A mostly useless table for exact goniometric functions

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Introduction

In school I learned the exact expressions for the sine, cosine and tangent of certain angles – 0, 30, 45, 60 and 90 degrees. These exact expressions are fairly simple, but the list is incomplete: with the formulae for the sine, cosine and tangent of the sum and the difference of two angles, it is easy to get expressions for 15 and 75 degrees as well.

Here is the result:

Angle degrees	*Angle radians*	*Sine*	*Cosine*	*Tangent*
0	0	0	1	0
15	$\frac{1}{12}\pi$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$1-\frac{1}{2}\sqrt{3}$
30	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
45	$\frac{1}{4}\pi$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1
60	$\frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$
75	$\frac{5}{12}\pi$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$
90	$\frac{1}{2}\pi$	1	0	∞

Can we refine this table? Yes, of course: we have the pentagon.

Extending the table via a pentagon

The regular pentagon has angles of 72 degrees or $\frac{2}{5}\pi$ radians between the centre and its vertices (`https://mathworld.wolfram.com/RegularPentagon.html`). Thanks to this figure, we know the exact sines and cosines of this angle:

$$\begin{aligned}\cos(\frac{2}{5}\pi) &= \frac{1}{4}(\sqrt{5}-1)\\ \sin(\frac{2}{5}\pi) &= \frac{1}{4}\sqrt{10+2\sqrt{5}}\end{aligned}$$

Together with the summation and difference formulae we can – in principle – extend the above table to multiples of 3 degrees! In principle, because the expressions are rather complicated for some of these angles and lose all practical relevance.

The angle of 3 degrees $= 75 - 72$ degrees has the following exact expressions for the sine and cosine:

$$
\begin{aligned}
\sin(3^\circ) &= \sin(75^\circ)\cos(72^\circ) - \cos(75^\circ)\sin(72^\circ) \\
&= \frac{1}{4}(\sqrt{6}+\sqrt{2}) \cdot \frac{1}{4}(\sqrt{5}-1) - \frac{1}{4}(\sqrt{6}-\sqrt{2}) \cdot \frac{1}{4}\sqrt{10+2\sqrt{5}} \\
\cos(3^\circ) &= \cos(75^\circ)\cos(72^\circ) + \sin(75^\circ)\sin(72^\circ) \\
&= \frac{1}{4}(\sqrt{6}-\sqrt{2}) \cdot \frac{1}{4}(\sqrt{5}-1) + \frac{1}{4}(\sqrt{6}+\sqrt{2}) \cdot \frac{1}{4}\sqrt{10+2\sqrt{5}}
\end{aligned}
$$

The possibilities to simplify these expressions are at best limited and the exact expression for $tan(3^\circ)$ is even more daunting!

For some of the multiples of 3 degrees (excluding the multiples of 15 degrees) there are relatively simple expressions, such as 12 degrees $= 72 - 60$ degrees:

$$
\begin{aligned}
\sin(12^\circ) &= \sin(72^\circ)\cos(60^\circ) - \cos(72^\circ)\sin(60^\circ) \\
&= \frac{1}{4}\sqrt{10+2\sqrt{5}} \cdot \frac{1}{2} - \frac{1}{4}(\sqrt{5}-1) \cdot \frac{1}{2}\sqrt{3} \\
&= \frac{1}{8}\left(\sqrt{10+2\sqrt{5}} - \sqrt{15} + \sqrt{3}\right)
\end{aligned}
$$

but even then it is completely impractical.

Extending the table via a cubic equation

An alternative approach is to use the trigonometric formulae for tripling an angle:

$$
\begin{aligned}
\cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha \\
\sin 3\alpha &= 3\sin^3\alpha - 4\sin^3\alpha
\end{aligned}
$$

which follow after some straightforward manipulation. With the help of these formulae we can determine the exact sine and cosine of angles like 5, 10 and 20 degrees. And, if we persevere in both approaches, the table could even be extended to a complete set of exact expressions for all angles 0, 1, 2, ... 360 degrees: $1^\circ = 6^\circ - 5^\circ$ for instance.

All it takes is finding the exact root of the cubic equations via Cardano's formula.[1] Unfortunately, this results in expressions with cubic roots and you need to be careful with selecting the right root, as there is either one real root and two imaginary ones or there are three real roots.

So, while this approach may lead to exact expressions of different angles, it does not lead to anything manageable either.

[1] `https://mathworld.wolfram.com/CubicFormula.html`