Proving the Impossibility of Constructions with Field Theory

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June 7, 2020

1 Introduction

In this paper, we will explore how field theory can be applied to show the impossibility of various straightedge and compass constructions. The idea of straightedge and compass constructions was first posed by ancient Greek mathematicians around the 4th century BC. For hundreds of years, many constructions which appeared to be impossible could not be proven to be impossible. With the development of field theory, however, we can now prove the impossibility of various constructions. First, we will define the set of constructible numbers. We will prove special properties about the set of constructible numbers and observe how they relate to our unmarked straightedge and compass constructions. We will then use the properties of the set of constructible numbers to prove that doubling the cube, squaring the circle, trisecting the angle, and constructing a regular 7-gon are impossible constructions. There are a few constructions that are important in helping prove the impossibility that we will not prove (we will only discuss briefly). These constructions include constructing perpendicular lines, constructing perpendicular bisectors, constructing parallel lines, and finding the midpoint of a segment. Many proofs and theorems in this paper are adapted from Joseph A. Gallian's *Contemporary Abstract Algebra*, Thomas W. Judson's *Abstract Algebra: Theory and Applications*, and John B. Fraleigh's *A First Course in Abstract Algebra*.

2 Constructible Numbers

First, let us discuss constructible numbers. Understanding the set of constructible numbers is vital in helping us prove the impossibility of various constructions. We will begin with the definition of a constructible number.

Definition 1. *A number α is* ***constructible*** *if, by means of unmarked straightedge and compass, and a line segment of length 1, we can construct a line segment of length $|\alpha|$ in a finite number of steps. [1]*

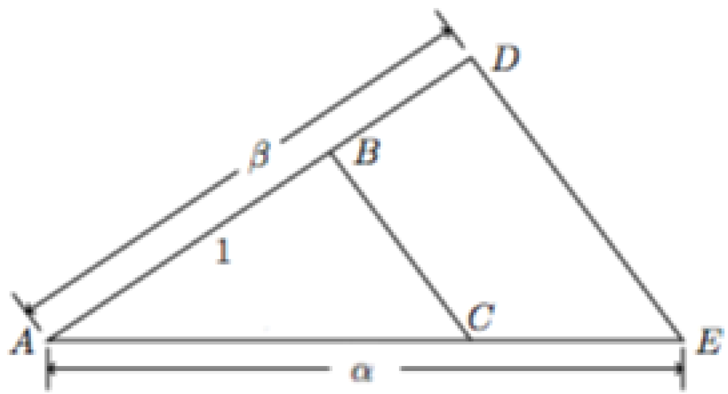

Figure 1: AC is a segment of length $\alpha\beta^{-1}$.[2]

Defining constructible numbers helps us observe which constructions can or cannot be completed with an unmarked straightedge and compass. Now, let us show a property of the set of constructible numbers.

Theorem 1. *The set of all constructible numbers is a subfield of the real numbers. [1]*

Proof. First, we know the set of real numbers is a field. Previously, we have observed that if F is a field with at least two elements and K is a subset of F, then K is a subfield of F if, for any $a, b \in K$ $b \neq 0$, $a - b$ and ab^{-1} belong to K. It is very clear that the set of constructible numbers is a subset of $\mathbb{R}$. So, let us take two elements from the set of constructible numbers, α and β, with $\beta \neq 0$. To show that the set of constructible numbers is a subfield of $\mathbb{R}$, we must show that $\alpha - \beta$ and $\alpha\beta^{-1}$ are both constructible.

First, let us show that $\alpha - \beta$ is constructible. We are given two line segments, one of length α and one of length β. We can use the compass by placing it on the starting point of the line of length β, and measuring out the distance of that segment. Next, maintaining the compass, we can can place the tip of our compass on the endpoint of the segment of length α. Now, we can use the compass to mark the length of β on our line with length α in the appropriate direction (towards the starting point if the signs of α and β have opposite signs and away from the starting point if they have the same signs). Thus, we can see the distance from our starting point of the segment length α to the new point we marked is $\alpha - \beta$. Using the compass, we can mark this length and then construct a line segment of that length. Thus, we see that we can construct a line segment of length $\alpha - \beta$ with only an unmarked straightedge and compass. Thus, we see that $\alpha - \beta$ is constructible.

Next, let us show that $\alpha\beta^{-1}$ is constructible. Consider the triangle in Figure 1. This triangle can be formed through the use of unmarked straightedge and compass constructions. First, create 2 sides of the triangle, AE and AD by using the segments of length, α and β. Then mark a point, B, on line AD one unit away from point A. Now, connect the D and E. Now, we can construct a line parallel to DE going through point B (We can do this by drawing a line from a point on DE through B. We can then measure the angle created by this new line and DE. We can copy this angle with our compass and straightedge to find a point on the parallel line. Then, by connecting this point with point B, we have our parallel line). Now, since BC and DE are parallel, we know that ABC and ADE are similar triangles. Thus, we know $\frac{|AB|}{|AD|} = \frac{|AC|}{|AE|}$. Substituting in, we see that $\frac{1}{\beta} = \frac{|AC|}{\alpha}$. Thus, we see that $|AC| = \frac{\alpha}{\beta}$. Now, using the compass and straightedge, we can create a segment of length $|AC| = \alpha\beta^{-1}$. Thus, we see $\alpha\beta^{-1}$ is constructible.

So, since we have observed that the set of constructible numbers is a subset of $\mathbb{R}$, $\alpha - \beta$ is constructible, and $\alpha\beta^{-1}$ is constructible, we can conclude that the set of constructible numbers is a subfield of $\mathbb{R}$. □

Defining constructible numbers as well as showing that the set is a subfield of the real numbers will help us in the following sections while proving that certain unmarked straightedge and compass constructions are impossible.

3 Theorems to Prove Impossibility

Before we can prove the impossibility of the constructions, we must prove a few important theorems. These theorems are vital in showing that all of the constructions we will discuss later are impossible.

Theorem 2. *Let F be a field of constructible numbers. Then the points determined by the intersections of lines and circles in F lie in the field $F(\sqrt{\alpha})$ for some $\alpha \in F$. [2]*

Proof. Let us consider a field of constructible numbers F. We know that there are only 3 ways to construct a new point in $\mathbb{R}$ with only an unmarked straightedge and compass. First, we can take the intersection of two lines which each have endpoints with coordinates in F. Next, we can take the intersection of a line with endpoints in F and a circle with a center and radius length in F. Finally, we can take the intersection of two circles whose centers and radii lengths are in F.

The first thing that we will show is that any line with endpoints with coordinates in F can be written in the form $y = mx + b$ where $m, b \in F$. First, let us consider 2 points which have coordinates in F, (x_1, y_1) and (x_2, y_2). Using the point-slope formula, we see $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$. Now, let $m = \frac{y_2 - y_1}{x_2 - x_1}$. By properties of fields, we know $m \in F$. Now, we see $y - y_1 = mx - mx_1$. Rearranging this, we see $y = mx - (mx_1 - y_1)$. Now, let $b = -(mx_1 - y_1)$. Again by properties of fields, we see $b \in F$. Thus, we see $y = mx + b$ for some $m, b \in F$. Thus, we have shown that any line between two points with coordinates in F can be represented as $y = mx + b$ with $m, b \in F$.

Now, consider the intersection of two lines. We know that we must have an equation for each line $y = m_1x + b_1$ and $y = m_2x + b_2$ with $m_1, m_2, b_1, b_2 \in F$. Now, using substitution, we can see $m_1x + b_1 = m_2x + b_2$. Thus, we see $x = \frac{b_2 - b_1}{m_1 - m_2}$. So, by properties of fields, we know $x \in F$. Now, since x is in F, we know by properties of fields that $y \in F$. Thus, we see the coordinates of the intersection of two lines with endpoints with coordinates in F are also in F.

Now, consider the case with the intersection of a line and a circle. This means that we know that the line has some equation $y = mx + b$ with $m, b \in F$ and the circle has some equation $(x-h)^2 + (y-k)^2 = r^2$ for some $h, k, r \in F$. Now, we can plug our linear equation into our equation for the circle. Thus, we see $(x-h)^2 + (mx + b - k)^2 = r^2$. By expanding and grouping, we can see $(1+m^2)x^2 + (-2h + 2bm - 2km)x + (h^2 + b^2 - 2bk + k^2 - r^2) = 0$. Now, let $A = 1 + m^2, B = -2h + 2bm - 2km$, and $C = h^2 + b^2 - 2bk + k^2 - r^2$. So, we see that we have the equation $Ax^2 + Bx + C = 0$. Now, by properties of fields (we know fields are maintained under addition, multiplication, division, and subtraction), we know $A, B, C \in F$. Now, using the quadratic formula, we know the roots of our polynomial are $\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, assuming the polynomial has a real root as the line and circle intersect. Now, clearly $-B$ and $2A$ are elements of F. But, $\sqrt{B^2 - 4AC}$ is not necessarily an element of F. Now, let $\alpha = B^2 - 4AC$. Again, by our properties of fields, we know $\alpha \in F$. Now, we know $\sqrt{\alpha} \in F(\sqrt{\alpha})$. Thus, we see the x coordinate must belong in the field $F(\sqrt{\alpha})$. Now let the x-coordinate of the intersection be some $d \in F(\sqrt{\alpha})$. We see $y = md + b$. Now, since $F \subset F(\sqrt{\alpha})$, $m, b \in F$, and $d \in F(\sqrt{\alpha})$ we know $y \in F(\sqrt{\alpha})$. So, we see any point determined by the intersection of a line (with endpoints have coordinates in F) and a circle (with a center that has coordinates in F and a radius that has length in F) has coordinates in $F(\sqrt{\alpha})$ for some $\alpha \in F$

Finally, let us consider the case with two circles. We will show that the case of two circles can be reduced to the case of a circle and a line. Let us consider two circles with centers that have coordinates in F and radii that have lengths in F. These circles can be defined by the equations $(x-h_1)^2 + (y-k_1)^2 = r_1^2$ and $(x-h_2)^2 + (y-k_2)^2 = r_2^2$ with $h_1, h_2, k_1, k_2, r_1, r_2 \in F$. Now, we can expand our equations to give us $x^2 + y^2 - 2h_1x - 2k_1y + h_1^2 + k_1^2 - r_1^2 = 0$ and $x^2 + y^2 - 2h_2x - 2k_2y + h_2^2 + k_2^2 - r_2^2 = 0$. Now, let $d_1 = -2h_1, d_2 = -2h_2, e_1 = -2k_1, e_2 = -2k_2, f_1 = h_1^2 + k_1^2 - r_1^2$ and $f_2 = h_2^2 + k_2^2 - r_2^2$. From our properties of fields, we see $d_1, d_2, e_1, e_2, f_1, f_2 \in F$. Now, we have the equations $x^2 + y^2 + d_1x + e_1y + f_1 = 0$ and $x^2 + y^2 + d_2x + e_2y + f_2 = 0$. Now, we can set the equations equal to each other to find the intersection of the circles. We see that $x^2 + y^2 + d_1x + e_1y + f_1 = x^2 + y^2 + d_2x + e_2y + f_2$. Now, by rearranging the equation, we see that $(d_2 - d_1)x + (e_2 - e_1)y + (f_2 - f_1) = 0$. Thus, we see the intersections of the two circles belong to the line $(d_2 - d_1)x + (e_2 - e_1)y + (f_2 - f_1) = 0$. So, we see the intersection of the two circles is the same as the intersection of the circle $x^2 + y^2 + d_1x + e_1y + f_1 = 0$ and the line $(d_2 - d_1)x + (e_2 - e_1)y + (f_2 - f_1) = 0$. Thus, we see we can reduce our two circles case into a case with a circle and a line, which we know has an intersection with coordinates in $F(\sqrt{\alpha})$.

Now, since $F \subset F(\sqrt{\alpha})$, we can conclude that the points determined by the intersection of lines and circles in F lie in the field $F(\sqrt{\alpha})$ for some $\alpha \in F$. $\square$

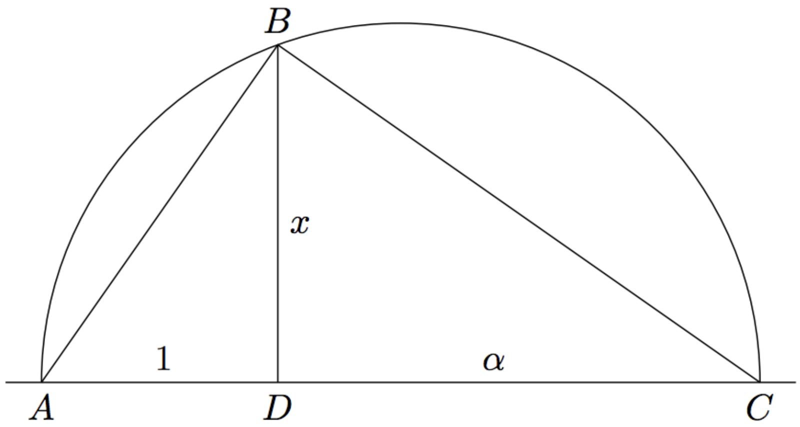

Figure 2: If α is constructible, then so is $\sqrt{\alpha}$. [2]

Now, another valuable property of the set of constructible numbers is that $\sqrt{\alpha}$ is constructible if α is constructible.

Theorem 3. *If α is a constructible number, then $\sqrt{\alpha}$ is a constructible number. [2]*

Proof. Consider Figure 2. This semi-circle can be constructed by extending our line of length α, by 1. Then, we can find the midpoint, O (not pictured), of the new line by using the compass from the endpoints of the line. From O, we can use the compass to form the semicircle with radius $|OC|$. Now, draw a line perpendicular to AC at point D (This can be done by using the compass to find two points on the line equal distance from D and applying the perpendicular bisector construction). Let this line intersect the semicircle at B. Then, using the straightedge, connect B to A and C. Thus, we have formed Figure 2. Now, by properties of geometry, we know $\triangle ABD, \triangle BCD$, and $\triangle ABC$ are similar triangles. So, we know $\frac{1}{x} = \frac{x}{\alpha}$ or $x^2 = \alpha$ or $\alpha = \sqrt{x}$. Thus, we see that if α is constructible, then so is $\sqrt{\alpha}$. $\square$

Now, using theorems 2 and 3, we can characterize exactly which elements of $\mathbb{R}$ are constructible.

Theorem 4. *A real number α is constructible if and only if there exists a sequence of fields*

$$\mathbb{Q} = F_0 \subset F_1 \subset ... \subset F_k$$

such that $F_i = F_{i-1}(\sqrt{\alpha_{i-1}})$ with $\alpha_{i-1} \in F_{i-1}$ and $\alpha \in F_k$. In particular, there exists an integer $k > 0$ such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^k$. [2]

Proof. First, let us assume that α is constructible. This means that we can construct α through some finite number, n, of steps of compass and straightedge constructions. Now, let E_i be the field generated by the lengths of the construction up to step i. So, $E_0 = \mathbb{Q}$. We can see that, by Theorem 2, $E_{i+1} = E_i$ or $E_{i+1} = E_i(\sqrt{\alpha_i})$ where $\alpha_i \in E_i$. Now, let F_i for $0 \leq i \leq k$ be the E_i's where we ignore the steps that do not change the field. We can see that $F_i = F_{i-1}(\sqrt{\alpha_{i-1}})$ for $\alpha_i \in F_i$ and $0 \leq i \leq k$. From this, we can clearly see that $F_0 \subset F_1 \subset ... \subset F_k$. We can also see that $\alpha \in F_k$. This proves the first direction.

Next, let us assume that $\mathbb{Q} = F_0 \subset F_1 \subset ... \subset F_k$ such that $F_i = F_{i-1}(\sqrt{\alpha_{i-1}})$ with $\alpha_{i-1} \in F_{i-1}$ and $\alpha \in F_k$. We will proceed to show α is constructible through induction on k. First, let us take the base case, $F_0 = \mathbb{Q}$. Now, we know that we can construct any value in $\mathbb{Q}$ as we have a segment of length 1, and the set of constructible numbers is a field. Now, let us assume the we can construct any value in F_i. We want to show that we can construct any value in F_{i+1}. Now, we know that $F_{i+1} = F_i(\sqrt{\alpha_i})$ for $\alpha_i \in F_i$. Now, by Theorem 3, since α_i is constructible, we know $\sqrt{\alpha_i}$ is constructible. So, since we can construct any value in F_i, $\sqrt{\alpha}_i$ is constructible, and F_{i+1} is a field, we know that we can construct any value in F_{i+1}. Thus, we see that by induction, we can construct any value in F_i, $0 \leq i \leq k$. Thus, we see that any value of F_k is constructible. Thus, we can conclude that α is constructible. So, since we have shown both ways, we can conclude that α is constructible if and only if there exists a sequence of fields $\mathbb{Q} = F_0 \subset F_1 \subset ... \subset F_k$ such that $F_i = F_{i-1}(\sqrt{\alpha_{i-1}})$ with $\alpha_i \in F_i$ and $\alpha \in F_k$.

Now from this, we can use our knowledge of fields to see that $[F_k : \mathbb{Q}] = [F_k : F_{k-1}][F_{k-1} : F_{k-2}]...[F_1 : \mathbb{Q}]$. We see the minimal polynomial's of $\sqrt{\alpha_i}$ in F_i is $x^2 - \alpha_i = 0$. Thus, we see $[F_{i+1} : F_i] = 2$ for all $0 \leq i < k$. Thus, we see $[F_k : \mathbb{Q}] = 2^k$. So, we can see that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^k$. □

Now that we have proved these theorems, we can proceed to show the impossibility of the constructions.

4 Doubling the Cube

The first unmarked straightedge and compass construction that we will prove is impossible is doubling the cube. The purpose of this construction is to create a cube with a volume double that of a cube you are given by only using an unmarked straightedge and compass. First, let us show that it is not possible to construct a segment of length $\sqrt[3]{2}$.

Theorem 5. *$\sqrt[3]{2}$ is not a constructible number. [2]*

Proof. Let us consider a polynomial in $\mathbb{Q}[x]$ that has $\sqrt[3]{2}$ as a root. We can clearly see that $\sqrt[3]{2}$ is a zero of the polynomial $x^3 - 2 \in \mathbb{Q}[x]$. Now, we will show that this polynomial is irreducible in $\mathbb{Q}[x]$. We can see that there exists a prime number, $p = 2$, such that $p \nmid 1, p \mid -2$, and $p^2 \nmid -2$. So, by Eisenstein's Criterion, we see $x^3 - 2$ is irreducible in $\mathbb{Q}$. Thus, we know it is the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$. Thus, we see that

$[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}] = 3$. But, by Theorem 4, if $\sqrt[3]{2}$ is constructible, there must exist some integer $k > 0$ such that $[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}] = 2^k$. Thus, we see that if $\sqrt[3]{2}$ is constructible, then there must be some $k > 0$ such that $2^k = 3$. Clearly, there is no integer k that satisfies this equation. Thus, we see $\sqrt[3]{2}$ is not constructible. □

Now that we have shown this, we have enough information to conduct our proof.

Theorem 6. *Doubling the cube is an impossible construction. [2]*

Proof. Let us consider a cube with a volume of 1. To perform the Doubling the Cube construction, we would need to construct a cube of volume 2. Now, we know our original cube with a volume of 1 has an edge length of 1. We also know that a cube with a volume of 2 must have edge length of $\sqrt[3]{2}$. But, according to Theorem 5, we know it is not possible to construct a segment of length $\sqrt[3]{2}$. Thus, we see that it is not possible to double the cube. □

5 Squaring the Circle

The next construction the we will show to be impossible is Squaring the Circle. This construction involves constructing a square with an area equal to that of a given circle. Before we can prove the impossibility of this construction, we must observe as special property of $\sqrt{\pi}$.

Theorem 7. *$\sqrt{\pi}$ is transcendental over $\mathbb{Q}$.*

Proof. First, we already know that π is transcendental over $\mathbb{Q}$. We will prove that $\sqrt{\pi}$ is transcendental through contradiction. Assume $\sqrt{\pi}$ is algebraic over $\mathbb{Q}$. This means that we know $[\mathbb{Q}(\sqrt{\pi}):\mathbb{Q}]$ is finite. Now, since we know fields are closed under multiplication, we know $\pi \in \mathbb{Q}(\sqrt{\pi})$. Thus, we see that $\mathbb{Q}(\pi) \subseteq \mathbb{Q}(\sqrt{\pi})$. Now, from our knowledge of the properties of fields, we see that $[\mathbb{Q}(\sqrt{\pi}):\mathbb{Q}] = [\mathbb{Q}(\sqrt{\pi}):\mathbb{Q}(\pi)][\mathbb{Q}(\pi):\mathbb{Q}]$. Thus, we see $[\mathbb{Q}(\pi):\mathbb{Q}]$ is finite. This means that π must be algebraic over $\mathbb{Q}$, a contradiction. Thus, we can conclude that $\sqrt{\pi}$ is transcendental over $\mathbb{Q}$. □

Now that we have shown that $\sqrt{\pi}$ is transcendental in $\mathbb{Q}$, we can show that squaring the circle is impossible.

Theorem 8. *Squaring the circle is an impossible construction. [2]*

Proof. Let us consider a circle with a radius of length 1. From our knowledge of geometry, we know that the area of this circle is π. To construct a square with the same area as the circle, we would need to construct a segment of length $\sqrt{\pi}$. Since $\sqrt{\pi}$ is transcendental over $\mathbb{Q}$, we know that $[\mathbb{Q}(\sqrt{\pi}):\mathbb{Q}]$ is not finite. Thus, we know that there does not exits any integer $k > 0$ such that $2^k = [\mathbb{Q}(\sqrt{\pi}):\mathbb{Q}]$. Thus, by Theorem 4, we see $\sqrt{\pi}$ is not a constructible number. So, we see that it is impossible to construct a square with length $\sqrt{\pi}$. Thus, we see that it is impossible to square the circle. □

6 Trisecting the Angle

Next, we will show that it is impossible to trisect an angle given only an unmarked straightedge and compass. To do this, we must find only one constructible angle that we can not trisect using an unmarked straightedge and compass. We will show that it is not possible to trisect a $60°$ into three $20°$ angles, as a $20°$ angle is not constructible. Before we do this, let us observe an important property of constructible angles. This will help us when we prove that it is impossible to construct a $20°$ angle.

Theorem 9. *θ is constructible if and only if $\cos\theta$ is constructible.*

Proof. First, let us assume that we have some angle θ that is know to be constructible. This means that we have 2 lines such that the angle between them is θ. Now, we can draw a segment perpendicular to one of the lines by taking two points on the line, and using a compass to find two points on the perpendicular line. We can extend the lines until we have created a right triangle in which one of the angles is θ. Now, we see that each side of the triangle is constructible. Using this, our knowledge that $\cos\theta$ is the length of the leg adjacent to θ divided by the hypotenuse of the triangle, and the result we proved in Section 2 ($\alpha\beta^{-1}$ is constructible if α and β are constructible), we can see $\cos\theta$ is constructible.

Next, let us assume that $\cos\theta$ is constructible. This means we have some segment of length $\cos\theta$. Now, using our straightedge and compass construct a segment perpendicular to our starting line. Now, using the length 1 we are given in all constructions, use the compass to join the other endpoint of our original segment with the perpendicular segment to create a new segment of length 1. Thus, we have created a right triangle in which one of the legs is length $\cos\theta$ and the hypotenuse is length 1. So, the angle between our original leg and the hypotenuse is θ. Thus, we see θ is constructible.

So, since we have shown both ways, we can conclude that θ is constructible if and only if $\cos\theta$ is constructible. $\square$

Now that we have observed this property of constructible angles, we can begin our proof to show that a $20°$ angle is not constructible.

Theorem 10. *It is impossible to construct a $20°$ angle. [2]*

Proof. Let us consider the triple angle formula for cosine. From our knowledge of trigonometric identities, we see that $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta = (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta = (2\cos^2\theta - 1)\cos\theta - 2(1-\cos^2\theta)\cos\theta = 4\cos^3\theta - 3\cos\theta$. So, we see that $4\cos^3\theta - 3\cos\theta = cos3\theta$. Now, let us take α to be $\cos 20$. When we plug 20 in for θ, we see $4\alpha^3 - \alpha = \frac{1}{2}$. From this, we can see that α is a zero of the polynomial $f(x) = 8x^3 - 6x - 1$. Now, we can use the Rational Root Theorem to check if the polynomial has any zeros. The Rational Root Theorem states that possible rational roots of $f(x) \in \mathbb{Z}[x]$ must be of the form $\pm\frac{a}{b}$, where a must divide the the constant term of $f(x)$ and b must divide the coefficient of the largest power of x in $f(x)$. From the Rational Root Theorem, we know the possible roots of $f(x) = 8x^3 - 6x - 1$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$, and $\pm\frac{1}{8}$. So, by plugging in, we see $f(1) = 1, f(-1) = -3, f(\frac{1}{2}) = -3, f(\frac{-1}{2}) = 1, f(\frac{1}{4}) = \frac{-19}{8}, f(\frac{-1}{4}) = \frac{3}{8}, (\frac{1}{8}) = \frac{-111}{64}$, and $f(\frac{-1}{8}) = \frac{-17}{64}$. Thus, we see $f(x)$ has no rational roots. So, we see that $8x^3 - 6x - 1$ is irreducible over $\mathbb{Q}$. Thus, we see it is the minimal polynomial. So, we see that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. But as we saw in the doubling the cube section, we know there does not exist an integer $k > 0$ such that $2^k = 3$. Thus, we see α is not constructible. Now, since we know θ can be constructed if and only if $\cos\theta$ is constructible (Theorem 9) and α is not constructible, we can conclude that a $20°$ angle can not be constructed. □

Theorem 11. *Trisecting the angle is an impossible construction. [2]*

Proof. We know, by Theorem 9, that, since $\cos 60 = 1/2 \in \mathbb{Q}$, a $60°$ angle is constructible. It follows from Theorem 10 that we can not trisect a $60°$ angle. Thus, trisecting the angle is an impossible construction. □

7 Constructing a Regular 7-gon

Finally, the last construction we will prove to be impossible with only an unmarked straightedge and compass is the construction of a regular 7-gon. Now, it is important to note that, to show any regular n-gon is not constructible, we can show that its exterior angle is not constructible. We can do this because, if we can consturct a regular n-gon, we could easily construct the exterior angle of the n-gon by extending one of the sides.

Theorem 12. *It is impossible to construct a $2\pi/7$ angle.*

Proof. We will start with $8\cos^3(2\pi/7) + 4\cos^2(2\pi/7) - 4\cos(2\pi/7) - 1 = 0$. This equation can be derived through the use of trigonometric identities which we will not discuss in this paper[1]. Now, let $\alpha = \cos(2\pi/7)$. We see that α is a zero of the polynomial $f(x) = 8x^3 + 4x^2 - 4x - 1$. Now, we can use the Rational Root Theorem to check if $f(x)$ is irreducible. By the Rational Root Theorem, we see the possible possible roots of $f(x)$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$, and $\pm\frac{1}{8}$. Now, we see that $f(1) = 7, f(-1) = -1, f(1/2) = -1, f(-1/2) = 1, f(1/4) = -13/8, f(-1/4) = 1/8, f(1/8) = -91/64$, and $f(-1/8) = -29/64$. Thus, we see $f(x)$ has no rational roots. So, since $f(x)$ is a third degree polynomial, we see $f(x)$ is irreducible in $\mathbb{Q}$. Thus, $f(x)$ is the minimal polynomial of α over $\mathbb{Q}$. So, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. But, according to Theorem 4, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^k$ for some integer $k > 0$. Thus, we see that $3 = 2^k$ for some integer $k > 0$. There clearly does not exist an integer k which satisfies this equation. Thus, we have a contradiction. Thus, we see α is not constructible. Now, by Theorem 9, we know the angle of $2\pi/7$ is also not constructible. □

Theorem 13. *It is impossible to construct a regular 7-gon.*

Proof. From our knowledge of geometry, we know that the exterior angle of a regular n-gon is $2\pi/n$. This is due to the fact that interior angles of a regular n-gon must all be equal and sum to $\pi(n-2)$. So, each interior angle of an n-gon is $\frac{\pi(n-2)}{n}$. Thus, the exterior angle is $\pi - \frac{\pi(n-2)}{n} = \frac{2\pi}{n}$. Thus, we see the exterior angle for a regular 7-gon must be $2\pi/7$. Since $2\pi/7$ is the exterior angle of a regular 7-gon, it follows from Theorem 12 that it is impossible to construct a regular 7-gon. $\square$

8 Conclusion

Throughout this paper, we discussed the set of constructible numbers as well as proved how certain unmarked straightedge and compass constructions can be proven to be impossible. We first defined what a constructible number is and then proved that the set of constructible numbers is a subfield of the set of real numbers. Then, we characterized exactly which numbers can be constructed. Next, we explored the doubling the cube construction. We found this to be impossible as $\sqrt[3]{2}$ is not a constructible number. Then, we explored the squaring a circle construction. We showed that this construction is impossible as $\sqrt{\pi}$ is not constructible as it is transcendental over $\mathbb{Q}$. We then explored the trisecting an angle construction. We showed that it is impossible by showing that we cannot trisect a 60° angle, as a 20° angle is not constructible. Finally, we explored the construction of a regular 7-gon. We proved that this was not possible by showing that it is impossible to construct its exterior angle, $2\pi/7$. Overall, we saw how field theory can be applied to show the impossibility of various geometric constructions.

References

[1] Joseph A. Gallian. *Contemporary Abstract Algebra*. Cengage Learning, Boston, Massachusetts, 2013.

[2] Thomas W. Judson *Abstract Algebra: Theory and Applications* Stephen F. Austin State University, Nacogdoches, Texas, 2019

[3] John B. Fraleigh *A First Course in Abstract Algebra* Pearson Education, Inc.