

# **RBE 500 Homework #2**

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### Problem 3.5

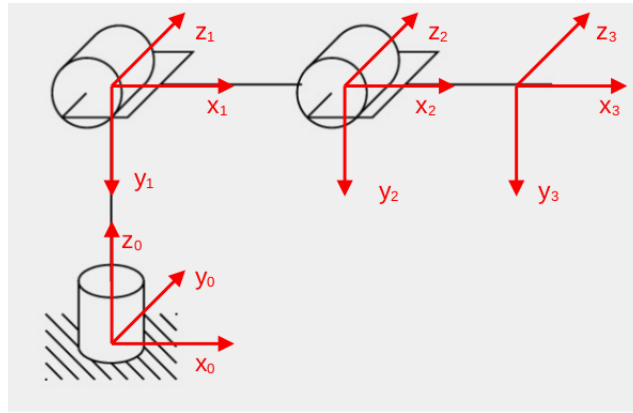
Consider the three-link articulated robot of Figure 3.16. Derive the forward kinematic equations using the DH convention.



Figure 3.16: Three-link articulated robot.

#### Solution

First we assign coordinate frames 0 through 3 (links 0 through 3). This is done as per the following figure.



Now, we create a table for quantities  $\alpha_i, a_i, \theta_i, d_i$  for links 1 through 3.

| Link | $\alpha_i$  | $a_i$ | $\theta_i$ | $d_i$ |
|------|-------------|-------|------------|-------|
| 1    | $-90^\circ$ | 0     | $\theta_1$ | $d_1$ |
| 2    | 0           | $a_2$ | $\theta_2$ | 0     |
| 3    | 0           | $a_3$ | $\theta_3$ | 0     |

Next, we use the matrix obtained from equation 3.10 of the textbook to calculate  $A_1, A_2, A_3$ .

$$A_1 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \cos(-90^\circ) & \sin \theta_1 \sin(-90^\circ) & 0 \cdot \cos \theta_1 \\ \sin \theta_1 & \cos \theta_1 \cos(-90^\circ) & -\cos \theta_1 \sin(-90^\circ) & 0 \cdot \sin \theta_1 \\ 0 & \sin(-90^\circ) & \cos(-90^\circ) & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where  $s_1 = \sin \theta_1$  and  $c_1 = \cos \theta_1$ . Similarly,

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we can find  $T_3^0 = A_1 A_2 A_3$ . We use the following MATLAB code to compute this.

```

1 % Calculation code for problem 3.5 of the RBE500 textbook (HW 2)
2
3 clear; close all; clc;
4
5 syms c1 s1 d1 c2 s2 a2 c3 s3 a3;
6 A1 = [c1 0 -s1 d1; s1 0 c1 0; 0 -1 0 d1; 0 -1 0 d1; 0 0 0 1];
7 A2 = [c2 -s2 0 a2*c2; s2 c2 0 a2*s2; 0 0 1 0; 0 0 0 1];
8 A3 = [c3 -s3 0 a3*c3; s3 c3 0 a3*s3; 0 0 1 0; 0 0 0 1];
9
10 T = A1*A2*A3;
11
12 % Show output matrix
13 disp(T)
14
15 % Generate LaTeX code
16 latex(T)

```

Therefore,

$$T_3^0 = \begin{bmatrix} c_1 c_2 c_3 - c_1 s_2 s_3 & -c_1 c_2 s_3 - c_1 c_3 s_2 & -s_1 & d_1 + a_2 c_1 c_2 - a_3 c_1 s_2 s_3 + a_3 c_1 c_2 c_3 \\ c_2 c_3 s_1 - s_1 s_2 s_3 & -c_2 s_1 s_3 - c_3 s_1 s_2 & c_1 & a_2 c_2 s_1 - a_3 s_1 s_2 s_3 + a_3 c_2 c_3 s_1 \\ -c_2 s_3 - c_3 s_2 & s_2 s_3 - c_2 c_3 & 0 & d_1 - a_2 s_2 - a_3 c_2 s_3 - a_3 c_3 s_2 \\ -c_2 s_3 - c_3 s_2 & s_2 s_3 - c_2 c_3 & 0 & d_1 - a_2 s_2 - a_3 c_2 s_3 - a_3 c_3 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

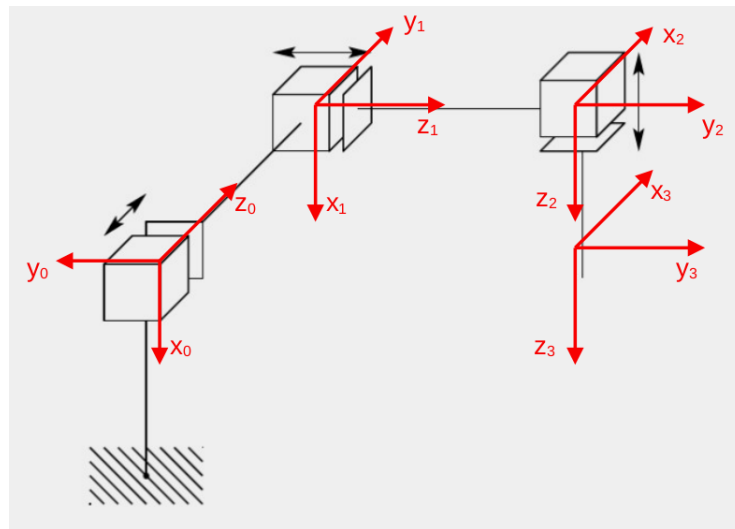
## Problem 3.6

Consider the three-link Cartesian manipulator of Figure 3.17. Derive the forward kinematic equations using the DH convention.



### Solution

First we assign coordinate frames 0 through 3 (links 0 through 3). This is done as per the following figure.



Now, we create a table for quantities  $\alpha_i, a_i, \theta_i, d_i$  for links 1 through 3.

| Link | $\alpha_i$ | $a_i$ | $\theta_i$ | $d_i$ |
|------|------------|-------|------------|-------|
| 1    | $90^\circ$ | 0     | 0          | $d_1$ |
| 2    | $90^\circ$ | 0     | $90^\circ$ | $d_2$ |
| 3    | 0          | 0     | 0          | $d_3$ |

Next, we use the matrix obtained from equation 3.10 of the textbook to calculate  $A_1, A_2, A_3$ .

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Problem -1

Write part of **Quick-Sort**(*list*, *start*, *end*)

```
1: function QUICK-SORT(list, start, end)
2:   if start  $\geq$  end then
3:     return
4:   end if
5:   mid  $\leftarrow$  PARTITION(list, start, end)
6:   QUICK-SORT(list, start, mid - 1)
7:   QUICK-SORT(list, mid + 1, end)
8: end function
```

Algorithm 1: Start of QuickSort

**Problem -1**

Suppose we would like to fit a straight line through the origin, i.e.,  $Y_i = \beta_1 x_i + e_i$  with  $i = 1, \dots, n$ ,  $E[e_i] = 0$ , and  $\text{Var}[e_i] = \sigma_e^2$  and  $\text{Cov}[e_i, e_j] = 0, \forall i \neq j$ .

**Part A**

Find the least squares estimator for  $\hat{\beta}_1$  for the slope  $\beta_1$ .

**Solution**

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 \\ &= \sum_{i=1}^n (Y_i - \hat{\beta}_1 x_i)^2 \end{aligned}$$

By taking the partial derivative in respect to  $\hat{\beta}_1$ , we get:

$$\frac{\partial}{\partial \hat{\beta}_1} (RSS) = -2 \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^n x_i Y_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^n x_i Y_i - \hat{\beta}_1 \sum_{i=1}^n x_i^2 \end{aligned}$$

Solving for  $\hat{\beta}_1$  gives the final estimator for  $\beta_1$ :

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

**Part B**

Calculate the bias and the variance for the estimated slope  $\hat{\beta}_1$ .

**Solution**

For the bias, we need to calculate the expected value  $E[\hat{\beta}_1]$ :

$$\begin{aligned} E[\hat{\beta}_1] &= E\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i E[Y_i]}{\sum x_i^2} \\ &= \frac{\sum x_i (\beta_1 x_i)}{\sum x_i^2} \\ &= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1 \end{aligned}$$

Thus since our estimator's expected value is  $\beta_1$ , we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned} \text{Var}[\hat{\beta}_1] &= \text{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \sigma^2 \\ &= \frac{\sigma^2}{\sum x_i^2} \end{aligned}$$



## Problem -1

Prove a polynomial of degree  $k$ ,  $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$  is a member of  $\Theta(n^k)$  where  $a_k \dots a_0$  are nonnegative constants.

*Proof.* To prove that  $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$ , we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are,  $n^k \leq a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$  even if  $c_1 = 1$  and  $n_0 = 1$ . This is because  $n^k \leq c_1 \cdot a_k n^k$  for any nonnegative constant,  $c_1$  and  $a_k$ .

Taking the second inequality, we prove it in the following way. By summation,  $\sum_{i=0}^k a_i$  will give us a new constant,  $A$ . By taking this value of  $A$ , we can then do the following:

$$\begin{aligned} a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0 &= \\ &\leq (a_k + a_{k-1} \dots a_1 + a_0) \cdot n^k \\ &= A \cdot n^k \\ &\leq c_2 \cdot n^k \end{aligned}$$

where  $n_0 = 1$  and  $c_2 = A$ .  $c_2$  is just a constant. Thus the proof is complete. □

**Problem 18**

Evaluate  $\sum_{k=1}^5 k^2$  and  $\sum_{k=1}^5 (k-1)^2$ .

**Problem -1**

Find the derivative of  $f(x) = x^4 + 3x^2 - 2$

**Problem 6**

Evaluate the integrals  $\int_0^1 (1-x^2)dx$  and  $\int_1^\infty \frac{1}{x^2} dx$ .