RBE 500 Group Assignment #1

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Task 1 — Gazebo Setup

Consider the following robot joint model

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = u(t) + d(t)$$

where J is the inertia of the link, B is the effective damping on the link, θ is the joint angle, u is the actuator torque (input), and d is the disturbance acting on the system.

First, assume that disturbance is zero and take $J = 2$, $B = 0.5$. Design a PD controller such that the closed loop system is critically damped, and settling time is 2 second. Do not do this by tuning the gains; calculate the K_p and K_d gains using natural frequency and damping ratio.

Solution

Since $d(t) = 0$, $J = 2$, $B = 0.5$, we have

$$2\ddot{\theta}(t) + 0.5\dot{\theta}(t) = u(t)$$

Transform to Laplace domain,

$$2\Theta(s)s^2 + 0.5\Theta(s)s = U(s)$$

$$\Theta(s)[2s^2 + 0.5s] = U(s) \tag{1}$$

$$\frac{\Theta(s)}{U(s)} = \frac{1}{2s^2 + 0.5s}$$

Let our PD controller model be

$$K_p e + K_d \dot{e} = u$$

Transform to Laplace domain,

$$K_p E(s) + K_d E(s)s = U(s)$$
$$E(s)[K_p + K_d s] = U(s)$$

Therefore, the transfer function for the PD controller is

$$\frac{U(s)}{E(s)} = K_p + K_d s \tag{2}$$

Now we can draw the block diagram, as shown below.
From the block diagram, we can see that

$$E = \Theta_r - \Theta$$

Using equation 2,

$$\frac{U(s)}{K_p + K_d s} = \Theta_r - \Theta$$

Furthermore, using equation 1,

$$\frac{\Theta(s)[2s^2+0.5s]}{K_p+K_ds} = \Theta_r - \Theta$$
$$\frac{\Theta[2s^2+0.5s]}{K_p+K_ds} + \Theta = \Theta_r$$
$$\Theta\left(\frac{2s^2+0.5s}{K_p+K_ds}+1\right) = \Theta$$
$$\Theta\left(\frac{2s^2+0.5s+K_p+K_ds}{K_p+K_ds}\right) = \Theta_r$$

Therefore,

$$\frac{\Theta}{\Theta_r} = \frac{K_p+K_ds}{2s^2+s(0.5+K_d)+K_p}$$

So our charateristic equation is,

$$2s^2+s(0.5+K_d)+K_p = 0$$
$$s^2+s\frac{(0.5+K_d)}{2}+\frac{K_p}{2} = 0$$

The general form of the charateristic equation is

$$s^2+(2\xi\omega_n)s+{\omega_n}^2 = 0$$

Where ξ is the damping ratio and ω_n is the natural frequency.

Hence, we have,

$${\omega_n}^2 = \frac{K_p}{2} \tag{3}$$

and

$$2\xi\omega_n = \frac{(0.5+K_d)}{2} \tag{4}$$

Also, we know that the natural frequency and settling time T_s are related by

$$\xi\omega_nT_s = 4$$

Since we are solving for a critically damped system, we set $\xi = 1$. We also want settling time $T_s = 2$ seconds.

So,

$$\xi\omega_nT_s = 4$$
$$1\cdot\omega_n\cdot 2 = 4$$
$$\omega_n = 2$$

Plugging this into equation 3, we have

$$(2)^2 = \frac{K_p}{2}$$
$$4 = \frac{K_p}{2}$$
$$\boxed{K_p = 8}$$

Also, plugging in values into equation 4, we have

$$
\begin{aligned}
2(1)(2) &= \frac{0.5 + K_d}{2} \\
8 &= 0.5 + K_d \\
\boxed{K_d = 7.5}
\end{aligned}
$$