Filling space with coloured cubes

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The question: three or more colours?

Using cubes with two different colours you can easily fill three-dimensional space in an elegant way, that is, the cubes of the same colour are all connected, the distribution is in a sense uniform and isotropic and the cubes of the same colour are equal in number. Well, that is using these terms above in a rather loose sense, but if you look at Figures 1 and 2, you will see what I mean.

The network of coloured cubes simply repeats itself in the three directions and since the unit cell is symmetric in the corners, there is no preferential direction. Also, the unit cell contains the same number of cubes for both colours, so that the resulting division is uniform in that respect.

The question I want to investigate here is: can the same be done for three colours? What about four or any number of colours, in fact? It turns out there are several ways of achieving a division of space in this fashion, but none are as elegant as the two-colours division.

An irregular method

With three colours we cannot quite use the same method as with two: select the cubes along each of the three axes of a block of three by thee by three cubes. The number of cubes of one colour would be 7 then, whereas we need 9 of each. But judiciously adding two of the remaining cubes and we are done (Figure 3).
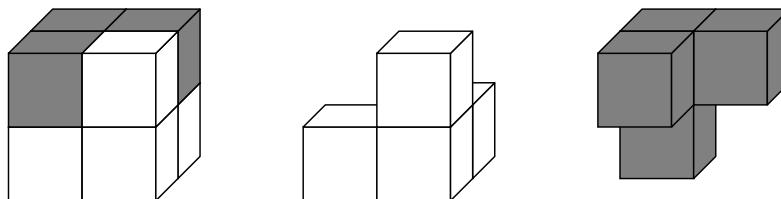

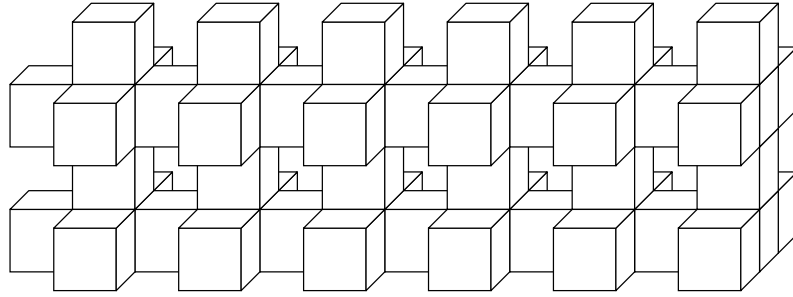Figure 1: Cubes in two colours – "unit cell" for filling space.

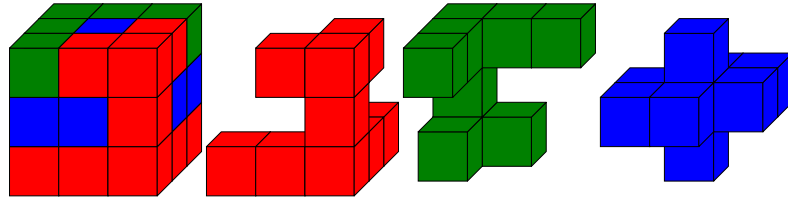Figure 2: Network of cubes, illustrating how the cubes fill space.

Figure 3: Cubes in three colours – "unit cell" for filling space.

The connectivity is guaranteed because the three cubes along each axis will touch the three cubes of a neighbouring unit cell, in much the same way as for the case of two colours.

Since the unit cell is a cube, we can fill space in the same way as before, just put multiple copies of the unit cube (in the same orientation) next to each other. The solution presented here is only one of many, but the drawback is clearly that for each colour you need a different configuration. Thus, it lacks something of the elegance of the two-colours division.

To get an impression of the number of possibilities:

- We can fix one cube, in any location of the 27 cubes making up the unit cell, as it does not matter which colour it gets. Therefore we need to assign colours to 26 cubes only.

- Using the rotational, reflexive and translational symmetries we can simply use that one cube as the centre of the unit cell. That gives us 24 rotations and 24 reflections.

- For the first colour we need to select 8 cubes out of all 26 that are still colourless, leaving 18 cubes. For the second colour we need to select 9 cubes out these 18 and for the third colour we have no further choices.
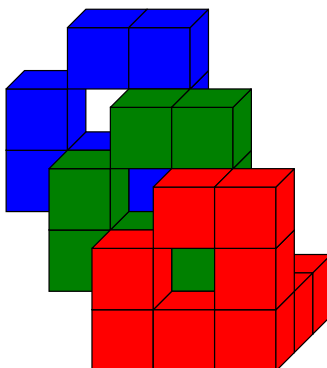
Figure 4: Translated copies – the "unit cell" is not depicted as a cube.

- The total number N of configurations of three colours is then:

$$N = \binom{26}{8} \cdot \binom{18}{9}$$
$$= 75957810500$$

With the 48 symmetries (some configurations are symmetric, so that complicates matters) you would end up with roughly $1.6 \cdot 10^9$ separate configurations. These all need to be examined for connectivity.

If we limit ourselves to configurations like the chosen one, where you have three cubes in every direction from one corner, we only need to select two more cubes for the first colour from the 13 uncoloured cubes and two for the second colour out of 11. This gives:

$$N_2 = \binom{13}{2} \cdot \binom{11}{2}$$
$$= 4290$$

which can be reduced by a factor 3 (rotational symmetry around the main diagonal joining the two corners) to 1430.

Translate copies of a single layout

An alternative is to design a layout that can be translated. One possibility is to extend the layout from the two-colours unit cell, select two cubes with some care to make up for the nine we need, and the result is a configuration like in Figure 4.

It is not immediately clear, but the unit cell as shown fills space in the same way as the ordinary blocks of 27 cubes from the first solution. And the configuration is the same for all three colours. Extending this design to four or more colours will require selecting the right cubes, though, just as in the first
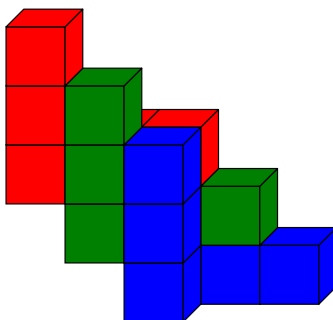
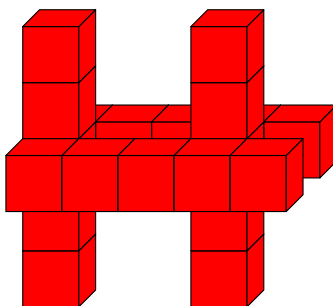Figure 5: Systematic configuration – the "unit cell" two deep, not three.

Figure 6: Systematic configuration – isolating a single colour.

solution. Since the block for n colours would consist of n^3 cubes and the sides take up only $n + 2(n - 1 = 3n - 2$ cubes, it will become progressively more complicated to select the $n^2 - 3n + 2$ needed.

A systematic solution

It turns out to be possible to divide space into three colours or actually n colours in a systematic way, such that the layouts for each colour is the same. Unfortunately, we lose one of aspects of the two-colours solution: it is no longer isometric.

The idea is that each colour forms large holes in which the other colours can be fitted. Isolating a single colour, the configuration is that of Figure 6.

While not isometric like the two-colours solution, it does have an elegant structure: you can distinguish two subsets of cubes, one consisting of vertical stacks of cubes and the other of horizontal rows. The other colours follow the same simple pattern, only shifted in the horizontal and vertical direction. The unit cell is $3 \times 3 \times 2$ instead of $3 \times 3 \times 3$, as with the other solutions.

Other dimensions

Most likely you can extend this type of division to other dimensions as well, although it is very difficult to visualise such divisions. For two dimensions there is a peculiar problem: periodic solutions, like the ones we have seen here with a finite unit cell, seem impossible. One alternative is a spiral-like design. This is intriguing. It might be possible to use the ideas behind Peano curves[1].

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[1] https://en.wikipedia.org/wiki/Peano_curve