RBE 500 Homework #6

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Question 1

Consider the following robot joint model

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = u(t) + d(t)$$

where J is the inertia of the link, B is the effective damping on the link, θ is the joint angle, u is the actuator torque (input), and d is the disturbance acting on the system.

First, assume that disturbance is zero and take $J = 2$, $B = 0.5$. Design a PD controller such that the closed loop system is critically damped, and settling time is 2 second. Do not do this by tuning the gains; calculate the K_p and K_d gains using natural frequency and damping ratio.

Solution

Since $d(t) = 0$, $J = 2$, $B = 0.5$, we have

$$2\ddot{\theta}(t) + 0.5\dot{\theta}(t) = u(t)$$

Transform to Laplace domain,

$$2\Theta(s)s^2 + 0.5\Theta(s)s = U(s)$$

$$\Theta(s)[2s^2 + 0.5s] = U(s) \tag{1}$$

$$\frac{\Theta(s)}{U(s)} = \frac{1}{2s^2 + 0.5s}$$

Let our PD controller model be

$$K_p e + K_d \dot{e} = u$$

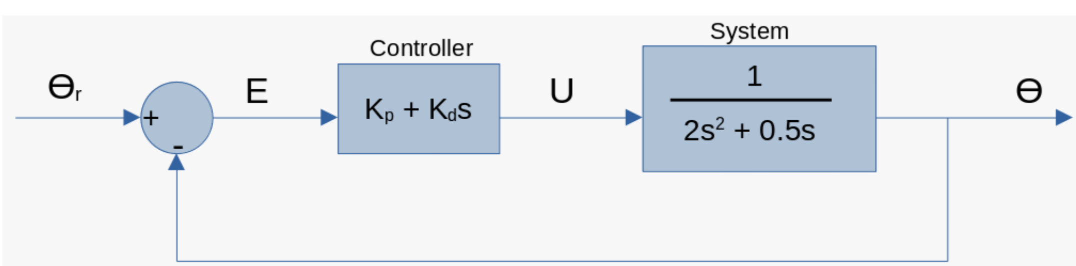
Transform to Laplace domain,

$$K_p E(s) + K_d E(s)s = U(s)$$
$$E(s)[K_p + K_d s] = U(s)$$

Therefore, the transfer function for the PD controller is

$$\frac{U(s)}{E(s)} = K_p + K_d s \tag{2}$$

Now we can draw the block diagram, as shown below.



From the block diagram, we can see that

$$E = \Theta_r - \Theta$$

Using equation 2,

$$\frac{U(s)}{K_p + K_d s} = \Theta_r - \Theta$$

Furthermore, using equation 1,

$$\begin{aligned}
\frac{\Theta(s)[2s^2 + 0.5s]}{K_p + K_d s} &= \Theta_r - \Theta \\
\frac{\Theta[2s^2 + 0.5s]}{K_p + K_d s} + \Theta &= \Theta_r \\
\Theta\left(\frac{2s^2 + 0.5s}{K_p + K_d s} + 1\right) &= \Theta \\
\Theta\left(\frac{2s^2 + 0.5s + K_p + K_d s}{K_p + K_d s}\right) &= \Theta_r
\end{aligned}$$

Therefore,

$$\frac{\Theta}{\Theta_r} = \frac{K_p + K_d s}{2s^2 + s(0.5 + K_d) + K_p}$$

So our charateristic equation is,

$$\begin{aligned}
2s^2 + s(0.5 + K_d) + K_p &= 0 \\
s^2 + s\frac{(0.5 + K_d)}{2} + \frac{K_p}{2} &= 0
\end{aligned}$$

The general form of the charateristic equation is

$$s^2 + (2\xi\omega_n)s + {\omega_n}^2 = 0$$

Where ξ is the damping ratio and ω_n is the natural frequency.

Hence, we have,

$${\omega_n}^2 = \frac{K_p}{2} \tag{3}$$

and

$$2\xi\omega_n = \frac{(0.5 + K_d)}{2} \tag{4}$$

Also, we know that the natural frequency and settling time T_s are related by

$$\xi\omega_n T_s = 4$$

Since we are solving for a critically damped system, we set $\xi = 1$. We also want settling time $T_s = 2$ seconds.

So,

$$\begin{aligned}
\xi\omega_n T_s &= 4 \\
1 \cdot \omega_n \cdot 2 &= 4 \\
\omega_n &= 2
\end{aligned}$$

Plugging this into equation 3, we have

$$\begin{aligned}(2)^2 &= \frac{K_p}{2} \\ 4 &= \frac{K_p}{2} \\ \boxed{K_p = 8}\end{aligned}$$

Also, plugging in values into equation 4, we have

$$\begin{aligned}2(1)(2) &= \frac{0.5 + K_d}{2} \\ 8 &= 0.5 + K_d \\ \boxed{K_d = 7.5}\end{aligned}$$

Question 2

Follow steps in the assignment PDF file. Explain the process and be sure to include the plot to your report.

Solution

First we write the following MATLAB script that contains the system and controller values of the closed loop system. Here, J is the intertia of the link in the system model, B is the effective damping on the link in the system model, K_p is the proportional gain for the controller, and K_d is the derivative gain for the controller.

```
% Code for Question 2 of HW6 for RBE500

clear; close all; clc;

% System model
J = 2;
B = 0.5;

% Controller
K_p = 8;
K_d = 7.5;
```

Next, we run our script so that the variables are loaded into our base workspace in MATLAB. We then type `simulink` into the MATLAB command window to launch Simulink. Now we use the Library Browser to begin to construct our closed-loop block diagram. Our process is as follows:

- First, we choose the step function as the input block. We double-click on this block and set the step time as 0 instead of 1, because we want to provide an instantaneous signal.
- Then we choose gain blocks to construct our PD controller. By double-clicking on the gain blocks, we are able to indicate the variables `K_p` and `K_d` from our base worksapce. In the case of `K_d`, we also add a derivative block before the gain block.
- Now we add a summation block to add these gains.
- Next, we add a transfer function block and write $[J\ B\ 0]$ in the denominator to accurately represent our transfer function.
- Now we connect the output of the controller summation into our system model.
- We can now add a scope to monitor the output of our controlled system. We connect the transfer function output to the scope.
- Now we add a summation after the input, and edit 22it by double-clicking the summation and making sure it says $| + -$.
- We can now represent our feedback error. For this, we right-click the connector leading to the scope, and draw a line into the minus part of the summation after the input.
- The input is fed into the plus part of the summation right after the input, so we draw that connector.

- The output of the summation right after the input feeds into our controller, so we draw that connector as well.
- We are almost ready to run our simulation. However as explained in the lectures, in order to be more accurate, we want to edit the solver in our model settings such that the solver-selection is fixed-step, and ode3 (Bogacki-Shampine).
- Now our closed loop system is ready to run. We click the play button to run the simulation.
- Our plot is now ready, which can be viewed by double-clicking the scope block.

Figure 1 below shows the end-result of the closed loop system we constructed in Simulink.

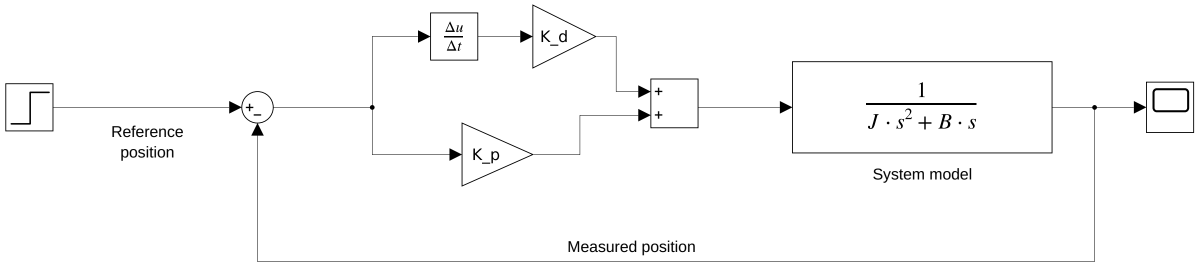


Figure 1: Block diagram for Question 2

After constructing the block diagram, we obtained the following plot (Figure 2).

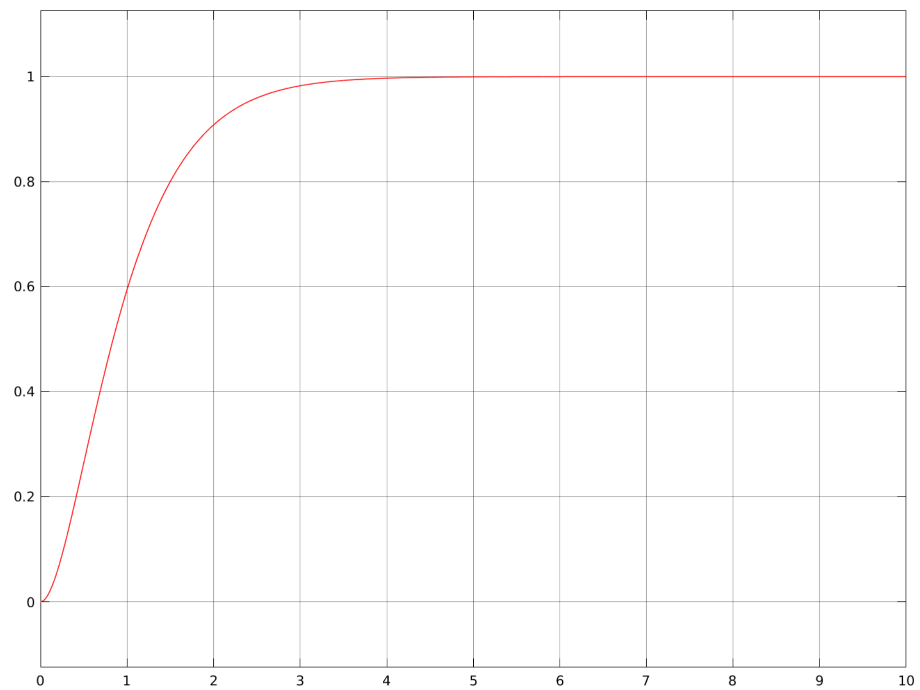


Figure 2: Generated plot for Question 2

Question 3

You may have noticed that the system performance (overshoot and settling time) is quite close to your goal, but it might not be exactly there. Report the current settling time in your report (you may need to zoom in to the plot to see the exact settling time). This is because the equations defining the relations between the damping ratio, natural frequency, settling time and overshoot, are approximations, not exact relations. Specifically, the settling time equation is only accurate when damping ratio is $\ll 1$. Now, tune the values of K_p and K_d so that you get closer to your goal performance. Please remember the K_p and K_d to the system performance from our slides and discussions. Explain your process and include a plot showing the results after tuning. If your values have already been good and do not require tuning, report that as well.

Solution

When we zoom into the plot we obtained in Question 2, we get the following plot (Figure 3).

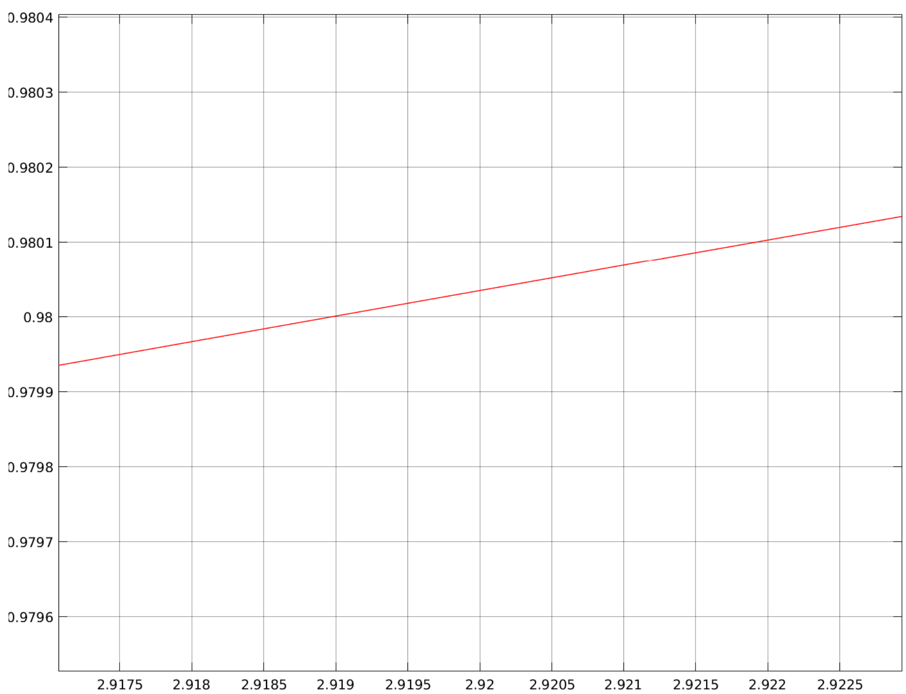


Figure 3: Zoomed-in plot for Question 3

It is clear from this plot that the system's response enters the 2% tolerance zone (i.e., above 0.98 and below 1.02) at 4.2 seconds. Therefore, the settling time T_s is 2.919 seconds.

Now we take the following steps to manually tune our K_p and K_d values to achieve our goal settling time of 2 seconds.

- First K_p was increased to 10. This decreased the settling time to about 2.2 seconds with no overshoot.
- Then K_p was incrased to 12. This decreased the settling time to about 1.6 seconds but introduced an overshoot of about 1%.
- K_p was reduced to 11. This decreased the settling time to 1.8, and reduced overshoot to 0.5%.
- Next, we increased K_d to 15. This drastically worsened our settling time (around 4 sec) while removing the overshoot.

Question 3 continued on next page. . .

- K_d was then decreased to 6. This decreased our settling time to 1.3, and caused an overshoot of 5%.
- We then set K_d to 8. This decreased our overshoot to about 0.1%, and decreased our settling time to 2.034 seconds.

Therefore, by tuning our controller gains to $\boxed{K_p = 11}$ and $\boxed{K_d = 8}$, we have achieved an improved controller.

The following plots show the improved system response and settling-time of the system.

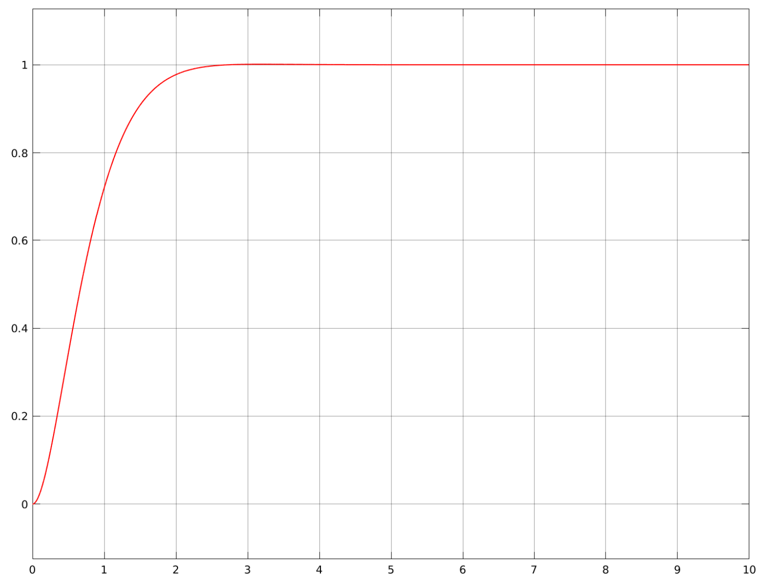


Figure 4: Tuned system reponse for Question 3

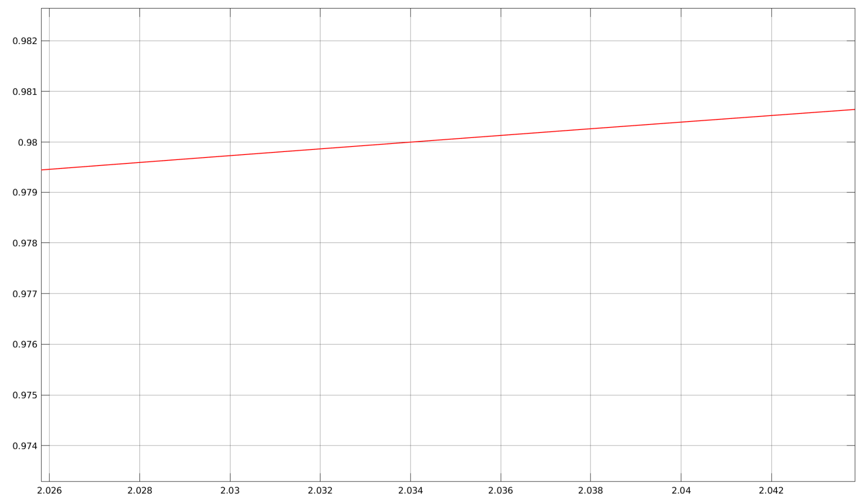


Figure 5: Zoomed-in plot to show settling time for Question 3