RBE 500 Midterm

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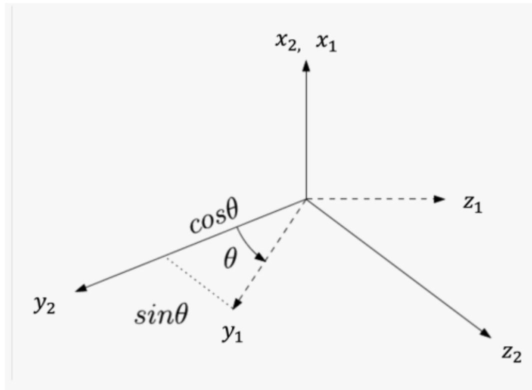
Question 1

Derive the rotation matrix R_2^1 (you can leave sines and cosines as is).



Solution

Since the x-axis remains the same in the rotation, we know this is a basic 3D rotation matrix representing a rotation about the x-axis. However, using the right-hand screw rule, we see that the angle θ here is negative. So, by using equation 2.7 (page 43) of our main textbook, the rotation matrix is given by,

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta) & -\sin(-\theta) \\ 0 & \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$\boxed{R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}$$

Question 2

Find the coordinates of point p expressed in frame 1 (i.e. p^1) given the following.

$$H_1^2 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0.9553 & 0.2955 & -0.9553 \\ 0 & -0.2955 & 0.9553 & 0.2955 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ p^2 = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

Solution

From our knowledge of homogeneous transformations, we know that

$$P^2 = H_1^2 P^1$$

Where $P^2 = \begin{bmatrix} p^2 \\ 1 \end{bmatrix}$ and $P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$.

However, we want to find P^1, so we apply the inverse of H to both sides,

$$P^2 = H_1^2 P^1$$
$$(H_1^2)^{-1} P^2 = P^1$$

We know that $H_1^2 = \begin{bmatrix} R_1^2 & d_1^2 \\ 0 & 1 \end{bmatrix}$, where $R_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9553 & 0.2955 \\ 0 & -0.2955 & 0.9553 \end{bmatrix}$ and $d_1^2 = \begin{bmatrix} -1 \\ -0.9553 \\ 0.2955 \end{bmatrix}$.

For accuracy while computing the inverse of H_1^2, we use equation 2.67 of the book (page 63). Therefore,

$$(H_1^2)^{-1} = \begin{bmatrix} (R_1^2)^T & -(R_1^2)^T d_1^2 \\ 0 & 1 \end{bmatrix}$$

. We use the following MATLAB code for this computation.

```
% Calculation code for problem 2 of the RBE500 Midterm

clear; close all; clc;

P2 = [2;5;0;1];

R2_1 = [1 0 0; 0 0.9553 0.2955; 0 -0.2955 0.9553];
d2_1 = [-1; -0.9553; 0.2955];
H_inv = [R2_1' (-R2_1'*d2_1); zeros(1,3) 1];

P1 = H_inv*P2
```

Which gives us the answer,

$$\boxed{P^1 = \begin{bmatrix} 3.0000 \\ 5.7764 \\ 1.4775 \\ 1.0000 \end{bmatrix}}$$

Question 3

If $R_1^0 = \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}$, $R_2^0 = \begin{bmatrix} 0 & 0.866 & 0.5 \\ 0 & 0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$, and $R_3^0 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$, calculate R_2^1.

Solution

Knowing the composition law for rotational transformations, we can write

$$
\begin{aligned}
R_2^1 &= R_0^1 R_2^0 \\
R_2^1 &= (R_1^0)^T R_2^0 \\
R_2^1 &= \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}^T \begin{bmatrix} 0 & 0.866 & 0.5 \\ 0 & 0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix} \\
R_2^1 &= \begin{bmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ 0.7071 & 0 & 0.7071 \end{bmatrix} \begin{bmatrix} 0 & 0.866 & 0.5 \\ 0 & 0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

We use the following MATLAB script to compute this multiplication.

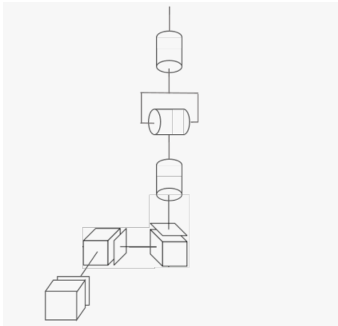
```
% Calculation code for problem 2 of the RBE500 Midterm

clear; close all; clc;

R01 = [0.7071 0 0.7071; 0 1 0; -0.7071 0 0.7071];
R02 = [0 0.866 0.5; 0 0.5 -0.866; -1 0 0];

R12 = R01'*R02
```

Therefore,

$$
\boxed{R_2^1 = \begin{bmatrix} 0.7071 & 0.6123 & 0.3535 \\ 0 & 0.5000 & -0.8660 \\ -0.7071 & 0.6123 & 0.3535 \end{bmatrix}}
$$

Question 4

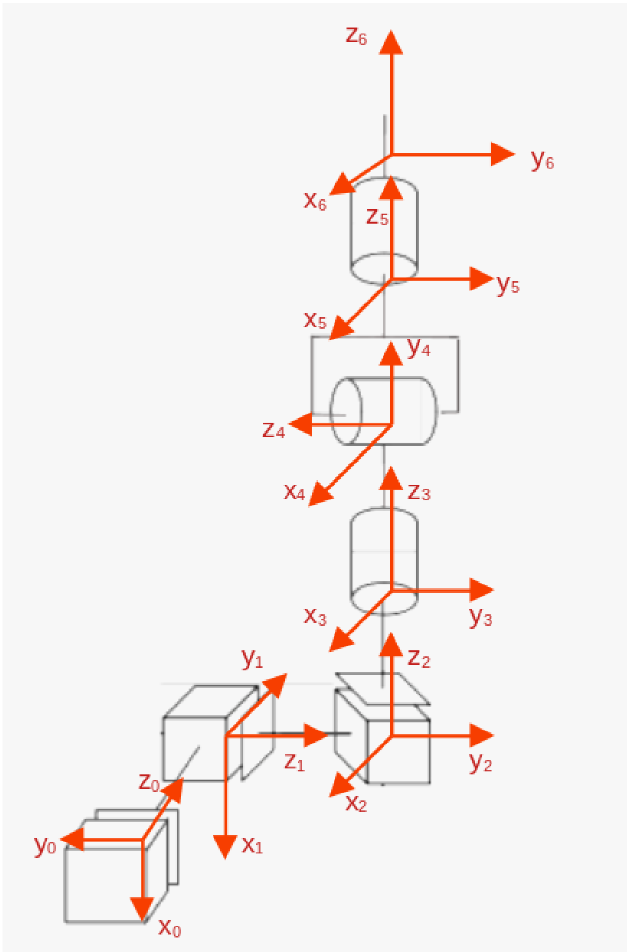
(a) Calculate Denavit Hertanberg parameters for the given manipulator (just filling out the Denavit Hertanberg table would suffice). For this question, you are expected to solve it parametrically, i.e. you can leave sines, cosines, joint values, and link lengths as parameters.



(b) Derive H_2^1. You can leave sines, cosines, joint values, and link lengths as parameters.

Solution for 4(a)

First we assign coordinate frames 0 through 5 (links 0 through 5). This is done as per the following figure.



Now, we create a table for quantities $\alpha_i, a_i, \theta_i, d_i$ for links 1 through 6. In this table, d_1, d_2, d_3, θ_4, θ_5, θ_6 are variable. However, l_4, l_5, l_6 are fixed (constants).

Link	α_i	a_i	θ_i	d_i
1	$90°$	0	0	d_1
2	$90°$	0	$-90°$	d_2
3	0	0	0	d_3
4	$90°$	0	θ_4	l_4
5	$-90°$	0	θ_5	l_5
6	0	0	θ_6	l_6

Solution for 4(b)

We know that $H_2^1 = A_2$, where A_2 is the DH matrix A_i with $i = 2$.

$$H_2^1 = A_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2\cos\alpha_2 & \sin\theta_2\sin\alpha_2 & a_i\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2\cos\alpha_2 & -\cos\theta_2\sin\alpha_2 & a_i\sin\theta_2 \\ 0 & \sin\alpha_2 & \cos\alpha_2 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-90°) & -\sin(-90°)\cos(90°) & \sin(-90°)\sin(90°) & a_i\cos(-90°) \\ \sin(-90°) & \cos(-90°)\cos(90°) & -\cos(-90°)\sin(90°) & a_i\sin(-90°) \\ 0 & \sin(90°) & \cos(90°) & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{H_2^1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Question 5

Calculate inverse kinematics for the manipulator in Question 4. Assume that all the forward kinematics information is available (i.e. all homogenous transformation matrices). Since there are no values given, you will be deriving your expressions parametrically, but please be sure to explicitly show, which homogeneous transformation matrix is required for the corresponding information, and which segment of the matrix is used to obtain that information. (e.g. in your derivations you can say something like: "to calculate this expression, I would need z_3^0, which is available to me at the 3^{rd} column of H_3^0").

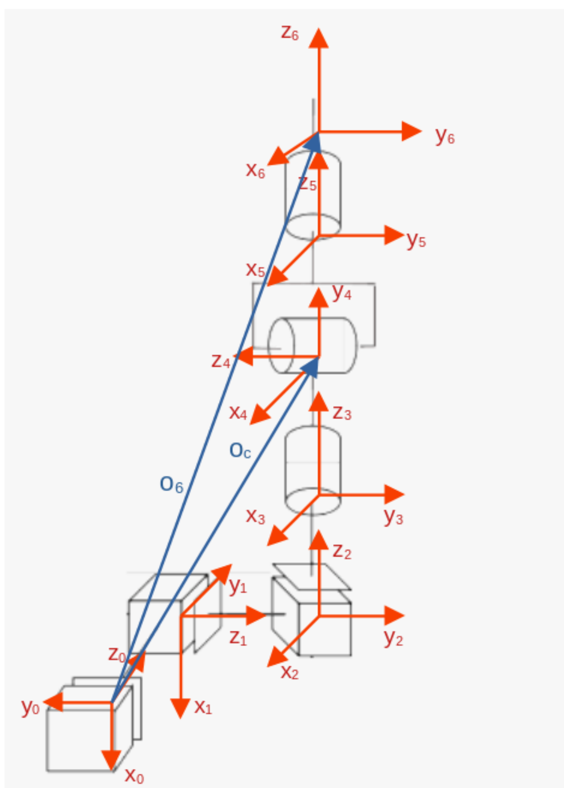
Solution

Before we begin, let us make a brief list of steps we need to take to solve the complete inverse kinematics problem for our particular manipulator's configuration.

1. Find wrist center o_c.
2. Find q_1, q_2, q_3.
3. Perform forward kinematics to arrive at $R_3^0 = (R_0^3)^T$.
4. Get $R_6^3 = R_0^3 R_6^0$.
5. Use R_6^3 to find ϕ, θ, ψ of Euler configuration to find q_4, q_5, q_6.

In essence, once we have found all joint variables given the end-effector's homogeneous transformation, we have solved the inverse kinematics problem.

Now, considering the figure we had in Question 4, let us draw vectors o_c and o_6 in it as shown below.



We have determined o_c as the wrist center. This is because joints 4, 5, 6 form a spherical wrist, and the z-axes of frames 3, 4, 5 intersect at o_c. Now let us proceed to Step 1.

Step 1 — Find the wrist center

The end-effector's homogeneous transformation is known to us as the 4×4 matrix

$$H_6^0 = \begin{bmatrix} R_6^0 & o_6^0 \\ 0 & 1 \end{bmatrix}$$

where

$$R_6^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, o_6^0 = \begin{bmatrix} x_6 \\ y_6 \\ z_6 \end{bmatrix}$$

Where o_6^0 is o_6 as shown in the diagram. As shown in the figure, we can establish a relationship between o_6 and o_c as

$$o_c = o_6 - (l_5 + l_6) R_6^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x_6 - (l_5 + l_6) r_{13} \\ y_6 - (l_5 + l_6) r_{23} \\ z_6 - (l_5 + l_6) r_{33} \end{bmatrix}$$

Where $(l_5 + l_6)$ is a scalar.

Step 2 — Find q_1, q_2, q_3.

In the figure we have drawn, we can see that $x_c = -(d_3 + l_4)$, where d_3 is variable and l_4 is a constant. Therfore, $d_3 = -x_c - l_4 = -(x_c + l_4)$.

We can also see that $y_c = -d_2$ and $z_c = d_1$.

Therefore, we have,

$$\boxed{\begin{aligned} q_1 &= d_1 = z_c = z_6 - (l_5 + l_6) r_{33} \\ q_2 &= d_2 = -y_c = -y_6 + (l_5 + l_6) r_{23} \\ q_3 &= d_3 = -(x_c + l_4) = -x_6 + (l_5 + l_6) r_{13} - l_4 \end{aligned}}$$

Step 3 — Perform forward kinematics

We perform forward kinematics for the first three joint variables. We already found the table in Question 4 as well as A_2. We write A_1 and A_3 as the following.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We obtain T_3^0 using the following MATLAB code.

```
% Calculation code for step 3 of Question 5 of the RBE500 midterm

clear; close all; clc;

syms d1 d2 d3;
A1 = [1 0 0 0; 0 0 -1 0; 0 1 0 d1; 0 0 0 1];
A2 = [0 0 -1 0; -1 0 0 0; 0 1 0 d2; 0 0 0 1];
A3 = [1 0 0 0; 0 1 0 0; 0 0 1 d3; 0 0 0 1];

T = A1*A2*A3;

% Generate LaTex code
latex(T)
```

Therefore,

$$T_3^0 = \begin{bmatrix} 0 & 0 & -1 & -d_3 \\ 0 & -1 & 0 & -d_2 \\ -1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From here, it is clear that

$$R_3^0 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Step 4 — Get $R_6^3 = R_0^3 R_6^0$

We know that $R_0^3 = (R_3^0)^T$. Given this fact, we use the following MATLAB code to calculate R_6^3.

```
% Calculation code for step 4 of Question 5 of the RBE500 midterm
clear; close all; clc;

syms c1 s1 r11 r12 r13 r21 r22 r23 r31 r32 r33;

R30 = [0 0 -1; 0 -1 0; -1 0 0];
R06 = [r11 r12 r13; r21 r22 r23; r31 r32 r33];

R36 = R30*R06;

latex(R36)
```

$$R_6^3 = \begin{bmatrix} -r_{31} & -r_{32} & -r_{33} \\ -r_{21} & -r_{22} & -r_{23} \\ -r_{11} & -r_{12} & -r_{13} \end{bmatrix}$$

Step 5 — Find q_4, q_5, q_6.

For the final step, we make use of the Euler angles matrix, where $q_4 = \phi, q_5 = \theta, q_6 = \psi$. The matrix for this, as given in the textbook, is

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}$$

Now, if we equate this with our matrix from Step 4, we get the third column as,

$$\begin{aligned} c_4 s_5 &= -r_{33} \\ s_4 s_5 &= -r_{23} \\ c_5 &= -r_{13} \end{aligned}$$

And the third row as,

$$\begin{aligned} -s_5 c_6 &= -r_{11} \\ s_5 s_6 &= -r_{12} \\ c_5 &= -r_{13} \end{aligned}$$

So, finally, using equations 2.29, 2.30, 2.31, and 2.32 of the textbook (page 54),

$$\boxed{q_5 = \theta_5 = Atan2(-r_{13}, \sqrt{1 - (-r_{13})^2})}$$

or

$$\boxed{q_5 = \theta_5 = Atan2(-r_{13}, -\sqrt{1 - (-r_{13})^2})}$$

where $Atan2$ is the two-argument algorithmic arctangent function defined in Appendix A of the textbook.

Also,

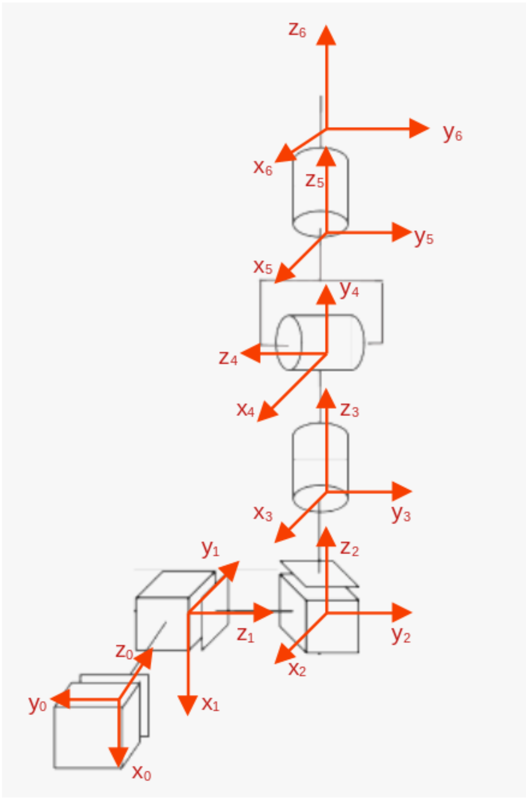
$$\boxed{q_4 = \theta_4 = Atan2(-r_{33}, -r_{23})}$$

$$\boxed{q_6 = \theta_6 = Atan2(r_{11}, -r_{12})}$$

Question 6

Calculate Jacobian matrix for the manipulator in Question 4. Assume that all the forward kinematics information is available (i.e. all homogenous transformation matrices). Since there are no values given, you will be deriving your Jacobian matrix parametrically, but please be sure to explicitly show which homogeneous transformation matrix is required for the corresponding information, and which segment of the matrix is used to obtain that information. (e.g. in your derivations you can say something like: "to calculate this expression, I would need z_3^0, which is available to me at the 3^{rd} column of H_3^0").

Solution



	Linear component	Angular component
Revolute joint Prismatic joint	$J_{v_i} = z_{i-1}^0 \times (o_n^0 - o_{i-1}^0)$ $J_{\omega_i} = z_{i-1}^0$	$J_{v_i} = z_{i-1}^0$ $J_{\omega_i} = 0$

Using this table, and the fact that the upper half of the Jacobian contains linear components while the bottom half contains angular components, we have

$$J = \begin{bmatrix} z_0 & z_1 & z_2 & z_3 \times (o_6 - o_3) & z_4 \times (o_6 - o_4) & z_5 \times (o_6 - o_5) \\ 0 & 0 & 0 & z_3 & z_4 & z_5 \end{bmatrix}$$

Question 7

The dynamic system below

$$a\ddot{x} + b\dot{x} + cx = u$$

Here u is the force applied to the system, x is the position of the system. a, b, and c are model parameters all of which are constant. There parameters are

$$a = 10, b = 3.5, c = 0.6$$

(a) (5 pts) Find the open loop transfer function for this system, i.e. $\frac{X(s)}{U(s)}$ in laplace domain.

(b) (3 pts) Draw a block diagram of a closed loop system with a PD controller for controlling the position of the system. Explicitly write the transfer functions (Laplace domain) of the system and the PD controller inside the blocks (leave K_p and K_d as parameters).

(c) (5 pts) Derive the closed loop transfer function i.e. $\frac{x}{x_r}$, where x_r is the position reference signal.

(d) (7pts) Find the values of the K_p and K_d gains for a critically damped system with 2 seconds settling time.

Show all your steps explicitly.

Solution for 7(a)

Transform the system model to Laplace domain on both sides,

$$\mathcal{L}\{a\ddot{x} + b\dot{x} + cx\} = \mathcal{L}\{u\}$$
$$aX(s)s^2 + bX(s)s + cX(s) = U(s)$$

$$X(s)[as^2 + bs + c] = U(s) \tag{1}$$

Therefore,

$$\boxed{\frac{X(s)}{U(s)} = \frac{1}{as^2 + bs + c} = \frac{1}{10s^2 + 3.5s + 0.6}}$$

Solution for 7(b)

First let us find the transfer function for the controller. Let our PD controller model be

$$K_p e + K_d \dot{e} = u$$

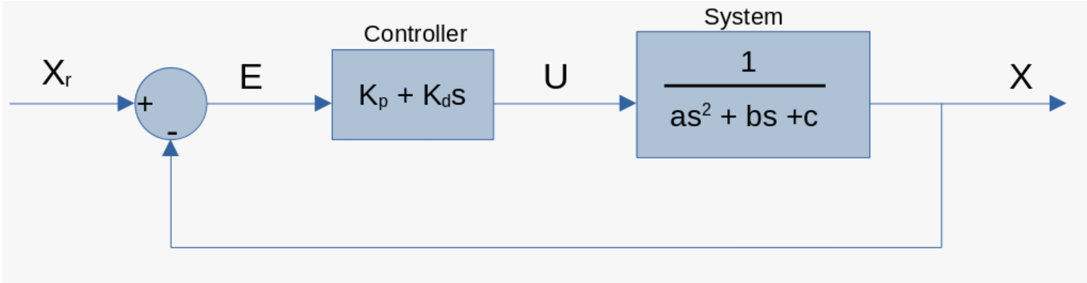
Transform to Laplace domain,

$$K_p E(s) + K_d E(s)s = U(s)$$
$$E(s)[K_p + K_d s] = U(s)$$

Therefore, the transfer function for the PD controller is

$$\frac{U(s)}{E(s)} = K_p + K_d s \tag{2}$$

Now we can draw the block diagram, as shown on the next page.



Solution for 7(c)

From the block diagram, we can see that

$$E = X_r - X$$

Using equation 2 from part (b),

$$\frac{U(s)}{K_p + K_d s} = X_r - X$$

Furthermore, using equation 1 from part (a),

$$\frac{X(s)[as^2 + bs + c]}{K_p + K_d s} = X_r - X$$
$$\frac{X[as^2 + bs + c]}{K_p + K_d s} + X = X_r$$
$$X\left(\frac{as^2 + bs + c}{K_p + K_d s} + 1\right) = X_r$$
$$X\left(\frac{as^2 + bs + c + K_p + K_d s}{K_p + K_d s}\right) = X_r$$

Therefore,

$$\boxed{\frac{X}{X_r} = \frac{K_p + K_d s}{as^2 + s(b + K_d) + (c + K_p)} = \frac{K_p + K_d s}{10s^2 + s(3.5 + K_d) + (0.6 + K_p)}}$$

Solution for 7(d)

Taking the denominator of $\frac{X}{X_r}$, we have the charateristic equation,

$$10s^2 + s(3.5 + K_d) + (0.6 + K_p) = 0$$
$$s^2 + s\frac{3.5 + K_d}{10} + \frac{0.6 + K_p}{10} = 0$$

The general form of the charateristic equation is

$$s^2 + (2\xi\omega_n)s + {\omega_n}^2 = 0$$

Where ξ is the damping ratio and ω_n is the natural frequency.

Hence, we have,

$$\omega_n{}^2 = \frac{0.6 + K_p}{10} \tag{3}$$

and

$$2\xi\omega_n = \frac{3.5 + K_d}{10} \tag{4}$$

Also, we know that the natural frequency and settling time T_s are related by

$$\xi\omega_n T_s = 4$$

Since we are solving for a critically damped system, we set $\xi = 1$. We also want settling time $T_s = 2$ seconds.

So,

$$\begin{aligned} \xi\omega_n T_s &= 4 \\ 1 \cdot \omega_n \cdot 2 &= 4 \\ \omega_n &= 2 \end{aligned}$$

Plugging this into equation 3, we have

$$\begin{aligned} (2)^2 &= \frac{0.6 + K_p}{10} \\ 40 &= 0.6 + K_p \\ &\boxed{K_p = 39.4} \end{aligned}$$

Also, plugging in values into equation 4, we have

$$\begin{aligned} 2(1)(2) &= \frac{3.5 + K_d}{10} \\ 40 &= 3.5 + K_d \\ &\boxed{K_d = 36.5} \end{aligned}$$

Question 8