Curves of infinite length but indistinguishable from a line piece?

Arjen Markus

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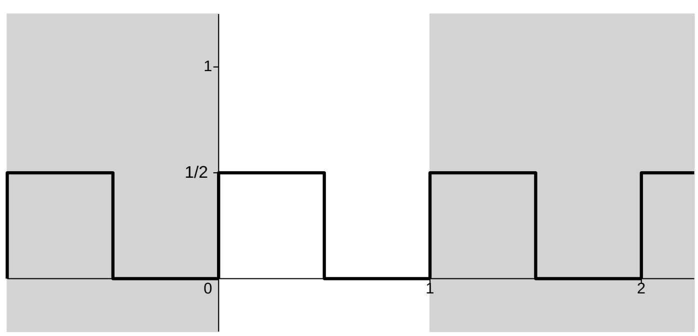
Introduction

Infinity tends to surprise us. Take fractals or space-filling curves. Or the theorem by Riemann that the order in which you sum the terms of a conditionally converging series matters for the outcome.

This short note is also about infinity, but in the form of a simple-looking geometric riddle. It involves the construction of a curve via an infinite iterative process whose length can take any value you want but which is not distinguishable from an ordinary line piece. I am not sure where I go wrong – if I go wrong, but, well, here it is.

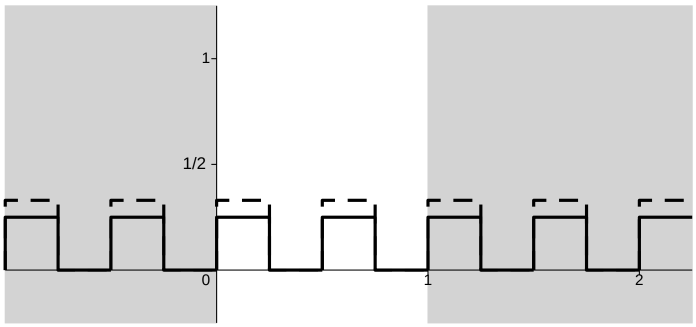
Construction of a curve

Consider the following curve:



The part we are interested in is the part with $0 \leq x < 1$ (the area with the white background in the picture, but excluding the vertical piece at $x = 1$). Its length is 2: the two horizontal pieces are each $\frac{1}{2}$ long and the two vertical pieces likewise. Now we can apply a simple linear map to bring in more "steps":

$(x, y) \mapsto (\frac{1}{2}x, \alpha y)$. With $\alpha = \frac{1}{2}$ we get the solid line in the figure below and with $\alpha = \frac{2}{3}$ we get the dashed curve:



The length of the two curves is: 2 for $\alpha = \frac{1}{2}$ and $2\frac{1}{3}$ for $\alpha = \frac{2}{3}$.

We can continue this mapping indefinitely and the result is that the length stays the same or grows indefinitely:

Step	$\alpha = \frac{1}{2}$	$\alpha = \frac{2}{3}$
1	$1 + 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2$	$1 + 4 \cdot \frac{1}{2} \cdot \frac{2}{3} = 2\frac{1}{3}$
2	$1 + 8 \cdot \frac{1}{2} \cdot \frac{1}{4} = 2$	$1 + 8 \cdot \frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 = 2\frac{7}{9}$
3	$1 + 16 \cdot \frac{1}{2} \cdot \frac{1}{8} = 2$	$1 + 16 \cdot \frac{1}{2} \cdot \left(\frac{2}{3}\right)^3 = 3\frac{10}{27}$
n	$1 + 2^{n-1} \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n = 2$	$1 + 2^n \cdot \frac{1}{2} \cdot \left(\frac{2}{3}\right)^n = 1 + \frac{1}{2} \cdot \left(\frac{4}{3}\right)^n$

With the iteration number n going to infinity the length for $\alpha = \frac{2}{3}$ goes to infinity too, but for $\alpha = \frac{1}{2}$ the length remains constant. With an α lower than $\frac{1}{2}$ the length will approach 1. The height of the "steps" is decreasing by a factor α with each iteration.

The case of $\alpha = \frac{1}{2}$ is clearly special, since the length can be tuned to any value. If we use a different initial height h for the steps, we can get any finite length for the case $\alpha = \frac{1}{2}$ we want. The length would be: $1 + 2h$.

The limit curve

As we continue the mapping, the height becomes less and less (for any factor α lower than 1, that is). The end result is a curve that can not be distinguished from a straight line piece – but the length is still strictly larger than the length of that line piece! Note that this is worse than the construction of a Koch snowflake. That gives a curve of infinite length too, but at least it does not approach a smooth curve.

The problem quite possibly is that we use the limit process in an inappropriate way.