# RBE 500 Homework #4

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# Problem 4.6

Given $R = R_{x,\theta} R_{y,\phi}$, compute $\frac{\partial R}{\partial \phi}$. Evaluate $\frac{\partial R}{\partial \phi}$ at $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$. First parametrically compute, then evaluate by plugging the values in.

## Solution

$$\frac{\partial}{\partial \phi}\left(R_{x,\theta} R_{y,\phi}\right) = R_{x,\theta} \frac{\partial}{\partial \phi}\left(R_{y,\phi}\right)$$

Using the the fact that $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(R_{y,\theta}\right) = S(j) R_{y,\theta}$,

$$\begin{aligned}
R_{x,\theta} \frac{\partial}{\partial \phi}\left(R_{y,\phi}\right) = R_{x,\theta} S(j) R_{y,\phi} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} -\sin(\phi) & 0 & \cos(\phi) \\ \cos(\phi)\sin(\theta) & 0 & \sin(\phi)\sin(\theta) \\ -\cos(\phi)\cos(\theta) & 0 & -\cos(\theta)\sin(\phi) \end{bmatrix}
\end{aligned}$$

Now, plugging in the values $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$, we get

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We performed the above computations using the following MATLAB code.

```matlab
% Calculation code for problem 4.6 of the RBE500 textbook (HW 4)

clear; close all; clc;

syms theta phi

% Define the matrices in our problem
rotx_theta = [1 0 0; 0 cos(theta) -sin(theta); 0 sin(theta) cos(theta)];
Sj = [0 0 1; 0 0 0; -1 0 0];
roty_phi = [cos(phi) 0 sin(phi); 0 1 0; -sin(phi) 0 cos(phi)];

% Multiply the matrices
product = rotx_theta*Sj*roty_phi;

% Get latex output
latex(product)

phi_val = pi/2;
theta_val = pi/2;

% Now plug in values
product_val = [ -sin(phi_val), 0, cos(phi_val);
cos(phi_val)*sin(theta_val), 0, sin(phi_val)*sin(theta_val);
-cos(phi_val)*cos(theta_val), 0, -cos(theta_val)*sin(phi_val)]
```

# Problem 4.10

Two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the homogeneous transformation

$$H = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A particle has velocity $v_1(t) = (3, 1, 0)$ relative to frame $o_1x_1y_1z_1$. What is the velocity of the particle in frame $o_0x_0y_0z_0$?

## Solution

The given H is the homogeneous transformation $H_1^0$. Therefore, $R_1^0 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $o_1^0 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

At any given point in time, the position of the particle with respect to frame $o_1x_1y_1z_1$ is given as $p^1(t)$. We know that

$$p^0(t) = R_1^0 p^1(t) + o_1^0$$

Taking the derivative of both sides and using the product rule, we get

$$v^0(t) = \dot{R}_1^0 p^1(t) + R_1^0 v^1(t) + 0$$

But, $\dot{R}_1^0 = 0$ since $R_1^0$ is a constant in time. Therefore,

$$v^0(t) = R_1^0 v^1(t)$$

$$v^0(t) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$v^0(t) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

Which was calculated by the following MATLAB script

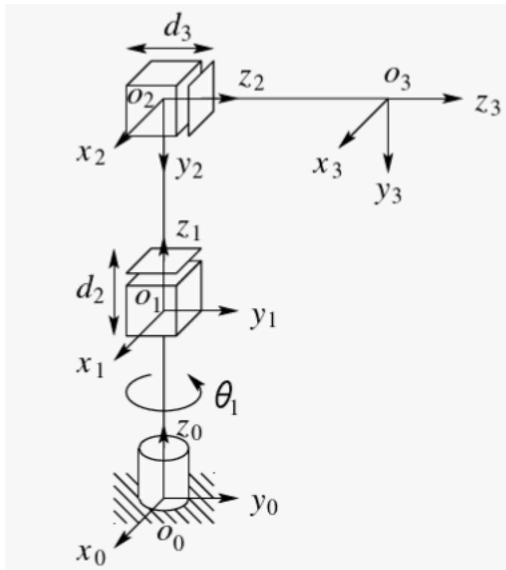
```
% Calculation code for problem 4.10 of the RBE500 textbook (HW 4)

clear; close all; clc;

R = [0 -1 0; 1 0 0; 0 0 1];
v1 = [3; 1; 0];

v0 = R*v1
```

# Problem 4.15

Find the $6 \times 3$ Jacobian for the three links of the cylindrical manipulator of Figure 3.7. Find the singular configurations for this arm.

## Solution

Figure 3.7 of our book is the following.




We can see that we have 3 joints, so $n = 3$. Let us also form the table given in the lecture videos:

| | Linear component | Angular component |
|---|---|---|
| Revolute joint | $J_{v_i} = z_{i-1}^0 \times (o_n^0 - o_{i-1}^0)$ | $J_{v_i} = z_{i-1}^0$ |
| Prismatic joint | $J_{\omega_i} = z_{i-1}^0$ | $J_{\omega_i} = 0$ |

Using this table, and the fact that the upper half of the Jacobian contains linear components while the bottom half contains angular components, we have

$$J = \begin{bmatrix} z_0 \times (o_3 - o_0) & z_1 & z_2 \\ z_0 & 0 & 0 \end{bmatrix}$$

In this Jacobian matrix, we know that $z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. To find $z_1$, $z_2$, and $o_3$, we need to find $T_n^0 = A_1 \dots A_n$ for $n = 1, 2, 3$. Following the DH convention (which Figure 3.7 already abides by), we have the following table for quantities $\alpha_i, a_i, \theta_i, d_i$.

| Link | $\alpha_i$ | $a_i$ | $\theta_i$ | $d_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $\theta_1$ | $d_1$ |
| 2 | -90° | 0 | 0 | $d_2$ |
| 3 | 0 | 0 | 0 | $d_3$ |

Problem 4.15 continued on next page. . . 

Which gives us the following $A_i$ matrices.

$$A_1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we can compute our $T$ matrices using the following MATLAB code,

```
% Calculation code for problem 4.15 (T matrices) of the RBE500 textbook (HW 4)

clear; close all; clc;

syms theta1 d1 d2 d3;

% Form the A matrices
A1 = [cos(theta1) -sin(theta1) 0 0; sin(theta1) cos(theta1) 0 0; 0 0 1 d1; 0 0 0 1];
A2 = [1 0 0 0; 0 0 1 0; 0 -1 0 d2; 0 0 0 1];
A3 = [1 0 0 0; 0 1 0 0; 0 0 1 d3; 0 0 0 1];

% Compute T matrices
T2 = A1*A2;
T3 = A1*A2*A3;

% Output to LaTex
latex(T2)
latex(T3)
```

$$T_1^0 = A_1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} \cos(\theta_1) & 0 & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & \cos(\theta_1) & 0 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} \cos(\theta_1) & 0 & -\sin(\theta_1) & -d_3\,\sin(\theta_1) \\ \sin(\theta_1) & 0 & \cos(\theta_1) & d_3\,\cos(\theta_1) \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From these $T$ matrices, we get

$$z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, z_2 = \begin{bmatrix} -\sin(\theta_1) \\ \cos(\theta_1) \\ 0 \end{bmatrix}, o_3 = \begin{bmatrix} -d_3\,\sin(\theta_1) \\ d_3\,\cos(\theta_1) \\ d_1 + d_2 \end{bmatrix}$$

Now we are ready to compute our Jacobian matrix. We do this using the following MATLAB code.

```
% Calculation code for problem 4.15 (Jacobian) of the RBE500 textbook (HW 4)

clear; close all; clc;

syms theta1 a3 d1 d2 d3;

% Declare vectors
z0 = [0; 0; 1];
z1 = z0;
z2 = [-sin(theta1); cos(theta1); 0];
o0 = [0; 0; 0];
o3 = [-d3*sin(theta1); d3*cos(theta1); d1+d2];

% Compute Jacobian
J = [cross(z0, (o3 - o0)) z1 z2; z0 zeros(3,1) zeros(3,1)]

% Output LaTex
latex(J)
```

Which gives us the following Jacobian,

$$J = \begin{bmatrix} -d_3\,\cos\left(\theta_1\right) & 0 & -\sin\left(\theta_1\right) \\ -d_3\,\sin\left(\theta_1\right) & 0 & \cos\left(\theta_1\right) \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$