Intersecting cubes and spheres

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Introduction

This note is about the volumes of included and excluded bits and pieces of cubes and spheres that intersect. As far as I know there is nothing practical about it, it just happened to attract my attention.

To define the problem: consider a cube and a sphere with different sizes but having the same centre. Parts of the cube or parts of the sphere may stick out and the question is how large is the volume of these parts? The answer depends on the relative size of the two and in some cases it will be awkward to calculate.

Figure 1 is an illustration in two dimensions of the problem.

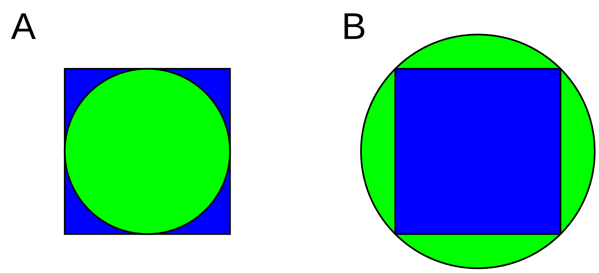

Figure 1: Intersection of a square (side 2) and a circle: (A) circle with radius 1, (B) circle with radius $\sqrt{2}$.

In the left half of the figure (A), the square is larger than the circle and it is easy to calculate the area that lies outside the circle:

$$\begin{aligned} A &= 2^2 - \pi \cdot 1^2 \\ &= 4 - \pi \\ &\approx 0.8584073464102069 \end{aligned}$$

In the right half (B), the circle touches the vertices of the square and the area of the circle outside the square is:

$$\begin{aligned} A &= \pi \cdot (\sqrt{2})^2 - 2^2 \\ &= 2\pi - 4 \\ &\approx 2.2831853071795862 \end{aligned}$$

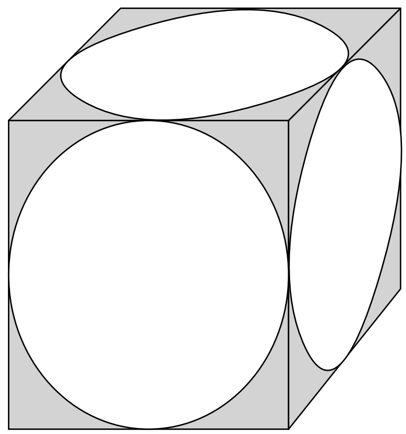

Figure 2: Rough sketch of a sphere touching the edges of the cube.

It is actually easy to calculate these areas for any radius of the circle (cf. Figure 3). For radii between 1 and $\sqrt{2}$ the area is not empty:

$$\begin{aligned} z &= \sqrt{R^2 - 1^2} \\ A &= \int_z^1 1 - \sqrt{R^2 - x^2} dx \end{aligned}$$

Evaluating the integral:

$$\begin{aligned} A &= (1 - z) - \frac{1}{2}\left[R^2 \arcsin \frac{x}{R} + x\sqrt{R^2 - x^2}\right]_z^1 \\ &= 1 - \sqrt{R^2 - 1} - \frac{1}{2}R^2\left(\arcsin \frac{1}{R} - \arcsin \frac{\sqrt{R^2 - 1}}{R}\right) \end{aligned}$$

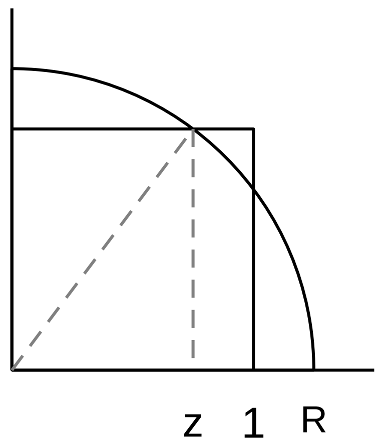

Figure 3: Calculation of the area between a circle and a square.

The three-dimensional case

For the intersection of a cube (of side 2) and a sphere we have three more or less special situations:

- The sphere touches the centres of the cube's faces – the radius is equal to half the side of the cube. It is akin to figure 1A. The volume outside the sphere is:

$$V = 8 - \frac{4}{3}\pi \cdot 1^3 \approx 3.8112097952136095$$

- The sphere touches the vertices of the cube – the radius is equal to $\sqrt{3}$. This is akin to figure 1B. The volume outside the cube is:

$$V = \frac{4}{3}\pi \cdot (\sqrt{3})^3 - 8 \approx 13.765592370810609$$

- The sphere touches the midpoints of the edges of the cube – the radius is equal to $\sqrt{2}$. This has no easy equivalent to the two-dimensional case and calculating the volumes is a bit more involved, but not overly difficult. (Radii between $\sqrt{2}$ and $\sqrt{3}$ create much more difficulties and are not included here.)

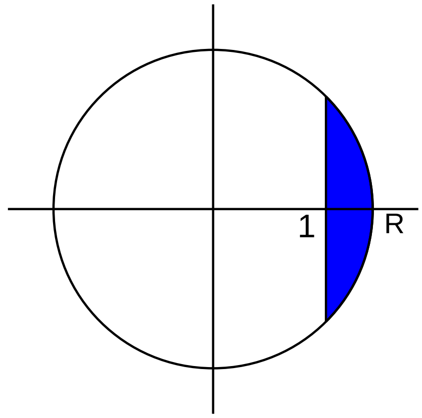

Figure 4: Sketch of the spherical cap. The radius of the sphere $R = \sqrt{2}$.

This last case requires some ingenuity:

- We first determine how much of the sphere sticks out. As illustrated in Figure 2 the parts that stick out are spherical caps. The volume is easy to calculate (see Figure 4):

$$\begin{aligned}
V_{cap} &= \int_1^{\sqrt{2}} \pi(2 - x^2)dx \\
&= \left[2\pi x - \frac{1}{3}\pi x^3\right]_1^{\sqrt{2}} \\
&= \frac{4}{3}\pi\sqrt{2} - \frac{5}{3}\pi \\
&\approx 0.6878561615614993
\end{aligned}$$

- As there are six such caps, the total volume of the sphere outside the cube is $8\pi\sqrt{2} - 10\pi$. This needs to be subtracted from the total volume of the sphere itself to get the volume included in the cube.

- The volume of the cube outside the intersection with the sphere is therefore:

$$\begin{aligned}
V &= 8 - \left(\frac{4}{3}\pi(\sqrt{2})^3 - (8\sqrt{2} - 10)\pi\right) \\
&= 8 + \frac{16}{3}\pi\sqrt{2} - 10\pi \\
&\approx 0.2794491342800214
\end{aligned}$$

As said, for the situation that the sphere's radius is between $\sqrt{2}$ and $\sqrt{3}$, this method does not work. The caps would partially overlap and you need to compensate for that to get the right answer. A brute-force method would be to determine the volume of the corners directly, but this leads to complicated three-dimensional integrals.