Doodles turned into geometric curves

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Introduction

I guess everyone draws "doodles" from time to time, meaningless little drawings to while away the time. Such drawings, four are shown in figure 1 below, may feel "geometric", at least, that is how I think about them. You could almost think of an equation or a parametric representation that would reproduce them and that is exactly what I intend with this note – construct geometric curves of the same general shape as the four doodles in the figure.

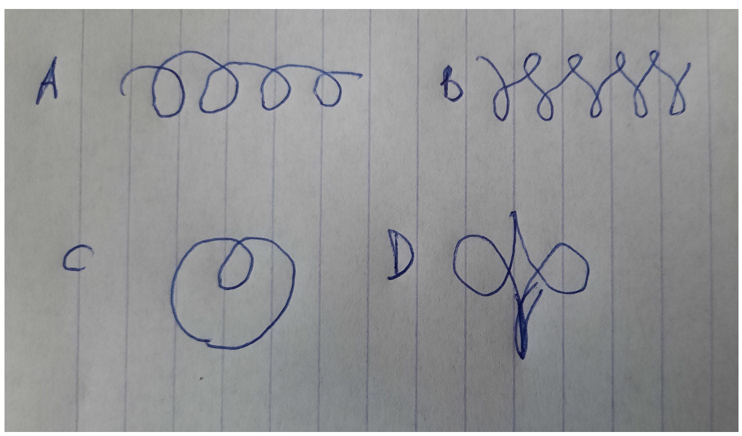

Figure 1: Four doodles

The top two doodles: travelling along the horizontal axis

The first of these, marked "A", is simply the trajectory of a point rotating on a circle while also travelling in the horizontal direction. So a parametric representation would be:

$$x = \sin t + At$$

$$y = \cos t$$

With a particular choice for the constant A you may get the curves in figure 2 – $A = 0.3$. If the constant is smaller, the loops get closer and overlap. With the constant $A = 1.0$ you get a cycloid.

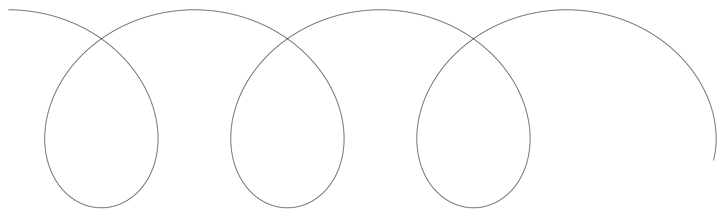

Figure 2: Doodle A: point on a circle and moving to the right

The second one, "B", might represent a point tracing a figure-eight curve and travelling in the horizontal direction as well. A figure eight could be a lemniscate[1], but it could also be a curve like (the parameter "B" allows us to control the width):

$$\begin{aligned} x &= B \sin 2t \\ y &= \cos t \end{aligned}$$

With a component for the horizontal motion you get (figure 3):

$$\begin{aligned} x &= B \sin 2t + At \\ y &= \cos t \end{aligned}$$

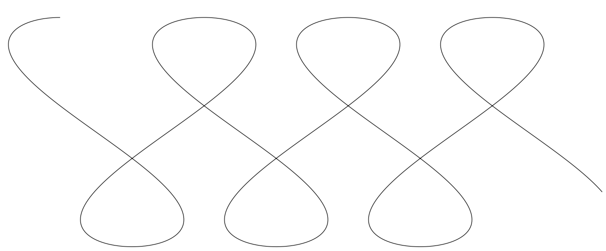

Figure 3: Doodle B: point on a figure-eight curve and moving to the right. Parameter values: $A = 0.2$, $B = -0.6$.

Note that for the curve to show the intended shape, we need to control the phase via the sign of parameter B, otherwise it takes a shape like in figure 4

[1]See `https://mathworld.wolfram.com/Lemniscate.html` for more information

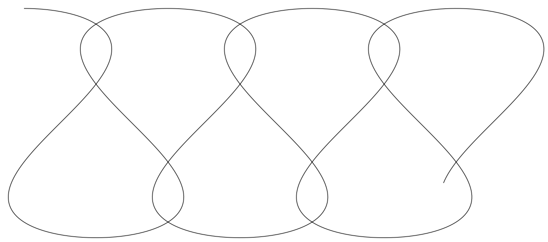

Figure 4: Alternative doodle B: point on a figure-eight curve and moving to the right. Parameter values: $A = 0.2$, $B = +0.6$.

The bottom two doodles: confined curves

To construct doodle C we need to be a trifle more systematic. By sketching the x- and y-coordinates as a function of a parameter t, we can get an idea of what the individual equations should be:

- If you start at the lowest point and follow the doodle counter-clockwise, then the x-coordinate first increases, then goes to zero and becomes slightly negative, to follow an opposite course and then to decrease again. This resembles a sine curve with an additional peak and dip around the value $t = \pi$.
- For the y-coordinate we get the following behaviour: from a minimum value it increases to a maximum, then decreases again, but remains above the minimum, increases again to the same maximum and finally decreases to the same minimum.

After some experimentation the result was figure 7 with equations (the scale factors are a consequence of keeping the curve within a square of side 2, just like the circle that would result if the second factors are left out):

$$\begin{aligned} x &= \sin t \cdot (1 + 7.6 \cos t)/8.6 \\ y &= -\cos t \cdot (1 + 4.6 \cos t)/5.6 \end{aligned}$$

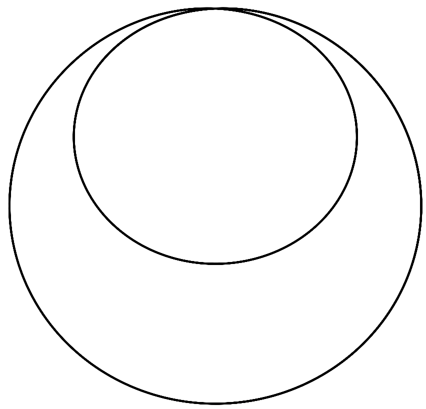

Figure 5: Doodle C: round curve with a smaller loop at the top.

While experimenting to get the curve in the shape I wanted, I also came across the curve in figure 6 with very similar equations:

$$
\begin{aligned}
x &= \sin t \cdot (1 + 2.6 \cos t)/3.6 \\
y &= \cos t \cdot (1 - 2.6 \cos t)/3.6
\end{aligned}
$$

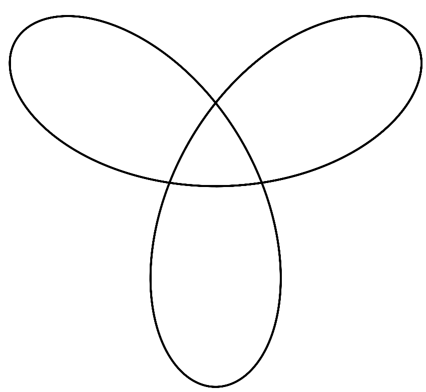

Figure 6: Alternative doodle C: curve resembling a trefoil knot.

Finally, the fourth doodle. This possesses two loops on the side and two cusps on top and bottom. In a similar way as for doodle C we can analyse the x- and y-coordinates to get an impression of the required functions that will produce this curve. The interesting parts are the two cusps. They can be represented by a function for x that changes very slowly there and a function for y that has a maximum. To get the cusp we need x to vary more slowly than y. The result is a curve with these parametric equations:

$$
\begin{aligned}
x &= \sin^3 t \\
y &= \cos t + \cos 3t
\end{aligned}
$$

The equations are surprisingly simple and elegant, in my opinion.

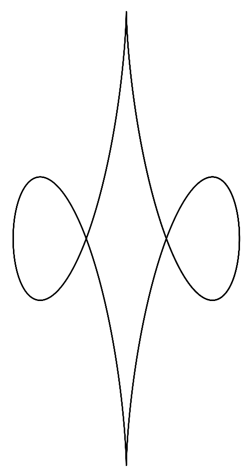

Figure 7: Doodle D: two loops and two cusps.