Permutations, derangements and variants

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Some recurrences

The topic of this note is something I stumbled upon while working on some simple combinatorical topics. As you know, the number of permutations grows fast with the number of items. If you put some constraints on the permutations, like requiring that no item remains in the same position, you still end up with a large number, but it is smaller than a regular factorial number ($n!$). Permutations where no item occupies its original position are called derangements (cf. `https://mathworld.wolfram.com/Derangement.html`). The number of derangements of n items, d_n, can be calculated via a straightforward recurrence relation:

$$\begin{aligned} d_1 &= 0 \\ d_{n+1} &= nd_n + (-1)^{n+1} \end{aligned}$$

While this number does not grow quite as fast as a factorial, it tends to a fraction of the factorial:

$$lim_{n \to \infty} \frac{d_{n+1}}{n!} = e$$

But what happens if you change the recurrence relation a bit? Instead of an alternating 1 and -1, always add 1 or subtract 1?

It is easy enough to write a small program that calculates these numbers:

$$\begin{aligned} (n+1)! &= (n+1)n! \\ a_{n+1} &= na_n - 1 \\ b_{n+1} &= nb_n + 1 \\ c_{n+1} &= nc_n + max(0, (-1)^{n+1}) \\ d_{n+1} &= nd_n + (-1)^{n+1} \end{aligned} \tag{1}$$

with initial conditions somewhat arbitrarily chosen as $a_1 = 1, b_1 = 1, c_1 = 0, d_1 = 0$.

The result for the first few values of n:

n	$n!$	a_n	b_n	c_n	d_n
1	1	1	1	0	0
2	2	1	3	1	1
3	6	2	10	3	2
4	24	7	41	13	9
5	120	34	206	65	44
6	720	203	1237	391	265
7	5040	1420	8660	2737	1854
8	40320	11359	69281	21897	14833
9	362880	102230	623530	197073	133496

Calculating the ratios with $n!$ gives the following results (rounded to four decimals):

$$\begin{aligned} a_n/n! &\rightarrow 0.2817 && (2)\\ b_n/n! &\rightarrow 1.718 && (3)\\ c_n/n! &\rightarrow 0.5431 && (4)\\ d_n/n! &\rightarrow 0.3679 && (5)\end{aligned}$$

The convergence is very fast – it takes only eight steps to reach the end result in four decimals. We can see that d_n (the number of derangements) converges to $1/e$, as expected. We can see that a_n likely converges to $3-e$ and b_n to $e-1$, judging from the numerical values.

The last one, c_n, is a bit more difficult, but:

$$\frac{0.5431}{e} \approx 0.19979 \tag{6}$$

so one could expect the exact result to be $\frac{1}{5}e$.

Analysis

The somewhat empirical considerations need to be substantiated and luckily it is not too difficult to analyse the results in a more thorough way. Let us take the following general relation:

$$\begin{aligned} A_1 &= \dots && (7)\\ A_{n+1} &= (n+1)A_n + k_n && (8)\end{aligned}$$

The ratio $A_{n+1}/(n+1)!$ can be written as:

$$\begin{aligned} \frac{A_{n+1}}{(n+1)!} &= \frac{(n+1)A_n + k_n}{(n+1)n!} && (9)\\ &= \frac{A_n}{n!} + \frac{k_n}{(n+1)!} && (10)\end{aligned}$$

Continuing this process we get the equation:

$$\frac{A_{n+1}}{(n+1)!} = A_1/1! + \sum_{i=2}^{n} \frac{k_i}{(i+1)!} \tag{11}$$

For $k_n = 1$ (b_n in the definition above), this leads to:

$$lim_{n\to\infty} \frac{b_{n+1}}{(n+1)!} = 1 + \sum_{i=2}^{n} \frac{1}{(i+1)!} \tag{12}$$

$$= 1 + e - 2 = e - 1 \tag{13}$$

For $k_n = -1$ (a_n), we get:

$$lim_{n\to\infty} \frac{a_{n+1}}{(n+1)!} = 1 - \sum_{i=2}^{n} \frac{1}{(i+1)!} \tag{14}$$

$$= 1 - (e-2) = 3 - e \tag{15}$$

Now the interesting case is c_n. The series is:

$$lim_{n\to\infty} \frac{c_{n+1}}{(n+1)!} = 0 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ... \tag{16}$$

so with the odd terms cancelled. If we add the series for e and e^{-1}, then we get a cancellation of these terms as well, so that (compensating for the first term, 1/0!:

$$lim_{n\to\infty} \frac{c_{n+1}}{(n+1)!} = \frac{1}{2}\left(e + e^{-1} - 2\right) \tag{17}$$

Numerically, the value is 0.5430806..., quite close to $\frac{1}{5}e = 0.543656....$ The expectation that the ratio converges to $\frac{1}{5}e$ is plainly the consequence of numerical similarity only.