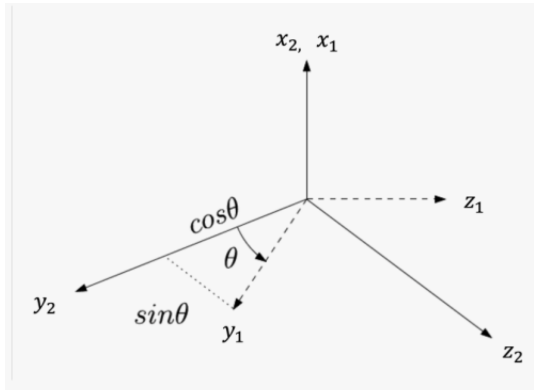
RBE 500 Midterm

Arjan Gupta

Question 1

Derive the rotation matrix R_2^1 (you can leave sines and cosines as is).



Solution

Since the x-axis remains the same in the rotation, we know this is a basic 3D rotation matrix representing a rotation about the x-axis. However, using the right-hand screw rule, we see that the angle θ here is negative. So, by using equation 2.7 (page 43) of our main textbook, the rotation matrix is given by,

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta) & -\sin(-\theta) \\ 0 & \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$\boxed{R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}$$

Question 2

Find the coordinates of point p expressed in frame 1 (i.e. p^1) given the following.

$$H_1^2 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0.9553 & 0.2955 & -0.9553 \\ 0 & -0.2955 & 0.9553 & 0.2955 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ p^2 = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

Solution

From our knowledge of homogeneous transformations, we know that

$$P^2 = H_1^2 P^1$$

Where $P^2 = \begin{bmatrix} p^2 \\ 1 \end{bmatrix}$ and $P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$.

However, we want to find P^1, so we apply the inverse of H to both sides,

$$\begin{aligned} P^2 &= H_1^2 P^1 \\ (H_1^2)^{-1} P^2 &= P^1 \end{aligned}$$

We know that $H_1^2 = \begin{bmatrix} R_1^2 & d_1^2 \\ 0 & 1 \end{bmatrix}$, where $R_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9553 & 0.2955 \\ 0 & -0.2955 & 0.9553 \end{bmatrix}$ and $d_1^2 = \begin{bmatrix} -1 \\ -0.9553 \\ 0.2955 \end{bmatrix}$.

For accuracy while computing the inverse of H_1^2, we use equation 2.67 of the book (page 63). Therefore,

$$(H_1^2)^{-1} = \begin{bmatrix} (R_1^2)^T & -(R_1^2)^T d_1^2 \\ 0 & 1 \end{bmatrix}$$

. We use the following MATLAB code for this computation.

```
% Calculation code for problem 2 of the RBE500 Midterm

clear; close all; clc;

P2 = [2;5;0;1];

R2_1 = [1 0 0; 0 0.9553 0.2955; 0 -0.2955 0.9553];
d2_1 = [-1; -0.9553; 0.2955];
H_inv = [R2_1' (-R2_1'*d2_1); zeros(1,3) 1];

P1 = H_inv*P2
```

Which gives us the answer,

$$\boxed{P^1 = \begin{bmatrix} 3.0000 \\ 5.7764 \\ 1.4775 \\ 1.0000 \end{bmatrix}}$$

Question 3

If $R_1^0 = \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}$, $R_2^0 = \begin{bmatrix} 0 & 0.866 & 0.5 \\ 0 & 0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$, and $R_3^0 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$, calculate R_2^1.

Solution

Knowing the composition law for rotational transformations, we can write

$$R_2^1 = R_0^1 R_2^0$$
$$R_2^1 = (R_1^0)^T R_2^0$$
$$R_2^1 = \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}^T \begin{bmatrix} 0 & 0.866 & 0.5 \\ 0 & 0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$$
$$R_2^1 = \begin{bmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ 0.7071 & 0 & 0.7071 \end{bmatrix} \begin{bmatrix} 0 & 0.866 & 0.5 \\ 0 & 0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$$

We use the following MATLAB script to compute this multiplication.

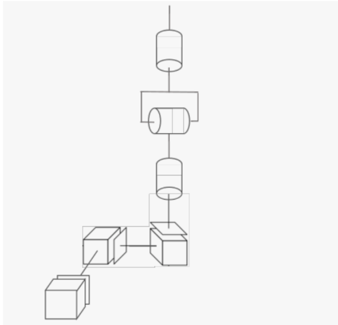
```
% Calculation code for problem 2 of the RBE500 Midterm

clear; close all; clc;

R01 = [0.7071 0 0.7071; 0 1 0; -0.7071 0 0.7071];
R02 = [0 0.866 0.5; 0 0.5 -0.866; -1 0 0];

R12 = R01'*R02
```

Therefore,

$$\boxed{R_2^1 = \begin{bmatrix} 0.7071 & 0.6123 & 0.3535 \\ 0 & 0.5000 & -0.8660 \\ -0.7071 & 0.6123 & 0.3535 \end{bmatrix}}$$

Question 4

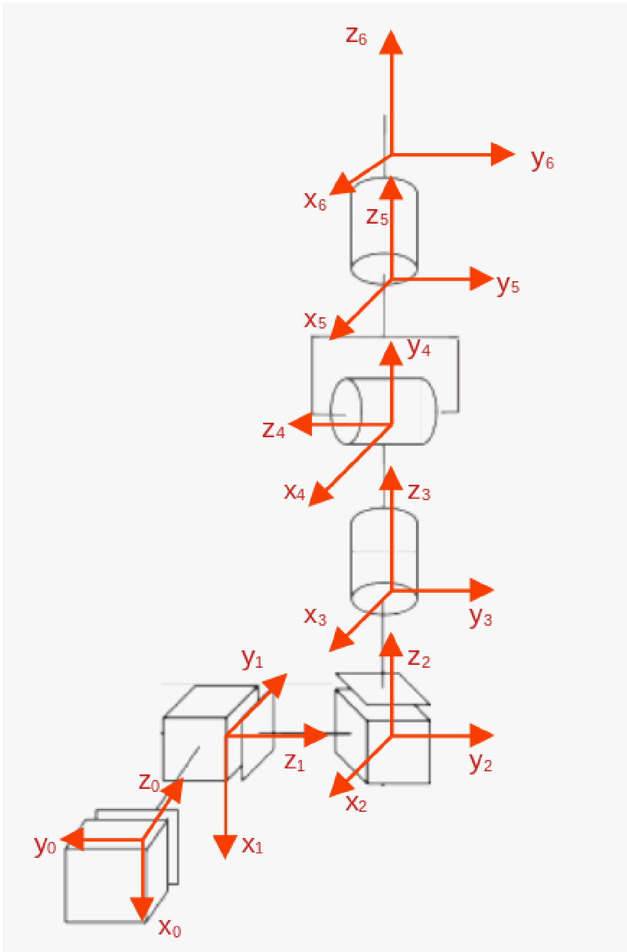
(a) Calculate Denavit Hertanberg parameters for the given manipulator (just filling out the Denavit Hertanberg table would suffice). For this question, you are expected to solve it parametrically, i.e. you can leave sines, cosines, joint values, and link lengths as parameters.



(b) Derive H_2^1. You can leave sines, cosines, joint values, and link lengths as parameters.

Solution for 4(a)

First we assign coordinate frames 0 through 5 (links 0 through 5). This is done as per the following figure.



Now, we create a table for quantities $\alpha_i, a_i, \theta_i, d_i$ for links 1 through 6. In this table, d_1, d_2, d_3, θ_4, θ_5, θ_6 are variable. However, l_4, l_5, l_6 are fixed (constants).

Link	α_i	a_i	θ_i	d_i
1	$90°$	0	0	d_1
2	$90°$	0	$-90°$	d_2
3	0	0	0	d_3
4	$90°$	0	θ_4	l_4
5	$-90°$	0	θ_5	l_5
6	0	0	θ_6	l_6

Solution for 4(b)

We know that $H_2^1 = A_2$, where A_2 is the DH matrix A_i with $i = 2$.

$$
\begin{aligned}
H_2^1 = A_2 &= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2\cos\alpha_2 & \sin\theta_2\sin\alpha_2 & a_i\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2\cos\alpha_2 & -\cos\theta_2\sin\alpha_2 & a_i\sin\theta_2 \\ 0 & \sin\alpha_2 & \cos\alpha_2 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(-90°) & -\sin(-90°)\cos(90°) & \sin(-90°)\sin(90°) & a_i\cos(-90°) \\ \sin(-90°) & \cos(-90°)\cos(90°) & -\cos(-90°)\sin(90°) & a_i\sin(-90°) \\ 0 & \sin(90°) & \cos(90°) & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\boxed{H_2^1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}
$$

Question 5

Calculate inverse kinematics for the manipulator in Question 4. Assume that all the forward kinematics information is available (i.e. all homogenous transformation matrices). Since there are no values given, you will be deriving your expressions parametrically, but please be sure to explicitly show, which homogeneous transformation matrix is required for the corresponding information, and which segment of the matrix is used to obtain that information. (e.g. in your derivations you can say something like: "to calculate this expression, I would need z_3^0, which is available to me at the 3^{rd} column of H_3^0").

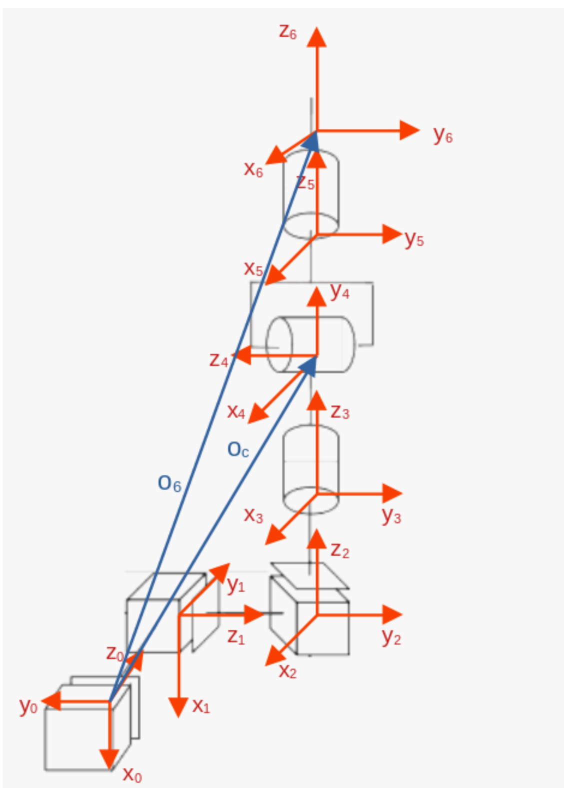
Solution

Before we begin, let us make a brief list of steps we need to take to solve the complete inverse kinematics problem for our particular manipulator's configuration.

1. Find wrist center o_c.
2. Find q_1, q_2, q_3.
3. Perform forward kinematics to arrive at $R_3^0 = (R_0^3)^T$.
4. Get $R_6^3 = R_0^3 R_6^0$.
5. Use R_6^3 to find ϕ, θ, ψ of Euler configuration to find q_4, q_5, q_6.

In essence, once we have found all joint variables given the end-effector's homogeneous transformation, we have solved the inverse kinematics problem.

Now, considering the figure we had in Question 4, let us draw vectors o_c and o_6 in it as shown below.



Question 5 continued on next page...

We have determined o_c as the wrist center. This is because joints 4, 5, 6 form a spherical wrist, and the z-axes of frames 3, 4, 5 intersect at o_c. Now let us proceed to Step 1.

Step 1 — Find the wrist center

The end-effector's homogeneous transformation is known to us as the 4×4 matrix

$$H_6^0 = \begin{bmatrix} R_6^0 & o_6^0 \\ 0 & 1 \end{bmatrix}$$

where

$$R_6^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, o_6^0 = \begin{bmatrix} x_6 \\ y_6 \\ z_6 \end{bmatrix}$$

Where o_6^0 is o_6 as shown in the diagram. As shown in the figure, we can establish a relationship between o_6 and o_c as

$$o_c = o_6 - (l_5 + l_6) R_6^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x_6 - (l_5 + l_6) r_{13} \\ y_6 - (l_5 + l_6) r_{23} \\ z_6 - (l_5 + l_6) r_{33} \end{bmatrix}$$

Where $(l_5 + l_6)$ is a scalar.

Step 2 — Find q_1, q_2, q_3.

In the figure we have drawn, we can see that $x_c = -(d_3 + l_4)$, where d_3 is variable and l_4 is a constant. Therfore, $d_3 = -x_c - l_4 = -(x_c + l_4)$.

We can also see that $y_c = -d_2$ and $z_c = d_1$.

Therefore, we have,

$$q_1 = d_1 = z_c = z_6 - (l_5 + l_6) r_{33}$$
$$q_2 = d_2 = -y_c = -y_6 + (l_5 + l_6) r_{23}$$
$$q_3 = d_3 = -(x_c + l_4) = -x_6 + (l_5 + l_6) r_{13} - l_4$$

Step 3 — Perform forward kinematics

We perform forward kinematics for the first three joint variables. We already found the table in Question 4 as well as A_2. We write A_1 and A_3 as the following.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$