# RBE 500 Homework #2

**Arjan Gupta**

# Problem 3.5

Consider the three-link articulated robot of Figure 3.16. Derive the forward kinematic equations using the DH convention.

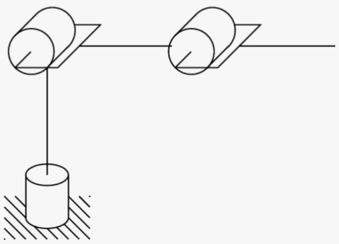
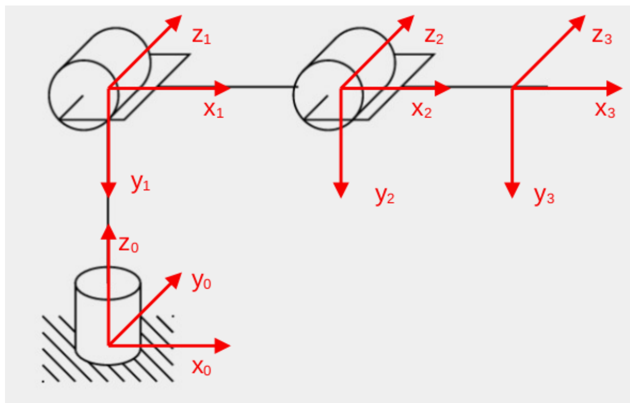


Figure 3.16: Three-link articulated robot.

### Solution

First we assign coordinate frames 0 through 3 (links 0 through 3). This is done as per the following figure.




Now, we create a table for quantities $\alpha_i, a_i, \theta_i, d_i$ for links 1 through 3.

| Link | $\alpha_i$ | $a_i$ | $\theta_i$ | $d_i$ |
|---|---|---|---|---|
| 1 | -90° | 0 | $\theta_1$ | $d_1$ |
| 2 | 0 | $a_2$ | $\theta_2$ | 0 |
| 3 | 0 | $a_3$ | $\theta_3$ | 0 |

Next, we use the matrix obtained from equation 3.10 of the textbook to calculate $A_1, A_2, A_3$.

$$A_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1\cos(-90°) & \sin\theta_1\sin(-90°) & 0\cdot\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1\cos(-90°) & -\cos\theta_1\sin(-90°) & 0\cdot\sin\theta_1 \\ 0 & \sin(-90°) & \cos(-90°) & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where $s_1 = \sin\theta_1$ and $c_1 = \cos\theta_1$. Similarly,

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we can find $T_3^0 = A_1A_2A_3$. We use the following MATLAB code to compute this.

```matlab
% Calculation code for problem 3.5 of the RBE500 textbook (HW 2)

clear; close all; clc;

syms c1 s1 d1 c2 s2 a2 c3 s3 a3;
A1 = [c1 0 -s1 d1; s1 0 c1 0; 0 -1 0 d1; 0 -1 0 d1; 0 0 0 1];
A2 = [c2 -s2 0 a2*c2; s2 c2 0 a2*s2; 0 0 1 0; 0 0 0 1];
A3 = [c3 -s3 0 a3*c3; s3 c3 0 a3*s3; 0 0 1 0; 0 0 0 1];

T = A1*A2*A3;

% Show output matrix
disp(T)

% Generate LaTex code
latex(T)
```

Therefore,

$$T_3^0 = \begin{bmatrix} c_1\,c_2\,c_3 - c_1\,s_2\,s_3 & -c_1\,c_2\,s_3 - c_1\,c_3\,s_2 & -s_1 & d_1 + a_2\,c_1\,c_2 - a_3\,c_1\,s_2\,s_3 + a_3\,c_1\,c_2\,c_3 \\ c_2\,c_3\,s_1 - s_1\,s_2\,s_3 & -c_2\,s_1\,s_3 - c_3\,s_1\,s_2 & c_1 & a_2\,c_2\,s_1 - a_3\,s_1\,s_2\,s_3 + a_3\,c_2\,c_3\,s_1 \\ -c_2\,s_3 - c_3\,s_2 & s_2\,s_3 - c_2\,c_3 & 0 & d_1 - a_2\,s_2 - a_3\,c_2\,s_3 - a_3\,c_3\,s_2 \\ -c_2\,s_3 - c_3\,s_2 & s_2\,s_3 - c_2\,c_3 & 0 & d_1 - a_2\,s_2 - a_3\,c_2\,s_3 - a_3\,c_3\,s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Problem 3.6

Consider the three-link Cartesian manipulator of Figure 3.17. Derive the forward kinematic equations using the DH convention.

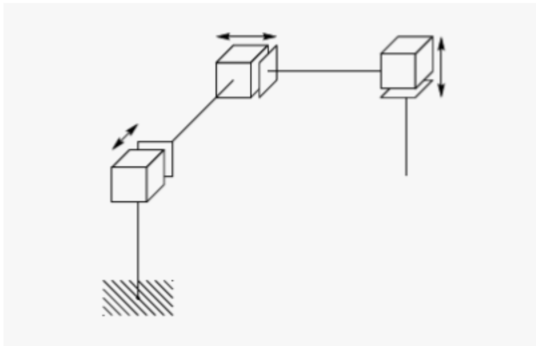
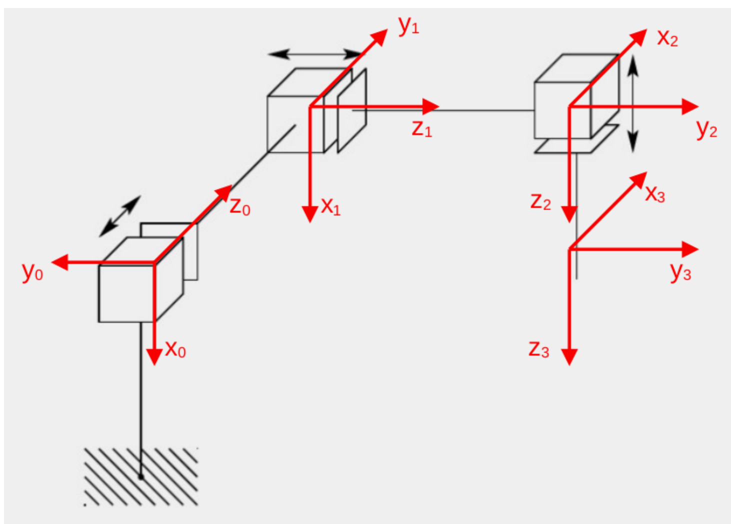


Figure 3.17: Three-link Cartesian robot.

### Solution

First we assign coordinate frames 0 through 3 (links 0 through 3). This is done as per the following figure.




Now, we create a table for quantities $\alpha_i, a_i, \theta_i, d_i$ for links 1 through 3.

| Link | $\alpha_i$ | $a_i$ | $\theta_i$ | $d_i$ |
|---|---|---|---|---|
| 1 | $90°$ | 0 | 0 | $d_1$ |
| 2 | $90°$ | 0 | $90°$ | $d_2$ |
| 3 | 0 | 0 | 0 | $d_3$ |

Next, we use the matrix obtained from equation 3.10 of the textbook to calculate $A_1, A_2, A_3$.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 =$$

# Problem -1

Write part of **Quick-Sort(**$list, start, end$**)**

```
function QUICK-SORT(list, start, end)
    if start ≥ end then
        return
    end if
    mid ← PARTITION(list, start, end)
    QUICK-SORT(list, start, mid − 1)
    QUICK-SORT(list, mid + 1, end)
end function
```

Algorithm 1: Start of QuickSort

# Problem -1

Suppose we would like to fit a straight line through the origin, i.e., $Y_i = \beta_1 x_i + e_i$ with $i = 1, \ldots, n$, $\mathrm{E}[e_i] = 0$, and $\mathrm{Var}[e_i] = \sigma_e^2$ and $\mathrm{Cov}[e_i, e_j] = 0, \forall i \neq j$.

## Part A

Find the least squares esimator for $\hat{\beta}_1$ for the slope $\beta_1$.

### Solution

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2 \\ &= \sum_{i=1}^{n} \left(Y_i - \hat{\beta}_1 x_i\right)^2 \end{aligned}$$

By taking the partial derivative in respect to $\hat{\beta}_1$, we get:

$$\frac{\partial}{\partial \hat{\beta}_1}(RSS) = -2\sum_{i=1}^{n} x_i(Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^{n} x_i(Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^{n} x_i Y_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^{n} x_i Y_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 \end{aligned}$$

Solving for $\hat{\beta}_1$ gives the final estimator for $\beta_1$:

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

Problem -1 continued on next page... 

## Part B

Calculate the bias and the variance for the estimated slope $\hat{\beta_1}$.

## Solution

For the bias, we need to calculate the expected value $\mathrm{E}[\hat{\beta_1}]$:

$$
\begin{aligned}
\mathrm{E}[\hat{\beta_1}] &= \mathrm{E}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i \mathrm{E}[Y_i]}{\sum x_i^2} \\
&= \frac{\sum x_i (\beta_1 x_i)}{\sum x_i^2} \\
&= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1
\end{aligned}
$$

Thus since our estimator's expected value is $\beta_1$, we can conclude that the bias of our estimator is 0.

For the variance:

$$
\begin{aligned}
\mathrm{Var}[\hat{\beta_1}] &= \mathrm{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \sigma^2 \\
&= \frac{\sigma^2}{\sum x_i^2}
\end{aligned}
$$

# Problem -1

Prove a polynomial of degree $k$, $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$ is a member of $\Theta(n^k)$ where $a_k \ldots a_0$ are nonnegative constants.

*Proof.* To prove that $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$, we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0,\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are, $n^k \leq a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$ even if $c_1 = 1$ and $n_0 = 1$. This is because $n^k \leq c_1 \cdot a_k n^k$ for any nonnegative constant, $c_1$ and $a_k$.

Taking the second inequality, we prove it in the following way. By summation, $\sum_{i=0}^{k} a_i$ will give us a new constant, $A$. By taking this value of $A$, we can then do the following:

$$\begin{aligned}
a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0 = & \\
& \leq (a_k + a_{k-1} \ldots a_1 + a_0) \cdot n^k \\
& = A \cdot n^k \\
& \leq c_2 \cdot n^k
\end{aligned}$$

where $n_0 = 1$ and $c_2 = A$. $c_2$ is just a constant. Thus the proof is complete. □

# Problem 18

Evaluate $\sum_{k=1}^{5} k^2$ and $\sum_{k=1}^{5}(k-1)^2$.

# Problem -1

Find the derivative of $f(x) = x^4 + 3x^2 - 2$

# Problem 6

Evaluate the integrals $\int_0^1 (1-x^2)\mathrm{d}x$ and $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$.