

RBE 500 Homework #3

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Problem 4.2

Verify Equation (4.7) by direct calculation.

$$S(a)p = a \times p \tag{4.7}$$

Solution

Suppose the vectors a and p are given as

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

By the definition of the cross-product, we know

$$a \times p = \begin{bmatrix} a_2p_3 - a_3p_2 \\ a_3p_1 - a_1p_3 \\ a_1p_2 - a_2p_1 \end{bmatrix} \tag{1}$$

Also, by the definition of skew-symmetric matrices, we know the form of $S(a)$, where a is the vector we have already defined,

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Hence, by normal matrix multiplication,

$$S(a)p = \begin{bmatrix} 0 - a_3p_2 + a_2p_3 \\ a_3p_1 + 0 - a_1p_3 \\ -a_2p_1 + a_1p_2 + 0 \end{bmatrix} = \begin{bmatrix} a_2p_3 - a_3p_2 \\ a_3p_1 - a_1p_3 \\ a_1p_2 - a_2p_1 \end{bmatrix}$$

Which is the same as (1). Therefore, Equation (4.7) is proved.

Problem 4.3

Prove the assertion given in Equation (4.9) that $R(a \times b) = Ra \times Rb$ for $R \in SO(3)$.

Solution

Let $v = R(a \times b)$, and $u = Ra \times Rb$. If v and u are to be proved as the same vector, then it must be shown that they have the exact same magnitude and direction. Let us consider magnitude and direction of v and u separately.

Magnitude

Since $R \in SO(3)$, $\det R = 1$, which means that the linear transformation R does not change the length (norm) of any vector that it transforms (rotates). Hence we can say

$$\|R(a \times b)\| = \|a \times b\| \quad (1)$$

$$\|Ra\| = \|a\| \quad (2)$$

$$\|Rb\| = \|b\| \quad (3)$$

Using the definition of the cross product along with (1), we can state that

$$\|v\| = \|R(a \times b)\| = \|a \times b\| = \|a\|\|b\| \sin \theta \quad (4)$$

Again using the definition of the cross product along with (2) and (3), we can state that

$$\|u\| = \|Ra \times Rb\| = \|Ra\|\|Rb\| \sin \theta = \|a\|\|b\| \sin \theta \quad (5)$$

We can see that (4) and (5) are equal. Therefore,

$$\|v\| = \|u\|$$

Which means that the magnitude of v and u are the same.

Direction

Since $R \in SO(3)$ is a merely a rotational transformation of the 3 dimensional vector space, any directional relationships given by the right-hand curl rule before R has been applied must remain preserved and obtainable by the right-hand curl rule even after R has been applied.

Now, let us assume that the vectors a and b lie in the plane P . Also, assume $a \times b$ lies in the direction given by \hat{q} . We know that \hat{q} can be obtained by applying the right-hand curl rule from a to b .

After R is applied, P becomes plane P_R , and \hat{q} becomes \hat{q}_R . This means that $v = R(a \times b)$ lies along \hat{q}_R . Also, vector a and b become vectors Ra and Rb , which now lie in plane P_R . However, we can still obtain \hat{q}_R by applying the right-hand curl rule from Ra to Rb , since this relationship is preserved. Hence, $u = Ra \times Rb$ lies in the direction \hat{q}_R . Therefore, v and u lie in the same direction.

Since v and u lie in the same direction and have the same magnitude, they are the same vector. Hence, (4.9) is proved.

Problem 4.5

Suppose that $a = (1, -1, 2)$ and that $R = R_{x,90}$. Show by direct calculation that $RS(a)R^T = S(Ra)$.

Solution

$$R = R_{x,90} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(90) & -\sin(90) \\ 0 & \sin(90) & \cos(90) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$S(a) = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} RS(a)R^T &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2 & -1 \\ -1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \end{aligned}$$

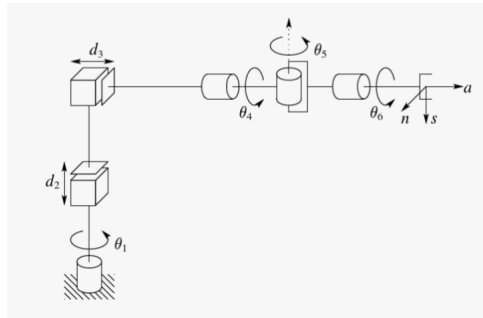
$$Ra = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$S(Ra) = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

As we can see, it has been shown that $RS(a)R^T = S(Ra)$.

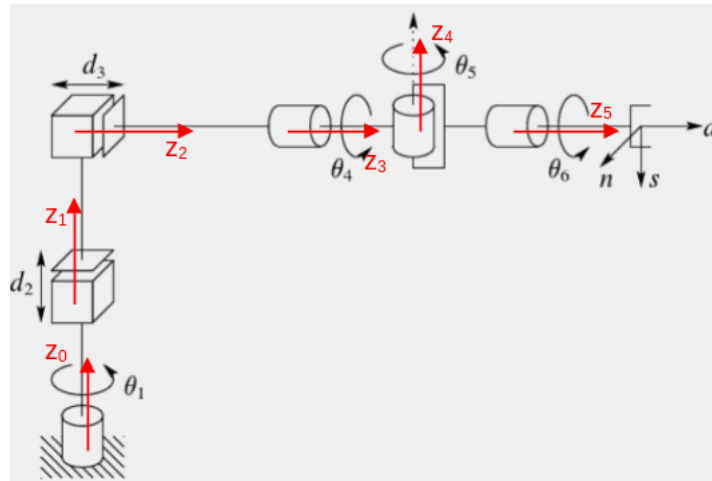
Jacobian Problem

Derive the Jacobian for the following image.



Solution

For easy visualization, let us draw the z axes for each joint.



We can see that we have 6 joints, so $n = 6$. Let us also form the table given in the lecture videos:

| | Linear component | Angular component |
|-----------------|--|----------------------------|
| Revolute joint | $J_{v_i} = z_{i-1}^0 \times (o_n^0 - o_{i-1}^0)$ | $J_{\omega_i} = z_{i-1}^0$ |
| Prismatic joint | $J_{\omega_i} = z_{i-1}^0$ | $J_{v_i} = 0$ |

Report for ROS2 Portion

For this week's ROS assignment portion, I created a subscriber just like how we did for last week's assignment. I also used the same `ros2` command type as the one last week, specifically, `ros2 topic pub --once ... <topic-name> <data-type> <data>`.

Futhermore, this week I set up my subscriber to accept 3 joint variable values from the publisher. These 3 values are floats that represent the angles for the three revolute joints $(\theta_1, \theta_2, \theta_3)$ of the robot manipulator we considered in Problem 3.5 of Homework #2. Using $\theta_1, \theta_2, \theta_3$, my subscriber calculated forward kinematics using the symbolic T matrix I calculated last week, given by

$$T_3^0 = \begin{bmatrix} c_1 c_2 c_3 - c_1 s_2 s_3 & -c_1 c_2 s_3 - c_1 c_3 s_2 & -s_1 & a_2 c_1 c_2 - a_3 c_1 s_2 s_3 + a_3 c_1 c_2 c_3 \\ c_2 c_3 s_1 - s_1 s_2 s_3 & -c_2 s_1 s_3 - c_3 s_1 s_2 & c_1 & a_2 c_2 s_1 - a_3 s_1 s_2 s_3 + a_3 c_2 c_3 s_1 \\ -c_2 s_3 - c_3 s_2 & s_2 s_3 - c_2 c_3 & 0 & d_1 - a_2 s_2 - a_3 c_2 s_3 - a_3 c_3 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

I imported the `numpy` python package to put this T_3^0 matrix in a `numpy.array`, and then I printed the matrix for the given joint variables. In my code, I made sure to name c_i and s_i exactly that way so that I could visually verify any issues in setting up the matrix. The `math.cos` and `math.sin` python functions were used to for assigning values to c_i and s_i . I also set d_1, a_2, a_3 as 1 to make sure I could easily verify that my code is working correctly.