RBE 500 Homework #3

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Problem 4.2

Verify Equation (4.7) by direct calculation.

$$S(a)p = a \times p \tag{4.7}$$

Solution

Suppose the vectors a and p are given as

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

By the definition of the cross-product, we know

$$a \times p = \begin{bmatrix} a_2p_3 - a_3p_2 \\ a_3p_1 - a_1p_3 \\ a_1p_2 - a_2p_1 \end{bmatrix} \tag{1}$$

Also, by the definition of skew-symmetric matrices, we know the form of $S(a)$, where a is the vector we have already defined,

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Hence, by normal matrix multiplication,

$$S(a)p = \begin{bmatrix} 0 - a_3p_2 + a_2p_3 \\ a_3p_1 + 0 - a_1p_3 \\ -a_2p_1 + a_1p_2 + 0 \end{bmatrix} = \begin{bmatrix} a_2p_3 - a_3p_2 \\ a_3p_1 - a_1p_3 \\ a_1p_2 - a_2p_1 \end{bmatrix}$$

Which is the same as (1). Therefore, Equation (4.7) is proved.

Problem 4.3

Prove the assertion given in Equation (4.9) that $R(a \times b) = Ra \times Rb$ for $R \in SO(3)$.

Solution

Let $v = R(a \times b)$, and $u = Ra \times Rb$. If v and u are the to be proved as the same vector, then it must be shown that they have the exact same magnitude and direction. Let us consider magnitude and direction of v and u separately.

Magnitude

Since $R \in SO(3)$, $\det R = 1$, which means that the linear transformation R does not change the length (norm) of any vector that it transforms (rotates). Hence we can say

$$\|R(a \times b)\| = \|a \times b\| \tag{1}$$

$$\|Ra\| = \|a\| \tag{2}$$

$$\|Rb\| = \|b\| \tag{3}$$

Using the definition of the cross product along with (1), we can state that

$$\|v\| = \|R(a \times b)\| = \|a \times b\| = \|a\|\|b\| \sin\theta \tag{4}$$

Again using the definition of the cross product along with (2) and (3), we can state that

$$\|u\| = \|Ra \times Rb\| = \|Ra\|\|Rb\| \sin\theta = \|a\|\|b\| \sin\theta \tag{5}$$

We can see that (4) and (5) are equal. Therefore,

$$\|v\| = \|u\|$$

Which means that the magnitude of v and u are the same.

Direction

$R \in SO(3)$ is a linear change of basis for the 3 dimensional vector space, and each column of R is a basis vector. Since R is a linear transformation, any two planes that are perpendicular by the right-hand curl rule before R has been applied must remain perpendicular by the right-hand curl rule even after R has been applied.

Given these facts, let us assume that the vectors a and b lie in the plane P. Also, assume $a \times b$ lies in the plane Q, and its direction is $\hat{q}$. By the definition of the cross product, $P \perp_{rhc} Q$, where $\perp_{rhc}$ is the perpendicular plane given by the right-hand curl rule. When we apply R, we obtain plane P_R for P and plane Q_R for Q. We already know that $P_R \perp_{rhc} Q_R$ by the facts established in the last paragraph.

Now, as we have described, $v = R(a \times b)$ lies on Q_R. Also, Ra and Rb lie on P_R, and hence $u = Ra \times Rb$ lies in Q_R.