RBE 501 Week 6: Physics by EL Assignment

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***Abstract*—This document provides an in-depth solution for the Week 6 Physics by Euler-Lagrange problems in RBE 501.**

***Index Terms*—physics, euler-lagrange equations**

I. Introduction

For Week 6, we are given two problems. The first problem shows us a mass m that is free to move on the surface of a frictionless table. Mass m is attached to another mass M via a string that goes through a hole in the table. We can assume the string has no spring-like properties, and is therefore always taut. The representation of this system is shown in Figure 1. Our first objective in this problem is to solve for the Euler-Lagrange equations of this system with respect to r and θ. Our second and third objectives here are to discuss the behaviors of r and θ in the EL equations. The fourth objective of the first problem is to give the conditions under which $\dot{r}$ and $\ddot{r}$ cause a circular motion for mass m.

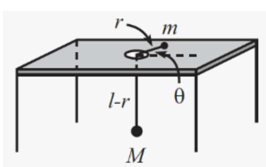

Fig. 1. Given figure for Problem 1

The second problem shows a mass m that is held at rest on ramp of mass M. The ramp has inclination θ, as shown in Figure 2. The surface of the ramp is frictionless, and the surface upon which the ramp rests is also frictionless. Our objective is to find the acceleration of the ramp caused by the release of mass m, and we must use Euler-Lagrange approach for this.

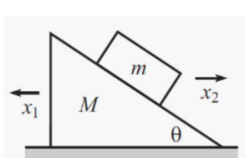

Fig. 2. Given figure for Problem 2

II. Materials and Methods

A. Approach for Problem I

Our first step toward solving Problem I is to find the overall kinetic energy (T) and overall potential energy (U) of the system. We then find the Lagrangian of the system as $L = T - U$. We can then use the Euler-Lagrangian equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \tag{1}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \tag{2}$$

to discuss the behaviors of r and θ.

1) Find overall kinetic energy of the system (T): In Fig. 1, we consider vector **r**. Say the *x-y* plane is the surface of the table. Since we consider an angle θ as shown, we can say,

$$\mathbf{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

Where the magnitude of r is dependent on theta. Differentiating with respect to θ to get velocity,

$$\dot{\mathbf{r}} = \begin{pmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{pmatrix}$$

To find the kinetic energy, we need the square of this velocity, which is simply the dot product of the vector with itself.

$$\begin{aligned}\dot{\mathbf{r}}^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} &= (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 \\ &= \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta \\ &\quad + \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta \\ &= \dot{r}^2(\cos^2\theta + \sin^2\theta) + r^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta)\end{aligned}$$

Hence,

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

Therefore the kinetic energy of the mass m moving on the table is given as,

$$T_m = \frac{1}{2}\, m\, (\dot{r}^2 + r^2\dot{\theta}^2) \tag{3}$$

Also, the length of the string that suspends mass M is $l - r$, where total length of the string is l. The suspended part of the string is also along the axes perpendicular to the table, which is the z axis. Therefore, the kinetic energy of the mass M moving perpendicular to the table is given as

$$T_M = \frac{1}{2}\ M\ \left(\frac{d}{dt}(l-r)\hat{\mathbf{z}}\right)^2 = \frac{1}{2}\ M\ \dot{r}^2 \tag{4}$$

So, using Equations 3 and 4, the total kinetic energy of the system (T) is,

$$\begin{aligned} T &= T_m + T_M \\ &= \frac{1}{2}\ m\ (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}\ M\ \dot{r}^2 \end{aligned}$$

Rearranging,

$$T = \frac{1}{2}\dot{r}^2(m+M) + \frac{1}{2}mr^2\dot{\theta}^2 \tag{5}$$

2) Find overall potential energy of system: The first step toward solving our problem is to redraw our robot in symbolic form, and assign frames for links 0 through 3. This is shown in Figure 3. Frame 0 ($x_0y_0z_0$) is assigned at the first joint, frame 1 ($x_1y_1z_1$) is assigned at the second joint, and frame 2 ($x_2y_2z_2$) is assigned at third joint. All joints in this case are revolute. Frame 3 ($x_3y_3z_3$) is assigned at the end of link 3.

3) Variables and constants: Now that we have assigned the frames, we can come up with some constants for the link lengths.

Assume that link 1 is of constant length L_1, link 2 is of constant length L_2, and link 3 is of constant length L_3. Additionally, we are dealing with 3 revolute joints, so when the first revolute joint moves, an angle subtended by x_1 from x_0 is the variable θ_1. Similarly, when the second and third revolute joints move, we get a variable angles of θ_2 and θ_3.

4) Find H matrices: We can now start formulating our homogeneous transformation matrices.

We first form the rotation matrices. The rotation matrices are given by the angle of rotation θ about the Y axis, given as the following.

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

In our Live Script, we use the following MATLAB code to formulate R_y.

All our link lengths take on the following form.

$$d = \begin{bmatrix} L\cos\theta \\ 0 \\ -L\sin\theta \end{bmatrix}$$

This can again be turned into a function, as follows.

Furthermore, we use another handy function to compute the homogeneous transformation matrices.

Using this function in conjunction with the previous ones, we found the following H matrices for the frame-to-frame transformations.

$$H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & L_1\cos\theta_1 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_1 & 0 & \cos\theta_1 & -L_1\sin\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 & L_2\cos\theta_2 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & -L_2\sin\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2 = \begin{bmatrix} R_3^2 & d_3^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_3 & 0 & \sin\theta_3 & L_3\cos\theta_3 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_3 & 0 & \cos\theta_3 & -L_3\sin\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we are ready to multiply these matrices to obtain the final result of our objective.

B. Approach for Problem 3–5

1) Restate objective in technical terms: In technical terms, we must first assign frames 0 through 3 for each link of the manipulator. We will then form a table of DH parameters. Using the table, and the general form of the DH matrix, we will find the A matrices for the manipulator i.e. we need to find A_1, A_2, and A_3. Using these, we need to find T_3^0 to give us our final answer. From our textbook [**?**], the DH Coordinate Frame Assumptions are,

- **(DH1)** The axis x_i is perpendicular to the axis z_{i-1}.
- **(DH2)** The axis x_i intersects the axis z_{i-1}.

2) Assign frames: The first step toward solving our problem is to redraw the robot manipulator in symbolic form, and assign frames for links 0 through 3. Since we are following the DH assumptions, we must follow the frame assignment style shown in Figure 4, which is from our textbook [**?**]. Our redrawn figure is shown in Figure 5.

As shown, frame 0 ($x_0y_0z_0$) is assigned at the first joint (revolute). Frame 1 ($x_1y_1z_1$) is also assigned at the first joint in order to have DH assumptions satisfied. Frame 2 ($x_2y_2z_2$) is assigned at third joint (revolute). Frame 3 ($x_3y_3z_3$) is assigned at the end of link 3, in a way that also satisfies DH assumptions.

3) Create DH table and set variables/constants: Now that we have assigned the frames, we can use Figure 4 to write the α_i, a_i, θ_i, d_i quantities for each link.

As seen in Table 1, we have chosen L_1 and L_3 to describe the constant link lengths of links 1 and 3, respectively. Additionally, q_2 describes the variable link length of link 2 (because of joint 2 being a prismatic joint). Furthermore, θ_1 and θ_3 describe the variable angles of revolute joints 1 and 3, respectively.

Link	α_i	a_i	θ_i	d_i
1	$-90°$	0	θ_1	0
2	$90°$	0	0	$L_1 + q_2$
3	0	L_3	$90° + \theta_3$	0

TABLE I
DENAVIT-HARTENBERG TABLE FOR PROBLEM 3–5

4) Find A matrices: As given in our textbook [**?**], the general form of an A_i matrix is,

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where c_{θ_i} is $\cos\theta_i$, s_{θ_i} is $\sin\theta_i$, c_{α_i} is $\cos\alpha_i$, and s_{α_i} is $\sin\alpha_i$. We can write a MATLAB function for this A_i matrix, as follows.

Using this function in our MATLAB Live Script, we produce the following A matrices.

$$A_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & L_1 + q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} \cos\left(\theta_3 + \frac{\pi}{2}\right) & -\sin\left(\theta_3 + \frac{\pi}{2}\right) & 0 & L_3 \cos\left(\theta_3 + \frac{\pi}{2}\right) \\ \sin\left(\theta_3 + \frac{\pi}{2}\right) & \cos\left(\theta_3 + \frac{\pi}{2}\right) & 0 & L_3 \sin\left(\theta_3 + \frac{\pi}{2}\right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we are ready to multiply these matrices to obtain the final result of our objective.

III. RESULTS

A. Result for Problem 3–2

For Problem 3–2, we obtained our final result by multiplying all three H matrices that we obtained in the Materials and Methods section. Hence, we obtained the following matrix,

$$H_3^0 = \begin{bmatrix} \sigma_1 & 0 & \sigma_5 & \lambda_1 \\ 0 & 1 & 0 & 0 \\ \sigma_4 & 0 & \sigma_1 & \lambda_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where,

$$\begin{aligned} \lambda_1 = L_1 \cos(\theta_1) + L_3 \cos(\theta_3)\ \sigma_2 \\ - L_3 \sin(\theta_3)\ \sigma_3 + L_2 \cos(\theta_1) \cos(\theta_2) \\ - L_2 \sin(\theta_1) \sin(\theta_2) \end{aligned}$$

$$\begin{aligned} \lambda_2 = -L_1 \sin(\theta_1) - L_3 \cos(\theta_3)\ \sigma_3 - L_3 \sin(\theta_3)\ \sigma_2 \\ - L_2 \cos(\theta_1) \sin(\theta_2) - L_2 \cos(\theta_2) \sin(\theta_1) \end{aligned}$$

and,

$$\begin{aligned} \sigma_1 &= \cos(\theta_3)\ \sigma_2 - \sin(\theta_3)\ \sigma_3 \\ \sigma_2 &= \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2) \\ \sigma_3 &= \cos(\theta_1) \sin(\theta_2) + \cos(\theta_2) \sin(\theta_1) \\ \sigma_4 &= -\cos(\theta_3)\ \sigma_3 - \sin(\theta_3)\ \sigma_2 \\ \sigma_5 &= \cos(\theta_3)\ \sigma_3 + \sin(\theta_3)\ \sigma_2 \end{aligned}$$

The significance of this H_3^0 matrix is that is provides a direct transformation matrix between the base frame (frame 0) and end-effector frame, which is a solution for forward kinematics.

B. Result for Problem 3–5

For Problem 3–5, we obtained our final result by multiplying all three A_i matrices that we obtained in the Materials and Methods section. Hence, we obtained the following matrix,

$$T_3^0 = \begin{bmatrix} \sigma_1 & \sigma_4 & 0 & \lambda_1 \\ \sigma_5 & \sigma_1 & 0 & \lambda_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where,

$$\begin{aligned} \lambda_1 &= L_3 \cos(\theta_1)\ \sigma_3 - \sin(\theta_1)\ (L_1 + q_2) - L_3 \sin(\theta_1)\ \sigma_2 \\ \lambda_2 &= \cos(\theta_1)\ (L_1 + q_2) + L_3 \cos(\theta_1)\ \sigma_2 + L_3\, \sigma_3 \sin(\theta_1) \end{aligned}$$

and,

$$\begin{aligned} \sigma_1 &= \cos(\theta_1)\ \sigma_3 - \sin(\theta_1)\ \sigma_2 \\ \sigma_2 &= \sin\left(\theta_3 + \frac{\pi}{2}\right) \\ \sigma_3 &= \cos\left(\theta_3 + \frac{\pi}{2}\right) \\ \sigma_4 &= -\cos(\theta_1)\ \sigma_2 - \sigma_3 \sin(\theta_1) \\ \sigma_5 &= \cos(\theta_1)\ \sigma_2 + \sigma_3 \sin(\theta_1) \end{aligned}$$

The significance of this T_3^0 matrix is that is provides a direct transformation matrix between the base frame (frame 0) and end-effector frame, which is a solution for forward kinematics.

IV. DISCUSSION

In the opinion of the author, this homework problem set was insightful. The first problem proved that we do not need to always use the DH convention when solving for forward kinematics in robotic manipulators. In fact, when using tools like MATLAB, manually executing a non-DH method of computing the forward kinematics is no more complex than using the DH method itself.

The second problem reinforced our learnings from RBE 500. We used the DH convention heavily in that class, so it was great to revisit that foundation as we move forward in this class.

A topic for further consideration could be, when would one prefer to use a non-DH method over the DH method? The DH convention can provide a minimal and efficient way to represent and compute the relationship between the base frame and the end effector in many cases, because it reduces the number of variables involved from 6 to 4. However, suppose we want to model the differential kinematics of a manipulator. The screw-based theory [**?**] can provide advantages in such a case. In the referenced paper for screw-based theory, it was found that, when various kinematic modelings for common manipulator configurations where compared, the screw-based theory did not provide any disadvantages in any case. The one noticeable difference was that it provided superior flexibility when differential kinematics was compared. The parameter identification is also a bit simpler in the screw-based theory, as compared to the DH-convention.

It was a great exercise the solve this week's problem set, and the author thanks the Professor for this.