RBE 500 Homework #3

Arjan Gupta

Problem 4.2

Verify Equation (4.7) by direct calculation.

$$S(a)p = a \times p \tag{4.7}$$

Solution

Suppose the vectors a and p are given as

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

By the definition of the cross-product, we know

$$a \times p = \begin{bmatrix} a_2 p_3 - a_3 p_2 \\ a_3 p_1 - a_1 p_3 \\ a_1 p_2 - a_2 p_1 \end{bmatrix} \tag{1}$$

Also, by the definition of skew-symmetric matrices, we know the form of $S(a)$, where a is the vector we have already defined,

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Hence, by normal matrix multiplication,

$$S(a)p = \begin{bmatrix} 0 - a_3 p_2 + a_2 p_3 \\ a_3 p_1 + 0 - a_1 p_3 \\ -a_2 p_1 + a_1 p_2 + 0 \end{bmatrix} = \begin{bmatrix} a_2 p_3 - a_3 p_2 \\ a_3 p_1 - a_1 p_3 \\ a_1 p_2 - a_2 p_1 \end{bmatrix}$$

Which is the same as (1). Therefore, Equation (4.7) is proved.

Problem 4.3

Prove the assertion given in Equation (4.9) that $R(a \times b) = Ra \times Rb$ for $R \in SO(3)$.

Solution

Let $v = R(a \times b)$, and $u = Ra \times Rb$. If v and u are the to be proved as the same vector, then it must be shown that they have the exact same magnitude and direction. Let us consider magnitude and direction of v and u separately.

Magnitude

Since $R \in SO(3)$, $\det R = 1$, which means that the linear transformation R does not change the length (norm) of any vector that it transforms. Hence we can say

$$\|R(a \times b)\| = \|a \times b\| \tag{1}$$

$$\|Ra\| = \|a\| \tag{2}$$

$$\|Rb\| = \|b\| \tag{3}$$

And by the definition of cross product, we know that

$$\|a \times b\| = \|a\|\|b\| \sin\theta$$

Using Equation (1) here, we have

$$\|R(a \times b)\| = \|a\|\|b\| \sin\theta \tag{4}$$

Direction