RBE 500 Homework #6

Arjan Gupta

Question 1

Consider the following robot joint model

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = u(t) + d(t)$$

where J is the inertia of the link, B is the effective damping on the link, θ is the joint angle, u is the actuator torque (input), and d is the disturbance acting on the system.

First, assume that disturbance is zero and take $J = 2$, $B = 0.5$. Design a PD controller such that the closed loop system is critically damped, and settling time is 2 second. Do not do this by tuning the gains; calculate the K_p and K_d gains using natural frequency and damping ratio.

Solution

Since $d(t) = 0$, $J = 2$, $B = 0.5$, we have

$$2\ddot{\theta}(t) + 0.5\dot{\theta}(t) = u(t)$$

Transform to Laplace domain,

$$2\Theta(s)s^2 + 0.5\Theta(s)s = U(s)$$

$$\Theta(s)[2s^2 + 0.5s] = U(s) \tag{1}$$

$$\frac{\Theta(s)}{U(s)} = \frac{1}{2s^2 + 0.5s}$$

Let our PD controller model be

$$K_p e + K_d \dot{e} = u$$

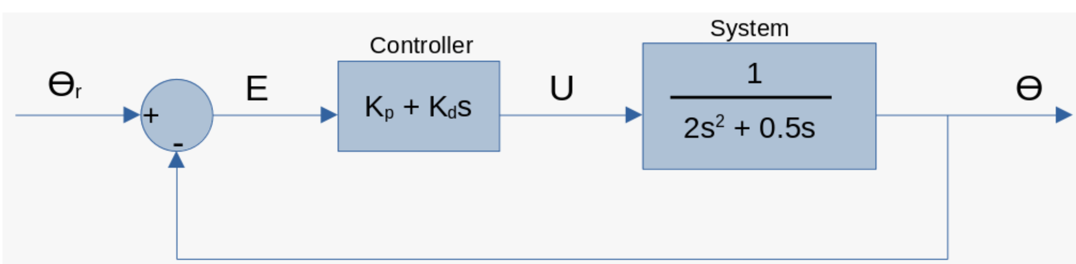
Transform to Laplace domain,

$$K_p E(s) + K_d E(s)s = U(s)$$
$$E(s)[K_p + K_d s] = U(s)$$

Therefore, the transfer function for the PD controller is

$$\frac{U(s)}{E(s)} = K_p + K_d s \tag{2}$$

Now we can draw the block diagram, as shown below.



From the block diagram, we can see that

$$E = \Theta_r - \Theta$$

Using equation 2,

$$\frac{U(s)}{K_p + K_d s} = \Theta_r - \Theta$$

Furthermore, using equation 1,

$$\begin{aligned}
\frac{\Theta(s)[2s^2 + 0.5s]}{K_p + K_d s} &= \Theta_r - \Theta \\
\frac{\Theta[2s^2 + 0.5s]}{K_p + K_d s} + \Theta &= \Theta_r \\
\Theta\left(\frac{2s^2 + 0.5s}{K_p + K_d s} + 1\right) &= \Theta \\
\Theta\left(\frac{2s^2 + 0.5s + K_p + K_d s}{K_p + K_d s}\right) &= \Theta_r
\end{aligned}$$

Therefore,

$$\frac{\Theta}{\Theta_r} = \frac{K_p + K_d s}{2s^2 + s(0.5 + K_d) + K_p}$$

So our charateristic equation is,

$$\begin{aligned}
2s^2 + s(0.5 + K_d) + K_p &= 0 \\
s^2 + s\frac{(0.5 + K_d)}{2} + \frac{K_p}{2} &= 0
\end{aligned}$$

The general form of the charateristic equation is

$$s^2 + (2\xi\omega_n)s + {\omega_n}^2 = 0$$

Where ξ is the damping ratio and ω_n is the natural frequency.

Hence, we have,

$${\omega_n}^2 = \frac{K_p}{2} \tag{3}$$

and

$$2\xi\omega_n = \frac{(0.5 + K_d)}{2} \tag{4}$$

Also, we know that the natural frequency and settling time T_s are related by

$$\xi\omega_n T_s = 4$$

Since we are solving for a critically damped system, we set $\xi = 1$. We also want settling time $T_s = 2$ seconds.

So,

$$\begin{aligned}
\xi\omega_n T_s &= 4 \\
1 \cdot \omega_n \cdot 2 &= 4 \\
\omega_n &= 2
\end{aligned}$$

Plugging this into equation 3, we have

$$\begin{aligned}(2)^2 &= \frac{K_p}{2} \\ 4 &= \frac{K_p}{2} \\ \Aboxed{K_p &= 8}\end{aligned}$$

Also, plugging in values into equation 4, we have

$$\begin{aligned}2(1)(2) &= \frac{0.5 + K_d}{2} \\ 8 &= 0.5 + K_d \\ \Aboxed{K_d &= 7.5}\end{aligned}$$

Question 2

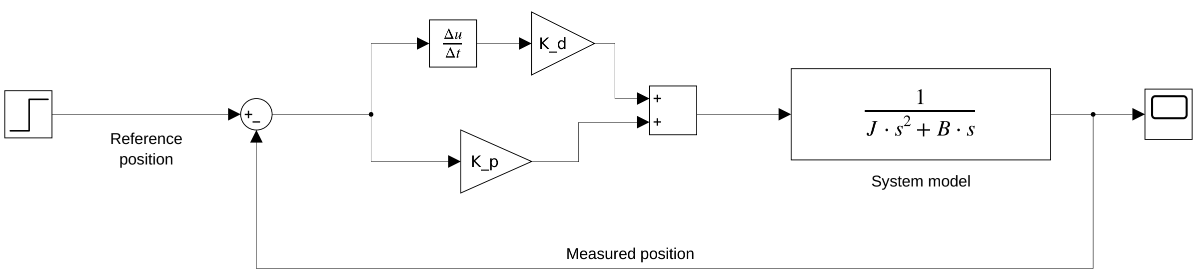
Follow steps in the assignment PDF file. Explain the process and be sure to include the plot to your report.

Solution

First we write the following MATLAB script that contains the system and controller values of the closed loop system. Here, J is the intertia of the link in the system model, B is the effective damping on the link in the system model, K_p is the proportional gain for the controller, and K_d is the derivative gain for the controller.

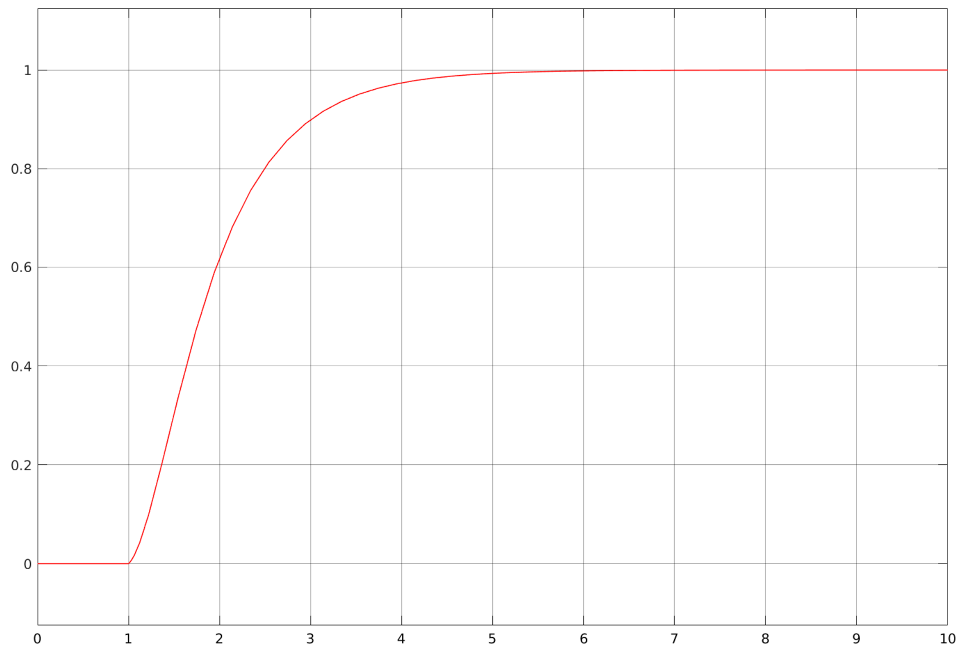
```
% Code for Question 2 of HW6 for RBE500

clear; close all; clc;

% System model
J = 2;
B = 0.5;

% Controller
K_p = 8;
K_d = 7.5;
```

Next, we run our script so that the variables are loaded into our base workspace in MATLAB. We then type `simulink`



Block diagram for Question 2

After constructing the system model, we obtained the following plot.



Generated plot for Question 2