

# RBE 500 Homework #6

Arjan Gupta

## Question 1

Consider the following robot joint model

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = u(t) + d(t)$$

where  $J$  is the inertia of the link,  $B$  is the effective damping on the link,  $\theta$  is the joint angle,  $u$  is the actuator torque (input), and  $d$  is the disturbance acting on the system.

First, assume that disturbance is zero and take  $J = 2$ ,  $B = 0.5$ . Design a PD controller such that the closed loop system is critically damped, and settling time is 2 second. Do not do this by tuning the gains; calculate the  $K_p$  and  $K_d$  gains using natural frequency and damping ratio.

## Solution

Since  $d(t) = 0$ ,  $J = 2$ ,  $B = 0.5$ , we have

$$2\ddot{\theta}(t) + 0.5\dot{\theta}(t) = u(t)$$

Transform to Laplace domain,

$$2\Theta(s)s^2 + 0.5\Theta(s)s = U(s)$$

$$\Theta(s)[2s^2 + 0.5s] = U(s) \quad (1)$$

$$\frac{\Theta(s)}{U(s)} = \frac{1}{2s^2 + 0.5s}$$

Let our PD controller model be

$$K_p e + K_d \dot{e} = u$$

Transform to Laplace domain,

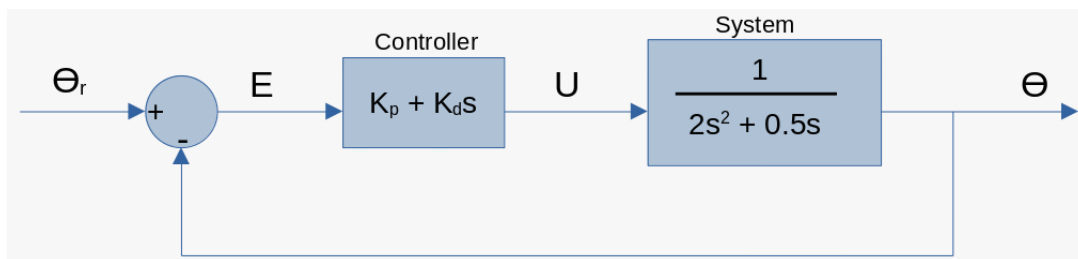
$$K_p E(s) + K_d E(s)s = U(s)$$

$$E(s)[K_p + K_d s] = U(s)$$

Therefore, the transfer function for the PD controller is

$$\frac{U(s)}{E(s)} = K_p + K_d s \quad (2)$$

Now we can draw the block diagram, as shown below.



From the block diagram, we can see that

$$E = \Theta_r - \Theta$$

Using equation 2,

$$\frac{U(s)}{K_p + K_d s} = \Theta_r - \Theta$$

Furthermore, using equation 1,

$$\begin{aligned}\frac{\Theta(s)[2s^2 + 0.5s]}{K_p + K_d s} &= \Theta_r - \Theta \\ \frac{\Theta[2s^2 + 0.5s]}{K_p + K_d s} + \Theta &= \Theta_r \\ \Theta \left( \frac{2s^2 + 0.5s}{K_p + K_d s} + 1 \right) &= \Theta_r \\ \Theta \left( \frac{2s^2 + 0.5s + K_p + K_d s}{K_p + K_d s} \right) &= \Theta_r\end{aligned}$$

Therefore,

$$\frac{\Theta}{\Theta_r} = \frac{K_p + K_d s}{2s^2 + s(0.5 + K_d) + K_p}$$

So our characteristic equation is,

$$\begin{aligned}2s^2 + s(0.5 + K_d) + K_p &= 0 \\ s^2 + s \frac{(0.5 + K_d)}{2} + \frac{K_p}{2} &= 0\end{aligned}$$

The general form of the characteristic equation is

$$s^2 + (2\xi\omega_n)s + \omega_n^2 = 0$$

Where  $\xi$  is the damping ratio and  $\omega_n$  is the natural frequency.

Hence, we have,

$$\omega_n^2 = \frac{K_p}{2} \quad (3)$$

and

$$2\xi\omega_n = \frac{(0.5 + K_d)}{2} \quad (4)$$

Also, we know that the natural frequency and settling time  $T_s$  are related by

$$\xi\omega_n T_s = 4$$

Since we are solving for a critically damped system, we set  $\xi = 1$ . We also want settling time  $T_s = 2$  seconds.

So,

$$\begin{aligned}\xi\omega_n T_s &= 4 \\ 1 \cdot \omega_n \cdot 2 &= 4 \\ \omega_n &= 2\end{aligned}$$

Plugging this into equation 3, we have

$$(2)^2 = \frac{K_p}{2}$$

$$4 = \frac{K_p}{2}$$

$$\boxed{K_p = 8}$$

Also, plugging in values into equation 4, we have

$$2(1)(2) = \frac{0.5 + K_d}{2}$$

$$8 = 0.5 + K_d$$

$$\boxed{K_d = 7.5}$$

## Question 2