

Analytical solution for a point source in a uniform ambient flow

Arjen Markus

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Introduction

This note concerns the analytical solution of an idealised discharge situation. It is a simplified model for localised sources of pollutants in an ambient flow, such occurs with outfalls in sea or the discharge of production water at oil platforms. More specifically: consider a uniform flow in an unbounded area. At the origin a point source of a pollutant is located with a discharge small enough not to disturb the ambient flow. To make things even simpler, we only consider a steady-state situation (see Fig. 1).

The area has a uniform depth H and the flow velocity is u . The discharge has a concentration C_s and a volume discharge of Q_s , so that the mass rate is $Q_s C_s$. Instead of the concentration of a polluting substance, one may also consider thermal pollution, so that the quantity of interest is the temperature increase. The (stationary) advection diffusion equation describing the spreading of the pollutant is:

$$\underline{u} \cdot \nabla C = D \nabla^2 C \quad (1)$$

The boundary conditions that apply are:

- At the origin there is a steady source of known strength, emitting the pollutant in a spherical-symmetric fashion.
- Far from the source the concentration approaches zero, as the pollutant is mixed with ambient water.

The flow field, represented here by the vector \underline{u} , may be thought to consist of two components: the ambient flow $u_0 \underline{e}_x$ and a radially outward component $u_r \underline{e}_r$. As long as the radial component is much less intense than the ambient flow one might consider the total flow to be a superposition of these two, idealised, flows. In much of the remainder of this note, however, we will assume that the radial component is negligible, as this makes the analytical solution much simpler, while preserving most of the physics.

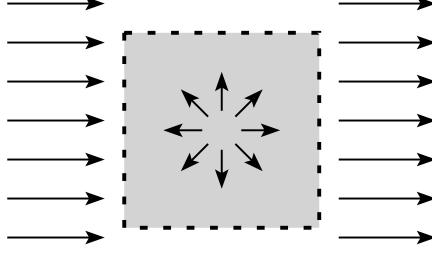


Figure 1: Schematic representation of the flow field around a source. The grey square indicates a control volume.

To analyse the problem we distinguish three regimes: the "overall" regime, a far-field regime where the pollutant is distributed uniformly over the vertical, a "box approach" (see Fig. 1). The concentration on the downstream side is approximately C and it is assumed that this concentration is also present on the two sides parallel to the flow. The concentration on the upstream side is approximately zero. The mass balance therefore reads:

Overall regime

Consider a box around the point source and assume that the pollutant is distributed uniformly over the vertical, a "box approach" (see Fig. 1). The concentration on the downstream side is approximately C and it is assumed that this concentration is also present on the two sides parallel to the flow. The concentration on the upstream side is approximately zero. The mass balance therefore reads:

$$BH(u_0 - u_r) \cdot 0 + BH(u_0 + u_r) \cdot C + 2BH u_r \cdot C = Q_s C_s \quad (2)$$

where $u_r = Q_s/4BH$ is the radial velocity component due to the volume of water discharged by the point source and B is the side of the box.

We can now estimate the "mean" concentration inside the box:

$$C = \frac{Q_s C_s}{BH(u_0 + 3u_r)} \quad (3)$$

To give an impression of the order of magnitude of this mean concentration:

- Take the side of the box as 10 m , the depth as 2 m .
- The ambient flow velocity is 0.2 m .
- The discharge rate is 0.2 m^3/s and the concentration is 50 mg/l .
- For the further analysis we need an estimate of the diffusion or dispersion coefficient as well. A practical estimate is:

$$D = \frac{1}{6} u_* H = \frac{1}{6} \frac{g^{1/2} u_0 H}{Ch} \quad (4)$$

where Ch is the Chézy coefficient, typically around $60 \text{ m}^{1/2}/\text{s}$. This gives a diffusion coefficient of $0.02 \text{ m}^2/\text{s}$

Then the mean concentration in the box is: $0.2 \cdot 50 / (2 \cdot 10 \cdot (0.2 + 3 \cdot 0.0025)) \approx 2.5 \text{ g/m}^3$

Far-field regime

The box approach has the disadvantage that we need to choose an arbitrary box width to get to a concentration estimate. A more convenient approach is the following: if there is no ambient flow, the pollutant will be spread in a axisymmetric pattern if it is well-mixed over the vertical or in a spherical symmetric pattern if it is far away from the bottom. We can transform Eq. 1 so that the resulting equation contains no advective term anymore:

$$C = e^{kx}\Gamma \quad (5)$$

and Eq. 1 becomes:

$$ku_0 e^{kx}\Gamma + u_0 e^{kx} \frac{\partial \Gamma}{\partial x} = D(k^2 e^{kx}\Gamma + 2k e^{kx} \frac{\partial \Gamma}{\partial x} + e^{kx}(\frac{\partial^2 \Gamma}{\partial x^2} + \frac{\partial^2 \Gamma}{\partial y^2})) \quad (6)$$

Choosing the parameter k equal to $u_0/2D$ means the first-order derivative terms cancel, yielding the equation (after substituting k and dividing by $\exp(kx)$):

$$D(\frac{\partial^2 \Gamma}{\partial x^2} + \frac{\partial^2 \Gamma}{\partial y^2}) - k^2 \Gamma = 0 \quad (7)$$

This equation, essentially a reaction-diffusion equation with a first-order decay term, allows both axisymmetric and spherical symmetric solutions in the form of modified Bessel functions and modified spherical Bessel functions. The first case represents the far-field approximation and the second the near-field.

The axisymmetric version of Eq. 7 is:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\Gamma}{dr} \right) - k^2 \Gamma = 0 \quad (8)$$

We seek a solution that remains bound if $r \rightarrow \infty$. This turns out to be a multiple of the modified Bessel function of the second kind, K_0 , as the modified Bessel function of the first kind, I_0 , increases as $r \rightarrow \infty$. All we need to do now is to find the multiplication factor. Since we know how much the point source discharges, we can equate that to the (diffusive) transport around the point source (the factor $\exp(kx)$ approaches 1 in the limit $r \rightarrow 0$):

$$-\int_0^{2\pi} r H D \frac{d\Gamma}{dr} d\phi = -\int_0^{2\pi} \gamma r H D \frac{dK(kr)}{dr} d\phi = Q_s C_s \quad (9)$$

Using the asymptotic relation $K(x) \rightarrow -\ln x$ for $x \rightarrow 0$ we can approximate this as follows:

$$2\pi\gamma DH = Q_s C_s \Rightarrow \gamma = \frac{Q_s C_s}{2\pi DH} \approx 40 \text{ g/m}^3 \quad (10)$$

For this regime to be valid, the pollutant must have spread uniformly over the vertical due to diffusion or dispersion. The time scale associated with this mixing process is: $T = H^2/D$. Due to the ambient flow the distance over which the pollutant will have been transported is:

$$L = u_0 T = u_0 H^2/D \quad (11)$$

With the above numerical values for the various quantities, this gives a length scale of 40 m.

Near-field regime

In the near-field regime we assume both the bottom and the surface are "far" away, so that the solution to Eq. 7 is spherically symmetric:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Gamma}{dr} \right) - k^2 \Gamma = 0 \quad (12)$$

The general solution can be expressed in terms of the modified spherical Bessel functions of the first and second kind, i_0 and k_0 respectively. As these are expressible in elementary functions, the solution is quite simply (implementing the first boundary condition, so that the term with i_0 disappears):

$$\Gamma = \gamma k_0(kr) = \gamma \frac{e^{-kr}}{kr} \quad (13)$$

The second boundary condition now is:

$$-\int_{-\pi/2}^{\pi/2} \cos\theta \int_0^{2\pi} r^2 D \frac{d\Gamma}{dr} d\phi d\theta = -\int_{-\pi/2}^{\pi/2} \cos\theta \int_0^{2\pi} \gamma r^2 D \frac{dk_0(kr)}{dr} d\phi d\theta = Q_s C_s \quad (14)$$

This simplifies to:

$$4\pi\gamma r^2 D \left(\frac{e^{-kr}}{r} + \frac{e^{-kr}}{kr^2} \right) = Q_s C_s \quad (15)$$

where $r \rightarrow 0$.

If we take the numerical example again with a diffusion coefficient of $0.02 \text{ m}^2/\text{s}$:

$$\gamma = \frac{k Q_s C_s}{4\pi D} = \frac{u_0 Q_s C_s}{8\pi D^2} \approx 200 \text{ g/m}^3 \quad (16)$$

The solution tends to infinity as the distance to the point source diminishes – this is the drawback of not including the radial component. However, a pragmatic argument could be that the solution becomes valid only if the distance is large enough, say larger than the distance where Γ becomes equal to C_s . In the numerical example that distance is the root of the equation:

$$\frac{e^{-kr}}{kr} = C_s/\gamma \Rightarrow kr \approx 1/2.5 \Rightarrow r = 1/2.5k \approx 8 \text{ cm} \quad (17)$$

Estimated concentration distribution

By combining the two regimes we obtain an approximate concentration distribution for the entire area. For the near-field regime the full solution is:

$$C = \frac{u_0 Q_s C_s}{8\pi D^2} \frac{e^{kx-kr}}{kr} \quad (18)$$

and for the far-field regime it is:

$$C = \frac{Q_s C_s}{2\pi DH} e^{kx} K_0(kr) \quad (19)$$

We can approximate these solutions on the x-axis as ($r \equiv x$) (see Fig. 2:

$$C = \frac{u_0 Q_s C_s}{8\pi D^2 kx} \quad (20)$$

For the far-field regime we take $r \rightarrow \infty$, so that the modified Bessel function can be approximated:

$$C = \frac{Q_s C_s}{2\pi DH} e^{kx} K_0(kr) \approx \frac{Q_s C_s}{2\pi DH} \sqrt{\frac{2}{\pi kx}} \quad (21)$$

As can be seen in the figure the two approximations give concentrations of the same order of magnitude. For the region $x \ll 40 \text{ m}$ the near-field approximation should be used and for the region $x \gg 40 \text{ m}$ the far-field approximation. In the intermediate region, the differences are fairly large. This gives an impression of the pollutant concentration or the temperature increase downstream of a source.

The profile along a line parallel to the y-axis can be approached by considering the region where $y \ll x$:

$$r = \sqrt{x^2 + y^2} \approx |x| \left(1 + \frac{y^2}{2|x|^2}\right) \quad (22)$$

Then in the near-field regime:

$$C = \frac{u_0 Q_s C_s}{8\pi D^2} kx e^{-ky^2/2x} \quad (23)$$

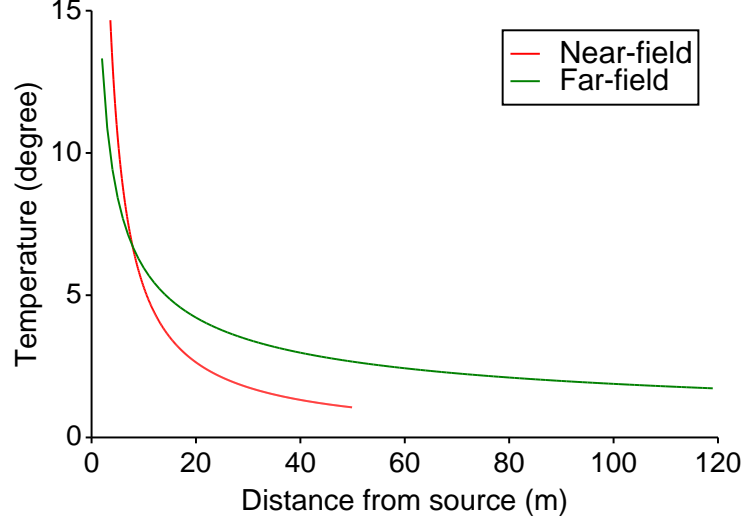


Figure 2: Concentration along the x-axis for the numerical example.

and in the far-field regime:

$$C \approx \frac{Q_s C_s}{2\pi D H} \sqrt{\frac{2}{\pi k x}} e^{-ky^2/2x} \quad (24)$$

So in both regimes the width of the plume is of the order $2\sqrt{x D/u_0}$. Using the numerical example again: at a distance of 1 km the plume reaches a width of 10 m, at a distance of 10 km it is 30 m wide, whereas the concentration is in the order of 0.5 respectively 0.15 g/m^3 . Thus such a plume remains fairly confined and this poses restrictions on the spatial resolution of numerical models.