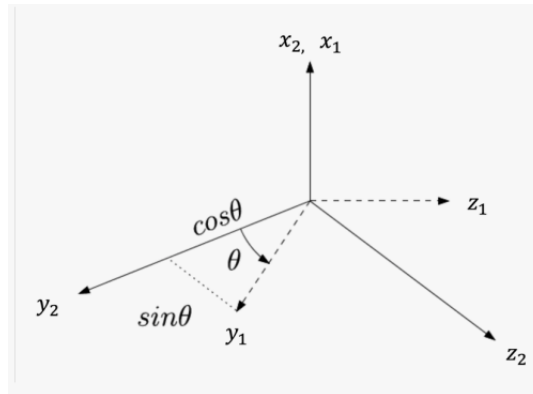


# **RBE 500 Midterm**

**Arjan Gupta**

## Question 1

Derive the rotation matrix  $R_2^1$  (you can leave sines and cosines as is).



## Solution

Since the x-axis remains the same in the rotation, we know this is a basic 3D rotation matrix representing a rotation about the x-axis. However, using the right-hand screw rule, we see that the angle  $\theta$  here is negative. So, by using equation 2.7 (page 43) of our main textbook, the rotation matrix is given by,

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta) & -\sin(-\theta) \\ 0 & \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}$$

## Question 2

Find the coordinates of point  $p$  expressed in frame 1 (i.e.  $p^1$ ) given the following.

$$H_1^2 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0.9553 & 0.2955 & -0.9553 \\ 0 & -0.2955 & 0.9553 & 0.2955 \\ 0 & 0 & 0 & 1 \end{bmatrix}, p^2 = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

## Solution

From our knowledge of homogeneous transformations, we know that

$$P^2 = H_1^2 P^1$$

Where  $P^2 = \begin{bmatrix} p^2 \\ 1 \end{bmatrix}$  and  $P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$ .

However, we want to find  $P^1$ , so we apply the inverse of  $H$  to both sides,

$$P^2 = H_1^2 P^1 \\ (H_1^2)^{-1} P^2 = P^1$$

We know that  $H_1^2 = \begin{bmatrix} R_1^2 & d_1^2 \\ 0 & 1 \end{bmatrix}$ , where  $R_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9553 & 0.2955 \\ 0 & -0.2955 & 0.9553 \end{bmatrix}$  and  $d_1^2 = \begin{bmatrix} -1 \\ -0.9553 \\ 0.2955 \end{bmatrix}$ .

For accuracy while computing the inverse of  $H_1^2$ , we use equation 2.67 of the book (page 63). Therefore,

$$(H_1^2)^{-1} = \begin{bmatrix} (R_1^2)^T & -(R_1^2)^T d_1^2 \\ 0 & 1 \end{bmatrix}$$

. We use the following MATLAB code for this computation.

```
1 % Calculation code for problem 2 of the RBE500 Midterm
2
3 clear; close all; clc;
4
5 P2 = [2;5;0;1];
6
7 R2_1 = [1 0 0; 0 0.9553 0.2955; 0 -0.2955 0.9553];
8 d2_1 = [-1; -0.9553; 0.2955];
9 H_inv = [R2_1' (-R2_1'*d2_1); zeros(1,3) 1];
10
11 P1 = H_inv*P2
```

Which gives us the answer,

$$P^1 = \begin{bmatrix} 3.0000 \\ 5.7764 \\ 1.4775 \\ 1.0000 \end{bmatrix}$$

### Question 3

If  $R_1^0 = \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}$ ,  $R_2^0 = \begin{bmatrix} 0 & 0.866 & 0.5 \\ 0 & 0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$ , and  $R_3^0 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ , calculate  $R_2^1$ .

### Solution

Knowing the composition law for rotational transformations, we can write

$$\begin{aligned}
 R_2^1 &= R_0^1 R_2^0 \\
 R_2^1 &= (R_1^0)^T R_2^0 \\
 R_2^1 &= \begin{bmatrix} 0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}^T \begin{bmatrix} 0 & 0.866 & 0.5 \\ 0 & 0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix} \\
 R_2^1 &= \begin{bmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ 0.7071 & 0 & 0.7071 \end{bmatrix} \begin{bmatrix} 0 & 0.866 & 0.5 \\ 0 & 0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

We use the following MATLAB script to compute this multiplication.

```

1 % Calculation code for problem 2 of the RBE500 Midterm
2
3 clear; close all; clc;
4
5 R01 = [0.7071 0 0.7071; 0 1 0; -0.7071 0 0.7071];
6 R02 = [0 0.866 0.5; 0 0.5 -0.866; -1 0 0];
7
8 R12 = R01'*R02

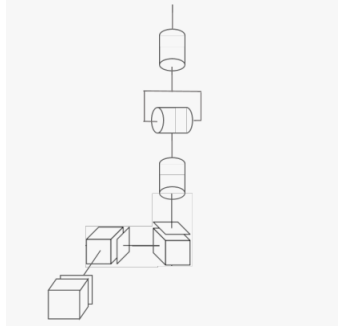
```

Therefore,

$$R_2^1 = \begin{bmatrix} 0.7071 & 0.6123 & 0.3535 \\ 0 & 0.5000 & -0.8660 \\ -0.7071 & 0.6123 & 0.3535 \end{bmatrix}$$

## Question 4

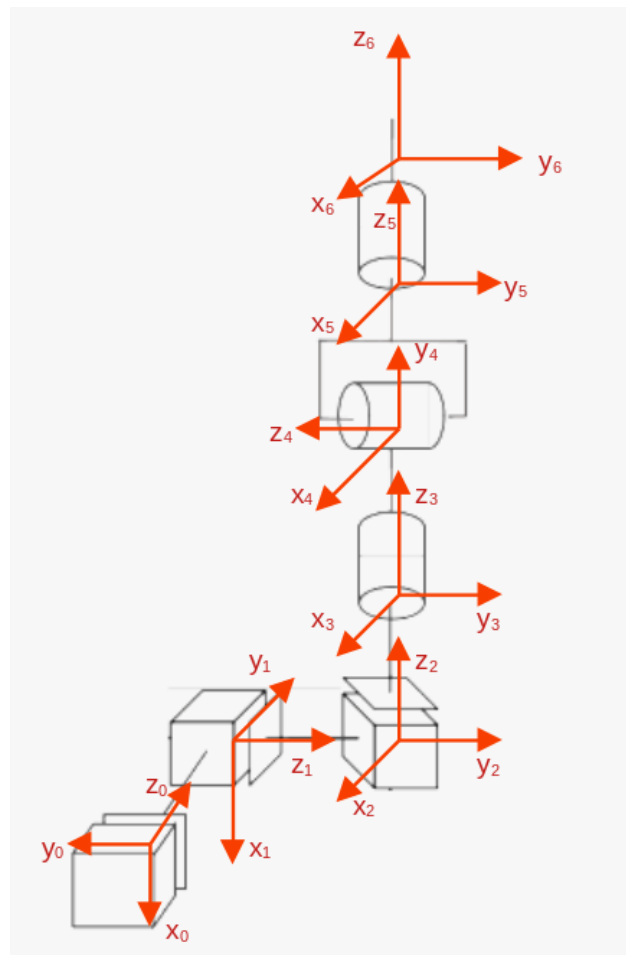
(a) Calculate Denavit Hertenberg parameters for the given manipulator (just filling out the Denavit Hertenberg table would suffice). For this question, you are expected to solve it parametrically, i.e. you can leave sines, cosines, joint values, and link lengths as parameters.



(b) Derive  $H_2^1$ . You can leave sines, cosines, joint values, and link lengths as parameters.

### Solution for 4(a)

First we assign coordinate frames 0 through 5 (links 0 through 5). This is done as per the following figure.



Now, we create a table for quantities  $\alpha_i, a_i, \theta_i, d_i$  for links 1 through 6. In this table,  $d_1, d_2, d_3, \theta_4, \theta_5, \theta_6$  are variable. However,  $l_4, l_5, l_6$  are fixed (constants).

Link	$\alpha_i$	$a_i$	$\theta_i$	$d_i$
1	$90^\circ$	0	0	$d_1$
2	$90^\circ$	0	$-90^\circ$	$d_2$
3	0	0	0	$d_3$
4	$90^\circ$	0	$\theta_4$	$l_4$
5	$-90^\circ$	0	$\theta_5$	$l_5$
6	0	0	$\theta_6$	$l_6$

### Solution for 4(b)

We know that  $H_2^1 = A_2$ , where  $A_2$  is the DH matrix  $A_i$  with  $i = 2$ .

$$\begin{aligned}
 H_2^1 = A_2 &= \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \cos \alpha_2 & \sin \theta_2 \sin \alpha_2 & a_i \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 \cos \alpha_2 & -\cos \theta_2 \sin \alpha_2 & a_i \sin \theta_2 \\ 0 & \sin \alpha_2 & \cos \alpha_2 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \cos(-90^\circ) & -\sin(-90^\circ) \cos(90^\circ) & \sin(-90^\circ) \sin(90^\circ) & a_i \cos(-90^\circ) \\ \sin(-90^\circ) & \cos(-90^\circ) \cos(90^\circ) & -\cos(-90^\circ) \sin(90^\circ) & a_i \sin(-90^\circ) \\ 0 & \sin(90^\circ) & \cos(90^\circ) & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \boxed{\begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}
 \end{aligned}$$

## Question 5

Calculate inverse kinematics for the manipulator in Question 4. Assume that all the forward kinematics information is available (i.e. all homogenous transformation matrices). Since there are no values given, you will be deriving your expressions parametrically, but please be sure to explicitly show, which homogeneous transformation matrix is required for the corresponding information, and which segment of the matrix is used to obtain that information. (e.g. in your derivations you can say something like: “to calculate this expression, I would need  $z_3^0$ , which is available to me at the 3<sup>rd</sup> column of  $H_3^0$ ”).

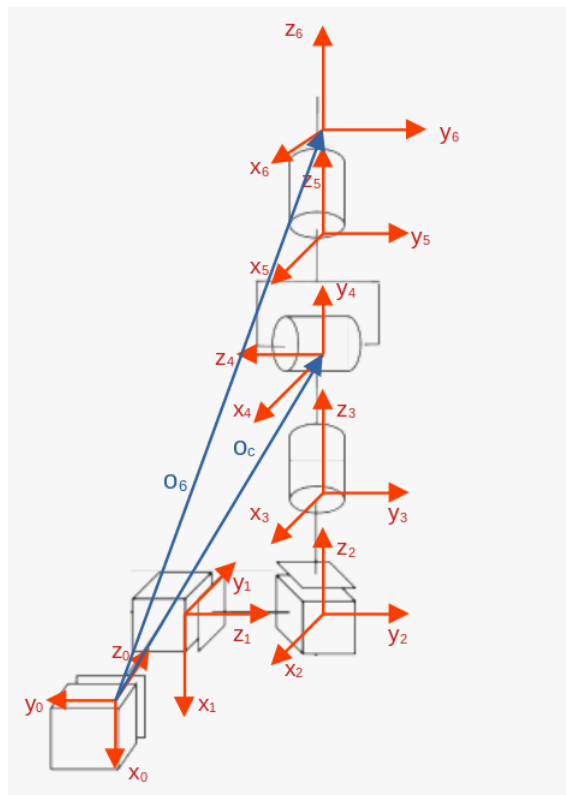
## Solution

Before we begin, let us make a brief list of steps we need to take to solve the complete inverse kinematics problem for our particular manipulator’s configuration.

1. Find wrist center  $o_c$ .
2. Find  $q_1, q_2, q_3$ .
3. Perform forward kinematics to arrive at  $R_3^0 = (R_0^3)^T$ .
4. Get  $R_6^3 = R_0^3 R_6^0$ .
5. Use  $R_6^3$  to find  $\phi, \theta, \psi$  of Euler configuration to find  $q_4, q_5, q_6$ .

In essence, once we have found all joint variables given the end-effector’s homogeneous transformation, we have solved the inverse kinematics problem.

Now, considering the figure we had in Question 4, let us draw vectors  $o_c$  and  $o_6$  in it as shown below.



We have determined  $o_c$  as the wrist center. This is because joints 4, 5, 6 form a spherical wrist, and the z-axes of frames 3, 4, 5 intersect at  $o_c$ . Now let us proceed to Step 1.

**Step 1** — Find the wrist center

The end-effector's homogeneous transformation is known to us as the  $4 \times 4$  matrix

$$H_6^0 = \begin{bmatrix} R_6^0 & o_6^0 \\ 0 & 1 \end{bmatrix}$$

where

$$R_6^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, o_6^0 = \begin{bmatrix} x_6 \\ y_6 \\ z_6 \end{bmatrix}$$

Where  $o_6^0$  is  $o_6$  as shown in the diagram. As shown in the figure, we can establish a relationship between  $o_6$  and  $o_c$  as

$$o_c = o_6 - (l_5 + l_6)R_6^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x_6 - (l_5 + l_6)r_{13} \\ y_6 - (l_5 + l_6)r_{23} \\ z_6 - (l_5 + l_6)r_{33} \end{bmatrix}$$

Where  $(l_5 + l_6)$  is a scalar.

**Step 2** — Find  $q_1, q_2, q_3$ .

In the figure we have drawn, we can see that  $x_c = -(d_3 + l_4)$ , where  $d_3$  is variable and  $l_4$  is a constant. Therefore,  $d_3 = -x_c - l_4 = -(x_c + l_4)$ .

We can also see that  $y_c = -d_2$  and  $z_c = d_1$ .

Therefore, we have,

$$\begin{aligned} q_1 &= d_1 = z_c = z_6 - (l_5 + l_6)r_{33} \\ q_2 &= d_2 = -y_c = -y_6 + (l_5 + l_6)r_{23} \\ q_3 &= d_3 = -(x_c + l_4) = -x_6 + (l_5 + l_6)r_{13} - l_4 \end{aligned}$$

**Step 3** — Perform forward kinematics

We perform forward kinematics for the first three joint variables. We already found the table in Question 4 as well as  $A_2$ . We write  $A_1$  and  $A_3$  as the following.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We obtain  $T_3^0$  using the following MATLAB code.



```

1 % Calculation code for step 3 of Question 5 of the RBE500 midterm
2
3 clear; close all; clc;
4
5 syms d1 d2 d3;
6 A1 = [1 0 0 0; 0 0 -1 0; 0 1 0 d1; 0 0 0 1];
7 A2 = [0 0 -1 0; -1 0 0 0; 0 1 0 d2; 0 0 0 1];
8 A3 = [1 0 0 0; 0 1 0 0; 0 0 1 d3; 0 0 0 1];
9
10 T = A1*A2*A3;
11
12 % Generate LaTeX code
13 latex(T)

```

Therefore,

$$T_3^0 = \begin{bmatrix} 0 & 0 & -1 & -d_3 \\ 0 & -1 & 0 & -d_2 \\ -1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From here, it is clear that

$$R_3^0 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

**Step 4** — Get  $R_6^3 = R_0^3 R_6^0$

We know that  $R_0^3 = (R_3^0)^T$ . Given this fact, we use the following MATLAB code to calculate  $R_6^3$ .

```

1 % Calculation code for step 4 of Question 5 of the RBE500 midterm
2 clear; close all; clc;
3
4 syms c1 s1 r11 r12 r13 r21 r22 r23 r31 r32 r33;
5
6 R30 = [0 0 -1; 0 -1 0; -1 0 0];
7 R06 = [r11 r12 r13; r21 r22 r23; r31 r32 r33];
8
9 R36 = R30*R06;
10
11 latex(R36)

```

$$R_6^3 = \begin{bmatrix} -r_{31} & -r_{32} & -r_{33} \\ -r_{21} & -r_{22} & -r_{23} \\ -r_{11} & -r_{12} & -r_{13} \end{bmatrix}$$

**Step 5** — Find  $q_4, q_5, q_6$ .

For the final step, we make use of the Euler angles matrix, where  $q_4 = \phi, q_5 = \theta, q_6 = \psi$ . The matrix for this, as given in the textbook, is

$$R_6^3 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 \\ -s_5 c_6 & s_5 s_6 & c_5 \end{bmatrix}$$

Now, if we equate this with our matrix from Step 4, we get the third column as,

$$c_4 s_5 = -r_{33}$$

$$s_4 s_5 = -r_{23}$$

$$c_5 = -r_{13}$$

And the third row as,

$$-s_5 c_6 = -r_{11}$$

$$s_5 s_6 = -r_{12}$$

$$c_5 = -r_{13}$$

So, finally, using equations 2.29, 2.30, 2.31, and 2.32 of the textbook (page 54),

$$q_5 = \theta_5 = \text{Atan2}(-r_{13}, \sqrt{1 - (-r_{13})^2})$$

or

$$q_5 = \theta_5 = \text{Atan2}(-r_{13}, -\sqrt{1 - (-r_{13})^2})$$

where  $\text{Atan2}$  is the two-argument algorithmic arctangent function defined in Appendix A of the textbook.

Also,

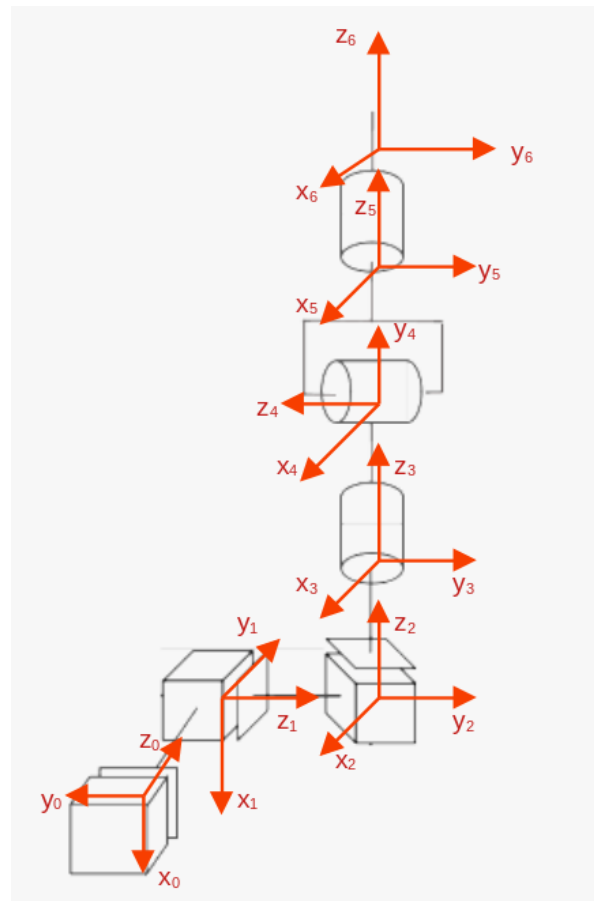
$$q_4 = \theta_4 = \text{Atan2}(-r_{33}, -r_{23})$$

$$q_6 = \theta_6 = \text{Atan2}(r_{11}, -r_{12})$$

## Question 6

Calculate Jacobian matrix for the manipulator in Question 4. Assume that all the forward kinematics information is available (i.e. all homogenous transformation matrices). Since there are no values given, you will be deriving your Jacobian matrix parametrically, but please be sure to explicitly show which homogeneous transformation matrix is required for the corresponding information, and which segment of the matrix is used to obtain that information. (e.g. in your derivations you can say something like: “to calculate this expression, I would need  $z_3^0$ , which is available to me at the 3<sup>rd</sup> column of  $H_3^0$ ”).

## Solution



	Linear component	Angular component
Revolute joint	$J_{v_i} = z_{i-1}^0 \times (o_n^0 - o_{i-1}^0)$	$J_{\omega_i} = z_{i-1}^0$
Prismatic joint	$J_{\omega_i} = z_{i-1}^0$	$J_{v_i} = 0$

Using this table, and the fact that the upper half of the Jacobian contains linear components while the bottom half contains angular components, we have

$$J = \begin{bmatrix} z_0 & z_1 & z_2 & z_3 \times (o_6 - o_3) & z_4 \times (o_6 - o_4) & z_5 \times (o_6 - o_5) \\ 0 & 0 & 0 & z_3 & z_4 & z_5 \end{bmatrix}$$

In this Jacobian matrix,

1.  $z_0$  is the third column of the base frame's rotation matrix, which is just the identity matrix. Therefore,

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2.  $z_1$  is the third column of  $R_1^0$ , which is the upper-left  $3 \times 3$  section of the  $H_1^0 = A_1$  matrix.
3.  $z_2$  is the third column of  $R_2^0$ , which is the upper-left  $3 \times 3$  section of the  $H_2^0 = T_2 = A_1 A_2$  matrix.
4.  $z_3$  is the third column of  $R_3^0$ , which is the upper-left  $3 \times 3$  section of the  $H_3^0 = T_3 = A_1 A_2 A_3$  matrix.
5.  $z_4$  is the third column of  $R_4^0$ , which is the upper-left  $3 \times 3$  section of the  $H_4^0 = T_4 = A_1 A_2 A_3 A_4$  matrix.
6.  $z_5$  is the third column of  $R_5^0$ , which is the upper-left  $3 \times 3$  section of the  $H_5^0 = T_5 = A_1 A_2 A_3 A_4 A_5$  matrix.
7.  $o_3$  is the  $3 \times 1$  vector in the upper-right section of  $T_3$ .
8.  $o_4$  is the  $3 \times 1$  vector in the upper-right section of  $T_4$ .
9.  $o_5$  is the  $3 \times 1$  vector in the upper-right section of  $T_5$ .
10.  $o_6$  is the  $3 \times 1$  vector in the upper-right section of  $H_6^0 = T_6 = A_1 A_2 A_3 A_4 A_5 A_6$  matrix.

We already found  $A_1$ ,  $A_2$ ,  $A_3$ , and  $T_3$  in the solutions for Question 4 and Question 5. Similarly, we write  $A_4$ ,  $A_5$  and  $A_6$  as follows.

$$A_4 = \begin{bmatrix} \cos(\theta_4) & 0 & \sin(\theta_4) & 0 \\ \sin(\theta_4) & 0 & -\cos(\theta_4) & 0 \\ 0 & 1 & 0 & l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_5 = \begin{bmatrix} \cos(\theta_5) & 0 & -\sin(\theta_5) & 0 \\ \sin(\theta_5) & 0 & \cos(\theta_5) & 0 \\ 0 & -1 & 0 & l_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} \cos(\theta_6) & -\sin(\theta_6) & 0 & 0 \\ \sin(\theta_6) & \cos(\theta_6) & 0 & 0 \\ 0 & 0 & 1 & l_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we write a MATLAB script that will utilize the  $A_i$  matrices we have found to find the  $T_i$  matrices. The script will then extract the required parts of the  $T_i$  matrices, place them into the Jacobian matrix and give us our final answer.

```

1  % Calculation code for Question 6 of the RBE500 midterm
2
3  clear; close all; clc;
4
5  % Declare symbolic variables/constants
6  syms d1 d2 d3 theta4 theta5 theta6 l4 l5 l6;
7
8  % Write out our A matrices
9  A1 = [1 0 0 0; 0 0 -1 0; 0 1 0 d1; 0 0 0 1];
10 A2 = [0 0 -1 0; -1 0 0 0; 0 1 0 d2; 0 0 0 1];
11 A3 = [1 0 0 0; 0 1 0 0; 0 0 1 d3; 0 0 0 1];
12 A4 = [cos(theta4) 0 sin(theta4) 0; sin(theta4) 0 -cos(theta4) 0; 0 1 0 l4; 0 0 0 1];
13 A5 = [cos(theta5) 0 -sin(theta5) 0; sin(theta5) 0 cos(theta5) 0; 0 -1 0 l5; 0 0 0 1];
14 A6 = [cos(theta6) -sin(theta6) 0 0; sin(theta6) cos(theta6) 0 0; 0 0 1 l6; 0 0 0 1];
15
16 % Compute T matrices
17 T2 = A1*A2;
18 T3 = T2*A3;
19 T4 = T3*A4;
20 T5 = T4*A5;
21 T6 = T5*A6;
22

```

```

23 R_base = eye(3);
24
25 % Extract required z vectors
26 z0 = R_base(:,3);
27 z1 = A1(1:3,3);
28 z2 = T2(1:3,3);
29 z3 = T3(1:3,3);
30 z4 = T4(1:3,3);
31 z5 = T5(1:3,3);
32
33 % Extract required o vectors
34 o3 = T3(1:3,4);
35 o4 = T4(1:3,4);
36 o5 = T5(1:3,4);
37 o6 = T6(1:3,4);
38
39 % Calculate the Jacobian matrix
40 J = [z0 z1 z2 cross(z3, (o6-o3)) cross(z4, (o6-o4)) cross(z5, (o6-o5)); zeros(3,1) zeros(3,1) ...
      zeros(3,1) z3 z4 z5];
41
42 % Export to LaTeX
43 latex(J)

```

Our Jacobian is as follows:

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 & \sin(\theta_4)(l_5 \cos(\theta_4) + l_6 \sin(\theta_4)) & \sin(\theta_5) - \cos(\theta_4)(l_5 \sin(\theta_4) - l_6 \cos(\theta_4) \sin(\theta_5)) & 0 \\ 0 & -1 & 0 & l_6 \cos(\theta_4) \sin(\theta_5) - l_5 \sin(\theta_4) & l_6 \cos(\theta_5) \sin(\theta_4) & 0 & 0 \\ 1 & 0 & 0 & -l_5 \cos(\theta_4) - l_6 \sin(\theta_4) \sin(\theta_5) & l_6 \cos(\theta_4) \cos(\theta_5) & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -\cos(\theta_5) & 0 \\ 0 & 0 & 0 & 0 & \cos(\theta_4) & \sin(\theta_4) \sin(\theta_5) & 0 \\ 0 & 0 & 0 & 0 & -\sin(\theta_4) & \cos(\theta_4) \sin(\theta_5) & 0 \end{pmatrix}$$

## Question 7

The dynamic system below

$$a\ddot{x} + b\dot{x} + cx = u$$

Here  $u$  is the force applied to the system,  $x$  is the position of the system.  $a$ ,  $b$ , and  $c$  are model parameters all of which are constant. There parameters are

$$a = 10, b = 3.5, c = 0.6$$

- (5 pts) Find the open loop transfer function for this system, i.e.  $\frac{X(s)}{U(s)}$  in laplace domain.
- (3 pts) Draw a block diagram of a closed loop system with a PD controller for controlling the position of the system. Explicitly write the transfer functions (Laplace domain) of the system and the PD controller inside the blocks (leave  $K_p$  and  $K_d$  as parameters).
- (5 pts) Derive the closed loop transfer function i.e.  $\frac{x}{x_r}$ , where  $x_r$  is the position reference signal.
- (7pts) Find the values of the  $K_p$  and  $K_d$  gains for a critically damped system with 2 seconds settling time.

Show all your steps explicitly.

### Solution for 7(a)

Transform the system model to Laplace domain on both sides,

$$\begin{aligned}\mathcal{L}\{a\ddot{x} + b\dot{x} + cx\} &= \mathcal{L}\{u\} \\ aX(s)s^2 + bX(s)s + cX(s) &= U(s)\end{aligned}$$

$$X(s)[as^2 + bs + c] = U(s) \tag{1}$$

Therefore,

$$\boxed{\frac{X(s)}{U(s)} = \frac{1}{as^2 + bs + c} = \frac{1}{10s^2 + 3.5s + 0.6}}$$

### Solution for 7(b)

First let us find the transfer function for the controller. Let our PD controller model be

$$K_p e + K_d \dot{e} = u$$

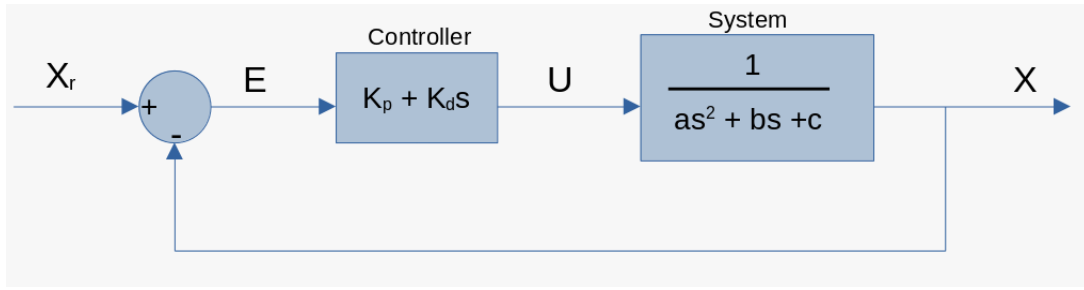
Transform to Laplace domain,

$$\begin{aligned}K_p E(s) + K_d E(s)s &= U(s) \\ E(s)[K_p + K_d s] &= U(s)\end{aligned}$$

Therefore, the transfer function for the PD controller is

$$\frac{U(s)}{E(s)} = K_p + K_d s \tag{2}$$

Now we can draw the block diagram, as shown on the next page.



### Solution for 7(c)

From the block diagram, we can see that

$$E = X_r - X$$

Using equation 2 from part (b),

$$\frac{U(s)}{K_p + K_d s} = X_r - X$$

Furthermore, using equation 1 from part (a),

$$\begin{aligned} \frac{X(s)[as^2 + bs + c]}{K_p + K_d s} &= X_r - X \\ \frac{X[as^2 + bs + c]}{K_p + K_d s} + X &= X_r \\ X \left( \frac{as^2 + bs + c}{K_p + K_d s} + 1 \right) &= X_r \\ X \left( \frac{as^2 + bs + c + K_p + K_d s}{K_p + K_d s} \right) &= X_r \end{aligned}$$

Therefore,

$$\boxed{\frac{X}{X_r} = \frac{K_p + K_d s}{as^2 + s(b + K_d) + (c + K_p)} = \frac{K_p + K_d s}{10s^2 + s(3.5 + K_d) + (0.6 + K_p)}}$$

### Solution for 7(d)

Taking the denominator of  $\frac{X}{X_r}$ , we have the characteristic equation,

$$\begin{aligned} 10s^2 + s(3.5 + K_d) + (0.6 + K_p) &= 0 \\ s^2 + s \frac{3.5 + K_d}{10} + \frac{0.6 + K_p}{10} &= 0 \end{aligned}$$

The general form of the characteristic equation is

$$s^2 + (2\xi\omega_n)s + \omega_n^2 = 0$$

Where  $\xi$  is the damping ratio and  $\omega_n$  is the natural frequency.

Hence, we have,

$$\omega_n^2 = \frac{0.6 + K_p}{10} \quad (3)$$

and

$$2\xi\omega_n = \frac{3.5 + K_d}{10} \quad (4)$$

Also, we know that the natural frequency and settling time  $T_s$  are related by

$$\xi\omega_n T_s = 4$$

Since we are solving for a critically damped system, we set  $\xi = 1$ . We also want settling time  $T_s = 2$  seconds.

So,

$$\begin{aligned}\xi\omega_n T_s &= 4 \\ 1 \cdot \omega_n \cdot 2 &= 4 \\ \omega_n &= 2\end{aligned}$$

Plugging this into equation 3, we have

$$\begin{aligned}(2)^2 &= \frac{0.6 + K_p}{10} \\ 40 &= 0.6 + K_p \\ \boxed{K_p = 39.4}\end{aligned}$$

Also, plugging in values into equation 4, we have

$$\begin{aligned}2(1)(2) &= \frac{3.5 + K_d}{10} \\ 40 &= 3.5 + K_d \\ \boxed{K_d = 36.5}\end{aligned}$$



## Question 8