_p = 8}
\end{aligned}
$$

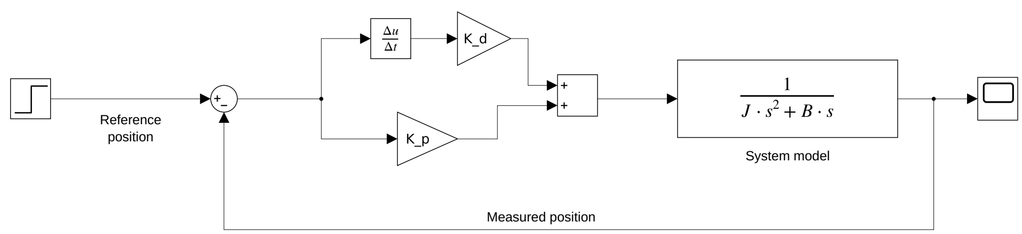
Also, plugging in values into equation 4, we have

$$
\begin{aligned}
2(1)(2) &= \frac{0.5 + K_d}{2} \\
8 &= 0.5 + K_d \\
\boxed{K_d = 7.5}
\end{aligned}
$$

Question 2

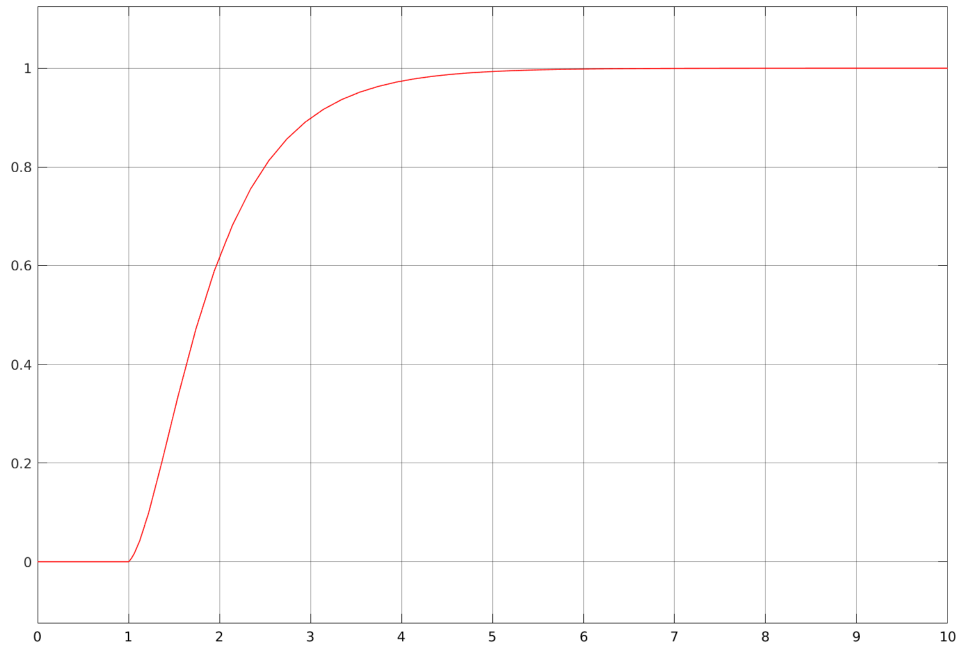
Follow steps in the assignment PDF file. Explain the process and be sure to include the plot to your report.

Solution



Block diagram for Question 2

After constructing the system model, here is the plot we obtained.



Generated plot for Question 2