Non-intuitive particle trajectories

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Introduction

Recently, while discussing the unexpected particle trajectories in the mouth of the Western Scheldt estuary between the Netherlands and Belgium, I was reminded of a phenomenon we encountered a long time ago that occurs along the Dutch coast. The residual flow in that part of the North Sea is northwards, but we have seen from salinity measurements that a part of the water from the river Rhine travels much further south than one would expect from the horizontal tidal excursion. In those days we were working on the Transport Atlas of the North Sea[1] and the calculations for this atlas were all done with the *residual* flow, that is, the tidally averaged flow field. To compensate for this strange phenomenon, we used a large dispersion (diffusion) coefficient to force the pattern of lower salinity to the south. This worked, but it was slightly disappointing: where did this come from?

My colleague, Leo Postma, found an explanation for this unexpected transport pattern using a simple schematic flow pattern. While it is overly simplistic, it does show that some non-intuitive phenomena are possible. It is not as dramatic, perhaps, as particles that either leave the estuary or are transported into it, depending on the phase of the tide that they are released at, as was the case with the Western Scheldt, but intruiging nonetheless.

Schematic flow and trajectories

The flow pattern that he used was something along these lines:

$$\frac{dx}{dt} = u_1 \cos \omega t \tag{1}$$

$$\frac{dy}{dt} = v_0 + v_1 \sin \omega t - v_2 x \sin \omega t \tag{2}$$

where x is the direction perpendicular to the shore and y is the direction along the shore.

[1] De Runter, W.P.M., Postma, L, de Kok, J.M. (1987). Transport Atlas of the Southern North Sea. RWS & Delft Hydraulics, The Hague/Delft, 33 pp. + floppy.

If you average the flow field at a fixed location, then the transport of matter will be along the y-axis in the direction dictated by v_0. However, a particle that is released at a location (x_0, y_0) at a time t_0 will be moved around and experience different flow velocities. This becomes clear if we solve the above equations of motion.

The solution for x is (x_0' corrects for the initial position at $t = t_0$):

$$x = x_0' + \frac{u_1}{\omega} \sin \omega t \tag{3}$$

so that the equation for y becomes:

$$\begin{aligned} \frac{dy}{dt} &= v_0 + (v_1 - x_0' v_2) \sin \omega t - \frac{u_1 v_2}{\omega} \sin^2 \omega t & (4) \\ &= v_0 - \frac{u_1 v_2}{2\omega} + (v_1 - x_0' v_2) \sin \omega t + \frac{u_1 v_2}{2\omega} \cos 2\omega t & (5) \end{aligned}$$

The last term in this equation is non-positive, so that with the right choice of parameters the net transport velocity for a particle will be opposite to the mean velocity v_0 at a fixed point. The equation has the following solution:

$$y = y_0' + \big(v_0 - \frac{u_1 v_2}{2\omega}\big)(t - t_0) - \frac{v_1 - x_0' v_2}{\omega} \cos \omega t + \frac{u_1 v_2}{4\omega^2} \sin 2\omega t \tag{6}$$

The condition to make sure that the particle is transported opposite to the Eulerian mean flow ia:

$$v_0 < \frac{u_1 v_2}{2\omega} \tag{7}$$

Sample trajectory

Now, take the parameters to be:

$u_1 = 0.2\ m/s$	the amplitude of the tidal flow perpendicular to the shore
$v_0 = 0.02\ m/s$	the residual flow (northward)
$v_1 = 0.7\ m/s$	the "nominal" amplitude of the tidal flow parallel to the shore
$v_2 = 0.025\ m/s.km$	the perpendicular gradient in the tidal amplitude
$\omega = 2\pi/45000\ s^{-1}$	the angular frequency of a diurnal tide

The nett velocity along the shore is (equation 6):

$$\begin{aligned} v_{nett} &= v_0 - \frac{u_1 v_2}{2\omega} \\ &= 0.02 - \frac{0.2 \cdot 0.025 \cdot 10^-3 \cdot 45000}{2\pi} \\ &\approx -0.016\ m/s \end{aligned}$$

in other words: the nett displacement of the particle is to the *south*.

The trajectory is illustrated in figure 1.

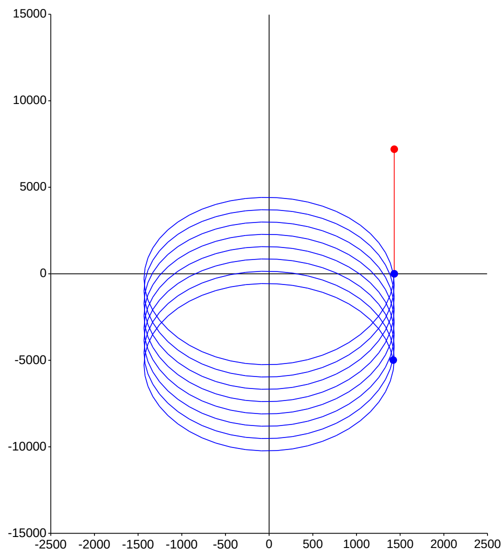

Figure 1: Trajectory of a particle released at the origin at time 0 (blue line). If the particle had been transported with the residual flow, then it would have followed the red line. The trajectories encompass eight tidal cycles.