Stevens Institute of Technology

Pricing and Hedging - Final Project

American Options on SPY

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1 Introduction

We will study the pricing of American options on SPY as of October 28, 2022. We will price the American options using a binomial tree implementation. The binomial asset pricing model (BAPM) is an asset valuation method that uses an iterative procedure, allowing for the points in time, during the time span between the valuation date and the option's expiration date. With the model, there are two possible outcomes with each iteration - a move up or a move down that follow a binomial tree. A binomial pricing tree has the parameters S_0, which represents the initial stock price, u, which is the up factor, d, which is the down factor, r which is the risk-free interest rate, and T, which is the time to maturity. The model reduces possibilities of price changes and removes the possibility for arbitrage.

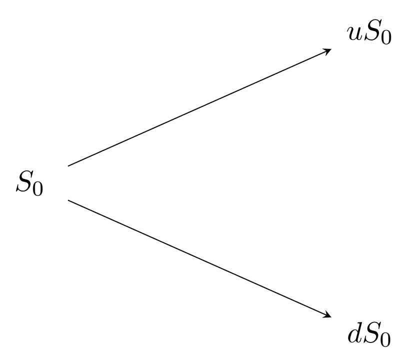
To prevent arbitrage, we enforce

$$0 < d < 1 + r < u$$

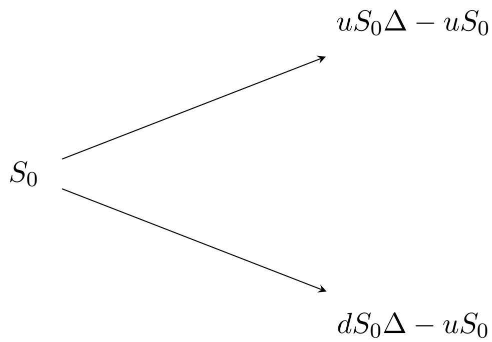
If this wasn't true, then we have the following two cases:

- $d \geq 1 + r$, then one can begin with zero wealth and borrow money to buy the stock at time zero. At time 1, the trader then will owe $(1 + r)S_0 \leq dS_0$, so the debt is paid even if the stock price goes down, but a profit is made if the stock price goes up.
- $u \leq 1 + r$, then one can short the stock and invest the money into the money market. The trader will earn $S_0(1 + r) \geq uS_0$, so the trader can then buy the stock to cover the short and if the stock price goes up, the debt is still paid.

A derivative lasts for time T and is dependent on a stock, S_0. A one-step binomial tree ($T = 1$) is demonstrated below.



The value of a portfolio that is long Δ shares and short 1 derivative is shown in a binomial tree below.



The portfolio is risk-less when $S_0u\Delta - uS_0 = S_0d\Delta - dS_0$ and solving for Δ yields,

$$\Delta = \frac{uS_0 - dS_0}{S_0u - S_0d}$$

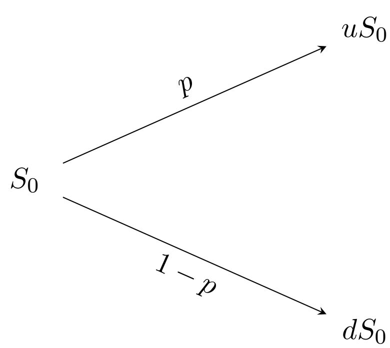
The value of a portfolio at maturity is $S_0u\Delta - uS_0$ and the value of a portfolio today is $(S_0u\Delta - uS_0)e^{-rT}$. We can also express the value of a portfolio today as $S_0\Delta - S_0$. Therefore,

$$S_0 = S_0\Delta - (S_0u\Delta - uS_0)e^{-rT}$$

If we substitute Δ, we then obtain

$$S_0 = [p \cdot uS_0 + (1-p) \cdot dS_0]e^{-rT} \quad \text{where } p = \frac{e^{rT} - d}{u - d}$$

We can interpret p and $1-p$ as probabilities of up and down movements. The value of a derivative is then the expected payoff in a risk-neutral world discounted at the risk-free rate.



When the probability of an up and down movement are p and $1-p$, then the expected stock price at time T is S_0e^{rT}. This shows that the stock price earns the risk-free rate. Binomial trees illustrate the general result that to value a derivative, we can assume the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate. This is known as using risk-neutral valuation.

We also need to consider the stock price volatility, σ when calculating the up and down factors. The stock price volatility is defined such that the standard deviation of the stock price over time T is $\sigma\sqrt{T}$ and the variance of the stock price is $\sigma^2 T$. When choosing u and d, we match the volatility to set

$$u = e^{rh+\sigma\sqrt{h}}$$
$$d = e^{rh-\sigma\sqrt{h}}$$

where σ is the volatility and h is the length (in years) of a period in the binomial tree.

Our research will focus on American-style options. American-style options allow the holder to exercise an option contract at any time before the expiry. European options, on the hand, can only be exercised at the expiry date. This means that for any given situation, American options demand a higher price than European options because of their greater flexibility.

As mentioned above, the binomial pricing tree has the following parameters: S_0, u, d, r, and T. Pricing the options in the binomial tree model requires that we determine/estimate these parameters. In addition, to calculate u and d, we must estimate σ.

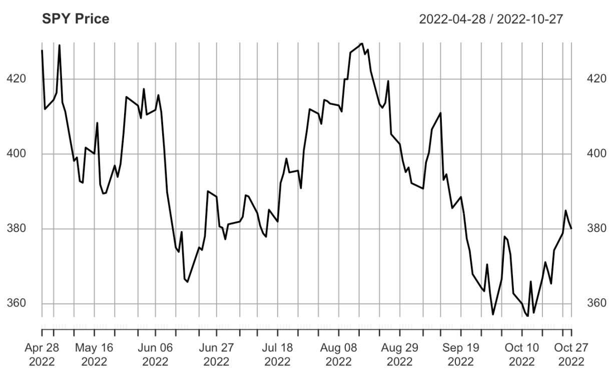
- Spot Price S_0. On October 28, 2022, the SPY closed at $S_0 = 394.03$.
- Volatility σ. We estimate volatility from the historical stock price data. This gives the historical volatility.
- Risk-free rate r. We estimate the risk-free rate from the three-month treasury bill.

2 Market Data Analysis

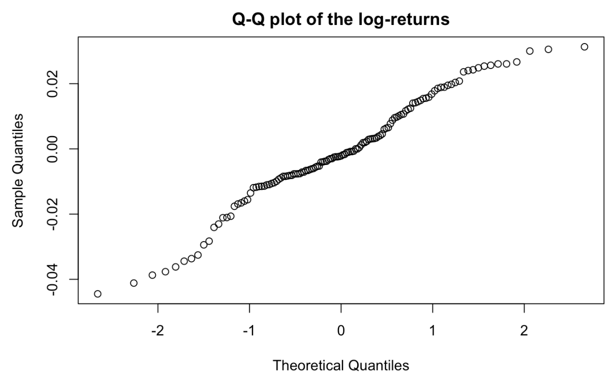
We download the market data for daily prices of SPY from Bloomberg. The R code is shown in the Appendix.

2.1 Time Series of SPY

The plot in Figure (a) shows the closing price of SPY over the past 6 months, ending at October 28, 2022. Once we obtain the daily closing prices, we compute the daily log returns, where $u_i = \ln \frac{S_i}{S_{i-1}}$, where S_i is the daily closing price at day i. We then examine the QQ plot of log returns. A straight line for the QQ plot indicates a normal distribution. We can perform a Jarque-Bera test and verify that the log returns are normally distributed.



(a) Historical Closing Price of SPY



(b) Q-Q Plot of Log Returns

Jarque-Bera Test		
X-Squared	df	P-Value
1.7168	2	0.4238

From the Jarque-Bera test, we notice that our p-value is not statistically significant. Therefore, we can conclude that the log returns are normally distributed.

2.2 Estimation of Volatility and Risk Free Rate

Volatility. We use the daily closing prices of SPY to calculate the volatility. We estimate the volatility as

$$\hat{\sigma} = \sqrt{252} \cdot \text{std}(u_i)$$

Our data dates back to May 18, 2022, which is 6 months. Thus, $\hat{\sigma} = 0.2619225$.

Risk Free Rate. The interest rate on a three-month U.S. Treasury bill (T-bill) is often used as the risk-free rate for U.S.-based investors. Thus we obtain the three-month U.S. Treasury bill values from July 28 to October 28, to proxy this rate.

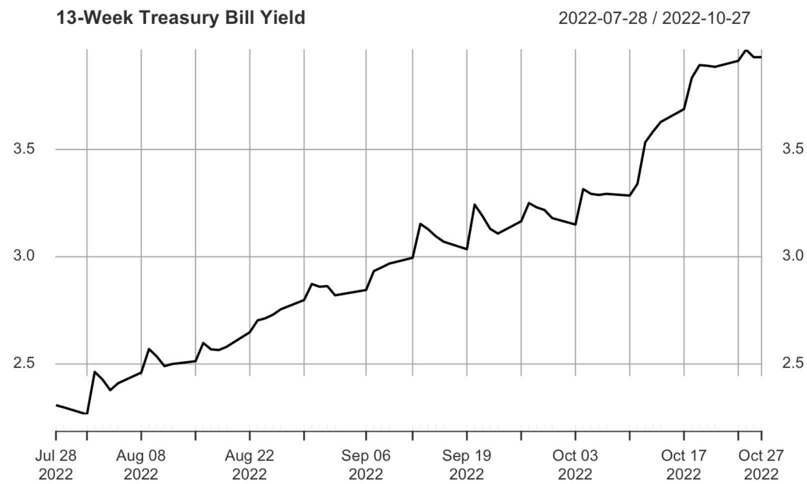


Figure 2: Treasury Bill from July 28 to October 28, 2022

From Figure 2, the closing value of the three-month Treasury bill on October 28, 2022 is 3.97. We can estimate the risk-free rate to be

$$r = 3.97\%$$

We will use this as the risk-free rate in our pricing of the American options for SPY.

3 Binomial Tree Pricing Model

For our binomial tree model, we calibrate our model using the parameters obtained in Section 2. We first gather the test data from Bloomberg for the options. We will use this data to compare our modeled results and see how accurate our model is.

Strike	18-Nov	16-Dec	30-Dec	20-Jan	17-Feb
382	14.45	19.32	20.70	22.88	27.14
383	14.01	18.51	19.68	22.23	26.54
384	13.22	18.25	19.40	21.74	25.89
385	12.71	17.44	18.74	21.39	25.28
386	11.89	16.67	17.97	20.75	24.69
387	11.34	16.43	17.37	19.06	24.11

Table 1: Data for Call Options on SPY

Strike	18-Nov	16-Dec	30-Dec	20-Jan	17-Feb
382	6.38	10.94	12.28	14.35	18.16
383	6.70	11.44	12.87	14.73	18.78
384	7.04	12.04	13.14	15.00	17.50
385	7.46	12.23	13.60	15.32	17.83
386	7.83	12.53	14.06	15.84	18.41
387	8.28	12.92	14.37	16.39	18.96

Table 2: Data for Put Options on SPY

We will use the Strike prices from Bloomberg in our model and the spot price of $S_0 = 394.03$ when pricing the call and put options for SPY. Our results from our own model are shown in Table 3.

Strike	Modeled Call Option	Modeled Put Option
382	14.300	6.415
383	13.696	6.811
384	13.092	7.207
385	12.489	7.609
386	11.963	8.082
387	11.437	8.556
388	10.911	9.031
389	10.385	9.507
390	9.864	9.992
391	9.418	10.546
392	8.971	11.100
393	8.525	11.656
394	8.079	12.212
395	7.636	12.777
396	7.266	13.409

Table 3: Results from Model for Call and Put Options Expiring November 18, 2022

From a glance, we can see that our model has a strong predictive power. This can be attributed to the fact that we used $n = 100$ steps, opposed to fewer steps. We know that the more number of steps leads to a higher accuracy when predicting the options call and put prices. We see that our model has some issues that we discuss later on. We proceed to compute the Greeks for our options to measure different factors, which could potentially affect the price of an options contract, which the binomial tree pricing model does not account for.

4 Greeks

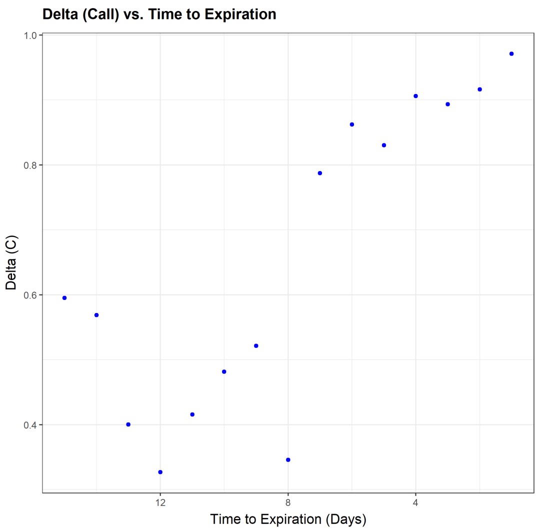
The Greeks refer to a set of calculations you can use to measure different factors that might affect the price of an options contract. With this information, an investor can make more informed decisions about which options to trade, and when to trade them. There are 3 Greeks that we will research in our study; however, we will describe in total 5 Greeks.

- **Delta (Δ)** - helps gauge the likelihood an option will expire in-the-money, meaning its strike price is below (for calls) or above (for puts) the underlying security's market price.
- **Gamma (Γ)** - helps estimate how much the Delta might change if the stock price changes.
- **Theta (Θ)** - helps measure how much value an option might lose each day as it approaches expiration.
- **Vega (υ)** - helps understand how sensitive an option might be to large price swings in the underlying stock.
- **Rho (ρ)** - helps simulate the effect of interest rate changes on an option.

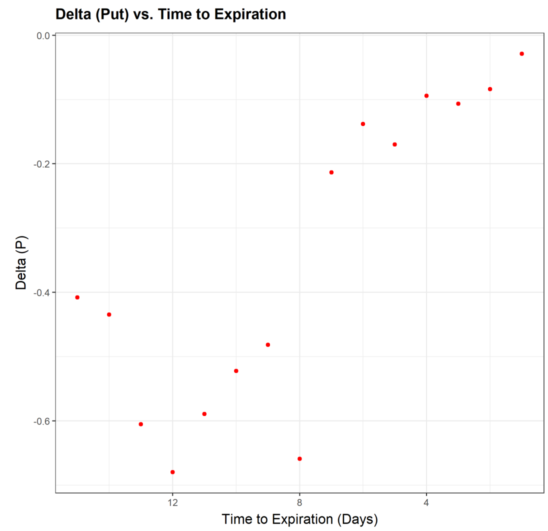
We will explore these Greeks in more detail below.

4.1 Delta

Delta measures how much an option's price can be expected to move for every \$1 change in the price of the underlying security or index. A Delta of 0.50 means that the option's price will in theory move \$0.50 for every \$1 change in the price of the underlying stock or index. The higher the Delta is, then the bigger the price change. Delta is often used to predict whether a given option will expire in-the-money. A Delta of 0.50 means that at that given moment in time, the option has about a 50% chance of being in-the-money at expiration.



(a) Delta Calls for SPY



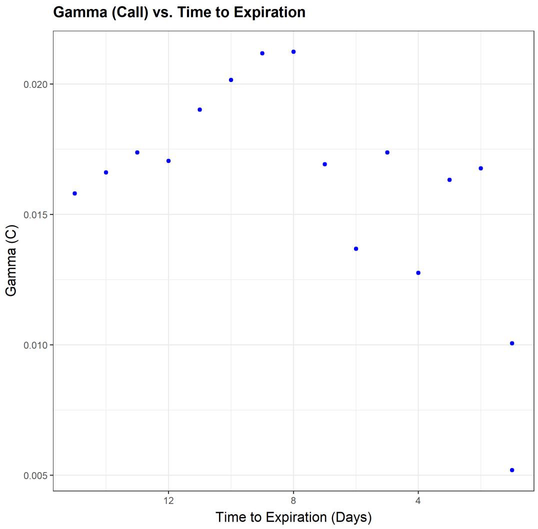
(b) Delta Puts for SPY

Figure 3: Plots for Delta

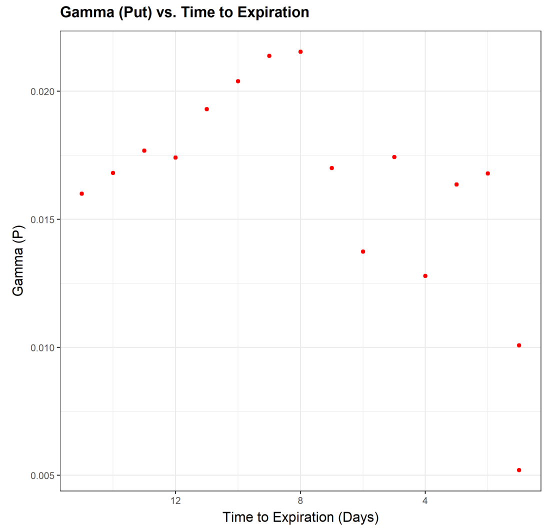
We can see that Delta approaches 1 for call options and Delta approaches 0 for put options. The Delta of ITM call options will get closer to 1.00 as expiration approaches. This agrees with our graph for the call option. The Delta of out-of-the-money put options will get closer to 0.00 as expiration approaches. This also agrees with our graph for the put option.

4.2 Gamma

Gamma measures the rate of change in an option's Delta over time. Gamma is the rate of change in an option's Delta per $1 change in the price of the underlying stock. From the example above, imagine an option with a Delta of 0.50. If the underlying stock moves $1 and the option moves $0.50 along with it, thus the option's Delta is no longer 0.50. This $1 move means the call option is now even deeper in-the-money, so the Delta should become closer to 1.00. We can assume the resulting Delta is now 0.65. This change in Delta from 0.50 to 0.65 is 0.15—this is the option's Gamma.



(a) Gamma Calls for SPY



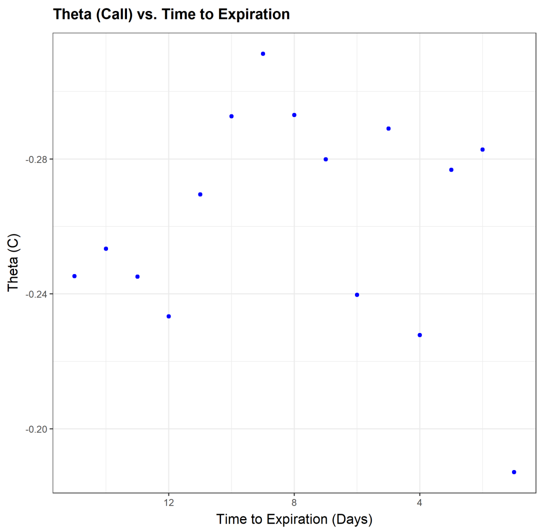
(b) Gamma Puts for SPY

Figure 4: Plots for Gamma

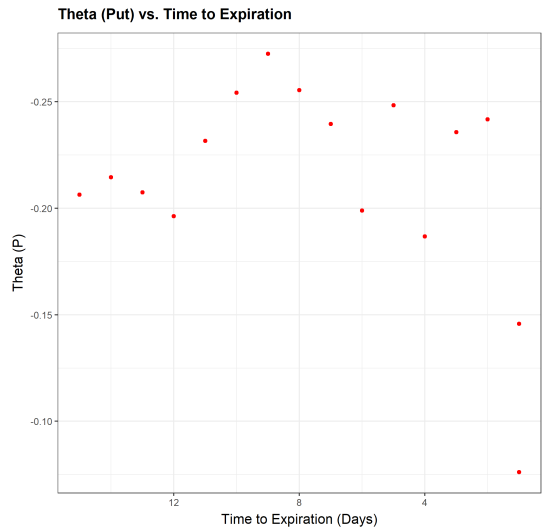
We can see that Gamma decreases as an option gets further ITM (call) or out of the money (put) and Delta approaches 1 (call) or 0 (put). This agrees with our graphs for both calls and puts because as the time to expiration nears, Gamma begins to decrease. The visualizations for Gamma for both calls and puts should be identical since Gamma measures the rate of change in an option's Delta over time, and based on the graphs for the Deltas, the Gammas make sense.

4.3 Theta

Theta measures how much the price of an option should decrease each day as the option nears expiration, if all other factors remain constant. This is known as time decay. Time-value erosion is non-linear, which implies that price erosion of at-the-money and in-the-money options tend to increase as expiration approaches, whereas out-of-the-money options tend to decrease as expiration approaches.



(a) Theta Calls for SPY



(b) Theta Puts for SPY

Figure 5: Plots for Theta

We can see that the graphs for Theta are not exactly the same for both calls and puts, but follow the same trend. As time to expiration nears, we see that Theta is slowly approaching 0.

4.4 Vega

Vega measures the rate of change in an option's price per one-percentage-point change in the implied volatility of the underlying stock. Implied volatility is a forecast of how volatile an underlying stock is expected to be in the future—but it is theoretical. Although we can forecast a stock's future moves by looking at historical volatility, implied volatility takes into account the upcoming earnings reports, pending product launches, and merger and acquisition rumors. Vega tells an investor how much an option's price should move when the volatility of the underlying security or index increases or decreases.

4.5 Rho

Rho measures the expected change in an option's price per one-percentage-point change in interest rates. It tells an investor how much the price of an option should rise or fall if the risk-free interest rate (in our case, the three-month U.S. Treasury-bills) increases or decreases. As interest rates increase, the value of call options will generally increase. As interest rates increase, the value of put options will usually decrease. Due to this, call options have positive Rho and put options have negative Rho. Rho is generally not a huge factor in the price of an option, but should be considered if prevailing interest rates are expected to change.

5 Delta Hedging

Delta hedging is an options trading strategy that aims to reduce, or hedge, the directional risk associated with price movements in the underlying asset. The approach uses options to offset the risk to either a single other option holding or an entire portfolio of holdings. The investor tries to reach a delta neutral state and not have a directional bias on the hedge.

By reducing directional risk, delta hedging can isolate volatility changes for an options trader. It can also protect profits from an option or stock position without unwinding the long-term holding.

Date	SPY Closing Price
2022-10-31	386.21
2022-11-01	384.52
2022-11-02	374.87
2022-11-03	371.01
2022-11-04	376.35
2022-11-07	379.95
2022-11-08	382.00
2022-11-09	374.13
2022-11-10	394.69
2022-11-11	398.51
2022-11-14	395.12
2022-11-15	398.49
2022-11-16	395.45
2022-11-17	394.24
2022-11-18	390.80

Table 4: Closing Prices of SPY From Oct 31-2022 to Nov-18-2022

From section 5, we computed the Greeks from the Binomial Tree Model. We use the Deltas to construct a dynamically hedged portfolio: for one put option, we purchase Δ shares of stock. We denote the price of the price of the option + stock hedged portfolio $D(t)$.

Daily price change of portfolio is as follows:

$$D(t) - D(t-1) = P(t) - P(t-1) + \Delta(t-1)(S(t) - S(t-1))$$

Maturity	Spot Price	Put Price	Δ	$P(t) - P(t-1)$	$D(t) - D(t-1)$
14/252	386.21	15.126	-0.6334		
13/252	384.52	15.945	-0.6662	0.8196	0.1082
12/252	374.87	22.878	-0.8151	6.9332	3.8923
11/252	371.01	25.959	-0.8722	3.0808	2.7380
10/252	376.35	21.263	-0.8177	-4.6961	-2.0340
9/252	379.95	18.145	-0.7777	-3.1181	-1.0820
8/252	382.00	16.307	-0.7565	-1.8377	1.0820
7/252	374.13	22.550	-0.8897	6.2426	6.1020
6/252	394.69	7.366	-0.5142	-15.1841	-4.5320
5/252	398.51	5.077	-0.4224	-2.2890	-1.0280
4/252	395.12	6.132	-0.5103	1.0559	0.8590
3/252	398.49	3.974	-0.4119	-2.1587	-1.2934
2/252	395.45	4.707	-0.5087	0.7331	0.2934
1/252	394.24	4.560	-0.5675	-0.1469	-0.1234
	390.80	5.935	-0.7861	1.3752	1.5920

Table 5: Hedging strategy for the put options on SPY with expiry Nov-18 2022 and strike K = 396. The last two columns show the daily price change of the option (unhedged) and the option hedged with Δ shares of stock.

The hedge is not perfect because of Theta (time change Greek) and Gamma contributions, and also because of time discretization errors (Delta hedging is perfect only when performed continuously in time).

6 Results and Discussion

We can construct a table similar to Table 3 in Section 3 to effectively compare our modeled options prices and the true options prices.

Strike Price	Call Option	Modeled Call	Put Option	Modeled Put	Call Difference	Put Difference	Call Relative Error	Put Relative Error
382	14.45	14.300	6.38	6.415	0.150	-0.035	0.010	0.006
383	14.01	13.696	6.70	6.811	0.314	-0.111	0.022	0.017
384	13.22	13.092	7.04	7.207	0.128	-0.167	0.010	0.024
385	12.71	12.489	7.46	7.609	0.221	-0.149	0.017	0.020
386	11.89	11.963	7.83	8.082	-0.073	-0.252	0.006	0.032
387	11.34	11.437	8.28	8.556	-0.097	-0.276	0.009	0.033
388	10.85	10.911	8.69	9.031	-0.061	-0.341	0.006	0.039
389	10.38	10.385	9.18	9.507	-0.005	-0.327	0.001	0.036
390	9.86	9.864	9.59	9.992	-0.004	-0.402	0.000	0.042
391	9.27	9.418	10.05	10.546	-0.148	-0.496	0.016	0.049
392	8.84	8.971	10.80	11.100	-0.131	-0.3	0.015	0.028
393	8.30	8.525	11.44	11.656	-0.225	-0.216	0.027	0.019
394	7.83	8.079	11.89	12.212	-0.249	-0.322	0.032	0.027
395	7.42	7.636	12.37	12.777	-0.216	-0.407	0.029	0.033
396	6.92	7.266	13.04	13.409	-0.346	-0.369	0.050	0.028

Table 6: Results from Model for Call and Put Options Expiring November 18, 2022

From the results we see that our model was able to predict the calls with a much higher accuracy as opposed to the puts. We also notice that as we near maturity, the relative errors for calls increases, whereas for puts it stays constant. At the middle, we see our model for calls is at the best, where we practically have no error between the model and the true call prices. The opposite can be said for puts as the difference between the model and the true put prices are the greatest in the middle. The average call relative error is 1.666% and the average put relative error is 2.833%.

7 Additional Topics

If there is extra time one could explore further along different directions.

- We would like to verify the relationship between the other types of Greeks (Vega and Rho) against the days to maturity.
- Improve the tree pricing using European options as a control variate. We would use the Black Scholes Model, where the formula is shown below for d_1, d_2, c, p, and then find a potential arbitrage strategy from the put-call parity. We would then compute the Greeks from the Black Scholes model using the formulas from the table below.

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$
$$c = S_0 \cdot N(d_1) - Ke^{-r \cdot T} \cdot N(d_2)$$
$$p = Ke^{-r \cdot T} \cdot N(-d_2) - S_0 \cdot N(-d_1)$$
$$c + Ke^{-r \cdot T} = p + S_0$$

		Calls	Puts
Delta	$\frac{\partial V}{\partial S}$	$N(d_1)$	$-N(-d_1) = N(d_1) - 1$
Gamma	$\frac{\partial^2 V}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$
Theta	$\frac{\partial V}{\partial t}$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$
Vega	$\frac{\partial V}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	$SN'(d_1)\sqrt{T-t}$
Rho	$\frac{\partial V}{\partial r}$	$K(T-t)e^{-r(T-t)}N(d_2)$	$-K(T-t)e^{-r(T-t)}N(-d_2)$

Table 7: Greeks for Black Scholes Model

- Use improved models such as Trinomial Pricing Tree to get more accurate results.