RBE 501 Week 6: Physics by EL Assignment

Arjan Gupta
Robotics Engineering
Worcester Polytechnic Institute
Worcester, MA, USA
agupta11@wpi.edu

***Abstract*—This document provides an in-depth solution for the Week 6 Physics by Euler-Lagrange problems in RBE 501.**

***Index Terms*—physics, euler-lagrange equations**

I. Introduction

For Week 6, we are given two problems. The first problem shows us a mass m that is free to move on the surface of a frictionless table. Mass m is attached to another mass M via a string that goes through a hole in the table. We can assume the string has no spring-like properties, and is therefore always taut. The representation of this system is shown in Figure 1. Our first objective in this problem is to solve for the Euler-Lagrange equations of this system with respect to r and θ. Our second and third objectives here are to discuss the behaviors of r and θ in the EL equations. The fourth objective of the first problem is to give the conditions under which $\dot{r}$ and $\ddot{r}$ cause a circular motion for mass m.

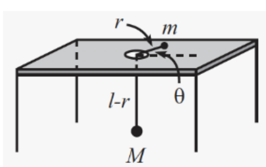

Fig. 1. Given figure for Problem 1

The second problem shows a mass m that is held at rest on ramp of mass M. The ramp has inclination θ, as shown in Figure 2. The surface of the ramp is frictionless, and the surface upon which the ramp rests is also frictionless. Our objective is to find the acceleration of the ramp caused by the release of mass m, and we must use Euler-Lagrange approach for this.

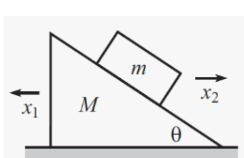

Fig. 2. Given figure for Problem 2

II. Materials and Methods

A. Approach for Problem I

Our first step toward solving Problem I is to find the overall kinetic energy (T) and overall potential energy (U) of the system. We then find the Lagrangian of the system as $L \equiv T - U$. We can then use the Euler-Lagrangian equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \tag{1}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \tag{2}$$

to discuss the behaviors of r and θ.

1) Find overall kinetic energy of the system (T): In Fig. 1, we consider vector $\mathbf{r}$. Say the x-y plane is the surface of the table. Since we consider an angle θ as shown, we can say,

$$\mathbf{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

Where the magnitude of r is dependent on theta. Differentiating with respect to θ to get velocity,

$$\dot{\mathbf{r}} = \begin{pmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{pmatrix}$$

To find the kinetic energy, we need the square of this velocity, which is simply the dot product of the vector with itself.

$$\begin{aligned}\dot{\mathbf{r}}^2 = \dot{\mathbf{r}}\cdot\dot{\mathbf{r}} &= (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 \\ &= \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + \\ &\quad \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta \\ &= \dot{r}^2(\cos^2\theta + \sin^2\theta) + r^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta)\end{aligned}$$

Hence,

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

Therefore the kinetic energy of the mass m moving on the table is given as,

$$T_m = \frac{1}{2}\, m\, (\dot{r}^2 + r^2\dot{\theta}^2) \tag{3}$$

Also, the length of the string that suspends mass M is $l - r$, where total length of the string is l. The suspended part of the string is also along the axes perpendicular to the table, which is the z axis. Therefore, the kinetic energy of the mass M moving perpendicular to the table is given as

$$T_M = \frac{1}{2}\ M\ \left(\frac{d}{dt}(l-r)\hat{\mathbf{z}}\right)^2 = \frac{1}{2}\ M\ \dot{r}^2 \tag{4}$$

So, using Equations 3 and 4, the total kinetic energy of the system (T) is,

$$\begin{aligned} T &= T_m + T_M \\ &= \frac{1}{2}\ m\ (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}\ M\ \dot{r}^2 \end{aligned}$$

Rearranging,

$$T = \frac{1}{2}\dot{r}^2(m+M) + \frac{1}{2}mr^2\dot{\theta}^2 \tag{5}$$

2) Find overall potential energy of system: The potential energy is also a sum of the potential energies of the masses m and M in the system. The only source of potential energy in this system is gravity caused by the earth. Since we are free to define the reference point from where we can count the height from earth, let us define the table-top in Figure 1 as height $h = 0$. Also, g is the constant acceleration due to gravity.

This means,

$$U_m = mgh = mg\ (0) = 0$$

and,

$$U_M = Mg(-(l-r)) = Mg(r-l)$$

So, the overall potential energy of the system is simply,

$$U = U_M + U_m = Mg(r-l) \tag{6}$$

3) Find Lagrangian of the system: The Lagrangian of the system is given as $L \equiv T - U$. Using Equations 5 and 6, we get

$$\begin{aligned} L &= \left(\frac{1}{2}\dot{r}^2(m+M) + \frac{1}{2}mr^2\dot{\theta}^2\right) - Mg(r-l) \\ L &= \frac{1}{2}\dot{r}^2(m+M) + \frac{1}{2}mr^2\dot{\theta}^2 - Mg(r-l) \end{aligned}$$

4) Find equations of motion for the system: We can use Equations 1 and 2 to find the equations of motion.

For the equations of motion with respect to r, we have,

$$\frac{\partial L}{\partial r} = m\ r\ \dot{\theta}^2 - Mg \tag{7}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{d}{dt}\left(\dot{r}(m+M)\right) = \ddot{r}(m+M) \tag{8}$$

And for the equations of motion with respect to θ, we have,

$$\frac{\partial L}{\partial \theta} = 0 \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m\ r^2\ \dot{\theta}\right) \tag{10}$$

While it is possible to show the expanded differentiated form of Equation 10, we have left it as is in order to aid our Discussion better.

5) Circular motion conditions: Problem 1 also asks to set the conditions under which mass m has circular motion, by setting $\dot{r} = 0$ and $\ddot{r} = 0$. We substitute these values in Equations 7 and 8. Also, we set $r = r_{circ}$ and $\dot{\theta} = \omega_{circ}$. This gives us the following,

$$\begin{aligned} m\ r_{circ}\ {\omega_{circ}}^2 - Mg &= (0)(m+M) \\ m\ r_{circ}\ {\omega_{circ}}^2 - Mg &= 0 \\ m\ r_{circ}\ {\omega_{circ}}^2 &= Mg \end{aligned}$$

So, we can write r_{circ} as,

$$r_{circ} = \frac{Mg}{m{\omega_{circ}}^2} \tag{11}$$

Also, under these conditions, we set Equations 9 and 10 with $\dot{\theta} = \omega_{circ}$ and $r = r_{circ}$ as follows,

$$\begin{aligned} \frac{d}{dt}\left(m\ {r_{circ}}^2\ \omega_{circ}\right) &= 0 \\ \int \left[\frac{d}{dt}\left(m\ {r_{circ}}^2\ \omega_{circ}\right)\right] dt &= \int (0) dt \end{aligned}$$

Which gives the well-known physical fact that angular momentum remains conserved in this system,

$$m\ {r_{circ}}^2\ \omega_{circ} = C_{am} \tag{12}$$

This is because the standard formula of the angular momentum, mvr, can easily be rewritten as $mr^2\omega$ because velocity is $r\omega$. We have chosen C_{am} to denote the constant angular momentum of the system. Futhermore, we can write ω_{circ} as,

$$\omega_{circ} = \frac{C_{am}}{m\ {r_{circ}}^2} \tag{13}$$

We can now substitute 13 into 11,

$$\begin{aligned} r_{circ} &= \frac{Mg(m\ {r_{circ}}^2)^2}{m\ {C_{am}}^2} \\ r_{circ} &= \frac{Mg\ m^2\ r_{circ}^4}{m\ {C_{am}}^2} \\ 1 &= \frac{Mg\ r_{circ}^3}{{C_{am}}^2} \end{aligned}$$

Isolating r_{circ},

$$r_{circ} = \sqrt[3]{\frac{{C_{am}}^2}{mMg}} \tag{14}$$

Additionally we can substitute 11 into 13.

$$\omega_{circ} = \frac{C_{am}(m\omega_{circ}{}^2)^2}{m\ (Mg)^2}$$
$$\omega_{circ} = \frac{C_{am}m^2\omega_{circ}{}^4}{m\ M^2g^2}$$
$$1 = \frac{C_{am}\ m\ \omega_{circ}{}^3}{M^2g^2}$$

Isolating ω_{circ},

$$\omega_{circ} = \sqrt[3]{\frac{M^2g^2}{m\ C_{am}}} \tag{15}$$

B. Approach for Problem 2

Like Problem 1, we first find T and U in order to give us $L \equiv T - U$. After that we solve for the following equations of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_3}}\right) = \frac{\partial L}{\partial x_3} \tag{16}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_2}}\right) = \frac{\partial L}{\partial x_2} \tag{17}$$

Where x_2 is the displacement as shown in Figure 2. Here, x_3 is the displacement down the ramp, in the reference frame of the ramp. The x_1 displacement, as shown in the figure, is simply the horizontal component of x_3.

1) Find overall kinematic energy of system: Let us first establish our vectors. With the directions shown in Figure 2, the velocity of the moving ramp is given as

$$\mathbf{v_2} = \dot{x_2}\hat{\mathbf{i}}$$

The velocity of the block sliding down the ramp is given as

$$\mathbf{v_1} = \dot{x_2}\hat{\mathbf{i}} + \dot{x_3}\hat{\mathbf{r}}$$

where $\dot{x_3}\hat{\mathbf{r}}$ is the vector showing the velocity down the ramp in the reference frame of the ramp, and $\hat{\mathbf{r}}$ is the direction down the ramp (which is angle θ down from the horizontal), in the reference frame of the ramp.

Thus, the kinetic energies are,

$$T_m = \frac{1}{2}m\mathbf{v_1}\cdot\mathbf{v_1} = \frac{1}{2}m\left(\dot{x_2}^2 + \dot{x_3}^2 + 2\dot{x_2}\dot{x_3}\cos\theta\right)$$

$$T_M = \frac{1}{2}M\mathbf{v_2}\cdot\mathbf{v_2} = \frac{1}{2}M\dot{x_2}^2$$

In T_m, θ is the angle of incline of the ramp. The overall kinetic energy is $T = T_m + T_M$.

2) Find overall potential energy of the system: The potential energies are,

$$U_m = -mg\ x_3\sin\theta$$

and,

$$U_M = Mg\ (0) = 0$$

For mass m, only the vertical component of the velocity vector down the ramp (in the reference frame of the ramp) affects the gravitational potential of the object. The mass M does not have any height change in our system, so we define its height as 0. The overall potential energy is therefore $U = U_m$.

3) Find Lagrangian: The Lagrangian, $L \equiv T - U$ for this system is

$$L = \frac{1}{2}m\left(\dot{x_2}^2 + \dot{x_3}^2 + 2\dot{x_2}\dot{x_3}\cos\theta\right) + \frac{1}{2}M\dot{x_2}^2 + mg\ x_3\sin\theta \tag{18}$$

4) Find equations of motion for the system: Using 17, we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_2}}\right) = \frac{\partial L}{\partial x_2}$$
$$\frac{d}{dt}\left[\frac{1}{2}m\left(2\dot{x_2} + 2\dot{x_3}\cos\theta\right) + M\dot{x_2}\right] = 0$$
$$\frac{d}{dt}\left[m\left(\dot{x_2} + \dot{x_3}\cos\theta\right) + M\dot{x_2}\right] = 0$$
$$m(\ddot{x_2} + \ddot{x_3}\cos\theta) + M\ddot{x_2} = 0$$
$$\ddot{x_2}(m + M) + m\ddot{x_3}\cos\theta = 0$$

Which gives us,

$$\ddot{x_2} = \frac{-m\ddot{x_3}\cos\theta}{m + M} \tag{19}$$

Using 16, we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x_3}}\right) = \frac{\partial L}{\partial x_3}$$
$$\frac{d}{dt}\left[m(\dot{x_3} + \dot{x_2}\cos\theta)\right] = mg\sin\theta$$
$$m(\ddot{x_3} + \ddot{x_2}\cos\theta) = mg\sin\theta$$
$$\ddot{x_3} + \ddot{x_2}\cos\theta = g\sin\theta$$

Which gives us,

$$\ddot{x_3} = g\sin\theta - \ddot{x_2}\cos\theta \tag{20}$$

Now, putting Equation 20 into Equation 19, we get,

$$\ddot{x_2} = \frac{-m(g\sin\theta - \ddot{x_2}\cos\theta)\cos\theta}{m+M}$$
$$\ddot{x_2} = \frac{-mg\sin\theta\cos\theta + m\ddot{x_2}\cos^2\theta}{m+M}$$
$$m\ddot{x_2} + M\ddot{x_2} = -mg\sin\theta\cos\theta + m\ddot{x_2}\cos^2\theta$$
$$m\ddot{x_2}(1-\cos^2\theta) + M\ddot{x_2} = -mg\sin\theta\cos\theta$$
$$m\ddot{x_2}\sin^2\theta + M\ddot{x_2} = -mg\sin\theta\cos\theta$$
$$\ddot{x_2}(m\sin^2\theta + M) = -mg\sin\theta\cos\theta$$

Isolating $\ddot{x_2}$, we get

$$\ddot{x_2} = \frac{-mg\sin\theta\cos\theta}{M + m\sin^2\theta} \tag{21}$$

III. Results

A. Results for Problem 1

Our first objective of Problem 1 was to find the equations of motion in terms of r and θ. From Equations 7, 8, 9, 10, these are as follows,

$$\boxed{m\ r\ \dot{\theta}^2 - M\ g\ l = \ddot{r}\ (m+M)} \tag{22}$$

and,

$$\boxed{\frac{d}{dt}\left(m\ r^2\ \dot{\theta}\right) = 0} \tag{23}$$

Our second and third objectives were to discuss Equations 22, 23, which we will do in our next section. The results of the fourth objective, by using 14 and 15, are as follows,

$$\boxed{r_{circ} = \sqrt[3]{\frac{{C_{am}}^2}{mMg}}} \tag{24}$$

and,

$$\boxed{\omega_{circ} = \sqrt[3]{\frac{M^2 g^2}{m\ C_{am}}}} \tag{25}$$

Where C_{am} is the constant angular momentum conserved (given by Equation 12) for r_{circ} and ω_{circ}, which are the radius and angular velocity values when the linear velocity and linear acceleration with regards to the vector **r** are 0.

B. Result for Problem 3–5

For Problem 2, we obtained the following result, as given in Equation 21,

$$\boxed{\ddot{x_2} = \frac{-mg\sin\theta\cos\theta}{M + m\sin^2\theta}}$$

Which shows the acceleration corresponding to the displacement x_2 in Figure 2.

IV. Discussion

The author found this assignment very fascinating. Being trained in mostly the circuits and electromagnetic side of physics (because a Computer Engineering degree), the author had never explored alternative formulations of classical mechanics besides Newtonian mechanics.

The Euler-Lagrange equations of the first problem reveal interesting properties of the system. The right-hand side and left-hand side of the E-L equations with respect to θ are shown in 9 and 10. Equating these, says the following,

$$\frac{d}{dt}\left(m\ r^2\ \dot{\theta}\right) = 0$$

Where $\dot{\theta}$ describes the angular velocity of the rotation of mass m. By looking at the dimensions of the expression inside the differentiation portion, we find that it is nothing but the formula for **angular momentum**. In fact, the entire E-L equation in this case is stating the well-known law that the angular momentum is *conserved*, since it will not vary given the passage of time. We can see this better by integrating both sides,

$$\int \left[\frac{d}{dt}\left(m\ r^2\ \dot{\theta}\right)\right] dt = \int 0 dt$$
$$m\ r^2\ \dot{\theta} = C_m$$

Which shows that for a given radius r and angular velocity $\dot{\theta}$, the angular momentum is conserved as some constant value C_m, and hence tweaking r will cause a compensation in $\dot{\theta}$, and vice-versa. One fascinating real-life application of the conservation of angular momentum is the spinning of a gyroscope. Spinning gyroscopes react to external forces by adjusting themselves in a direction that is given by the cross product of the force applied and their angular momentum. This often gives the appearance that gyroscopes 'resist' forces, which is in some way true, since it can be seen as the angular momentum 'fighting' to conserve itself. In modern day sensors, gyroscopic sensors are used to tell the orientation in something. Specifically, in robotics, these sensors can help a robot know its orientation, which is essential for a real-time controller to correct its error in path planning and kinematics.

Let us also discuss the E-L equations with respect to r. These are given by Equations 7 and 8 as

$$m\ r\ \dot{\theta}^2 - Mg = \ddot{r}(m+M)$$

In this equation, the constants are m, M, and g. The only variables are r, $\dot{\theta}$, and $\ddot{r}$. The variable r is the length of the portion of string on the table. The angular speed is $\dot{\theta}$, which is causing rotations in mass m, and the acceleration at which the string is slipping off the table is $\ddot{r}$. So, analyzing the causes of the changes in these variables, we can state -
If the linear acceleration of the string ($\ddot{r}$) changes, then naturally r will change proportionately. However, since the

equality in this equation must remain true, the mass m will start spinning faster or slower (change in magnitude of $\dot{\theta}$), according to the magnitude of the linear acceleration.

This statement demonstrates that, fascinatingly, if the mass m spins fast enough, it can completely lift mass M. Mass M can be much greater than m as well, and this will still work. In fact, humans have been using this simple concept in windmill turbines for over a thousand years. Imagine a simple windmill with a pipe attached to the back of it. The pipe goes deep into the ground, and reaches into potable water. Inside the pipe is a weight, which when pulled up by the rotating blades, sucks water up into the pipe. At a certain height the water can now be siphoned into a storage tank. This way, effortlessly, without electricity, humans used to pump water for storage and usage. They could even irrigate their crops using this type of a pumping system.

The author thanks the Professor for an engaging and nourishing assignment.